OPTIMIZATION ALGORITHM FOR RECONSTRUCTING INTERFACE CHANGES OF A CONDUCTIVITY INCLUSION FROM MODAL MEASUREMENTS

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Abstract. In this paper, we propose an original and promising optimization approach for reconstructing interface changes of a conductivity inclusion from measurements of eigenvalues and eigenfunctions associated with the transmission problem for the Laplacian. Based on a rigorous asymptotic analysis, we derive an asymptotic formula for the perturbations in the modal measurements that are due to small changes in the interface of the inclusion. Using fine gradient estimates, we carefully estimate the error term in this asymptotic formula. We then provide a key dual identity which naturally yields to the formulation of the proposed optimization problem. The viability of our reconstruction approach is documented by a variety of numerical results. The resolution limit of our algorithm is also highlighted.

1. Introduction

Let Ω be a smooth domain and D be an inclusion contained in Ω whose boundary is also assumed to be smooth. Shape deformation of D causes a perturbation of modal parameters. The aim of this paper is to show how this information can be used to reconstruct the unknown deformation. For doing so, we rigorously derive an asymptotic formula for the perturbations in the eigenvalues of the transmission problem for the Laplacian that are due to small deformations of the interface of an inclusion. Based on this formula, we design an efficient reconstruction algorithm from modal measurements. Our algorithm consists on minimizing a functional whose minimizer yields certain geometric properties of the unknown inclusion. It naturally follows from a key identity that is in some sense dual to the asymptotic formula. Numerical experiments showing the viability of our algorithm are presented.

Our asymptotic formula for the perturbations in the eigenvalues due to changes of the shape of the inclusion D that is inside a background domain Ω is in connection with the more classical ones established under variation of the background domain Ω. There have been several interesting works on the eigenvalue perturbation problem under variation of the domain since the seminal formula of Hadamard [13]. See for example the works by Garabedian and Schiffer [11], Kato [14], Sanchez Hubert and Sanchez Palencia [21], and Kozlov [15]. Convergence results for the eigenvalues and eigenvectors under boundary variations have been also proved by abstract methods as for example those from [9, 22, 23].

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Our results in this paper extend those established in the context of small volume inclusions as well as those for the conductivity interface problem. In fact, on one hand, in a series of recent papers [8, 6, 4, 5] we have derived high-order asymptotic expansions of the eigenvalue perturbations due to the presence of small inclusions and used them for locating the inclusions and identifying some of their geometric features. On the other hand, in [7], we have derived high-order terms in the asymptotic expansions of the boundary perturbations of steady-state voltage potentials resulting from small perturbations of the shape of a conductivity inclusion. Based on these derivations, we have designed an effective algorithm to determine some geometric features of the shape perturbation of the inclusion based on boundary measurements.

In this paper, the asymptotic formula for the perturbations in the modal measurements that are due to small changes in the interface of an inclusion is original. Fine gradient estimates are used for its derivation. Indeed, careful estimates of the error term in this formula are provided and a systematic way for deriving the dual identity that yields to the optimization problem is presented. The case of multiple eigenvalues is rigorously handled as well.

The paper is organized as follows. In the next section we derive an asymptotic formula for the eigenvalue perturbations due to shape deformation. We provide in section 3 a functional whose minimizer yields the interface of the inclusion. For doing so, we provide a key dual identity which naturally yields to the formulation of the proposed optimization problem. In section 4, we consider the case of a multiple eigenvalue. In section 5, we perform numerical experiments to test the viability of the algorithm. Many applications of our results in this paper are expected, especially in structural vibration testing [20].

2. Asymptotic Formula

Throughout this paper, let $\mathcal{C}^{k,\alpha}$ denote the Hölder space which consists of functions having derivatives up to order k and such that the kth derivative is Hölder continuous with exponent α, where $0 < \alpha \leq 1$. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\mathcal{C}^{2,1}$ boundary and let D be an open subset of Ω such that $\mathrm{dist}(\partial\Omega, \partial D) \geq d_0 > 0$. The boundary ∂D of D is also assumed to be $\mathcal{C}^{2,1}$. Suppose that the conductivity (or the dielectric constant) of the background is γ_e while that of the inclusion D is γ_i. So the conductivity profile denoted by γ_D is given by

$$\gamma_D = \gamma_e \chi_{\Omega\setminus D} + \gamma_i \chi_D, \tag{1}$$

where χ_D is the characteristic function of D. Let (u_0, ω_0^2) be a solution of the following eigenvalue problem:

$$\begin{cases} \nabla \cdot (\gamma_D \nabla u_0) = -\omega_0^2 u_0 \quad \text{in} \quad \Omega, \\ \gamma_D \dfrac{\partial u_0}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \\ \displaystyle\int_\Omega u_0^2 = 1. \end{cases} \tag{2}$$

Let

$$u_0^e = u_0|_{\Omega\setminus D} \quad \text{and} \quad u_0^i = u_0|_{\overline{D}}. \tag{3}$$

Then along the interface ∂D, the following transmission conditions hold:

$$\begin{cases} u_0^i = u_0^e \\ \gamma_i \dfrac{\partial u_0^i}{\partial \nu} = \gamma_e \dfrac{\partial u_0^e}{\partial \nu} \end{cases} \quad \text{on } \partial D, \tag{4}$$

where $\frac{\partial}{\partial \nu}$ denotes the normal derivative with respect to the outward unit normal to ∂D. The first condition in (4) represents the continuity of the potential while the second one does that of the flux. We emphasize that since u_0^i and u_0^e are $\mathcal{C}^{1,\alpha}$ for some $0 < \alpha < 1$, as will be proven in the next section, these conditions hold in the pointwise sense. From the first condition, we also have

$$\frac{\partial u_0^i}{\partial \tau} = \frac{\partial u_0^e}{\partial \tau} \quad (= \frac{\partial u_0}{\partial \tau}), \tag{5}$$

where $\frac{\partial}{\partial \tau}$ denotes the tangential derivative along ∂D.

In this section as well as in the next one, we will assume that ω_0^2 is a simple eigenvalue. This will make our arguments more readable. In section 4 we will state our result in the case of multiple eigenvalues.

Now let us consider D_ε an ε-perturbation of the domain D with

$$\partial D_\varepsilon = \left\{ \tilde{x} : \tilde{x} = x + \varepsilon h(x)\nu(x), x \in \partial D \right\},$$

where $\nu(x)$ is the unit outer normal vector to ∂D at x, $h \in \mathcal{C}^{1,1}(\partial D)$ with $\|h\|_{\mathcal{C}^{1,1}} \leq H$ for some positive constant H, and ε is a positive small parameter.

Let $\gamma_{D_\varepsilon} = \gamma_e \chi_{\Omega \setminus D_\varepsilon} + \gamma_i \chi_{D_\varepsilon}$ and consider the following eigenvalue problem on the perturbed domain:

$$\begin{cases} \nabla \cdot (\gamma_{D_\varepsilon} \nabla u_\varepsilon) = -\omega_\varepsilon^2 u_\varepsilon \quad \text{in} \quad \Omega, \\ \gamma_{D_\varepsilon} \dfrac{\partial u_\varepsilon}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \\ \displaystyle\int_\Omega u_\varepsilon^2 = 1. \end{cases} \tag{6}$$

Our main result in this section is the following theorem.

Theorem 2.1. *Let ω_0^2 be a simple eigenvalue of (2). Then, as $\varepsilon \to 0$, the eigenvalue ω_ε^2 of (6) has the following asymptotic expansion:*
(7)

$$\omega_\varepsilon^2 - \omega_0^2 = -\varepsilon(\gamma_i - \gamma_e) \int_{\partial D} h(x) \left(\left(\frac{\partial u_0^e}{\partial \tau}(x) \right)^2 + \frac{\gamma_e}{\gamma_i} \left(\frac{\partial u_0^e}{\partial \nu}(x) \right)^2 \right) d\sigma_x + O(\varepsilon^{1+\beta})$$

for some $\beta > 0$, where (u_0, ω_0^2) is the solution to (2).

It is worth noticing that if h has a constant sign on ∂D than there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, $\omega_\varepsilon^2 - \omega_0^2$ has the same sign as $(\gamma_e - \gamma_i)h$.

We will prove Theorem 2.1 using Osborn's result in [18] concerning estimates for the eigenvalues of a sequence of self-adjoint compact operators. More precisely, let X be a real Hilbert space and let $T : X \to X$ and $T_\varepsilon : X \to X$ be compact, self-adjoint linear operators such that $\{T_\varepsilon\}_{\varepsilon > 0}$ are collectively compact and $T_\varepsilon \to T$ pointwise as $\varepsilon \to 0$. Let μ_0 be a nonzero eigenvalue of T with multiplicity m. Then, for ε small, T_ε has a set of m eigenvalues (counted according to their multiplicity) such that $\mu_\varepsilon^j \to \mu_0$ for each $j = 1, \ldots, m$, as $\varepsilon \to 0$. Let $\bar{\mu}_\varepsilon = \frac{1}{m} \sum_{j=1}^m \mu_\varepsilon^j$. If

$\{u_{0,1}, u_{0,2}, \ldots, u_{0,m}\}$ is an orthonormal basis for $\mathrm{Ker}(T-\mu_0 I)$, then there exists a constant C (independent of ε) such that

$$
\left|\mu_0 - \bar{\mu}_\varepsilon - \frac{1}{m}\sum_{j=1}^{m} < (T-T_\varepsilon)u_{0,j}, u_{0,j} > \right| \leq C \left\| (T-T_\varepsilon)_{|\mathrm{Ker}(T-\mu_0 I)} \right\|^2_{X\to X}, \tag{8}
$$

where the right hand side of (8) denotes the operator norm of $T - T_\varepsilon$ on the subspace $\mathrm{Ker}(T-\mu_0 I) \subset X$. Moreover, for each $j = 1, \ldots, m$, there is an eigenfunction $u_{\varepsilon,j}$ corresponding to μ_ε^j, such that $\|u_{\varepsilon,j}\|_X = 1$, and

$$
\|u_{\varepsilon,j} - u_{0,j}\|_X \leq C \left\| (T-T_\varepsilon)_{|\mathrm{Ker}(T-\mu_0 I)} \right\|_{X\to X}. \tag{9}
$$

If μ_0 is a simple eigenvalue, for ε small, there is a simple eigenvalue μ_ε for T_ε such that

$$
\left|\mu_0 - \mu_\varepsilon - < (T-T_\varepsilon)u_0, u_0 > \right| \leq C \left\| (T-T_\varepsilon)(u_0) \right\|^2_X. \tag{10}
$$

Furthermore, let u_ε be the eigenfunction corresponding to μ_ε and such that $\|u_\varepsilon\|_X = 1$, then

$$
\|u_\varepsilon - u_0\|_X \leq C \|(T-T_\varepsilon)(u_0)\|_X.
$$

Let us consider $X = \{f \in L^2(\Omega) : \int_\Omega f = 0\}$ with the usual inner product of $L^2(\Omega)$ and $T : X \to X$ the linear operator given by $Tf = v_0$ where v_0 is the solution to

$$
\begin{cases}
\nabla \cdot (\gamma_D \nabla v_0) = f & \text{in} \quad \Omega, \\
\gamma_D \dfrac{\partial v_0}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \\
\displaystyle\int_\Omega v_0 = 0.
\end{cases} \tag{11}
$$

We define $T_\varepsilon : X \to X$ similarly, *i.e.*, by $T_\varepsilon f = v_\varepsilon$, where v_ε is the solution to

$$
\begin{cases}
\nabla \cdot (\gamma_{D_\varepsilon} \nabla v_\varepsilon) = f & \text{in} \quad \Omega, \\
\gamma_{D_\varepsilon} \dfrac{\partial v_\varepsilon}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \\
\displaystyle\int_\Omega v_\varepsilon = 0.
\end{cases} \tag{12}
$$

Then T and T_ε are compact self-adjoint operators. We now prove that $\{T_\varepsilon\}_{\varepsilon\geq 0}$, ($T_0 = T$) are collectively compact and that $T_\varepsilon \to T$ pointwise as $\varepsilon \to 0$.

(i) $\{T_\varepsilon\}_{\varepsilon\geq 0}$ are collectively compact, *i.e.* $\{T_\varepsilon f : \|f\|_X \leq 1, \varepsilon \geq 0\}$ is sequentially compact: If $v \in \{T_\varepsilon f : \|f\|_X \leq 1, \varepsilon \geq 0\}$, then from energy estimates and the Poincaré inequality we have that $\|v\|_{H^1(\Omega)} \leq C$ where C is independent of ε. Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, this guarantees that $\{T_\varepsilon f : \|f\|_X \leq 1, \varepsilon \geq 0\}$ is sequentially compact in $L^2(\Omega)$.

(ii) $T_\varepsilon \to T$ pointwise as $\varepsilon \to 0$ in X: For $f \in X$, let $v_\varepsilon = T_\varepsilon f$ and $v_0 = Tf$. Then, for any $w \in H^1(\Omega)$,

$$
\int_\Omega \gamma_{D_\varepsilon} \nabla v_\varepsilon \cdot \nabla w = -\int_\Omega f w
$$

and

$$\int_\Omega \gamma_D \nabla v_0 \cdot \nabla w = -\int_\Omega f w.$$

Hence, choosing $w = v_\varepsilon - v_0$ and subtracting these two equations we get

$$\int_\Omega (\gamma_{D_\varepsilon} \nabla v_\varepsilon - \gamma_D \nabla v_0) \cdot \nabla (v_\varepsilon - v_0) = 0,$$

which gives

$$\int_\Omega \gamma_{D_\varepsilon} \nabla (v_\varepsilon - v_0) \cdot \nabla (v_\varepsilon - v_0) = -\int_\Omega (\gamma_{D_\varepsilon} - \gamma_D) \nabla v_0 \cdot \nabla (v_\varepsilon - v_0).$$

Hence

$$\|\nabla (v_\varepsilon - v_0)\|^2_{L^2(\Omega)} \leq C \|\nabla v_0\|_{L^2(D_\varepsilon \triangle D)} \|\nabla (v_\varepsilon - v_0)\|_{L^2(\Omega)},$$

where $\triangle$ denotes the symmetric difference, which implies

$$\|\nabla (v_\varepsilon - v_0)\|_{L^2(\Omega)} \leq C \|\nabla v_0\|_{L^2(D_\varepsilon \triangle D)}.$$

It then follows by the Poincaré inequality that

$$\|v_\varepsilon - v_0\|_{H^1(\Omega)} \leq C \|\nabla v_0\|_{L^2(D_\varepsilon \triangle D)}. \tag{13}$$

Finally, using the last inequality and the fact that $|D_\varepsilon \triangle D| \to 0$ as $\varepsilon \to 0$ and that $\nabla v_0 \in L^2(\Omega)$ we obtain that $T_\varepsilon \to T$ pointwise as $\varepsilon \to 0$ in $L^2(\Omega)$.

We can now apply Osborn's result to conclude that, for small ε, there is an eigenvalue μ_ε of T_ε such that $\mu_\varepsilon \to \mu_0$ and

$$\Big| \mu_0 - \mu_\varepsilon - < (T_\varepsilon - T) u_0, u_0 > \Big| \leq C \|(T_\varepsilon - T) u_0\|^2_{L^2(\Omega)}, \tag{14}$$

where u_0 is such that $T u_0 = \mu_0 u_0$ and $\int_\Omega u_0^2 = 1$. (Note that the compatibility condition $\int_\Omega u_0 = 0$ is also satisfied). Moreover

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C \|(T_\varepsilon - T) u_0\|_{L^2(\Omega)}, \tag{15}$$

where u_ε is the eigenfunction corresponding to μ_ε such that $\int_\Omega u_\varepsilon^2 = 1$. The eigenfunctions u_0 and u_ε solve respectively the problems

$$\begin{cases} \nabla \cdot (\gamma_D \nabla u_0) = \dfrac{u_0}{\mu_0} & \text{in} \quad \Omega, \\ \gamma_D \dfrac{\partial u_0}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \\ \displaystyle\int_\Omega u_0^2 = 1, \end{cases} \tag{16}$$

and

$$\begin{cases} \nabla \cdot (\gamma_{D_\varepsilon} \nabla u_\varepsilon) = \dfrac{u_\varepsilon}{\mu_\varepsilon} & \text{in} \quad \Omega, \\ \gamma_{D_\varepsilon} \dfrac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \\ \displaystyle\int_\Omega u_\varepsilon^2 = 1. \end{cases} \tag{17}$$

Let us now consider some regularity facts on the functions u_ε and u_0. From [17], it follows that, for some $\alpha \in (0,1)$, $u_0 \in \mathcal{C}^{1,\alpha}(\bar{D}) \cap \mathcal{C}^{1,\alpha}(\Omega \backslash \bar{D})$ and

$$\|u_0\|_{\mathcal{C}^{1,\alpha}(\bar{D})} \leq C (\|u_0\|_{L^2(\Omega)} + \|u_0\|_{L^\infty(\Omega)}), \tag{18}$$

where $C = C(\gamma_e, \gamma_i, \Omega, D, \mu_0, \alpha)$ and analogously, since $\text{dist}(\partial\Omega, \partial D) \geq d_0 > 0$ then

$$\|u_0\|_{\mathcal{C}^{1,\alpha}(\Omega_{d_0/2}\setminus D)} \leq C(\|u_0\|_{L^2(\Omega)} + \|u_0\|_{L^\infty(\Omega)}), \tag{19}$$

where $C = C(\gamma_e, \gamma_i, \Omega, D, \mu_0, d_0, \alpha)$ and $\Omega_{d_0/2} = \{x \in \Omega : \text{dist}(x, \partial\Omega) > d_0/2\}$. Note that the right-hand sides in (18) and (19) are bounded by $C\|u\|_{L^\infty(\Omega)}$. Recalling that u_0 is solution of a homogeneous Neumann problem we have, by global estimates for weak solutions of elliptic equations in divergence form with bounded coefficients (De Giorgi-Nash method, cf. [12, Theorem 8.24]), that

$$\|u_0\|_{L^\infty(\Omega)} \leq C\|u_0\|_{L^2(\Omega)}. \tag{20}$$

It then follows from (18) and (19) that

$$\|u_0\|_{\mathcal{C}^{1,\alpha}(\bar{D})} \leq C\|u_0\|_{L^2(\Omega)},$$

where $C = C(\gamma_e, \gamma_i, \Omega, D, \mu_0)$ and

$$\|u_0\|_{\mathcal{C}^{1,\alpha}(\Omega_{d_0/2}\setminus D)} \leq C\|u_0\|_{L^2(\Omega)},$$

where $C = C(\gamma_e, \gamma_i, \Omega, D, \mu_0, d_0)$. Since $\|u_0\|_{L^2(\Omega)} = 1$, we have that

$$\|u_0\|_{\mathcal{C}^{1,\alpha}(\bar{D})} \leq C, \tag{21}$$

and

$$\|u_0\|_{\mathcal{C}^{1,\alpha}(\Omega_{d_0/2}\setminus D)} \leq C. \tag{22}$$

Similarly we get for u_ε

$$\|u_\varepsilon\|_{\mathcal{C}^{1,\alpha}(\bar{D}_\varepsilon)} \leq C, \tag{23}$$

where $C = C(\gamma_e, \gamma_i, \Omega, D, \mu_0, H, \alpha)$ and

$$\|u_\varepsilon\|_{\mathcal{C}^{1,\alpha}(\Omega_{d_0/2}\setminus D_\varepsilon)} \leq C, \tag{24}$$

where here $C = C(\gamma_e, \gamma_i, \Omega, D, \mu_0, d_0, H, \alpha)$. It is worth emphasizing that the constant C in all the four above estimates is independent of ε.

Let us now evaluate the right-hand side of inequality (15). We know that $Tu_0 = \mu_0 u_0$ and $T_\varepsilon u_0 = \tilde{v}_\varepsilon$, where $\tilde{v}_\varepsilon$ is the solution to

$$\begin{cases} \nabla \cdot (\gamma_{D_\varepsilon} \nabla \tilde{v}_\varepsilon) = u_0 & \text{in} \quad \Omega, \\ \gamma_{D_\varepsilon} \dfrac{\partial \tilde{v}_\varepsilon}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \\ \displaystyle\int_\Omega \tilde{v}_\varepsilon = 0. \end{cases} \tag{25}$$

Since u_0 satisfies (16), we may use exactly the same argument as the one for deriving (13) to show that

$$\|\tilde{v}_\varepsilon - \mu_0 u_0\|_{H^1(\Omega)} \leq C\|\nabla u_0\|_{L^2(D_\varepsilon \triangle D)}.$$

It then follows from (21) and (22) that

$$\|\tilde{v}_\varepsilon - \mu_0 u_0\|_{H^1(\Omega)} \leq C|D_\varepsilon \triangle D|^{1/2}.$$

Observe that $\tilde{v}_\varepsilon - \mu_0 u_0$ is a solution to

$$\begin{cases} \nabla \cdot (\gamma_{D_\varepsilon} \nabla(\tilde{v}_\varepsilon - \mu_0 u_0)) = \mu_0 \nabla \cdot ((\gamma_D - \gamma_{D_\varepsilon}) \nabla u_0) \quad \text{in} \quad \Omega, \\ \gamma_{D_\varepsilon} \dfrac{\partial(\tilde{v}_\varepsilon - \mu_0 u_0)}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \\ \displaystyle\int_\Omega (\tilde{v}_\varepsilon - \mu_0 u_0) = 0. \end{cases}$$

Applying Lemma A.1 in [2] for $V = \{v \in H^1(\Omega) : \int_\Omega v = 0\}, \phi = \tilde{v}_\varepsilon - \mu_0 u_0$ and $F = \mu_0(\gamma_D - \gamma_{D_\varepsilon})\nabla u_0$ we then get

$$\|\tilde{v}_\varepsilon - \mu_0 u_0\|_{L^2(\Omega)} \leq C|D_\varepsilon \triangle D|^{1/2+\eta},$$

for some $\eta > 0$ where $C = C(\gamma_e, \gamma_i, \mu_0, \Omega, H, \eta)$ but is otherwise independent of ε. Hence we have

$$\|(T_\varepsilon - T)u_0\|_{L^2(\Omega)} \leq C\varepsilon^{1/2+\eta} \tag{26}$$

and from (15)

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon^{1/2+\eta}. \tag{27}$$

Furthermore, observing that $\|\tilde{v}_\varepsilon\|_{H^1(\Omega)} \leq \|u_0\|_{L^2(\Omega)}$ and using the gradient estimates of [17], we obtain that

$$\|\tilde{v}_\varepsilon\|_{\mathcal{C}^{1,\alpha}(\bar{D}_\varepsilon)} \leq C, \tag{28}$$

and analogously

$$\|\tilde{v}_\varepsilon\|_{\mathcal{C}^{1,\alpha}(\Omega_{d_0/2} \setminus D_\varepsilon)} \leq C, \tag{29}$$

where $C = C(\gamma_e, \gamma_i, \Omega, D, \mu_0, d_0, H, \alpha)$.

As in (3), let us put

$$\tilde{v}_\varepsilon^e := \tilde{v}_\varepsilon|_{\Omega \setminus D_\varepsilon} \quad \text{and} \quad \tilde{v}_\varepsilon^i := \tilde{v}_\varepsilon|_{\overline{D}_\varepsilon}.$$

The following lemma holds.

Lemma 2.2. *Let $\alpha > 0$ be the same Hölder exponent as in (29). There exists a constant C independent of ε such that*

$$\|\nabla(\tilde{v}_\varepsilon^e - \mu_0 u_0^e)\|_{L^\infty(\partial D_\varepsilon \setminus D)} + \|\nabla(\tilde{v}_\varepsilon^i - \mu_0 u_0^i)\|_{L^\infty(\partial D_\varepsilon \cap \overline{D})} \leq C\varepsilon^{\frac{\alpha}{2\alpha+2}}, \tag{30}$$

where $C = C(\gamma_e, \gamma_i, \Omega, D, \mu_0, d_0, H, \alpha)$.

Proof. Let $2\varepsilon < d < d_0/2$ and let $\Omega_d^\varepsilon = \{x \in \Omega \setminus (D \cup D_\varepsilon) : \text{dist}(x, \partial(\Omega \setminus D \cup D_\varepsilon)) > d\}$. Since $\nabla(\tilde{v}_\varepsilon - \mu_0 u_0)$ is harmonic in $\Omega \setminus D \cup D_\varepsilon$ we may apply the mean value theorem to points $y \in \Omega_d^\varepsilon$:

$$\nabla(\tilde{v}_\varepsilon^e - \mu_0 u_0^e)(y) = \frac{1}{|B_{d/2}|} \int_{B_{d/2}(y)} \nabla(\tilde{v}_\varepsilon - \mu_0 u_0) dx$$

to get

$$\begin{aligned} \|\nabla(\tilde{v}_\varepsilon^e - \mu_0 u_0^e)\|_{L^\infty(\Omega_d^\varepsilon)} &\leq Cd^{-1}\|\nabla(\tilde{v}_\varepsilon - \mu_0 u_0)\|_{L^2(\Omega)} \\ &\leq Cd^{-1}\varepsilon^{1/2}. \end{aligned} \tag{31}$$

Now, let $y \in \partial D_\varepsilon \setminus D$ and let y_d denote the closest point to y in the set Ω_d^ε. By (29) we obtain

$$|\nabla \tilde{v}_\varepsilon^e(y) - \nabla \tilde{v}_\varepsilon^e(y_d)| \leq Cd^\alpha.$$

Combining (31) and (32) gives

$$\begin{aligned}|\nabla(\tilde v^e_\varepsilon-\mu_0u^e_0)(y)| &\le |\nabla\tilde v^e_\varepsilon(y)-\nabla\tilde v^e_\varepsilon(y_d)|+|\nabla\tilde v^e_\varepsilon(y_d)-\nabla(\mu_0u^e_0)(y_d)|\\ &\quad+|\nabla\mu_0u^e_0(y_d)-\nabla\mu_0u^e_0(y)|\\ &\le C(d^\alpha+d^{-1}\varepsilon^{1/2}).\end{aligned}$$

Here we also used the gradient estimates for u_0. By choosing $d=\varepsilon^{\frac{1}{2(\alpha+1)}}$ we get

$$\|\nabla(\tilde v^e_\varepsilon-\mu_0u^e_0)\|_{L^\infty(\partial D_\varepsilon\setminus D)}\le C\varepsilon^{\frac{\alpha}{2\alpha+2}}.$$

In a similar way one can prove that

$$\|\nabla(\tilde v^i_\varepsilon-\mu_0u^i_0)\|_{L^\infty(\partial D_\varepsilon\cap D)}\le C\varepsilon^{\frac{\alpha}{2\alpha+2}}$$

to complete the proof of the lemma. □

We are now ready to compute the term $<(T-T_\varepsilon)u_0,u_0>$ in (14). We proceed with

$$\begin{aligned}<(T-T_\varepsilon)u_0,u_0> &= <\mu_0u_0-\tilde v_\varepsilon,u_0>\\ &= \mu_0\int_\Omega u_0^2-\int_\Omega u_0\tilde v_\varepsilon\\ &= -\mu_0\int_\Omega(\gamma_{D_\varepsilon}-\gamma_D)\nabla\tilde v_\varepsilon\cdot\nabla u_0\\ &= -\mu_0\int_{D_\varepsilon\setminus D}(\gamma_i-\gamma_e)\nabla\tilde v^i_\varepsilon\cdot\nabla u^e_0+\mu_0\int_{D\setminus D_\varepsilon}(\gamma_i-\gamma_e)\nabla\tilde v^e_\varepsilon\cdot\nabla u^i_0.\end{aligned}$$

Let $x_t:=x+th(x)\nu(x)$ for $x\in\partial D$ and $t\in[0,\varepsilon]$. Then the Jacobian determinant of the change of variables $(x,t)\in\partial D\times[0,\varepsilon]\mapsto x_t\in D_\varepsilon\triangle D$ is $|h(x)|+O(\varepsilon)$ for ε small enough, and hence we get

$$\begin{aligned}&-\mu_0\int_{D_\varepsilon\setminus D}(\gamma_i-\gamma_e)\nabla\tilde v^i_\varepsilon\cdot\nabla u^e_0dx\\ &=-\mu_0\int_0^\varepsilon\int_{\partial D\cap\{h>0\}}h(x)(\gamma_i-\gamma_e)\nabla\tilde v^i_\varepsilon(x_t)\cdot\nabla u^e_0(x_t)\,d\sigma_x\,dt+O(\varepsilon^2),\end{aligned}\tag{32}$$

and

$$\begin{aligned}&\mu_0\int_{D\setminus D_\varepsilon}(\gamma_i-\gamma_e)\nabla\tilde v^e_\varepsilon\cdot\nabla u^i_0\,dx\\ &=-\mu_0\int_0^\varepsilon\int_{\partial D\cap\{h<0\}}h(x)(\gamma_i-\gamma_e)\nabla\tilde v^e_\varepsilon(x_t)\cdot\nabla u^i_0(x_t)\,d\sigma_x\,dt+O(\varepsilon^2).\end{aligned}\tag{33}$$

Using the gradient estimates (21), (22), (28), and (29), we have for $t\in[0,\varepsilon]$

$$\begin{aligned}\nabla\tilde v^i_\varepsilon(x_t)\cdot\nabla u^e_0(x_t) &= \nabla\tilde v^i_\varepsilon(x_\varepsilon)\cdot\nabla u^e_0(x_\varepsilon)+O(\varepsilon^\alpha)\\ &= \frac{\partial\tilde v^i_\varepsilon}{\partial\tau}(x_\varepsilon)\frac{\partial u^e_0}{\partial\tau}(x_\varepsilon)+\frac{\partial\tilde v^i_\varepsilon}{\partial\nu}(x_\varepsilon)\frac{\partial u^e_0}{\partial\nu}(x_\varepsilon)+O(\varepsilon^\alpha)\\ &= \frac{\partial\tilde v^e_\varepsilon}{\partial\tau}(x_\varepsilon)\frac{\partial u^e_0}{\partial\tau}(x_\varepsilon)+\frac{\gamma_e}{\gamma_i}\frac{\partial\tilde v^e_\varepsilon}{\partial\nu}(x_\varepsilon)\frac{\partial u^e_0}{\partial\nu}(x_\varepsilon)+O(\varepsilon^\alpha).\end{aligned}$$

Here we used the transmission conditions (4) and (5). We then use (22) and (30) to have

$$\begin{aligned}\nabla\tilde{v}^i_\varepsilon(x_t)\cdot\nabla u^e_0(x_t) &= \mu_0\frac{\partial u^e_0}{\partial\tau}(x_\varepsilon)\frac{\partial u^e_0}{\partial\tau}(x_\varepsilon)+\mu_0\frac{\gamma_e}{\gamma_i}\frac{\partial u^e_0}{\partial\nu}(x_\varepsilon)\frac{\partial u^e_0}{\partial\nu}(x_\varepsilon)+O(\varepsilon^{\frac{\alpha}{2\alpha+2}})\\ &= \mu_0\frac{\partial u^e_0}{\partial\tau}(x)\frac{\partial u^e_0}{\partial\tau}(x)+\mu_0\frac{\gamma_e}{\gamma_i}\frac{\partial u^e_0}{\partial\nu}(x)\frac{\partial u^e_0}{\partial\nu}(x)+O(\varepsilon^{\frac{\alpha}{2\alpha+2}}).\end{aligned}$$

It then follows from (32) that

$$\begin{aligned}&-\mu_0\int_{D_\varepsilon\setminus D}(\gamma_i-\gamma_e)\nabla\tilde{v}^i_\varepsilon\cdot\nabla u^e_0dx\\ &= -\mu_0^2\varepsilon(\gamma_i-\gamma_e)\int_{\partial D\cap\{h>0\}}h\left[\left(\frac{\partial u^e_0}{\partial\tau}\right)^2+\frac{\gamma_e}{\gamma_i}\left(\frac{\partial u^e_0}{\partial\nu}\right)^2\right]d\sigma+O(\varepsilon^{1+\beta}),\end{aligned}\tag{34}$$

where $\beta=\frac{\alpha}{2\alpha+2}$.

Analogously, we get from (33) that

$$\begin{aligned}&\mu_0\int_{D\setminus D_\varepsilon}(\gamma_i-\gamma_e)\nabla\tilde{v}^e_\varepsilon\cdot\nabla u^i_0dx\\ &= -\mu_0^2\varepsilon(\gamma_i-\gamma_e)\int_{\partial D\cap\{h<0\}}h\left[\left(\frac{\partial u^i_0}{\partial\tau}\right)^2+\frac{\gamma_i}{\gamma_e}\left(\frac{\partial u^i_0}{\partial\nu}\right)^2\right]d\sigma+O(\varepsilon^{1+\beta})\\ &= -\mu_0^2\varepsilon(\gamma_i-\gamma_e)\int_{\partial D\cap\{h<0\}}h\left[\left(\frac{\partial u^e_0}{\partial\tau}\right)^2+\frac{\gamma_e}{\gamma_i}\left(\frac{\partial u^e_0}{\partial\nu}\right)^2\right]d\sigma+O(\varepsilon^{1+\beta}),\end{aligned}\tag{35}$$

where the last equality follows from the transmission conditions (4) and (5).

It now follows from (34) and (35) that for ε small enough,

$$\begin{aligned}&<(T-T_\varepsilon)u_0,u_0>\\ &= -\mu_0^2\varepsilon(\gamma_i-\gamma_e)\int_{\partial D}h\left[\left(\frac{\partial u^e_0}{\partial\tau}\right)^2+\frac{\gamma_e}{\gamma_i}\left(\frac{\partial u^e_0}{\partial\nu}\right)^2\right]d\sigma+O(\varepsilon^{1+\beta}).\end{aligned}$$

In view of (10) and (26), we finally obtain

$$\mu_\varepsilon-\mu_0=-\mu_0^2\varepsilon(\gamma_i-\gamma_e)\int_{\partial D}h\left[\left(\frac{\partial u^e_0}{\partial\tau}\right)^2+\frac{\gamma_e}{\gamma_i}\left(\frac{\partial u^e_0}{\partial\nu}\right)^2\right]d\sigma+O(\varepsilon^{1+\beta})$$

for some $\beta>0$. Since μ_0,μ_ε are negative we can set $\mu_0^{-1}=-\omega_0^2$ and $\mu_\varepsilon^{-1}=-\omega_\varepsilon^2$. With this substitution and simple manipulations we get (7) and Theorem 2.1 is proved.

3. Reconstruction Method

In order to reconstruct the perturbation εh from modal measurements, a first idea is to minimize the difference between the measured and the computed eigenvalues by using a least-square approach. This yields a laborious reconstruction algorithm which may not converge if we start away from the solution. Another idea, which sounds more attractive is to take advantage of the smallness of ε and minimize over εh the quantity

$$\left|\omega_\varepsilon^2-\omega_0^2+(\gamma_i-\gamma_e)\int_{\partial D}\varepsilon h(x)\left[\left(\frac{\partial u^e_0}{\partial\tau}(x)\right)^2+\frac{\gamma_e}{\gamma_i}\left(\frac{\partial u^e_0}{\partial\nu}(x)\right)^2\right]d\sigma_x\right|.$$

A problem with this approach is that oscillations in h can not be determined effectively. This comes from the fact that the application

$$h(x) \mapsto \int_{\partial D} h(x) \left[\left(\frac{\partial u_0^e}{\partial \tau}(x) \right)^2 + \frac{\gamma_e}{\gamma_i} \left(\frac{\partial u_0^e}{\partial \nu}(x) \right)^2 \right] d\sigma_x \tag{36}$$

acts like a filter.

In this section, we rigorously establish a reconstruction formula for the function h which allows to determine h with better resolution by less filtering of oscillations. Based on this dual formula, we then formulate the reconstruction of h as an optimization problem.

3.1. **Dual Asymptotic Formula.** Let u_0 be the eigenfunction of (2). For $g \in L^2(\partial\Omega)$ satisfying $\int_{\partial\Omega} g u_0 = 0$, let w_g be the solution to

$$\begin{cases} \nabla \cdot (\gamma_D \nabla w_g) = -\omega_0^2 w_g & \text{in} \quad \Omega, \\ \gamma_D \dfrac{\partial w_g}{\partial \nu} = g & \text{on} \quad \partial\Omega, \\ \displaystyle\int_\Omega w_g u_0 = 1. \end{cases} \tag{37}$$

Multiplying the first equation in (37) by u_ε and integrating over Ω, we get from the divergence theorem

$$\int_{\partial\Omega} g u_\varepsilon + \omega_0^2 \int_\Omega w_g u_\varepsilon = \int_\Omega \gamma_D \nabla u_\varepsilon \cdot \nabla w_g.$$

Since $\int_{\partial\Omega} g u_0 = 0$ and

$$\omega_\varepsilon^2 \int_\Omega w_g u_\varepsilon = \int_\Omega \gamma_{D_\varepsilon} \nabla u_\varepsilon \cdot \nabla w_g,$$

we obtain

$$\int_{\partial\Omega} g(u_\varepsilon - u_0) + (\omega_0^2 - \omega_\varepsilon^2) \int_\Omega w_g u_\varepsilon = -\int_\Omega (\gamma_{D_\varepsilon} - \gamma_D) \nabla u_\varepsilon \cdot \nabla w_g dx.$$

By Theorem 2.1 we have that for ε small enough, $\omega_0^2 - \omega_\varepsilon^2 = O(\varepsilon)$. Furthermore, since $u_\varepsilon \to u_0$ in $L^2(\Omega)$ as $\varepsilon \to 0$, we derive

$$\int_{\partial\Omega} g(u_\varepsilon - u_0) + (\omega_0^2 - \omega_\varepsilon^2) \int_\Omega w_g u_0 = -\int_\Omega (\gamma_{D_\varepsilon} - \gamma_D) \nabla u_\varepsilon \cdot \nabla w_g dx + O(\varepsilon^{1+\beta}), \tag{38}$$

for some $\beta > 0$.

We now prove the following theorem in the same way as in the previous section. The asymptotic formula in the theorem can be regarded as a dual formula to that of $\omega_\varepsilon^2 - \omega_0^2$. It plays a key role in our reconstruction procedure.

Theorem 3.1. *The following asymptotic formula holds as $\varepsilon \to 0$:*

$$\begin{aligned} &\int_{\partial\Omega} g(u_\varepsilon - u_0) + (\omega_0^2 - \omega_\varepsilon^2) \int_\Omega w_g u_0 \\ &= \varepsilon(\gamma_i - \gamma_e) \int_{\partial D} h(x) \left(\frac{\partial u_0^e}{\partial \tau}(x) \frac{\partial w_g^e}{\partial \tau}(x) + \frac{\gamma_e}{\gamma_i} \frac{\partial u_0^e}{\partial \nu}(x) \frac{\partial w_g^e}{\partial \nu}(x) \right) d\sigma_x + O(\varepsilon^{1+\beta}) \end{aligned}$$

for some $\beta > 0$.

Proof. In view of (38), it suffices to show that

$$\begin{aligned}(39)\qquad &\int_\Omega (\gamma_{D_\varepsilon} - \gamma_D)\nabla u_\varepsilon \cdot \nabla w_g \\ &= -(\gamma_i - \gamma_e)\varepsilon \int_{\partial D} h(x) \left(\frac{\partial u_0^e}{\partial \tau}(x) \frac{\partial w_g^e}{\partial \tau}(x) + \frac{\gamma_e}{\gamma_i} \frac{\partial u_0^e}{\partial \nu}(x) \frac{\partial w_g^e}{\partial \nu}(x) \right) d\sigma_x \\ &\quad + O(\varepsilon^{1+\beta}).\end{aligned}$$

To prove the lemma we use the gradient estimates for u_ε (see (23) and (24)). We can show that the same kind of estimates hold for w_g:

$$(40)\qquad \|w_g\|_{\mathcal{C}^{1,\alpha}(\bar{D})} \leq C,$$

and

$$(41)\qquad \|w_g\|_{\mathcal{C}^{1,\alpha}(\Omega_{d_0/2}\setminus D)} \leq C.$$

These estimates follow immediately from [17] since on one hand, the operator $\nabla \cdot \gamma_D \nabla + \omega_0^2$ in Ω with Neumann boundary conditions on $\partial\Omega$ is well-posed on the subspace of $H^1(\Omega)$ orthogonal to u_0 and on the other hand, u_0 itself satisfies such estimates.

Proceeding similarly as we did for estimating $\int_{D_\varepsilon \setminus D} \nabla \tilde{v}_\varepsilon^i \cdot \nabla u_0^e dx$ in the previous section, we have

$$\int_\Omega (\gamma_{D_\varepsilon} - \gamma_D)\nabla u_\varepsilon \cdot \nabla w_g = \int_{D_\varepsilon \setminus D} (\gamma_i - \gamma_e)\nabla u_\varepsilon^i \cdot \nabla w_g^e - \int_{D \setminus D_\varepsilon} (\gamma_i - \gamma_e)\nabla u_\varepsilon^e \cdot \nabla w_g^i.$$

Changing variables and using the gradient estimates for u_ε and w_g we obtain

$$\int_{D_\varepsilon \setminus D} (\gamma_i - \gamma_e)\nabla u_\varepsilon^i \cdot \nabla w_g^e dx = \int_0^\varepsilon \int_{\partial D \cap \{h>0\}} h(x)(\gamma_i - \gamma_e)\nabla u_\varepsilon^i(x_\varepsilon) \cdot \nabla w_g^e(x_\varepsilon)\, d\sigma_x\, dt + O(\varepsilon^{1+\beta}),$$

and analogously

$$\int_{D \setminus D_\varepsilon} (\gamma_i - \gamma_e)\nabla u_\varepsilon^e \cdot \nabla w_g^i dx = \int_0^\varepsilon \int_{\partial D \cap \{h<0\}} (-h(x))(\gamma_i - \gamma_e)\nabla u_\varepsilon^e(x_\varepsilon) \cdot \nabla w_g^i(x_\varepsilon)\, d\sigma_x\, dt + O(\varepsilon^{1+\beta}).$$

Using the transmission conditions, we get

$$\begin{aligned}(42)\qquad &\int_{D_\varepsilon \setminus D} (\gamma_i - \gamma_e)\nabla u_\varepsilon^i \cdot \nabla w_g^e dx \\ &= \varepsilon \int_{\partial D \cap \{h>0\}} h(x)(\gamma_i - \gamma_e) \left(\frac{\partial u_\varepsilon^e}{\partial \tau}(x_\varepsilon) \frac{\partial w_g^e}{\partial \tau}(x_\varepsilon) + \frac{\gamma_e}{\gamma_i} \frac{\partial u_\varepsilon^e}{\partial \nu}(x_\varepsilon) \frac{\partial w_g^e}{\partial \nu}(x_\varepsilon) \right) d\sigma_x \\ &\quad + O(\varepsilon^{1+\beta})\end{aligned}$$

and

$$\begin{aligned}(43)\qquad &\int_{D \setminus D_\varepsilon} (\gamma_i - \gamma_e)\nabla u_\varepsilon^e \cdot \nabla w_g^i dx = \\ &= -\varepsilon \int_{\partial D \cap \{h<0\}} h(x)(\gamma_i - \gamma_e) \left(\frac{\partial u_\varepsilon^i}{\partial \tau}(x_\varepsilon) \frac{\partial w_g^i}{\partial \tau}(x_\varepsilon) + \frac{\gamma_i}{\gamma_e} \frac{\partial u_\varepsilon^i}{\partial \nu}(x_\varepsilon) \frac{\partial w_g^i}{\partial \nu}(x_\varepsilon) \right) d\sigma_x \\ &\quad + O(\varepsilon^{1+\beta}).\end{aligned}$$

In order to replace u_ε with u_0 in (42) and (43), we shall show that

$$(44)\qquad \|\nabla(u_\varepsilon^e - u_0^e)\|_{L^\infty(\partial D_\varepsilon \setminus D)} + \|\nabla(u_\varepsilon^i - u_0^i)\|_{L^\infty(\partial D_\varepsilon \cap \overline{D})} \leq C\varepsilon^{\frac{\alpha}{2\alpha+2}}$$

for some constant C independent of ε, following the same arguments as those in the proof of Lemma 2.2.

For doing so, let $2\varepsilon < d < d_0/2$ and let $\Omega_d^\varepsilon = \{x \in \Omega\backslash(D \cup D_\varepsilon) : \mathrm{dist}(x, \partial(\Omega\backslash D \cup D_\varepsilon) > d\}$. Since $\theta_\varepsilon = \nabla(u_\varepsilon - u_0)$ is solution of the following equation in $\Omega\backslash D \cup D_\varepsilon$

$$\Delta\theta_\varepsilon - \mu_\varepsilon\theta_\varepsilon = (\mu_\varepsilon - \mu_0)\nabla u_0,$$

we may apply Theorem 8.17 of [12] to obtain

$$\|\nabla(u_\varepsilon^e - u_0^e)\|_{L^\infty(\Omega_d^\varepsilon)} \le C(\|\nabla(u_\varepsilon - u_0)\|_{L^2(\Omega)} + |\mu_\varepsilon - \mu_0|\|\nabla u_0\|_{L^2(\Omega)}).$$

Using the energy estimates and the fact that $|\mu_\varepsilon - \mu_0| \le C\varepsilon$ we get

$$\|\nabla(u_\varepsilon^e - u_0^e)\|_{L^\infty(\Omega_d^\varepsilon)} \le \frac{C}{d}\sqrt{\varepsilon}.$$

Now, let $y \in \partial D_\varepsilon\backslash D$ and let y_d denote the closest point to y in the set Ω_d^ε. From (23) and (24) it follows that

$$|\nabla u_\varepsilon^e(y) - \nabla u_\varepsilon^e(y_d)| \le Cd^\alpha, \tag{45}$$

which yields

$$\begin{aligned} |\nabla(u_\varepsilon^e - u_0^e)(y)| &\le |\nabla u_\varepsilon^e(y) - \nabla u_\varepsilon^e(y_d)| + |\nabla u_\varepsilon^e(y_d) - \nabla u_0^e(y_d)| \\ &\quad + |\nabla u_0^e(y_d) - \nabla u_0^e(y)| \\ &\le C(d^\alpha + d^{-1}\varepsilon^{1/2}) \ . \end{aligned}$$

Here we also used the gradient estimates for u_0. By choosing $d = \varepsilon^{\frac{1}{2(\alpha+1)}}$ we get

$$\|\nabla(u_\varepsilon^e - u_0^e)\|_{L^\infty(\partial D_\varepsilon\backslash D)} \le C\varepsilon^{\frac{\alpha}{2\alpha+2}} \ .$$

In a similar way one can prove that

$$\|\nabla(u_\varepsilon^i - u_0^i)\|_{L^\infty(\partial D_\varepsilon\cap\overline{D})} \le C\varepsilon^{\frac{\alpha}{2\alpha+2}} \ .$$

Finally, inserting this into (42) and (43) and using the gradient estimates for u_0 we get

$$\begin{aligned} &\int_{D_\varepsilon\backslash D} (\gamma_i - \gamma_e)\nabla u_\varepsilon^i \cdot \nabla w_g^e dx \\ &= \varepsilon\int_{\partial D\cap\{h>0\}} h(x)(\gamma_i - \gamma_e)\left(\frac{\partial u_0^e}{\partial\tau}(x)\frac{\partial w_g^e}{\partial\tau}(x) + \frac{\gamma_e}{\gamma_i}\frac{\partial u_0^e}{\partial\nu}(x)\frac{\partial w_g^e}{\partial\nu}(x)\right)d\sigma_x + O(\varepsilon^{1+\beta}) \end{aligned}$$

and

$$\begin{aligned} &\int_{D\backslash D_\varepsilon} (\gamma_i - \gamma_e)\nabla u_\varepsilon^e \cdot \nabla w_g^i dx \\ &= -\varepsilon\int_{\partial D\cap\{h<0\}} h(x)(\gamma_i - \gamma_e)\left(\frac{\partial u_0^i}{\partial\tau}(x)\frac{\partial w_g^i}{\partial\tau}(x) + \frac{\gamma_i}{\gamma_e}\frac{\partial u_0^i}{\partial\nu}(x)\frac{\partial w_g^i}{\partial\nu}(x)\right)d\sigma_x + O(\varepsilon^{1+\beta}). \end{aligned}$$

Applying the transmission conditions (4) and (5) once more, summing up the two integrals and inserting the sum into (38) we obtain (39). This completes the proof.
□

3.2. **Optimization Problem.** In view of Theorem 3.1, the reconstruction method is rather apparent. With the measurements $(\omega_\varepsilon^2 - \omega_0^2, (u_\varepsilon - u_0)|_{\partial\Omega})$ and a finite number of linearly independent functions $g_1, \ldots, g_L$ on $\partial\Omega$ satisfying $\int_{\partial\Omega} g_l u_0 d\sigma = 0$, define the functional $J(h)$ by

$$J(h) := \sum_{l=1}^{L} \Bigg| \int_{\partial\Omega} g_l (u_\varepsilon - u_0) + (\omega_0^2 - \omega_\varepsilon^2) \int_\Omega w_{g_l} u_0 \\ - \varepsilon \int_{\partial D} h(x)(\gamma_i - \gamma_e) \left(\frac{\partial u_0^e}{\partial \tau}(x) \frac{\partial w_{g_l}^e}{\partial \tau}(x) + \frac{\gamma_e}{\gamma_i} \frac{\partial u_0^e}{\partial \nu}(x) \frac{\partial w_{g_l}^e}{\partial \nu}(x) \right) d\sigma_x \Bigg|^2 .$$

The method for reconstructing the shape deformation is to minimize $J(h)$ over h. If the small parameter ε is known, then by minimizing $J(h)$ over h we can reconstruct h. If ε is unknown, we may consider the functional J as a function of εh instead of h to obtain the deformation εh.

It is worth emphasizing that the integral

$$\varepsilon \int_{\partial D} h(x) \left(\frac{\partial u_0^e}{\partial \tau}(x) \frac{\partial w_g^e}{\partial \tau}(x) + \frac{\gamma_e}{\gamma_i} \frac{\partial u_0^e}{\partial \nu}(x) \frac{\partial w_g^e}{\partial \nu}(x) \right) d\sigma_x$$

filters less oscillations in h than the one in (36) because of the flexibility we have in choosing w_g and therefore, minimizing $J(h)$ allows better reconstruction of h.

The best choice of $g_1, \ldots, g_L$ is such that the functions

$$v_g = \frac{\partial u_0^e}{\partial \tau} \frac{\partial w_g^e}{\partial \tau} + \frac{\gamma_e}{\gamma_i} \frac{\partial u_0^e}{\partial \nu} \frac{\partial w_g^e}{\partial \nu} \quad \text{for } g = g_1, \ldots, g_L$$

on ∂D are highly oscillating. To formalize this, introduce the operator Λ defined for $g \in V(\partial\Omega) = \{g \in L^2(\partial\Omega) : \int_{\partial\Omega} g u_0 = 0\}$ by $\Lambda(g) = w_g|_{\partial\Omega}$, where w_g is the solution to (37). The best choice is then to take $\{g_1, \ldots, g_L\}$ as a basis of the image space of $\Lambda^*\Lambda$, where $\Lambda^* : L^2(\partial D) \to V(\partial\Omega)$ is the adjoint of Λ. Indeed, the only changes that we can reconstruct are linear combinations of $v_g|_{\partial D}$ for $g \in \text{Image}(\Lambda^*\Lambda)$. See [3].

Nevertheless, since computing $\text{Image}(\Lambda^*\Lambda)$ is costly, we just take in our numerical examples Ω and D to be disks and the functions $g_1, \ldots, g_L$ to be cosine and sine functions. See (49). Note that if there is an index i for which the ith Fourier coefficients of $v_{g_l}|_{\partial D}$, $l = 1, \ldots, L$ are all zero then the ith Fourier coefficient of h can not be reconstructed by our algorithm.

4. Case of a Multiple Eigenvalue

In this section we consider the case of a multiple eigenvalue. We first derive an averaged approximation formula for a multiple eigenvalue. This is based on a standard argument from [18, 14] that the mean of the cluster resulting from the eigenvalue splitting of converging eigenvalues better approximates the limit eigenvalue than any of the individual eigenvalues from the cluster. Then we provide a reconstruction formula for the interface changes that has the same general form as the one in the simple eigenvalue case.

Let ω_0^2 denote an eigenvalue of the problem for (2) with geometric multiplicity m and let $\{u_{0,j}\}_{j=1,\ldots,m}$ be L^2-orthonormal eigenfunctions corresponding to ω_0^2. Let $(\omega_\varepsilon^j)^2$ be the eigenvalues of problem (6) for $\varepsilon > 0$ that are generated by splitting from ω_0^2 and let u_ε^j be the associated eigenfunction (normalized with respect to L^2) such that $u_\varepsilon^j \to u_{0,j}$ as $\varepsilon \to 0$.

Then, by (8) and (9) and proceeding similarly as in the proof of Theorem 2.1 we get the following result.

Theorem 4.1. *As $\varepsilon \to 0$, the following asymptotic expansion holds:*

$$\begin{aligned}&\frac{1}{\omega_0^2} - \frac{1}{m}\sum_{j=1}^m \frac{1}{(\omega_\varepsilon^j)^2}\\ &= -\frac{\varepsilon(\gamma_i-\gamma_e)}{m\omega_0^4}\sum_{j=1}^m \int_{\partial D} h(x)\left((\frac{\partial u_{0,j}^e}{\partial \tau}(x))^2 + \frac{\gamma_e}{\gamma_i}(\frac{\partial u_{0,j}^e}{\partial \nu}(x))^2\right) d\sigma_x + O(\varepsilon^{1+\beta}),\end{aligned}$$

for some $\beta > 0$.

Using Theorem 4.1, we can adapt the algorithm described in the previous section to reconstruct the shape deformation in the case of a multiple eigenvalue. For $g \in L^2(\partial\Omega)$ satisfying $\int_{\partial\Omega} g u_{0,j} = 0$ for $j = 1,\dots,m$, let w_g be the solution to

$$\left\{\begin{aligned}\nabla\cdot(\gamma_D \nabla w_g) &= -\omega_0^2 w_g \quad \text{in } \Omega,\\ \gamma_D \frac{\partial w_g}{\partial \nu} &= g \quad \text{on } \partial\Omega,\\ \int_\Omega w_g u_{0,j} &= 1, \quad j=1,\dots,m.\end{aligned}\right. \tag{46}$$

The method for reconstructing the shape deformation in the case of a multiple eigenvalue ω_0^2 is to minimize the functional $J(h)$ over h where J is given by

$$\begin{aligned}J(h) := \sum_{l=1}^L \Bigg| &\frac{1}{m}\sum_{j=1}^m \int_{\partial\Omega} g_l(u_\varepsilon^j - u_{0,j}) - \frac{1}{m}\sum_{j=1}^m (\omega_\varepsilon^j)^2 + \omega_0^2\\ &- \frac{\varepsilon}{m}\sum_{j=1}^m \int_{\partial D} h(x)(\gamma_i-\gamma_e)\left(\frac{\partial u_{0,j}^e}{\partial \tau}(x)\frac{\partial w_{g_l}^e}{\partial \tau}(x) + \frac{\gamma_e}{\gamma_i}\frac{\partial u_{0,j}^e}{\partial \nu}(x)\frac{\partial w_{g_l}^e}{\partial \nu}(x)\right) d\sigma_x\Bigg|^2,\end{aligned}$$

where $g_1,\dots,g_L$ are linearly independent functions.

5. Numerical Examples

We now present numerical examples of the shape deformation reconstruction method described in the previous section. In the following examples, the background domain Ω is assumed to be the unit disk centered at the origin and the (unperturbed) inclusion D is the disk centered at $(0,-0.2)$ with radius 0.4. We fix the conductivities:

$$\gamma_e = 1 \quad \text{and} \quad \gamma_i = 1.5.$$

In order to acquire (simulated) data, we use a boundary integral method. Let $\mathcal{S}_D^\omega$ and $\mathcal{D}_D^\omega$ be the single and double layer potentials on ∂D defined by the fundamental (outgoing) solution $\Gamma_\omega(x) = -\frac{i}{4}H_0^{(1)}(\omega|x|)$ to the operator $\Delta + \omega^2$:

$$\begin{aligned}\mathcal{S}_D^\omega[\varphi](x) &= \int_{\partial D}\Gamma_\omega(x-y)\varphi(y)d\sigma(y),\ x\in\mathbb{R}^2,\\ \mathcal{D}_D^\omega[\varphi](x) &= \int_{\partial D}\frac{\partial \Gamma_\omega(x-y)}{\partial \nu_y}\varphi(y)d\sigma(y),\ x\in\mathbb{R}^2\setminus\partial D,\end{aligned}$$

for $\varphi \in L^2(\partial D)$. Here $H_0^{(1)}$ is the Hankel function of first kind and order 0.

We define analogously $\mathcal{S}_\Omega^\omega$ and $\mathcal{D}_\Omega^\omega$ to be the single and double layer potentials on $\partial\Omega$.

The solution u_0 to (2) can be represented as

$$u_0(x) = \begin{cases} \mathcal{S}_D^{\frac{\omega}{\sqrt{\gamma_i}}}[\varphi](x), & x \in D, \\ \mathcal{S}_D^{\omega}[\psi](x) + \mathcal{D}_\Omega^{\omega}[f](x) - \mathcal{S}_\Omega^{\omega}[g](x), & x \in \Omega \setminus \bar{D}, \end{cases} \tag{47}$$

where $f = u_0|_{\partial\Omega}$, $g = \frac{\partial u_0}{\partial \nu}|_{\partial\Omega} = 0$, and $(\varphi, \psi) \in L^2(\partial D) \times L^2(\partial D)$ is a solution to the following integral equation:

$$\begin{cases} \mathcal{S}_D^{\frac{\omega}{\sqrt{\gamma_i}}}[\varphi] = \mathcal{S}_D^{\omega}[\psi] + \mathcal{D}_\Omega^{\omega}[f] & \text{on } \partial D, \\ k\dfrac{\partial}{\partial \nu}(\mathcal{S}_D^{\frac{\omega}{\sqrt{\gamma_i}}}[\varphi])^i = \dfrac{\partial}{\partial \nu}(\mathcal{S}_D^{\omega}[\psi] + \mathcal{D}_\Omega^{\omega}[f])^e & \text{on } \partial D, \\ \mathcal{S}_D^{\omega}[\psi] + (\mathcal{D}_\Omega^{\omega}[f])^i = f & \text{on } \partial\Omega. \end{cases} \tag{48}$$

The computations of the eigenvalues are performed numerically. We discretize (48) to have a linear system, which is singular when ω is an eigenvalue, and obtain $(\varphi, \psi, f(= u_0|_{\partial\Omega}))$ which corresponds to the first and second eigenvalues. For the perturbed solution, we use the same method with D_ϵ instead of D. Both unperturbed and perturbed eigenvalues have multiplicity 2. For each of these eigenvalues, the interior value of corresponding eigenfunctions are calculated using (47). The computations of the eigenvalues and the eigenfunctions on the boundary $\partial\Omega$ are performed with an accuracy much higher than ϵ because otherwise this would affect dramatically the reconstruction algorithm and would make the reconstruction of h inaccurate.

Note that (for both the perturbed and the unperturbed problem) there are many almost zero eigenvalues of the discrete system corresponding to (48). But all the eigenfunctions except two are nearly zeros on $\partial\Omega$. The two eigenfunctions are obtained from the eigenfunctions which have non-zero values on $\partial\Omega$. Moreover, by a linear transformation, we can insure that the eigenfunctions associated with the perturbed eigenvalue converge to those associated with the unperturbed one.

In a similar way, the function w_{g_l} is calculated for

$$g_l = a_l + b_l \cos\theta + c_l \sin(l+1)\theta + d_l \cos(l+1)\theta, \ 1 \le l \le L\,(= 8), \tag{49}$$

where a_l, b_l, c_l, d_l are constants chosen for g_l to satisfy the normalization condition in (46). Problem (46) is solved using a boundary integral method with a minimal nodal point distance of order ϵ. This is to insure a quality of the solution to the auxiliary problem (46) enough to have an accurate reconstruction of h.

We simulate the reconstruction method for the perturbation function h given by

$$h(\theta) = 1 - 2\sin(j\theta), \ j = 0, 3, 6, 9, \ \text{and } \epsilon = 0.02, \ 0.04.$$

As j increases, oscillations in h become higher and the reconstruction problem becomes more and more difficult to solve.

In the reconstruction algorithm, h is approximated as follows:

$$h(\theta) \approx h_0 + \sum_{p=1}^{9}(h_{2p-1}\cos p\theta + h_{2p}\sin p\theta) =: \sum_{p=0}^{18} h_p \Phi_p(\theta),$$

where

$$\Phi_0(\theta) = 1, \Phi_{2p-1}(\theta) = \cos p\theta, \ \Phi_{2p}(\theta) = \sin p\theta, \quad p = 1, \dots, 9.$$

For $1 \leq j \leq 2$, $1 \leq l \leq 8$, and $0 \leq p \leq 18$, define a matrix M_s as

$$M_s(2(j-1)+l,p) := \varepsilon(k-1)\int_{\partial D}\Phi_p(x)\left(\frac{\partial u^e_{0,j}}{\partial \tau}(x)\frac{\partial w^e_{g_l}}{\partial \tau}(x)+\frac{1}{k}\frac{\partial u^e_{0,j}}{\partial \nu}(x)\frac{\partial w^e_{g_l}}{\partial \nu}(x)\right)d\sigma_x,$$

where $s = 1$ and $s = 2$ stands for the first and second eigenvalue, respectively. The measurement data vector B is

$$B_s(2(j-1)+l) = \int_{\partial\Omega} g_l(u^j_\varepsilon - u_{0,j})d\sigma_x - ((\omega^j_\epsilon)^2 - \omega_0^2).$$

For $s = 1, 2$, coefficients h^s_p are obtained by

$$(h^s_0, \ldots, h^s_{18}) = \left(M_s^T M_s + \delta I_{19}\right)^{-1} M_s^T B_s,$$

where I_{19} is the 19×19 identity matrix and the regularization parameter δ is one of the following numbers $1, 0.01, 0.001$. Now, stack two matrices M_1 and M_2 and data vectors B_1 and B_2, vertically, and compute the new coefficients h^3_p by

$$(h^3_0, \ldots, h^3_{18}) = \left(\text{Stack}(M_1, M_2)^T\text{Stack}(M_1, M_2) + \delta I_{19}\right)^{-1}\text{Stack}(M_1, M_2)^T\text{Stack}(B_1, B_2).$$

A similar formula could be derived if the eigenvalue is more than double.

Example 1. In this example, $h(\theta) = 1 - 2\sin(j\theta)$, $j = 0, 3, 6, 9$, and $\epsilon = 0.02$. Here and in the following examples, we assume that ε is known and reconstruct h. In Figure 1, h is approximated from the data corresponding to the first eigenvalue in the first column, and second eigenvalue in the second column, and first and second eigenvalues in third column. The regularization parameter δ is taken to be equal to 0.001 except for $j = 3$ and 6 on the third column. In these two exceptional cases, δ is chosen to be 0.01. Figure 1 shows that first eigenvalue measurements work very well for not highly oscillating perturbations h, but it is not enough for higher oscillatory perturbation. This clearly indicates the resolution limit of our algorithm and shows that it is function of the modal measurements we use. However, the quality of image is increased when second eigenvalue measurements are used as well.

Example 2. In this example, $h(\theta) = 1 - 2\sin(j\theta)$, $j = 0, 3, 6, 9$, and $\epsilon = 0.04$. Regularization parameter δ is 1 for all cases in this example. Reconstruction results are shown in Figure 2.

Example 3. The example in Figure 3 shows the reconstruction of an inclusion which is shifted 0.2 to the right. First eigenvalue measurements are used, and regularization parameter δ is 0.01. In this example, the obtained image is very close to the real one.

Example 4. The example in Figure 4 shows the reconstruction of an inclusion which is perturbed and shifted to the right, i.e., D is perturbed as $\epsilon h(\theta) = 0.02(1 - 2\sin 6\theta)$ and then shifted 0.2 to the right. Regularization parameter δ is 1. In this example we used the first and third eigenvalues.

6. Concluding Remarks

In this paper we have introduced an optimization procedure for reconstructing interface changes of an inclusion from modal measurements. Our procedure takes advantage of the smallness of the changes. It is based on the dual asymptotic formula in Theorem 3.1. We have presented numerical experiments that show that our reconstruction procedure from eigenvalue measurements works pretty well for reconstructing perturbations of the interface. We have also pointed out the

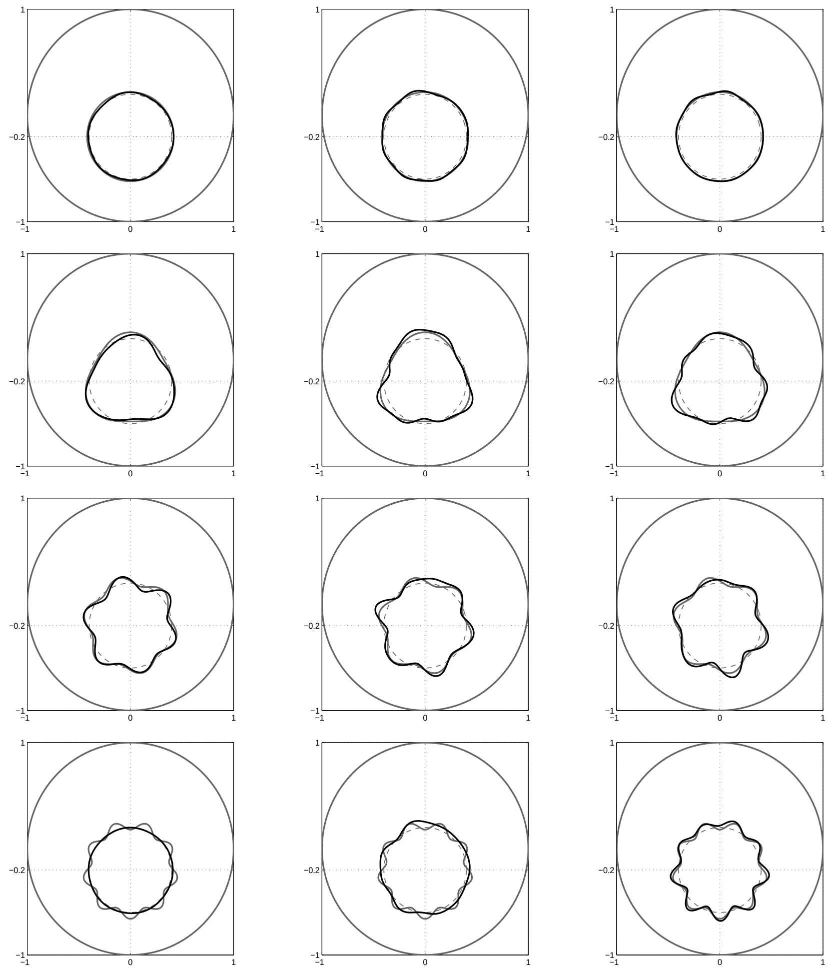

FIGURE 1. The solid grey curves represent the inclusions, which are perturbations of disks, given by the dashed grey curves. The black curves are the reconstructed inclusions. The perturbation is given by ϵh where $\epsilon = 0.02$.

resolution limit of our procedure and observed how it increases as the used eigenfrequency increases. Indeed, we have showed that multi-modal measurements yield better reconstruction than those obtained by only one pair of modal parameters. Very recently, all of the results of this paper have been extended to linear elasticity in [1].

To conclude this paper, we make few remarks. We first note that Theorem 2.1 may be used to compute the shape derivative of objective functionals involving eigenvalues of (2). Recall that if we consider the perturbation under the map θ:

$$D_\theta = \Big\{ x + \theta(x) : x \in D \Big\},$$

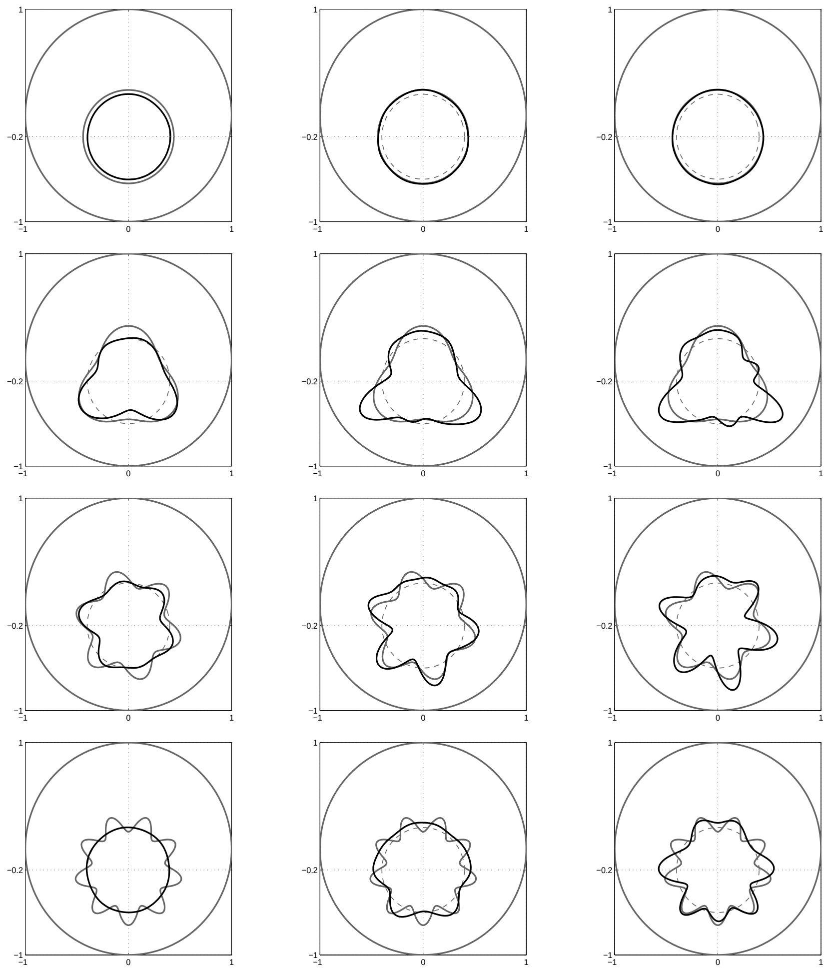

FIGURE 2. Reconstruction result when $\epsilon = 0.04$.

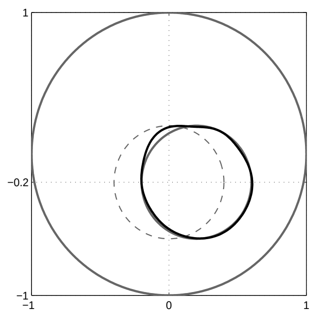

FIGURE 3. Reconstruction of a shifted inclusion.

where $\theta \in W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$ is such that $||\theta||_{W^{1,\infty}} < 1$, then the shape derivative of an objective functional $J(D)$ at D is defined as the Fréchet differential of $\theta \mapsto J(D_\theta)$ at 0, which depends only on $\theta \cdot \nu$ on the boundary ∂D.

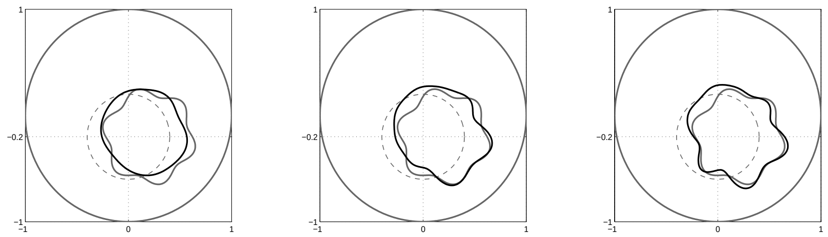

FIGURE 4. Reconstruction of a perturbed and shifted inclusion using the first eigenvalue (on the left), the third eigenvalue (in the middle) and both the first and the third eigenvalues (on the right).

Indeed, based again on Theorem 2.1, the level set approach developed by Osher and Santosa in [19] for solving the acoustic drum problem can be immediately generalized to the inclusion problem $\inf_D J(D)$. See for instance [5, 3].

Finally, it would be interesting to study the limit of (7) as γ_e tends to 0. In this case, $\omega_\varepsilon^2/\gamma_i$ and ω_0^2/γ_i approach to the Neumann eigenvalues for D_ε and D, respectively. Recall that the formula for the Neumann eigenvalue perturbation due to small deformation of the boundary is well-known and the leading-order term is given by

$$\varepsilon\left(\int_{\partial D} h \left|\nabla v_0\right|^2 \, d\sigma - \omega_0^2 \int_{\partial D} h \left|v_0\right|^2 \, d\sigma \right), \tag{50}$$

where v_0 is the (normalized) Neumann eigenfunction. See, for example, section 5.6 in [21]. It would be interesting to show rigorously that the first order term of the expansion of $\omega_\varepsilon^2/\gamma_i - \omega_0^2/\gamma_i$ in (7) converges to the one in (50) as $\gamma_e \to 0$.

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