Asymptotic expansion for the Helmholtz equation and Applications
(Based on Joint Works with
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Layer Potentials for the Laplacian

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi} \ln |x|, & d = 2, \\ \dfrac{1}{(2-d)\omega_d} |x|^{2-d}, & d \geq 3, \end{cases}$$

where ω_d is the area of $(d-1)$ dimensional unit sphere. The single and double layer potentials of the density function ϕ on B are defined by

$$\mathcal{S}_B \phi(x) := \int_{\partial B} \Gamma(x-y)\phi(y)d\sigma(y), \quad x \in \mathbb{R}^d,$$

$$\mathcal{D}_B \phi(x) := \int_{\partial B} \frac{\partial}{\partial \nu_y} \Gamma(x-y)\phi(y)d\sigma(y), \quad x \in \mathbb{R}^d \setminus \partial B.$$

Trace formula (Fabes-Jodeit-Riviere, Verchota):

$$\frac{\partial}{\partial \nu^{\pm}} \mathcal{S}_B \phi(x) = (\pm\frac{1}{2} I + \mathcal{K}_B^*)\phi(x),$$

$$(\mathcal{D}_B \phi)|_{\pm} = (\mp\frac{1}{2} I + \mathcal{K}_B)\phi(x), \quad x \in \partial B,$$

where

$$\mathcal{K}_B \phi(x) = \frac{1}{\omega_d} \text{p.v.} \int_{\partial B} \frac{\langle x-y, \nu_y \rangle}{|x-y|^d} \phi(y) d\sigma(y)$$

and $\mathcal{K}_B^*$ is the L^2-adjoint of $\mathcal{K}_B$.

Polarization Tensor

B: a Lipschitz bounded domain in $\mathbb{R}^d$

The conductivity of B is k ($k \neq 1$), and that of background is 1.

The polarization tensor is $M = (m_{ij})$, $1 \leq i, j \leq d$, is defined by

$$m_{ij} := (1 - \frac{1}{k}) \left[\delta_{ij}|B| + (k-1) \int_{\partial B} y_i \frac{\partial \psi_j}{\partial \nu^+}(y) d\sigma(y) \right],$$

where ψ_j is the unique solution of

$$\begin{cases} \Delta \psi_j(x) = 0, \quad x \in B \cup \mathbb{R}^d \setminus \overline{B}, \\ \psi_j|_+ - \psi_j|_- = 0 \quad \text{on } \partial B, \\ \dfrac{\partial}{\partial \nu^+}\psi_j - k \dfrac{\partial}{\partial \nu^-}\psi_j = \nu_j \quad \text{on } \partial B, \\ \psi_j(x) \to 0 \text{ as } |x| \to \infty. \end{cases}$$

(Polya-Szego-Schiffer, Cedio.Fenya-Moskow-Vogelius, Friedman-Vogelius)

Theorem 1.1 *M is symmetric and positive-definite.*

[Cedio.Fenya-Moskow-Vogelius, Movchan-Serkov]

$$\psi_j = \frac{1}{k-1}\mathcal{S}_B(\lambda I - \mathcal{K}_B^*)^{-1}(\nu_j), \quad \lambda := \frac{k+1}{2(k-1)}.$$

Thus

$$\begin{aligned}
&(k-1)\int_{\partial B} y_i \frac{\partial}{\partial \nu^+}\psi_j(y)d\sigma(y) \\
&= \int_{\partial B} y_i(\frac{1}{2}I + \mathcal{K}_B^*)(\lambda I - \mathcal{K}_B^*)^{-1}(\nu_j)(y)d\sigma(y) \\
&= -\int_{\partial B} y_i\nu_j d\sigma(y) + (\lambda + \frac{1}{2})\int_{\partial B} y_i(\lambda I - \mathcal{K}_B^*)^{-1}(\nu_j)(y)d\sigma(y) \\
&= -\delta_{ij}|B| + \frac{k}{k-1}\int_{\partial B} y_i(\lambda I - \mathcal{K}_B^*)^{-1}(\nu_j)(y)d\sigma(y).
\end{aligned}$$

Therefore we prove that the polarization tensor M associated with B and k is given by

$$m_{ij} = \int_{\partial B} y_i(\lambda I - \mathcal{K}_B^*)^{-1}(\nu_j)(y)d\sigma(y). \tag{1.1}$$

Generalized Polarization Tensor

$\alpha = (\alpha_1, \ldots, \alpha_d), \beta = (\beta_1, \cdots, \beta_d) \in \mathbb{N}^d$, define the $M_{\alpha\beta}$ by

$$M_{\alpha\beta} := \int_{\partial B} y^\beta \phi_\alpha(y)d\sigma(y),$$

where

$$\phi_\alpha(x) := (\lambda I - \mathcal{K}_B^*)^{-1}(\nu_y \cdot \nabla y^\alpha)(x), \quad x \in \partial B.$$

Properties of GPT

Theorem 1.2 (Symmetry) *Suppose that a_α and b_β are constants such that $\sum_\alpha a_\alpha y^\alpha$ and $\sum_\beta b_\beta y^\beta$ are harmonic polynomials. Then*

$$\sum_{\alpha,\beta} a_\alpha b_\beta m_{\alpha\beta} = \sum_{\alpha,\beta} a_\alpha b_\beta m_{\beta\alpha}.$$

Theorem 1.3 (Positivity) *There exists a constant C depending only on the Lipschitz character of B such that if $\sum_{\alpha\in I} a_\alpha x^\alpha$ is a harmonic polynomial, then*

$$\int_B |\nabla(\sum_{\alpha\in I} a_\alpha x^\alpha)|^2 dx \leq \frac{k+1}{|k-1|} \Big| \sum_{\alpha,\beta\in I} a_\alpha a_\beta m_{\alpha\beta} \Big| \leq C \int_B |\nabla(\sum_{\alpha\in I} a_\alpha x^\alpha)|^2 dx.$$

In particular, if $|\alpha| = |\beta| = 1$, then

$$|B| \leq \frac{k+1}{|k-1|} \Big| \sum_{\alpha,\beta\in I} a_\alpha a_\beta m_{\alpha\beta} \Big| \leq C|B|.$$

Theorem 1.4 (Center of Mass) *Let B be a Lipschitz domain and x^* the center of mass of B. Let $\alpha_j := e_j$ and $\beta_j := 2e_j$, $j = 1, \ldots, d$. Then there exists C which depends only on the Lipschitz character of B such that*

$$\left| \frac{m_{\alpha_j \beta_j}}{m_{jj}} - x_j^* \right| \leq C \frac{|k-1|}{k+1} diam(B).$$

Theorem 1.5 (Dirichlet-to-Neumann map) *Let Ω be a domain compactly containing $\overline{B}$. Then the GTP uniquely determines the Dirichlet-to-Neumann map on $\partial\Omega$, and hence k and B.*

Asymptotic Expansion of Voltage Potential

- Ω : conductor in $\mathbb{R}^d$ (with a connected Lipschitz boundary),
- Electric inhomogeneity D in Ω :

$$D = \cup_{j=1}^m D_j = \cup_{j=1}^m (\epsilon B_j + z_j)$$

  where B_j is a bounded Lipschitz domain in $\mathbb{R}^d$ and z_j represents the location of D_j, and ϵ is the common order of magnitude.

- D_j has conductivity k_j
- D_j are well-separated: there exists $d_0 > 0$ such that

$$\inf_{x \in D} \mathrm{dist}(x, \partial\Omega) > d_0, \quad |z_i - z_j| > d_0.$$

Let u_ϵ be the solution to

$$\begin{cases} \nabla \cdot \Big(\chi(\Omega \setminus \bigcup_{l=1}^{m} \overline{D_l}) + \sum_{l=1}^{m} k_l \chi(D_l) \Big) \nabla u_\epsilon = 0 \quad \text{in } \Omega, \\ \dfrac{\partial u_\epsilon}{\partial \nu}\Big|_{\partial\Omega} = g. \end{cases}$$

Theorem 1.6 (Asymptotic Expansion) *On* $\partial\Omega$

$$\begin{aligned} u_\epsilon(x) &= U(x) \\ &- \epsilon^{d-2} \sum_{j=1}^{m} \sum_{|\alpha|=1}^{d} \sum_{|\beta|=1}^{n-|\alpha|+1} \frac{\epsilon^{|\alpha|+|\beta|}}{\alpha!\beta!} \partial^\alpha U(z_j) M^j_{\alpha\beta} \partial_z^\beta N(x, z_j) \\ &\quad + O(\epsilon^{2d}), \end{aligned}$$

where U *is the background solution,* $M^j_{\alpha\beta} = M_{\alpha\beta}(k_j, B_j)$ *are GPT, and* $N(x, z)$ *is the Neumann function.*

2 Helmholtz Equation

$D = z + \delta B$. Consider

$$\nabla \cdot (\frac{1}{\mu_\delta} \nabla u) + \omega^2 \varepsilon_\delta u = 0 \quad \text{in } \Omega, \tag{2.1}$$

with the boundary condition $u = f$ on $\partial\Omega$, where $\omega > 0$ is a given frequency. Here μ_δ and ε_δ denote the constitutive parameters of the inhomogeneity defined by

$$\mu_\delta(x) = \begin{cases} \mu_0, & x \in \Omega \setminus \overline{D}, \\ \mu, & x \in D, \end{cases} \tag{2.2}$$

$$\varepsilon_\delta(x) = \begin{cases} \varepsilon_0, & x \in \Omega \setminus \bar{D}, \\ \varepsilon, & x \in D, \end{cases} \tag{2.3}$$

where μ, μ_0, ε, and ε_0 are positive constants. If we allow the degenerate case $\delta = 0$, then the functions $\mu_\delta(x)$ and $\varepsilon_\delta(x)$ equal the constants μ_0 and ε_0. Problem (2.1) can be written as

$$\begin{cases} (\Delta + \omega^2 \varepsilon_0 \mu_0) u = 0 \quad \text{in } \Omega \setminus \overline{D}, \\ (\Delta + \omega^2 \varepsilon \mu) u = 0 \quad \text{in } D, \\ \frac{1}{\mu} \frac{\partial u}{\partial \nu}\Big|_- - \frac{1}{\mu_0} \frac{\partial u}{\partial \nu}\Big|_+ = 0 \quad \text{on } \partial D, \\ u|_- - u|_+ = 0 \quad \text{on } \partial D, \\ u = f \quad \text{on } \partial \Omega. \end{cases} \tag{2.4}$$

Layer Potential. Let $k_0 := \omega\sqrt{\varepsilon_0\mu_0}$ and $k := \omega\sqrt{\varepsilon\mu}$. Let $\Phi_k(x)$ be the fundamental solution for $\Delta + k^2$, that is for $x \neq 0$,

$$\Phi_k(x) = \begin{cases} \frac{i}{4}H_0^1(k|x-y|), & d = 2, \\ \dfrac{e^{ik|x-y|}}{4\pi|x-y|}, & d = 3, \end{cases}$$

where H_0^1 is the Hankel function of the first kind of order 0. Let

$$\Phi(x) = \Phi_0(x).$$

Let

$$\mathcal{S}_D^k\varphi(x) = \int_{\partial D} \Phi_k(x-y)\varphi(y)d\sigma(y), \quad x \in \mathbb{R}^d,$$
$$\mathcal{D}_D^k\varphi(x) = \int_{\partial D} \frac{\partial\Phi_k(x-y)}{\partial\nu(y)}\varphi(y)d\sigma(y), \quad x \in \mathbb{R}^d \setminus \partial D,$$

Jump Relation:

$$\frac{\partial(\mathcal{S}_D^k\varphi)}{\partial\nu}\Big|_{\pm}(x) = \Big(\pm\frac{1}{2}I + (\mathcal{K}_D^k)^*\Big)\varphi(x), \quad \text{a.e. } x \in \partial D,$$
$$(\mathcal{D}_D^k\varphi)|_{\pm} = (\mp\frac{1}{2}I + \mathcal{K}_D^k)\varphi(x), \quad \text{a.e. } x \in \partial D,$$

where

$$\mathcal{K}_D^k\varphi(x) = \text{p.v.}\int_{\partial D} \frac{\partial\Phi_k(x,y)}{\partial\nu(y)}\varphi(y)d\sigma(y).$$

Theorem 2.1 *Suppose that k_0^2 is not a Dirichlet eigenvalue for the Laplacian on D. For each $(F, G) \in H^1(\partial D) \times L^2(\partial D)$, there exists a unique solution $(f, g) \in L^2(\partial D) \times L^2(\partial D)$ to the integral equation*

$$\left\{ \begin{array}{l} \mathcal{S}_D^k f - \mathcal{S}_D^{k_0} g = F \\ \dfrac{1}{\mu} \dfrac{\partial (\mathcal{S}_D^k f)}{\partial \nu} \Big|_- - \dfrac{1}{\mu_0} \dfrac{\partial (\mathcal{S}_D^{k_0} g)}{\partial \nu} \Big|_+ = G \end{array} \right. \quad \text{on } \partial D.$$

There exists a constant C independent of F and G such that

$$\|f\|_{L^2(\partial D)} + \|g\|_{L^2(\partial D)} \leq C(\|F\|_{H^1(\partial D)} + \|G\|_{L^2(\partial D)}).$$

Moreover, if k_0 and k go to zero, then the constant C can be chosen independently of k_0 and k.

Representation of Solutions

Theorem 2.2 *Suppose that k_0^2 is not a Dirichlet eigenvalue for the Laplacian on D. Let u be the solution of (2.4) and $g := \frac{\partial u}{\partial \nu}|_{\partial\Omega}$. Define*

$$H(x) := \mathcal{S}_\Omega^{k_0}(g)(x) - \mathcal{D}_\Omega^{k_0}(f)(x), \quad x \in \mathbb{R}^d \setminus \partial\Omega,$$

and $(\varphi, \psi) \in L^2(\partial D) \times L^2(\partial D)$ be the unique solution of

$$\begin{cases} \mathcal{S}_D^k \varphi - \mathcal{S}_D^{k_0} \psi = H \\ \dfrac{1}{\mu} \dfrac{\partial (\mathcal{S}_D^k \varphi)}{\partial \nu}\Big|_- - \dfrac{1}{\mu_0} \dfrac{\partial (\mathcal{S}_D^{k_0} \psi)}{\partial \nu}\Big|_+ = \dfrac{1}{\mu_0} \dfrac{\partial H}{\partial \nu} \end{cases} \quad \text{on } \partial D.$$

Then u can be represented as

$$u(x) = \begin{cases} H(x) + \mathcal{S}_D^{k_0} \psi(x), & x \in \Omega \setminus \overline{D}, \\ \mathcal{S}_D^k \varphi(x), & x \in D. \end{cases}$$

Moreover, there exists $C > 0$ independent of H such that

$$\|\varphi\|_{L^2(\partial D)} + \|\psi\|_{L^2(\partial D)} \leq C\big(\|H\|_{L^2(\partial D)} + \|\nabla H\|_{L^2(\partial D)}\big).$$

Let $G(x,y)$ be the Dirichlet Green function for $\Delta + k_0^2$ in Ω, i.e., for each $y \in \Omega$,

$$\begin{cases} (\Delta + k_0^2) G(x,y) = \delta_y(x), & x \in \Omega, \\ G(x,y) = 0, & x \in \partial\Omega. \end{cases}$$

Define

$$G_D \varphi(x) := \int_{\partial D} G(x,y)\varphi(y) d\sigma(y), \quad x \in \overline{\Omega}.$$

Theorem 2.3 *Let ψ be the function defined before. Then*

$$\frac{\partial u}{\partial \nu}(x) = \frac{\partial u_0}{\partial \nu}(x) - \frac{\partial (G_D \psi)}{\partial \nu}(x), \quad x \in \partial\Omega.$$

Asymptotic Formula

Theorem 2.4 *The following pointwise asymptotic expansion on $\partial\Omega$ holds for $d = 2, 3$:*

$$\frac{\partial u}{\partial \nu}(x) = \frac{\partial u_0}{\partial \nu}(x) - \delta^{d-2} \sum_{|\beta|=0}^{n+1} \sum_{|\alpha|=0}^{n+1-|\beta|} \frac{\delta^{|\alpha|+|\beta|}}{\alpha!\beta!} \times$$

$$\Big[\Big((I + \sum_{p=1}^{n+2-|\alpha|-|\beta|-d} \delta^{d+p-1} \mathcal{Q}_p)(\partial^\gamma u_0(z))\Big)_\alpha \frac{\partial \partial_z^\beta G(x,z)}{\partial \nu(x)} W_{\alpha\beta}\Big]$$

$$+O(\delta^{n+d}),$$

where the remainder $O(\delta^{d+n})$ is dominated by $C\delta^{d+n}\|f\|_{H^{1/2}(\partial\Omega)}$ for some C independent of $x \in \partial\Omega$.

Here,

$$W_{\alpha\beta} := \int_{\partial B} w^\beta \psi_\alpha(w) d\sigma(w),$$

$$\begin{cases} \mathcal{S}_B^{k\delta} \varphi_\alpha - \mathcal{S}_B^{k_0\delta} \psi_\alpha = x^\alpha \\ \dfrac{1}{\mu} \dfrac{\partial(\mathcal{S}_B^{k\delta} \varphi_\alpha)}{\partial \nu}\Big|_- - \dfrac{1}{\mu_0} \dfrac{\partial(\mathcal{S}_B^{k_0\delta} \psi_\alpha)}{\partial \nu}\Big|_+ = \dfrac{1}{\mu_0} \dfrac{\partial x^\alpha}{\partial \nu} \end{cases} \quad \text{on } \partial B.$$

Important Fact.

$$W_{\alpha\beta} = m_{\alpha\beta}(\frac{\mu}{\mu_0}) + O(\delta).$$

If $D = \cup_{s=1}^m (\delta B_s + z_s)$, well separated. The magnetic permeability and electric permittivity of the inclusion $\delta B_s + z_s$ are μ_s and ϵ_s, $s = 1, ..., m$.

Theorem 2.5 *The following pointwise asymptotic expansion on $\partial\Omega$ holds for $d = 2, 3$:*

$$\frac{\partial u}{\partial \nu}(x) = \frac{\partial u_0}{\partial \nu}(x)$$
$$- \delta^{d-2} \sum_{s=1}^{m} \sum_{|\beta|=0}^{d+1} \sum_{|\alpha|=0}^{d+1-|\beta|} \frac{\delta^{|\alpha|+|\beta|}}{\alpha!\beta!} \partial^\alpha u_0(z) \frac{\partial \partial_z^\beta G(x,z)}{\partial \nu(x)} W_{\alpha\beta}^s$$
$$+ O(\delta^{2d}).$$

Here $W_{\alpha\beta}^s$ corresponds to B_s, μ_s, ϵ_s.

The first order term:

$$\frac{\partial u}{\partial \nu}(x) = \frac{\partial u_0}{\partial \nu}(x)$$
$$- \delta^d \Big(\nabla u_0(z) M(\frac{\mu}{\mu_0}) \frac{\partial \nabla_z G(x,z)}{\partial \nu(x)}$$
$$+ \omega^2 \mu_0 (\epsilon - \epsilon_0) |B| u_0(z) \frac{\partial G(x,z)}{\partial \nu(x)} \Big)$$
$$+ O(\delta^{d+1}).$$

This formula is obtained by Vogelius-Volkov.

3 Detection of Inclusions

Inverse Problem. Given f, measure $\frac{\partial u}{\partial \nu}$. Using $(f, \frac{\partial u}{\partial \nu})$, determine the location and size of inclusions.

We apply plane waves:

$$f = e^{ik\theta\cdot x}, \quad |\theta| = 1.$$

Let u_δ be the corresponding solution.

Goal: Reconstruct the electromagnetic inhomogeneities $\{D_l\}_{l=1}^m$ from limited current-to-voltage pairs

$$\Big(e^{ik\theta\cdot x}|_{\partial\Omega}, \frac{\partial u_\delta}{\partial \nu}|_{\partial\Omega}\Big).$$

Define $A_\delta(\frac{x}{|x|}, \theta, k)$ by

$$\begin{aligned} &\mathcal{S}_\Omega(\frac{\partial u_\delta}{\partial \nu}|_{\partial\Omega})(x) - \mathcal{D}_\Omega(e^{ik\theta\cdot y}|_{\partial\Omega})(x) \\ &= A_\delta(\frac{x}{|x|}, \theta, k)\frac{e^{ik|x|}}{4\pi|x|} + O(\frac{1}{|x|^2}) \end{aligned}$$

as $|x| \to \infty$.

Note that $A_\delta(\frac{x}{|x|}, \theta, k)$ is directly computed from the current-to voltage pairs $(e^{ik\theta\cdot y}|_{\partial\Omega}, \frac{\partial u_\delta}{\partial \nu}|_{\partial\Omega})$.

Theorem 3.1

$$A_\delta(\frac{x}{|x|}, \theta, k) = \delta^3 k^2 \sum_{l=1}^{m} \Big[\frac{x}{|x|} \cdot M_l(\frac{\mu_l}{\mu_0}) \cdot \theta$$
$$+(\frac{\epsilon_l}{\epsilon_0} - 1)|B_l| \Big] e^{ik(\theta - \frac{x}{|x|})\cdot z_l} + O(\delta^4),$$

for any $\frac{x}{|x|}$ *and* $\theta \in S^2$*, where* $O(\delta^4)$ *is independent of the set of points* $\{z_l\}_{l=1}^m$.

Reconstruction of Single Inclusion

Magnitude:

$$|A_\delta(-\theta, \theta, k)| \approx \delta^3.$$

Location:

$$e^{i4k\theta\cdot z_1} = \frac{A_\delta(\theta, -\theta, k)}{A_\delta(-\theta, \theta, k)} + O(\delta).$$

Multiple Inclusions

Assume that B_l, for $l = 1, \ldots, m$, are balls.

$$M_l\left(\frac{\mu_l}{\mu_0}\right) = (1 - \frac{\mu_l}{\mu_0})m_l I_3,$$

where I_3 is the 3×3 identity matrix and

$$m_l = 8\pi|B_l|\frac{\mu_l}{\mu_l + \mu_0}.$$

1. Let

$$g(\frac{x}{|x|}, \theta) := \frac{1}{k^2\delta^3} A_\delta(\frac{x}{|x|}, \theta, k)$$
$$= \sum_{l=1}^{m} e^{ik(\theta - \frac{x}{|x|})\cdot z_l} \left[(1 - \frac{\mu_l}{\mu_0})m_l \frac{x}{|x|} \cdot \theta + (\frac{\epsilon_l}{\epsilon_0} - 1)|B_l|\right],$$
$$(\frac{x}{|x|}, \theta) \in S^2 \times S^2.$$

2. Let M be the analytic variety

$$M = \{\xi \in C^3, \xi \cdot \xi = 1\}$$

S^2 is a totally real submanifold of M. On $M \times M$,

$$g(\xi_1, \xi_2) = \sum_{l=1}^{m} e^{-ik(\xi_1 - \xi_2)\cdot z_l} \left[(1 - \frac{\mu_l}{\mu_0})m_l \xi_1 \cdot \xi_2 + (\frac{\epsilon_l}{\epsilon_0} - 1)|B_l|\right].$$

This is a unique analytic continuation of g.

3. Idea of Calderón and Sylvester-Uhlmann: for any $\xi \in \mathbb{R}^3$ there exist $\xi_1, \xi_2 \in M$ such that $\xi = \frac{\xi_1 - \xi_2}{k}$. Since

$$\xi_1 \cdot \xi_2 = 1 - \frac{1}{2} k^2 |\xi|^2,$$

we can rewrite g as follows

$$g(\xi_1, \xi_2) = \sum_{l=1}^{m} e^{-i\xi \cdot z_l} \left[(1 - \frac{\mu_l}{\mu_0}) m_l (1 - \frac{1}{2} k^2 |\xi|^2) + (\frac{\epsilon_l}{\epsilon_0} - 1)|B_l| \right].$$

Define

$$\widetilde{g}(\xi) = g(\xi_1, \xi_2),$$

Then

$$\mathcal{F}^{-1}(\widetilde{g}(\xi)) = \sum_{l=1}^{m} L_l(\delta_{z_l}),$$

where L_l are, second order differential operators with constant coefficients.

4 The Full Maxwell's Equations

Let E_δ denote the electric field in the presence of the imperfections. It is the solution to full Maxwell's equations

$$\nabla \times \left(\frac{1}{\mu_\delta} \nabla \times E_\delta \right) - \omega^2 \epsilon_\delta E_\delta = 0 \quad , \quad \text{in } \Omega \ ,$$

with

$$E_\delta \times \nu = f \quad , \quad \text{on } \partial\Omega \ .$$

Let

$$\mathbf{\Gamma}(x,y) = \Gamma(x,y) I + \frac{1}{k^2} \nabla_x \nabla_x \cdot (\Gamma(x,y) I).$$

Apply

$$f(x) = e^{ik\theta\cdot x} \theta' \times \nu.$$

Define $A_\delta(\frac{x}{|x|}, \theta, \theta', k)$ by

$$\int_{\partial\Omega} \nabla \times \mathbf{\Gamma} \times \nu \cdot E_\delta - \int_{\partial\Omega} \nabla \times E_\delta \times \nu \cdot \mathbf{\Gamma}$$
$$:= A_\delta(\frac{x}{|x|}, \theta, \theta', k) \frac{e^{ik|x|}}{|x|} + O(\frac{1}{|x|^2})$$

as $|x| \to \infty$,

Using an asymptotic expansion formula of Ammari-Vogelius-Volkov,

Theorem 4.1

$$
\begin{aligned}
&A_\delta(\frac{x}{|x|}, \theta, \theta', k) \\
&= ik^3\delta^3 \sum_{j=1}^{l} \Big[(M_l(\frac{\mu_0}{\mu_l})(\theta \times (\theta \times \theta'))) \times \frac{x}{|x|} \\
&+ (1 - \frac{\varepsilon_0}{\varepsilon_l})(I - \frac{x}{|x|}\frac{x}{|x|}^t) M_l(\frac{\varepsilon_0}{\varepsilon_l})(\theta \times \theta) \Big] e^{ik(\theta - \frac{x}{|x|})\cdot z_l} \\
&+ O(\delta^4).
\end{aligned}
$$

for any $\frac{x}{|x|}$, θ, *and* $\theta' \in S^2$, *where the remainder* $O(\delta^4)$ *is independent of the set of points* $\{z_l\}_{l=1}^m$.