강의계획서 실해석학(대학원), 2000년 1학기

강좌번호 : 301.512
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강의시간 및 강의실 : 화, 목 2:30-3:45, 24-210

개요: 이 강의에서는 고급해석학을 공부하는데 필요한 이론 중에서 다음에 열거하는 몇가지 중요한 주제를 다룬다.

(1) Review of the theory of measure and integration

(2) Fourier transform and function spaces- Sobolev spaces, Hölder spaces, various relations among function spaces, basic inequalities

(3) Compact operators and Fredholm theory- motivated by the eigenvalue problem

이 강의는 1999년에 변경된 강의계획에 따라 이루어지는 것으로 measure 와 적분이론을 공부하지 않은 학생은 학부 4학년 실해석학을 먼저 수강하거나 병행하여 수강하도록 권장한다.

평가방법 : 과제물 (30%), 중간고사 (30%), 기말고사 (40%)

참고문헌 :

[1] 김성기, 계승혁, 실해석학, 서울대학교 출판부, 1999

[2] L.C.Evans and R.F.Gariepy, Measure theory and fine properties of functions, CRC Press, 1992

[3] G.B.Folland, Real analysis, John Wiley & Sons, 1984

[4] L.C.Evans, Partial Diffential Equations, AMS, 1998

[5] W.Rudin, Real and Complex Analysis, 3rd Ed., McGraw-Hill, 1987

숙제 I 실해석학(대학원), 2000년 1학기

숙제 제출일: 3월 17일

1. Let f be an increasing function on $\mathbb{R}^1$. For each subset A of $\mathbb{R}^1$, define

$$\mu(A) := \inf \sum_{i=1}^{n} (f(b_i) - f(a_i))$$

where the infimum is taken over all covering such that $A \subset \cup_{i=1}^{n} (a_i, b_i)$.

(1) Show that μ is an (outer) measure.

(2) Show that μ is a Radon measure.

(3) Show that

$$\lim_{h \to 0} \frac{\mu(x-h, x+h)}{2h} = f'(x)$$

if f is differentiable at x.

2. If μ is a Radon measure and $f \in L^1(\mu)$ is nonnegative, show that

$$\nu(A) := \int_A f d\mu$$

is also a Radon measure.

3. Let μ be a Radon measure on $\mathbb{R}^n$ such that $\mu(\{x\}) = 0$ for each $x \in \mathbb{R}^n$. Let A be a set such that $0 < \mu(A) < \infty$.

(1) Show that for all α, β with $0 < \alpha < \beta < \mu(A)$ there exists a Borel set $B \subset A$ such that $\alpha < \mu(B) < \beta$.

(2) Show that for all α with $0 < \alpha < \mu(A)$ there exists a Borel set $B \subset A$ such that $\mu(B) = \alpha$.

숙제 제출일: 3월 23일

4. Show by an example that Egoroff's Theorem does not hold if $\mu(A) = \infty$. If $\mu(A) = \infty$ and $|f_n| \leq g$ for all n where $g \in L^1(\mu)$, show that the statement of Egoroff's Theorem is true.

5. Let E be a set of positive Lebesgue measure. Show that $E - E$ contains an interval. Here $E - E = \{x - y | x, y \in E\}$.

6. Show that an increasing function $f : \mathbb{R}^1 \to \mathbb{R}^1$ is Lebesgue measurable.

7. RCA Ch.1, 9 (RCA: Real and Complex Analysis, 3rd Ed., by W. Rudin)

8. RCA Ch.1, 12

숙제 제출일: 3월 30일

9. RCA Ch.3, 4
10. RCA Ch.3, 5
11. RCA Ch.3, 12
12. RCA Ch.3, 14
13. RCA Ch.3, 15
14. Let μ be a σ-finite measure on X and f be a nonnegative measurable function on X. Show that the set $G := \{(x, y)|x \in X,\ 0 \leq y \leq f(x)\}$ is $\mu \times m$-measurable. Here m is the 1-dimensional Lebesgue measure. Show that

$$(\mu \times m)(G) = \int_X f d\mu.$$

숙제 제출일: 4월 11일

15. Let

$$Tf(x) = \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n/2}} f(y)dy.$$

   (1) Show that

$$\|Tf\|_\infty \le C(\|f\|_q + \|f\|_{q^*})$$

   if $1 \le q < 2$ and $1/q + 1/q^* = 1$.

   (2) If $q = 2$, show that above inequality does not hold.

16. Find the necessary and sufficient condition on real numbers $\alpha \ge 0$ and $\beta \ge 0$ so that the function $\dfrac{|x|^\alpha}{|x|^\beta + 1}$ belongs to $L^p(\mathbb{R}^n)$ $(1 \le p \le \infty)$.

17. If $f \in L^1(\mathbb{R}^n)$ and $\|f\|_{L^1} \ne 0$, show that $Mf \notin L^1(\mathbb{R}^n)$. Here Mf is the Hardy Lettlewood maximal function.

18. Let μ be a measure on $\mathbb{R}^n$ and f be a nonnegative μ-measurable function. Let

$$\omega(\lambda) = \mu(\{x | f(x) \ge \lambda\}).$$

   (1) Show that ω is measurable on $[0, \infty)$.

   (2) Show that for $1 \le p < \infty$

$$\int_{\mathbb{R}^n} f^p d\mu = p \int_0^\infty \lambda^{p-1} \omega(\lambda) d\lambda.$$

   (3) Explain the meaning of the identity in (2).

숙제 제출일: 4월 20일

19. Prove the following: Let

$$\tilde{M}f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy.$$

   (1) Show that $\tilde{M}$ is of weak $(1,1)$. [Hint: Compare $\tilde{M}f$ with Mf, the usual Hardy Littlewwod maximal function.]

   (2) Show without using Corollary 2 of 1.7.2 that

$$\lim_{B \downarrow \{x\}} \frac{1}{|B|} \int_B f(y) dy = f(x) \quad \text{for } \mathcal{L}^n - a.e.\ x.$$

20. Use the Lebesgue density theorem to prove

   (1) If E is a Lebesgue measurable set of positive Lebesgue measure, then $E - E$ contains an interval.

   (2) If E and E' are Lebesgue measurable sets of positive Lebesgue measure, then $E - E'$ contains an interval.

21. Let μ be a Radon measure on $\mathbb{R}^n$ and Λ be a bounded linear functional on $L^p(\mu)$ $(1 \le p < \infty)$. Answer the following questions. (We are about to prove $(L^p)^* = L^q$. So do not use this fact.)

   (1) Show that there exists a Radon measure ν on $\mathbb{R}^n$ and a function σ such that $|\sigma| = 1$ μ-a.e. and

$$\Lambda(\varphi) = \int_{\mathbb{R}^n} \varphi \sigma d\nu \qquad \text{for all } \varphi \in C_c(\mathbb{R}^n).$$

   (2) Show that $\nu \ll \mu$.

   (3) Let $g = D_\mu \nu$ (the Radon Nikodym derivative). Show that $g \in L^q$ $(1/p + 1/q = 1)$.

   (4) Show by an example that $L^\infty(\mathbb{R}^n)^* \neq L^1(\mathbb{R}^n)$.

22. (1) Let μ be a Radon measure on $\mathbb{R}^n$. Show that $L^p(\mu)$ is separable if $1 \le p < \infty$. A normed space is separable if it has a countable dense subset.

   (2) Show that $L^\infty(\mathbb{R}^n, \mathcal{L}^n)$ is not separable.

   (3) Show that $C_c(\mathbb{R}^n)$ with the uniform norm is separable. (Hint. Start with the Weierstrass Approximation Theorem.)

숙제 제출일: 4월 27일

23. Show by an example that the weak compactness does not hold in L^1.
24. Show by an example that on $L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ the weak convergence does not imply the strong convergence.

숙제 제출일: 5월 16일

25. Let u be a real valued and $u \in L^2(0,1)$. Find the maximum of

$$\int_0^1 tu(t)dt$$

under the assumption that

$$\int_0^1 u(t)^2 dt = 1, \qquad \int_0^1 u(t)dt = 0.$$

Find a function u which gives the maximum value. (Consider an orthogonal projection.)

26. (revised) For $n = 0, 1, 2, 3, \cdots$, the Rademacher function r_n on $(0,1)$ is defined as follows: Let $r_0 = 1$. For $x \in (0,1)$ with the binary expansion $x = 0.x_1x_2\cdots x_n\cdots$,

$$r_n(x) := \begin{cases} 1 & \text{if } x_n = 0 \\ -1 & \text{if } x_n = 1 \end{cases}$$

(1) Show that $\{r_n\}$ is an orthonormal system for $L^2(0,1)$.

(2) Show that $\{r_{n_1}r_{n_2}\cdots r_{n_p} \mid 0 \le n_1 < n_2 < \cdots < n_p,\ p = 1, 2, \cdots\}$ is a complete orthonormal system for $L^2(0,1)$. (Hint. To prove the completeness, consider $(1 + r_{n_1})\cdots(1 + r_{n_p})(1 - r_{m_1})\cdots(1 - r_{m_q})$.)

(3) Let $\psi(x) = r_1(x)$ for $x \in [0,1]$ and $\psi(x) = 0$ for other $x \in \mathbb{R}^1$. For $j, k = 0, \pm 1, \pm 2, \cdots$, define

$$\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k).$$

Show that $\{\psi_{j,k}\}$ is a complete orthonormal system for $L^2(\mathbb{R})$. Such a function ψ is called a WAVELET.

27. (1) Let Ω be a measurable subset of $\mathbb{R}^n$ and $K(x,y)$ be a measurable function on $\Omega \times \Omega$. Suppose that there are non-negative measurable functions $G_1(x,y)$ and $G_2(x,y)$ such that

$$|K(x,y)| \le G_1(x,y)G_2(x,y)$$
$$\sup_{x\in\Omega}\int_\Omega G_1(x,y)^2 dy < +\infty$$
$$\sup_{y\in\Omega}\int_\Omega G_2(x,y)^2 dx < +\infty.$$

Define

$$Tf(x) := \int_\Omega K(x,y)f(y)dy, \quad x \in \Omega$$

for $f \in L^2(\Omega)$. Show that T is bounded on $L^2(\Omega)$, i.e., that there exists a constant C such that

$$\|Tf\|_2 \le C\|f\|_2 \quad \text{for all } f \in L^2(\Omega).$$

(Such an operator T is called a Hilbert-Schmidt operator.)

(2) Define

$$Au(t) := \int_0^\infty \frac{u(s)}{t+s} ds \quad t \in (0,\infty).$$

Show that A is bounded on $L^2(0,\infty)$.

28. Show that $C^{k,\alpha}(\overline{U})$ is a Banach space.

시험일: 4월 20일

1. Let μ be a Radon measure on $\mathbb{R}^n$. Show that $L^1(\mu)$ is nontrivial, i.e., there exists a nonzero $f \in L^1(\mu)$.

2. Show that if $f \in L^p(\mathbb{R}^n, \mu)$ $(1 \le p < \infty)$, then for each $\lambda > 0$
$$\mu(\{|f| > \lambda\}) \le \left(\frac{\|f\|_p}{\lambda}\right)^p.$$

3. Let f be a non-negative Lebesgue measurable function on $\mathbb{R}^n$ and suppose the measure μ satisfies
$$\mu(E) = \int_E f dx$$
for all Lebesgue measurable set E.

   (3-1) If g is μ-measurable, show that gf is Lebesgue measurable.

   (3-2) Show that
$$\int_{\mathbb{R}^n} g d\mu = \int_{\mathbb{R}^n} gf dx$$
   for all non-negative μ-measurable function g.

4. Let μ be a Borel measure on $\mathbb{R}^n$ and U be an open set in $\mathbb{R}^n$. Use Besicovitch's Covering Theorem to show that there exists a countable collection $\mathcal{G}$ of disjoint closed balls such that
$$\cup_{B \in \mathcal{G}} B \subset U$$
and
$$\mu\left(U - \cup_{B \in \mathcal{G}} B\right) = 0.$$

5. Use the Lebesgue density theorem to prove that if E and E' are Lebesgue measurable sets of positive Lebesgue measure, then $E - E'$ contains an interval.

6. Let B_r be a closed ball of radius r centered at 0. Let $1 \le p < \infty$. For $n = 1, 2, \cdots$, let
$$f_n(x) := \frac{1}{|B_{1/n}|^{1/p}} \chi_{B_{1/n}}(x).$$

   (6-1) If $1 < p < \infty$, show that f_n converges weakly in L^p. What is the limit?

   (6-2) If $p = 1$, does f_n have a weakly convergent subsequence?

기말고사 실해석학(대학원), 2000년 1학기

시험일: 6월 22일

1. Fix $r > 0$ and let $B := \{x \in \mathbb{R}^n \,|\, |x| \le r\}$. Define
$$M_r f(x) := \frac{1}{|B|}\int_B f(x+y)dy, \quad f \in C^{0,\alpha}(\mathbb{R}^n) \ (\alpha < 1).$$
Show that
$$\|M_r f\|_{C^{0,\alpha}(\mathbb{R}^n)} \le \|f\|_{C^{0,\alpha}(\mathbb{R}^n)} \quad \text{for all } f \in C^{0,\alpha}(\mathbb{R}^n).$$

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2. If $f \in C_c^1(\mathbb{R}^n)$, show that
$$|\hat{f}(\xi)| \le \frac{C}{1+|\xi|}$$
for some constant C.

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3. Let φ be a non-negative function (not necessarily compactly supported) such that $\int_{\mathbb{R}^n} \varphi dx = A < +\infty$. For $\epsilon > 0$, define $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(\frac{x}{\epsilon})$. Show that for each $f \in L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$, $f * \varphi_\epsilon \to Af$ in L^p as $\epsilon \to 0$. (You don't have to prove the fact that $\|f(\cdot - y) - f\|_{L^p} \to 0$ as $y \to 0$.)

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4. Let ϕ be a C^1 function on $\mathbb{R}^{n-1}$ such that $\|D\phi\|_{C^1} < +\infty$. Let $\Omega := \{x = (x_1, \cdots, x_n) \mid x_n > \phi(x_1, \cdots, x_{n-1})\}$. For each $f \in W^{1,p}(\Omega)$ $(1 \le p \le +\infty)$, show that there exists $F \in W^{1,p}(\mathbb{R}^n)$ such that $F(x) = f(x)$ for $x \in \Omega$ and
$$\|F\|_{W^{1,p}(\mathbb{R}^n)} \le C\|f\|_{W^{1,p}(\Omega)}$$
for some C independent of f.

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5. Let $I = [0, 1]$.

   (5-1) If $u \in W^{1,p}(I)$ with $1 < p < +\infty$, show that u is equal a.e. to an absolutelty continuous function.

   (5-2) If $u \in W^{1,\infty}(I)$, show that u is Lipschitz continuous on I.
