Boundary Integral Methods for Boundary Value Problems on Lipschitz Domains
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Contents

Chapter 1

Introduction

In the first part of this lecture we study the layer potential methods to solve the classical Dirichlet and Neumann problems developed in last 30 years.

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded Lipschitz domain with a connected boundary. A domain is called a Lipschitz domain if its boundary is locally given by a Lipschitz curve. We consider the classical boundary value problems, Dirichlet and Neumann problems:

$$DP[f] : \begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

and

$$NP[g] : \begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega, \quad \displaystyle\int_{\partial\Omega} u = 0. \end{cases}$$

Lax-Milgram Theorem guarantee the existence of unique solutions $u \in H^1(\Omega)$ for the Dirichlet problem $DP[f]$ with data $f \in H^{1/2}(\partial\Omega)$ and the Neumann problem $NP[g]$ with data $g \in H^{-1/2}(\partial\Omega)$, respectively.

To find the explicit solution of the boundary value problems, we will write down the solution in integral forms. To this end, it is necessary to introduce the *fundamental solution* of the Laplace's equation: for $x \in \mathbb{R}^d$, $x \neq 0$,

$$\Gamma(x) := \begin{cases} \dfrac{1}{2\pi} \log |x| & d = 2, \\ \dfrac{1}{(2-d)\omega_d} |x|^{2-d} & d \geq 3 \end{cases} \tag{1.1}$$

where ω_d is the surface area of the $d-1$ dimensional unit sphere. Then $-\Delta\Gamma(x) = \delta(x)$ in the distributional sense where δ is the Dirac delta function. The *double layer potential* and the *single layer potential* with density

g on Ω is defined to be:

$$\mathcal{S}_\Omega g(x) := \int_{\partial\Omega} \Gamma(x-y)g(y)d\sigma_y, \qquad x \in \mathbb{R}^n, \tag{1.2}$$

$$\mathcal{D}_\Omega g(x) := \int_{\partial\Omega} \langle \nu_y, \nabla_y \Gamma(x-y)\rangle f(y)d\sigma_y, \qquad x \in \mathbb{R}^n \setminus \partial\Omega \tag{1.3}$$

where ν_y is the outer unit normal vector to $\partial\Omega$ at $y \in \partial\Omega$. By the property of the fundamental solution Γ,

$$\mathcal{D}_\Omega f \text{ and } \mathcal{S} g \text{ are harmonic in } \mathbb{R}^n \setminus \partial\Omega.$$

Therefore to solve $DP[f]$ it suffices to solve the following integral equation

$$\text{Find } \phi \in L^2(\partial\Omega) \text{ so that } \mathcal{D}_\Omega \phi|_{\partial\Omega} = f \text{ on } \partial\Omega. \tag{1.4}$$

This simple question involves a great deal of hard analysis and it is the purpose of this note to explain the theory to solve (1.4).

Chapter 2

Boundary Value Problem on C^2-Domain

2.1 Layer Potentials on C^2-domain.

Let Ω be a C^2-domain. The main advantage of the C^2 case over the Lipschitz case in dealing with Dirichlet or Neumann problems is the following fact; If Ω is a C^2-domain, then

$$\langle x - y, \nu_y \rangle = O(|x - y|^2) \quad \forall x, y \in \partial\Omega, \tag{2.1}$$

and hence

$$\left| \frac{\partial}{\partial \nu_y} \Gamma(x, y) \right| + \left| \frac{\partial}{\partial \nu_x} \Gamma(x, y) \right| \leq \frac{C}{|x - y|^{d-2}}. \tag{2.2}$$

Since $\partial\Omega$ is a manifold of dimension $d - 1$, it thus follows that

$$\int_{\partial\Omega} \left| \frac{\partial}{\partial \nu_y} \Gamma(x, y) \right| d\sigma(y) \leq C \tag{2.3}$$

independently of $x \in \partial\Omega$. This makes the theory for C^2-domains much easier than that for C^1 or Lipschitz domains. You may notice that if the given domain has $C^{1,\alpha}$ boundary for some $\alpha > 0$, then (2.2) holds with the power $d - 2$ in the denominator of RHS replaced with $d - 1 + \alpha$. So what will be said in this chapter is true even if the domain is $C^{1,\alpha}$. But we will continue to assume that the domain is C^2 for simplicity.

To see (2.1), we may assume, after rotation and translation if necessary, that $y = 0$ and near 0 $(x', x_d) \in \Omega$ is given by $x_d > \varphi(x')$, where φ is a defining function for Ω near 0 such that $\varphi(0) = 0$ and $\nabla\varphi(0) = 0$. Then $\nu_0 = (0, -1)$ and it is easy to see (2.1). We make note of

$$\frac{\partial}{\partial \nu_y} \Gamma(x - y) = \frac{1}{\omega_d} \frac{\langle y - x, \nu_d \rangle}{|x - y|^d}, \quad x, y \in \partial\Omega.$$

Define the boundary integral operator $\mathcal{K}_\Omega$ by

$$\mathcal{K}_\Omega f(x) = \frac{1}{\omega_d} \int_{\partial\Omega} \frac{\langle y - x, \nu_y \rangle}{|x-y|^d} f(y) d\sigma_y, \quad x \in \partial\Omega.$$

Let us fix notations: for a function defined in $\mathbb{R}^d \setminus \partial\Omega$, set

$$u|_\pm(x) := \lim_{t \to +0} u(x + t\nu_x), \quad x \in \partial\Omega,$$

when the limit exists. So the subscript + and − denote the approach from outside and inside Ω, respectively.

Theorem 2.1 *Let $f \in C(\partial\Omega)$. Then*

$$\mathcal{D}_\Omega f|_\pm(P) = (\mp \frac{1}{2} I + \mathcal{K}_\Omega) f(P), \quad P \in \partial\Omega. \tag{2.4}$$

Proof. We first observe that

$$\int_{\partial\Omega} \frac{\partial}{\partial \nu_y} \Gamma(x,y) d\sigma(y) = \begin{cases} 1 & \text{if } x \in \Omega \\ 1/2 & \text{if } x \in \partial\Omega \\ 0 & \text{if } x \in \mathbb{R}^d \setminus \overline{\Omega}. \end{cases} \tag{2.5}$$

(2.5) can be proved using the Green theorem. We leave the proofs as an exercise.

If $x \in \Omega$, then by (2.5)

$$\mathcal{D}_\Omega f(x) = \int_{\partial\Omega} \frac{\partial}{\partial \nu_y} \Gamma(x-y)[f(y) - f(P)] d\sigma(y) + f(P).$$

Let $w(x)$ be the first function in the RHS of the above. If $x = P - t\nu_P$, then $w(x) \to w(P)$ as $t \to 0$. To prove this, for a given $\epsilon > 0$, let $\delta > 0$ be such that $|f(y) - f(P)| < \epsilon$ whenever $|y - P| < \delta$. Then

$$\begin{aligned} w(x) - w(P) &= \int_{\partial\Omega \cap B_\delta} \frac{\partial}{\partial \nu_y} \Gamma(x-y)[f(y) - f(P)] d\sigma(y) \\ &\quad - \int_{\partial\Omega \cap B_\delta} \frac{\partial}{\partial \nu_y} \Gamma(P-y)[f(y) - f(P)] d\sigma(y) \\ &\quad + \int_{\partial\Omega \setminus B_\delta} \left[\frac{\partial}{\partial \nu_y} \Gamma(x-y) - \frac{\partial}{\partial \nu_y} \Gamma(P-y) \right] [f(y) - f(P)] d\sigma(y) \\ &= I_1 + I_2 + I_3. \end{aligned}$$

It easily follows from (2.3) that

$$|I_2| \le C\epsilon \tag{2.6}$$

Since

$$\left|\frac{\partial}{\partial \nu_y}\Gamma(x-y) - \frac{\partial}{\partial \nu_y}\Gamma(P-y)\right| \leq C\frac{|x-P|}{|y-P|^d}, \quad \forall y \in \partial\Omega,$$

we get

$$|I_3| \leq CM|x-P|, \tag{2.7}$$

where M is the maximum of f on $\partial\Omega$. To estimate I_1 we assume that $P = 0$ and near P, Ω is given by $y = (y', y_d)$ with $y_d > \varphi(y')$ where φ is a C^2-function such that $\varphi(0) = 0$ and $\nabla\varphi(0) = 0$. With these coordinates, one can show that

$$\left|\frac{\partial}{\partial \nu_y}\Gamma(x-y)\right| \leq C\frac{|y'|^2+t}{(|y'|^2+t^2)^{d/2}},$$

and hence

$$|I_1| \leq C\epsilon. \tag{2.8}$$

Combining (2.6), (2.7), and (2.8), we can see that

$$\limsup_{t\to 0} |w(x) - w(P)| \leq C\epsilon.$$

Since ϵ is arbitrary, we obtain

$$\mathcal{D}_\Omega f|_-(P) = (\frac{1}{2}I + \mathcal{K}_\Omega)f(P).$$

To see the other identity in (2.5), it suffices to notice that if $x \in \mathbb{R}^d \setminus \overline{\Omega}$, then

$$\mathcal{D}_\Omega f(x) = \int_{\partial\Omega} \frac{\partial}{\partial \nu_y}\Gamma(x-y)[f(y) - f(P)]d\sigma(y),$$

which follows from (2.3). The rest of arguments are the same. This completes the proof. □

Let $\mathcal{K}_\Omega^*$ be the adjoint operator on $L^2(\partial\Omega)$. Then

$$\mathcal{K}_\Omega f(x) = \frac{1}{\omega_d}\int_{\partial\Omega} \frac{\langle y-x, \nu_x\rangle}{|x-y|^d} f(y)d\sigma_y, \quad x \in \partial\Omega.$$

Then in a similar way one can prove

Theorem 2.2 *Let $f \in C(\partial\Omega)$. Then*

$$\left.\frac{\partial(\mathcal{S}_\Omega f)}{\partial \nu}\right|_\pm (P) = (\pm\frac{1}{2}I + \mathcal{K}_\Omega^*)f(P), \quad P \in \partial\Omega. \tag{2.9}$$

In order to solve $DP[f]$ and $NP[g]$, it is now enough to solve the following integral equation:

$$(\frac{1}{2}I + \mathcal{K}_\Omega)\phi = f \quad \text{on } \partial\Omega,$$

and

$$(-\frac{1}{2}I + \mathcal{K}_\Omega)\phi = g \quad \text{on } \partial\Omega.$$

Another advantage we can use for C^2-domains is that the operator $\mathcal{K}_\Omega$ is compact. In fact, this follows from (2.1). More generally, we have the following theorem:

Theorem 2.3 *For each $\alpha > 0$, the operator T_α defined by*

$$T_\alpha f(x) := \int_{\partial\Omega} \frac{f(y)}{|x-y|^{d-1-\alpha}} d\sigma(y), \quad x \in \partial\Omega,$$

is compact on $L^2(\partial\Omega)$.

Thanks to Theorem 2.3, we can use the Fredholm alternative to investigate the invertibility of the operator $\pm\frac{1}{2}I + \mathcal{K}_\Omega$.

Theorem 2.4 (Fredholm Alternative) *Suppose that K is a compact operator on a Hilbert space X. Then, $I + K$ is onto if and only if $I + K$ is one to one.*

For proofs of Theorem 2.3 and Theorem 2.4, see [9].

Theorem 2.5 *Let X be one of $L^2(\partial\Omega)$, $H^{1/2}(\partial\Omega)$, and $C(\partial\Omega)$, and let X_0 be the space of $f \in X$ satisfying $\int_{\partial\Omega} f d\sigma = 0$. Then, $\frac{1}{2}I + \mathcal{K}_\Omega$ is invertible on X and $-\frac{1}{2}I + \mathcal{K}_\Omega$ is invertible on X_0*

Proof. To prove $\frac{1}{2}I + \mathcal{K}_\Omega$ is onto $L^2(\partial\Omega)$, we prove that $\frac{1}{2}I + \mathcal{K}_\Omega^*$ is one to one. Suppose that

$$(\frac{1}{2}I + \mathcal{K}_\Omega^*)\phi = 0 \quad \text{on } \partial\Omega. \tag{2.10}$$

We first observe that $\mathcal{K}_\Omega(1) = 1/2$ which follows from (2.1) and (2.5). Thus

$$0 = \int_{\partial\Omega} (\frac{1}{2}I + \mathcal{K}_\Omega^*)\phi d\sigma = \int_{\partial\Omega} (\frac{1}{2}I + \mathcal{K}_\Omega)(1)\phi d\sigma = \int_{\partial\Omega} \phi d\sigma.$$

Let $u(x) := \mathcal{S}_\Omega\phi(x), \quad x \in \mathbb{R}^d \setminus \overline{\Omega}$. Then u satisfies

$$\begin{cases} \Delta u = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ \left.\dfrac{\partial u}{\partial \nu}\right|_+ = 0 \quad \text{on } \partial\Omega, \\ u(x) = O(|x|^{1-d}) \quad \text{as } |x| \to \infty. \end{cases}$$

In fact, the second follows from (2.2) and (2.10) while the third can be shown as follows: Since $\int_{\partial\Omega} \phi d\sigma = 0$,

$$\mathcal{S}_\Omega\phi(x) = \int_{\partial\Omega} [\Gamma(x-y) - \Gamma(x)]\phi(y) d\sigma(y) = O(|x|^{1-d}), \quad |x| \to \infty.$$

We now prove that $u = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}$. Since

$$\int_{\mathbb{R}^d \setminus \overline{\Omega}} |\nabla u|^2 = -\int_{\partial\Omega} u \frac{\partial u}{\partial \nu}\Big|_+ d\sigma = 0,$$

u is constant and this constant must be 0. Now since $\mathcal{S}_\Omega \phi$ is continuous in $\mathbb{R}^d$ and harmonic in Ω, we get $\mathcal{S}_\Omega \phi = 0$ in Ω and hence in $\mathbb{R}^d$. It then follows from (2.2) that

$$\phi = \frac{\partial}{\partial \nu} \mathcal{S}_\Omega \phi \Big|_+ - \frac{\partial}{\partial \nu} \mathcal{S}_\Omega \phi \Big|_- = 0.$$

To prove $\frac{1}{2} I + \mathcal{K}_\Omega$ is onto $L^2_0(\partial\Omega)$, it suffices to prove that $(\frac{1}{2} I + \mathcal{K}^*_\Omega)\phi = 0$ and $\phi \in L^2_0(\partial\Omega)$, then $\phi = 0$. However the proof is almost the same. In fact, we first prove that $\mathcal{S}_\Omega \phi = 0$ in Ω, and then using the fact $\phi \in L^2_0(\partial\Omega)$ we prove $\mathcal{S}_\Omega \phi = 0$ in $\mathbb{R}^d$.

To prove the invertibility on the spaces $H^{1/2}(\partial\Omega)$ and $C(\partial\Omega)$, it suffices to notice that $\mathcal{K}_\Omega$ is improving regularity (see the following exercise). □

Exercise. For this we suppose $d = 3$ for simplicity. If $\partial\Omega$ is C^2, prove the following.

(1) $\mathcal{K}_\Omega : L^2(\partial\Omega) \to H^{1/2}(\partial\Omega)$ bounded.

(2) $\mathcal{K}_\Omega : L^2(\partial\Omega) \to L^6(\partial\Omega)$, $L^6(\partial\Omega) \to L^\infty(\partial\Omega)$ bounded.

(2) $\mathcal{K}_\Omega : L^\infty(\partial\Omega) \to C^\alpha(\partial\Omega)$ bounded ($\alpha < 1$).

Notice that the spaces are not optimal. (Hint. First localize the operator as in the following section. Then you see that you end up with a convolution operator. Then you can apply the generalized Young's inequality, etc.)

2.2 Lipschitz domain

Before we move to the next section, let us take a look at the operator $\mathcal{K}_\Omega$ when $\partial\Omega$ is only Lipschitz continuous. The main cause of serious difficulties is the failure of (2.2) for the Lipschitz domains. For those, the following holds:

$$\left| \frac{\partial}{\partial \nu_y} \Gamma(x, y) \right| + \left| \frac{\partial}{\partial \nu_x} \Gamma(x, y) \right| \le \frac{C}{|x - y|^{d-1}}, \quad x, y \in \partial\Omega. \tag{2.11}$$

In order to see the type of operators we will be considering, let us localize the operator $\mathcal{K}_\Omega$. Let $\{\zeta_j : j = 1, ..., M\}$ be a partition of unity for $\partial\Omega$. We further assume that for each j, the set $\cup(\mathrm{supp}(\zeta_k))$, where the union is taken over all k such that $\mathrm{supp}(\zeta_k) \cap \mathrm{supp}(\zeta_j) \neq \emptyset$, is represented by a

Lipschitz φ as $x_d = \varphi(x')$ after rotation and translation if necessary, where $x' = (x_1, ..., x_{d-1})$. Then

$$\mathcal{K}_\Omega f(x) = \sum_{j,k} \zeta_k(x)\mathcal{K}_\Omega(\zeta_j f)(x) := \sum_{j,k} \mathcal{K}_{jk}(f)(x).$$

For those j, k with $\mathrm{supp}(\zeta_k) \cap \mathrm{supp}(\zeta_j) = \emptyset$, it is easy to see that $\mathcal{K}_{jk}$ is bounded on $L^2(\partial\Omega)$. But for those j, k with $\mathrm{supp}(\zeta_k) \cap \mathrm{supp}(\zeta_j) \neq \emptyset$, it becomes a completely different story. For such j, k the kernel of the operator $\mathcal{K}_{jk}$ takes the form, after rotation and translation,

$$\frac{1}{\omega_d}\frac{\langle y - x, \nu_y\rangle}{|x-y|^d} = \frac{1}{\omega_d}\frac{(x'-y')\cdot\nabla\varphi(y') + (\varphi(y') - \varphi(x'))}{[|x'-y'|^2 + |\varphi(x') - \varphi(y')|^2]^{\frac{d}{2}}}\frac{1}{\sqrt{1+|\nabla\varphi(y')|^2}}.$$

Therefore, the type kernels are

$$\frac{x_j - y_j}{[|x'-y'|^2 + |\varphi(x') - \varphi(y')|^2]^{\frac{d-1}{2}}} \quad \text{or} \quad \frac{\varphi(x') - \varphi(y')}{[|x'-y'|^2 + |\varphi(x') - \varphi(y')|^2]^{\frac{d}{2}}}$$

where φ is a Lipschitz function. More generally,

$$\frac{A(x') - A(y')}{[|x'-y'|^2 + |\varphi(x') - \varphi(y')|^2]^{\frac{d}{2}}} \tag{2.12}$$

where A and φ are Lipschitz functions.

The major part of the theory for this kind of operators is L^2 boundedness. In this lecture, we will prove a beautiful theorem of Coifman-McIntosh-Meyer [2]. Their result was further generalized to the celebrated $T1$-theorem due to David-Journé [6]. There are many prerequisites to understand the CMM theorem. Among them are classical theory of singular integral operators, maximal functions, Carleson measures, BMO.

Chapter 3

Calderon-Zygmund Theory of SIO

3.1 Preliminary

In this chapter, we study the Calderón-Zygmund theory of singular integral operators. We first state two major theorems to be used in this chapter and throughout this note, without proofs. For proofs, we refer to [15]

Theorem 3.1 (Marcinkiewicz Interpolation Theorem) *Suppose that*

(1) $T : L^1 + L^\infty \to L^1 + L^\infty$ *sublinear, i.e.,* $|T(f_1 + f_2)| \le |Tf_1| + |Tf_2|$,

(2) T *is of weak type* (p_i, q_i) $\;(i = 1, 2,\ 1 \le p_i \le q_i \le \infty)$, *i.e., there are constants* C_i *such that for all positive number* λ *and* $f \in L^{p_i}$,

$$|\{Tf > \lambda\}| \le \left(\frac{C_i \|f\|_{p_i}}{\lambda}\right)^{q_i}$$

Let $p = (1-\theta)\frac{1}{p_1} + \theta\frac{1}{p_2}$ *and* $q = (1-\theta)\frac{1}{q_1} + \theta\frac{1}{q_2}$ $(0 < \theta < 1)$. *Then* T *is of (strong) type* (p, q), *i.e, there is a constant* C *depending only on* C_1, C_2, *and* θ *such that*

$$\|Tf\|_q \le C\|f\|_p.$$

Remark. When $q = \infty$, the weak type (p, q) means the strong type (p, q).

Another important ingredient is the Hardy-Littlewood maximal operator. For an integrable function f, define

$$\mathcal{M}f(x) := \sup_{r>0} \frac{1}{|C_r(x)|}\int_{C_r(x)} |f(y)|dy, \quad x \in \mathbb{R}^n, \tag{3.1}$$

where $C_r(x)$ is either $B_r(x)$, a ball centered at x with radius r, or $Q_r(x)$, a cube centered at x with the side length r.

Theorem 3.2 *The Hardy-Littlewood maximal operator $\mathcal{M}$ is of weak type $(1,1)$ and (∞,∞), and hence is of strong type (p,p) for all p, $1 < p \le \infty$.*

Lemma 3.3 (Calderon-Zygmund Decomposition) *Let $f \ge 0$, $\|f\|_1 < \infty$, and $\alpha > 0$ be a fixed number. Then there exists non-overlapping dyadic cubes $\{Q_i\}$ such that*

(1) $f \le \alpha$ *a.e.* $x \in \mathbb{R}^n \setminus \cup_j Q_j$,

(2) $\alpha < \displaystyle⨍_{Q_j} f \le 2^n \alpha$,

where $\displaystyle⨍_{Q_j} f = \frac{1}{|Q_j|}\int_{Q_j} f = f_{Q_j}$.

Before proving Lemma 3.3, let us state another lemma which is equivalent to Lemma 3.3.

Let Q_j be the cubes in CZ-decomposition. Let

$$g := f\chi_{\mathbb{R}^n\setminus\cup Q_j} + \sum_j f_{Q_j}\chi_{Q_j} \quad \text{and} \quad b = \sum_j b_j := \sum_j (f - f_{Q_j})\chi_{Q_j}$$

where χ_{Q_j} is the characteristic function of Q_j. Then $f = g + b$. Each b_j satisfies

$$\|b_j\|_1 \le \int_{Q_j} |f| + |f_{Q_j}| \le 2^{n+1}\alpha|Q_j|.$$

We also have

$$\sum_j |Q_j| \le \sum_j \frac{1}{\alpha}\int_{Q_j} |f| \le \frac{1}{\alpha}\int_{\cup Q_j} |f| \le \frac{1}{\alpha}\|f\|_1,$$

and hence

$$\begin{aligned}
\|g\|_2^2 &= \int_{\mathbb{R}^n} |g|^2 \\
&\le 2\Big[\int_{\mathbb{R}^n\setminus\cup Q_j} |f|^2 + \sum_{j=1}^{\infty}\int_{Q_j} |f_{Q_j}|^2\Big] \\
&\le 2\Big[\alpha\int_{\mathbb{R}^n\setminus\cup Q_j} |f| + 2^{2n}\alpha^2\sum_j |Q_j|\Big] \\
&\le 2\Big[\alpha\|f\|_1 + 2^{2n}\alpha\|f\|_1\Big] \\
&= 2(2^{2n}+1)\alpha\|f\|_1.
\end{aligned}$$

So we have the following lemma which is, in fact, equivalent to the CZ-decomposition.

Lemma 3.4 (CZ-Decomposition) *Let $f \in L^1$ and $\alpha > 0$. Then f can be decomposed as $f = g + b = g + \sum_{j=1}^{\infty} b_j$ so that*

(1) $\|g\|_2^2 \le 2^{2n+2}\alpha\|f\|_1$,

(2) $\operatorname{supp} b_j \subset Q_j$ *and* $\{Q_j\}$ *is mutually non-overlapping,*

(3) $\|b_j\|_1 \le 2^{n+1}\alpha|Q_j|$,

(4) $\int_{Q_j} b_j = 0 \ \ \forall j$,

(5) $\sum_j |Q_j| \le \frac{1}{\alpha}\|f\|_1$.

Proof of Lemma 3.3. For any integer k, let $\mathcal{D}_k$ be the collection of all dyadic cubes with side length 2^{-k}. So each $Q \in \mathcal{D}_k$ is a closed cube whose corners are of the form $(l_1 2^{-k}, ..., l_n 2^{-k})$ where $l_1, ...l_n$ are integers. Observe that any two different cubes in $\mathcal{D}_k$ are mutually non-overlapping, i.e., they only share, if any, sides which is of measure zero. We also observe that each Q in $\mathcal{D}_k$ contains exactly 2^n cubes in $\mathcal{D}_{k+1}$, while each cube in $\mathcal{D}_{k+1}$ is contained in exactly one cube in $\mathcal{D}_k$.

Let $\alpha > 0$ be given. Since $f \in L^1$, there exists j such that

$$\fint_Q f < \alpha$$

for all $Q \in \mathcal{D}_j$. Assume $j = 0$ without loss of generality. Let

$$\mathcal{F}_1 = \{Q \in \mathcal{D}_1 : \fint_Q f > \alpha\}.$$

If $Q \in \mathcal{D}_1 \setminus \mathcal{F}_1$, then bisect the sides of Q to have 2^n sub-cubes. Define

$$\mathcal{F}_2 = \{Q \in \mathcal{D}_2 : \fint_Q f > \alpha, \text{ and } Q \nsubseteq \tilde{Q} \text{ for any } \tilde{Q} \in \mathcal{F}_1\}.$$

Repeat this procedure indefinitely (if necessary) to have the classes $\mathcal{F}_k$, $k = 1, 2, ...$ Enumerate all members of $\cup_k \mathcal{F}_k$ by $\{Q_j\}$. If $Q_j \in \mathcal{F}_k$ for some k, then there exists $\tilde{Q} \in \mathcal{D}_{k-1}$ containing Q_j. Since $\tilde{Q} \notin \mathcal{F}_{k-1}$, we have

$$\alpha \le \frac{1}{|Q_j|}\int_{Q_j} f = \frac{2^n}{|\tilde{Q}|}\int_{Q} f \le 2^n \fint_{\tilde{Q}} f \le 2^n \alpha.$$

If $x \in \mathbb{R}^n \setminus \cup_j Q_j$, then there exists a sequence $\{C_j\}$ of cubes such that

$$C_1 \supset C_2 \supset \cdots, \quad \bigcap_j C_j = \{x\}, \quad \text{and} \quad C_j \in \mathcal{D}_j \setminus \mathcal{F}_j.$$

By definition of $\mathcal{F}_j$, $\fint_{C_j} f < \alpha$ for all j. It then follows from the Lebesgue differentiation theorem that

$$f(x) = \lim_{j\to\infty} \frac{1}{|C_j|} \int_{C_j} f(y)dy \le \alpha \qquad \text{a.e. } x.$$

Note that the Lebesgue differentiation theorem used in the proof is slightly different from the usual differentiation theorem which asserts that

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy = f(x) \qquad \text{a.e. } x.$$

Such difference causes no trouble. In fact, if we define a maximal function

$$\mathcal{M}_1 f(x) := \sup_{\substack{Q:cube \\ x\in Q}} \frac{1}{|Q|} \int_Q |f(y)|dy,$$

one can easily shows that $\mathcal{M}_1 f(x) \le C\mathcal{M}f(x)$ for some constant C depending only on the dimension n. Following the proof of the usual Lebesgue differentiation theorem (e.g., [13]), one can prove the desired differentiation theorem. □

3.2 Singular Integral Operators

The singular integral operators are defined as follows.

Definition 3.5 *An integral kernel $k(x,y)$ $(x,y \in \mathbb{R}^n)$ is called a standard kernel if for $x,y \in \mathbb{R}^n$,*

(1) $|k(x,y)| \le \dfrac{C}{|x-y|^n}$,

(2) $|\nabla_x k(x,y)| + |\nabla_y k(x,y)| \le \dfrac{C}{|x-y|^{n+1}}$

for some constant C.

Observe that the kernel of the type (2.12) in which we are interested is a standard kernel on $\mathbb{R}^{d-1}$. Moreover it is skew symmetric, i.e.,

$$k(y,x) = -k(x,y), \quad x,y \in \mathbb{R}^{d-1}.$$

We will assume throughout this lecture that the kernel $k(x,y)$ is skew symmetric.

The singular integral operator (SIO) T corresponding to the kernel $k(x,y)$ is defined defined as a Cauchy principal value: for each $f \in C_0^\infty(\mathbb{R}^n)$

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} k(x,y)f(y)dy = \lim_{\epsilon\to 0} \int_{|x-y|>\epsilon} k(x,y)f(y)dy. \tag{3.2}$$

We first prove that the limit exists for each $f \in C_0^\infty(\mathbb{R}^n)$. For each $\epsilon > 0$, let

$$k_\epsilon(x,y) = k(x,y)\chi_{\{|x-y|>\epsilon\}},$$

and let T_ϵ be the integral operator defined by $k_\epsilon(x,y)$. Then $k_\epsilon(x,y)$ is also skew-symmetric. So we get for all $f, g \in C_0^\infty(\mathbb{R}^n)$

$$\langle T_\epsilon f, g\rangle = \frac{1}{2}\iint k_\epsilon(x,y)\Big[f(x)g(y) - f(y)g(x)\Big]dxdy.$$

Since $|f(x)g(y) - f(y)g(x)| \le C|x-y|$, it is now clear that they converge as $\epsilon \to 0$.

The main theorem of this chapter is the following which is already classical.

Theorem 3.6 *Let T be a SIO. If T is bounded on L^2, then*

(1) $|\{|Tf| > \lambda\}| \le \dfrac{C\|f\|_1}{\lambda}, \quad \forall f \in L^1, \forall \lambda > 0,$

(2) *T is bounded on L^p,* $\;1 < p < \infty.$

Remark. The meaning of Tf for $f \in L^1$ is not clear yet. In view of the CZ-decomposition, it is reasonable to define it by

$$Tf = Tg + \sum_{j=1}^\infty Tb_j, \quad \text{when } f = g + \sum_{j=1}^\infty b_j.$$

Since $g \in L^2$, Tg makes sense. We will give a meaning to Tb_j after introducing the notion of BMO in the next chapter.

Theorem 3.6 says that an SIO which is bounded on L^2 is automatically of weak type $(1,1)$, and hence bounded on L^p, $1 < p < \infty$.

Proof of Theorem 3.6. Let $f \in L^1$ and $\lambda > 0$. Let $f = g + b$ be the CZ-decomposition with respect to λ and $\{Q_j\}$ be those cubes in Lemma 3.4. Note that

$$|\{|Tf| > \lambda\}| \le |\{|Tg| > \frac{\lambda}{2}\}| + |\{|Tb| > \frac{\lambda}{2}\}|.$$

By Lemma 3.4 (1), we have

$$\begin{aligned} |\{|Tg| > \frac{\lambda}{2}\}| &\le \int_{\{|Tg|>\frac{\lambda}{2}\}} \frac{|Tg|^2}{(\frac{\lambda}{2})^2}dx \le \frac{4}{\lambda^2}\|Tg\|_2^2 \\ &\le \frac{C}{\lambda^2}\|g\|_2^2 \le \frac{C}{\lambda^2}2^{2n+2}\lambda\|f\|_1 = \frac{C}{\lambda}\|f\|_1. \end{aligned}$$

Let $A = \{x \notin \cup_{j=1}^{\infty}(2Q_j) : |Tb| > \lambda/2\}$. Then, $\{|Tb| > \lambda/2\} \subset \cup_{j=1}^{\infty}(2Q_j) \cup A$. By Lemma 3.4 (5), we have

$$|\bigcup_{j=1}^{\infty}(2Q_j)| \leq \sum_{j=1}^{\infty}|2Q_j| = 2^n \sum_{j=1}^{\infty}|Q_j| \leq \frac{2^n}{\lambda}\|f\|_1.$$

Suppose $x \notin 2Q_j$ and let y^j be the center of Q_j. By Lemma 3.4 (4), we have

$$\begin{aligned} Tb_j(x) &= \int_{Q_j} k(x,y)b_j(y)dy \\ &= \int_{Q_j} [k(x,y) - k(x,y^j)]b_j(y)dy \\ &= \int_{Q_j} \nabla_y k(x,\xi) \cdot (y - y^j)b_j(y)dy \end{aligned}$$

for some point $\xi \in Q_j$. Since $x \notin 2Q_j$, $|x - y^j| \approx |x - \xi|$ independently of x and hence

$$|\nabla_y k(x,\xi)| \leq \frac{C}{|x-\xi|^{n+1}} \leq \frac{C}{|x-y^j|^{n+1}}.$$

Thus Lemma 3.4 (3) leads to

$$\begin{aligned} |Tb_j(x)| &\leq C\int_{Q_j} \frac{|y-y^j|}{|x-y^j|^{n+1}}|b_j(y)|dy \\ &\leq \frac{C}{|x-y^j|^{n+1}} l(Q_j) \int_{Q_j} |b_j(y)|dy \\ &\leq \frac{C}{|x-y^j|^{n+1}} \lambda |Q_j|^{1+\frac{1}{n}}, \end{aligned}$$

where $l(Q_j)$ denotes the side length of Q_j. It then follows that

$$\begin{aligned} \int_{\mathbb{R}^n \setminus 2Q_j} |Tb_j(x)|dx &\leq C\lambda|Q_j|^{1+\frac{1}{n}} \int_{\mathbb{R}^n \setminus 2Q_j} \frac{1}{|x-y^j|^{n+1}}dx \\ &\leq C\lambda|Q_j|^{1+\frac{1}{n}} \int_{|x|>Cl(Q_j)} \frac{1}{|x|^{n+1}}dx \leq C\lambda|Q_j|. \end{aligned}$$

As a consequence, we have from Lemma 3.4 (5)

$$\begin{aligned} |A| &\leq \frac{2}{\lambda}\int_A |Tb| \\ &\leq \frac{2}{\lambda}\sum_{j=1}^{\infty}\int_{\mathbb{R}^n \setminus (2Q_j)} |Tb_j(x)| \leq \frac{C}{\lambda}\lambda\sum_{j=1}^{\infty}|Q_j| \\ &\leq \frac{C}{\lambda}\|f\|_1. \end{aligned}$$

This proves the weak $(1,1)$ property of T.

The strong (p,p) property for $1 < p < 2$ follows from the Marcinkiewicz Interpolation Theorem.

If $2 < p < \infty$, let T^* be the adjoint operator of T. Then T^* is also a CZO. Thus $\|T^* f\|_q \le C_q \|f\|_q$, $1 < q < 2$. By duality, we have boundedness of T on L^p. In fact,

$$|(Tf,g)| = |(f,T^*g)| \le \|f\|_p \|T^*g\|_q \le C_q \|f\|_p \|g\|_q$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Hence

$$\|Tf\|_p = \sup_g \frac{|(Tf,g)|}{\|g\|_q} \le C_q \|f\|_p.$$

This completes the proof. □

Define

$$T_* f(x) = \sup_{\epsilon > 0} |T_\epsilon f(x)|, \quad x \in \mathbb{R}^n.$$

The following lemma is due to Cotlar.

Lemma 3.7 *Suppose T is bounded on L^2. Then there is a constant $C > 0$ such that for all $f \in C_0^\infty(\mathbb{R}^n)$*

$$T_* f(x) \le C(\mathcal{M} f(x) + \mathcal{M} T f(x)) \quad x \in \mathrm{R}^n \tag{3.3}$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal function. As a consequence, we have

$$\|T_* f\|_p \le C_p \|f\|_p \quad \forall f \in L^p,\ 1 < p < \infty. \tag{3.4}$$

Proof. Suppose that $x = 0$ without loss of generality. If $y \in B_{\epsilon/2}(0)$, then

$$\begin{aligned} T_\epsilon f(y) - T_\epsilon f(0) &= \int_{|y-z|>\epsilon} k(y,z) f(z) dz - \int_{|z|>\epsilon} k(0,z) f(z) dz \\ &= \int_{|z|>\epsilon} [k(y,z) - k(0,z)] f(z) dz \\ &\quad + \int_{B_\epsilon(0)\backslash B_\epsilon(y)} k(y,z) f(z) dz - \int_{B_\epsilon(y)\backslash B_\epsilon(0)} k(y,z) f(z) dz \\ &:= I_1 + I_2 + I_3 \end{aligned}$$

For all $y \in B_{\epsilon/2}(0)$ and $z \in (B_\epsilon(0)\backslash B_\epsilon(y)) \cup (B_\epsilon(y)\backslash B_\epsilon(0))$, $|y - z| \ge \frac{\epsilon}{2}$, and hence

$$|I_2| + |I_3| \le \frac{C}{\epsilon^n} \int_{B_\epsilon(0)} |f(z)| dz + \frac{C}{\epsilon^n} \int_{B_{2\epsilon}(0)} |f(z)| dz \le C\mathcal{M} f(0).$$

By mean value theorem, for $y \in B_{\epsilon/2}(0)$ and $z \notin B_\epsilon(0)$,

$$|k(y,z) - k(0,z)| \le \frac{C|y|}{|z|^{n+1}} \le \frac{C\epsilon}{|z|^{n+1}}.$$

It then follows that

$$\begin{aligned} |I_1| &\le C\epsilon \int_{|z|>\epsilon} \frac{|f(z)|}{|z|^{n+1}} dz \\ &= C\epsilon \sum_{j=0}^{\infty} \int_{2^j\epsilon<|z|\le 2^{j+1}\epsilon} \frac{|f(z)|}{|z|^{n+1}} dz \le C\mathcal{M}f(0). \end{aligned}$$

Thus we have for $y \in B_{\epsilon/2}(0)$,

$$|T_\epsilon f(0)| \le |T_\epsilon f(y)| + |T_\epsilon f(y) - T_\epsilon f(0)| \le |T_\epsilon f(y)| + C\mathcal{M}f(0).$$

If $|T_\epsilon f(y)| \le \frac{1}{2}|T_\epsilon f(0)|$ for some $y \in B_{\epsilon/2}(0)$, (3.3) follows.

Suppose $|T_\epsilon f(y)| > \frac{1}{2}|T_\epsilon f(0)|$ for all $y \in B_{\epsilon/2}(0)$. Let χ be the characteristic function of $B_\epsilon(0)$. Since $T_\epsilon f(y) = Tf(y) - T(f\chi)(y)$, we have

$$B_{\epsilon/2}(0) \subset E_1 \cup E_2$$

where

$$\begin{aligned} E_1 &= \{y \in B_{\epsilon/2}(0) : |Tf(y)| > \frac{1}{4}|T_\epsilon f(0)|\} \\ E_2 &= \{y \in B_{\epsilon/2}(0) : |T(f\chi)(y)| > \frac{1}{4}|T_\epsilon f(0)|\}. \end{aligned}$$

One can easily get

$$\frac{1}{4}|T_\epsilon f(0)||E_1| \le \int_{B_{\epsilon/2}(0)} |Tf(y)|dy.$$

Since T is of weak type $(1,1)$, we have

$$\frac{1}{4}|T_\epsilon f(0)||E_2| \le C \int_{B_\epsilon(0)} |f(y)|dy.$$

Hence

$$\begin{aligned} |B_{\epsilon/2}(0)|\frac{1}{4}|T_\epsilon(0)| &\le \frac{1}{4}|T_\epsilon(0)|(|E_1| + |E_2|) \\ &\le C(\int_{B_{\epsilon/2}(0)} |Tf(y)|dy + \int_{B_\epsilon(0)} |f(y)|dy). \end{aligned}$$

It thus follows that

$$T_\epsilon f(0) \le C(\mathcal{M}Tf(0) + \mathcal{M}f(0))$$

for all $\epsilon > 0$. This completes the proof. □

As a consequence of (3.4) we can prove that the limit in (3.2) exists for all $f \in L^p$, $1 < p < \infty$.

Lemma 3.8 *Let $f \in L^p$, $1 < p < \infty$. Then*

$$Tf(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} k(x,y) f(y) dy, \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Proof. Let $\lambda > 0$ and

$$A := \left\{ x : \limsup_{\epsilon \to 0} |T_\epsilon f(x) - Tf(x)| > \lambda \right\}.$$

For a given $\delta > 0$, choose $g \in C_0^\infty(\mathbb{R}^n)$ such that $\|f - g\|_p \leq \delta$. Then

$$\limsup_{\epsilon \to 0} |T_\epsilon f(x) - Tf(x)| \leq |T_*(f-g)(x)| + |T(f-g)(x)|,$$

and hence

$$A \subset \{|T_*(f-g)(x)| > \lambda/2\} \cup \{|T(f-g)(x)| > \lambda/2\}.$$

It then follows from (3.4) that

$$|A| \leq C \left(\frac{\delta}{\lambda} \right)^p.$$

Since δ is arbitrary, $|A| = 0$. This completes the proof. □

3.3 Convolution Operators

Theorem 3.6 says that for the L^p-boundedness of a SIO, the main question is L^2-boundedness. We list some conditions on the kernel which guarantee L^2-boundedness of the SIO of the convolution type $Tf(x) = (k * f)(x)$. An essential property is "the cancellation property". Since for convolution operators one may apply Fourier transform and Plancherel identity, L^2-boundedness of those operators can be derived without much difficulty. Proofs of the following theorems can be found in [13].

Theorem 3.9 *If $k(x)$ satisfies*

(1) $|k(x)| \leq \dfrac{C}{|x|^n}$,

(2) $\displaystyle\int_{|x|>2|y|} |k(x-y) - k(x)| dx \leq C$ *for all* $y \neq 0$ (*Hörmander condition*),

(3) $\displaystyle\int_{R_1<|x|<R_2} k(x) dx = 0$ *for all* $0 < R_1 < R_2 < \infty$ (*Cancellation*),

then T is bounded on L^p $(1 < p < \infty)$.

Theorem 3.10 *Let* $\Omega \in C^1(S^{n-1})$ *and* $\int_{S^{n-1}} \Omega(x) d\sigma(x) = 0$. *Define* $\Omega(x) = \Omega(\frac{x}{|x|})$ *for* $x \neq 0$. *Then the operator* T *defined by*

$$Tf(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^n} f(x-y) dy$$

is bounded on L^p $(1 < p < \infty)$.

Here are two important convolution operator which fall in the case of Theorem 3.10.

- Hilbert transform.

$$Hf(x) := \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}^1} \frac{1}{y} f(x-y) dy.$$

- Riesz transform.

$$R_j(x) = c_n \text{p.v.} \int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+1}} f(x-y) dy.$$

Observe that the operator (2.12) is not a convolution type. The L^2-boundedness of the non-convolution type SIO is a very hard question and this problem has been one of the central theme in the harmonic analysis and potential theory.

For the operators of type (2.12) there is a impressive result due to Coifman-McIntosh-Meyer [2]. The main purpose of this lecture note is to reproduce, with details, their proof. The method of CMM was further developed to produce the celebrated $T1$-theorem by David-Journé [6]. If time permits, we will discuss about the $T1$-theorem. But I don't think time would.

Chapter 4

Carleson Measures and BMO

4.1 Carleson Measure

The concept of Calreson measures came out in solving the follwing problem which was solved by Carleson.

Problem. Characterize those positive measures μ on $\mathbb{R}^{n+1}_+ = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^1 : y > 0\}$ for which the following holds;

$$\iint_{\mathbb{R}^n_+} |P_t f(x)|^2 d\mu(x,t) \le C \int_{\mathbb{R}^n} |f(x)|^2 dx \quad \forall f \in L^2(\mathbb{R}^n), \tag{4.1}$$

where $P_t f$ is the Poisson extention of f in $\mathbb{R}^{n+1}_+$

A necessary condition can be easily found: Let Q be a cube in $\mathbb{R}^n$ and $f = \chi_{2Q}$. If $x \in Q$ and $t \le l = l(Q)$, then since $B(x,l) \subset 2Q$ we have

$$\begin{aligned} P_t f(x) &= c_n \int_{\mathbb{R}^n} \frac{t}{[|x-y|^2 + t^2]^{\frac{n+1}{2}}} f(y) dy \\ &\ge c_n \int_{B(x,l)} \frac{t}{[|x-y|^2 + t^2]^{\frac{n+1}{2}}} dy \\ &= C \int_{|y| \le l} \frac{t}{[|y|^2 + t^2]^{\frac{n+1}{2}}} dy \\ &= C \int_{|y| \le l/t} \frac{1}{[|y|^2 + 1]^{\frac{n+1}{2}}} dy \end{aligned}$$

Since $t \le l$, it follows that $P_t f(x) \ge C$ for some constant C. Therefore, if (4.1) holds, then

$$\begin{aligned} \mu(Q \times [0,l]) &\le C \int_{Q \times [0,l]} |P_t f(x)|^2 d\mu(x,t) \\ &\le C \int_{\mathbb{R}^n} |f(x)|^2 dx \\ &\le C|Q| \, . \end{aligned}$$

For each cube $Q \subset \mathbb{R}^n$, define the tent over Q by

$$T(Q) := Q \times [0, l(Q)] \subset \mathbb{R}^{n+1}_+.$$

We have seen that if (4.1) holds, then $\mu(T(Q)) \leq C|Q|$.

Definition 4.1 *A positive measure μ on $\mathbb{R}^{n+1}_+$ is called a Carleson measure if there is a constant $C > 0$ such that*

$$\mu(T(Q)) \leq C|Q| \quad \textit{for every cube } Q \subset \mathbb{R}^n.$$

If μ is a Carleson measure, the Carleson norm is defined to be

$$\|\mu\|_{\mathcal{C}} := \sup_Q \frac{\mu(T(Q))}{|Q|}$$

For example, $d\mu(x,t) = \varphi(t)dxdt$ is a Carleson measure if and only if $\varphi \in L^1(\mathbb{R}_+)$. In particular, $\frac{1}{t}dxdt$ is not a Carleson measure

We will prove that being a Carleson measure is also sufficient for μ to satisfy (4.1).

Lemma 4.2 (Whitney decomposition Lemma) *Let Ω be an open set in $\mathbb{R}^n$ such that $\Omega^c \neq \emptyset$. Then $\Omega = \cup_{j=1}^{\infty} Q_j$ where*

(1) *$\mathcal{F} = \{Q_j\}$ is mutually non-overlapping dyadic cubes,*

(2) *There are constants C_1 and C_2 so that*

$$C_1 l(Q_j) \leq dist(Q_j, \Omega^c) \leq C_2 l(Q_j) \ \textit{for all } j.$$

Proof. For each integer j, let $\Omega_j := \{x \in \Omega : 2\sqrt{n}\ 2^{-j} < \text{dist}(x, \Omega^c) \leq 4\sqrt{n}\ 2^{-j}\}$, $\mathcal{D}_j$ be the collection of all dyadic cubes with side length 2^{-j}, $\mathcal{F}_j := \{Q_j : Q \cap \Omega_j \neq \emptyset\}$, and $\mathcal{F}' = \cup_j \mathcal{F}_j$. Then $\cup_{Q \in \mathcal{F}'} = \Omega$. If $Q \in \mathcal{F}'$, then there is $x \in Q \cap \Omega_j$ where $Q \in \mathcal{D}_j$, and hence $\text{dist}(x, \Omega^c) \geq 2\sqrt{n}2^{-j}$. It thus follows that

$$\begin{aligned}\text{dist}(Q, \Omega^c) &\geq \text{dist}(x, \Omega^c) - \sqrt{n}\, l(Q) \\ &\geq 2\sqrt{n}2^{-j} - \sqrt{n}l(Q) \geq \sqrt{n}l(Q).\end{aligned}$$

And

$$\text{dist}(Q, \Omega^c) \leq \text{dist}(x, \Omega^c) + \sqrt{n}l(Q) \leq 5\sqrt{n}l(Q).$$

Since $\mathcal{F}'$ consist of dyadic cubes, any two of members of $\mathcal{F}'$ are either mutually non-overlapping or one contains the other. So, for each $Q \in \mathcal{F}'$

there exists $\widetilde{Q} \in \mathcal{F}'$ which is maximal with respect to the inclusion relation. In fact, if $Q, \tilde{Q} \in \mathcal{F}'$ and $Q \subset \tilde{Q}$, then

$$\begin{aligned} l(\tilde{Q}) &\leq \frac{1}{C_1}\mathrm{dist}(\tilde{Q}, \Omega^c) \\ &\leq \frac{1}{C_1}\mathrm{dist}(Q, \Omega^c) \leq \frac{C_2}{C_1} l(Q). \end{aligned}$$

Let $\mathcal{F}$ be the collection of all maximal elements of $\mathcal{F}'$. This $\mathcal{F}$ does the job. □

For functions u defined on $\mathbb{R}^{n+1}_+$, define the non-tangential maximal function by

$$\mathcal{N}u(x) = \sup_{(y,t)\in\Gamma(x)} |u(y,t)| \quad (x \in \mathbb{R}^n)$$

where $\Gamma(x)$ is the cone defined by $\Gamma(x) = \{(y,t) : t > |y - x|\}$.

Let us prove a useful lemma.

Lemma 4.3

$$\mathcal{N}(P_t f)(x) \leq C\mathcal{M}f(x), \quad x \in \mathbb{R}^n. \tag{4.2}$$

Proof. Put

$$\begin{aligned} P_t f(y) &= c_n \int_{\mathbb{R}^n} \frac{t}{[|y-z|^2 + t^2]^{\frac{n+1}{2}}} f(z)dz \\ &= c_n \left(\int_{|z-y|\leq t} + \sum_{j=1}^{\infty} \int_{2^{j-1}t<|z-y|\leq 2^j t} \right) \\ &:= c_n(I_0 + \sum_{j=1}^{\infty} I_j). \end{aligned}$$

If $(y,t) \in \Gamma(x)$ and $|z - y| \leq t$, then $|x - z| \leq 2t$, and hence

$$|I_0| \leq \frac{1}{t^n} \int_{|z-y|\leq 2t} |f(z)|dz \leq C\mathcal{M}f(x).$$

If $(y,t) \in \Gamma(x)$ and $2^{j-1}t < |z - y| \leq 2^j t$, then $|z - x| \leq 2^{j+1}t$, and hence

$$|I_j| \leq \frac{1}{2^{j-1}} \frac{1}{(2^{j-1}t)^n} \int_{|z-y|\leq 2^{j+1}t} |f(z)|dz \leq \frac{C}{2^{j-1}} \mathcal{M}f(x),$$

for each j. This completes the proof. □

Theorem 4.4 *If μ is a Carleson measure and u is continuous in $\mathbb{R}^{n+1}_+$. Then*

$$\mu(\{(x,t) : |u(x,t)| > \lambda\}) \leq C|\{x : \mathcal{N}u(x) > \lambda\}|. \tag{4.3}$$

Proof. For $\lambda > 0$, let $G_\lambda := \{x : |\mathcal{N}u(x)| > \lambda\}$. Since u is continuous, G_λ is open. We may assume $G_\lambda \neq \mathbb{R}^n$ since otherwise there is nothing to prove. Let $\{Q_j\}$ be the cubes in the Whitney decomposition lemma for G_λ. Suppose $|u(x,t)| > \lambda$. Then $x \in G_\lambda$ and hence $x \in Q_j$ for some j. Thus there exists $y_j \in G_\lambda^c$ such that

$$C_1 l(Q_j) \leq \mathrm{dist}(y_j, Q_j) \leq C_2 l(Q_j)$$

and hence

$$C_1 l(Q_j) \leq |y_j - x| \leq C_3 l(Q_j).$$

Since $y_j \notin G_\lambda$, $(x,t) \notin \Gamma(y_j)$. Thus

$$t < |x - y_j| \leq C_3 l(Q_j).$$

Therefore, $(x,t) \in Q_j \times [0, C_3 l(Q_j)]$. Since μ is a Carleson measure, it follows that

$$\begin{aligned}
\mu(\{(x,t) : |u(x,t)| > \lambda\}) &\leq \mu(\bigcup_j Q_j \times [0, C_3 l(Q_j)]) \\
&\leq \sum_j \mu(Q_j \times [0, C_3 l(Q_j)]) \\
&\leq C \sum_j |Q_j| = C|G_\lambda|.
\end{aligned}$$

This completes the proof. □

Finally, we are ready to prove

Theorem 4.5 *μ is a Calreson measure if and only if* (4.1) *holds.*

Proof. Recall that

$$\int_X |u(x)|^p d\mu = p \int_0^\infty \lambda^{p-1} \mu(\{x \in X : |u(x)| > \lambda\}) d\lambda,$$

for any positive measure on a measurable space X if $1 \leq p < \infty$. So it follows from (4.2) and (4.3) that

$$\begin{aligned}
\iint_{\mathbb{R}^{n+1}_+} |P_t f(x)|^p d\mu(x,t) &= p \int_0^\infty \lambda^{p-1} \mu(\{(x,t) : |P_t f(x)| > \lambda\}) d\lambda \\
&\leq Cp \int_0^\infty \lambda^{p-1} |\{\mathcal{N}(P_t f)(x) > \lambda\}| d\lambda \\
&\leq Cp \int_0^\infty \lambda^{p-1} |\{(Mf > \lambda\}| d\lambda \\
&= C\|\mathcal{M}f\|_p^p \leq Cp\|f\|_p^p.
\end{aligned}$$

This completes the proof. □

4.2 Bounded Mean Oscillation

Definition 4.6 *A function* $f \in L^1_{loc}(\mathbb{R}^n)$ *is called a function of bounded mean oscillation* (BMO) *if*

$$\|f\|_* = \sup_Q \fint_Q |f(x) - f_Q| dx < \infty.$$

If this is the case, $\|f\|_*$ *is called the BMO-norm of* f.

Remark Let us observe a few facts on BMO functions.

1. It is easy to see that f is constant if and only if $\|f\|_* = 0$. If we define an equivalence relation $\sim$ by

   $$f \sim g \Longleftrightarrow f - g = \text{constant a.e},$$

   then $BMO/\sim$ is a Banach space.

2. If $\alpha \in \mathbb{C}$, then

   $$\fint_Q |f - f_Q| \le \fint_Q |f - \alpha| + \fint_Q |\alpha - f_Q| \le 2 \fint_Q |f - \alpha|.$$

   Thus we have

   $$\frac{1}{2} \fint_Q |f - f_Q| \le \inf_\alpha \fint_Q |f - \alpha| \le \fint_Q |f - f_Q|.$$

   Therefore

   $$\|f\|'_* := \sup_Q \inf_{\alpha \in \mathbb{C}} \fint_Q |f - \alpha|$$

   defines an equivalent norm for BMO.

3. $L^\infty \subset BMO$. In fact, $\|f\|_* \le 2\|f\|_\infty$.

4. $\log|x| \in BMO(\mathbb{R}^n)$. We give a proof for the case $n = 1$. Let $Q = [a, b]$ and assume that $-b < a < b, b > 0$. (The other case can be treated in similar ways.)

   $$\begin{aligned} \frac{1}{|Q|} \int_Q |f - f(b)| dx &= \frac{1}{b-a} \int_a^b |\log|x| - \log b| dx \\ &= -\frac{1}{b-a} \int_a^b \log \frac{|x|}{b} dx \\ &= -\frac{b}{b-a} \int_{\frac{a}{b}}^1 \log|y| dy. \end{aligned}$$

If $\frac{a}{b} > \frac{1}{2}$, then $\log|y|$ is bounded and hence

$$I \leq C\frac{b}{b-a}\int_{\frac{a}{b}}^{1} dx \leq C.$$

If $\frac{a}{b} \leq \frac{1}{2}$, then $\frac{b}{b-a} \leq 2$ and hence

$$I \leq 2\int_{-1}^{1} \log|y| dx \leq C.$$

5. $signx \cdot \log|x| \notin BMO(\mathbb{R}^1)$. In general, $|f| \in BMO$ does not imply $f \in BMO$. Being a BMO function is *not* simply a size condition.

Theorem 4.7 (John-Nirenberg inequality) *There are constants* $C_1, C_2 >$ 0 *such that for all* $f \in BMO$, *cube* Q , $\lambda > 0$,

$$|\{x \in Q : |f(x) - f_Q| > \lambda\}| \leq C_1|Q|\exp(-\frac{C_2}{\|f\|_*}\lambda). \tag{4.4}$$

Proof. Fix a cube Q. By considering $g = C(f - f_Q)\chi_Q$ if necessary, we may assume $f_Q = 0$ and $\|f\|_* = 1$. Here"dyadic" means dyadic with respect to Q. Apply CZ-decomposition with $\alpha = 2$ to obtain mutually non-overlapping dyadic cubes $\{Q_j^1\}$ such that

(1) $|f(x)| \leq 2$ a.e. on $E_1 := Q \backslash \bigcup_j Q_j^1$,

(2) $2 < \fint_{Q_j^1} |f| \leq 2^{n+1}$ for all j,

(3) $\sum_j |Q_j^1| \leq \frac{1}{2}\int_Q |f| = \frac{1}{2}\int_Q |f - f_Q| \leq \frac{1}{2}\|f\|_*|Q| = \frac{1}{2}|Q|.$

To each $(f - f_{Q_j^1})\chi_{Q_j^1}$ apply CZ-decomposition with $\alpha = 2$ to obtain mutually non-overlapping dyadic cubes $\{Q_j^2\}$ such that

(1') $|f - f_{{Q^1}_j}| \leq 2$ a.e. on $E_2 := \bigcup_j Q_j^1 \backslash \bigcup_k Q_k^2$,

(2') $2 < \fint_{Q_k^2} |f - f_{Q_j^1}| \leq 2^{n+1}$ for all j, k such that $Q_k^2 \subset Q_j^1$,

(3') $$\sum_k |Q_k^2| = \sum_j \sum_{Q_k^2 \subset Q_k^1} |Q_k^2| \leq \frac{1}{2}\sum_j \int_{Q_j^1} |f - f_{Q_j^1}|$$
$$\leq \frac{1}{2}\sum_j |Q_j^1| = \frac{1}{2^2}|Q|.$$

Note that for almost all $x \in E_2$,

$$|f(x)| \le |f(x) - f_{Q_j^1}| + |f_{Q_j^1}| \le (2^n + 1) \cdot 2.$$

Repeat this process to obtain E_k and $\{Q_j^k\}$ $(k = 1, 2, \cdots)$ so that for almost all $x \in E_k$,

$$\begin{aligned} |f(x)| &\le |f(x) - f_{Q_j^{k-1}}| + |f_{Q_j^{k-1}} - f_{Q_j^{k-2}}| + \cdots + |f_{Q_j^1}| \\ &\le 2 + ⨍_{Q_j^{k-1}} |f - f_{Q_j^{k-2}}| + \cdots + |f_{Q_j^1}| \\ &\le 2 + (k-1)2^{n+1} = (1 + (k-1)2^n) \cdot 2. \end{aligned}$$

We also have

$$\begin{aligned} |Q \backslash \bigcup_{l=1}^{k} E_l| = \big| \bigcap_{l=1}^{k} (Q \backslash E_l) \big| &\le \big| \bigcup_j Q_j^k \big| \\ &\le \sum_j |Q_j^k| \le 2^{-k} |Q| \qquad \forall k. \end{aligned}$$

Let $\lambda > 0$ be a number. If $\lambda < 4$, there is nothing to prove. Suppose $\lambda \ge 4$ and choose k so that

$$2((k-1)2^n + 1) \le \lambda < (k \cdot 2^n + 1).$$

Then

$$\begin{aligned} |\{x \in Q : |f(x)| > \lambda\}| &\le |\{x \in Q : |f(x)| > 2((k-1)2^n + 1)\}| \\ &\le |Q \backslash \bigcup_{l=1}^{k} E_l| \\ &\le 2^{-k}|Q| \le e^{-c_2 \lambda}|Q|. \end{aligned}$$

This completes the proof. □

The following is the original version of John-Nirenberg inequality.

Corollary 4.8 *There exist constants $C, \alpha > 0$ such that for all cube Q and $f \in BMO$,*

$$⨍_Q \exp\left(\frac{\alpha}{\|f\|_*} |f(x) - f_Q| \right) dx \le C. \tag{4.5}$$

Proof. Fix Q and for $\lambda > 0$ let

$$\begin{aligned} E_\lambda &:= \{x \in Q : |f(x) - f_Q| > \lambda\} \\ &= \left\{ x \in Q : \exp\left(\frac{\alpha |f(x) - f_Q|}{\|f\|_*} \right) > \exp\left(\frac{\alpha \lambda}{\|f\|_*} \right) \right\}. \end{aligned}$$

Let $\eta = \exp(\frac{\alpha\lambda}{\|f\|_*})$. Then it follows from (4.4) that

$$\begin{aligned}
\fint_Q \exp(\frac{\alpha}{\|f\|_*}|f(x)-f_Q|)dx &= \frac{1}{|Q|}\int_0^\infty |E_\lambda| d\eta \\
&\le \frac{1}{|Q|}\int_0^\infty C_1|Q|\exp\left(\frac{-C_2\lambda}{\|f\|_*}\right)\exp\left(\frac{\alpha\lambda}{\|f\|_*}\right)\frac{\alpha}{\|f\|_*}d\lambda \\
&< C
\end{aligned}$$

if $\alpha < C_2$. This completes the proof. □

Corollary 4.9 *For* $1 \le p < \infty$*, let*

$$\|f\|_{p,*} = \sup\left(\fint_Q |f-f_Q|^p\right)^{\frac{1}{p}}.$$

Then $\|f\|_{p,*} \approx \|f\|_*$.

Proof. It follows from Jensen's inequality that

$$\|f\|_* \le \|f\|_{p,*}.$$

If $\|f\|_*=1$, then

$$\fint_Q |f-f_Q|^p \le C(p,\alpha)\fint_Q e^{\alpha|f-f_Q|}dx \le C_p.$$

Hence $\|f\|_{p,*} \le C_p$, and the proof is complete. □

4.3 BMO and Carleson Measures

Let $\psi \in C^\infty(\mathbb{R}^n)$ be such that

$$\begin{cases} |\psi(x)| + |\nabla\psi(x)| \le C(1+|x|)^{-n-1}, \\ \int_{\mathbb{R}^n} \psi(x)dx = 0. \end{cases} \tag{4.6}$$

For $t > 0$, define

$$\psi_t(x) = t^{-n}\psi(t^{-1}x),$$

and

$$Q_t f(x) = (f * \psi_t)(x).$$

We are going to prove

Theorem 4.10 *If $f \in$BMO, then*

$$d\mu(x,t) = \frac{|Q_t f(x)|^2}{t} dxdt$$

is a Carleson measure and the Carleson norm

$$C_\mu \leq C||f||_*^2.$$

Notice that since $\frac{dxdt}{t}$ is *not* a Carleson measure, the estimate $|Q_t f(x)| \leq C\|f\|_*$ is not enough to prove Theorem 4.10.

For example, let

$$\phi(x) := P_1(x) = c_n \frac{1}{(1+|x|^2)^{\frac{n+1}{2}}},$$

the Poisson kernel. Then $P_t(x) = \phi_t(x)$ and hence the Poisson extension is given by

$$P_t f(x) = (f * \phi_t)(x).$$

Let

$$\psi(x) := \nabla\phi(x) = (\psi^1(x), \ldots, \psi^n(x)).$$

Then each ψ^j satisfies the condition (4.6). Let

$$Q_t^j f(x) = (f * \psi_t^j)(x), \qquad j = 1, \cdots, n.$$

It follows from Theorem 4.10 that if $f \in BMO$, then $|Q_t^j f(x)|^2 \frac{dxdt}{t}$ is a Carleson measure. Note that

$$\nabla_x \phi_t(x) = t^{-1}\psi_t(x),$$

and hence

$$|\nabla_x P_t f(x)|^2 = |\nabla_x \phi_t * f(x)|^2 = \frac{1}{t^2}\sum_{j=1}^n |Q_t^j f(x)|^2.$$

So we have the following Theorem from Theorem 4.10.

Theorem 4.11 *If $f \in$BMO, then $t|\nabla_x P_t f(x)|^2 dxdt$ is a Carleson measure and its Carleson norm is less than $C||f||_*^2$.*

In order to prove Theorem 4.10, we need the following Lemmas

Lemma 4.12 *If $\psi \in C^\infty(\mathbb{R}^n)$ satisfies* (4.6)*, then there exists a constant C depending only on the constants in (4.6) such that*

$$|\hat{\psi}(\xi)| \leq C\frac{|\xi|^{\frac{1}{n+2}}}{1+|\xi|}. \tag{4.7}$$

Proof. If $|\xi| \le 1$, then

$$
\begin{aligned}
|\hat{\psi}(\xi)| &= \left| \int_{\mathbb{R}^n} \psi(x) e^{-2\pi i x\cdot\xi} dx \right| \\
&= \left| \int_{\mathbb{R}^n} \psi(x) [e^{-2\pi i x\cdot\xi} - 1] dx \right| \\
&\le C \int_{\mathbb{R}^n} |\psi(x)| \min(|x||\xi|, 1) dx \\
&= C\Big[\int_{|x|<\delta} + \int_{|x|\ge\delta} \Big] \\
&:= I_1 + I_2.
\end{aligned}
$$

Here $\delta > 0$ is to be chosen later. We have

$$
\begin{aligned}
I_1 &\le \int_{|x|<\delta} |\psi(x)||x||\xi| dx \\
&\le C|\xi| \int_{|x|<\delta} \frac{|x|}{(1+|x|)^{n+1}} dx \\
&\le C\delta^{n+1}|\xi|.
\end{aligned}
$$

On the other hand, we get

$$
I_2 \le \int_{|x|\ge\delta} |\psi(x)| dx \le C \int_{|x|\ge\delta} |x|^{-n-1} dx \le C\delta^{-1}.
$$

Choose $\delta = |\xi|^{-\frac{1}{n+2}}$ to obtain (4.7) for $|\xi| \le 1$.

If $|\xi| > 1$, assume without loss of generality that $|\xi_1| \ge \dfrac{1}{\sqrt{n}}|\xi|$. Write $x = (x_1, x') = (x_1, x_2, \ldots, x_n)$. Then

$$
\begin{aligned}
\hat{\psi}(\xi) &= \int_{\mathbb{R}^n} \psi(x) e^{-2\pi i x\cdot\xi} dx \\
&= \int_{x'\in\mathbb{R}^{n-1}} \Big[\int_{-\infty}^{\infty} \psi(x) e^{-2\pi i x_1\cdot\xi_1} dx_1 \Big] e^{-2\pi i x'\cdot\xi'} dx'.
\end{aligned}
$$

Integration by parts yields

$$
\begin{aligned}
\int_{-\infty}^{\infty} \psi(x) e^{-2\pi i x_1 \xi_1} dx_1 &= \int_{-\infty}^{\infty} \frac{-1}{2\pi i \xi_1} \frac{\partial}{\partial x_1} e^{-2\pi i x_1\cdot\xi_1} \psi(x) dx_1 \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi i \xi_1} \frac{\partial}{\partial x_1} \psi(x) e^{-2\pi i x_1\cdot\xi_1} dx_1.
\end{aligned}
$$

Thus we get

$$
|\hat{\psi}(\xi)| \le \frac{C}{|\xi_1|} \int_{\mathbb{R}^n} |\nabla\psi(x)| dx \le \frac{C}{|\xi|}.
$$

This completes the proof. $\square$

Corollary 4.13 *There is $C > 0$ independent of ξ such that*

$$\int_0^\infty |\hat{\psi}(t\xi)|^2 \frac{dt}{t} \le C.$$

Proof. Let $s = t|\xi|$. Then

$$\begin{aligned} \int_0^\infty |\hat{\psi}(t\xi)|^2 \frac{dt}{t} &= \int_0^\infty |\hat{\psi}(s\frac{\xi}{|\xi|})|^2 \frac{ds}{s} \\ &\le C\Big(\int_0^1 s^{\frac{2}{n+2}} \frac{ds}{s} + \int_1^\infty \frac{1}{s^2} dx\Big) \le C. \end{aligned}$$

This completes the proof. □

Lemma 4.14 *If ψ satisfies* (4.6)*, then there is C depending only on the constants in (4.6) such that*

$$\int_0^\infty \int_{\mathbb{R}^n} |Q_t f(x)|^2 \frac{dxdt}{t} \le C \int_{\mathbb{R}^n} |f(x)|^2 dx \qquad \forall f \in L^2(\mathbb{R}^n).$$

Proof. Since $\widehat{\psi_t}(\xi) = \hat{\psi}(t\xi)$, $\widehat{Q_t f}(\xi) = \hat{\psi}(t\xi)\hat{f}(\xi)$. Thus

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^n} |Q_t f(x)|^2 \frac{dxdt}{t} &= \int_0^\infty \int_{\mathbb{R}^n} |\hat{\psi}(t\xi)|^2 |\hat{f}(\xi)|^2 d\xi \frac{dt}{t} \\ &\le C \int_{\mathbb{R}^n} |f(x)|^2 dx. \end{aligned}$$

This completes the proof. □

Lemma 4.15 *There is $C > 0$ such that for $f \in$BMO and cube Q with the center at* 0*,*

$$\int_{(2Q)^c} \frac{|f(y) - f_{2Q}|}{|y|^{n+1}} dy \le C \frac{1}{l(Q)} ||f||_*.$$

Proof. On $2^{k+1}Q \backslash 2^k Q$, $|y| \approx 2^k\ l(Q)$. Therefore

$$\begin{aligned} \int_{(2Q)^c} \frac{|f(y) - f_{2Q}|}{|y|^{n+1}} dy &= \sum_{k=1}^\infty \int_{2^{k+1}Q \backslash 2^k Q} \frac{|f(y) - f_{2Q}|}{|y|^{n+1}} dy \\ &\le C \sum_{k=1}^\infty \frac{1}{(2^k\ l(Q))^{n+1}} \int_{2^{k+1}Q} |f(y) - f_{2Q}| dy \end{aligned}$$

The triangular inequality yields

$$\begin{aligned}
&\int_{2^{k+1}Q} |f(y) - f_{2Q}|dy \\
&\quad \leq \int_{2^{k+1}Q} |f(y) - f_{2^{k+1}Q}| + \sum_{j=1}^{k} \int_{2^{k+1}Q} |f_{2^{j+1}Q} - f_{2^jQ}| \\
&\quad \leq ||f||_*|2^{k+1}Q| + |2^{k+1}Q| \sum_{j=1}^{k} |f_{2^{j+1}Q} - f_{2^jQ}|.
\end{aligned}$$

However,

$$|f_{2^{j+1}Q} - f_{2^jQ}| \leq \fint_{2^jQ} |f(y) - f_{2^{j+1}Q}| \leq C||f||_*,$$

and hence

$$\int_{2^{k+1}Q} |f(y) - f_{2Q}|dy \leq C|2^{k+1}Q|||f||_*(1 + k).$$

It thus follow that

$$\int_{(2Q)^c} \frac{|f(y) - f_{2Q}|}{|y|^{n+1}} dy \leq C \sum_{k=1}^{\infty} \frac{1 + k}{2^k} \cdot \frac{1}{l(Q)} ||f||_* \leq \frac{C}{l(Q)} ||f||_*.$$

This completes the proof. □

Proof of Theorem 4.10. Let Q be a cube and assume $Q = Q_r(0)$ without loss of generality. Let $f \in BMO$. Since $\int_{\mathbb{R}^n} \psi_t(x)dx = 0$,

$$Q_t f(x) = Q_t(f - f_{2Q})(x).$$

Let $f_1 = (f - f_{2Q})\chi_{2Q}$ and $f_2 = (f - f_{2Q})\chi_{(2Q)^c}$. Then $Q_t f = Q_t f_1 + Q_t f_2$. Thus

$$\begin{aligned}
d\mu &= |Q_t f(x)|^2 \frac{dxdt}{t} \leq 2\Big(|Q_t f_1(x)|^2 \frac{dxdt}{t} + |Q_t f_2(x)|^2 \frac{dxdt}{t}\Big) \\
&:= 2(d\mu_1 + d\mu_2).
\end{aligned}$$

By Lemma 4.14, we have

$$\begin{aligned}
\mu_1(Q \times [0, r]) &\leq \int_0^\infty \int_{\mathbb{R}^n} |Q_t f_1(x)|^2 \frac{dxdt}{t} \\
&\leq C \int_{\mathbb{R}^n} |f_1(x)|^2 dx \\
&= C \int_{2Q} |f(x) - f_{2Q}|^2 dx \\
&\leq C||f||_*^2 |Q|.
\end{aligned}$$

For $d\mu_2$, we first observe that

$$\begin{aligned}|Q_t f_2(x)| &= \left| \int_{(2Q)^c} t^{-n}\psi(t^{-1}(x-y))(f(y)-f_{2Q})dy \right| \\ &\leq C\int_{(2Q)^c} t^{-n}\cdot \frac{1}{(1+\frac{|x-y|}{t})^{n+1}}|f(y)-f_{2Q}|dy \\ &\leq Ct\int_{(2Q)^c} \frac{1}{|x-y|^{n+1}}|f(y)-f_{2Q}|dy.\end{aligned}$$

If $x \in Q$ and $0 \leq t \leq r = l(Q)$, and $y \in (2Q)^c$, then $|x-y| \approx |y|$. Therefore,

$$|Q_t f_2(x)| \leq Ct\int_{(2Q)^c} \frac{|f(y)-f_{2Q}|}{|y|^{n+1}}dy.$$

It then follows from Lemma 4.15 that

$$\begin{aligned}\mu_2(Q\times[0,r]) &= \int_0^r \int_Q |Q_t f_2(x)|^2 \frac{dxdt}{t} \\ &\leq C\int_0^r \int_Q t^2 \frac{1}{r^2}||f||_*^2 \frac{dxdt}{t} \\ &= C||f||_*^2|Q|.\end{aligned}$$

This completes the proof. □

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