Analysis of plasmon resonance on smooth domains using spectral properties of the Neumann-Poincaré operator[*]

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January 11, 2015

Abstract

We investigate in a quantitative way the plasmon resonance at eigenvalues and the essential spectrum (the accumulation point of eigenvalues) of the Neumann-Poincaré operator on smooth domains. We first extend the symmetrization principle so that the single layer potential becomes a unitary operator from $H^{-1/2}$ onto $H^{1/2}$. We then show that the resonance at the essential spectrum is weaker than that at eigenvalues. It is shown that anomalous localized resonance occurs at the essential spectrum on ellipses, but cloaking does not occur on ellipses unlike the core-shell structure considered in [20]. It is shown that resonance does not occur at the essential spectrum on three dimensional balls.

AMS subject classifications. 35J47 (primary), 35P05 (secondary)

Key words. Neumann-Poincaré operator, symmetrization, plasmonic resonance, anomalous localized resonance, eigenvalues, essential spectrum

1   Introduction

The Neumann-Poincaré (NP) operator is a boundary integral operator which arises naturally when solving classical Dirichlet and Neumann boundary value problems using layer potentials. This operator can be realized as a self-adjoint operator using Plemelj's symmetrization principle (see the next section). If the boundary of the domain is smooth, the operator is compact (see [9, 16]) and its spectrum consists of the point spectrum (eigenvalues) and the essential spectrum which is an accumulation point of the eigenvalues. The purpose of this paper is to investigate resonance at eigenvalues and at the essential spectrum, and compare them in a quantitatively precise way. The resonance at the eigenvalues of the NP operator is the plasmon resonance [10]. We show that the resonance at the essential spectrum (on ellipses) is the anomalous localized resonance which was first discovered on a concentric core-shell structure in [22].

To be more precise, suppose that a bounded simply connected domain Ω in $\mathbb{R}^d$ ($d = 2, 3$) is occupied with a plasmonic material of negative dielectric constant. In general the material property of the domain Ω is represented by $\epsilon_c + i\delta$ where $\epsilon_c < 0$ is the dielectric

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constant and $\delta > 0$ indicates the dissipation. Let $\epsilon_m > 0$ be the dielectric constant of the matrix $\mathbb{R}^d \setminus \overline{\Omega}$. So the distribution of the dielectric constant of the structure is given by

$$\epsilon = \begin{cases} \epsilon_c + i\delta & \text{in } \Omega, \\ \epsilon_m & \text{in } \mathbb{R}^d \setminus \overline{\Omega}. \end{cases} \tag{1.1}$$

We quantify the resonance through the following equation:

$$\begin{cases} \nabla \cdot \epsilon \nabla u = a \cdot \nabla \delta_z & \text{in } \mathbb{R}^d, \\ u(x) = O(|x|^{1-d}) & \text{as } |x| \to \infty, \end{cases} \tag{1.2}$$

where a is a constant vector and δ_z is the Dirac mass at $z \in \mathbb{R}^d \setminus \overline{\Omega}$. If u_δ is the solution to (1.2), the resonance is characterized by the blow-up of $\|\nabla u_\delta\|_{L^2(\Omega)}$:

$$\|\nabla u_\delta\|_{L^2(\Omega)} \to \infty \quad \text{as } \delta \to 0. \tag{1.3}$$

We are particularly interested in the blow-up rate of $\|\nabla u_\delta\|_{L^2(\Omega)}$ in terms of δ when the resonance takes place. We emphasize that as $\delta \to 0$ ϵ tends to $\epsilon_c < 0$ in Ω, and so the problem is not an elliptic one. Resonance is connected to the spectrum of the NP operator because of the transmission conditions (continuity of the potential and the flux) along the interface $\partial\Omega$:

$$u_\delta|_+ = u_\delta|_-, \quad \epsilon_m \frac{\partial u_\delta}{\partial \nu}\Big|_+ = (\epsilon_c + i\delta)\frac{\partial u_\delta}{\partial \nu}\Big|_- \quad \text{on } \partial\Omega, \tag{1.4}$$

where the subscripts $+$ and $-$ denote limits to $\partial\Omega$ from outside and inside Ω, respectively. We will make this connection clear in section 3.

The findings of this paper show that the generic rate of the resonance at the eigenvalues is δ^{-1} while that at the essential spectrum is weaker than δ^{-1}. We then show that resonance occurs at the essential spectrum (0) on ellipses and exact rate of resonance is provided. It turns out that it is anomalous localized resonance (see subsection 5.1 for precise statements). We also show that resonance does not occur at the essential spectrum on the three dimensional balls. For the purpose of analysis on resonance we extend the symmetrization principle and show as a result that the single layer potential is a unitary operator from $H^{-1/2}(\partial\Omega)$ onto $H^{1/2}(\partial\Omega)$ (H^s is a Sobolev space). We also derive an expansion formula for the fundamental solution of the Laplacian in terms of eigenfunctions of the NP operator which seems of independent interest. It is worth mentioning that it is recently found in [21] that on a disk a complete resonance occurs at 0. This happens because 0 is the only eigenvalue of the NP operator on a disk.

Recently there is rapidly growing interest in the spectral theory of the NP operator in relation to plasmonics. In [17] the Poincaré's variational program was revisited with modern prospective and symmetrization of the NP operator was proved. Eigenvalues on disks, ellipses, and balls were computed [4, 14]. Asymptotics of eigenvalues of the NP operator associated with closely located two-dimensional convex domain has been obtained in [5, 6]. There has been progress on spectral theory of the NP operator on non-smooth domains. A bound on the essential spectrum on the two-dimensional curvilinear domains has been obtained [23]. Quite recently, the complete spectral resolution of the NP operator on intersecting disks has been derived which in particular shows that only the absolutely continuous spectrum exists [13]. Interestingly, the results in [13] shows that the spectral bound obtained in [23] is optimal in the case of intersecting disks. The spectral theory of the NP operator has been applied to analysis of cloaking by anomalous localized resonance on the plasmonic structure [1] and to the study of uniformity of elliptic estimates [12].

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This paper is organized as follows. In section 2 we review and extend the symmetrization of the NP operator. In section 3 we derive an expansion formula for the fundamental solution of the Laplacian in terms of the eigenfunctions of the NP operator and obtain a representation of the solution to (1.2). Resonance at eigenvalues and at essential spectrum is studied in section 4 and section 5, respectively. In Appendix we derive explicit expansion formula for the fundamental solution in terms of spherical harmonics and elliptic harmonics.

2   Neumann-Poincaré operator and symmetrization

Let us first fix some notation.

- We denote by $H^{-1/2}(\partial\Omega)$ the dual space of $H^{1/2}(\partial\Omega)$, and by $\langle\cdot,\cdot\rangle$ the duality pairing of $H^{-1/2}$ and $H^{1/2}$, and $\|\cdot\|_s$ denotes the H^s norm on $\partial\Omega$. Let $H_0^{-1/2}(\partial\Omega)$ be the space of $\psi \in H^{-1/2}(\partial\Omega)$ satisfying $\langle\psi,1\rangle = 0$.

- The notation $A \lesssim B$ means that $A \le CB$ for some constant C, and $A \approx B$ means that both $A \lesssim B$ and $B \lesssim A$ hold.

- Let $f(\delta)$ and $g(\delta)$ be positive quantities depending on δ. We write $f(\delta) \sim g(\delta)$ as $\delta \to 0$ if there are constants C_1 and C_2 such that

$$C_1 < \frac{f(\delta)}{g(\delta)} < C_2. \tag{2.1}$$

Let $\Gamma(x)$ be the fundamental solution to the Laplacian on $\mathbb{R}^d$ ($d = 2, 3$), *i.e.*,

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi}\ln|x|, & d = 2, \\ -\dfrac{1}{4\pi}|x|^{-1}, & d = 3. \end{cases}$$

The Neumann-Poincaré (NP) operator on $\partial\Omega$, denoted by $\mathcal{K}_{\partial\Omega}$, is defined by

$$\mathcal{K}_{\partial\Omega}[\varphi](x) := \int_{\partial\Omega} \frac{\partial}{\partial\nu_x}\Gamma(x-y)\varphi(y)\,d\sigma(y), \quad x \in \partial\Omega. \tag{2.2}$$

where $\frac{\partial}{\partial\nu_x}$ indicates the outward normal derivative in x-variable. The adjoint operator $\mathcal{K}_{\partial\Omega}^*$ on $L^2(\partial\Omega)$ will be called the adjoint NP operator.

Importance of the adjoint operator in dealing with interface problems lies in its relation with the single layer potential $\mathcal{S}_{\partial\Omega}[\varphi]$ defined by

$$\mathcal{S}_{\partial\Omega}[\varphi](x) := \int_{\partial\Omega} \Gamma(x-y)\varphi(y)\,d\sigma(y), \quad x \in \mathbb{R}^d. \tag{2.3}$$

The following jump relation holds:

$$\frac{\partial}{\partial\nu}\mathcal{S}_{\partial\Omega}[\varphi]\big|_{\pm}(x) = \left(\pm\frac{1}{2}I + \mathcal{K}_{\partial\Omega}^*\right)[\varphi](x), \quad x \in \partial\Omega, \tag{2.4}$$

where the subscripts $\pm$ indicate the limits (to $\partial\Omega$) from outside and inside of Ω, respectively.

The NP operator is not self-adjoint on the usual L^2-space, unless the domain Ω is a disc or a ball (see [19]). However, it is found in [17] (see also [11]) that the adjoint NP operator $\mathcal{K}_{\partial\Omega}^*$ can be symmetrized using Plemelj's symmetrization principle (also known as Calderón's identity):

$$\mathcal{S}_{\partial\Omega}\mathcal{K}_{\partial\Omega}^* = \mathcal{K}_{\partial\Omega}\mathcal{S}_{\partial\Omega}. \tag{2.5}$$

Define, for $\varphi, \psi \in H^{-1/2}(\partial\Omega)$,

$$\langle \varphi, \psi \rangle_{\mathcal{H}^*} := -\langle \varphi, \mathcal{S}_{\partial\Omega}[\psi] \rangle. \tag{2.6}$$

Since $\mathcal{S}_{\partial\Omega}$ maps $H^{-1/2}(\partial\Omega)$ into $H^{1/2}(\partial\Omega)$, the right hand side of (2.6) is well-defined. It is known that $\langle \cdot, \cdot \rangle_{\mathcal{H}^*}$ is an inner product on $H_0^{-1/2}(\partial\Omega)$, and the norm $\| \cdot \|_{\mathcal{H}^*}$ induced by this inner product is equivalent to the $H^{-1/2}(\partial\Omega)$ norm, namely,

$$\|\varphi\|_{\mathcal{H}^*} \approx \|\varphi\|_{-1/2} \tag{2.7}$$

for all $\varphi \in H_0^{-1/2}(\partial\Omega)$ (see [12]). Let $\mathcal{H}_0^*$ be the space $H_0^{-1/2}(\partial\Omega)$ equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}^*}$. Then the symmetrization principle (2.5) shows that $\mathcal{K}_{\partial\Omega}^*$ is self-adjoint on $\mathcal{H}_0^*$.

Let us now consider symmetrization of $\mathcal{K}_{\partial\Omega}$. The NP operator $\mathcal{K}_{\partial\Omega}$ can be symmetrized using (2.5) expressed in a different form:

$$\mathcal{K}_{\partial\Omega}^* \mathcal{S}_{\partial\Omega}^{-1} = \mathcal{S}_{\partial\Omega}^{-1} \mathcal{K}_{\partial\Omega}, \tag{2.8}$$

provided that $\mathcal{S}_{\partial\Omega}^{-1}$ exists. However, it was proved in [24] that $\mathcal{S}_{\partial\Omega}^{-1}$ exists only in three dimensions (or higher), and there are domains Ω in two dimensions where $\mathcal{S}_{\partial\Omega}^{-1}$ does not exist. Here we present a simple way to overcome this difficulty.

To symmetrize $\mathcal{K}_{\partial\Omega}$, we use, as a replacement of $\mathcal{S}_{\partial\Omega}$ in (2.5), the operator $\mathcal{A} : H^{-1/2}(\partial\Omega) \times \mathbb{C} \to H^{1/2}(\partial\Omega) \times \mathbb{C}$ defined by

$$\mathcal{A}(\psi, a) := (\mathcal{S}_{\partial\Omega}[\psi] + a, \langle \psi, 1 \rangle). \tag{2.9}$$

It is proved in [3, Theorem 2.13] and [12] that $\mathcal{A}$ is invertible for $d \geq 2$. For $f \in H^{1/2}(\partial\Omega)$ let

$$(\psi_f, a_f) = \mathcal{A}^{-1}(f, 0). \tag{2.10}$$

Then $\psi_f \in H_0^{-1/2}(\partial\Omega)$ and it holds that

$$\mathcal{S}_{\partial\Omega}[\psi_f] + a_f = f \tag{2.11}$$

Moreover, we have

$$\|\psi_f\|_{-1/2} + |a_f| \approx \|f\|_{1/2}. \tag{2.12}$$

This shows in particular that the mapping $f \mapsto a$ is a bounded linear functional on $H^{1/2}(\partial\Omega)$. So, there is a unique $\varphi_0 \in H^{-1/2}(\partial\Omega)$ such that

$$a_f = \langle \varphi_0, f \rangle \tag{2.13}$$

for all $f \in H^{1/2}(\partial\Omega)$. Since such a (ψ_f, a_f) is unique, we see that $a_f = 1$ if $f = 1$, in other words,

$$\langle \varphi_0, 1 \rangle = 1. \tag{2.14}$$

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Since $\mathcal{K}_{\partial\Omega}[1] = 1/2$, by applying $\mathcal{K}_{\partial\Omega}$ to both sides of (2.11) we see that

$$\mathcal{K}_{\partial\Omega}\mathcal{S}_{\partial\Omega}[\psi_f] + \frac{1}{2}a_f = \mathcal{K}_{\partial\Omega}[f].$$

It then follows from (2.5) that

$$\mathcal{S}_{\partial\Omega}\mathcal{K}_{\partial\Omega}^*[\psi_f] + \frac{1}{2}a_f = \mathcal{K}_{\partial\Omega}[f],$$

which implies that $a_{\mathcal{K}_{\partial\Omega}[f]} = \frac{1}{2}a_f$, in other words,

$$\langle \varphi_0, \mathcal{K}_{\partial\Omega}[f] \rangle = \langle \varphi_0, f \rangle / 2.$$

So, we have

$$\mathcal{K}_{\partial\Omega}^*[\varphi_0] = \frac{1}{2}\varphi_0, \tag{2.15}$$

namely, φ_0 is an eigenfunction of $\mathcal{K}_{\partial\Omega}^*$ (on $H^{-1/2}(\partial\Omega)$ corresponding to the eigenvalue $1/2$ normalized by (2.14). We can infer from the jump formula (2.4) and (2.15) that $\mathcal{S}_{\partial\Omega}[\varphi_0]$ is constant in Ω. We emphasize that the function φ_0 already appeared in literatures. It is proved in [24] that $\mathcal{S}_{\partial\Omega}[\varphi_0] = 0$ in Ω for some domain Ω in two dimensions, which is why $\mathcal{S}_{\partial\Omega}$ is not invertible.

We now define a variant of the single layer potential on $\mathcal{H}^*$ by

$$\widetilde{\mathcal{S}}_{\partial\Omega}[\varphi] = \begin{cases} \mathcal{S}_{\partial\Omega}[\varphi] & \text{if } \langle \varphi, 1 \rangle = 0, \\ 1 & \text{if } \varphi = \varphi_0. \end{cases} \tag{2.16}$$

Then we have an extension of (2.5):

$$\widetilde{\mathcal{S}}_{\partial\Omega}\mathcal{K}_{\partial\Omega}^* = \mathcal{K}_{\partial\Omega}\widetilde{\mathcal{S}}_{\partial\Omega}. \tag{2.17}$$

Using this we can extend the inner product (2.6) (defined on $H_0^{-1/2}(\partial\Omega)$) to $H^{-1/2}(\partial\Omega)$. For $\varphi, \psi \in H^{-1/2}(\partial\Omega)$, define

$$\langle \varphi, \psi \rangle_{\mathcal{H}^*} := -\langle \varphi, \widetilde{\mathcal{S}}_{\partial\Omega}[\psi] \rangle. \tag{2.18}$$

We also have a new inner product on $H^{1/2}(\partial\Omega)$:

$$\langle \varphi, \psi \rangle_{\mathcal{H}} := -\langle \varphi, \widetilde{\mathcal{S}}_{\partial\Omega}^{-1}[\psi] \rangle. \tag{2.19}$$

If we define $\mathcal{H}$ to be $H^{1/2}(\partial\Omega)$ with this inner product, then $\mathcal{K}_{\partial\Omega}$ is self-adjoint on $\mathcal{H}$. We emphasize that norm $\| \cdot \|_{\mathcal{H}}$ is equivalent to $\| \cdot \|_{1/2}$. Moreover, $\widetilde{\mathcal{S}}_{\partial\Omega} : \mathcal{H}^* \to \mathcal{H}$ is a unitary operator. Observe that if $\{\psi_j\}_{j=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}_0^*$, then $\{\psi_j\}_{j=1}^{\infty} \cup \{\varphi_0\}$ is an orthonormal basis of $\mathcal{H}^*$ and $\{\mathcal{S}_{\partial\Omega}[\psi_j]\}_{j=1}^{\infty} \cup \{1\}$ is an orthonormal basis of $\mathcal{H}$.

3 Representation of the solution

If $\partial\Omega$ is $\mathcal{C}^{1,\alpha}$-smooth for some $\alpha > 0$, then it is known that $\mathcal{K}_{\partial\Omega}^*$ is a compact operator on $\mathcal{H}^*$ (see [16]). Since $\mathcal{K}_{\partial\Omega}^*$ is self-adjoint on $\mathcal{H}_0^*$, its eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ are real and accumulates to 0. We emphasize that $|\lambda_j| < 1/2$ (see [7, 24]). Let $\{\psi_j\}_{j=1}^{\infty}$ with $\|\psi_j\|_{\mathcal{H}^*} = 1$ be the corresponding (real valued) eigenfunctions counting the multiplicities. Then we have shown in the previous section that $\{\mathcal{S}_{\partial\Omega}[\psi_j]\}_{j=1}^{\infty} \cup \{1\}$ is an orthonormal basis of $\mathcal{H}$.

Fix $z \in \mathbb{R}^d \setminus \overline{\Omega}$. Then $\Gamma(\cdot - z)$ belongs to $H^{1/2}(\partial\Omega)$, and so admits the following decomposition:

$$\Gamma(x - z) = \sum_{j=1}^{\infty} c_j(z)\mathcal{S}_{\partial\Omega}[\psi_j](x) + c_0(z), \quad x \in \partial\Omega, \tag{3.1}$$

for some constants $c_j(z)$ (depending on z) satisfying

$$\sum_{j=1}^{\infty} |c_j(z)|^2 < \infty. \tag{3.2}$$

Since $-\langle \mathcal{S}_{\partial\Omega}[\psi_j], \psi_i \rangle = \delta_{ij}$ where δ_{ij} is the Kronecker's delta, we see that

$$c_j(z) = -\int_{\partial\Omega} \Gamma(x - z)\psi_j(x)d\sigma(x) = -\mathcal{S}_{\partial\Omega}[\psi_j](z), \quad j = 1, 2, 3, \dots.$$

We also see from (2.14) that

$$c_0(z) = \mathcal{S}_{\partial\Omega}[\varphi_0](z), \quad j = 1, 2, 3, \dots.$$

So, we obtain the following formula:

$$\Gamma(x - z) = -\sum_{j=1}^{\infty} \mathcal{S}_{\partial\Omega}[\psi_j](z)\mathcal{S}_{\partial\Omega}[\psi_j](x) + \mathcal{S}_{\partial\Omega}[\varphi_0](z), \quad x \in \partial\Omega, \tag{3.3}$$

Observe that

$$\left\| \sum_{j=1}^{\infty} \mathcal{S}_{\partial\Omega}[\psi_j](z)\mathcal{S}_{\partial\Omega}[\psi_j] \right\|_{\mathcal{H}}^2 = \sum_{j=1}^{\infty} |\mathcal{S}_{\partial\Omega}[\psi_j](z)|^2 < \infty. \tag{3.4}$$

Since $\| \cdot \|_{\mathcal{H}}$ is equivalent to $\| \cdot \|_{1/2}$ as shown in the previous section, we find from the trace theorem that $\sum_{j=1}^{\infty} \mathcal{S}_{\partial\Omega}[\psi_j](z)\mathcal{S}_{\partial\Omega}[\psi_j]$ converges in $H^1(\Omega)$ and is harmonic in Ω. So, we obtain the following theorem on an expansion of the fundamental solution to the Laplacian.

Theorem 3.1 *It holds that*

$$\Gamma(x - z) = -\sum_{j=1}^{\infty} \mathcal{S}_{\partial\Omega}[\psi_j](z)\mathcal{S}_{\partial\Omega}[\psi_j](x) + \mathcal{S}_{\partial\Omega}[\varphi_0](z), \quad x \in \overline{\Omega}, \ z \in \mathbb{R}^d \setminus \overline{\Omega}. \tag{3.5}$$

If Ω is a ball, then $\mathcal{S}_{\partial\Omega}[\psi_j]$ is a spherical harmonics (see subsection 5.2). Therefore (3.5) is the expansion of $\Gamma(x - z)$ in terms of the spherical harmonics, which is well-known (see, for example, [16]). We will show explicit formula for (3.5) when Ω is a unit ball in $\mathbb{R}^3$ or an ellipse in $\mathbb{R}^2$ in the Appendix. The formula (3.5) shows that the fundamental solution can be expanded on any smooth domain in terms of the localized plasmon (the single layer potential of an eigenfunction is called a localized plasmon). The estimate (3.4) shows that $\mathcal{S}_{\partial\Omega}[\psi_j](z) \to 0$ as $j \to \infty$ for all $z \notin \overline{\Omega}$. So, if j is large then $\mathcal{S}_{\partial\Omega}[\psi_j]$ is localized near Ω. It explains why $\mathcal{S}_{\partial\Omega}[\psi_j]$ is called a *localized* plasmon.

We now derive a representation of the solution to (1.2). Let

$$F_z(x) := -a \cdot \nabla_x \Gamma(x - z), \quad x \neq z. \tag{3.6}$$

Then $\Delta F_z(x) = a \cdot \nabla \delta_z(x)$, and hence we see from transmission conditions (1.4) on $\partial\Omega$ and (2.4) that the solution u_δ to (1.2) can be represented as

$$u_\delta(x) = F_z(x) + \mathcal{S}_{\partial\Omega}[\varphi_\delta](x), \quad x \in \mathbb{R}^d, \tag{3.7}$$

where the potential $\varphi_\delta \in \mathcal{H}_0^*(\partial\Omega)$ is the solution to the integral equation

$$(\lambda I - \mathcal{K}_{\partial\Omega}^*)[\varphi_\delta] = \partial_\nu F_z \quad \text{on } \partial\Omega \tag{3.8}$$

($\partial_\nu F_z$ denotes the normal derivative of F_z). Here

$$\lambda := \frac{\epsilon_c + \epsilon_m + i\delta}{2(\epsilon_c - \epsilon_m) + 2i\delta}. \tag{3.9}$$

Note that $\lambda \to \frac{\epsilon_c + \epsilon_m}{2(\epsilon_c - \epsilon_m)}$ as $\delta \to 0$. The number ϵ_c/ϵ_m such that $\frac{\epsilon_c + \epsilon_m}{2(\epsilon_c - \epsilon_m)}$ is an eigenvalue of $\mathcal{K}_{\partial\Omega}^*$ is called a plasmonic eigenvalue [10]. We emphasize that ϵ_c is negative if and only if $\frac{\epsilon_c + \epsilon_m}{2(\epsilon_c - \epsilon_m)}$ lies in $(-1/2, 1/2)$ where the spectrum of $\mathcal{K}_{\partial\Omega}^*$ lies.

Since $\mathcal{K}_{\partial\Omega}^*$ admits the spectral decomposition

$$\mathcal{K}_{\partial\Omega}^* = \sum_{j=1}^{\infty} \lambda_j \psi_j \otimes \psi_j, \tag{3.10}$$

the solution φ_δ to (3.8) can be expressed as

$$\varphi_\delta = \sum_{j=1}^{\infty} \frac{\alpha_j(z)}{\lambda - \lambda_j} \psi_j, \tag{3.11}$$

where

$$\alpha_j(z) := \langle \partial_\nu F_z, \psi_j \rangle_{\mathcal{H}^*}. \tag{3.12}$$

We can see from (3.6) that

$$\alpha_j(z) = -a \cdot \nabla \int_{\partial\Omega} \frac{\partial}{\partial \nu_x} \Gamma(x - z) \mathcal{S}_{\partial\Omega}[\psi_j](x) \, d\sigma(x).$$

It can seen from (2.4) and (3.5) that

$$\frac{\partial}{\partial \nu_x} \Gamma(x - z) = -\sum_{j=1}^{\infty} \mathcal{S}_{\partial\Omega}[\psi_j](z) \frac{\partial}{\partial \nu} \mathcal{S}_{\partial\Omega}[\psi_j](x) = \sum_{j=1}^{\infty} \left(\frac{1}{2} - \lambda_j\right) \mathcal{S}_{\partial\Omega}[\psi_j](z) \psi_j(x).$$

It then follows that

$$\alpha_j(z) = \left(\lambda_j - \frac{1}{2}\right) a \cdot \nabla \mathcal{S}_{\partial\Omega}[\psi_j](z). \tag{3.13}$$

4 Resonance at eigenvalues

We investigate the behavior of the solution u_δ when λ approaches to one of eigenvalues $\lambda_l \neq 0$ as $\delta \to 0$, namely, when

$$\frac{\epsilon_c + \epsilon_m}{2(\epsilon_c - \epsilon_m)} = \lambda_l. \tag{4.1}$$

We show that

$$\|\nabla u_\delta\|_{L^2(\Omega)} \sim \delta^{-1} \quad \text{as } \delta \to 0, \tag{4.2}$$

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as one may expect.

We first show that

$$\|\nabla \mathcal{S}_{\partial\Omega}[\varphi]\|_{L^2(\Omega)}^2 \approx \|\varphi\|_{\mathcal{H}^*}^2 \tag{4.3}$$

for all $\varphi \in \mathcal{H}_0^*$. In fact, we see from (2.4) and (3.10) that

$$\begin{aligned}
\|\nabla \mathcal{S}_{\partial\Omega}[\varphi]\|_{L^2(\Omega)}^2 &= \int_{\partial\Omega} \mathcal{S}_{\partial\Omega}[\varphi] \overline{\frac{\partial}{\partial\nu}\mathcal{S}_{\partial\Omega}[\varphi]}\Big|_{-} d\sigma \\
&= \left\langle \varphi, (-\frac{1}{2}I + \mathcal{K}_{\partial\Omega}^*)[\varphi]\right\rangle_{\mathcal{H}^*} \\
&= \sum_{j=1}^{\infty}(\frac{1}{2} - \lambda_j)|\langle\varphi, \psi_j\rangle_{\mathcal{H}^*}|^2.
\end{aligned}$$

Since $|\lambda_j| < 1/2$ and accumulates to 0, we have (4.3).

We now see from (3.11) that

$$\|\nabla(u_\delta - F_z)\|_{L^2(\Omega)}^2 \approx \|\varphi_\delta\|_{\mathcal{H}^*}^2 = \sum_{\lambda_j=\lambda_l} \frac{|\alpha_j(z)|^2}{|\lambda - \lambda_l|^2} + \sum_{\lambda_j \neq \lambda_l} \frac{|\alpha_j(z)|^2}{|\lambda - \lambda_j|^2}.$$

If $\lambda_j \neq \lambda_l$, then $|\lambda_j - \lambda_l| > C$ for some positive constant C since $\lambda_l \neq 0$, and so the second term on the righthand side above is bounded as $\delta \to 0$. Since $|\lambda - \lambda_l| \sim \delta$ as $\delta \to 0$, we obtain

$$\|\nabla(u_\delta - F_z)\|_{L^2(\Omega)} \sim \delta^{-1}.$$

Since $\|\nabla F_z\|_{L^2(\Omega)}$ is bounded, (4.2) follows.

5 Resonance at the essential spectrum

In this section we assume that 0 is not an eigenvalue of $\mathcal{K}_{\partial\Omega}^*$. Since eigenvalues of $\mathcal{K}_{\partial\Omega}^*$ converges to 0, $\{0\}$ is the essential spectrum of $\mathcal{K}_{\partial\Omega}^*$. It is worth mentioning that we are not aware of any domain other than disks on which the NP operator has 0 as an eigenvalue. If Ω is a disk, then $\mathcal{K}_{\partial\Omega}^* \equiv 0$ on $\mathcal{H}_0^*$.

We consider the resonance when $\lambda \to 0$ as $\delta \to 0$, in other words, when

$$\epsilon_c + \epsilon_m = 0. \tag{5.1}$$

In this case, we assume that $\lambda = i\delta$ for simplicity of presentation.

We first obtain the following theorem which shows that the resonance, if it occurs, is never at the rate of δ^{-1} unlike the resonance at eigenvalues.

Theorem 5.1 *It holds that*

$$\lim_{\delta\to 0} \delta\|\nabla u_\delta\|_{L^2(\Omega)} = 0. \tag{5.2}$$

Proof. We obtain from (3.11) and (4.3) that

$$\|\nabla(u_\delta - F_z)\|_{L^2(\Omega)}^2 \sim \sum_{j=1}^{\infty} \frac{|\alpha_j(z)|^2}{\delta^2 + \lambda_j^2}. \tag{5.3}$$

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We then decompose the summation into two parts as

$$\sum_{j=1}^{\infty} \frac{|\alpha_j(z)|^2}{\delta^2 + \lambda_j^2} = \sum_{|\lambda_j| \le \delta} \frac{|\alpha_j(z)|^2}{\delta^2 + \lambda_j^2} + \sum_{|\lambda_j| > \delta} \frac{|\alpha_j(z)|^2}{\delta^2 + \lambda_j^2} =: S_1 + S_2. \tag{5.4}$$

Since $\sum |\alpha_j(z)|^2 < \infty$, we have

$$\delta^2 S_1 \le \sum_{|\lambda_j| \le \delta} |\alpha_j(z)|^2 \to 0 \text{ as } \delta \to 0.$$

To show that $\delta^2 S_2 \to 0$, we express S_2 as

$$S_2 = \sum_{k=0}^{\infty} \sum_{2^k \delta < |\lambda_j| \le 2^{k+1}\delta} \frac{|\alpha_j(z)|^2}{\delta^2 + \lambda_j^2}.$$

Then we see that

$$\delta^2 S_2 \le \sum_{k=0}^{\infty} \frac{1}{1 + 2^{2k}} \sum_{2^k \delta < |\lambda_j| \le 2^{k+1}\delta} |\alpha_j(z)|^2 \le \sum_{k=0}^{\infty} \frac{1}{1 + 2^{2k}} \sum_{|\lambda_j| \le 2^{k+1}\delta} |\alpha_j(z)|^2.$$

For each fixed k, $\sum_{|\lambda_j| \le 2^{k+1}\delta} \alpha_j(z)^2 \to 0$ as $\delta \to 0$. So we infer that $S_2 \to 0$ as $\delta \to 0$ by Lebesgue dominated convergence theorem. The proof is complete. $\square$

In order to derive estimates for the resonance, we need to investigate the asymptotic behavior of the quantity on the righthand side of (5.3) as $\delta \to 0$. This seems a quite difficult task since it depends on the behavior of λ_j and $\alpha_j(z)$ as $j \to \infty$. So, we deal with two specific domains, ellipses and three dimensional balls, where eigenvalues and eigenfunctions are known. It is worth mentioning that the disks are out of consideration since the NP operator on disks are 0.

5.1 Anomalous localized resonance on ellipses

We first consider the resonance when Ω is an ellipse in $\mathbb{R}^2$. The elliptic coordinates $x = (x_1, x_2) = (x_1(\rho, \omega), x_2(\rho, \omega))$ is given by

$$x_1(\rho, \omega) = R \cos \omega \cosh \rho, \quad x_2(\rho, \omega) = R \sin \omega \sinh \rho, \quad \rho > 0, \ 0 \le \omega < 2\pi.$$

When the above holds, we denote $\rho = \rho_x$ and ω_x. Then we can represent $\partial \Omega$ as

$$\partial \Omega = \{x \in \mathbb{R}^2; \rho_x = \rho_0\} \tag{5.5}$$

for some $\rho_0 > 0$. The number ρ_0 is called the elliptic radius of Ω. The position z of the source is denote by ρ_z and ω_z in elliptic coordinates. We obtain the following theorem whose proof will be given in the next section.

Theorem 5.2 *Suppose that Ω is an ellipse given by (5.5). Then we have*

$$\|\nabla u_\delta\|_{L^2(\Omega)}^2 \sim \begin{cases} \delta^{-3 + \rho_z/\rho_0} |\log \delta| & \text{if } \rho_0 < \rho_z < 3\rho_0, \\ |\log \delta|^2 & \text{if } \rho_z = 3\rho_0, \\ 1 & \text{if } \rho_z > 3\rho_0, \end{cases} \tag{5.6}$$

as $\delta \to 0$.

The quantity $E_\delta := \delta\|\nabla u_\delta\|^2_{L^2(\Omega)}$ is of particular interest since it represents the imaginary part of the energy, namely,

$$\delta\|\nabla u_\delta\|^2_{L^2(\Omega)} = \Im \int_{\mathbb{R}^d} \epsilon|\nabla u_\delta|^2 dx.$$

Physically it represents the electro-magnetic energy dissipated into heat. Estimates (5.6) shows that $E_\delta \to \infty$ if $\rho_0 < \rho_z \le 2\rho_0$ while it tends to 0 as $\delta \to 0$ if $\rho_z > 2\rho_0$. So the critical (elliptic) radius is $2\rho_0$. This phenomenon is reminiscent of the anomalous localized resonance (ALR) discovered in [22, 20] (see also [1, 8, 18]). There it is shown that a disk (a core) is coated by a concentric disk (a shell) of plasmonic material, then ALR occurs if the source is located within a critical radius, and does not occur for sources outside the radius. This is exactly what (5.6) shows. It also in accordance with the result in [8] where the core-shell structure of confocal ellipses of radii $\rho_e > \rho_i$ is considered. There it is shown that the critical (elliptic) radius is given by

$$\rho_* = \begin{cases} (3\rho_e - \rho_i)/2, & \text{if } \rho_e \le 3\rho_i, \\ 2(\rho_e - \rho_i), & \text{if } \rho_e > 3\rho_i. \end{cases}$$

If the core shrinks, in other words ρ_i tends to 0, then ρ_* tends to $2\rho_e$ which is the critical radius found in this paper.

ALR also requires the solution to be bounded outside a bounded set. This requirement is satisfied on ellipse as the following theorem shows. Theorem 5.2 and Theorem 5.3 show that ALR may occur not only on the coated structures but also on simply connected structure. So ALR may be regarded as resonance at the accumulation points of eigenvalues. We emphasize that the NP operator on the coated disk has 0 as its essential spectrum as proved in [1].

Theorem 5.3 *Let Ω be an ellipse given by (5.5). It holds for all x satisfying $\rho_x + \rho_z - 4\rho_0 > 0$ that*

$$|u_\delta(x) - F_z(x)| \lesssim \sum_{n=1}^{\infty} e^{-n(\rho_x + \rho_z - 4\rho_0)}. \tag{5.7}$$

In particular, let $\overline{\rho} > 0$ be such that $\overline{\rho} > 4\rho_0 - \rho_z$, then there exists some $C = C_{\overline{\rho}} > 0$ such that

$$\sup_{\rho_x \ge \overline{\rho}} |u_\delta(x) - F_z(x)| < C. \tag{5.8}$$

To prove Theorem 5.2 and Theorem 5.3 let us recall some facts on the NP operator on ellipses. It is proved in [4] and [8] that eigenvalues of $\mathcal{K}^*_{\partial\Omega}$ are

$$\lambda_n = \frac{1}{2e^{2n\rho_0}}, \quad n = 1, 2, \cdots, \tag{5.9}$$

and corresponding eigenfunctions are

$$\phi^c_n(\omega) := \Xi(\rho_0, \omega)^{-1} \cos n\omega, \quad \phi^s_n(\omega) := \Xi(\rho_0, \omega)^{-1} \sin n\omega, \quad n = 1, 2, \cdots, \tag{5.10}$$

where

$$\Xi = \Xi(\rho_0, \omega) := R\sqrt{\sinh^2 \rho_0 + \sin^2 \omega}. \tag{5.11}$$

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It is also proved in the same papers that

$$\mathcal{S}_{\partial\Omega}[\phi_n^c](x) = \begin{cases} -\dfrac{\cosh n\rho}{n} e^{-n\rho_0} \cos n\omega, & \rho < \rho_0, \\ -\dfrac{\cosh n\rho_0}{n} e^{-n\rho} \cos n\omega, & \rho \geq \rho_0, \end{cases} \tag{5.12}$$

and

$$\mathcal{S}_{\partial\Omega}[\phi_n^s](x) = \begin{cases} -\dfrac{\sinh n\rho}{n} e^{-n\rho_0} \sin n\omega, & \rho < \rho_0, \\ -\dfrac{\sinh n\rho_0}{n} e^{-n\rho} \sin n\omega, & \rho \geq \rho_0. \end{cases} \tag{5.13}$$

Here and afterwards (ρ, ω) is the elliptic coordinates of x. Since the length element $d\sigma$ on $\partial\Omega$ is given by

$$d\sigma = \Xi d\omega, \tag{5.14}$$

one can see that

$$\|\phi_n^c\|_{\mathcal{H}^*}^2 = -\langle \phi_n^c, \mathcal{S}_{\partial\Omega}[\phi_n^c] \rangle = \frac{\pi \cosh n\rho_0}{n e^{n\rho_0}},$$

and

$$\|\phi_n^s\|_{\mathcal{H}^*}^2 = \frac{\pi \sinh n\rho_0}{n e^{n\rho_0}}.$$

So, the normalized eigenfunctions are

$$\psi_n^c := \sqrt{\frac{n e^{n\rho_0}}{\pi \cosh n\rho_0}} \phi_n^c, \quad \psi_n^s := \sqrt{\frac{n e^{n\rho_0}}{\pi \sinh n\rho_0}} \phi_n^s. \tag{5.15}$$

We see from (5.12)

$$\mathcal{S}_{\partial\Omega}[\psi_n^c](x) = \begin{cases} -\sqrt{\dfrac{e^{-n\rho_0}}{n\pi \cosh n\rho_0}} \cosh n\rho \cos n\omega, & \rho < \rho_0, \\ -\sqrt{\dfrac{e^{n\rho_0} \cosh n\rho_0}{n\pi}} e^{-n\rho} \cos n\omega, & \rho \geq \rho_0, \end{cases} \tag{5.16}$$

and from (5.13)

$$\mathcal{S}_{\partial\Omega}[\psi_n^s](x) = \begin{cases} -\sqrt{\dfrac{e^{-n\rho_0}}{n\pi \sinh n\rho_0}} \sinh n\rho \sin n\omega, & \rho < \rho_0, \\ -\sqrt{\dfrac{e^{n\rho_0} \sinh n\rho_0}{n\pi}} e^{-n\rho} \sin n\omega, & \rho \geq \rho_0. \end{cases} \tag{5.17}$$

Proof of Theorem 5.2. We see from (3.13) and (5.3) that

$$\|\nabla(u_\delta - F_z)\|_{L^2(\Omega)}^2 \sim \sum_{j=1}^{\infty} \frac{|\alpha_j(z)|^2}{\delta^2 + \lambda_j^2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{\delta^2 + \lambda_n^2} \left[|a \cdot \nabla \mathcal{S}_{\partial\Omega}[\psi_n^c](z)|^2 + |a \cdot \nabla \mathcal{S}_{\partial\Omega}[\psi_n^s](z)|^2 \right]$$

$$= \sum_{n=1}^{\infty} \frac{1}{\delta^2 + \lambda_n^2} \left[\frac{e^{n\rho_0} \cosh n\rho_0}{n\pi} |a \cdot \nabla_z \left(e^{-n\rho_z} \cos n\omega_z \right)|^2 + \frac{e^{n\rho_0} \sinh n\rho_0}{n\pi} |a \cdot \nabla_z \left(e^{-n\rho_z} \sin n\omega_z \right)|^2 \right].$$

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Since $\cosh n\rho_0 \approx \sinh n\rho_0 \approx e^{n\rho_0}$, we see that

$$\sum_{j=1}^{\infty} \frac{|\alpha_j(z)|^2}{\delta^2 + \lambda_j^2} \sim \sum_{n=1}^{\infty} \frac{1}{\delta^2 + \lambda_n^2} \cdot \frac{e^{2n\rho_0}}{n} \left[\left| a \cdot \nabla_z \left(e^{-n\rho_z} \cos n\omega_z \right) \right|^2 + \left| a \cdot \nabla_z \left(e^{-n\rho_z} \sin n\omega_z \right) \right|^2 \right]$$

(5.18)

as $\delta \to 0$.

Let $U(\omega)$ be the rotation by the angle ω, namely,

$$U(\omega) = \begin{bmatrix} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{bmatrix}.$$

Using the change of variables formula

$$\frac{\partial}{\partial x_1} = \frac{R}{\Xi(\rho, \omega)^2} \left(\cos \omega \sinh \rho \frac{\partial}{\partial \rho} - \sin \omega \cosh \rho \frac{\partial}{\partial \omega} \right)$$

and

$$\frac{\partial}{\partial x_2} = \frac{R}{\Xi(\rho, \omega)^2} \left(\sin \omega \cosh \rho \frac{\partial}{\partial \rho} + \cos \omega \sinh \rho \frac{\partial}{\partial \omega} \right)$$

where $\Xi(\rho, \omega)$ is given by (5.11), we see through tedious but straightforward computations (we omit the computations) that

$$a \cdot \nabla \left(e^{-n\rho} \cos n\omega \right) = \frac{-Rne^{-n\rho}}{\Xi(\rho, \omega)^2} a \cdot U(n\omega)b(\rho, \omega)$$

and

$$a \cdot \nabla \left(e^{-n\rho} \sin n\omega \right) = \frac{-Rne^{-n\rho}}{\Xi(\rho, \omega)^2} a \cdot U(n\omega - \pi/2)b(\rho, \omega),$$

where

$$b(\rho, \omega) = \begin{bmatrix} \cos \omega \sinh \rho \\ \sin \omega \cosh \rho \end{bmatrix}.$$

Let θ_n be the angle between the vectors a and $U(n\omega)b(\rho, \omega)$. Then we have

$$|a \cdot U(n\omega)b(\rho, \omega)|^2 = |a|^2 |b(\rho, \omega)|^2 \cos^2 \theta_n$$

and

$$|a \cdot U(n\omega - \pi/2)b(\rho, \omega)|^2 = |a|^2 |b(\rho, \omega)|^2 \sin^2 \theta_n,$$

which implies

$$\left| a \cdot \nabla \left(e^{-n\rho} \cos n\omega \right) \right|^2 + \left| a \cdot \nabla \left(e^{-n\rho} \sin n\omega \right) \right|^2 = \frac{n^2 e^{-2n\rho} |a|^2 |b(\rho, \omega)|^2}{R^2 \left(\sinh^2 \rho + \sin^2 \omega \right)^2}.$$

(5.19)

It then follows from (5.18) that

$$\sum_{j=1}^{\infty} \frac{|\alpha_j(z)|^2}{\delta^2 + \lambda_j^2} \sim \sum_{n=1}^{\infty} \frac{n e^{2n\rho_0} e^{-2n\rho_z}}{\delta^2 + \lambda_n^2}$$

(5.20)

as $\delta \to 0$.

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We now investigate the asymptotic behavior of the righthand side of (5.20) as $\delta \to 0$. Let

$$N = \left[-\frac{1}{2\rho_0} \log 2\delta \right], \tag{5.21}$$

which is the first number such that $\delta > \frac{1}{2} e^{-2N\rho_0}$. Then one can easily see that

$$\sum_{n=1}^{\infty} \frac{ne^{2n\rho_0} e^{-2n\rho_z}}{\delta^2 + \lambda_n^2} = \sum_{n \leq N} + \sum_{n > N} \sim \sum_{n \leq N} \frac{ne^{2n\rho_0} e^{-2n\rho_z}}{e^{-4n\rho_0}} + \frac{1}{\delta^2} \sum_{n > N} ne^{-2n(\rho_z - \rho_0)}.$$

Observe that

$$\sum_{n \leq N} \frac{ne^{2n\rho_0} e^{-2n\rho_z}}{e^{-4n\rho_0}} \sim \sum_{n \leq N} ne^{2n(3\rho_0 - \rho_z)} \sim \begin{cases} 1 & \text{if } \rho_z > 3\rho_0, \\ |\log \delta|^2 & \text{if } \rho_z = 3\rho_0, \\ |\log \delta|\delta^{-3+\rho_z/\rho_0} & \text{if } \rho_0 < \rho_z < 3\rho_0. \end{cases}$$

On the other hand, we have

$$\frac{1}{\delta^2} \sum_{n > N} ne^{-2n(\rho_z - \rho_0)} \sim |\log \delta|\delta^{-3+\rho_z/\rho_0}.$$

So we infer that

$$\sum_{n=1}^{\infty} \frac{ne^{2n\rho_0} e^{-2n\rho_z}}{\delta^2 + \lambda_n^2} \sim \begin{cases} 1 & \text{if } \rho_z > 3\rho_0, \\ |\log \delta|^2 & \text{if } \rho_z = 3\rho_0, \\ |\log \delta|\delta^{-3+\rho_z/\rho_0} & \text{if } \rho_0 < \rho_z < 3\rho_0. \end{cases}$$

Since $\|\nabla F_z\|_{L^2(\Omega)}^2$ is bounded, we obtain (5.6). □

Proof of Theorem 5.3. One can see from (3.7), (3.11) and (3.13) that

$$u_\delta(x) - F_z(x) = \sum_{n=1}^{\infty} \frac{1}{i\delta - \lambda_n} \left(\frac{1}{2} - \lambda_n \right) \Big[(a \cdot \nabla_z \mathcal{S}_{\partial\Omega}[\psi_n^c](z)) \, \mathcal{S}_{\partial\Omega}[\psi_n^c](x)$$
$$+ (a \cdot \nabla_z \mathcal{S}_{\partial\Omega}[\psi_n^s](z)) \, \mathcal{S}_{\partial\Omega}[\psi_n^s](x) \Big].$$

It then follows from (5.16) and (5.17) that

$$u_\delta(x) - F_z(x) = \sum_{n=1}^{\infty} \frac{1}{i\delta - \lambda_n} \left(\frac{1}{2} - \lambda_n \right)$$
$$\times \Big[\frac{e^{n\rho_0} \cosh n\rho_0}{n\pi} \left(a \cdot \nabla_z \left(e^{-n\rho_z} \cos n\omega_z \right) \right) e^{-n\rho_x} \cos n\omega_x$$
$$+ \frac{e^{n\rho_0} \sinh n\rho_0}{n\pi} \left(a \cdot \nabla_z \left(e^{-n\rho_z} \sin n\omega_z \right) \right) e^{-n\rho_x} \sin n\omega_x \Big]$$

where (ρ_z, ω_z) is the elliptic coordinates of z. Therefore we have

$$|u_\delta(x) - F_z(x)| \lesssim \sum_{n=1}^{\infty} \frac{e^{4n\rho_0}}{n} \Big[\left| a \cdot \nabla_z \left(e^{-n\rho_z} \cos n\omega_z \right) \right| + \left| a \cdot \nabla_z \left(e^{-n\rho_z} \sin n\omega_z \right) \right| \Big] e^{-n\rho_x}$$

We then see from (5.19) that

$$|u_\delta(x) - F_z(x)| \lesssim \sum_{n=1}^{\infty} \frac{e^{4n\rho_0}}{n} ne^{-n\rho_z} e^{-n\rho_x} = \sum_{n=1}^{\infty} e^{-n(\rho_x + \rho_z - 4\rho_0)}.$$

(5.8) is an immediate consequence of (5.7). This completes the proof. □

5.2 Anomalous localized resonance on balls

The following theorem shows that resonance at the essential spectrum does not occur if Ω is a three dimensional ball.

Theorem 5.4 *Suppose that Ω is a three dimensional ball and $f(x) = a \cdot \nabla \delta_z(x)$ for some $z \in \mathbb{R}^3 \setminus \overline{\Omega}$. Then there is a constant C such that*

$$\|\nabla u_\delta\|_{L^2(\Omega)} \leq C \tag{5.22}$$

for all δ.

Proof. Assume that Ω is the unit ball centered at 0 for convenience. Then eigenvalue of $\mathcal{K}_{\partial\Omega}^*$ are

$$\lambda_n = \frac{1}{2(2n+1)}, \quad n = 1, 2, \ldots, \tag{5.23}$$

and corresponding eigenfunctions are $Y_n^m(\hat{x})$, $m = -n, -n+1, \cdots, n-1, n$, the spherical harmonics of degree n, where $\hat{x} = x/|x|$ (see [2]). It is also proven in the same paper that

$$\mathcal{S}_{\partial\Omega}[Y_n^m](x) = \begin{cases} -\dfrac{1}{2n+1} r^n Y_n^m(\hat{x}), & \text{for } |x| = r < 1, \\ -\dfrac{1}{2n+1} r^{-(n+1)} Y_n^m(\hat{x}), & \text{for } |x| = r \geq 1. \end{cases} \tag{5.24}$$

So the eigenfunctions normalized in $\mathcal{H}^*$ are

$$\psi_n^m(x) = \frac{1}{\sqrt{2n+1}} Y_n^m(x), \quad m = -n, -n+1, \cdots, n-1, n. \tag{5.25}$$

According to (3.13) and (5.3), we have

$$\sum_{j=1}^{\infty} \frac{\alpha_j(z)^2}{\delta^2 + \lambda_j^2} \approx \sum_{n=1}^{\infty} \frac{1}{\delta^2 + \lambda_n^2} \sum_{m=-n}^{n} \left| a \cdot \nabla \mathcal{S}_{\partial\Omega}[\psi_n^m](z) \right|^2. \tag{5.26}$$

We see from (5.24) that $\mathcal{S}_{\partial\Omega}[\psi_n^m]$ is a homogeneous harmonic polynomial of degree $-(n+1)$ in $\mathbb{R}^3 \setminus \overline{\Omega}$. So, we have

$$\frac{\partial}{\partial x_j} \mathcal{S}_{\partial\Omega}[\psi_n^m](x) = r^{-(n+2)} \sum_{m=-n-1}^{n+1} a_{nm}^j Y_{n+1}^m(\hat{x}), \quad |x| > 1 \tag{5.27}$$

for some constants a_{nm}^j, $j = 1, 2, 3$. We then have

$$\int_{\mathbb{R}^3 \setminus \Omega} \left| \nabla \mathcal{S}_{\partial\Omega}[\psi_n^m](x) \right|^2 dx = \frac{1}{2(n+2)} \sum_{j=1}^{3} \sum_{m=-n-1}^{n+1} |a_{nm}^j|^2.$$

We then infer from (4.3) that

$$\frac{1}{2(n+2)} \sum_{j=1}^{3} \sum_{m=-n-1}^{n+1} |a_{nm}^j|^2 \lesssim \|\psi_n^m\|_{\mathcal{H}^*} = 1.$$

It then follows from (5.27) that

$$\left|\nabla \mathcal{S}_{\partial\Omega}[\psi_n^m](z)\right|^2 = r_0^{-2(n+2)} \Big(\sum_{j=1}^{3} \sum_{m=-n-1}^{n+1} |a_{nm}^j|^2 \Big) \Big(\sum_{m=-n-1}^{n+1} |Y_n^m(\hat{z})|^2 \Big)$$

$$\lesssim (n+1) r_0^{-2(n+2)} \sum_{m=-n-1}^{n+1} |Y_n^m(\hat{z})|^2,$$

where $r_0 = |z|$. Using the Unsöld's theorem

$$\sum_{m=-n}^{n} |Y_n^m(\hat{z})|^2 = \frac{2n+1}{4\pi}, \ \ n = 0, 1, 2, \cdots,$$

we obtain

$$\left|\nabla \mathcal{S}_{\partial\Omega}[\psi_n^m](z)\right|^2 \lesssim n^2 r_0^{-2(n+2)}. \tag{5.28}$$

We now see from (5.26) that

$$\|\nabla(u_\delta - F_z)\|_{L^2(\Omega)}^2 \lesssim \sum_{n=1}^{\infty} \frac{n^3 r_0^{-2(n+2)}}{\delta^2 + \lambda_n^2} \lesssim \sum_{n=1}^{\infty} n^5 r_0^{-2(n+2)} \lesssim 1$$

for any δ. Since $\|\nabla F_z\|_{L^2(\Omega)}^2$ is finite, the proof is completed. $\quad\square$

Appendix: Expansions of the fundamental solution

In this appendix we write down the formula (3.5) explicitly when Ω is a unit ball in $\mathbb{R}^3$ or an ellipse in $\mathbb{R}^2$ to demonstrate the connection of (3.5) with the known expansion formula of the fundamental solution for the Laplace equation.

Let Ω be the unit ball in $\mathbb{R}^3$. We see from (5.25) that the eigenfunctions in $\mathcal{H}_0^*$ are

$$\{\frac{1}{\sqrt{2n+1}} Y_n^m; \ n = 1, 2, \ldots \text{ and } m = -n, -n+1, \ldots, n-1, n, \}.$$

One can easily see that $\varphi_0(x) = \frac{1}{4\pi}$ and

$$\mathcal{S}_{\partial\Omega}[\varphi_0](x) = \begin{cases} -\dfrac{1}{4\pi}, & \text{for } |x| = r < 1, \\ -\dfrac{1}{4\pi}\dfrac{1}{r}, & \text{for } |x| = r \geq 1. \end{cases} \tag{5.29}$$

Then, using (5.24) and (5.29), we have

$$\Gamma(x-z) = -\sum_{n=1}^{\infty} \frac{r^n r_z^{-n-1}}{(2n+1)^3} \sum_{m=-n}^{n} Y_{n,m}(\hat{x}) Y_{n,m}(\hat{z}) - \frac{1}{4\pi}\frac{1}{r_z}, \quad r \leq 1 < r_z, \tag{5.30}$$

where $r = |x|$, $r_z = |z|$ and $\hat{x} = x/|x|$, $\hat{z} = z/|z|$. As mentioned right after Theorem 3.1, (5.30) is the well-known expansion of the fundamental solution in $\mathbb{R}^3$ in terms of spherical harmonics.

Now let Ω be an ellipse with the elliptic radius ρ_0 in $\mathbb{R}^2$. Then eigenfunctions in $\mathcal{H}_0^*$ are given by $\{\psi_n^c, \psi_n^s\}_{n=1}^\infty$ defined by (5.15). We can see from (5.16) and (5.17) that $\Xi(\rho_0, \omega)^{-1}$ is orthogonal to $\{\psi_n^c, \psi_n^s\}_{n=1}^\infty$ in $\mathcal{H}^*$. So we have from (2.14) that

$$\varphi_0(\omega) = \frac{1}{2\pi \Xi(\rho_0, \omega)}.$$

Moreover, one can see that

$$\mathcal{S}_{\partial\Omega}[\varphi_0](x) = \begin{cases} \dfrac{1}{2\pi}(\rho_0 + \ln R - \ln 2), & \rho < \rho_0, \\ \dfrac{1}{2\pi}(\rho + \ln R - \ln 2), & \rho \geq \rho_0, \end{cases} \tag{5.31}$$

where (ρ, ω) be the elliptic coordinates of $x \in \mathbb{R}^2$. In fact, as $\rho \to \infty$, we have

$$|x| = R\sqrt{\cosh^2 \rho \cos^2 \omega + \sinh^2 \rho \sin^2 \omega} = \frac{R}{2}e^\rho + O(1),$$

which implies that

$$\rho + \ln R - \ln 2 = \ln |x| + O(|x|^{-1}) \quad \text{as } |x| \to \infty.$$

Note that ρ is a harmonic function. So, the righthand side of (5.31) is constant in $\rho < \rho_0$, harmonic in $\rho > \rho_0$, and behaves like $\ln |x|$ as $|x| \to \infty$. So the equality in (5.31) holds. Using (5.16), (5.17) and (5.31), we now have

$$\begin{aligned} \Gamma(x-z) = &-\sum_{n=1}^\infty \frac{1}{n\pi}\left(\cosh n\rho \cos n\rho\, e^{-n\rho_z} \cos n\rho_z + \sinh n\rho \sin n\rho\, e^{-n\rho_z} \sin n\rho_z\right) \\ &+ \frac{1}{2\pi}\left(\rho_z + \ln\left(\frac{R}{2}\right)\right), \quad \rho \leq \rho_0 < \rho_z, \end{aligned} \tag{5.32}$$

where (ρ, ω) is the elliptic coordinate of $x \in \overline{\Omega}$ and (ρ_z, ω_z) is that of $z \in \mathbb{R}^2 \setminus \overline{\Omega}$. An expansion of a derivative of $\Gamma(x-z)$ is also obtained in [8, (4.6)].

References

[1] H. Ammari, G. Ciraolo, H. Kang, H. Lee and G.W. Milton, Spectral theory of a Neumann-Poincaré-type operator and analysis of cloaking due to anomalous localized resonance, Arch. Ration. Mech. An. 208 (2013), 667–692.

[2] H. Ammari, G. Ciraolo, H. Kang, H. Lee and G.W. Milton, Spectral theory of a Neumann-Poincaré-type operator and analysis of anomalous localized resonance II, Contemporary Math. 615, 1–14, 2014.

[3] H. Ammari and H. Kang, *Polarization and moment tensors*, Applied Mathematical Sciences, 162, Springer, New York. 2007.

[4] H. Ammari, H. Kang, and H. Lee, A boundary integral method for computing elastic moment tensors for ellipses and ellipsoids, J. Comp. Math. 25 (1) (2007), 2-12.

[5] E. Bonnetier and F. Triki, Pointwise bounds on the gradient and the spectrum of the Neumann-Poincaré operator: The case of 2 discs, Contemp. Math. 577 (2012), 79–90.

[6] E. Bonnetier and F. Triki, On the spectrum of Poincaré variational problem for two close-to-touching inclusions in 2D, Arch. Ration. Mech. An. 209 (2013), 541–567.

[7] T. Chang and K. Lee, Spectral properties of the layer potentials on Lipschitz domains, Illinois J. Math. 52 (2008), 463–472.

[8] D. Chung, H. Kang, K. Kim and H. Lee, Cloaking due to anomalous localized resonance in plasmonic structures of confocal ellipses, SIAM J. Appl. Math. 74 (2014), 1691–1707.

[9] G. B. Folland, *Introduction to partial differential equations*, 2nd Ed., Princeton Univ. Press, Princeton, 1995.

[10] D. Grieser, The plasmonic eigenvalue problem, Rev. Math. Phys. 26 (2014), 1450005.

[11] H. Kang, Layer potential approaches to interface problems, *Inverse problems and imaging, Panoramas et Syntheses*, Societe Mathematique de France, to appear.

[12] H. Kang, K. Kim, H. Lee and J. Shin, Spectral properties of the Neumann-Poincaré operator and uniformity of estimates for the conductivity equation with complex coefficients, submitted, arXiv 1406.3873.

[13] H. Kang, M. Lim and S. Yu, Spectral resolution of the Neumann-Poincaré operator on intersecting disks and analysis of plasmonic resonance, preprint.

[14] H. Kang and J.-K. Seo, The inverse conductivity problem with one measurement: uniqueness for balls, SIAM J. of Applied Math. Vol 59 (1999), 1533–1539.

[15] H. Kang and J-K. Seo, Recent progress in the inverse conductivity problem with single measurement, in Inverse Problems and Related Fields, CRC Press, Boca Raton, FL, 2000, 69–80.

[16] O. D. Kellogg, *Foundations of potential theory*, Dover, New York, 1953 (Reprint from the first edition of *Die Grundlehren der Mathematischen Wissenschaften*, Band 31, Springer-Verlag, Berlin-New York, 1929).

[17] D. Khavinson, M. Putinar, and H. S. Shapiro, Poincaré's variational problem in potential theory, Arch. Ration. Mech. An. 185 (2007), 143–184.

[18] R. V. Kohn, J. Lu, B. Schweizer and M. I. Weinstein, A variational perspective on cloaking by anomalous localized resonance, Comm. Math. Phys. 328 (2014), 1–27.

[19] M. Lim, Symmetry of a boundary integral operator and a characterization of a ball, Illinois J. Math. 45 (2001), 537–543.

[20] G.W. Milton and N.-A.P. Nicorovici, On the cloaking effects associated with anomalous localized resonance, Proc. R. Soc. A 462 (2006), 3027–3059.

[21] H.M. Nguyen and L.H. Nguyen, Localized and complete resonance in plasmonic structres, arXiv:1310.3633.

[22] N.-A.P. Nicorovici, R.C. McPhedran, and G.W. Milton, Optical and dielectric properties of partially resonant composites, Phys. Rev. B 49 (1994), 8479–8482.

[23] K. Perfekt and M. Putinar, Spectral bounds for the Neumann-Poincaré operator on planar domains with corners, arXiv:1209.3918.

[24] G.C. Verchota, Layer potentials and boundary value problems for Laplace's equation in Lipschitz domains, J. Funct. Anal. 59 (1984), 572–611.