Asymptotic Expansion of Solutions to the Lamé System in the Presence of Inclusions and Applications (Based on a Joint Work with Habib Ammari, Gen Nakamura, and Kazumi Tanuma)

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”Imperfection in the metallic structure can lead to a significant reduction in the performance of a given item, but worse still can be ’inclusion’ or ’defect’ (small particles of other materials trapped in the metal). Metallic items normally ultimately fail by cracking and inclusions can act as the starting points for cracks - the larger the inclusion, the larger the crack and the quicker it will grow. In aerospace applications, inclusions as small as 1-hundredths of a millimetre are important. To put this in perspective an inclusion of about 20 millionth of a gramme can lead to failure in a component a metre long.

On 19 July 1989, United Airlines Flight 232, a wide-bodied DC-10, crashed at Sioux City, Iowa, ultimately resulting in 112 deaths (Randall, 1991). This crash was a direct consequence of a fatigue failure initiated by the presence of a ’hard alpha’ inclusion in a titanium alloy engine component. Ensuring the safe performance of such components is therefore of paramount importance. However, it is not just the aerospace industry which requires predictable long life from significantly stressed components - in both the medical and offshore industries, the effects of component failure could be disastrous.”

Source : www.irc.bham.ac.uk/theme1/plasma/production.htm

Contents

1 Problem

- Ω : elastic body $\mathbb{R}^3$ (with a connected Lipschitz boundary),
- (λ, μ) : Lamé coefficients (constant) of Ω,
- Elastic inhomogeneity D in Ω :

$$D = \cup_{j=1}^m D_j = \cup_{j=1}^m (\epsilon B_j + z_j)$$

  where B_j is a bounded Lipschitz domain in $\mathbb{R}^3$ and z_j represents the location of D_j, and ϵ is the common order of magnitude.

- $(\widetilde{\lambda}_j, \widetilde{\mu}_j)$: Lamé constants of D_j
- Assume

$$\widetilde{\mu}_j > 0, \quad 3\widetilde{\lambda}_j + 2\widetilde{\mu}_j > 0, \quad (\lambda - \widetilde{\lambda}_j)(\mu - \widetilde{\mu}_j) > 0.$$

- D_j are well-separated: there exists $d_0 > 0$ such that

$$\inf_{x \in D} \operatorname{dist}(x, \partial\Omega) > d_0, \quad |z_i - z_j| > d_0.$$

Consider the transmission problem:

$$\begin{cases} \displaystyle\sum_{j,k,l=1}^{3} \frac{\partial}{\partial x_j}\left(C_{ijkl}\frac{\partial u_k}{\partial x_l}\right) = 0 \quad \text{in } \Omega, \quad i = 1,2,3, \\ \displaystyle\frac{\partial \vec{u}}{\partial \nu}|_{\partial\Omega} = \vec{g}, \end{cases}$$

where

$$\begin{aligned} C_{ijkl} := \ & \Big(\lambda\chi(\Omega\setminus D) + \sum_{s=1}^{m} \widetilde{\lambda}_s \chi(D_s)\Big)\delta_{ij}\delta_{kl} \\ & + \Big(\mu\chi(\Omega\setminus D) + \sum_{s=1}^{m} \widetilde{\mu}_s \chi(D_s)\Big)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \end{aligned}$$

($\chi(D)$ is the characteristic function of D),

$\frac{\partial}{\partial \nu}$ denotes the conormal derivative:

$$\frac{\partial \vec{u}}{\partial \nu} := \lambda(\text{div } \vec{u})N + \mu(\nabla\vec{u} + \nabla\vec{u}^T)N \quad \text{on } \partial D,$$

(N: outward unit normal to ∂D, T: the transpose),

$\vec{g}$ satisfies the usual compatibility condition:

$$\int_{\partial D} \vec{g}\cdot\vec{\psi} d\sigma = 0 \text{ for all } \vec{\psi} \in \Psi$$

where Ψ is the set of all $\vec{\psi}$ satisfying

$$\partial_i\psi_j + \partial_j\psi_i = 0, \quad 1 \le i,j \le 3.$$

Or equivalently,

$$
\begin{cases}
\mathcal{L}_{\lambda,\mu}\vec{u} = 0 \quad \text{in } \Omega \setminus \overline{D}, \\
\mathcal{L}_{\widetilde{\lambda}_j,\widetilde{\mu}_j}\vec{u} = 0 \quad \text{in } D_j, \\
\vec{u}|_+ = \vec{u}|_- \quad \text{on } \partial D_j, \\
\dfrac{\partial \vec{u}}{\partial \widetilde{\nu}}\Big|_+ = \dfrac{\partial \vec{u}}{\partial \nu}\Big|_- \quad \text{on } \partial D_j, \\
\dfrac{\partial \vec{u}}{\partial \nu}|_{\partial\Omega} = \vec{g}, \quad (\vec{u}\big|_{\partial\Omega} \perp \Psi),
\end{cases}
$$

$$\mathcal{L}_{\lambda,\mu}\vec{u} := \mu\Delta\vec{u} + (\lambda+\mu)\nabla \text{div } \vec{u}.$$

Problem. Derive an asymptotic expansion of $\vec{u}$ as $\epsilon \to 0$ in terms of ϵ and the background solution $\vec{U}$, i.e., the solution without inhomogeneities:

$$
\begin{cases}
\displaystyle\sum_{j,k,l=1}^{3} \frac{\partial}{\partial x_j}\left(C^0_{ijkl}\frac{\partial \vec{U}_k}{\partial x_l}\right) = 0 \quad \text{in } \Omega, \quad i = 1,2,3, \\
\dfrac{\partial \vec{U}}{\partial \nu}|_{\partial\Omega} = \vec{g},
\end{cases}
$$

where

$$C^0_{ijkl} := \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

2 Asymptotic Formula

Theorem 2.1

$$\vec{u}(x) = \vec{U}(x)$$
$$+ \sum_{s=1}^{m} \sum_{j=1}^{3} \sum_{|\alpha|=1}^{3} \sum_{|\beta|=1}^{4-|\alpha|} \frac{\epsilon^{|\alpha|+|\beta|+1}}{\alpha!\beta!} (\partial^\alpha U_j)(z) \partial_z^\beta N(x,z) M_{\alpha\beta}^{(s)j}$$
$$+O(\epsilon^6), \quad \textit{uniformly } x \in \partial\Omega.$$

where $N(x, y)$ *be the Neumann function (matrix) for* $\mathcal{L}_{\lambda,\mu}$ *in* Ω*:*

$$\begin{cases} \mathcal{L}_{\lambda,\mu} N(x,y) = -\delta_y(x) I \quad \textit{in } \Omega, \\ \dfrac{\partial N}{\partial \nu}\Big|_{\partial\Omega} = -\dfrac{1}{|\partial\Omega|} I, \\ N(\cdot, y) \perp \Psi \quad \textit{for each } y \in \Omega, \end{cases}$$

where the differentiations act on the x*-variables, and* $M_{\alpha\beta}^{(s)j}$ *is the (generalized) Elastic Moment Tensor (Pólya-Szegö tensor).*

Remark. 1. A complete expansion formula is obtained.
2. Other related works :

- Conductivity : Cedio-Fenya-Moskow-Vogelius (first order term), Ammari-Kang (complete expansion)
- Maxwell System : Ammari-Vogelius-Volkov (first order term)
- Elasticity : Maz'ya-Nazarov (first order term for cavity or hard inclusion). Cavity: $\widetilde{\lambda} = \widetilde{\mu} = 0$, Hard inclusion: $\widetilde{\lambda} = \widetilde{\mu} = \infty$

3 Layer Potentials for the Lamé System

The Kelvin matrix of fundamental solutions $\Gamma = (\Gamma_{ij})$ for the Lamé system corresponding to the Lamé parameters (λ, μ):

$$\Gamma_{ij}(x) := \frac{A}{4\pi}\frac{\delta_{ij}}{|x|} + \frac{B}{4\pi}\frac{x_i x_j}{|x|^3}, \quad x \in \mathbb{R}^3,\ x \neq 0,$$

where

$$A = \frac{1}{2}\left(\frac{1}{\mu} + \frac{1}{2\mu + \lambda}\right) \quad \text{and} \quad B = \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{2\mu + \lambda}\right).$$

The single and double layer potentials of the density function $\vec{\phi}$ on D associated with the Lamé parameters (λ, μ) are defined by

$$\mathcal{S}_D\vec{\phi}(x) := \int_{\partial D} \Gamma(x-y)\vec{\phi}(y)d\sigma(y), \quad x \in \mathbb{R}^3,$$

$$\mathcal{D}_D\vec{\phi}(x) := \int_{\partial D} \frac{\partial}{\partial \nu_y}\Gamma(x-y)\vec{\phi}(y)d\sigma(y), \quad x \in \mathbb{R}^3 \setminus \partial D,$$

Lemma 3.1 (Dahlberg-Kenig-Verchota)

$$\mathcal{D}_D\vec{\phi}|_\pm = (\mp\frac{1}{2}I + \mathcal{K}_D)\vec{\phi}, \quad on\ \partial D,$$
$$\frac{\partial}{\partial\nu}\mathcal{S}_D\vec{\phi}\Big|_\pm = (\pm\frac{1}{2}I + \mathcal{K}_D^*)\vec{\phi}, \quad on\ \partial D,$$

where $\mathcal{K}_D$ *is defined by*

$$\mathcal{K}_D\vec{\phi}(x) := p.v. \int_{\partial D} \frac{\partial}{\partial\nu_y}\Gamma(x-y)\vec{\phi}(y)d\sigma(y), \quad x \in \partial D,$$

and $\mathcal{K}_D^*$ *is the adjoint operator of* $\mathcal{K}_D$ *on* $L^2(\partial D)$. *Here and throughout this paper* $\vec{u}|_+$ *and* $\vec{u}|_-$ *denote the limit from inside* D *and outside* D, *respectively.*

Theorem 3.2 ([Dahlberg-Kenig-Verchota) *] The operators* $\frac{1}{2}I + \mathcal{K}_D^*$ *and* $-\frac{1}{2}I + \mathcal{K}_D^*$ *are invertible on* $L^2_\Psi(\partial D)$ *and* $L^2(\partial D)$, *respectively.*

Corollary 3.3 *The null space of* $\frac{1}{2}I + \mathcal{K}_D$ *on* $L^2(\partial D)$ *is* Ψ.

4 Transmission Problem

$D = \epsilon B + z$ with the Lamé parameters $(\widetilde{\lambda}, \widetilde{\mu})$.

Theorem 4.1 (Escauriaza-Seo) *Suppose that* $(\lambda - \widetilde{\lambda})(\mu - \widetilde{\mu}) \geq 0$. *For any given* $(\vec{F}, \vec{G}) \in L^2_1(\partial D) \times L^2(\partial D)$, *there exists a unique pair* $(\vec{f}, \vec{g}) \in L^2(\partial D) \times L^2(\partial D)$ *such that*

$$\left\{ \begin{array}{ll} \widetilde{\mathcal{S}}_D \vec{f}|_+ - \mathcal{S}_D \vec{g}|_- = \vec{F} & \text{on } \partial D, \\ \dfrac{\partial}{\partial \widetilde{\nu}} \widetilde{\mathcal{S}}_D \vec{f}\Big|_+ - \dfrac{\partial}{\partial \nu} \mathcal{S}_D \vec{g}\Big|_- = \vec{G} & \text{on } \partial D, \end{array} \right.$$

and there exists a constant C *depending only on* λ, μ, $\widetilde{\lambda}$, $\widetilde{\mu}$, *and the Lipschitz character of* D *such that*

$$\|\vec{f}\|_{L^2(\partial D)} + \|\vec{g}\|_{L^2(\partial D)} \leq C(\|\vec{F}\|_{L^2_1(\partial D)} + \|\vec{G}\|_{L^2(\partial D)}).$$

Moreover, if $\vec{G} \in L^2_\Psi(\partial D)$, *then* $\vec{g} \in L^2_\Psi(\partial D)$.

Theorem 4.2 *There exists a unique pair* $(\vec{\varphi}, \vec{\psi}) \in L^2(\partial D) \times L^2_\Psi(\partial D)$ *such that the solution* $\vec{u}$ *is represented by*

$$\vec{u}(x) = \begin{cases} \vec{H}(x) + \mathcal{S}_D \vec{\psi}(x), & x \in \Omega \setminus \overline{D}, \\ \widetilde{\mathcal{S}}_D \vec{\varphi}(x), & x \in D, \end{cases}$$

where $\vec{H}$ *is defined by*

$$\vec{H}(x) = \mathcal{S}_\Omega(\vec{g})(x) - \mathcal{D}_\Omega(\vec{f})(x), \quad \vec{f} := \vec{u}|_{\partial\Omega},$$

There exists C *such that*

$$\|\vec{\varphi}\|_{L^2(\partial D)} + \|\vec{\psi}\|_{L^2(\partial D)} \le C \|\vec{H}\|_{L^2_1(\partial D)}.$$

For each integer n *there exists* C_n *depending only on* d_0 *and* λ, μ *(not on* $\widetilde{\lambda}, \widetilde{\mu}$*) such that*

$$\|\vec{H}\|_{C^n(\overline{D})} \le C_n \|\vec{g}\|_{L^2(\partial\Omega)}.$$

Moreover,

$$\vec{H}(x) = -\mathcal{S}_D \vec{\psi}(x), \quad x \in \mathbb{R}^3 \setminus \overline{\Omega}.$$

The following lemma relates the fundamental solution with the Neumann function.

Lemma 4.3 *For $z \in \Omega$ and $x \in \partial\Omega$, let $\Gamma_z(x) := \Gamma(x-z)$ and $N_z(x) := N(x,z)$. Then*

$$(\frac{1}{2}I + \mathcal{K}_\Omega)(N_z)(x) = \Gamma_z(x) \quad mod\ \Psi,$$

or to be more precise, for any simply connected Lipschitz domain D compactly contained in Ω and for any $\vec{g} \in L^2_\Psi(\partial D)$, we have

$$\int_{\partial D} (\frac{1}{2}I + \mathcal{K}_\Omega)(N_z)(x)\vec{g}(z)d\sigma(z)$$
$$= \int_{\partial D} \Gamma_z(x)\vec{g}(z)d\sigma(z), \quad \forall x \in \partial\Omega.$$

Let

$$N_D\vec{f}(x) := \int_{\partial D} N(x,y)\vec{f}(y)d\sigma(y), \quad x \in \overline{\Omega}.$$

Theorem 4.4

$$\vec{u}(x) = \vec{U}(x) + N_D\vec{\psi}(x), \quad x \in \partial\Omega,$$

where $\vec{\psi}$ is defined in Theorem 4.2

5 Elastic Moment Tensors

We now introduce the notion of elastic moment tensors.

Definition 5.1 *(Elastic Moment Tensors). For multi-index $\alpha \in \mathbb{N}^3$ and $j = 1, 2, 3$, let $\vec{f}_\alpha^j$ and $\vec{g}_\alpha^j$ in $L^2(\partial B)$ be the solution of*

$$\begin{cases} \widetilde{\mathcal{S}}_B \vec{f}_\alpha^j|_+ - \mathcal{S}_B \vec{g}_\alpha^j|_- = x^\alpha e_j|_{\partial B}, \\ \dfrac{\partial}{\partial \widetilde{\nu}} \widetilde{\mathcal{S}}_B \vec{f}_\alpha^j|_+ - \dfrac{\partial}{\partial \nu} \mathcal{S}_B \vec{g}_\alpha^j|_- = \dfrac{\partial (x^\alpha e_j)}{\partial \nu}|_{\partial B}. \end{cases}$$

For $\beta \in \mathbb{N}^3$, the elastic moment tensor (EMT) $M_{\alpha\beta}^j$ associated with the domain B and Lamé parameters (λ, μ) for the background and $(\widetilde{\lambda}, \widetilde{\mu})$ for B is defined by

$$M_{\alpha\beta}^j = (m_{\alpha\beta 1}^j, m_{\alpha\beta 2}^j, m_{\alpha\beta 3}^j) = \int_{\partial B} y^\beta \vec{g}_\alpha^j(y) d\sigma(y).$$

Remark.

- The first order EMT is the elastic version of the polarization tensor in electro-magnetism introduced by Pólya-Schiffer-Szegö
- In the case of cavities and hard inclusions, the first order EMT was introduced by Maz'ya-Nazarov, and studied by Lewiński-Sokolowski, Movchan-Serkov, and a lot more.
- Our definition includes non-cavity cases and higher order tensors.
- Polarization Tensors of all orders and their properties (conductivity case): Ammari-Kang
  - Polarization tensors of all orders determine the Dirichlet-to-Neumann map.
  - First order tensor - volume, second order - center of mass
- Anisotropic Polarization Tensor : Kang-Kim-Kim.

When $\alpha = e_i$ and $\beta = e_p$ $(i, p = 1, 2, 3)$, put

$$m_{pq}^{ij} := m_{\alpha\beta q}^{j}, \quad p, j = 1, 2, 3.$$

Lemma 5.2 Properties of EMT

- *EMT is symmetric:* $m_{pq}^{ij} = m_{qp}^{ij}$, $m_{pq}^{ij} = m_{pq}^{ji}$, *and* $m_{pq}^{ij} = m_{ij}^{pq}$, $p, q, i, j = 1, 2, 3$.
- *EMT is positive definite on the space of symmetric matices.*
- *Suppose* $i \neq j$ *and that* B *satisfies* $diam(B)|\partial B| \leq C_0|B|$ *for some* C_0*. Then there exists* $C = C(\lambda, \mu, \widetilde{\lambda}, \widetilde{\mu}, C_0)$ *such that*

$$\mu \left| \frac{\mu - \widetilde{\mu}}{\mu + \widetilde{\mu}} \right| |B| \leq |m_{ij}^{ij}| \leq C|B|.$$

6 Application: Detection of an Inclusion

Inverse Problem Given a Neumann data $\vec{g}$, measure $\vec{u}$ on $\partial\Omega$. Determine the location and size (or other geometry) of inclusions by means of $(\vec{u}|_{\partial\Omega}, \vec{g})$.

For a given Neumann data $\vec{g}$, let

$$\vec{H}[\vec{g}](x) := \mathcal{S}_\Omega(\vec{g})(x) - \mathcal{D}_\Omega(\vec{u}|_{\partial\Omega})(x), \quad x \in \mathbb{R}^3 \setminus \overline{\Omega}.$$

As a consequence of the asymptotic expansion of $\vec{u}$,

Theorem 6.1 *For $x \in \mathbb{R}^3 \setminus \overline{\Omega}$,*

$$\vec{H}[\vec{g}](x) = \sum_{j=1}^{3} \sum_{|\alpha|=1}^{3} \sum_{|\beta|=1}^{4-|\alpha|} \frac{\epsilon^{|\alpha|+|\beta|+1}}{\alpha!\beta!} (\partial^\alpha U_j)(z) \partial^\beta \Gamma(x-z) M_{\alpha\beta}^j + O(\frac{\epsilon^6}{|x|^2}),$$

where $M_{\alpha\beta}^j$ are the elastic moment tensors and Γ is the Kelvin matrix of fundamental solutions corresponding to the Lamé parameters (λ, μ).

Remember! $\vec{H}[\vec{g}](x)$ $(x \in \mathbb{R}^3 \setminus \overline{\Omega})$ can be computed from the measured data $(\vec{u}|_{\partial\Omega}, \vec{g})$.

[Reconstruction Procedure]

Let

$$E_{uv} = (\delta_{iu}\delta_{jv})_{i,j=1}^{3} \quad \text{and} \quad \vec{g}_{uv} := \frac{\partial(E_{uv}\vec{x})}{\partial\nu}|_{\partial\Omega}.$$

Step 1 (Detection of EMT) Compute

$$h_{kl}^{uv} := \lim_{t\to\infty} t^2 H_k[\vec{g}_{uv}](te_l), \quad k,l,u,v = 1,2,3.$$

Then the entries m_{kl}^{uv}, $u,v,k,l = 1,2,3$ of the elastic moment tensor can be recovered, modulo $O(\epsilon^6)$, as follows:

$$\epsilon^3 m_{ii}^{vu} = -\frac{8\pi\mu(\lambda+2\mu)}{3\lambda+5\mu}\left[\frac{\lambda+\mu}{2\mu}\sum_{k=1}^{3} h_{kk}^{uv} + h_{ii}^{uv}\right], \quad u,v,i = 1,2,3,$$

$$\epsilon^3 m_{kl}^{vu} = -4\pi(\lambda+2\mu)h_{kl}^{uv}, \quad u,v,k,l = 1,2,3, \; k \neq l.$$

Step 2 (Detection of Size) Having found $\epsilon^3 m_{kp}^{uv}$,

$$|\epsilon^3 m_{ij}^{ij}| \approx \epsilon^3 |B|, \quad i \neq j$$

gives the order of magnitude of D.

Step 3 (Detection of Center) The idea is as follows: From $\vec{H}[\vec{g}_{uv}]$, we can recover $\nabla\Gamma(x-z)$. It means that, basically, we can recover $\dfrac{x-z}{|x-z|^3}$ for x near ∞. From this information we can recover z.

Step 3' (Detection of Center) We can use second order homogeneous solution and proceed as Step 1 to detect the center.

Another important application: Effective Moduli of Dilute Materials

7 Numerical Results