Boundary Layer Techniques for Deriving the Effective Properties of Composite Materials

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Abstract

In this paper we present mathematically rigorous derivations of asymptotic expansions of the effective electrical conductivity of periodic dilute composites in terms of the volume fraction occupied by the inclusions. Our derivations are based on layer potential techniques, and valid for high contrast mixtures and inclusions with Lipschitz boundaries. They are motivated by the practically important inverse problem of determining the volume fraction of a suspension of complicated shaped particles from boundary measurements of voltage potentials.

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1 Introduction

One of the classical problems in physics is the determination of the effective or macroscopic property of a two-phase medium consisting of inclusions of one material of known shape embedded homogeneously into a continuous matrix of another having physical properties different from its own. When the inclusions are well-separated d-dimensional spheres and their amount is small, the effective electrical conductivity, $\widetilde{\sigma}$, of the composite medium is

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given by the well-known Maxwell-Garnett formula [45]

$$\widetilde{\sigma} = \sigma_0 \Big[1 + f \frac{d(\sigma - \sigma_0)}{(\sigma - \sigma_0) + d\sigma_0} + df^2 \frac{(\sigma - \sigma_0)^2}{((\sigma - \sigma_0) + d\sigma_0)^2} \Big] I + o(f^2), \qquad (1)$$

where σ_0 and σ are the electrical conductivity of the continuous matrix phase and the inclusions respectively, and f is the volume fraction of the d-spheres.

Despite the importance of calculating the effective properties of composites there has been very little work addressing itself to the influence of inclusion shape. Most theoretical treatments focus on generalizing (1) to finite concentrations ($f = O(1)$). The methods include bounds on the effective properties of the mixtures and many effective medium type models have been proposed (see the book of Milton [42] and the extensive list of references therein). Indeed, there are effective medium calculations that attempt to extend (1) to higher power of f, but only for the case of d-dimensional spherical inclusions. See Jefferey [29], Sangani [45], and the references therein.

Until recently, ellipsoids are the only family of inclusions that could be rigorously and accurately estimated [49]. Douglas and Garboczi [21, 26, 39] made an important advance in treating more complicated shape inclusions by formally finding that the leading order term in the expansion of the effective conductivity (and other effective properties) in terms of the volume fraction, f, could be expressed by means of the polarization tensors of the inclusion shape which are defined in Section 4. See also the review paper of Douglas and Friedman [20], and the works of Zhikov *et al* [30], Greengard and Moura [27], Movchan and Serkov [43], and Capdeboscq and Vogelius [13].

Our objective in this paper is to present a general unified layer potentials technique for rigorously deriving very accurate asymptotic expansions of electrical effective properties of dilute media for non-spherical Lipschitz inclusions. Our approach is valid for high contrast mixtures and inclusions with Lipschitz boundaries. We shall also emphasize the fact that it gives us any higher-order term in the asymptotic expansion of the effective conductivity. To the best of our knowledge, this is the first work to rigorously justify the approximations derived by Douglas and Garboczi [21, 26, 39] using a layer potentials technique. Indeed, we go further to obtain higher-order terms in the expansions. The present work is motivated by the practically important inverse problem of determining the volume fraction of a suspension of complicated shaped particles from boundary measurements of voltage potentials.

Our approach can be easily extended to other equations such as the anisotropic conductivity problem, Stokes, the Maxwell's and the Lamé systems. Recent progress for understanding the effect of small anisotropic conductivity inhomogeneities, dielectric, electromagnetic, and elastic inclusions has been achieved in [25], [14], [48], [31], [6], [7], [5], [10], and [13]. Our method is expected to have a great potential for rigorously deriving very accurate approximations for other mixtures properties such as the effective viscosity, $\widetilde{\eta}$, of a suspension of general shaped obstacles suspended in a viscous fluid and the shear modulus, $\widetilde{\mu}$, of an elastic medium (incompressible) with arbitrary shaped elastic inclusions. We refer to Einstein [22] and Batchelor and Green [8] for approximations of $\widetilde{\eta}$ which correspond to a suspension of hard spheres in a viscous fluid. See Haber and Brenner [28] for the leading order-term in the expansion of $\widetilde{\mu}$ when the elastic inclusions are hard spheres. The derivations of high-order approximations for $\widetilde{\eta}$ and $\widetilde{\mu}$ for arbitrary shaped objects is much more difficult than the one presented here for the effective conductivity, $\widetilde{\sigma}$, because of the tensorial nature of the periodic Green's functions of the steady state Navier-Stokes equation and the Lamé system.

The combination of the high-order asymptotic expansions of $\widetilde{\sigma}$ in terms of the volume fraction f, which are derived in this paper, together with the earlier results of Bruno [11, 12] about the analyticity of $\widetilde{\sigma}$ with respect to the variable $\tau = \sigma/\sigma_0$ suggests that introducing Padé approximants of $\widetilde{\sigma}$ in the two variables f and τ for general shaped inclusions and values of f and τ (not necessary ~ 0 and ~ 1) would give a very accurate algorithm for computing $\widetilde{\sigma}$. The implementation of this new method is being under consideration and numerical results will be reported elsewhere.

2 Problem Formulation

For the sake of clarity, we will focus our attention on two-phase periodic composite materials, that are composites obtained by mixing periodically two different constituents. These composites are only considered in the context of electrical conductivity.

To set up this problem mathematically, we set Ω to be a bounded domain in $\mathbb{R}^2$, with a connected Lipschitz boundary $\partial\Omega$. We consider a periodic dilute composite filling Ω. The material consists of a matrix of constant conductivity $\sigma_0 > 0$ containing a periodic array of small conductivity inhomogeneities "centered" in those period cells that fall inside some smooth subdomain $\Omega' \subset\subset \Omega$. The periodic array has period ρ, and each period contains an inclusion of constant conductivity $\sigma > 0$ which is of the form $\rho^{1+\beta} B$ for some $\beta > 0$. Here B is a bounded Lipschitz domain in $\mathbb{R}^2$ containing the origin. As $\rho \to 0$ the volume fraction of the inhomogeneities is $O(\rho^{2\beta})$.

Let $Y =]-1/2, 1/2[^2$ denote the unit cell and $D = \rho^\beta B$. The effective conductivity matrix $\widetilde{\sigma} = (\widetilde{\sigma}_{ij})_{i,j=1,2}$ of Ω is defined by (see for instance [30, 42])

$$\widetilde{\sigma}_{ij} := \int_Y \sigma_\rho \nabla u_i \cdot \nabla u_j,$$

where $\sigma_\rho = \sigma_0 + (\sigma - \sigma_0)\chi(D)$ and u_i, for $i = 1, 2$, is the unique solution to

$$\begin{cases} \nabla \cdot \sigma_\rho \nabla u_i = 0 \quad \text{in } Y, \\ u_i - y_i \text{ periodic}, \\ \displaystyle\int_Y u_i = 0. \end{cases} \tag{2}$$

Here $\chi(D)$ denotes the characteristic function of D. Using the Green's formula we can rewrite $\widetilde{\sigma}$ in the following form:

$$\widetilde{\sigma}_{ij} = \sigma_0 \int_{\partial Y} u_j \frac{\partial u_i}{\partial \nu}, \tag{3}$$

where $\partial/\partial\nu$ is the outward normal derivative to ∂Y. The matrix $\widetilde{\sigma}$ depends on ρ as a parameter, and cannot be written explicitly.

The organization of the paper is as follows. We give in the next section some useful facts on the periodic Green's function G and periodic layer potentials. In Section 4 we recall the concept of generalized polarization tensors. In Section 5 we establish a representation formula for the unique solution to (2). This formula generalizes the formula proved by Kang and Seo in [32, 33]. Our aim in Section 6 is to rigorously derive a complete asymptotic expansion of $\widetilde{\sigma}$ in terms of ρ for any shape B of the inclusion and any prescribed conductivity σ. Our technique is in the spirit of [3]. The derivation simply follows from the Taylor expansion of $G(x)-1/(2\pi)\log|x|$ at 0. Our asymptotic expansions are valid for high contrast mixtures and inclusions with Lipschitz boundaries. The concept of generalized polarization tensors is employed for these derivations. Section 7 is devoted to the case of multiple closely spaced inclusions.

3 Periodic Layer Potentials and Representation Formula

Let $\Gamma(x)$ be the fundamental solution of the Laplacian Δ in $\mathbb{R}^2$:

$$\Gamma(x) = \frac{1}{2\pi}\ln|x|. \tag{4}$$

Let us introduce the periodic Green's function through its Fourier representation:

$$G(x) = -\sum_{n\in\mathbb{Z}^2\setminus\{0\}} \frac{e^{i2\pi n\cdot x}}{4\pi^2|n|^2}. \tag{5}$$

Then we get, in the sense of distributions,

$$\Delta G(x) = \sum_{n\in\mathbb{Z}^2\setminus\{0\}} e^{i2\pi n\cdot x} = \sum_{n\in\mathbb{Z}^2} e^{i2\pi n\cdot x} - 1.$$

It then follows from the Poisson summation formula

$$\sum_{n\in\mathbb{Z}^2} e^{i2\pi n\cdot x} = \sum_{n\in\mathbb{Z}^2} \delta(x+n),$$

that

$$\Delta G(x) = \sum_{n\in\mathbb{Z}^2} \delta(x+n) - 1. \tag{6}$$

The distribution $\sum_{n\in\mathbb{Z}^2}\delta(x+n)$ is the periodic Dirac function. Note that because of the condition of charge neutrality that the right-hand side of the Poisson equation associated with (5) must satisfy there is no Green's function for the bare Coulomb potential, i.e., a periodic solution G' to $\Delta G'(x) = \sum_{n\in\mathbb{Z}^2}\delta(x+n)$ in the unit cell Y. We refer to [15] for another representation of G.

The following lemma on the behavior of the periodic Green's function $G(x)$ as $x \to 0$ is important for later use.

Lemma 3.1 *There exists a harmonic function $R(x)$ in the unit cell Y such that*

$$G(x) = \frac{1}{2\pi}\log|x| + R(x). \tag{7}$$

Moreover, the Taylor expansion of $R(x)$ at 0 is given by

$$R(x) = R(0) - \frac{1}{4}(x_1^2 + x_2^2) + O(|x|^4). \tag{8}$$

Proof. We have

$$
\begin{aligned}
G(x) &= -\sum_{n\in\mathbb{Z}^2\setminus\{0\}} \frac{e^{i2\pi n\cdot x}}{4\pi^2|n|^2} = -\frac{1}{4\pi^2}\sum_{n\in\mathbb{Z}^2\setminus\{0\}} \frac{\cos 2\pi n_1x_1\cos 2\pi n_2x_2}{n_1^2+n_2^2}\\
&= -\frac{1}{2\pi^2}\sum_{n_1=0}^{+\infty}\cos 2\pi n_1x_1\sum_{n_2=1}^{+\infty}\frac{\cos 2\pi n_2x_2}{n_1^2+n_2^2} - \frac{1}{2\pi^2}\sum_{n_2=0}^{+\infty}\cos 2\pi n_2x_2\sum_{n_1=1}^{+\infty}\frac{\cos 2\pi n_1x_1}{n_1^2+n_2^2}\\
&:= G_1+G_2.
\end{aligned}
$$

Let us invoke three summation identities (see for instance [17], pp. 813-814):

$$
\sum_{n_2=1}^{+\infty}\frac{\cos 2\pi n_2x_2}{n_1^2+n_2^2} = \begin{cases} -\dfrac{1}{2n_1^2}+\dfrac{\pi}{2n_1}\dfrac{\cosh\pi(2x_2-1)n_1}{\sinh\pi n_1} & \text{if } n_1\neq 0,\\ \dfrac{\pi^2}{6}-\pi^2x_2+\pi^2x_2^2 & \text{if } n_1=0,\end{cases}
$$

$$
\sum_{n_1=1}^{+\infty}\frac{\cos 2\pi n_1x_1}{n_1}e^{-2\pi n_1x_2} = \pi x_2-\log 2-\frac{1}{2}\log\Big(\sinh^2\pi x_2+\sin^2\pi x_1\Big).
$$

We then compute

$$
\begin{aligned}
G_1 &= -\frac{1}{2\pi^2}\sum_{n_2=1}^{+\infty}\frac{\cos 2\pi n_2x_2}{n_2^2} - \frac{1}{2\pi^2}\sum_{n_1=1}^{+\infty}\cos 2\pi n_1x_1\Big(-\frac{1}{2n_1^2}+\frac{\pi}{2n_1}\frac{\cosh\pi(2x_2-1)n_1}{\sinh\pi n_1}\Big)\\
&= -\frac{1}{2\pi^2}\sum_{n_2=1}^{+\infty}\frac{\cos 2\pi n_2x_2}{n_2^2} + \frac{1}{4\pi^2}\sum_{n_1=1}^{+\infty}\frac{\cos 2\pi n_1x_1}{n_1^2}\\
&\qquad -\frac{1}{4\pi}\sum_{n_1=1}^{+\infty}\frac{\cos 2\pi n_1x_1}{n_1}\frac{\cosh\pi(2x_2-1)n_1}{\sinh\pi n_1}\\
&= -\frac{1}{12}+\frac{1}{2}x_2-\frac{1}{2}x_2^2+\frac{1}{24}-\frac{1}{4}x_1+\frac{1}{4}x_1^2-\frac{1}{4\pi}\sum_{n_1=1}^{+\infty}\frac{\cos 2\pi n_1x_1}{n_1}e^{-2\pi n_1x_2}\\
&-\frac{1}{4\pi}\sum_{n_1=1}^{+\infty}\frac{\cos 2\pi n_1x_1}{n_1}\Big(\frac{\cosh\pi(2x_2-1)n_1}{\sinh\pi n_1}-e^{-2\pi n_1x_2}\Big)
\end{aligned}
$$

to arrive at

$$
G_1 = -\frac{1}{24}+\frac{\log 2}{4\pi}+\frac{1}{4}(x_2-x_1)-\frac{1}{4}(2x_2^2-x_1^2)+\frac{1}{8\pi}\log\Big(\sinh^2\pi x_2+\sin^2\pi x_1\Big)+r_1(x),
$$

where the function $r_1(x)$ is given by

$$
\begin{aligned}
r_1(x) &= -\frac{1}{4\pi}\sum_{n_1=1}^{+\infty}\frac{\cos 2\pi n_1x_1}{n_1}\Big(\frac{\cosh\pi(2x_2-1)n_1}{\sinh\pi n_1}-e^{-2\pi n_1x_2}\Big)\\
&= -\frac{1}{4\pi}\sum_{n_1=1}^{+\infty}\frac{\cos 2\pi n_1x_1}{n_1}\frac{e^{2\pi n_1x_2}+e^{-2\pi n_1x_2}}{e^{2\pi n_1}-1}.
\end{aligned}
$$

Because of the term $e^{-\pi n_1}$, one can easily see that r_1 is a $\mathcal{C}^\infty$-function.

In the same way we can derive

$$G_2 = -\frac{1}{24} + \frac{\log 2}{4\pi} + \frac{1}{4}(x_1 - x_2) - \frac{1}{4}(2x_1^2 - x_2^2) + \frac{1}{8\pi}\log\left(\sinh^2\pi x_1 + \sin^2\pi x_2\right) + r_2(x),$$

where

$$r_2(x) = -\frac{1}{4\pi}\sum_{n_1=1}^{+\infty}\frac{\cos 2\pi n_1 x_2}{n_1}\frac{e^{2\pi n_1 x_1} + e^{-2\pi n_1 x_1}}{e^{2\pi n_1} - 1}.$$

By the Taylor expansion, one can see that

$$\log\left(\sinh^2\pi x_2 + \sin^2\pi x_1\right) + \log\left(\sinh^2\pi x_1 + \sin^2\pi x_2\right) = 4\log(\pi) + 2\log(x_1^2 + x_2^2) + r_3(x),$$

where $r_3(x)$ is a C^∞ function with $r_3(x) = O(|x|^4)$ as $|x| \to 0$. In short, we obtain

$$G(x) = \frac{1}{2\pi}\log|x| + R(x),$$

where

$$R(x) = C - \frac{1}{4}(x_1^2 + x_2^2) + r_1(x) + r_2(x) + r_3(x)$$

for some constant C. By the Taylor expansion again, one can see that $r_1(x) + r_2(x) = C + O(|x|^4)$ as $|x| \to 0$, for some constant C. That R is harmonic follows from (6). This concludes the proof. □

We can expand $R(x)$ even further to get

$$R(x) = R(0) - \frac{1}{4}(x_1^2 + x_2^2) + \sum_{k=3}^{m} R_k(x) + O(|x|^{m+1}), \quad \text{as } |x| \to 0,$$

where R_k is a harmonic polynomial of homogeneous degree k. Since $R(-x_1, x_2) = R(x_1, x_2)$ and $R(x_1, -x_2) = R(x_1, x_2)$, $R_k \equiv 0$ if k is odd, and hence

$$R(x) = R(0) - \frac{1}{4}(x_1^2 + x_2^2) + \sum_{k=2}^{m} R_{2k}(x) + O(|x|^{m+2}), \quad \text{as } |x| \to 0. \tag{9}$$

Let $L_0^2(\partial D) := \{f \in L^2(\partial D) : \int_{\partial D} f\, ds = 0\}$. The periodic single layer potential of the density function $\varphi \in L_0^2(\partial D)$ is defined for $x \in \mathbb{R}^2$ by

$$\mathcal{S}_D\varphi(x) := \int_{\partial D} G(x - y)\varphi(y)\, ds(y).$$

Let $\widetilde{\mathcal{S}}_D$ be the (non-periodic) single layer potential, i.e.,

$$\widetilde{\mathcal{S}}_D\varphi(x) := \int_{\partial D} \Gamma(x - y)\varphi(y)\, ds(y).$$

Then by Lemma 3.1, we have

$$\mathcal{S}_D\varphi(x) = \widetilde{\mathcal{S}}_D\varphi(x) + \int_{\partial D} R(x - y)\varphi(y)\, ds(y), \qquad \text{for any } \varphi \in L_0^2(\partial D),$$

and the second operator on the right hand side is a smoothing operator.

For a function u defined on $\mathbb{R}^2 \setminus \partial D$, we denote

$$\frac{\partial u}{\partial \nu}\bigg|_{\pm}(x) = \lim_{t\to 0} \nu(x)\cdot \nabla u(x \pm t\nu(x)) \text{ for } x \in \partial D$$

if the limit exists. Here $\nu(x)$ is the outward unit normal to ∂D at x. The following theorem is proved in [23, 47].

Theorem 3.1 **(i)** *For $\varphi \in L^2(\partial D)$. The following trace formula holds:*

$$\frac{\partial}{\partial \nu}\widetilde{\mathcal{S}}_D\varphi\bigg|_{\pm}(x) = (\pm\frac{1}{2}I + \widetilde{\mathcal{K}}_D^*)\varphi(x) \quad \text{on } \partial D, \tag{10}$$

where the singular bounded integral operator $\widetilde{\mathcal{K}}_D^* : L^2(\partial D) \to L^2(\partial D)$ is given by

$$\widetilde{\mathcal{K}}_D^*\varphi(x) = p.v. \int_{\partial D} \frac{\partial}{\partial \nu(x)}\Gamma(x-y)\varphi(y)\, ds(y). \tag{11}$$

(ii) *If $|\lambda| \geq \frac{1}{2}$, then the operator $\lambda I - \widetilde{\mathcal{K}}_D^*$ is invertible on $L_0^2(\partial D)$.*

As a consequence of Theorem 3.1, we get the following Lemma on the periodic layer potential.

Lemma 3.2 **(i)** *Let $\varphi \in L_0^2(\partial D)$. The following trace formula holds:*

$$\frac{\partial}{\partial \nu}\mathcal{S}_D\varphi\bigg|_{\pm}(x) = (\pm\frac{1}{2}I + \mathcal{K}_D^*)\varphi(x) \text{ on } \partial D, \tag{12}$$

where $\mathcal{K}_D^* : L_0^2(\partial D) \to L_0^2(\partial D)$ is given by

$$\mathcal{K}_D^*\varphi(x) = p.v. \int_{\partial D} \frac{\partial}{\partial \nu(x)}G(x-y)\varphi(y)\, ds(y).$$

(ii) *If $\varphi \in L_0^2(\partial D)$, then $\mathcal{S}_D\varphi$ is harmonic in D and $Y \setminus \overline{D}$.*

(iii) *If $|\lambda| \geq \frac{1}{2}$, then the operator $\lambda I - \mathcal{K}_D^*$ is invertible on $L_0^2(\partial D)$.*

Proof. Since $\mathcal{K}_D^* = \widetilde{\mathcal{K}}_D^* + R_D$ where the smoothing operator R_D is defined by

$$R_D\varphi(x) = \int_{\partial D} \frac{\partial}{\partial \nu(x)}R(x-y)\varphi(y)\, ds(y), \tag{13}$$

(i) immediately follows from Theorem 3.1 (i).
(ii) follows from (6) and the fact that $\varphi \in L_0^2(\partial D)$.
As a consequence of (i) and (ii), it follows that $\lambda I - \mathcal{K}_D^*$ maps $L_0^2(\partial D)$ into $L_0^2(\partial D)$. To prove (iii), we observe that R_D is a smoothing operator. In particular, R_D maps $L^2(\partial D)$ into $H^1(\partial D)$, and hence it is a compact operator on $L^2(\partial D)$. Since $\lambda I - \widetilde{\mathcal{K}}_D^*$ is invertible on $L_0^2(\partial D)$, by the Fredholm alternative, it suffices to show that $\lambda I - \mathcal{K}_D^*$ is one-to-one on

$L^2_0(\partial D)$. To do so, let $|\lambda| \geq \frac{1}{2}$, and suppose that $\varphi \in L^2_0(\partial D)$ satisfies $(\lambda I - \mathcal{K}^*_D)\varphi = 0$ and $\varphi \neq 0$. Let

$$A := \int_D |\nabla \mathcal{S}_D \varphi|^2 \, dx, \quad B := \int_{Y \setminus \overline{D}} |\nabla \mathcal{S}_D \phi|^2 \, dx \, .$$

Then $A \neq 0$. In fact, if $A = 0$, then $\mathcal{S}_D \varphi$ is constant in D. Therefore $\mathcal{S}_D \varphi$ in $Y \setminus \overline{D}$ satisfies that $\mathcal{S}_D \varphi|_{\partial D} =$ constant and periodic. Hence $\mathcal{S}_D \varphi =$ constant in $Y \setminus \overline{D}$. Therefore, by (i), we get

$$\varphi = \frac{\partial}{\partial \nu} \mathcal{S}_D \varphi \Big|_+ - \frac{\partial}{\partial \nu} \mathcal{S}_D \varphi \Big|_- = 0,$$

which contradicts our assumption. In a similar way, one can show that $B \neq 0$.

On the other hand, using the divergence theorem and periodicity, we have

$$A = \int_{\partial D} (-\frac{1}{2} I + \mathcal{K}^*_D) \phi \; \mathcal{S}_D \phi \, d\sigma, \quad B = -\int_{\partial D} (\frac{1}{2} I + \mathcal{K}^*_D) \phi \; \mathcal{S}_D \phi \, d\sigma \, .$$

Since $(\lambda I - \mathcal{K}^*_D)\phi = 0$, it follows that

$$\lambda = \frac{1}{2} \frac{B - A}{B + A} \, .$$

Thus, $|\lambda| < 1/2$, which is a contradiction. This completes the proof. □

The following explicit representation formula will be very useful for the derivation of a complete asymptotic expansion for $\widetilde{\sigma}$. This type of formulae was first obtained by Kang and Seo [32, 33] for the conductivity problem in a bounded domain. The following lemma holds.

Lemma 3.3 *Let u_i be the unique solution to the transmission problem (2). Then u_i can be expressed as follows*

$$u_i(x) = x_i + C_i + \mathcal{S}_D (\lambda I - \mathcal{K}^*_D)^{-1} (\nu_i)(x) \quad \text{in } Y, \quad i = 1, 2, \tag{14}$$

where C_i is a constant,

$$\lambda = \frac{1}{2} \frac{\sigma + \sigma_0}{\sigma - \sigma_0}, \tag{15}$$

and ν_i is the i-component of the outward unit normal ν to ∂D.

Before proving this lemma, it is worth observing that x_i is the harmonic part of the solution u_i and $\mathcal{S}_D (\lambda I - \mathcal{K}^*_D)^{-1}(\nu_i)$ carries the information on the reflection. Moreover, this decomposition of u_i into a harmonic part and a reflection part depends only on the inclusion D.

Proof. Observe that (2) is equivalent to

$$\begin{cases} \Delta u_i = 0 \quad \text{in } D \cup (Y \setminus \overline{D}), \\ u_i|_+ - u_i|_- = 0 \quad \text{on } \partial D, \\ \sigma_0 \dfrac{\partial u_i}{\partial \nu} \Big|_+ - \sigma \dfrac{\partial u_i}{\partial \nu} \Big|_- = 0 \quad \text{on } \partial D, \\ u_i - y_i \text{ periodic}, \\ \displaystyle\int_Y u_i = 0. \end{cases}$$

Define $V_i(x) = \mathcal{S}_D(\lambda I - \mathcal{K}_D^*)^{-1}(\nu_i)(x)$ in Y. Then

$$\begin{cases} \Delta V_i = 0 \text{ in } D \cup (Y \setminus \overline{D}), \\ V_i|_+ - V_i|_- = 0 \text{ on } \partial D, \\ \frac{\partial V_i}{\partial \nu}|_+ - \frac{2\lambda+1}{2\lambda-1}\frac{\partial V_i}{\partial \nu}|_- = \frac{2}{2\lambda-1}\nu_i \text{ on } \partial D, \\ V_i \text{ periodic.} \end{cases} \tag{16}$$

Thus by choosing C_i so that $\int_Y u_i dx = 0$, we get (14). This completes the proof. □

4 Generalized Polarization Tensors

4.1 Single Inclusion

We now recall the concept of generalized polarization tensors. Let B be a bounded Lipschitz domain in $\mathbb{R}^2$ and $0 < \sigma \neq \sigma_0 < +\infty$.

Definition 4.1 *The generalized polarization tensor of order $(l, l') \in \mathbb{N}^2 \times \mathbb{N}^2$ associated to (B, σ) is given by:*

$$M^{l,l'}(\lambda, B) = \int_{\partial B} y^l (\lambda I - \widetilde{\mathcal{K}}_B^*)^{-1}(\frac{\partial y^{l'}}{\partial \nu})\, ds(y), \tag{17}$$

where the constant λ and the operator $\widetilde{\mathcal{K}}_B^*$ are given by (15) and (11), respectively.

It is worth to note that there exists an equivalent definition for the generalized polarization tensor that is

$$M^{l,l'}(\lambda, B) = (\frac{1}{\sigma_0} - \frac{1}{\sigma})(\int_{\partial B} y^l \frac{\partial y^{l'}}{\partial \nu}) d\sigma_y + (\sigma - \sigma_0)\frac{\partial \psi^{l'}}{\partial \nu}(y)|_+ d\sigma_y),$$

where $\psi^{l'}$ is the solution of the problem:

$$\begin{cases} \Delta \psi^{l'} = 0 \text{ in } B \cup (\mathbb{R}^2 \setminus \overline{B}), \\ \psi^{l'}|_+ - \psi^{l'}|_- = 0 \text{ on } \partial B, \\ \sigma_0 \frac{\partial \psi^{l'}}{\partial \nu}|_+ - \sigma \frac{\partial \psi^{l'}}{\partial \nu}|_- = \nu \cdot \nabla x^{l'} \quad \partial B, \\ \psi^{l'}(x) + \frac{1}{2\pi}(\sigma_0 - \sigma)\log|x| \int_{\partial B} \nu \cdot \nabla y^{l'} \to 0. \end{cases}$$

Furthermore, Definition 4.1 of the generalized polarization tensors is valid even for the extreme cases when $\sigma = 0$ or ∞. If $\sigma = 0$, namely, if B is an insulating inclusion, then

$$M^{l,l'}(-\frac{1}{2}, B) = \int_{\partial B} y^l (-\frac{1}{2} I - \widetilde{\mathcal{K}}_B^*)^{-1}(\frac{\partial y^{l'}}{\partial \nu})\, ds(y),$$

while if $\sigma = +\infty$, namely, if B is perfectly conducting, then

$$M^{l,l'}(\frac{1}{2}, B) = \int_{\partial B} y^l (\frac{1}{2} I - \widetilde{\mathcal{K}}_B^*)^{-1}(\frac{\partial y^{l'}}{\partial \nu})\, ds(y).$$

The generalized polarization tensors enjoy some important properties such as symmetry and positivity and their eigenvalues can be estimated in terms of the volume of the inclusion B. We refer the reader to [2, 1] for exact statements and rigorous proofs of these properties. The following lemma from [2] will be of use to us.

Lemma 4.1 *Suppose that a_l and $b_{l'}$ are constants such that $\sum_l a_l y^l$ and $\sum_{l'} b_{l'} y^{l'}$ are harmonic polynomials. Then*

$$\sum_{l,l'} a_l b_{l'} M^{l,l'}(\lambda, B) = \sum_{l,l'} a_l b_{l'} M^{l',l}(\lambda, B).$$

We will refer to the polarization tensor $M^{l,l'}$ when $|l| = |l'| = 1$ as the Pólya-Szegö polarization tensor. This classical polarization tensor has been extensively studied in the literature [14, 25, 35, 44, 46, 19]. When $\sigma = 0$ and $|l| = |l'| = 1$, $M^{l,l'}$ is called the virtual mass. From the definition (17) it follows that the polarization tensor of an insulating inclusion is related to the one of a perfectly conducting inclusion of the same shape by a change of sign:

$$\text{If } |l| = |l'| = 1 \text{ then } M^{l,l'}(-\frac{1}{2}, B) = -M^{l,l'}(\frac{1}{2}, B). \tag{18}$$

The following lemma is proved in [37].

Lemma 4.2 *Suppose that a_l and $b_{l'}$ are constants such that $\sum_l a_l y^l$ and $\sum_{l'} b_{l'} y^{l'}$ are harmonic polynomials of homogeneous degree. If B is a disk, then*

$$\sum_{l,l'} a_l b_{l'} M^{l,l'}(\lambda, B) = 0 \quad \textit{if } \sum_l a_l y^l \neq \sum_{l'} b_{l'} y^{l'}.$$

We will denote $M^{l,l'}$ by $M_{l,i}$ if $l' = e_i$. We also use the notation M_{ij} for $M^{l,l'}$ if $l = e_i$ and $l' = e_j$. Then it follows from Lemma 4.1 and Lemma 4.2 that if a_l are constants such that $\sum_l a_l y^l$ is a harmonic polynomial without a linear term, then

$$\sum_l a_l M_{i,l}(\lambda, B) = \sum_l a_l M_{l,i}(\lambda, B) = 0, \quad i = 1, 2. \tag{19}$$

4.2 Multiple Inclusions

Let B_s for $s = 1, \ldots, m$ be a bounded Lipschitz domain in $\mathbb{R}^2$. Suppose that:

(H1) there exist positive constants C_1 and C_2 such that

$$C_1 \leq \operatorname{diam} B_s \leq C_2, \quad \text{and} \quad C_1 \leq \operatorname{dist}(B_s, B_{s'}) \leq C_2, \quad s \neq s' ;$$

(H2) the conductivity of the inclusion B_s for $s = 1, \ldots, m$ is equal to some positive constant $\sigma_s \neq \sigma_0$.

The following definition is introduced in [4].

Definition 4.2 *Let $l = (l_1, l_2), l' = (l'_1, l'_2) \in \mathbb{N}^2$ be multi-indices. For $s = 1, \ldots, m$, let $\varphi_{l'}^{(s)}$ be the solution of*

$$(\lambda_s I - \widetilde{\mathcal{K}}_{B_s}^*)\varphi_{l'}^{(s)} - \sum_{s' \neq s} \frac{\partial \widetilde{\mathcal{S}}_{B_{s'}} \varphi_{l'}^{(s')}}{\partial \nu^{(s)}}\Big|_{\partial B_s} = \frac{\partial y^{l'}}{\partial \nu^{(s)}}\Big|_{\partial B_s} \quad \textit{on } \partial B_s, \tag{20}$$

where $\lambda_s = (\sigma_s + \sigma_0)/(2(\sigma_s - \sigma_0))$ and $\nu^{(s)}$ denotes the outward unit normal to ∂B_s. Then the polarization tensor $\mathcal{M}^{l,l'}$ of the multiple inclusions $\cup_{s=1}^m B_s$ is defined by

$$\mathcal{M}^{l,l'} = \sum_{s=1}^{m} \int_{\partial B_s} y^l \varphi_{l'}^{(s)}(y)\, ds(y)\ . \tag{21}$$

If $|l| = |l'| = 1$, we denote $\mathcal{M}_{ij} = \mathcal{M}^{l,l'}$ where $l = e_i$ and $l' = e_j$, and $\mathcal{M} = (\mathcal{M}_{ij})_{i,j=1}^2$. This 2×2 matrix $\mathcal{M}$ is called the first-order polarization tensor.

We define the overall conductivity $\bar{\sigma}$ of $B = \cup_{s=1}^m B_s$ by

$$\frac{\bar{\sigma} - 1}{\bar{\sigma} + 1} \sum_{s=1}^{m} |B_s| := \sum_{s=1}^{m} \frac{\sigma_s - 1}{\sigma_s + 1} |B_s|. \tag{22}$$

It is shown in [4] that we can represent and visualize the multiple inclusions $\cup_{s=1}^m B_s$ by means of an ellipse, $\mathcal{E}$ with the same first-order polarization tensor. We call $\mathcal{E}$ the equivalent ellipse of $\cup_{s=1}^m B_s$. The following useful lemma is also proved in [4].

Lemma 4.3 *Suppose that a_l and $b_{l'}$ are constants such that $\sum_l a_l y^l$ and $\sum_{l'} b_{l'} y^{l'}$ are harmonic polynomials. Then*

$$\sum_{l,l'} a_l b_{l'} \mathcal{M}^{l,l'} = \sum_{l,l'} a_l b_{l'} \mathcal{M}^{l',l}.$$

5 Asymptotic Expansion of the Effective Conductivity

For simplicity, we set for $i = 1, 2$

$$\varphi_i(y) = (\lambda I - \mathcal{K}_D^*)^{-1}(\nu_i)(y) \quad \text{for } y \in \partial D. \tag{23}$$

Substituting the representation formula (14) into (3) yields

$$\widetilde{\sigma}_{ij} = \sigma_0 \int_{\partial Y} (y_j + C + \mathcal{S}_D \varphi_j(y)) \frac{\partial}{\partial \nu}(y_i + \mathcal{S}_D \varphi_i(y)) ds.$$

Because of periodicity of $\mathcal{S}_D \varphi_j$, we get

$$\int_{\partial Y} \frac{\partial}{\partial \nu} \mathcal{S}_D \varphi_j ds = \int_{\partial Y} \nu_j \mathcal{S}_D \varphi_i ds = \int_{\partial Y} \mathcal{S}_D \varphi_j(y) \frac{\partial}{\partial \nu} \mathcal{S}_D \varphi_i(y) ds = 0,$$

and hence we have

$$\widetilde{\sigma}_{ij} = \sigma_0 \Big[\delta_{ij} + \int_{\partial Y} y_j \frac{\partial}{\partial \nu} \mathcal{S}_D \varphi_i(y)\, ds(y) \Big]. \tag{24}$$

Let

$$\psi_i(y) = \varphi_i(\rho^\beta y) \quad \text{for } y \in \partial B.$$

Lemma 5.1 *For $i, j = 1, 2$, the following identity holds*

$$\int_{\partial Y} y_j \frac{\partial}{\partial \nu} \mathcal{S}_D \varphi_i(y)\, ds(y) = \rho^{2\beta} \int_{\partial B} y_j \psi_i(y)\, ds(y). \tag{25}$$

Proof. Periodicity of $\mathcal{S}_D\varphi_i$ and the divergence theorem applied on $Y \setminus \overline{D}$ yield

$$\int_{\partial Y} y_j \frac{\partial}{\partial \nu}\mathcal{S}_D\varphi_i(y)\, ds = \int_{\partial D} y_j \frac{\partial}{\partial \nu}\mathcal{S}_D\varphi_i\big|_+(y)\, ds - \int_{\partial D} \nu_j \mathcal{S}_D\varphi_i(y)\, ds$$
$$\int_{\partial D} y_j \frac{\partial}{\partial \nu}\mathcal{S}_D\varphi_i\big|_+(y)\, ds - \int_{\partial D} y_j \frac{\partial}{\partial \nu}\mathcal{S}_D\varphi_i\big|_-(y)\, ds.$$

By (12), we get

$$\int_{\partial Y} y_j \frac{\partial}{\partial \nu}\mathcal{S}_D\varphi_i(y)\, ds = \int_{\partial D} y_j \varphi_i(y)\, ds.$$

Now by scaling $y \to \rho^\beta y$, we obtain (25) and the proof is complete. □

Let

$$p_{ij} := \int_{\partial B} y_j \psi_i(y)\, ds(y), \quad i, j = 1, 2, \tag{26}$$

and $P := (P_{ij})$. Then by (24) and (25), we obtain

$$\widetilde{\sigma} = \sigma_0[I + \rho^{2\beta} P]. \tag{27}$$

In order to derive an asymptotic expansion of $\widetilde{\sigma}$ we now expand P in terms of ρ. In view of (7), the integral equation (23) can be rewritten as

$$(\lambda I - \widetilde{\mathcal{K}}_D^*)\varphi_i(x) - \int_{\partial D} \frac{\partial}{\partial \nu(x)} R(x-y)\varphi_i(y)\, ds(y) = \nu_i(x), \quad x \in \partial D,$$

which, by an obvious change of variables, yields

$$(\lambda I - \widetilde{\mathcal{K}}_B^*)\psi_i(x) - \rho^\beta \int_{\partial B} \frac{\partial}{\partial \nu(x)} R(\rho^\beta(x-y))\psi_i(y)\, ds(y) = \nu_i(x), \quad x \in \partial B. \tag{28}$$

To illustrate our method, we first restrict for simplicity ourselves to the derivations of the $\rho^{4\beta}$-order terms in the asymptotic expansion of $\widetilde{\sigma}$.

To derive our asymptotic formula we use (8) to write

$$\nu \cdot \nabla R(\rho^\beta(x-y)) = -\frac{\rho^\beta}{2}\nu \cdot (x-y) + O(\rho^{3\beta}) \tag{29}$$

uniformly in $x, y \in \partial B$. Since $\int_{\partial B} \psi_i(y)\, ds(y) = 0$, then we get

$$(\lambda I - \widetilde{\mathcal{K}}_B^*)\psi_i(x) - \frac{\rho^{2\beta}}{2}\nu(x) \cdot \int_{\partial B} y\psi_i(y)\, ds(y) + O(\rho^{4\beta}) = \nu_i(x), \quad x \in \partial B,\ i = 1, 2. \tag{30}$$

Therefore, we obtain

$$\psi_i = (\lambda I - \widetilde{\mathcal{K}}_B^*)^{-1}(\nu_i) + \frac{\rho^{2\beta}}{2}\sum_{k=1}^{2}(\lambda I - \widetilde{\mathcal{K}}_B^*)^{-1}(\nu_k) \cdot \int_{\partial B} y_k \psi_i(y)\, ds(y)$$
$$+O(\rho^{4\beta}) \quad \text{on } \partial B. \tag{31}$$

Let $\widetilde{\psi}_i := (\lambda I - \widetilde{\mathcal{K}}_B^*)^{-1}(\nu_i)$, $i = 1, 2$. Then $M_{ij} = \int_{\partial B} y_j \widetilde{\psi}_i(y) ds(y)$, etc. By iterating the formula (31), we get

$$\psi_i = \widetilde{\psi}_i + \frac{\rho^{2\beta}}{2} \sum_{k=1}^{2} \widetilde{\psi}_k \int_{\partial B} y_k \widetilde{\psi}_i(y)\, ds(y) + O(\rho^{4\beta}) \quad \text{on } \partial B.$$

It then follows from the definition (26) of P that

$$P_{ij} = M_{ij} + \frac{\rho^{2\beta}}{2} \sum_{k=1}^{2} M_{kj} M_{ik} + O(\rho^{4\beta}), \tag{32}$$

and then we obtain from (27) the following theorem.

Theorem 5.1 *We have*

$$\widetilde{\sigma} = \sigma_0 \Big[I + \rho^{2\beta} M + \frac{\rho^{4\beta}}{2} M^2 \Big] + O(\rho^{6\beta}), \tag{33}$$

where $M = M^{1,1}(\lambda, B)$ is the Pólya-Szegö polarization tensor associated to the scaled inclusion B and the conductivity $\sigma = \sigma_0(2\lambda + 1)/(2\lambda - 1)$.

In the case of spherical inclusions the Pólya-Szegö polarization tensor $M^{1,1}$ is known exactly:

$$M = mI, \quad m = \frac{2(\sigma - \sigma_0)}{\sigma + \sigma_0} |B|, \tag{34}$$

and therefore, (33) yields the well-known Maxwell-Garnett formula (1):

$$\widetilde{\sigma} = \sigma_0 \Big[1 + f \frac{2(\sigma - \sigma_0)}{\sigma + \sigma_0} + 2 f^2 \frac{(\sigma - \sigma_0)^2}{(\sigma + \sigma_0)^2} \Big] I + o(f^2),$$

where $f = \rho^{2\beta}|B|$ is the volume fraction occupied by the conductivity inhomogeneities within the unit cell.

Note also that in view of (33), identity (18) that asserts that the polarization tensor of an insulating inclusion is related to the one of a perfectly conducting inclusion of the same shape by simply a change of sign is nothing else than the Keller-Mendelson inversion theorem [34, 41]. Moreover, performing an expansion of the classical Hashin-Shtrikman bounds for $\widetilde{\sigma}$ in terms of f, as it has been done for instance in [36], we immediately obtain, in view of formula (33), optimal bounds on the trace of the Pólya-Szegö polarization tensor. See [9] and [38].

To end this section, we show how to derive further terms in the asymptotic expansion of $\widetilde{\sigma}$ following the same arguments as in Theorem 5.1 and how the generalized polarization tensors naturally occur there.

By the (higher-order) Taylor expansion

$$R(x) = R(0) - \frac{1}{4}(x_1^2 + x_2^2) + R_4(x) + O(|x|^6) \tag{35}$$

given in (9), we get

$$\nu \cdot \nabla R(\rho^\beta (x - y)) = -\frac{\rho^\beta}{2} \nu \cdot (x - y) + \rho^{3\beta} \nu \cdot \nabla_x R_4(x - y) + O(\rho^{5\beta}) \tag{36}$$

uniformly in $x, y \in \partial B$. Write

$$R_4(x-y) = \sum_{|l|+|l'|=4} c_{l,l'} x^l y^{l'}.$$

Then for each fixed l', $\sum_l c_{l,l'} x^l$ is harmonic since

$$\sum_l c_{l,l'} x^l = \frac{1}{l'!} \partial_y^{l'} (R_4(x-y))\big|_{y=0}.$$

It follows from (28) that for $i = 1, 2$

$$\begin{aligned}(\lambda I - \widetilde{\mathcal{K}}_B^*)\psi_i(x) - \frac{\rho^{2\beta}}{2}\nu(x)\cdot \int_{\partial B} y\psi_i(y)\,ds(y) \\ -\rho^{4\beta} \sum_{|l|+|l'|=4} c_{l,l'}(\nu\cdot\nabla x^l)\int_{\partial B} y^{l'}\psi_i(y)\,ds(y) + O(\rho^{6\beta}) = \nu_i(x), \quad x \in \partial B.\end{aligned} \tag{37}$$

Since $\int_{\partial B} \psi_i ds = 0$ for $i = 1, 2$, we get

$$\begin{aligned}\psi_i = (\lambda I - \widetilde{\mathcal{K}}_B^*)^{-1}(\nu_i) + \frac{\rho^{2\beta}}{2}\sum_{k=1}^2 (\lambda I - \widetilde{\mathcal{K}}_B^*)^{-1}(\nu_k)\int_{\partial B} y_k\psi_i(y)\,ds(y) \\ +\rho^{4\beta} \sum_{\substack{|l|+|l'|=4\\|l'|>0}} c_{l,l'}(\lambda I - \widetilde{\mathcal{K}}_B^*)^{-1}(\nu\cdot\nabla x^l)\int_{\partial B} y^{l'}\psi_i(y)\,ds(y) + O(\rho^{6\beta}).\end{aligned} \tag{38}$$

Let $\widetilde{\psi}^l := (\lambda I - \widetilde{\mathcal{K}}_B^*)^{-1}(\nu\cdot\nabla x^l)$ and if $l = e_i$, let $\widetilde{\psi}_i := \widetilde{\psi}^l$, $i = 1, 2$. Then (38) takes the form

$$\begin{aligned}\psi_i = \widetilde{\psi}_i + \frac{\rho^{2\beta}}{2}\sum_{k=1}^2 \widetilde{\psi}_k \int_{\partial B} y_k\psi_i(y)\,ds(y) \\ +\rho^{4\beta} \sum_{\substack{|l|+|l'|=4\\|l'|>0}} c_{l,l'}\widetilde{\psi}^l \int_{\partial B} y^{l'}\psi_i(y)\,ds(y) + O(\rho^{6\beta}) \quad \text{on } \partial B.\end{aligned} \tag{39}$$

In particular, we get

$$\psi_i = \widetilde{\psi}_i + O(\rho^{2\beta}). \tag{40}$$

Substituting (40) into (39), we get

$$\begin{aligned}\psi_i = \widetilde{\psi}_i + \frac{\rho^{2\beta}}{2}\sum_{k=1}^2 \widetilde{\psi}_k \int_{\partial B} y_k\psi_i(y)\,ds(y) \\ +\rho^{4\beta} \sum_{\substack{|l|+|l'|=4\\|l'|>0}} c_{l,l'}\widetilde{\psi}^l \int_{\partial B} y^{l'}\widetilde{\psi}_i(y)\,ds(y) + O(\rho^{6\beta}) \quad \text{on } \partial B.\end{aligned}$$

It then follows from the definitions (26) of P and (17) of the generalized polarization tensor that

$$P_{ij} = M_{ij} + \frac{\rho^{2\beta}}{2}\sum_{k=1}^2 P_{ik}M_{kj} + \rho^{4\beta} \sum_{\substack{|l|+|l'|=4\\|l|>0,|l'|>0}} c_{l,l'} M_{l,j} M_{i,l'} + O(\rho^{6\beta}).$$

Let A be the 2×2 matrix defined by

$$A_{ij} = \sum_{\substack{|l|+|l'|=4 \\ |l|>0, |l'|>0}} c_{l,l'} M_{l,j} M_{i,l'}, \quad i, j = 1, 2. \tag{41}$$

We then get

$$P = M + \frac{\rho^{2\beta}}{2} PM + \rho^{4\beta} A + O(\rho^{6\beta}),$$

and hence

$$P = M(I - \frac{\rho^{2\beta}}{2} M)^{-1} + \rho^{4\beta} A + O(\rho^{6\beta}).$$

Finally, we arrive at the following theorem:

Theorem 5.2 *The effective conductivity $\widetilde{\sigma}$ has an asymptotic expansion as $\rho \to 0$:*

$$\widetilde{\sigma} = \sigma_0 \Big[I + \rho^{2\beta} M(I - \frac{\rho^{2\beta}}{2} M)^{-1} + \rho^{6\beta} A \Big] + O(\rho^{8\beta}), \tag{42}$$

where M is the (first order) polarization tensor and A is given by (41).

Let us consider a special but interesting case: the case when B is a disk. If we fix l' so that $|l'| = 1$ or 2, then $\sum_{|l|=4-|l'|} c_{l,l'} y^l$ is a harmonic polynomial of degree 2 or 3, and hence we get from (19) that

$$\sum_{|l| \geq 2} c_{l,l'} M_{l,i} = 0.$$

Therefore

$$A = \sum_{|l|=1} M_{l,j} \sum_{|l'|=3} c_{l,l'} M_{i,l'}.$$

Using (19), we can show that $A = 0$. Therefore, we get

$$\widetilde{\sigma} = \sigma_0 \Big[I + \rho^{2\beta} (I - \frac{\rho^{2\beta}}{2} M)^{-1} M \Big] + O(\rho^{8\beta}). \tag{43}$$

If B is a disk, we can even go further to obtain the full asymptotic expansion for the effective conductivity. Given an integer m, let R_{2k} be the polynomial defined in (9). Write

$$\sum_{s=2}^{m} R_{2s}(x - y) = \sum_{s=2}^{m} \sum_{|l|+|l'|=2s} c_{l,l'} x^l y^{l'}.$$

Then for each fixed l', $\sum_{|l|=2s-|l'|} c_{l,l'} x^l$ is harmonic. The same argument as before and (28) yield that for $i = 1, 2$

$$(\lambda I - \widetilde{\mathcal{K}}_B^*) \psi_i(x) - \frac{\rho^{2\beta}}{2} \nu(x) \cdot \int_{\partial B} y \psi_i(y) \, ds(y)$$
$$- \sum_{s=2}^{m} \rho^{2s\beta} \sum_{|l|+|l'|=2s} c_{l,l'} (\nu \cdot \nabla x^l) \int_{\partial B} y^{l'} \psi_i(y) \, ds(y) + O(\rho^{2(m+1)\beta}) = \nu_i(x),$$

and hence

$$
\begin{aligned}
\psi_i = \widetilde{\psi}_i + \frac{\rho^{2\beta}}{2} \sum_{k=1}^{2} \widetilde{\psi}_k \int_{\partial B} y_k \psi_i(y)\, ds(y) \\
+ \sum_{s=2}^{m} \rho^{2s\beta} \sum_{|l|+|l'|=2s} c_{l,l'} \widetilde{\psi}^l \int_{\partial B} y^{l'} \psi_i(y)\, ds(y) + O(\rho^{2(m+1)\beta}).
\end{aligned}
\tag{44}
$$

Let $P_{i,l} := \int_{\partial B} y^{l'} \psi_i(y)\, ds(y)$. We then get from (44)

$$
\begin{aligned}
\psi_i = \widetilde{\psi}_i + \frac{\rho^{2\beta}}{2} \sum_{k=1}^{2} P_{ik} \widetilde{\psi}_k + \sum_{s=2}^{m} \rho^{2s\beta} \sum_{\substack{|l|=2s-1 \\ |l'|=1}} c_{l,l'} P_{i,l'} \widetilde{\psi}^l \\
+ \sum_{s=2}^{m} \rho^{2s\beta} \sum_{\substack{|l|+|l'|=2s \\ |l'|>1}} c_{l,l'} \widetilde{\psi}^l \int_{\partial B} y^{l'} \psi_i(y)\, ds(y) + O(\rho^{2(m+1)\beta}).
\end{aligned}
\tag{45}
$$

In particular, we get

$$
\psi_i = \widetilde{\psi}_i + \frac{\rho^{2\beta}}{2} \sum_{k=1}^{2} P_{ik} \widetilde{\psi}_k + \sum_{s=2}^{m} \rho^{2s\beta} \sum_{\substack{|l|=2s-1 \\ |l'|=1}} c_{l,l'} P_{i,l'} \widetilde{\psi}^l + O(\rho^{4\beta}). \tag{46}
$$

Since $\int_{\partial B} y^{l'} \widetilde{\psi}^l(y)\, ds(y) = 0$ if $|l| = 1$ and $|l'| > 1$ by (19), we obtain by substituting (46) into (45)

$$
\psi_i = \widetilde{\psi}_i + \frac{\rho^{2\beta}}{2} \sum_{k=1}^{2} P_{ik} \widetilde{\psi}_k + \sum_{s=2}^{m} \rho^{2s\beta} \sum_{\substack{|l|=2s-1 \\ |l'|=1}} c_{l,l'} P_{i,l'} \widetilde{\psi}^l + O(\rho^{8\beta}).
$$

By iterating this argument we get

$$
\psi_i = \widetilde{\psi}_i + \frac{\rho^{2\beta}}{2} \sum_{k=1}^{2} P_{ik} \widetilde{\psi}_k + \sum_{s=2}^{m} \rho^{2s\beta} \sum_{\substack{|l|=2s-1 \\ |l'|=1}} c_{l,l'} P_{i,l'} \widetilde{\psi}^l + O(\rho^{2(m+1)\beta}).
$$

It then follows from the definitions (26) of P and (17) of the generalized polarization tensor that

$$
P_{ij} = M_{ij} + \frac{\rho^{2\beta}}{2} \sum_{k=1}^{2} P_{ik} M_{kj} + \sum_{s=2}^{m} \rho^{2s\beta} \sum_{\substack{|l|=2s-1 \\ |l'|=1}} c_{l,l'} P_{i,l'} \int_{\partial B} x_j \widetilde{\psi}^l(x)\, ds(x) + O(\rho^{2(m+1)\beta}).
$$

Observe that since $\sum_{|l|=2s-1} c_{l,l'} y^l$ is harmonic,

$$
\sum_{|l|=2s-1} c_{l,l'} \int_{\partial B} x_j \widetilde{\psi}^l(x)\, ds(x) = \sum_{|l|=2s-1} c_{l,l'} M_{l,j} = 0.
$$

Therefore we finally get

$$
P_{ij} = M_{ij} + \frac{\rho^{2\beta}}{2} \sum_{k=1}^{2} P_{ik} M_{kj} + O(\rho^{2(m+1)\beta}),
$$

or equivalently,

$$P = M(I - \frac{\rho^{2\beta}}{2}M)^{-1} + O(\rho^{2(m+1)\beta}).$$

In conclusion, we get the following theorem.

Theorem 5.3 *If B is a disk, then the effective conductivity $\widetilde{\sigma}$ has an asymptotic expansion as $\rho \to 0$: for any integer m,*

$$\widetilde{\sigma} = \sigma_0 \Big[I + \rho^{2\beta} M (I - \frac{\rho^{2\beta}}{2} M)^{-1} \Big] + O((\rho^{2\beta})^{m+1}), \tag{47}$$

where M is the polarization tensor.

Using (34) we can rewrite (47) as

$$\widetilde{\sigma} = \sigma_0 \Big[1 + \frac{2f \frac{\sigma - \sigma_0}{\sigma + \sigma_0}}{1 - f \frac{\sigma - \sigma_0}{\sigma + \sigma_0}} \Big] I + O(f^{m+1}), \tag{48}$$

for each m where $f = \rho^{2\beta}|B|$.

6 Derivation of the Effective Conductivity for Closely Spaced Small Inclusions

An asymptotic formula similar to (33) can be rigorously obtained for closely spaced inclusions. In this section we present the formula and its derivation in brief.

Let D denote a set of m closely spaced inhomogeneities inside Ω:

$$D = \cup_{s=1}^{m} D_s := \cup_{s=1}^{m} (\rho^\beta B_s + z) ,$$

where $z \in Y$, $\beta > 0$ is small and B_s for $s = 1, \ldots, m$ is a bounded Lipschitz domain in $\mathbb{R}^2$. We suppose in addition to (H1) and (H2) in Section 4 that the set D is well-separated from the boundary ∂Y, *i.e.*, $\mathrm{dist}(D, \partial Y) > c_0 > 0$. Define $\sigma_\rho = \sigma_0 + \sum_{s=1}^{m} (\sigma_s - \sigma_0)\chi(\rho^\beta B_s + z)$.

Based on the arguments given in Lemma 3.3, the following Lemma holds.

Lemma 6.1 *The solution u of the problem (2) can be represented as*

$$u_i(x) = x_i + C_i + \sum_{s=1}^{m} \mathcal{S}_{D_s} \varphi_i^{(s)}(x) , \quad x \in Y, i = 1, 2, \tag{49}$$

where $\varphi_i^{(s)} \in L_0^2(\partial D_s)$, $s = 1, \cdots, m$, satisfies the integral equation

$$(\lambda_s I - \mathcal{K}_{D_s}^*)\varphi_i^{(s)} - \sum_{s' \neq s} \frac{\partial (\mathcal{S}_{D_{s'}} \varphi_i^{(s')})}{\partial \nu^{(s)}} \Big|_{\partial D_s} = \nu_i^{(s)} \quad \text{on } \partial D_s. \tag{50}$$

One can also prove the following Lemma.

Lemma 6.2 *Let*

$$\psi_i^{(s)}(y) = \varphi_i^{(s)}(\rho^\beta y) \quad \text{for } y \in \partial B_s.$$

The following identity holds

$$\widetilde{\sigma}_{ij} = \sigma_0 \Big[\delta_{ij} + \rho^{2\beta} \sum_{s=1}^m \int_{\partial B_s} y_j \psi_i^{(s)}(y)\, ds(y) \Big]. \tag{51}$$

By change of variables, identity (50) leads to

$$(\lambda I - \widetilde{\mathcal{K}}_{B_s}^*)\psi_i^{(s)}(x) - \sum_{s' \neq s} \frac{\partial (\widetilde{\mathcal{S}}_{B_{s'}} \psi_i^{(s')})}{\partial \nu^{(s)}} \Big|_{\partial B_s}$$
$$- \rho^\beta \sum_{s'=1}^m \int_{\partial B_{s'}} \frac{\partial}{\partial \nu^{(s)}(x)} R(\rho^\beta (x-y)) \psi_i^{(s')}(y)\, ds'(y) = \nu_i^{(s)}(x),$$

for $x \in \partial B_s$. Following the same lines of the proof as in the previous section and making use of Lemma 4.3, we now obtain the following theorem.

Theorem 6.1 *We have*

$$\widetilde{\sigma} = \sigma_0 \Big[I + \rho^{2\beta} \mathcal{M} + \frac{\rho^{4\beta}}{2} \mathcal{M}^2 \Big] + O(\rho^{6\beta}), \tag{52}$$

where $\mathcal{M}$ is the first-order polarization tensor of the equivalent ellipse of $\cup_{s=1}^m B_s$, which is defined by (21).

We conclude this paper by noticing that our asymptotic formula shows that the Pólya-Szegö polarization tensor of the small inhomogeneities in the unit cell is the only information that can be reconstructed from boundary measurements. No other type of information or details of the composite can be obtained (if the noise level is sufficiently large).

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