Lecture Notes*
PDE Summer School, Kyoungju, Korea
Generalized Polarization Tensors, Inverse Problems, and Dilute Composite Materials

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This is a lecture note for the PDE summer school held in Kyungju, Korea. Here we explain recent developments in inverse problems to detect diametrically small electric inclusions and calculation of effective properties of dilute composite materials. The central concept in these developments is the generalized polarization tensor and the main tool is the boundary integral expressed in terms of layer potentials. Most of materials except the last lecture are from the recent book [10].

Contents

*This lecture note is available at www.math.snu.ac.kr/∼hkang/lecture/LectureNotes/LectureNotes.htm. Preprints of papers quoted in this note are available at www.math.snu.ac.kr/∼hkang/paper/paper.htm

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1 Leture 1
Layer Potentials and Transmission Problems

Let us begin our study with the following boundary value problem:

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega \, , \\ \displaystyle \frac{\partial u}{\partial \nu}\Big|_{\partial \Omega} = g \, . \end{cases} \tag{1.1}$$

Here Ω is a bounded domain representing a conductive body, the function g represents the applied boundary current; it belongs to $L^2(\partial\Omega)$ and has mean value zero, and the solution u represents the voltage distribution in Ω generated by g. Here and throughout this note $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$ and ν is the outward normal to $\partial\Omega$.

We seek the solution to (1.1) in the form

$$u(x) = \int_{\partial\Omega} \Gamma(x-y)\phi(y) d\sigma(y), \quad x \in \Omega, \tag{1.2}$$

where u is a fundamental solution to the Laplacian and given by

$$\Gamma(x) = \begin{cases} \displaystyle \frac{1}{2\pi} \ln |x| \, , & d = 2 \, , \\ \displaystyle \frac{1}{(2-d)\omega_d} |x|^{2-d} \, , & d \geq 3 \, . \end{cases} \tag{1.3}$$

The function u defined by (1.2) is harmonic in Ω (also in $\mathbb{R}^d \setminus \overline{\Omega}$). Thus for u to be the solution to (1.1), it suffices to have the boundary condition. For this purpose, we have to investigate the boundary behavior of

$$\frac{\partial}{\partial \nu} \int_{\partial\Omega} \Gamma(x-y)\phi(y) d\sigma(y),$$

as x approaches to $\partial\Omega$.

1.1 Layer potentials for Laplacian

Given a bounded Lipschitz domain D in $\mathbb{R}^d, d \geq 2$, we will denote the single and double layer potentials of a function $\phi \in L^2(\partial D)$ as $\mathcal{S}_D\phi$ and $\mathcal{D}_D\phi$, respectively, where

$$\mathcal{S}_D\phi(x) := \int_{\partial D} \Gamma(x-y)\phi(y)\, d\sigma(y) \, , \quad x \in \mathbb{R}^d, \tag{1.4}$$

$$\mathcal{D}_D\phi(x) := \int_{\partial D} \frac{\partial}{\partial \nu_y} \Gamma(x-y)\phi(y)\, d\sigma(y) \, , \quad x \in \mathbb{R}^d \setminus \partial D \, . \tag{1.5}$$

Observe that both $\mathcal{S}_D\phi$ and $\mathcal{D}_D\phi$ are harmonic in D and $\mathbb{R}^d \setminus \overline{D}$. For a function u defined on $\mathbb{R}^d \setminus \partial D$, we denote

$$u\big|_{\pm}(x) := \lim_{t \to 0^+} u(x \pm t\nu_x), \quad x \in \partial D \, ,$$

and

$$\frac{\partial}{\partial \nu_x} u \bigg|_{\pm} (x) := \lim_{t \to 0^+} \langle \nabla u(x \pm t\nu_x), \nu_x \rangle \ , \quad x \in \partial D \ ,$$

if the limits exist. Here ν_x is the outward unit normal to ∂D at x, and $\langle , \rangle$ denotes the scalar product in $\mathbb{R}^d$.

We want to investigate $\frac{\partial}{\partial \nu} \mathcal{S}_D \phi \big|_{\pm}(x)$ and $\mathcal{D}_D \phi \big|_{\pm}(x)$ for $x \in \partial D$. Fix $z \in \partial D$. Since

$$\langle \nabla \Gamma(x-y), \nu(z) \rangle = \frac{1}{\omega_d} \frac{\langle x-y, \nu_z \rangle}{|x-y|^d}, \quad x \in D, \ y \in \partial D,$$

one may expect that

$$\frac{\partial}{\partial \nu} \int_{\partial D} \Gamma(x-y) \phi(y) d\sigma(y) \to \frac{1}{\omega_d} \int_{\partial D} \frac{\langle z-y, \nu_z \rangle}{|z-y|^d} \phi(y) d\sigma(y), \quad \text{as } x \to z.$$

But this is not the case. The main difficulty is that since $|\frac{\langle z-y, \nu_z \rangle}{|z-y|^d}| \le C |z-y|^{1-d}$ and ∂D is a manifold of dimension $d-1$, the righthand side of the above is not absolutely integrable. On the other hand, $\frac{1}{\omega_d} \frac{\langle x-y, \nu_z \rangle}{|x-y|^d}$ has a structure of the Poisson kernel. To see this, let us suppose that $d=2$, $z=0$, ∂D is x-axis near 0, and D is a part below x-axis. In this case, if $x = z - \epsilon \nu_z = (0, -\epsilon)$, then for $y = (y_1, 0) \in \partial D$, we have

$$\langle \nabla \Gamma(x-y), \nu(z) \rangle = -\frac{1}{2\pi} \frac{\epsilon}{|y_1|^2 + \epsilon^2},$$

which is exactly $-\frac{1}{2}$ times the Poisson kernel. So we can expect that $\int_{\partial D} \langle \nabla \Gamma(x-y), \nu(z) \rangle \phi(y) d\sigma(y)$ will pick up $-\frac{1}{2}\phi(z)$ as $x \to z$ from inside D. The following theorem shows that it is indeed the case. Proofs of the following theorem can be found in [50] for when D has the $\mathcal{C}^2$-boundary, and in [93] for D with the Lipschitz boundary.

Theorem 1.1 (Jump formula) *Let D be a bounded Lipschitz domain in $\mathbb{R}^d$. For $\phi \in L^2(\partial D)$*

$$\mathcal{S}_D \phi \big|_{+}(x) = \mathcal{S}_D \phi \big|_{-}(x) \quad a.e. \ x \in \partial D \ , \tag{1.6}$$

$$\frac{\partial}{\partial \nu} \mathcal{S}_D \phi \bigg|_{\pm}(x) = \left(\pm \frac{1}{2} I + \mathcal{K}_D^* \right) \phi(x) \quad a.e. \ x \in \partial D \ , \tag{1.7}$$

$$(\mathcal{D}_D \phi) \big|_{\pm} = \left(\mp \frac{1}{2} I + \mathcal{K}_D \right) \phi(x) \quad a.e. \ x \in \partial D \ , \tag{1.8}$$

where $\mathcal{K}_D$ is defined by

$$\mathcal{K}_D \phi(x) = \frac{1}{\omega_d} p.v. \int_{\partial D} \frac{\langle y-x, \nu_y \rangle}{|x-y|^d} \phi(y) \, d\sigma(y) \tag{1.9}$$

and $\mathcal{K}_D^*$ is the L^2-adjoint of $\mathcal{K}_D$, i.e.,

$$\mathcal{K}_D^* \phi(x) = \frac{1}{\omega_d} p.v. \int_{\partial D} \frac{\langle x-y, \nu_x \rangle}{|x-y|^d} \phi(y) \, d\sigma(y) \ . \tag{1.10}$$

Here p.v. *denotes the Cauchy principal value.*

This theorem is based on the fact that the operators $\mathcal{K}_D$ is well-defined, which is guaranteed by the following celebrated theorem of Coifman-McIntosh-Meyer [42].

Theorem 1.2 *The operators $\mathcal{K}_D$ and $\mathcal{K}_D^*$ are singular integral operators and bounded on $L^2(\partial D)$.*

We strongly recommend the readers to read [42] to have feeling of power and beauty of the classical harmonic analysis. If D has $\mathcal{C}^2$ boundary, or $\mathcal{C}^{1,\alpha}$ ($\alpha > 0$) boundary for that matter, then there is extra orthogonality, which is absent for the Lipschitz boundary, between $x - y$ ($x, y \in \partial D$) and ν_x. Using this orthogonality, one can show that

$$\left| \frac{\langle x - y, \nu_x \rangle}{|x - y|^d} \right| \leq \frac{C}{|x - y|^{d-2}} \qquad \left(\leq \frac{C}{|x - y|^{d-1-\alpha}} \text{ if } \partial D \text{ is } \mathcal{C}^{1,\alpha} \right),$$

and hence $\mathcal{K}_D^*$ becomes a compact operator on $L^2(\partial D)$. See [50] for this.

Let us state further mapping properties of layer potentials, whose proof can be found in [93]. (See also [10])

Theorem 1.3 *Let D be a bounded Lipschitz domain in $\mathbb{R}^d$. Then*

(i) *$\mathcal{S}_D : L^2(\partial D) \to W_1^2(\partial D)$ bounded.*

(ii) *$\mathcal{K}_D : W_1^2(\partial D) \to W_1^2(\partial D)$ bounded.*

Here $W_1^2(\partial D)$ is the L^2 Sobolev space of order 1 on ∂D.

Observe that if D is a two dimensional disk with radius r, then, as was observed in [61],

$$\frac{\langle x - y, \nu_x \rangle}{|x - y|^2} = \frac{1}{2r} \quad \forall\, x, y \in \partial D, x \neq y \ ,$$

and therefore, for any $\phi \in L^2(\partial D)$,

$$\mathcal{K}_D^* \phi(x) = \mathcal{K}_D \phi(x) = \frac{1}{4\pi r} \int_{\partial D} \phi(y)\, d\sigma(y) \ , \tag{1.11}$$

for all $x \in \partial D$. For $d \geq 3$, if D denotes a sphere with radius r, then, since

$$\frac{\langle x - y, \nu_x \rangle}{|x - y|^d} = \frac{1}{2r} \frac{1}{|x - y|^{d-2}} \quad \forall\, x, y \in \partial D, x \neq y \ ,$$

we have, as shown by Lemma 2.3 of [63], that for any $\phi \in L^2(\partial D)$,

$$\mathcal{K}_D^* \phi(x) = \mathcal{K}_D \phi(x) = \frac{(2 - d)}{2r} \mathcal{S}_D \phi(x) \tag{1.12}$$

for all $x \in \partial D$. In particular, if D is a ball in $\mathbb{R}^d$, $d \geq 2$, then $\mathcal{K}_D$ is a self-adjoint operator on $L^2(\partial D)$. The converse is also true: Let D be a Lipschitz domain. If $\mathcal{K}_D$ is self-adjoint, then D is a ball. This was proved by Lim [76].

There is a conjecture which has not been resolved completely. Observe that $\mathcal{K}_D(1) = \frac{1}{2}$, and hence $\mathcal{K}_D^*(1) = \frac{1}{2}$ provided that D is a ball. The conjecture is that if $\mathcal{K}_D^*(1) = \frac{1}{2}$ and D is a Lipschitz domain, then D is a ball. The conjecture has been proved to be true for some important classes of domains: Piecewise smooth domains in $\mathbb{R}^2$ by Martensen [81], star-shaped $\mathcal{C}^{2,\alpha}$-domains by Payne and Philippin [85] and Philippin [87], and $\mathcal{C}^{2,\alpha}$-domains by Reichel [88, 89]. Recently Mendez and Reichel proved the conjecture for bounded Lipschitz domains in $\mathbb{R}^2$ and bounded Lipschitz convex domains in $\mathbb{R}^d$ ($d \geq 3$) [82].

Let us now go back to the problem (1.1). If the solution u takes the form $\mathcal{S}_\Omega \phi$ for some ϕ in Ω, then by the boundary condition we need to have

$$\frac{\partial}{\partial \nu} \mathcal{S}_\Omega \phi \Big|_{-} = g \quad \text{on } \partial\Omega.$$

It then follows from (1.7) that

$$(-\frac{1}{2} I + \mathcal{K}_D^*)\phi = g \quad \text{on } \partial\Omega.$$

In other words we need to invert the operator $(-\frac{1}{2}I + \mathcal{K}_D^*)$ on $L^2(\partial\Omega)$.

If D is a $\mathcal{C}^{1,\alpha}$ domain, then $\mathcal{K}_D^*$ is a compact operator, and hence $(-\frac{1}{2}I + \mathcal{K}_D^*)$ is a Fredholm operator of the second kind (is it the first kind?). Therefore, by the Fredholm alternative, the question of invertibility is reduced to that of injectivity, and injectivity of $(-\frac{1}{2}I + \mathcal{K}_D^*)$ can be proved without much effort. See [50]. But if ∂D is merely Lipschitz, $\mathcal{K}_D^*$ is not a compact operator and invertibility of $(-\frac{1}{2}I + \mathcal{K}_D^*)$ is a serious matter. The following theorem is due to Verchota [93] for $|\lambda| = \frac{1}{2}$ and Escauriaza *et al* [49] for $|\lambda| > \frac{1}{2}$.

Theorem 1.4 *The operator $\lambda I - \mathcal{K}_D^*$ is invertible on $L_0^2(\partial D)$ if $|\lambda| \geq \frac{1}{2}$, and for $\lambda \in (-\infty, -\frac{1}{2}] \cup (\frac{1}{2}, \infty)$, $\lambda I - \mathcal{K}_D^*$ is invertible on $L^2(\partial D)$.*

Here and throughout this note,

$$L_0^2(\partial D) := \left\{ \phi \in L^2(\partial D) : \int_{\partial D} \phi \, d\sigma = 0 \right\}.$$

The key fact in proving Theorem 1.4 is the following: For a function u the tangential derivative of u along ∂D is defined to be

$$\frac{\partial u}{\partial T} := \sum_{p=1}^{d-1} \frac{\partial u}{\partial T_p} T_p,$$

where $T_1, \ldots, T_{d-1}$ is an orthonormal basis for the tangent plane to ∂D at x.

Lemma 1.5 *Let D be a bounded Lipschitz domain in $\mathbb{R}^d, d \geq 2$. Let u be a function such that either*

(i) u is Lipschitz in $\overline{D}$ and $\Delta u = 0$ in D, or

(ii) u is Lipschitz in $\mathbb{R}^d \setminus D$, $\Delta u = 0$ in $\mathbb{R}^d \setminus \overline{D}$, and $|u(x)| = O(1/|x|^{d-2})$ when $d \geq 3$ and $|u(x)| = O(1/|x|)$ when $d = 2$ as $|x| \to +\infty$.

Then there exists a positive constant C depending only on the Lipschitz character of D such that

$$\frac{1}{C} \left\| \frac{\partial u}{\partial T} \right\|_{L^2(\partial D)} \leq \left\| \frac{\partial u}{\partial \nu} \right\|_{L^2(\partial D)} \leq C \left\| \frac{\partial u}{\partial T} \right\|_{L^2(\partial D)}. \tag{1.13}$$

Lemma 1.5 says that the L^2 norms of the normal and tangential derivatives of a harmonic function are comparable, and can be proved using the Rellich identity [93]. (See also [10].) Let us briefly see how Lemma 1.5 leads us to Theorem 1.4.

Let $u(x) = \mathcal{S}_D f(x)$, where $f \in L^2_0(\partial D)$. Because of the jump formula (1.7), we have

$$\frac{\partial u}{\partial \nu}\Big|_{\pm} = (\pm\frac{1}{2}I + \mathcal{K}_D^*)f \ ,$$

and by (1.13)

$$\left\|\frac{\partial u}{\partial \nu}\Big|_{-}\right\|_{L^2(\partial D)} \simeq \left\|\frac{\partial u}{\partial \nu}\Big|_{+}\right\|_{L^2(\partial D)} \ ,$$

or equivalently

$$\begin{aligned} \frac{1}{C}\ ||(\frac{1}{2}I + \mathcal{K}_D^*)f||_{L^2(\partial D)} &\leq ||(\frac{1}{2}I - \mathcal{K}_D^*)f||_{L^2(\partial D)} \\ ||(\frac{1}{2}I - \mathcal{K}_D^*)f||_{L^2(\partial D)} &\leq C\ ||(\frac{1}{2}I + \mathcal{K}_D^*)f||_{L^2(\partial D)} \ . \end{aligned} \tag{1.14}$$

Since $f = (\frac{1}{2}I + \mathcal{K}_D^*)f + (\frac{1}{2}I - \mathcal{K}_D^*)f$, (1.14) yields that

$$||(\frac{1}{2}I + \mathcal{K}_D^*)f||_{L^2(\partial D)} \geq C||f||_{L^2(\partial D)} \ . \tag{1.15}$$

We also have the following estimate from [8].

Lemma 1.6 *There exists a constant C depending only on the Lipschitz character of D such that for any $|\lambda| \geq 1/2$*

$$\|\phi\|_{L^2(\partial D)} \leq C\frac{1}{|\lambda|}\ \|(\lambda I - \mathcal{K}_D^*)\,\phi\|_{L^2(\partial D)} \tag{1.16}$$

for all $\phi \in L^2_0(\partial D)$.

1.2 Neumann and Dirichlet Functions

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d, d \geq 2$. Let $N(x,z)$ be the Neumann function for Δ in Ω corresponding to a Dirac mass at z. That is, N is the solution to

$$\begin{cases} \Delta_x N(x,z) = -\delta_z \quad \text{in } \Omega \ , \\ \displaystyle\frac{\partial N}{\partial \nu_x}\Big|_{\partial\Omega} = -\frac{1}{|\partial\Omega|} \ , \int_{\partial\Omega} N(x,z)\, d\sigma(x) = 0 \quad \text{for } z \in \Omega \ . \end{cases} \tag{1.17}$$

Note that the Neumann function $N(x,z)$ is defined as a function of $x \in \overline{\Omega}$ for each fixed $z \in \Omega$. The operator defined by $N(x,z)$ is the solution operator for the Neumann problem

$$\begin{cases} \Delta U = 0 \quad \text{in } \Omega \ , \\ \displaystyle\frac{\partial U}{\partial \nu}\Big|_{\partial\Omega} = g \ , \end{cases} \tag{1.18}$$

namely, the function U defined by

$$U(x) := \int_{\partial\Omega} N(x,z)g(z)d\sigma(z)$$

is the solution to (1.18) satisfying $\int_{\partial\Omega} U\, d\sigma = 0$.

For D, a subset of Ω, let

$$\mathcal{N}_D f(x) := \int_{\partial D} N(x,y)f(y)\, d\sigma(y) \; .$$

The following lemma from [7] relates the fundamental solution with the Neumann function.

Lemma 1.7 *For $z \in \Omega$ and $x \in \partial\Omega$, let $\Gamma_z(x) := \Gamma(x-z)$ and $N_z(x) := N(x,z)$. Then*

$$\left(-\frac{1}{2}I + \mathcal{K}_\Omega\right)(N_z)(x) = \Gamma_z(x) \quad \textit{modulo constants}, \quad x \in \partial\Omega \; , \tag{1.19}$$

or, to be more precise, for any simply connected Lipschitz domain D compactly contained in Ω and for any $g \in L^2_0(\partial D)$, we have for any $x \in \partial\Omega$

$$\int_{\partial D}\left(-\frac{1}{2}I + \mathcal{K}_\Omega\right)(N_z)(x)g(z)\, d\sigma(z) = \int_{\partial D} \Gamma_z(x)g(z)\, d\sigma(z) \; . \tag{1.20}$$

Observe that we can express (1.19) in the following form: for any $g \in L^2_0(\partial D)$

$$\mathcal{N}_D g(x) = \left(-\frac{1}{2}I + \mathcal{K}_\Omega\right)^{-1}\big((\mathcal{S}_D g)|_{\partial\Omega}\big)(x), \quad x \in \partial\Omega. \tag{1.21}$$

We have a similar formula for the Dirichlet function. Let $G(x,z)$ be the Green's function for the Dirichlet problem in Ω, that is, the unique solution to

$$\begin{cases} \Delta_x G(x,z) = -\delta_z \quad \text{in } \Omega \; , \\ G(x,z) = 0 \quad \text{on } \partial\Omega \; , \end{cases}$$

and let $G_z(x) = G(x,z)$. Then for any $x \in \partial\Omega$, and $z \in \Omega$ we can prove that

$$\left(\frac{1}{2}I + \mathcal{K}^*_\Omega\right)^{-1}\left(\frac{\partial\Gamma_z(y)}{\partial\nu_y}\right)(x) = -\frac{\partial G_z}{\partial\nu_x}(x) \; . \tag{1.22}$$

1.3 Transmission Problems

Let Ω be a bounded domain in $\mathbb{R}^d$ with a connected Lipschitz boundary and conductivity equal to 1. Consider a bounded domain $D \subset\subset \Omega$ with a connected Lipschitz boundary and conductivity $0 < k \neq 1 < +\infty$. Let $g \in L^2_0(\partial\Omega)$, and consider the following transmission problem.

$$\begin{cases} \nabla \cdot \Big(1 + (k-1)\chi(D)\Big)\nabla u = 0 \quad \text{in } \Omega \; , \\ \displaystyle\frac{\partial u}{\partial\nu}\Big|_{\partial\Omega} = g \; , \\ \displaystyle\int_{\partial\Omega} u(x)\, d\sigma(x) = 0 \; , \end{cases} \tag{1.23}$$

where $\chi(D)$ is the characteristic function of D. By integrating the equation against functions with compact supports, one can easily see that the problem (1.23) can be rephrased as

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^d \setminus \partial\Omega \, , \\ u\big|_- - u\big|_+ = 0 & \text{on } \partial\Omega \, , \\ k\dfrac{\partial u}{\partial \nu}\Big|_- - \dfrac{\partial u}{\partial \nu}\Big|_+ = 0 & \text{on } \partial\Omega \, , \\ \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = g \, , \\ \displaystyle\int_{\partial\Omega} u(x)\, d\sigma(x) = 0 \, . \end{cases} \tag{1.24}$$

At this point we have all the necessary ingredients to state and prove a decomposition formula of the steady-state voltage potential u into a harmonic part and a refraction part which will be the main tool for deriving the asymptotic expansion, for designing efficient reconstruction algorithms, and for calculating effective properties of composite materials in later sections. This decomposition formula is unique and inherits geometric properties of the inclusion D.

The following theorem was proved in [61, 62, 64]. The original formula in [61] is rather complicated even if it contains all the idea. The following simple formula was appeared in [64] with a simpler proof.

Theorem 1.8 *Suppose that D is a domain compactly contained in Ω with a connected Lipschitz boundary and conductivity $0 < k \neq 1 < +\infty$. Then the solution u of the Neumann problem (1.23) is represented as*

$$u(x) = H(x) + \mathcal{S}_D\phi(x), \quad x \in \Omega \, , \tag{1.25}$$

where the harmonic function H is given by

$$H(x) = -\mathcal{S}_\Omega(g)(x) + \mathcal{D}_\Omega(f)(x), \quad x \in \Omega \, , \quad f := u|_{\partial\Omega} \, , \tag{1.26}$$

and $\phi \in L^2_0(\partial D)$ satisfies the integral equation

$$\left(\frac{k+1}{2(k-1)} I - \mathcal{K}_D^*\right)\phi = \frac{\partial H}{\partial \nu}\Big|_{\partial D} \quad \text{on } \partial D \, . \tag{1.27}$$

The decomposition (1.25) into a harmonic part and a refraction part is unique. Moreover, $\forall\ n \in \mathbb{N}$, there exists a constant $C_n = C(n, \Omega, dist(D, \partial\Omega))$ independent of $|D|$ and the conductivity k such that

$$\|H\|_{\mathcal{C}^n(\overline{D})} \leq C_n \|g\|_{L^2(\partial\Omega)} \, . \tag{1.28}$$

Furthermore, the following holds

$$H(x) + \mathcal{S}_D\phi(x) = 0, \quad \forall\, x \in \mathbb{R}^d \setminus \overline{\Omega} \, . \tag{1.29}$$

Similar formula can be derived when there are several inclusions. Consider the following problem in the presence of multiple inclusions $D_1, \ldots, D_m$:

$$\begin{cases} \nabla \cdot \Big(\chi(\Omega \setminus \bigcup_{s=1}^m \overline{D_s}) + \sum_{s=1}^m k_s \chi(D_s) \Big) \nabla u = 0 \quad \text{in } \Omega \, , \\ \displaystyle \frac{\partial u}{\partial \nu}\Big|_{\partial \Omega} = g \, , \\ \displaystyle \int_{\partial \Omega} u(x) \, d\sigma(x) = 0 \, . \end{cases} \tag{1.30}$$

The following theorem was first proved in [71].

Theorem 1.9 *Let H be a harmonic function in Ω as defined in (1.26). There are unique functions $\varphi^{(l)} \in L^2_0(\partial D_l)$, $l = 1, \ldots, m$, such that the solution u to (1.30) can be represented as*

$$u(x) = H(x) + \sum_{l=1}^m \mathcal{S}_{D_l} \varphi^{(l)}(x) \, . \tag{1.31}$$

The potentials $\varphi^{(l)}$, $l = 1, \ldots, m$, are the unique solution to the integral equations

$$(\lambda_l I - \mathcal{K}^*_{D_l})\varphi^{(l)} - \sum_{s \neq l} \frac{\partial (\mathcal{S}_{D_s} \varphi^{(s)})}{\partial \nu^{(l)}}\Big|_{\partial D_l} = \frac{\partial H}{\partial \nu^{(l)}}\Big|_{\partial D_l} \quad \text{on } \partial D_l \, , \tag{1.32}$$

where $\nu^{(l)}$ denotes the outward unit normal to ∂D_l and

$$\lambda_l = \frac{k_l + 1}{2(k_l - 1)} \, .$$

Let $\delta := \min\{\text{dist}(D_i, D_j) : i \neq j\}$. Then since $\varphi^{(l)} \in L^2_0(\partial D_l)$, we have

$$\frac{\partial (\mathcal{S}_{D_s} \varphi^{(s)})}{\partial \nu^{(l)}}(x) = \int_{\partial D_s} \left[\frac{\partial}{\partial \nu^{(l)}(x)} \Gamma(x - y) - \frac{\partial}{\partial \nu^{(l)}(x)} \Gamma(x - y_0) \right] \varphi^{(s)} d\sigma(y),$$

for some $y_0 \in \partial D_s$, and hence if $l \neq s$, then

$$\frac{\partial (\mathcal{S}_{D_s} \varphi^{(s)})}{\partial \nu^{(l)}}(x) = O(\delta^{-d}).$$

It then follows from

$$\varphi^{(l)} = (\lambda_l I - \mathcal{K}^*_{D_l})^{-1} \left(\frac{\partial H}{\partial \nu^{(l)}}\Big|_{\partial D_l} \right) + O(\delta^{-d}), \quad \text{on } \partial D_l \, . \tag{1.33}$$

Thus if the inclusions are far apart, in other words, if δ is large, then

$$\varphi^{(l)} \approx (\lambda_l I - \mathcal{K}^*_{D_l})^{-1} \left(\frac{\partial H}{\partial \nu^{(l)}}\Big|_{\partial D_l} \right).$$

We note that the formula (1.27) and (1.31) are valid even for extreme conductivities $k = 0, \infty$ [62]. If $k_l = 0$, then from (1.7) and (1.31) we obtain

$$\frac{\partial u}{\partial \nu^{(l)}}\bigg|_+ = \frac{\partial H}{\partial \nu^{(l)}} + (\frac{1}{2}I + \mathcal{K}^*_{D_l})\varphi^{(l)} + \sum_{s\neq l} \frac{\partial(\mathcal{S}_{D_s}\varphi^{(s)})}{\partial \nu^{(l)}}\big|_{\partial D_l} \quad \text{on } \partial D_l \ .$$

Since $\lambda_l = -1/2$, we have

$$\frac{\partial u}{\partial \nu^{(l)}}\bigg|_+ = 0 \quad \text{on } \partial D_l \ ,$$

which means that ∂D_l is insulated. On the other hand, if $k_l = \infty$, then by the same argument, one can see that

$$\frac{\partial u}{\partial \nu^{(l)}}\bigg|_- = 0 \quad \text{on } \partial D_l \ ,$$

and hence $u =$ constant in D_l, which means that D_l is a perfect conductor.

Observe that the harmonic function H depends on f and hence on the inclusion D. We now present another decomposition of the solution where the harmonic part is independent of D, which was derived in [7]. Let U be the solution without the inclusion D, *i.e.*, the solution to

$$\begin{cases} \Delta U = 0 \quad \text{in } \Omega \ , \\ \displaystyle \frac{\partial U}{\partial \nu}\bigg|_{\partial\Omega} = g \ , \\ \displaystyle \int_{\partial\Omega} U(x)\, d\sigma(x) = 0 \ , \end{cases} \tag{1.34}$$

Then U can be written as

$$U(y) = \mathcal{N}_\Omega g(y) = \int_{\partial\Omega} N(x,y) g(x)\, d\sigma(x) \ .$$

Theorem 1.10 *The solution u of* (1.30) *can be represented as*

$$u(x) = U(x) - \sum_{l=1}^m \mathcal{N}_{D_l}\varphi^{(l)}(x), \quad x \in \partial\Omega \ , \tag{1.35}$$

where $\varphi^{(l)}$ is defined in (1.32).

Recall that the Neumann-to-Dirichlet (NtD) map $\Lambda_D : L^2_0(\partial\Omega) \to W^2_{\frac{1}{2}}(\partial\Omega)$ corresponding to the conductivity distribution $\chi(\Omega \setminus \bigcup_{s=1}^m \overline{D_s}) + \sum_{s=1}^m k_s \chi(D_s)$ is defined by, for $g \in L^2_0(\partial\Omega)$,

$$\Lambda_D(g) := u|_{\partial\Omega},$$

where u is the unique weak solution to (1.30). It then follows from (1.26) and (1.31) that

$$-u(x) + \mathcal{D}_\Omega(u|_{\partial\Omega})(x) = \mathcal{S}_\Omega(g)|_{\partial\Omega} - \sum_{j=1}^m \mathcal{S}_{B_j}\varphi^{(j)}|_{\partial\Omega}, \quad x \in \Omega,$$

and hence, by (1.8), we get

$$(-\frac{1}{2}I + \mathcal{K}_\Omega)(u|_{\partial\Omega})(x) = \mathcal{S}_\Omega(g)|_{\partial\Omega} - \sum_{j=1}^m \mathcal{S}_{B_j}\varphi^{(j)}|_{\partial\Omega}, \quad x \in \partial\Omega,$$

We then have from (1.21) that

$$\Lambda_D(g) = (-\frac{1}{2}I + \mathcal{K}_\Omega)^{-1}\left(\mathcal{S}_\Omega(g)|_{\partial\Omega} - \sum_{j=1}^m \mathcal{S}_{D_j}\varphi^{(j)}|_{\partial\Omega}\right) = \Lambda_0(g) - \sum_{l=1}^m \mathcal{N}_{D_l}\varphi^{(l)}(x), \quad (1.36)$$

which is exactly the formula (1.35).

We have a similar representation for solutions of the Dirichlet problem. Let $f \in W^2_{\frac{1}{2}}(\partial\Omega)$, and let v and V be the (variational) solutions of the Dirichlet problems:

$$\begin{cases} \nabla \cdot \Big(1 + (k-1)\chi(D)\Big)\nabla v = 0 \quad \text{in } \Omega\ , \\ v = f \quad \text{on } \partial\Omega\ , \end{cases} \tag{1.37}$$

and

$$\begin{cases} \Delta V = 0 \quad \text{in } \Omega\ , \\ V = f \quad \text{on } \partial\Omega\ . \end{cases} \tag{1.38}$$

The following representation theorem holds.

Theorem 1.11 *Let v and V be the solution of the Dirichlet problems (1.37) and (1.38). Then $\partial v/\partial\nu$ on ∂D can be represented as*

$$\frac{\partial v}{\partial \nu}(x) = \frac{\partial V}{\partial \nu}(x) - \frac{\partial}{\partial \nu} G_D\phi(x), \quad x \in \partial\Omega\ , \tag{1.39}$$

where ϕ is defined in (1.27) *with H given by* (1.26) *and $g = \partial v/\partial\nu$ on $\partial\Omega$, and*

$$G_D\phi(x) := \int_{\partial D} G(x,y)\phi(y)\, d\sigma(y)\ .$$

1.4 Periodic transmission problem

We now consider the following periodic transmission problem for the use in calculating effective properties of dilute composite materials: Let $Y =]-1/2, 1/2[^2$ denote the unit cell and $\overline{D} \subset Y$. Consider

$$\begin{cases} \nabla \cdot \Big(1 + (k-1)\chi(D)\Big)\nabla u_i = 0 \quad \text{in } Y, \\ u_i - y_i \text{ periodic}, \\ \displaystyle\int_Y u_i = 0. \end{cases} \tag{1.40}$$

In order to derive a representation formula for the solution to (1.40), we need a periodic Green function. Let

$$G(x) = -\sum_{n \in \mathbb{Z}^d \setminus \{0\}} \frac{e^{i2\pi n \cdot x}}{4\pi^2 |n|^2}. \tag{1.41}$$

Then we get, in the sense of distributions,

$$\Delta G(x) = \sum_{n \in \mathbb{Z}^d \setminus \{0\}} e^{i2\pi n \cdot x} = \sum_{n \in \mathbb{Z}^d} e^{i2\pi n \cdot x} - 1.$$

It then follows from the Poisson summation formula

$$\sum_{n \in \mathbb{Z}^2} e^{i2\pi n \cdot x} = \sum_{n \in \mathbb{Z}^2} \delta(x+n),$$

that

$$\Delta G(x) = \sum_{n \in \mathbb{Z}^2} \delta(x+n) - 1. \tag{1.42}$$

Appearance of the constant 1 in (1.42) may be a little peculiar. It is the volume of Y and an integration by parts shows that it should be there. In fact,

$$\int_Y \Delta G(x) dx = \int_{\partial Y} \frac{\partial G}{\partial \nu} d\sigma,$$

and the righthand side is 0 because of the periodicity.

The following results were obtained in [20].

Lemma 1.12 *Suppose that $d = 2$. There exists a harmonic function $R(x)$ in the unit cell Y such that*

$$G(x) = \frac{1}{2\pi} \log |x| + R(x). \tag{1.43}$$

Moreover, the Taylor expansion of $R(x)$ at 0 is given by

$$R(x) = R(0) - \frac{1}{4}(x_1^2 + x_2^2) + O(|x|^4). \tag{1.44}$$

The periodic single layer potential of the density function $\varphi \in L_0^2(\partial D)$ is defined by

$$\mathcal{G}_D \varphi(x) := \int_{\partial D} G(x-y)\varphi(y)\, ds(y), \quad x \in \mathbb{R}^2.$$

Lemma 1.12 shows that , we have

$$\mathcal{G}_D \varphi(x) = \mathcal{S}_D \varphi(x) + \mathcal{R}_D \varphi(x), \tag{1.45}$$

where $\mathcal{R}_D$ is a smoothing operator defined by

$$\mathcal{R}_D \varphi(x) := \int_{\partial D} R(x-y)\varphi(y)\, ds(y)$$

Thanks to (1.45), we have

$$\frac{\partial}{\partial \nu} \mathcal{G}_D \varphi \Big|_{\pm}(x) = \frac{\partial}{\partial \nu} \mathcal{S}_D \varphi \Big|_{\pm}(x) + \frac{\partial}{\partial \nu} \mathcal{R}_D \varphi(x), \quad x \in \partial D.$$

Thus we can understand $\frac{\partial}{\partial \nu} \mathcal{G}_D \varphi|_{\pm}$ as a compact perturbation of $\frac{\partial}{\partial \nu} \mathcal{S}_D \varphi|_{\pm}$. Based on this idea, we obtain the following results.

Lemma 1.13 **(i)** *Let $\varphi \in L^2_0(\partial D)$. The following trace formula holds:*

$$\frac{\partial}{\partial \nu} \mathcal{S}_D \varphi \Big|_{\pm} (x) = (\pm \frac{1}{2} I + \mathcal{B}^*_D) \varphi(x) \text{ on } \partial D, \tag{1.46}$$

where $\mathcal{B}^*_D : L^2_0(\partial D) \to L^2_0(\partial D)$ is given by

$$\mathcal{B}^*_D \varphi(x) = p.v. \int_{\partial D} \frac{\partial}{\partial \nu(x)} G(x-y) \varphi(y)\, ds(y).$$

(ii) *If $\varphi \in L^2_0(\partial D)$, then $\mathcal{S}_D \varphi$ is harmonic in D and $Y \setminus \overline{D}$.*

(iii) *If $|\lambda| \geq \frac{1}{2}$, then the operator $\lambda I - \mathcal{B}^*_D$ is invertible on $L^2_0(\partial D)$.*

Theorem 1.14 *Let u_i be the unique solution to the transmission problem (1.40). Then u_i can be expressed as follows*

$$u_i(x) = x_i + C_i + \mathcal{G}_D (\lambda I - \mathcal{B}^*_D)^{-1} (\nu_i)(x) \quad \text{in } Y, \quad i = 1, 2, \tag{1.47}$$

where C_i is a constant and ν_i is the i-component of the outward unit normal ν to ∂D.

2 Lecture 2 Generalized Polarization Tensors

2.1 Generalized Polarization Tensors

For this section we suppose that $B = \cup_{l=1}^m B_l \subset \mathbb{R}^d$ where $\overline{B_l}$'s are mutually disjoint, and the conductivity of B_l is k_l. We first define the overall conductivity $\bar{k}$ and the center $\bar{z}$ of B as follows.

$$\frac{\bar{k}-1}{\bar{k}+1}\sum_{s=1}^m |B_s| := \sum_{s=1}^m \frac{k_s-1}{k_s+1}|B_s| \ , \tag{2.1}$$

and

$$\frac{\bar{k}-1}{\bar{k}+1}\bar{z}\sum_{s=1}^m |B_s| = \sum_{s=1}^m \frac{k_s-1}{k_s+1}\int_{B_s} x\, dx \ . \tag{2.2}$$

Note that if k_s is the same for all s then $\bar{k} = k_s$ and $\bar{z}$ is the center of mass of B.

Suppose that $\bar{z} = 0$ for simplicity of presentation. In order define the generalized polarization tensor (GPT), let us consider the following transmission problem on $\mathbb{R}^d$:

$$\begin{cases} \nabla\cdot\Big(\chi(\Omega \setminus \bigcup_{l=1}^m \overline{B_l}) + \sum_{l=1}^m k_l \chi(B_l)\Big)\nabla u = 0 \quad \text{in } \mathbb{R}^d, \\ u(x) - H(x) = O(|x|^{1-d}) \quad \text{as } |x| \to \infty, \end{cases} \tag{2.3}$$

where H is a harmonic function in $\mathbb{R}^d$. The solution u to (2.3) is the electric charge distribution caused by the dipole moment ∇H at infinity in the presence of inclusions $\cup_{l=1}^m B_l$. By Theorem 1.9, the solution u to (2.3) is represented as

$$u(x) = H(x) + \sum_{l=1}^m \mathcal{S}_{B_l}\varphi^{(l)}(x), \quad x \in \mathbb{R}^d, \tag{2.4}$$

where $\varphi^{(l)} \in L_0^2(\partial B_l)$, $l = 1, \ldots, m$, is the solution of the integral equation (1.32).

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, let $\partial^\alpha f = \partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d} f$ and $x^\alpha := x_1^{\alpha_1}\cdots x_d^{\alpha_d}$. Let $\alpha = (\alpha_1, \ldots, \alpha_d), \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d$ be multi-indices. For $l = 1, \ldots, m$ let $\varphi_\alpha^{(l)}$ be the solution of

$$(\lambda_l I - \mathcal{K}_{B_l}^*)\varphi_\alpha^{(l)} - \sum_{k\neq l}\frac{\partial(\mathcal{S}_{B_k}\varphi_\alpha^{(k)})}{\partial\nu^{(l)}}\Big|_{\partial B_l} = \frac{\partial x^\alpha}{\partial\nu^{(l)}}\Big|_{\partial B_l} \quad \text{on } \partial B_l. \tag{2.5}$$

Since $H(x) = \sum_\alpha \frac{1}{\alpha!}\partial^\alpha H(0)x^\alpha$ where the series converges uniformly and absolutely on any compact set, we get

$$\varphi^{(l)} = \sum_{|\alpha|\geq 1}\frac{1}{\alpha!}\partial^\alpha H(0)\varphi_\alpha^{(l)}, \quad l = 1, \ldots, m. \tag{2.6}$$

By Taylor expansion, we have

$$\Gamma(x-y) = \sum_\beta \frac{(-1)^{|\beta|}}{\beta!}\partial^\beta\Gamma(x)y^\beta, \quad y \text{ in a compact set}, \quad |x| \to \infty. \tag{2.7}$$

It then follows from (1.31), (2.6), and (2.7) that

$$u(x) = H(x) + \sum_{|\alpha|\geq 1} \sum_{\beta} \frac{(-1)^{|\beta|}}{\alpha!\beta!} \partial^\alpha H(0) \partial^\beta \Gamma(x) \sum_{l=1}^{m} \int_{\partial B_l} y^\beta \varphi_\alpha^{(l)}(y) d\sigma(y), \quad |x| \text{ large.} \tag{2.8}$$

Observe that

$$\sum_{|\alpha|\geq 1} \frac{1}{\alpha!} \partial^\alpha H(0) \sum_{l=1}^{m} \int_{\partial B_l} \varphi_\alpha^{(l)}(y) d\sigma(y) = \sum_{l=1}^{m} \int_{\partial B_l} \varphi^{(l)}(y) d\sigma(y) = 0.$$

It then follows from (2.8) that

$$u(x) = H(x) + \sum_{|\alpha|\geq 1} \sum_{\beta} \frac{(-1)^{|\beta|}}{\alpha!\beta!} \partial^\alpha H(0) \partial^\beta \Gamma(x) M_{\alpha\beta}, \quad |x| \to \infty, \tag{2.9}$$

where

$$M_{\alpha\beta} := \sum_{l=1}^{m} \int_{\partial B_l} y^\beta \varphi_\alpha^{(l)}(y) d\sigma(y).$$

The asymptotic expansion formula (2.9) shows that the perturbation of the electric potential in $\mathbb{R}^d$ due to the presence of inclusions are completely described by $M_{\alpha\beta}$.

Definition 2.1 *Suppose that $B = \cup_{l=1}^m B_l \subset \mathbb{R}^d$ where $\overline{B}_l$'s are mutually disjoint, and the conductivity of B_l is k_l. Suppose that the center of B in the sense of (2.2) is $\bar{z}$. The generalized polarization tensor (GPT) associated with the inclusions $B = \cup_{l=1}^m B_l$ $M_{\alpha\beta}$, $\alpha, \beta \in \mathbb{N}^d$, is defined by*

$$M_{\alpha\beta} := \sum_{l=1}^{m} \int_{\partial B_l - \bar{z}} y^\beta \varphi_\alpha^{(l)}(y) d\sigma(y), \tag{2.10}$$

where $\varphi_\alpha^{(l)}$ is the solution of (2.5) with B_l replaced with $B_l - \bar{z}$. If $|\alpha| = |\beta| = 1$, we denote $M_{\alpha\beta}$ by M_{pq}, $p, q = 1, \ldots, d$ and call $M = (M_{pq})_{p,q=1}^d$ the polarization tensor of Pólya–Szegö or the first-order polarization tensor.

It should be observed that $M_{\alpha\beta}$ depends on the conductivities k_l as well as B_l. Thus it would be more precise to say that the GPT is associated with the conductivity distribution $\chi(\Omega \setminus \bigcup_{l=1}^m \overline{B_l}) + \sum_{l=1}^m k_l \chi(B_l)$. We also note that since $\int_{\partial B_l} \varphi_\alpha^{(l)} d\sigma = 0$ if $|\alpha| = 1$, the first order PT is invariant under translations of the inclusion.

When there is a single inclusion B whose centroid is the origin, then

$$\phi_\alpha(y) := (\lambda I - \mathcal{K}_B^*)^{-1}\big(\nu_x \cdot \nabla x^\alpha\big)(y), \quad y \in \partial B\ , \tag{2.11}$$

and hence we can rephrase the definition in the following way.

$$M_{\alpha\beta} := \int_{\partial B} y^\beta (\lambda I - \mathcal{K}_B^*)^{-1}\big(\nu_x \cdot \nabla x^\alpha\big)(y)\, d\sigma(y)\ . \tag{2.12}$$

We can represent the GPT in terms of solutions to transmission problems.

Lemma 2.2 *For all $\alpha, \beta \in \mathbb{N}^d$, $M_{\alpha\beta}$ associated with the single inclusion B can be rewritten in the following form:*

$$M_{\alpha\beta} = (k-1)\int_{\partial B} x^j \frac{\partial x^i}{\partial \nu}\, d\sigma(x) + (k-1)^2 \int_{\partial B} x^j \frac{\partial \psi_i}{\partial \nu}\bigg|_{-}(x)\, d\sigma(x)\ , \tag{2.13}$$

where ψ_i is the unique solution of the transmission problem

$$\begin{cases} \Delta \psi_i(x) = 0, \quad x \in B \cup (\mathbb{R}^d \setminus \overline{B})\ , \\ \psi_i\Big|_{+}(x) - \psi_i\Big|_{-}(x) = 0 \quad x \in \partial B\ , \\ \dfrac{\partial \psi_i}{\partial \nu}\bigg|_{+}(x) - k\dfrac{\partial \psi_i}{\partial \nu}\bigg|_{-}(x) = \nu \cdot \nabla x^i \quad x \in \partial B\ , \\ \psi_i(x) \to 0 \ \textit{as}\ |x| \to \infty \quad \textit{if}\ d = 3\ , \\ \psi_i(x) - \dfrac{1}{2\pi}\log|x| \displaystyle\int_{\partial B} \nu \cdot \nabla y^i\, d\sigma(y) \to 0 \ \textit{as}\ |x| \to \infty \quad \textit{if}\ d = 2\ . \end{cases} \tag{2.14}$$

The concepts of higher-order polarization tensors generalize those of classical Pólya–Szegö polarization tensors which was introduced by Pólya, Szegö, and Schiffer in relation to certain problems arising in hydrodynamics and in electrostatics [90, 92] in the form in Lemma 2.2. If the conductivity k is zero, namely, if B is insulated, the polarization tensor of Pólya–Szegö is called the virtual mass. The notion of Pólya–Szegö polarization tensor has been extensively studied in the literature by many authors for various purposes [34, 20, 37, 48, 79, 78, 51, 69, 70, 86, 45]. The concept of polarization tensors also occurs in several other interesting contexts, in particular in asymptotic models of dilute composites [84, 20, 47] and in low-frequency scattering of acoustic and electromagnetic waves [69, 44]. Recently the first order PT was defined for arbitrary inclusions [34]. Our definition contains all the higher order tensors. One significant advantage of using boundary integrals for defining PT is that all the computation can be carried out on ∂B without going to unbounded spaces.

The polarization tensor M of Pólya–Szegö can be explicitly computed for disks and ellipses in the plane and balls and ellipsoids in three-dimensional space [69]. If, for example, B is an ellipse whose focal line is on either the x_1- or the x_2-axis, its semi-major axis is of length a, and its semi-minor axis is of length b then its polarization tensor of Pólya–Szegö M takes the form

$$M = (k-1)|B| \begin{pmatrix} \dfrac{a+b}{a+kb} & 0 \\ 0 & \dfrac{a+b}{b+ka} \end{pmatrix}\ , \tag{2.15}$$

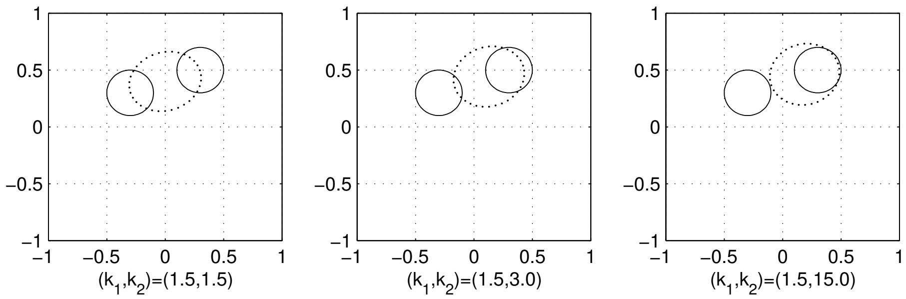
where $|B|$ denotes the volume of B. All the higher order tensors for disks, balls, and ellipses were computed in [77].

Lemma 2.3 *Let B' be a domain and $B = \mathcal{R}B'$ where $\mathcal{R}$ is a unitary transformation, and let M and M' be the first order PT associated with M and M', respectively. Then*

$$M = \mathcal{R}M'\mathcal{R}^T. \tag{2.16}$$

The GPT's seem to carry important geometric and potential theoretic properties of the domain B. Let us see one example illustrating what kind of information the PT carries, leaving other important properties to later sections.

Let us look at Figures 3.3 and 2 from [13, 10]. We consider two inclusions of disk shape. We compute PT associated with two inclusions following Definition 2.1 and find the ellipse whose PT is the computed one. We call such an ellipse the *equivalent ellipse*. Figures 1 and 2 show how the equivalent ellipse changes as the conductivities and the sizes of the inclusions B_s vary. The solid line represents the actual inclusions and the dashed lines are the equivalent ellipses. It represents overall property of the multiple inclusions as a conductor immerse in a material of conductivity 1.



$a_0^i, a_1^i, a_2^i, b_0^i, b_1^i, b_2^i$	k_i	$\bar{k}$	a	b	θ	$\bar{z}$
	1.5 1.5	1.5	0.313	0.256	0.322	(-0.000, 0.400)
-0.3,0.2,0, 0.3,0.2,0 0.3,0.2,0, 0.5,0.2,0	1.5 3	2.077	0.307	0.261	0.322	(0.129, 0.443)
	1.5 15	3.324	0.301	0.266	0.322	(0.188, 0.463)

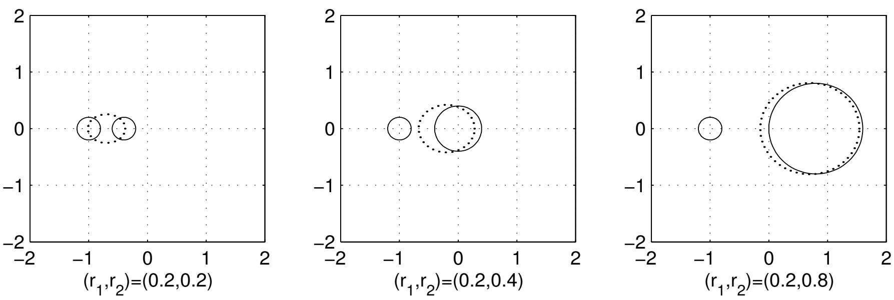
Figure 1: When the two disks have the same radius and the conductivity of the one on the right-hand side is increasing, the equivalent ellipse moves toward the right inclusion. In the table $\bar{k}$ and $\bar{z}$ are the overall conductivity and center defined by (2.1) and (2.2) and a, b, θ are the semi-axes lengths and angle of orientation measured in radians of the equivalent ellipse.

If the inclusions are far apart from each other so that the interaction between inclusions are negligible, then we get from (1.33) that

$$\varphi_j^{(s)} = (\lambda_l I - \mathcal{K}_{D_s}^*)^{-1}(\nu_j) + O(\delta^{-d}), \quad \text{on } \partial D_s, \quad j = 1, \dots d \,,$$

and hence

$$M = \sum_{s=1}^{m} M^{(s)} + O(\delta^{-d})$$



k_i	$a_0^i, a_1^i, a_2^i, b_0^i, b_1^i, b_2^i$	$\bar{k}$	a	b	θ	$\bar{z}$
	-1,0.2,0, 0,0.2,0 -0.4,0.2,0, 0,0.2,0	1.5	0.317	0.254	0	(-0.700 , 0.000)
1.5 1.5	-1,0.2,0, 0,0.2,0 0,0.2,0, 0,0.2,0	1.5	0.478	0.420	0	(-0.200, 0.000)
	-1,0.2,0, 0,0.2,0 0.8,0.2,0, 0,0.2,0	1.5	0.844	0.806	0	(0.694, 0.000)

Figure 2: When the conductivities of the two disks are the same and the radius of the disk on the right-hand side is increasing, the equivalent ellipse moves toward the right inclusion.

where $M^{(s)}$ is the first order tensor associated with B_s.

2.2 NtD map, Symmetry and Positivity of GPT

We now explain some important properties of GPT obtained in [8, 13, 11]. We first show

Theorem 2.4 *Let Ω be a domain with smooth boundary such that $\cup_{l=1}^m \overline{B_l} \subset \Omega$. Then $M_{\alpha\beta}$ uniquely determines the Neumann-to-Dirichlet map on Ω, and hence $\cup_{l=1}^m B_l$ and $k_1, \ldots, k_m$.*

Proof of Theorem 2.4. Let u be the solution of (1.30) in the form (1.31). Let Ω' be a relatively compact subset of Ω containing $\cup_{l=1}^m \overline{B}_l$. Choose H_n, $n = 1, 2, \ldots$, harmonic functions in $\mathbb{R}^d$ so that H_n converges uniformly to H in Ω'. Let $\psi_n^{(j)} \in L_0^2(\partial B_j)$ be the solution of (1.27) with H replaced with H_n. By the uniqueness and solvability of (1.27), $\psi_n^{(j)}$ converges to $\psi^{(j)}$ in $L^2(\partial B_j)$ as $n \to \infty$. Let

$$u_n(x) := H_n(x) + \sum_{j=1}^m \mathcal{S}_{B_j} \psi_n^{(j)}(x), \quad x \in \mathbb{R}^d.$$

Then u_n is the solution of (2.3) with H replaced with H_n, and u_n converges to u uniformly on Ω'. In view of (2.4), $u_n(x)$ is completely determined by $M_{\alpha\beta}$ for $|x| > R$, and hence $x \in \mathbb{R}^d \setminus \cup_{l=1}^m \overline{B}_l$ by the unique continuation. Since u_n converges to u uniformly on Ω', u is now uniquely determined by $M_{\alpha\beta}$ on $\Omega' \setminus \cup_{l=1}^m \overline{B}_l$. By the unique continuation, u is uniquely determined by $M_{\alpha\beta}$ on $\Omega \setminus \cup_{l=1}^m \overline{B}_l$. Therefore NtD map Λ_D is uniquely determined by $M_{\alpha\beta}$. By the uniqueness result of Isakov [55], Λ_D uniquely determines $\cup_{l=1}^m B_l$ and $k_1, \ldots, k_m$. This completes the proof. $\square$

Theorem 2.4 shows that all the GPT completely characterize the domain and conductivity. However, it seem to be difficult to characterize those information of the domain carried by individual $M_{\alpha\beta}$. We will see that the first order tensor carries information on the volume and orientation of the inclusion, and higher order ones carry information on the weighted volume.

Theorem 2.5 (Symmetry) *The first order polarization tensor M is symmetric. Moreover, if a_α and b_β are constants such that $\sum_\alpha a_\alpha y^\alpha$ and $\sum_\beta b_\beta y^\beta$ are harmonic polynomials, then*

$$\sum_{\alpha,\beta} a_\alpha b_\beta m_{\alpha\beta} = \sum_{\alpha,\beta} a_\alpha b_\beta m_{\beta\alpha}. \tag{2.17}$$

We also have a theorem on the positivity of GPT.

Theorem 2.6 (Positivity) *Suppose that either $k_l - 1 > 0$ or $k_l - 1 < 0$ for all $l = 1, \ldots, m$. Let*

$$\kappa := \max_{1 \le l \le m} \left| 1 - \frac{1}{k_l} \right|.$$

For any a_α such that $\sum_\alpha a_\alpha y^\alpha$ is harmonic,

$$\left| \sum_{\alpha,\beta} a_\alpha a_\beta m_{\alpha\beta} \right| \ge \frac{|\kappa - 1|}{m+1} \sum_{l=1}^m |k_l - 1| \int_{B_l} \left| \nabla (\sum_\alpha a_\alpha y^\alpha) \right|^2 dy. \tag{2.18}$$

In particular, if $k_l - 1 > 0$ (resp. < 0) for all $l = 1, \ldots, m$, then M is positive (resp. negative) definite.

2.3 Optimal Bounds

We now explain refined estimates of the eigenvalues of the polarization tensors associated with a single inclusion.

Theorem 2.7 *Suppose that B is a single inclusion and $\sum_{\alpha \in J} a_\alpha x^\alpha$ is a harmonic polynomial. Then we get the following inequalities:*

$$\frac{1}{k}(k-1) \int_B |\nabla (\sum_{\alpha \in J} a_\alpha x^\alpha)|^2 \le \sum_{\alpha,\beta \in J} a_\alpha a_\beta M_{\alpha\beta} \le (k-1) \int_B |\nabla (\sum_{\alpha \in J} a_\alpha x^\alpha)|^2. \tag{2.19}$$

In particular, if κ is an eigenvalue of $M = (M_{\alpha\beta})_{|\alpha|=|\beta|=1}$, then

$$\frac{1}{k}(k-1)|B| \le \kappa \le (k-1)|B|. \tag{2.20}$$

Proofs of Theorem 2.7 will be given in the next section since they are not available. Optimal estimates for the more general anisotropic case were obtained in [66]. See also [15].

The estimates (2.20) are optimal as estimates for eigenvalues. This can be seen from Figure 3. We do not know whether (2.19) is also optimal. Estimates (2.20) can be improved in the following way.

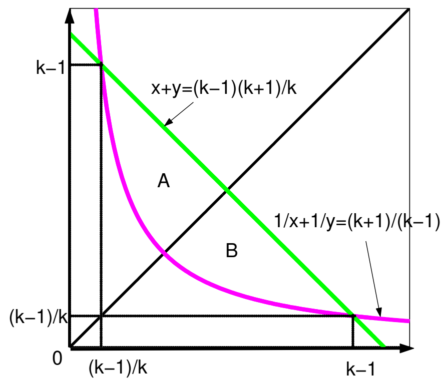


Figure 3: The optimal bounds for the polarization tensor in $\mathbb{R}^2$.

Theorem 2.8 *Let M be the PT associated with the domain B whose volume $|B| = 1$. Then,*

$$Trace(M) \le (k-1)(d-1+\frac{1}{k}), \tag{2.21}$$

and

$$Trace(M^{-1}) \le \frac{d-1+k}{k-1}. \tag{2.22}$$

In particular, if $d = 2$ and λ_1 and λ_2 are two eigenvalues of M, then

$$\lambda_1 + \lambda_2 \le \frac{(k-1)(k+1)}{k}, \tag{2.23}$$

and

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} \le \frac{k+1}{k-1}. \tag{2.24}$$

Figure 3 shows these bounds graphically. The square box $[(k-1)/k, k-1] \times [(k-1)/k, k-1]$ in Figure 3 represents the estimates (2.20). This box is called a Wiener box in the community of composite materials.

The bounds in (2.21) and (2.22) are called the Hashin-Shtrikman bounds since they first derive, using variational methods, similar kinds of geometry independent bounds for composite materials. The estimates (2.21) and (2.22) for PT was obtained by Capdeboscq and Vogelius [35]. They also showed that the bounds are optimal in two dimensions in the sense that each point inside the bounds is a pair of eigenvalues of the PT associated with a domain. They showed that each point inside the bounds is attained as a PT associated with a coated ellipse, or a washer of elliptic shape [36]. In fact, every point on the lower bound $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = \frac{k+1}{k-1}$ corresponds to an ellipse as one can see from (2.15), and as ellipses get thinner, corresponding points on the lower bound move to the upper or lower corner. If we start from an ellipse corresponding to a point on the lower bound, and make confocal washers of elliptic shape, then corresponding points move toward the upper bound following a certain curve as the washers get thinner and larger. These curves make foliations and cover all regions inside the bounds except the upper bound. This result is exact and PT for the elliptic washer can be computed using elliptic coordinates. This optimal estimates were efficiently used to estimate the size of unknown inclusions [35].

On the other hand, it is shown numerically that each point inside the bounds is attained by a simply connected domain [14]. It turns out that if start from a disk and vary the domain to make thin and long cross as in the Figure 4, the corresponding eigenvalues move from the intersection point of the lower hyperbola and the line $\lambda_1 = \lambda_2$ toward the the intersection point of the upper bound and the line $\lambda_1 = \lambda_2$ following the line $\lambda_1 = \lambda_2$. Note that the intersection point of the lower hyperbola and the line $\lambda_1 = \lambda_2$ is the pair of eigenvalues of the PT associated with the disk. We also note that the cross-shaped domain in Figure 4 is invariant under the rotation by $\frac{\pi}{2}$, and hence the corresponding PT is of the form λI for some λ where I is the 2×2 identity matrix. Thus by interpolating a cross-shape domain and an ellipse, we can obtain foliation of the region inside the bounds (2.23) and (2.24). It should be noted that the result of [14] is numerical; exact computation of PTs associated with cross shape domains seems unlikely.

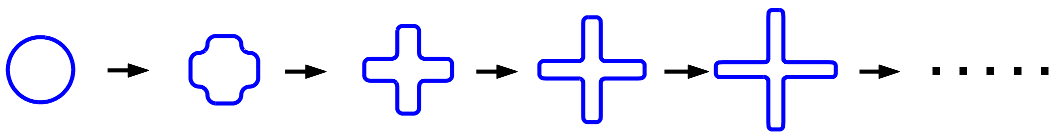


Figure 4: The variation of cross-domains starting from disk.

Before completing introduction, we briefly mention about a conjecture of Pólya-Szegö. Observe that among the points in HS-bounds, the intersection point the lower bound and the line $\lambda_1 = \lambda_2$, $(\frac{2(k-1)}{k+1}, \frac{2(k-1)}{k+1})$, has the minimal trace, and this is the eigenvalue of the PT associated with the disk of the unit area. Pólya and Szegö [90] conjectured that the disk (or the sphere for three dimensions) is a unique domain with the minimal trace, and this conjecture has not been proved yet.

2.4 Proofs of Theorem 2.7

Put

$$f(y) := \sum_{\alpha} a_\alpha y^\alpha \quad \text{and} \quad \phi := \sum_{\alpha} a_\alpha \phi_\alpha = (\lambda I - \mathcal{K}_B^*)^{-1}(\frac{\partial f}{\partial \nu}),$$

where ϕ_α is defined in (2.11). We then obtain from (2.12) that

$$\sum_{\alpha,\beta} a_\alpha a_\beta M_{\alpha\beta} = \int_{\partial B} \sum_{\beta} b_\beta y^\beta \sum_{\alpha} a_\alpha \phi_\alpha(y)\, d\sigma(y) = \int_{\partial B} f\phi\, d\sigma.$$

Then $\mathcal{S}_B\phi$ satisfies the transmission condition

$$\frac{\partial}{\partial\nu}\mathcal{S}_B\phi|_+ - k\frac{\partial}{\partial\nu}\mathcal{S}_B\phi|_- = (k-1)\frac{\partial f}{\partial\nu} \quad \text{on } \partial B.$$

It then follows that

$$\int_{\partial B} f\phi\, d\sigma = \int_{\partial B} f \left[\frac{\partial \mathcal{S}_B\phi}{\partial\nu}\bigg|_+ - \frac{\partial \mathcal{S}_B\phi}{\partial\nu}\bigg|_- \right] d\sigma = (k-1)\int_{\partial B} f\frac{\partial}{\partial\nu}(\mathcal{S}_B\phi + f)\bigg|_- d\sigma\,. \quad (2.25)$$

We also obtain from (2.25) and the transmission condition that

$$\begin{aligned}
\int_{\partial B} f\phi \, d\sigma &= (k-1)\int_{\partial B} (\mathcal{S}_B\phi + f)\frac{\partial}{\partial \nu}(\mathcal{S}_B\phi + f)\Big|_- \, d\sigma \\
&\quad - (k-1)\int_{\partial B} \mathcal{S}_B\phi \frac{\partial}{\partial \nu}\mathcal{S}_B\phi\Big|_- \, d\sigma - (k-1)\int_{\partial B} \mathcal{S}_B\phi \frac{\partial f}{\partial \nu}\, d\sigma \\
&= (k-1)\int_{\partial B} (\mathcal{S}_B\phi + f)\frac{\partial}{\partial \nu}(\mathcal{S}_B\phi + f)\Big|_- \, d\sigma \\
&\quad - \left(1-\frac{1}{k}\right)\int_{\partial B} \mathcal{S}_B\phi \frac{\partial}{\partial \nu}\mathcal{S}_B\phi\Big|_+ \, d\sigma - \left(1-\frac{1}{k}\right)\int_{\partial B} \mathcal{S}_B\phi \frac{\partial f}{\partial \nu}\, d\sigma \, . \qquad (2.26)
\end{aligned}$$

Define quadratic forms $Q_D(u)$ by

$$Q_D(u) := \int_D |\nabla u|^2 \, dx \, . \qquad (2.27)$$

Then, by equating (2.25) and (2.26), we obtain

$$\int_{\partial B} \mathcal{S}_B\phi \frac{\partial f}{\partial \nu}\, d\sigma = \frac{k}{k+1} Q_B(\mathcal{S}_B\phi + f) + \frac{1}{k+1} Q_{\mathbb{R}^d\setminus B}(\mathcal{S}_B\phi) - \frac{k}{k+1} Q_B(f) \, .$$

Substituting this identity into (2.25), we get

$$\int_{\partial B} f\phi \, d\sigma = \frac{k(k-1)}{k+1} Q_B(\mathcal{S}_B\phi + f) + \frac{k-1}{k+1} Q_{\mathbb{R}^d\setminus B}(\mathcal{S}_B\phi) + \frac{k-1}{k+1} Q_B(f) \, .$$

So we obtain the following theorem of positivity, which was first proved in [8].

Theorem 2.9 *Suppose that a_α, $i \in I$, where I is a finite index set, are constants such that $f(y) = \sum_{i\in I} a_\alpha y^\alpha$ is a harmonic polynomial. Let $\phi = (\lambda I - \mathcal{K}_B^*)^{-1}(\frac{\partial f}{\partial \nu})$. Then*

$$\sum_{\alpha,\beta\in I} a_\alpha a_\beta M_{\alpha\beta} = \frac{k-1}{k+1}\Big[k Q_B(\mathcal{S}_B\phi + f) + Q_{\mathbb{R}^d\setminus B}(\mathcal{S}_B\phi) + Q_B(f)\Big] \, . \qquad (2.28)$$

Theorem 2.9 says that if $k > 1$, then GPT's are positive definite, and they are negative definite if $0 < k < 1$.

Set $w_B = \mathcal{S}_B\phi$. Then

$$\begin{cases} \nabla \cdot (1 + (k-1)\chi(B))\nabla(w_B + f) = 0 \quad \text{in } \mathbb{R}^d, \\ w_B = O(|x|^{d-1}), \end{cases}$$

and hence

$$\int_{\mathbb{R}^d} (1 + (k-1)\chi(B))(\nabla w_B + (1-\frac{1}{k})\chi(B)\nabla f)\cdot \nabla v = 0, \quad \forall\, v \in H^1(\mathbb{R}^d). \qquad (2.29)$$

Therefore, w_B is the minimizer of the functional

$$\begin{aligned}
I_B(v) \;&:=\; \int_{\mathbb{R}^d} (1 + (k-1)\chi(B))\Big|\nabla v + (1-\frac{1}{k})\chi(B)\nabla f\Big|^2 \\
&=\; \int_{\mathbb{R}^d} (1 + (k-1)\chi(B))|\nabla v|^2 + \frac{(k-1)^2}{k}\int_B |\nabla f|^2 + 2(k-1)\int_B \nabla v \cdot \nabla f. \qquad (2.30)
\end{aligned}$$

It then follows from (2.28) that

$$\sum_{\alpha,\beta} a_\alpha a_\beta M_{\alpha\beta} = I_B(w_B) + (1 - \frac{1}{k}) \int_B |\nabla f|^2 = \inf_{v \in H^1(\mathbb{R}^d)} I_B(v) + (1 - \frac{1}{k}) \int_B |\nabla f|^2. \quad (2.31)$$

Substituting $v = w_B$ in (2.29), we get

$$-\int_{\mathbb{R}^d} (1 + (k-1)\chi(B))|\nabla w_B|^2 = (k-1) \int_B \nabla f \cdot \nabla w_B.$$

Thus,

$$I_B(w) = -\int_{\mathbb{R}^d} (1 + (k-1)\chi(B))|\nabla w_B|^2 + \frac{(k-1)^2}{k} \int_B |\nabla f|^2,$$

and we therefore arrive at

$$\sum_{\alpha,\beta} a_\alpha a_\beta M_{\alpha\beta} = -\int_{\mathbb{R}^d} (1 + (k-1)\chi(B))|\nabla w_B|^2 + (k-1) \int_B |\nabla f|^2. \quad (2.32)$$

Combining (2.31) and (2.32) we obtain Theorem 2.7. This completes the proof.

2.5 Further results

There is a concept of polarization tensor for the isotropic linear elasticity, which is called the Elastic Moment Tensor (EMT). It is a 4-tensor, not a matrix (2-tensor), and enjoys properties of symmetry and positivity. See [84, 18]. EMT has been effectively used in detection of small elastic inclusions [58].

3 Lecture 3
Reconstruction of small inclusions

We now consider the inverse problem of reconstruction of diametrically small inclusions via boundary measurements. Let us begin with the description of the configuration of the medium.

Configuration of Medium. The conductor Ω, which is a bounded domain in $\mathbb{R}^d$, $d = 2$ or 3, contains collections of multiple inclusions $D = \cup_{s=1}^m D_s$ which satisfies the following conditions:

(H1) Each D_s is well-separated from each other and the boundary of Ω, i.e. there exists $d_0 > 0$ such that $dist(D_s, D_t) > d_0$, $s \neq t$, and $dist(D, \partial\Omega) > d_0$.

(H2) Each D_s consists of multiple closely spaced inclusions:

$$D_s = \cup_{l=1}^{m_s} D_s^l = \epsilon_s \cup_{l=1}^{m_s} B_s^l + z_s,$$

where ϵ_s is a small number representing the common order of magnitude of D_s, and z_s denotes the location of D_s. Here B_s^l is a bounded Lipschitz domain in $\mathbb{R}^d$ such that there exist positive constants C_1 and C_2 such that

$$C_1 \leq \operatorname{diam} B_s^l \leq C_2, \quad \text{and} \quad C_1 \leq \operatorname{dist}(B_s^l, B_s^{l'}) \leq C_2, \quad l \neq l',$$

for each $s = 1, \ldots, m$.

So the inclusion D consists of several collections of closely spaced multiple inclusions and takes the form

$$D = \cup_{s=1}^m D_s = \cup_{s=1}^m \big(\cup_{l=1}^{m_s} D_s^l \big) = \cup_{s=1}^m \big(\epsilon_s \cup_{l=1}^{m_s} B_s^l + z_s \big). \tag{3.1}$$

See Figure 1. Detection of such inclusions has some practical significance since some anomalies inside a body may located very closely to each other.

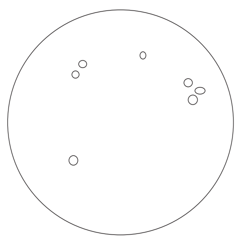


Figure 5: Configuration of Medium.

Suppose that the conductivity of D_s^l is $k_s^l > 0$ ($k_s^l \neq 1$). The conductivity profile of Ω is then

$$\gamma(x) = \begin{cases} k_s^l, & x \in D_s^l, \quad l = 1, \ldots, m_s, \; s = 1, \ldots, m, \\ 1, & x \in \Omega \setminus \overline{D}. \end{cases} \tag{3.2}$$

We consider the following Neumann problem:

$$\begin{cases} \nabla \cdot (\gamma(x)\nabla u) = 0 \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = g. \end{cases} \tag{3.3}$$

The problem is then to detect the inclusion D by means of finitely many boundary measurements, g and $u|_{\partial\Omega}$ where u is the solution to (3.3).

We note that we do not consider those inclusions which are very close to the boundary. In such cases, interaction between the boundary and the inclusions is not negligible, and to describe the interaction a somewhat different idea is required. For this we we refer to [2].

3.1 Asymptotic expansions

Let u be the solution of (3.3). According to Theorem 1.10, u on $\partial\Omega$ or $\Lambda_D(g)$ can be represented as

$$\Lambda_D(g)(x) = \Lambda_0(g)(x) - (-\frac{1}{2}I + \mathcal{K}_\Omega)^{-1}\left(\sum_{s=1}^{m}\sum_{l=1}^{m_s} \mathcal{S}_{D_s^l}\psi_l^{(s)}|_{\partial\Omega}\right)(x), \quad x \in \partial\Omega, \tag{3.4}$$

where $\psi_l^{(s)}$ is the solution of

$$(\lambda_s^l I - \mathcal{K}_{D_s^l}^*)\psi_l^{(s)} - \sum_{(t,k)\neq(s,l)} \frac{\partial(\mathcal{S}_{D_t^k}\psi_k^{(t)})}{\partial\nu_l^{(s)}}\Big|_{\partial D_s^l} = \frac{\partial H}{\partial \nu_l^{(s)}}\Big|_{\partial D_s^l} \quad \text{on } \partial D_s^l \tag{3.5}$$

and

$$H(x) = -\mathcal{S}_\Omega(g)(x) + \mathcal{D}_\Omega(\Lambda(g))(x), \quad x \in \Omega. \tag{3.6}$$

Recalling that $D_s^l = \epsilon_s B_s^l + z_s$, we make a change of variables $x = \epsilon_s x' + z_s$ for a fixed s. Observe that for $x = \epsilon_s x' + z_s \in \partial D_s^l$

$$\frac{\partial(\mathcal{S}_{D_t^k}\psi_k^{(t)})}{\partial\nu_l^{(s)}}(\epsilon_s x' + z_s) = \frac{1}{\omega_d}\int_{\partial D_t^k} \frac{\langle \epsilon_s x' + z_s - y, \nu_l^{(s)}(x')\rangle}{|\epsilon_s x' + z_s - y|^d}\psi_k^{(t)}(y)d\sigma(y),$$

where $\nu_l^{(s)}(x')$ denotes the unit normal to ∂B_s^l.

If $t = s$, by making change of variables $y = \epsilon_s y' + z_s$, one can see that

$$\frac{\partial(\mathcal{S}_{D_s^k}\psi_k^{(s)})}{\partial\nu_l^{(s)}}(\epsilon_s x' + z_s) = \frac{\partial(\mathcal{S}_{B_s^k}\widetilde{\psi}_k^{(s)})}{\partial\nu_l^{(s)}}(x'), \quad x' \in \partial B_s^k, \ \ k = 1, \ldots, m_s,$$

where

$$\widetilde{\psi}_k^{(s)})(x') := \psi_k^{(s)}(\epsilon_s x' + z_s), \quad x' \in \partial B_s^k.$$

On the other hand, if $t \neq s$, then we have

$$\begin{aligned} &\frac{\partial(\mathcal{S}_{D_t^k}\psi_k^{(t)})}{\partial\nu_l^{(s)}}(\epsilon_s x' + z_s) \\ &= \frac{1}{\omega_d}\int_{\partial D_t^k}\left[\frac{\langle \epsilon_s x' + z_s - y, \nu_l^{(s)}(x')\rangle}{|\epsilon_s x' + z_s - y|^d} - \frac{\langle \epsilon_s x' + z_s - y_0, \nu_l^{(s)}(x')\rangle}{|\epsilon_s x' + z_s - y_0|^d}\right]\psi_k^{(t)}(y)d\sigma(y) \end{aligned}$$

for some $y_0 \in \partial D_t^k$ since $\int_{\partial D_t^k} \psi_k^{(t)} d\sigma = 0$. Since D_t^k and D_s^l are well-separated, in other words, $|z_s - y| > d_0$ for all $y \in \partial D_t^k$, it follows from the mean value theorem that

$$\left| \frac{\partial (\mathcal{S}_{D_t^k} \psi_k^{(t)})}{\partial \nu_l^{(s)}} (\epsilon_s x' + z_s) \right| \le C \int_{\partial D_t^k} |y - y_0| |\psi_k^{(t)}(y)| d\sigma(y) \le C \epsilon_s^d \| \widetilde{\psi}_k^{(t)} \|_{L^2(\partial B_t^k)}.$$

It means that the impact of D_t^k observed at D_s^l is $O(\epsilon_s^d)$ which is negligible provided that they are far apart. We now rephrase the integral equation (3.5) as

$$(\lambda_s^l I - \mathcal{K}_{B_s^l}^*) \widetilde{\psi}_l^{(s)}(x') - \sum_{k \ne l} \frac{\partial (\mathcal{S}_{B_s^k} \widetilde{\psi}_k^{(s)})}{\partial \nu_l^{(s)}} (x') = \frac{\partial H}{\partial \nu_l^{(s)}} (\epsilon_s x' + z_s) + O(\epsilon^d), \quad x' \in \partial B_s^l, \tag{3.7}$$

where $\epsilon = \max \epsilon_s$. For each multi-index α, let $\varphi_{l\alpha}^{(s)}$, $l = 1, \ldots, m_s$, be the solution to

$$(\lambda_s^l I - \mathcal{K}_{B_s^l}^*) \varphi_{l\alpha}^{(s)}(x') - \sum_{k \ne l} \frac{\partial (\mathcal{S}_{B_s^k} \varphi_k^{(s)})}{\partial \nu_{l\alpha}^{(s)}} (x') = \frac{\partial x^\alpha}{\partial \nu_l^{(s)}} (x'), \quad x' \in \partial B_s^l$$

as before. Since

$$\frac{\partial H}{\partial \nu_l^{(s)}} (\epsilon_s x' + z_s) = \sum_{|\alpha| \ge 1} \epsilon_s^{|\alpha|-1} \partial^\alpha H(z_s) \frac{\partial x^\alpha}{\partial \nu_l^{(s)}} (x'), \quad x' \in \partial B_s^l,$$

we have from the linearity of the equation that

$$\widetilde{\psi}_l^{(s)} = \sum_{|\alpha| \ge 1} \epsilon_s^{|\alpha|-1} \partial^\alpha H(z_s) \varphi_{l\alpha}^{(s)} + O(\epsilon^d).$$

In a similar way to the derivation of (2.4) one can show that for $x \in \partial\Omega$

$$\begin{aligned} \sum_{l=1}^{m_s} \mathcal{S}_{D_s^l} \psi_l^{(s)}(x) &= \epsilon_s^{d-1} \sum_{l=1}^{m_s} \int_{\partial B_s^l} \Gamma(x - \epsilon y' - z_s) \widetilde{\psi}_l^{(s)}(y') d\sigma(y') \\ &= \sum_{|\alpha|=1}^\infty \sum_{|\beta|=1}^\infty \frac{\epsilon_s^{|\alpha|+|\beta|+d-2}}{\alpha! \beta!} (\partial^\alpha H)(z_s) \partial_z^\beta \Gamma(x - z_s) M_{\alpha\beta}^{(s)} + O(\epsilon^{2d}) \end{aligned}$$

where $M_{\alpha\beta}^{(s)}$, $s = 1, \ldots, m$, is the polarization tensor associated with $B_s = \cup_{l=1}^{m_s} B_s^l$. Observe that the error term is $O(\epsilon^{2d})$, not $O(\epsilon^{2d-1})$. It is because $\int_{\partial B_s^l} \widetilde{\psi}_l^{(s)} d\sigma = 0$.

Since $(-\frac{1}{2} I + \mathcal{K}_\Omega)(N(\cdot - z))(x) = \Gamma(x - z)$ for $z \in \Omega$ and $x \in \mathbb{R}^d \setminus \overline{\Omega}$ by (1.19), it follows from (3.4) that

$$\Lambda_D(g) = \Lambda_0(g) - \sum_{s=1}^m \sum_{|\alpha|=1}^\infty \sum_{|\beta|=1}^\infty \frac{\epsilon_s^{|\alpha|+|\beta|+d-2}}{\alpha! \beta!} (\partial^\alpha H)(z_s) \partial_z^\beta N(x, z_s) M_{\alpha\beta}^{(s)} + O(\epsilon^{2d}). \tag{3.8}$$

Since the formula is written in terms of H and H is dependent upon ϵ, we now want to remove this dependence. Observe from (3.6) and Green's representation, $U(x) = -\mathcal{S}_\Omega(g)(x) +$

$\mathcal{D}_\Omega(\Lambda_0(g))(x)$, $x \in \Omega$, that

$$\begin{aligned} H(x) - U(x) &= \mathcal{D}_\Omega(\Lambda(g) - \Lambda_0(g))(x) \\ &= \mathcal{D}_\Omega(-\frac{1}{2}I + \mathcal{K}_\Omega)^{-1} \left(\sum_{s=1}^{m} \sum_{l=1}^{m_s} \mathcal{S}_{D_s^l} \psi_l^{(s)}|_{\partial\Omega} \right)(x), \quad x \in \Omega. \end{aligned}$$

Since $\sum_{s=1}^{m} \sum_{l=1}^{m_s} \mathcal{S}_{D_s^l} \psi_l^{(s)}(x) = O(\epsilon^d)$ for $x \in \partial\Omega$, we get

$$\|H - U\|_{C^n(K)} = O(\epsilon^d)$$

for any integer n and a compact subset K of Ω. Thus we obtain the following asymptotic expansion formula

Theorem 3.1 *Suppose that the inclusion D is of the form $D = \cup_{s=1}^m D_s$, and each D_s consists of multiple closely spaced inclusions, i.e., $D_s = \epsilon \cup_{l=1}^{m_s} B_s^l + z_s$. Then, the following asymptotic expansion holds uniformly on $\partial\Omega$.*

$$\begin{aligned} &\Lambda_D(g)(x) - \Lambda_0(g)(x) \\ &= -\sum_{s=1}^{m} \sum_{|\alpha|=1}^{d} \sum_{|\beta|=1}^{d} \frac{\epsilon_s^{|\alpha|+|\beta|+d-2}}{\alpha!\beta!} (\partial^\alpha U)(z_s) \partial_z^\beta N(x, z_s) M_{\alpha\beta}^{(s)} + O(\epsilon^{2d}), \quad x \in \partial\Omega. \end{aligned} \tag{3.9}$$

We remark that in order to find terms higher than ϵ^{2d}, the interaction between inclusions and $\partial\Omega$ should be considered. If we only look at the leading order terms, the formula (3.9) takes the following simpler form:

$$\Lambda_D(g)(x) - \Lambda_0(g)(x) = -\sum_{s=1}^{m} \epsilon_s^d \nabla U(z_s) M^{(s)} \nabla_z N(x, z_s) + O(\epsilon^{d+1}), \quad x \in \partial\Omega, \tag{3.10}$$

where $M^{(s)}$ is the the polarization tensor of Pólya–Szegö associated with B_s.

Asymptotic expansion of the perturbation due to the presence of inclusions is the basic ingredient in designing algorithms to detect unknown inclusions. Several asymptotic expansion formula for various equations have been obtained [7, 12, 18, 21, 23, 37, 94].

3.2 Detection of inclusions-a single inclusion

Here ‘single’ means a single collection of multiple closely spaced inclusions.

Note that we have $\Lambda_D(g)$ as our data collected from the measurements on $\partial\Omega$. Neglecting the small term $O(\epsilon^{d+1})$ in (3.10), we have

$$\sum_{s=1}^{m} \epsilon_s^d \nabla U(z_s) M^{(s)} \nabla_z N(x, z_s), \quad x \in \partial\Omega.$$

in our hand. Now the task is to find the locations z_s, the polarization tensors $M^{(s)}$ from this data.

For a given Neumann data (or current) $g \in L_0^2(\partial\Omega)$, define the harmonic function $\Phi[g](x)$, $x \in \mathbb{R}^d \setminus \overline{\Omega}$, by

$$\Phi[g](x) := -\mathcal{S}_\Omega(g)(x) + \mathcal{D}_\Omega(\Lambda_D(g))(x), \quad x \in \mathbb{R}^d \setminus \bar{\Omega}. \tag{3.11}$$

By the Green's theorem $-\mathcal{S}_\Omega(g)(x) + \mathcal{D}_\Omega(\Lambda_0(g))(x) = 0$ for all $x \in \mathbb{R}^d \setminus \bar{\Omega}$, and $\mathcal{D}_\Omega(N(\cdot - z))(x) = \Gamma(x-z)$ for $z \in \Omega$ and $x \in \mathbb{R}^d \setminus \overline{\Omega}$ by (1.19). Therefore, we obtain by substituting (3.10) into (3.11)

$$
\begin{aligned}
\Phi[g](x) &= \mathcal{D}_\Omega\big(\Lambda_D(g) - \Lambda_0(g)\big)(x) \\
&= -\sum_{s=1}^{m} \epsilon_s^d \nabla U(z_s) M^{(s)} \nabla_z \Gamma(x - z_s) + O(\epsilon^{d+1}), \quad x \in \mathbb{R}^d \setminus \overline{\Omega}
\end{aligned}
\tag{3.12}
$$

We now have $\Phi[g](x)$, $x \in \mathbb{R}^d \setminus \overline{\Omega}$, as data for reconstruction of $M^{(s)}$ and z_s.

Suppose now that there is a single collection of inclusions, i.e., $m = 1$. So, D takes the form $D = \epsilon \cup_{l=1}^k B^l + z$. Then $H[g](x)$ becomes, in its first order term,

$$
\Phi[g](x) = -\epsilon^d \nabla U(z) M \nabla_z \Gamma(x - z) + O(\epsilon^{d+1}), \tag{3.13}
$$

where $M = (M_{ij})$ is the polarization tensor associated with $\cup_{l=1}^k B^l$. Observe that $\nabla_z \Gamma(x-z)$ behaves like $-\frac{1}{\omega_d} \frac{x}{|x|^d}$ as $|x| \to \infty$. Therefore by applying linear harmonic functions for U, we can recover M. Then by applying quadratic harmonic functions for U, we can recover z. Assume for the sake of simplicity that $d = 2$. The reconstruction procedure is the following.

[Quadratic algorithm]

Step 1: For $g_p = \partial x_p / \partial \nu$, $p = 1, 2$, measure $u|_{\partial\Omega}$.

Step 2: Compute the first-order polarization tensor $\epsilon^2 M = \epsilon^2 (M_{pq})_{p,q=1}^d$ for D by

$$
\epsilon^2 M_{pq} = \lim_{t \to \infty} 2\pi t H[g_p](t e_q) .
$$

The estimate (2.18) yields an estimate for $\epsilon^2 |B| = |D|$.

Step 3: Compute $h_p = \lim_{t\to\infty} 2\pi t H[g_3](t e_p)$ for $g_3 = \frac{\partial (x_1 x_2)}{\partial \nu}, p = 1, 2$. Then the center is estimated by solving

$$
z = (h_1, h_2)(\epsilon^2 M)^{-1} .
$$

Step 4: Let the overall conductivity $\bar{k} = \infty$ if the polarization tensor M is positive definite. Otherwise assume $\bar{k} = 0$. Use (2.15) to obtain the shape of the equivalent ellipse.

Observe that we only use $2d - 1$ measurements. By visualizing the detected polarization tensor M by an ellipse, we can find a geometric figure representing $D = \epsilon \cup_{l=1}^k B^l + z$. Figure 2 shows a result of numerical experiments. We note that the detected ellipse and location are in very good match with the equivalent ellipse and the center of $D = \cup_{l=1}^k D^l := \epsilon \cup_{l=1}^k B^l + z$ as defined in (2.1) and (2.2). For complete results, we refer to [13, 10]. There is a method using only linear currents. See [72, 22].

3.3 Detection of inclusions-Multiple inclusions

Detection of a single inclusion using (3.13) was a rather simple matter. However detection of multiple inclusions is dramatically different. One can expect this by looking at the formula (3.12). The summation in (3.12) is a mixture of information coming from all the inclusions and we need to extract individual information from the mixture.

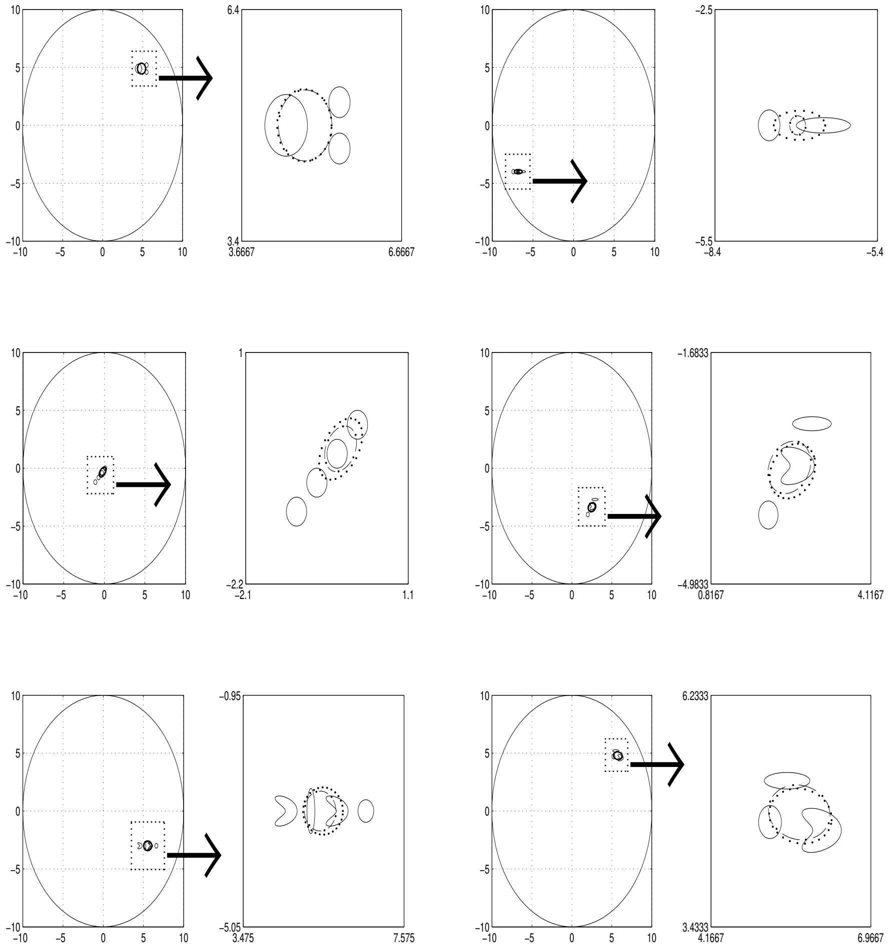


Figure 6: Reconstruction of closely spaced small inclusions. The dashed line is the equivalent ellipse and the dash-dot line is the detected ellipse. The numerical values are given in Table 1.

The first method to detect multiple inclusion is an iterative least square method of Cedio-Fenya *et al* [37]. Here we explain non-iterative direct methods.

Variational Method. This method is due to Ammari *et al* [21] based on the original idea of Calderón [33].

For arbitrary $\eta \in \mathbb{R}^d$, let $g_\eta(y) = i(\eta+i\eta^\perp)\cdot\nu_y e^{i(\eta+i\eta^\perp)\cdot y}$ on $\partial\Omega$ so that the corresponding background potential is given by $U(y) = e^{i(\eta+i\eta^\perp)\cdot y}$, where $\eta^\perp \in \mathbb{R}^d$ is orthogonal to η with

k_i	$a_0^i, a_1^i, a_2^i, b_0^i, b_1^i, b_2^i$	$\bar{k}$	$\bar{a}$	$\bar{b}$	$\bar{\theta}$	$\bar{z}$
		k	a	b	θ	z
100	5.5, 0.2, 0, 5.2, 0.2, 0	60.079	0.511	0.468	0.000	(4.838, 4.900)
100	5.5, 0.2, 0, 4.6, 0.2, 0	∞	0.502	0.461	0.000	(4.856, 4.899)
50	4.5, 0.4, 0, 4.9, 0.4, 0					
1.5	-7.4, 0.2, 0, -4, 0.2, 0	1.5	0.474	0.190	0.000	(-6.844, -4.000)
1.5	-6.4, 0.5, 0, -4, 0.1, 0	∞	0.146	0.123	0.000	(-6.875, -4.000)
100	0.1, 0.2, 0, 0, 0.2, 0					
100	-0.3, 0.2, 0, -0.4, 0.2, 0	3.88	0.511	0.315	0.785	(-0.236, -0.336)
1.5	-0.7, 0.2, 0, -0.8, 0.2, 0	∞	0.355	0.267	0.785	(-0.233, -0.333)
1.5	-1.1, 0.2, 0,-1.2, 0.2, 0					
5	2.9, 0.4, 0, -2.7, 0.1, 0	18.655	0.491	0.365	0.443	(2.494, -3.375)
100	2.5, 0.25, 0.2, -3.3, 0.25, 0.05	∞	0.458	0.351	0.443	(2.434, -3.321)
50	2.0, 0.2, 0, -4.0, 0.2, 0					
5	4.5, 0.15, 0.2, -3, 0.25, 0.05					
5	5.2, 0.1, 0, -3, 0.4, 0	5	0.507	0.419	-0.000	(5.502, -3.000)
5	5.8, 0.15, 0.2, -3, 0.25,0.05	∞	0.401	0.353	-0.000	(5.436, -3.000)
5	6.6, 0.2, 0, -3, 0.2, 0					
100	6.0, 0.25, 0.2, 4.6, 0.25, 0.05	100	0.549	0.331	-0.089	(5.728, 4.772)
100	5.5, 0.4, 0, 5.2, 0.1, 0	∞	0.540	0.329	-0.089	(5.712, 4.817)
100	5.2, 0.2, 0, 4.7, 0.2, 0					

Table 1: Table for Fig. 6. Here $\bar{k}, \bar{z}$ are the overall conductivity and center defined by (2.1) and (2.2). $\bar{a}$, $\bar{b}$, and $\bar{\theta}$ are semi axis lengths and the angle of orientation of the equivalent ellipse while a, b, and θ are those of detected ellipse assuming $k = \infty$. The point z is the detected center.

$|\eta| = |\eta^\perp|$. Using (3.10) and the definition of the Neumann function, we get

$$\begin{aligned} \mathcal{E}(\eta) \quad &:= \int_S \left[\Lambda_D(g)(y) - \Lambda_0(g)(x)\right] \frac{\partial}{\partial \nu} e^{i(\eta - i\eta^\perp)\cdot y}\, d\sigma(y) \\ &= \sum_{s=1}^m \epsilon_s^d (\eta + i\eta^\perp) \cdot M^s \cdot (\eta - i\eta^\perp) e^{2i\eta\cdot z_s} + O(\epsilon^{d+1}) \; . \end{aligned} \tag{3.14}$$

Recall that the function $e^{2i\eta\cdot z_s}$ (up to a multiplicative constant) is exactly the Fourier transform of the Dirac function δ_{-2z_s} (a point mass located at $-2z_s$). Multiplication by powers of η in the Fourier space corresponds to differentiation of the Dirac function. The function $\mathcal{E}(\eta)$ is therefore (approximately) the Fourier transform of a linear combination of derivatives of point masses, or

$$\check{\mathcal{E}} \simeq \sum_{s=1}^m \epsilon_s^d L_s \delta_{-2z_s} \; ,$$

where L_s is a second-order constant coefficient, differential operator whose coefficients depend on the polarization tensor of Pólya–Szegö M^s, and $\check{\mathcal{E}}$ represents the inverse Fourier transform of $\mathcal{E}(\eta)$. Thus by computing the inverse Fourier transform of $\mathcal{E}$ one can recover points z and the operator L_s from which $M^{(s)}$ can be computed. $\check{\mathcal{E}}$ can be computed using FFT with sample points determined by Shannon's sampling theorem.

Linear Sampling Method. We now describe the interesting approach proposed by Brühl *et al* in their recent paper [31]. This approach is related to the linear sampling method of Colton and Kirsch [43] (see also [67] and [30]) and allows one to reconstruct small inclusions by taking measurements only on some portion of $\partial\Omega$. It has also some similarities to a MUltiple Signal Classification (MUSIC)-type algorithm developed by Devaney [46, 73] for estimating the locations of a number of pointlike scatterers. We refer to Cheney [38] and Kirsch [68] for detailed discussions of the connection between MUSIC algorithm and the linear sampling method. Even if the method in [31] works with partial measurement, or partial NtD map, we describe the method with full NtD map for simplicity. The partial NtD map on $\partial\omega \subset \partial\Omega$ is defined to be $\Lambda_D(g) = u|_{\partial\omega}$ for g with support in $\partial\omega$.

Suppose that $\epsilon_s = \epsilon$ for all s. Then (3.10) reads

$$\Lambda_D(g) - \Lambda_0(g) = -\epsilon^d T(g) + O(\epsilon^{d+1}), \quad g \in L^2_0(\partial\Omega), \tag{3.15}$$

where

$$T(g)(x) = \sum_{s=1}^m \nabla U(z_s) M^{(s)} \nabla_z N(x, z_s), \quad x \in \partial\Omega. \tag{3.16}$$

The NtD maps Λ_D and Λ_0 are self-adjoint and by Theorem 1.10 $\Lambda_D - \Lambda_0$ is compact as a map on $L^2_0(\partial\Omega)$. It is also not hard to see that T is a self-adjoint compact operator on $L^2_0(\partial\Omega)$.

Introduce now the linear operator $\mathcal{G} : L^2_0(\partial\Omega) \to \mathbb{R}^{d\times m}$ defined by

$$\mathcal{G}g = (\nabla U(z_1), \dots, \nabla U(z_m)) \ .$$

Endowing $\mathbb{R}^{d\times m}$ with the standard Euclidean inner product,

$$\langle a, b\rangle = \sum_{s=1}^m a_s \cdot b_s \text{ for } a = (a_1, \dots, a_m), b = (b_1, \dots, b_m), a_s, b_s \in \mathbb{R}^d \ ,$$

we then obtain

$$\langle \mathcal{G}g, a\rangle = \sum_{s=1}^m a_s \cdot \nabla U(z_s) = \int_{\partial\Omega} \bigg(\sum_{s=1}^m a_s \cdot \nabla_z N(x, z_s) \bigg) g(x)\, d\sigma(x) \ ,$$

for arbitrary $a = (a_1, \dots, a_m) \in \mathbb{R}^{d\times m}$. Therefore, the adjoint $\mathcal{G}^* : \mathbb{R}^{d\times m} \to L^2_0(\partial\omega)$ is given by

$$\mathcal{G}^* a = \sum_{s=1}^m a_s \cdot \nabla N(\cdot, z_s).$$

We also introduce

$$\mathcal{M}a = \bigg(M(k_1, B_1)a_1, \dots, M(k_m, B_m)a_m \bigg), \quad a = (a_1, \dots, a_m) \in \mathbb{R}^{d\times m} \ .$$

Then one can immediately see that T can be decomposed as

$$T = \mathcal{G}^* \mathcal{M} \mathcal{G}.$$

Note that

$$\text{Range}(T) = \left\{ \sum_{s=1}^{m} a_s \cdot \nabla_z N(\cdot, z_s) : \ a_s \in \mathbb{R}^d \right\},$$

since Range$(T) =$ Range$(\mathcal{G}^* \mathcal{M} \mathcal{G}) =$ Range$(\mathcal{G}^*)$ and $\mathcal{M}$ and $\mathcal{G}$ are surjective.

Theorem 3.2 *Let $v \in \mathbb{R}^d \setminus \{0\}$. A point $z \in \Omega$ belongs to the set $\{z_s : s = 1, \ldots, m\}$ if and only if $v \cdot \nabla_z N(\cdot, z)\big|_{\partial\omega} \in$ Range (T).*

Proof. It suffices to consider the if part. $v \cdot \nabla_z N(\cdot, z)\big|_{\partial\omega} \in$ Range (T) means that there are $a_s \in \mathbb{R}^d$, $s = 1, \ldots, d$, such that

$$\sum_{s=1}^{m} a_s \cdot \nabla_z N(x, z_s) = v \cdot \nabla_z N(x, z) \quad x \in \partial\Omega,$$

which implies that

$$\sum_{s=1}^{m} a_s \cdot \nabla_z U(z_s) = v \cdot \nabla_z U(z)$$

for any harmonic function U in Ω. Since harmonic functions separate points, z must be one of z_s. □

Since T is self-adjoint, injective, and of rank dm, it is an orthogonal projection, and hence Theorem 3.2 is equivalent to the following statement:

$$z \in \{z_s : s = 1, \ldots, m\} \quad \text{iff} \quad \cot\theta(z) = +\infty ,$$

where the angle $\theta(z) \in [0, \pi/2)$ is defined by

$$\cot\theta(z) = \frac{||T(v \cdot \nabla_z N(\cdot, z)|_{\partial\Omega})||_{L^2(\partial\Omega)}}{||(I - T)(v \cdot \nabla_z N(\cdot, z)|_{\partial\Omega})||_{L^2(\partial\Omega)}} .$$

In view of (3.15) we can use $\Lambda_D - \Lambda_0$ instead of T in computing $\cot\theta(z)$. This method has been nicely implemented in [31].

We now present another method using finitely many measurements assuming that all the inclusions are of disk shape. Note that above mentioned methods work for more general situation, but they use infinite measurements.

Simple Pole Method. This method is due to Kang and Lee [60]. Suppose that the inclusion is of the form $D = \cup_{s=1}^m D_s$ and each D_s is a disk, or $D_s = \epsilon_s B$ where B is a unit disk. In this case the polarization tensor M associated with B is given by

$$M = \pi \frac{2(k_s - 1)}{k_s + 1} I,$$

where I is the identity matrix. Therefore, in this case we get from (3.12)

$$\Phi[g](x) = \sum_{s=1}^{m} \frac{\epsilon_s^2 (k_s - 1)}{k_s + 1} \frac{\nabla U(z_s) \cdot (x - z_s)}{|x - z_s|^2} + O(\epsilon^3), \quad x \in \mathbb{R}^2 \setminus \overline{\Omega} \tag{3.17}$$

Thus, we get get

$$\Phi[\nu_1](x) + i\Phi[\nu_2](x) = \sum_{s=1}^{m} \frac{\beta_s}{z - \alpha_s} + O(\epsilon^3), \quad z \in \mathbb{C} \setminus \overline{\Omega}, \tag{3.18}$$

where $\beta_s := \frac{\epsilon_s^2(k_s-1)}{k_s+1}$ and $\alpha_s = (z_s)_1 + i(z_s)_2$. Therefore our inclusion detection problem is reduced to the problem of finding the simple poles α_s and the residues β_s from the knowledge of a meromorphic function $f(z) = \sum_{s=1}^m \frac{\beta_s}{z-\alpha_s}$ on a circle $|z| = R$ enclosing all the poles.

It turns out that there is a nice way to locate the simple poles. By the Cauchy integral formula, we have

$$\frac{1}{2\pi i} \int_{|z|=R} z^n f(z) dz = \sum_{j=1}^{m} \beta_j \alpha_j^n.$$

The method of identification of simple poles is based on the following simple observations.

Lemma 3.3 *Suppose that the sequence $\{c_n\}$ takes the form $c_n = \sum_{j=1}^k \beta_j \alpha_j^n$, $n = 0, 1, \ldots$. If $l_1, \ldots, l_k$ satisfies the generating equation*

$$c_{n+k} + l_1 c_{n+k-1} + \cdots + l_k c_n = 0, \quad n = 0, 1, \ldots, k-1 \,, \tag{3.19}$$

then $\alpha_1, \alpha_2, \cdots, \alpha_k$ are solutions of

$$z^k + l_1 z^{k-1} + \cdots + l_k = 0. \tag{3.20}$$

The converse is also true. Furthermore, if (3.19) holds, then it holds for all n.

Lemma 3.4 *Let c_n be as in the Lemma 3.3 and let*

$$D_n = \det \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-2} \end{pmatrix}. \tag{3.21}$$

Then

$$D_n = \begin{cases} \beta_1 \beta_2 \cdots \beta_k \prod_{i<j} (\alpha_i - \alpha_j)^2 & \text{if } n = k, \\ 0 & \text{if } n > k. \end{cases} \tag{3.22}$$

Lemma 3.3 and Lemma 3.4 suggest that if we know the number m of the simple poles, then we first solve the system of equations (3.19) to find $l_1, \ldots, l_m$. We then solve the equation (3.20) to find the poles $\alpha_1, \ldots, \alpha_m$. Once we find $\alpha_1, \ldots, \alpha_m$, it is a simple matter to find $\beta_1, \ldots, \beta_k$. The most serious difficulty in finding poles comes from the fact that we do not know the number of poles aforehand. In order to determine the number, we use the formula (3.22). That is, we start with the bound N of the number and compute the determinants in (3.22) for $n = N, N-1, \ldots$ until it becomes nonzero. The first number n where the determinant is nonzero is the number of simple poles we seek for. Of course, these computations are performed within a given tolerance. A result of numerical test in a simple situation is given in [60].

We note that an EIT problem of reconstructing multiple thin inclusions can be reduced to a problem of finding simple poles for which the method of this paper may be applied. See a recent paper by Ammari *et al* [3].

3.4 Further results and remarks

There have been many works on asymptotic expansions of the perturbation due to the presence of small well-separated inclusions and to reconstruct them from boundary measurements in various contexts: for conductivity [2, 7, 13, 21, 22, 31, 34, 35, 37, 51, 60, 72], for anisotropic conductivity [59], for Helmholtz equations [5, 6, 12], for Maxwell equations [9, 23, 24, 94, 95], for thin inclusions [3, 4, 26, 27, 28], for linear elasticity [1, 18, 58]. Recently the layer potential method has been used to derive asymptotic expansion of the voltage potential due to the presence of internal corrosion [19].

4 Lecture 4
Effective properties and estimates of stress

4.1 Effective properties of composites

Let us begin by defining the concept of effective property in a rigorous way. Let $Y =]-1/2, 1/2[^2$ denote the unit cell and $D = \rho\Omega$, $\rho < 1$, where Ω is a reference domain containing 0 whose volume, $|\Omega|$, is 1. Let γ be a conductivity periodic with the periodic cell Y, and on Y we set

$$\gamma = \sigma_0 \chi(Y \setminus D) + \sigma \chi(D).$$

In previous lectures we set $\sigma_0 = 1$ and $\sigma = k$ for convenience. For a small parameter ϵ, $\gamma_\epsilon(x) := \gamma(\frac{x}{\epsilon})$ makes a highly oscillating conductivity and represent the material property of the composite. Figure 7 shows a geometry of a composite.

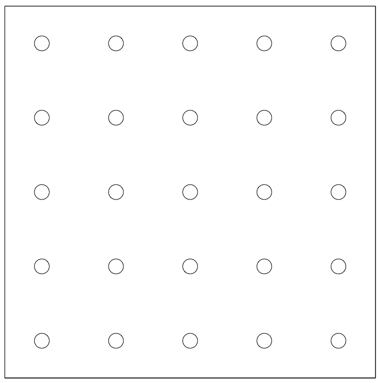


Figure 7: Periodic composite material. Inclusion $D = \rho^{1+\beta}B$, distance between inclusions ϵ, the volume fraction $f = \rho^2$. Conductivity of inclusion$= \sigma$, conductivity of background$= \sigma_0$.

Let Ω be a bounded domain in $\mathbb{R}^d$ and consider a conductivity equation

$$\nabla \cdot \gamma_\epsilon \nabla u_\epsilon = 0 \quad \text{in } \Omega$$

with an appropriate boundary condition. It is known that the solution u_ϵ converges weakly to a function u_0 in $W^{1,2}(\Omega)$ and there exists a conductivity, γ^*, such that

$$\nabla \cdot \gamma^* \nabla u_0 = 0 \quad \text{in } \Omega.$$

This γ^* is called an effective conductivity or a homogeneized coefficient. It represents overall macroscopic material property of the periodic composite material. See [57]. For a more intuitive approach, see [83]. See also [40] for applications to the structural optimization.

In general effective conductivities can not be computed exactly except for a few laminates. We consider the problem to determine the effective property of the composite when the volume fraction $f = |D|$ goes to zero. In other words, the inclusions has much less volume than the matrix. This kind of material is called a dilute material.

The determination of the effective or macroscopic property of a two-phase medium has been one of the classical problems in physics. When the inclusion D is a disk or sphere, then the effective electrical conductivity, γ^*, of the composite medium is given by the well-known Maxwell-Garnett formula[1] [91]

$$\widetilde{\sigma} = \sigma_0 \Big[1 + f \frac{d(\sigma - \sigma_0)}{(\sigma - \sigma_0) + d\sigma_0} + df^2 \frac{(\sigma - \sigma_0)^2}{((\sigma - \sigma_0) + d\sigma_0)^2} \Big] I + o(f^2), \tag{4.1}$$

where f is the volume fraction of the inclusions, *i.e.*, $f = |D|$.

This formula has been generalized in many directions: To include higher power terms of f for spherical inclusions, [56, 91]; To include other shape of the inclusion such as ellipses, [96, 48, 52, 79, 47, 57, 53, 84, 34]; To include the case when $f = O(1)$, see [83] and the references therein.

4.2 Computation of effective conductivity

We now explain a scheme way to derive asymptotic expansion of the effective property of the dilute composite material. This result is from [20].

The effective conductivity matrix $\gamma^* = (\gamma^*_{ij})_{i,j=1,2}$ of Ω is defined by (see for instance [57, 83])

$$\gamma^*_{ij} := \int_Y (\sigma_0 \chi(Y \setminus D) + \sigma \chi(D)) \nabla u_i \cdot \nabla u_j,$$

where u_i, for $i = 1, 2$, is the unique solution to

$$\begin{cases} \nabla \cdot (\sigma_0 \chi(Y \setminus D) + \sigma \chi(D)) \nabla u_i = 0 \quad \text{in } Y, \\ u_i - y_i \text{ periodic}, \\ \displaystyle\int_Y u_i = 0. \end{cases} \tag{4.2}$$

Using Green's formula we can rewrite γ^* in the following form:

$$\gamma^*_{ij} = \sigma_0 \int_{\partial Y} u_j \frac{\partial u_i}{\partial \nu}. \tag{4.3}$$

The matrix γ^* depends on ρ as a parameter, and cannot be written explicitly.

According to Theorem 1.14, the solution to (4.2) can be written as

$$u_i(x) = x_i + C_i + \mathcal{G}_D (\lambda I - \mathcal{B}^*_D)^{-1}(\nu_i)(x) \quad \text{in } Y, \quad i = 1, 2.$$

For simplicity, we set for $i = 1, 2$

$$\varphi_i(y) = (\lambda I - \mathcal{B}^*_D)^{-1}(\nu_i)(y) \quad \text{for } y \in \partial D. \tag{4.4}$$

Thus we get from (4.3)

$$\gamma^*_{ij} = \sigma_0 \int_{\partial Y} (y_j + C + \mathcal{G}_D \varphi_j(y)) \frac{\partial}{\partial \nu} (y_i + \mathcal{G}_D \varphi_i(y)) ds.$$

Because of periodicity of $\mathcal{G}_D \varphi_j$, we get

$$\int_{\partial Y} \frac{\partial}{\partial \nu} \mathcal{G}_D \varphi_j ds = \int_{\partial Y} \nu_j \mathcal{G}_D \varphi_i ds = \int_{\partial Y} \mathcal{G}_D \varphi_j(y) \frac{\partial}{\partial \nu} \mathcal{G}_D \varphi_i(y) ds = 0,$$

[1]To this formula several different pairs of names are attached. For this see [83]

and hence we have

$$\gamma_{ij}^* = \sigma_0\Big[\delta_{ij} + \int_{\partial Y} y_j \frac{\partial}{\partial \nu}\mathcal{G}_D\varphi_i(y)\,ds(y)\Big]. \tag{4.5}$$

Periodicity of $\mathcal{G}_D\varphi_i$ and the divergence theorem applied on $Y \setminus \overline{D}$ yield

$$\begin{aligned}
\int_{\partial Y} y_j \frac{\partial}{\partial \nu}\mathcal{G}_D\varphi_i(y)\,ds &= \int_{\partial D} y_j \frac{\partial}{\partial \nu}\mathcal{G}_D\varphi_i\big|_+(y)\,ds - \int_{\partial D} \nu_j \mathcal{G}_D\varphi_i(y)\,ds \\
&= \int_{\partial D} y_j\varphi_i(y)\,ds + \int_{\partial D} y_j \frac{\partial}{\partial \nu}\mathcal{G}_D\varphi_i\big|_-(y)\,ds - \int_{\partial D} \nu_j \mathcal{G}_D\varphi_i(y)\,ds \\
&= \int_{\partial D} y_j\varphi_i(y)\,ds.
\end{aligned}$$

Let

$$\psi_i(y) = \varphi_i(\rho y) \quad \text{for } y \in \partial B.$$

Then by (4.5), we obtain

$$\widetilde{\sigma} = \sigma_0[I + fP], \tag{4.6}$$

where $f = \rho^2$ is the volume fraction of D and $P := (P_{ij})$ is given by

$$p_{ij} := \int_{\partial B} y_j \psi_i(y)\,ds(y), \quad i,j = 1,2. \tag{4.7}$$

In order to derive an asymptotic expansion of γ^* we now expand P in terms of ρ. In view of (1.43), the integral equation (4.4) can be rewritten as

$$(\lambda I - \mathcal{K}_D^*)\varphi_i(x) - \int_{\partial D} \frac{\partial}{\partial \nu(x)} R(x-y)\varphi_i(y)\,ds(y) = \nu_i(x), \quad x \in \partial D,$$

which, by an obvious change of variables, yields

$$(\lambda I - \mathcal{K}_B^*)\psi_i(x) - \rho\int_{\partial B} \frac{\partial}{\partial \nu(x)} R(\rho(x-y))\psi_i(y)\,ds(y) = \nu_i(x), \quad x \in \partial B. \tag{4.8}$$

Because of (1.44) we get

$$\nu \cdot \nabla R(\rho(x-y)) = -\frac{\rho}{2}\nu\cdot(x-y) + O(\rho^{3\beta})$$

uniformly in $x, y \in \partial B$. Since $\int_{\partial B} \psi_i(y)\,ds(y) = 0$, we now have

$$(\lambda I - \mathcal{K}_B^*)\psi_i(x) - \frac{\rho^2}{2}\nu(x)\cdot\int_{\partial B} y\psi_i(y)\,ds(y) + O(\rho^4) = \nu_i(x), \quad x \in \partial B.$$

Therefore, we obtain

$$\psi_i = (\lambda I - \mathcal{K}_B^*)^{-1}(\nu_i) + \frac{\rho^2}{2}\sum_{k=1}^{2}(\lambda I - \mathcal{K}_B^*)^{-1}(\nu_k)\cdot\int_{\partial B} y_k\psi_i(y)\,ds(y) + O(\rho^4). \tag{4.9}$$

Let $\widetilde{\psi}_i := (\lambda I - \widetilde{\mathcal{K}}_B^*)^{-1}(\nu_i)$, $i = 1,2$. Then $M_{ij} = \int_{\partial B} y_j\widetilde{\psi}_i(y)ds(y)$, etc. By iterating the formula (4.9), we get

$$\psi_i = \widetilde{\psi}_i + \frac{\rho^2}{2}\sum_{k=1}^{2}\widetilde{\psi}_k\int_{\partial B} y_k\widetilde{\psi}_i(y)\,ds(y) + O(\rho^4) \quad \text{on } \partial B.$$

It then follows from the definition (4.7) of P that

$$P_{ij} = M_{ij} + \frac{\rho^2}{2}\sum_{k=1}^{2} M_{kj}M_{ik} + O(\rho^4), \tag{4.10}$$

and then we obtain from (4.6) the following theorem.

Theorem 4.1 *We have*

$$\gamma^* = \sigma_0\Big[I + fM(I - \frac{f}{2}M)^{-1}] + O(f^3), \tag{4.11}$$

where M is the Pólya-Szegö polarization tensor associated to the scaled inclusion B and the conductivity $\sigma = \sigma_0(2\lambda+1)/(2\lambda-1)$.

In the case of spherical inclusions the Pólya-Szegö polarization tensor M is known exactly:

$$M = mI, \quad m = \frac{2(\sigma-\sigma_0)}{\sigma+\sigma_0}|B|, \tag{4.12}$$

and therefore, (4.11) yields the well-known Maxwell-Garnett formula (4.1):

$$\widetilde{\sigma} = \sigma_0\Big[1 + f\frac{2(\sigma-\sigma_0)}{\sigma+\sigma_0} + 2f^2\frac{(\sigma-\sigma_0)^2}{(\sigma+\sigma_0)^2}\Big]I + o(f^2).$$

The method presented above basically enables us to derive all the higher order terms of the asymptotic expansion of the effective conductivity. For example, by the (higher-order) Taylor expansion

$$R(x) = R(0) - \frac{1}{4}(x_1^2 + x_2^2) + R_4(x) + O(|x|^6), \tag{4.13}$$

we get

$$\nu\cdot\nabla R(\rho(x-y)) = -\frac{\rho}{2}\nu\cdot(x-y) + \rho^{3\beta}\nu\cdot\nabla_x R_4(x-y) + O(\rho^{5\beta}) \tag{4.14}$$

uniformly in $x, y \in \partial B$. Write

$$R_4(x-y) = \sum_{|l|+|l'|=4} c_{l,l'}x^l y^{l'}.$$

Then for each fixed l', $\sum_l c_{l,l'}x^l$ is harmonic since

$$\sum_l c_{l,l'}x^l = \frac{1}{l'!}\partial_y^{l'}(R_4(x-y))\big|_{y=0}.$$

By the same way one can derive the following theorem:

Theorem 4.2 *The effective conductivity γ^* has an asymptotic expansion as $\rho \to 0$:*

$$\gamma^* = \sigma_0\Big[I + fM(I - \frac{f}{2}M)^{-1} + f^3A\Big] + O(f^4), \tag{4.15}$$

where M is the (first order) polarization tensor and A is given by

$$A_{ij} = \sum_{\substack{|l|+|l'|=4\\ |l|>0,|l'|>0}} c_{l,l'}M_{l,j}M_{i,l'}, \quad i,j = 1,2. \tag{4.16}$$

4.3 Estimates of the stress

In this section we consider

$$\begin{cases} \nabla \cdot \Big(\chi(\Omega \setminus \bigcup_{s=1}^{2} \overline{B_s}) + \sum_{s=1}^{2} k_s \chi(B_s) \Big) \nabla u = 0 \quad \text{in } \Omega \, , \\ \dfrac{\partial u}{\partial \nu} \Big|_{\partial \Omega} = g \, , \\ \displaystyle\int_{\partial \Omega} u(x) \, d\sigma(x) = 0 \, . \end{cases} \tag{4.17}$$

So the composite materials Ω contains two inclusions B_1 and B_2 with conductivities k_1 and k_2, respectively. The question we consider is whether ∇u can be arbitrarily large as two inclusions get closer. We are especially interested in the case of extreme conductivities $k_s = \infty$ (a perfect conductor) or $k_s = 0$ (an insulator). This question arises in the study of composite media. Frequently in composites, the inclusions are very closely spaced and may even touch, see [25].

If $0 < C_1 < k_s < C_2 < +\infty$ for some constants C_1 and C_2, it has been shown by Bonnetier and Vogelius in [29] that the gradient of u remains bounded for circular touching inclusions of comparable radii. Li and Vogelius derived in [75] a uniform upper bound on the gradient of u in much more general setting. On the other hand, for two identical perfectly conducting circular inclusions (with $k_1 = k_2 = +\infty$) which are δ apart, it has been shown in [32] (see also [80] and [65]) that the gradient generally becomes unbounded as the distance δ approaches zero. The rate at which this gradient becomes unbounded has actually been calculated in [32], for a special solution. For this special solution, the rate turns out to be $\delta^{-1/2}$.

Here we derive very precise estimates for $\|\nabla u\|_{L^\infty(\Omega)}$ for arbitrary conductivities when inclusions are two dimensional disks.

According to Theorem 1.9, the solution u to (4.17) can be represented as

$$u(x) = H(x) + \mathcal{S}_{B_1} \varphi^{(1)}(x) + \mathcal{S}_{B_2} \varphi^{(2)}(x), \quad x \in \Omega, \tag{4.18}$$

where H is the harmonic function in Ω as defined in (1.26) and $\varphi^{(l)} \in L^2_0(\partial D_l)$, $l = 1, 2$, are the unique solution to the integral equations

$$(\lambda_l I - \mathcal{K}^*_{B_l}) \varphi^{(l)} - \frac{\partial (\mathcal{S}_{B_s} \varphi^{(s)})}{\partial \nu^{(l)}} \Big|_{\partial D_l} = \frac{\partial H}{\partial \nu^{(l)}} \Big|_{\partial D_l} \quad \text{on } \partial D_l \, , \quad l = 1, 2, \quad s \neq l, \tag{4.19}$$

where

$$\lambda_l = \frac{k_l + 1}{2(k_l - 1)} \, .$$

Since B_l are disks, $\mathcal{K}^*_{B_l} \varphi^{(l)} = 0$ by (1.11), and hence we get

$$\lambda_l \varphi^{(l)} - \frac{\partial (\mathcal{S}_{B_s} \varphi^{(s)})}{\partial \nu^{(l)}} \Big|_{\partial D_l} = \frac{\partial H}{\partial \nu^{(l)}} \Big|_{\partial D_l} \quad \text{on } \partial D_l \, , \quad l = 1, 2, \quad s \neq l, \tag{4.20}$$

Let $B_i = B(z_i, r_i)$ be the disk with center z_i and radius r_i for $i = 1, 2$. We assume that the disks B_1 and B_2 are separated: $\overline{B_1} \cap \overline{B_2} = \emptyset$. Let R_i, $i = 1, 2$, be the reflection with respect to the disk B_i, *i.e.*,

$$R_i(x) := \frac{r_i^2 (x - z_i)}{|x - z_i|^2} + z_i.$$

We also define the reflection of a function f by

$$(R_i f)(x) = f(R_i(x)), \quad x \in \mathbb{R}^2, \quad i = 1, 2.$$

The following lemma has been proved in [13].

Lemma 4.3 *We have*

$$\begin{aligned} \varphi_1 &= \frac{1}{\lambda_1} \sum_{m=0}^{\infty} \frac{1}{(4\lambda_1\lambda_2)^m} \frac{\partial}{\partial \nu^{(1)}} \left[(R_2 R_1)^m (I - \frac{1}{2\lambda_2} R_2) H \right] \Big|_{\partial B_1}, \\ \varphi_2 &= \frac{1}{\lambda_2} \sum_{m=0}^{\infty} \frac{1}{(4\lambda_1\lambda_2)^m} \frac{\partial}{\partial \nu^{(2)}} \left[(R_1 R_2)^m (I - \frac{1}{2\lambda_1} R_1) H \right] \Big|_{\partial B_2}, \end{aligned} \tag{4.21}$$

where the series in (4.21) converge absolutely and uniformly.

Precise estimates of the explicit formula (4.21) lead us to the following theorem.

Theorem 4.4 *There exists a constant C depending only on H such that*

$$||\nabla u||_{L^\infty(\Omega)} \geq \frac{C}{4\lambda_1\lambda_2 - 1 + \sqrt{\frac{\delta}{r_*}}}\ , \tag{4.22}$$

where $\delta = dist(B_1, B_2)$ and $r_* := \inf\{r_1, r_2\}$.

If $k_1 = k_2 = 0$ or $k_1 = k_2 = \infty$, then $\lambda_1\lambda_2 = 1/4$, and hence

$$||\nabla u||_{L^\infty(\Omega)} \geq C\sqrt{\frac{r_*}{\delta}}\ ,$$

so it blows up at the rate of $\delta^{-1/2}$. We can also see that if the radius of the disks are of order δ, then the gradient does not blow up, which should be the case as our asymptotic formula says.

4.4 Further results

The asymptotic expansion formula has been generalized to include anisotropic materials [15] and elastic composite [17].

Li and Nirenberg proved that the stress for the linear elasticity stays bounded if the Lamé parameters are bounded [74]. It would be very interesting to characterize the blow-up rate of the stress when the inclusions are either hard ones or holes, i.e., when the Young modulus is either ∞ or zero.

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