중간고사 I **해석개론, 2000년 1학기**

시험일 : 4월 11일

1. (1-1) Write down the definitions of
   - Open Set:
   - Interior Point:
   - Accumulation Point:

   (1-2) Show that a subset F of $\mathbb{R}^n$ is closed (i.e., $\mathbb{R}^n - F$ is open) if and only if every accumulation point of F lies in F.

2. (2-1) Find the interior, closure, and boundary of the following subsets of $\mathbb{R}^2$.
   - $A = \{(x, y) \mid x^2 + y^2 \leq 1\}$
   - $A = \{(x, y) \mid x = 0,\ y \in \mathbb{Q}\}$

   (2-2) Find the accumulation points of the set
   $\{(\frac{m}{n}, \frac{1}{n}) \mid m, n \text{ integers},\ n \neq 0\}$

3. Determine whether the following statements are true or false. If your answer is true, give a brief explanation. If it is false, give a counter-example.

   (3-1) $\limsup_{n\to\infty}(x_n + y_n) = \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$.

   (3-2) If x is a cluster point of a sequence $\{x_n\}$, there exists a subsequence of $\{x_n\}$ converging to x.

   (3-3) $\text{cl}(\cup_{i=1}^{\infty} A_i) = \cup_{i=1}^{\infty} \text{cl}(A_i)$.

   (3-4) The series $\frac{1}{2} + \frac{3}{4} + (\frac{1}{2})^2 + (\frac{3}{4})^2 + \cdots$ converges.

4. (4-1) State the least upper bound property.

   (4-2) Use the least upper bound property to prove the monotone sequence property.

5. (5-1) Let $\{x_n\}$ be a sequence of real numbers and r be a number such that $0 \leq r < 1$. If

   $$|x_{n+1} - x_n| \leq r|x_n - x_{n-1}| \quad \text{for all } n \geq 2,$$

   show that $\{x_n\}$ is a Cauchy sequence.

   (5-2) Define a sequence $\{x_n\}$ by $x_1 = 1$ and

   $$x_{n+1} = (8 + x_n)^{1/3}, \quad n = 1, 2, ...$$

   Show that $\{x_n\}$ converges. (Hint: Use the mean value theorem and (5-1).)

6. (6-1) Let $\{a_n\}$ be a sequence of non-negative real numbers. If the sequence a_n does not converge to 0, show that there exist a number $\epsilon > 0$ and a subsequence $\{a_{n_j}\}$ such that

   $$a_{n_j} > \epsilon \quad \text{and} \quad n_{j+1} - n_j \geq \frac{n_{j+1}}{2}, \quad j = 1, 2, ...$$

   (6-2) Let $\{b_n\}$ be a sequence of non-negative real numbers such that

(1) $b_n \geq b_{n+1}$ for all $n = 1, 2, ...$

(2) $\sum_{n=1}^{\infty} b_n$ converges.

Show that $nb_n \to 0$ as $n \to \infty$.

중간고사 II 해석개론, 2000년 1학기

시험일 : 5월 23일

1. Determine if the following sets are compact.

   (1-1) $A = \{\frac{1}{n} : n = 1, 2, ...\}$

   (1-2) $B = A \cup \{0\}$

2. (2-1) Let F_k be nested connected compact sets, i.e., $F_{k+1} \subset F_k$ $(k = 1, 2, \cdots)$. Prove that $\cap_{k=1}^{\infty} F_k$ is connected.

   (2-2) Show that in $\mathbb{R}^1$ the statment in (2-1) is true without compactness, that is, $\cap_{k=1}^{\infty} F_k$ is connected if F_k are nested connected sets in $\mathbb{R}^1$.

   (2-3) Give an example to show that the statment in (2-1) is <u>not</u> true without compactness, that is, $\cap_{k=1}^{\infty} F_k$ may not be connected even if F_k are nested connected sets.

3. (3-1) Let A be a subset of $\mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ be a uniformly continuous mapping. If $\{x_k\}$ is a Cauchy sequence in A, show that $\{f(x_k)\}$ is also a Cauchy sequence.

   (3-2) Give an example to show that the statment in (3-1) is <u>not</u> true if f is merely continuous on A.

4. Let M be a compact subset of $\mathbb{R}^n$ and $f : M \to M$ be a continuous mapping such that
$$\|f(x) - f(y)\| < \|x - y\| \quad \text{for all } x, y \in M.$$
   Show that there is a point $x \in M$ such that $f(x) = x$. (Hint: Consider $g(x) = \|x - f(x)\|$.)

5. For a nonempty subset A of $\mathbb{R}^n$, the function $f_A : \mathbb{R}^n \to \mathbb{R}^1$ is defined by
$$f_A(x) := \inf\{\|x - y\| : y \in A\}.$$
   Show that

   (5-1) f_A is unformly continuous on $\mathbb{R}^n$.

   (5-2) $\mathrm{cl}(A) = \{x \in \mathbb{R}^n : f_A(x) = 0\}$.

6. Let f be a differentiable function on $\mathbb{R}^1$. If f' is bounded on $\mathbb{R}^1$, show that f is uniformly continuous on $\mathbb{R}^1$.

7. Write down the Fundamental Theorem of Calculus.

8. Show that a monotone function on $[a, b]$ is integrable on $[a, b]$.

9. Let f be a continuous function from $[a, b]$ into $\mathbb{R}^1$. If $f \geq 0$ on $[a, b]$ and $\int_a^b f(x)dx = 0$, show that $f \equiv 0$.

시험일 : 6월 19일

1. Let $\{f_n\}$ be a sequence of continuous functions on $X \subset \mathbb{R}^n$ converging uniformly to f on X. Show that f is continuous on X.

2. (2-1) Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is continuous on $[-1, 1]$.

   (2-2) Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is differentiable in $(-1, 1)$.

3. Let $\{f_n\}$ be a sequence of continuous functions on $M \subset \mathbb{R}^n$ converging uniformly to f on M.

   (3-1) If $x_n \in M$ converges to $x \in M$, show that

   $$\lim_{n\to\infty} f_n(x_n) = f(x).$$

   (3-2) Show by an example that above statement is false under the assumption that f_n converges pointwise to f.

4. Let g be a continuous function on $[0, 1]$ with $g(1) = 0$. Show that the sequence $\{g(x)x^n\}$ converges uniformly on $[0, 1]$.

5. Is the family of functions $\{x^n : n = 1, 2, 3, \cdots\}$ equicontinuous on $[0, 1]$? Prove your answer.

6. Let K be a compact subset of $\mathbb{R}^n$ and $\{f_n\}$ a sequence of continuous functions on K which converges uniformly on K. Show that

   (6-1) $\{f_n\}$ is uniformly bounded on K, i.e., there exists $M > 0$ such that $\|f(x)\| \leq M$ for all $x \in K$ and for all $n \in \mathbb{N}$,

   (6-2) $\{f_n\}$ is equicontinuous on K.

7. (7-1) State the Arzela-Ascoli Theorem.

   (7-2) Let $f_n : [a, b] \to \mathbb{R}$ be unformly bounded continuous functions. Set

   $$F_n(x) = \int_a^x f_n(t)dt, \quad a \leq x \leq b.$$

   Prove that F_n has a uniformly convergent subsequence.

8. Let X be a compact subset of $\mathbb{R}^n$ and $f : X \to X$ be a contraction. Show that $\cap_{n=1}^{\infty} f^n(X)$ is a single point, where $f^n = f \circ \cdots \circ f$ (n times).