Transient Elasticity Imaging and Time Reversal[*]

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Abstract

In this paper we consider a purely quasi-incompressible elasticity model. We rigorously establish asymptotic expansions of near- and far-field measurements of the transient elastic wave induced by a small elastic anomaly. Our proof uses layer potential techniques for the modified Stokes system. Based on these formulas, we design asymptotic imaging methods leading to a quantitative estimation of elastic and geometrical parameters of the anomaly.

1   Introduction

An interesting approach to assessing elasticity is to use the acoustic radiation force of an ultrasonic focused beam to remotely generate mechanical vibrations in organs [11, 14]. The acoustic force is due to the momentum transfer from the acoustic wave to the medium. The radiation force acts as a dipolar source. A spatio-temporal sequence of the propagation of the induced transient wave can be acquired, leading to a quantitative estimation of the viscoelastic parameters of the studied medium in a source-free region [6, 7].

The Voigt model has been chosen to describe the viscoelastic properties of tissues. Catheline *et al.* [8] have shown that this model is well adapted to describe the viscoelastic response of tissues to low-frequency excitations.

In this paper, we neglect the viscosity effect and only consider a purely quasi-incompressible elasticity model. We derive asymptotic expansions of the perturbations of the elastic wavefield that are due to the presence of a small anomaly in both the near- and far-field regions as the size of the anomaly goes to zero. Then we design an asymptotic imaging method leading to a quantitative estimation of the shear modulus and shape of the anomaly from near-field measurements. Using time-reversal, we show how to reconstruct the location and geometric features of the anomaly from the far-field measurements. We put a particular emphasis on the difference between the acoustic and the elastic cases, namely, the anisotropy of the focal

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spot and the birth of a near fieldlike effect by time reversing the perturbation due to an elastic anomaly.

The results of this paper extend those in [3] to transient wave propagation in elastic media.

The paper is organized as follows. In Section 2 we rigorously derive asymptotic formulas for quasi-incompressible elasticity and estimate the dependence of the remainders in these formulas with respect to the frequency. Based on these estimates, we obtain in Section 3 formulas for the transient wave equation that are valid after truncating the high-frequency components of the elastic fields. These formulas describe the effect of the presence of a small elastic anomaly in both the near- and far-field. We then investigate in Section 4 the use of time-reversal for locating the anomaly and detecting its overall geometric and material parameters via the viscous moment tensor. An optimization problem is also formulated for reconstructing geometric parameters of the anomaly and its shear modulus from near-field measurements.

2 Asymptotic expansions

We suppose that an elastic medium occupies the whole space $\mathbb{R}^3$. Let the constants λ and μ denote the Lamé coefficients of the medium, that are the elastic parameters in absence of any anomaly. With these constants, $\mathcal{L}_{\lambda,\mu}$ denotes the linear elasticity system, namely

$$\mathcal{L}_{\lambda,\mu}\mathbf{u} := \mu\Delta\mathbf{u} + (\lambda + \mu)\nabla\nabla \cdot \mathbf{u}. \tag{1}$$

The traction on a smooth boundary $\partial\Omega$ is given by the conormal derivative $\partial\mathbf{u}/\partial\nu$ associated with $\mathcal{L}_{\lambda,\mu}$,

$$\frac{\partial\mathbf{u}}{\partial\nu} := \lambda(\nabla \cdot \mathbf{u})\mathbf{N} + \mu\widehat{\nabla}\mathbf{u}\mathbf{N}, \tag{2}$$

where $\mathbf{N}$ denotes the outward unit normal to $\partial\Omega$. Here $\widehat{\nabla}$ denotes the symmetric gradient, i.e.,

$$\widehat{\nabla}\mathbf{u} := \nabla\mathbf{u} + \nabla\mathbf{u}^T, \tag{3}$$

where the superscript T denotes the transpose.

The time-dependent linear elasticity system is given by

$$\partial_t^2\mathbf{u} - \mathcal{L}_{\lambda,\mu}\mathbf{u} = 0. \tag{4}$$

The fundamental solution for the system (4) is given by $\mathbf{G} = (G_{ij})$ where

$$G_{ij} = \frac{1}{4\pi}\frac{3\gamma_i\gamma_j - \delta_{ij}}{r^3}H_\alpha^{\sqrt{\mu}}(x,t) + \frac{1}{4\pi\alpha^2}\frac{\gamma_i\gamma_j}{r}\delta_{t=\frac{r}{\alpha}} - \frac{1}{4\pi\mu}\frac{\gamma_i\gamma_j - \delta_{ij}}{r}\delta_{t=\frac{r}{\sqrt{\mu}}}. \tag{5}$$

Here $r = |x|, \gamma_i = x_i/r, \alpha = \sqrt{\lambda + 2\mu}$, δ_{ij} denotes the Kronecker symbol, δ denotes the Dirac delta function, and $H_\alpha^\beta(x,t)$ is defined by

$$H_\alpha^\beta(x,t) := \begin{cases} t & \text{if } \dfrac{r}{\alpha} < t < \dfrac{r}{\beta}, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Note that $(1/r^3)H_\alpha^\beta(x,t)$ behaves like $1/r^2$ for times $(r/\alpha) < t < (r/\beta)$. See [1].

2

Suppose that there is an elastic anomaly D, given by $D = \epsilon B + z$, which has the elastic parameters $(\tilde{\lambda}, \tilde{\mu})$. Here B is a $\mathcal{C}^2$-bounded domain containing the origin, z the location of the anomaly, and ϵ a small positive parameter representing the order of magnitude of the anomaly size.

For a given point source $\bar{y}$ away from the anomaly D and a constant vector $\mathbf{a}$, we consider the following transient elastic wave problem in the presence of an anomaly:

$$\begin{cases} \partial_t^2 \mathbf{u} - \mathcal{L}_{\lambda,\mu} \mathbf{u} = \delta_{t=0} \delta_{x=\bar{y}} \mathbf{a} \quad \text{in} \quad (\mathbb{R}^3 \setminus \overline{D}) \times \mathbb{R}, \\ \partial_t^2 \mathbf{u} - \mathcal{L}_{\tilde{\lambda},\tilde{\mu}} \mathbf{u} = 0 \quad \text{in} \quad D \times \mathbb{R}, \\ \mathbf{u}\big|_+ - \mathbf{u}\big|_- = 0 \quad \text{on} \quad \partial D \times \mathbb{R}, \\ \dfrac{\partial \mathbf{u}}{\partial \nu}\big|_+ - \dfrac{\partial \mathbf{u}}{\partial \tilde{\nu}}\big|_- = 0 \quad \text{on} \quad \partial D \times \mathbb{R}, \\ \mathbf{u}(x,t) = 0 \quad \text{for } x \in \mathbb{R}^3 \quad \text{and } t \ll 0, \end{cases} \tag{7}$$

where $\partial \mathbf{u}/\partial \nu$ and $\partial \mathbf{u}/\partial \tilde{\nu}$ denote the conormal derivatives on ∂D associated respectively with $\mathcal{L}_{\lambda,\mu}$ and $\mathcal{L}_{\tilde{\lambda},\tilde{\mu}}$. Here and throughout this paper the subscripts $\pm$ denote the limit from outside and inside D, respectively.

As was observed in [11, 13], the Poisson ratio of human tissues is very close to $1/2$, which amounts to λ/μ and $\tilde{\lambda}/\tilde{\mu}$ being very large. So we seek for a good approximation of the problem (7) as λ and $\tilde{\lambda}$ go to $+\infty$. To this end, let

$$p := \begin{cases} \lambda \nabla \cdot \mathbf{u} \quad \text{in} \quad (\mathbb{R}^3 \setminus \overline{D}) \times \mathbb{R}, \\ \tilde{\lambda} \nabla \cdot \mathbf{u} \quad \text{in} \quad D \times \mathbb{R}. \end{cases}$$

One can show by modifying a little the argument in [4] that as λ and $\tilde{\lambda}$ go to $+\infty$ with $\tilde{\lambda}/\lambda$ of order one, the displacement field $\mathbf{u}$ can be represented in the form of the following series:

$$\mathbf{u}(x,t) = \mathbf{u}_0(x,t) + \left(\frac{1}{\lambda}\chi(\mathbb{R}^3 \setminus \overline{D}) + \frac{1}{\tilde{\lambda}}\chi(D)\right)\mathbf{u}_1(x,t)$$

$$+ \left(\frac{1}{\lambda^2}\chi(\mathbb{R}^3 \setminus \overline{D}) + \frac{1}{\tilde{\lambda}^2}\chi(D)\right)\mathbf{u}_2(x,t) + \dots,$$

$$p = p_0 + \left(\frac{1}{\lambda}\chi(\mathbb{R}^3 \setminus \overline{D}) + \frac{1}{\tilde{\lambda}}\chi(D)\right)p_1 + \left(\frac{1}{\lambda^2}\chi(\mathbb{R}^3 \setminus \overline{D}) + \frac{1}{\tilde{\lambda}^2}\chi(D)\right)p_2 + \dots,$$

where the leading-order term $(\mathbf{u}_0(x,t), p_0(x))$ is solution to the following homogeneous time-dependent Stokes system

$$\begin{cases} \partial_t^2 \mathbf{u}_0 - \nabla \cdot (\tilde{\mu}\chi(D) + \mu\chi(\mathbb{R}^3 \setminus \overline{D}))\nabla \mathbf{u}_0 - \nabla p_0 = \delta_{t=0}\delta_{x=y}\mathbf{a} \quad \text{in } \mathbb{R}^3 \times \mathbb{R}, \\ \nabla \cdot \mathbf{u}_0 = 0 \quad \text{in } \mathbb{R}^3 \times \mathbb{R}, \\ \mathbf{u}_0(x,t) = 0 \quad \text{for } x \in \mathbb{R}^3 \quad \text{and } t \ll 0. \end{cases} \tag{8}$$

The inverse problem considered in this paper is to image an anomaly D with shear modulus $\tilde{\mu}$ inside a background medium of shear modulus $\mu \neq \tilde{\mu}$ from near-field or far-field measurements of the transient elastic wave $\mathbf{u}(x,t)$ (approximated by $\mathbf{u}_0(x,t)$) that is the solution to (7) (approximated by (8)).

In order to design an accurate and robust algorithm to detect the anomaly D incorporating the fact that D is of small size of order ϵ, we will derive an asymptotic expansion of $\mathbf{u}_0$ as $\epsilon \to 0$. As shown in [2], this scale separation methodology yields to efficient medical imaging algorithms.

2.1 Layer potentials for the Stokes system

We begin by reviewing some basic facts on layer potentials for the Stokes system, which we shall use in the next subsection. Relevant derivations or proofs of these facts can be found in [12] and [4].

We consider the following modified Stokes system:

$$\begin{cases} (\Delta + \kappa^2)\mathbf{v} - \nabla q = 0, \\ \nabla \cdot \mathbf{v} = 0. \end{cases} \qquad (9)$$

Let $\partial_i = \frac{\partial}{\partial x_i}$. The fundamental tensor $\mathbf{\Gamma}^\kappa = (\Gamma_{ij}^\kappa)_{i,j=1}^3$ and $\mathbf{F} = (F_1, F_2, F_3)$ to (9) in three dimensions are given by

$$\begin{cases} \Gamma_{ij}^\kappa(x) = -\dfrac{\delta_{ij}}{4\pi} \dfrac{e^{\sqrt{-1}\kappa|x|}}{|x|} - \dfrac{1}{4\pi\kappa^2} \partial_i \partial_j \dfrac{e^{\sqrt{-1}\kappa|x|} - 1}{|x|}, \\ \\ F_i(x) = -\dfrac{1}{4\pi} \dfrac{x_i}{|x|^3}. \end{cases} \qquad (10)$$

If $\kappa = 0$, let

$$\Gamma_{ij}^0(x) = -\frac{1}{8\pi}\left(\frac{\delta_{ij}}{|x|} + \frac{x_i x_j}{|x|^3}\right). \qquad (11)$$

Then $\mathbf{\Gamma}^0 = (\Gamma_{ij}^0)$ together with $\mathbf{F}$ is the fundamental tensor for the standard Stokes system given by

$$\begin{cases} \Delta\mathbf{v} - \nabla q = 0, \\ \nabla \cdot \mathbf{v} = 0. \end{cases}$$

One can easily see that

$$\Gamma_{ij}^\kappa(x) = \Gamma_{ij}^0(x) - \frac{\delta_{ij}\kappa\sqrt{-1}}{6\pi} + O(\kappa^2) \qquad (12)$$

uniformly in x as long as $|x|$ is bounded.

For a bounded $\mathcal{C}^2$-domain D and $\kappa \geq 0$, let

$$\begin{cases} \mathcal{S}_D^\kappa[\varphi](x) := \displaystyle\int_{\partial D} \mathbf{\Gamma}^\kappa(x-y)\varphi(y)d\sigma(y), \\ \\ \mathcal{Q}_D[\varphi](x) := \displaystyle\int_{\partial D} \mathbf{F}(x-y)\cdot\varphi(y)\,d\sigma(y), \end{cases} \qquad x \in \mathbb{R}^3 \qquad (13)$$

for $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in L^2(\partial D)^3$. When $\kappa = 0$, $\mathcal{S}_D^0$ is the single layer potential for the Stokes system. It is worth emphasizing that $\mathcal{S}_D^\kappa[\varphi](x)$ is a vector while $\mathcal{Q}_D[\varphi](x)$ is a scalar, and the pair $(\mathcal{S}_D^\kappa[\varphi], \mathcal{Q}_D[\varphi])$ is a solution to (9).

By abuse of notation, let

$$\frac{\partial\mathbf{u}}{\partial\mathbf{N}} = (\widehat{\nabla}\mathbf{u})\mathbf{N} \quad \text{on } \partial D.$$

We define the conormal derivative $\partial/\partial n$ (for the Stokes system) on ∂D by

$$\left.\frac{\partial\mathbf{v}}{\partial n}\right|_\pm = \left.\frac{\partial\mathbf{v}}{\partial\mathbf{N}}\right|_\pm - q\big|_\pm \mathbf{N}$$

for a pair of solutions $(\mathbf{v}, q)$ to (9). It is well-known that

$$\left.\frac{\partial \mathcal{S}_D^\kappa[\varphi]}{\partial n}\right|_\pm = (\pm\frac{1}{2}I + (\mathcal{K}_D^\kappa)^*)[\varphi] \quad \text{a.e. on } \partial D, \tag{14}$$

where $\mathcal{K}_D^\kappa$ is the boundary integral operator defined by

$$\mathcal{K}_D^\kappa[\varphi](x) := \text{p.v.} \int_{\partial D} \left[\frac{\partial}{\partial \mathbf{N}(y)}(\mathbf{\Gamma}^\kappa(x-y)\varphi(y)) + \mathbf{F}(x-y)\mathbf{N}(y) \cdot \varphi(y)\right] d\sigma(y) \tag{15}$$

for almost all $x \in \partial D$ and $(\mathcal{K}_D^\kappa)^*$ is the L^2-adjoint operator of $\mathcal{K}_D^{-\kappa}$:

$$(\mathcal{K}_D^\kappa)^*[\varphi](x) := \text{p.v.} \int_{\partial D} \left[\frac{\partial}{\partial \mathbf{N}(x)}(\mathbf{\Gamma}^\kappa(x-y)\varphi(y)) + \mathbf{F}(x-y) \cdot \varphi(y)\mathbf{N}(x)\right] d\sigma(y). \tag{16}$$

Here p.v. denotes the Cauchy principal value.

Let $H^1(\partial D) := \{\varphi \in L^2(\partial D), \partial\varphi/\partial\tau \in L^2(\partial D)\}$, $\partial/\partial\tau$ being the tangential derivative. The operator $\mathcal{S}_D^0$ is bounded from $L^2(\partial D)^3$ into $H^1(\partial D)^3$ and invertible in three dimensions. Moreover, one can see that for κ small

$$\|\mathcal{S}_D^\kappa[\varphi] - \mathcal{S}_D^0[\varphi]\|_{H^1(\partial D)} \leq C\kappa\|\varphi\|_{L^2(\partial D)} \tag{17}$$

for all $\varphi \in L^2(\partial D)^3$, where C is independent of κ. It is also well-known that the singular integral operator $(\mathcal{K}_D^0)^*$ is bounded on $L^2(\partial D)^3$. Similarly to (17), one can see that for κ small

$$\|(\mathcal{K}_D^{-\kappa})^*[\varphi] - (\mathcal{K}_D^0)^*[\varphi]\|_{L^2(\partial D)} \leq C\kappa\|\varphi\|_{L^2(\partial D)}$$

for some constant C independent of κ, which in view of (14) yields

$$\left\|\left.\frac{\partial(\mathcal{S}_D^\kappa[\varphi])}{\partial n}\right|_\pm - \left.\frac{\partial(\mathcal{S}_D^0[\varphi])}{\partial n}\right|_\pm\right\|_{L^2(\partial D)} \leq C\kappa\|\varphi\|_{L^2(\partial D)}. \tag{18}$$

2.2 Derivation of asymptotic expansions

Recall that $\bar{y}$ is a point source in $\mathbb{R}^3$ such that $|\bar{y} - z| \gg \epsilon$. Taking the Fourier transform of (8) in the t-variable yields

$$\begin{cases} (\Delta + \dfrac{\omega^2}{\mu})\hat{\mathbf{u}}_0 - \dfrac{1}{\mu}\nabla\hat{p}_0 = \dfrac{1}{\mu}\delta_{x=\bar{y}}\,\mathbf{a} & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\[2mm] (\Delta + \dfrac{\omega^2}{\tilde{\mu}})\hat{\mathbf{u}}_0 - \dfrac{1}{\tilde{\mu}}\nabla\hat{p}_0 = 0 & \text{in } D, \\[2mm] \hat{\mathbf{u}}_0|_+ - \hat{\mathbf{u}}_0|_- = 0 & \text{on } \partial D, \\[2mm] (\hat{p}_0|_- - \hat{p}_0|_+)\mathbf{N} + \mu\dfrac{\partial\hat{\mathbf{u}}_0}{\partial\mathbf{N}}\Big|_+ - \tilde{\mu}\dfrac{\partial\hat{\mathbf{u}}_0}{\partial\mathbf{N}}\Big|_- = 0 & \text{on } \partial D, \\[2mm] \nabla \cdot \hat{\mathbf{u}}_0 = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{19}$$

subject to the radiation condition:

$$\begin{cases} \hat{p}_0(x) \to 0 & \text{as } r = |x| \to +\infty, \\[2mm] \partial_r\nabla \times \hat{\mathbf{u}}_0 - \sqrt{-1}\dfrac{\omega}{\sqrt{\mu}}\nabla \times \hat{\mathbf{u}}_0 = o(\dfrac{1}{r}) & \text{as } r = |x| \to +\infty \text{ uniformly in } \dfrac{x}{|x|}, \end{cases} \tag{20}$$

5

where $\hat{\mathbf{u}}_0$ and $\hat{p}_0$ denote the Fourier transforms of $\mathbf{u}_0$ and of p_0, respectively. We say that $(\hat{\mathbf{u}}_0, \hat{p}_0)$ satisfies the radiation condition if (20) holds.

Let

$$\hat{\mathbf{U}}_0(x, \omega) := \frac{1}{\mu}\mathbf{\Gamma}^{\frac{\omega}{\sqrt{\mu}}}(x - \bar{y})\mathbf{a}, \tag{21}$$

$$\hat{q}_0(x) := \mathbf{F}(x - \bar{y}) \cdot \mathbf{a}. \tag{22}$$

Then the pair $(\hat{\mathbf{U}}_0(x, \omega), \hat{q}_0(x))$ satisfies

$$\begin{cases} (\Delta + \dfrac{\omega^2}{\mu})\hat{\mathbf{U}}_0 - \dfrac{1}{\mu}\nabla\hat{q}_0 = \dfrac{1}{\mu}\delta_{x=\bar{y}}\,\mathbf{a} & \text{in } \mathbb{R}^3, \\ \nabla \cdot \hat{\mathbf{U}}_0 = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{23}$$

In view of (19) and (23), it is proper to expect that $\hat{\mathbf{u}}_0$ converges to $\hat{\mathbf{U}}_0$ as ϵ tends to 0. We shall derive an asymptotic expansion for $\hat{\mathbf{u}}_0 - \hat{\mathbf{U}}_0$ as ϵ tends to zero and carefully estimate the dependence of the remainder on the frequency ω.

Let $\mathbf{w} = \hat{\mathbf{u}}_0 - \hat{\mathbf{U}}_0$ and introduce

$$p := \begin{cases} \dfrac{1}{\mu}(\hat{p}_0 - \hat{q}_0) & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ \dfrac{1}{\tilde{\mu}}(\hat{p}_0 - \hat{q}_0) & \text{in } D. \end{cases}$$

Then the pair $(\mathbf{w}, p)$ satisfies

$$\begin{cases} (\Delta + \dfrac{\omega^2}{\mu})\mathbf{w} - \nabla p = 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ (\Delta + \dfrac{\omega^2}{\tilde{\mu}})\mathbf{w} - \nabla p = (\dfrac{1}{\mu} - \dfrac{1}{\tilde{\mu}})(\omega^2\hat{\mathbf{U}}_0 - \nabla\hat{q}_0) & \text{in } D, \\ \mathbf{w}|_+ - \mathbf{w}|_- = 0 & \text{on } \partial D, \\ \mu(\dfrac{\partial\mathbf{w}}{\partial\mathbf{N}}\Big|_+ - p|_+\mathbf{N}) - \tilde{\mu}(\dfrac{\partial\mathbf{w}}{\partial\mathbf{N}}\Big|_- - p|_-\mathbf{N}) = (\tilde{\mu} - \mu)\dfrac{\partial\hat{\mathbf{U}}_0}{\partial\mathbf{N}} & \text{on } \partial D, \\ \nabla \cdot \mathbf{w} = 0, \\ (\mathbf{w}, p) \text{ satisfies the radiation condition.} \end{cases} \tag{24}$$

Therefore, we can represent $(\mathbf{w}, p)$ as

$$\mathbf{w}(x) = \begin{cases} (\dfrac{1}{\mu} - \dfrac{1}{\tilde{\mu}})\displaystyle\int_D \mathbf{\Gamma}^{\frac{\omega}{\sqrt{\mu}}}(x - y)(\omega^2\hat{\mathbf{U}}_0(y) - \nabla\hat{q}_0(y))\,dy + \mathcal{S}_D^{\frac{\omega}{\sqrt{\mu}}}[\varphi](x) & \text{in } D, \\ \mathcal{S}_D^{\frac{\omega}{\sqrt{\mu}}}[\psi](x) & \text{in } \mathbb{R}^3 \setminus \overline{D}, \end{cases} \tag{25}$$

and

$$p(x) = \begin{cases} (\dfrac{1}{\mu} - \dfrac{1}{\tilde{\mu}})\displaystyle\int_D \mathbf{F}(x - y) \cdot (\omega^2\hat{\mathbf{U}}_0(y) - \nabla\hat{q}_0(y))\,dy + \mathcal{Q}_D[\varphi](x) & \text{in } D, \\ \mathcal{Q}_D[\psi](x) & \text{in } \mathbb{R}^3 \setminus \overline{D}, \end{cases} \tag{26}$$

6

where (φ, ψ) is the solution to the following system of integral equations

$$
\begin{cases}
\mathcal{S}_D^{\frac{\omega}{\sqrt{\mu}}}[\varphi](x) - \mathcal{S}_D^{\frac{\omega}{\sqrt{\tilde{\mu}}}}[\psi](x) = (\frac{1}{\mu} - \frac{1}{\tilde{\mu}}) \int_D \mathbf{\Gamma}^{\frac{\omega}{\sqrt{\tilde{\mu}}}}(x-y)(\omega^2 \hat{\mathbf{U}}_0(y) - \nabla \hat{q}_0(y)) \, dy, \\[2mm]
\mu \dfrac{\partial \mathcal{S}_D^{\frac{\omega}{\sqrt{\mu}}}[\varphi]}{\partial n}\Big|_+(x) - \tilde{\mu} \dfrac{\partial \mathcal{S}_D^{\frac{\omega}{\sqrt{\mu}}}[\psi]}{\partial n}\Big|_-(x) = (\tilde{\mu} - \mu) \dfrac{\partial \hat{\mathbf{U}}_0}{\partial \mathbf{N}} \\[2mm]
\qquad + (\dfrac{\tilde{\mu}}{\mu} - 1) \dfrac{\partial}{\partial \mathbf{N}} \int_D \mathbf{\Gamma}^{\frac{\omega}{\sqrt{\tilde{\mu}}}}(x-y)(\omega^2 \hat{\mathbf{U}}_0(y) - \nabla \hat{q}_0(y)) \, dy \\[2mm]
\qquad - (\dfrac{\tilde{\mu}}{\mu} - 1) \int_D \mathbf{F}(x-y) \cdot (\omega^2 \hat{\mathbf{U}}_0(y) - \nabla \hat{q}_0(y)) \, dy \, \mathbf{N}.
\end{cases} \tag{27}
$$

In order to prove the unique solvability of (27), let us make a change of variables: Recalling that D is of the form $D = \epsilon B + z$, we put

$$
\tilde{\varphi}(\tilde{x}) = \varphi(\epsilon \tilde{x} + z), \quad \tilde{x} \in \partial B, \tag{28}
$$

and define similarly $\tilde{\psi}$. Then after scaling, (27) takes the form

$$
\begin{cases}
\mathcal{S}_B^{\frac{\epsilon \omega}{\sqrt{\mu}}}[\tilde{\varphi}](\tilde{x}) - \mathcal{S}_B^{\frac{\epsilon \omega}{\sqrt{\mu}}}[\tilde{\psi}](\tilde{x}) = \mathbf{A}(\tilde{x}), \\[2mm]
\tilde{\mu} \dfrac{\partial \mathcal{S}_B^{\frac{\epsilon \omega}{\sqrt{\mu}}}[\tilde{\varphi}]}{\partial n}\Big|_-(\tilde{x}) - \mu \dfrac{\partial \mathcal{S}_D^{\frac{\epsilon \omega}{\sqrt{\mu}}}[\tilde{\psi}]}{\partial n}\Big|_+(\tilde{x}) = \mathbf{B}(\tilde{x}),
\end{cases} \quad \tilde{x} \in \partial B \tag{29}
$$

where $\mathbf{A} = (A_1, A_2, A_3)$ and $\mathbf{B} = (B_1, B_2, B_3)$ are defined in an obvious way, namely

$$
\mathbf{A}(\tilde{x}) = \epsilon (\frac{1}{\mu} - \frac{1}{\tilde{\mu}}) \int_B \mathbf{\Gamma}^{\frac{\epsilon \omega}{\sqrt{\tilde{\mu}}}}(\tilde{x} - \tilde{y})(\omega^2 \hat{\mathbf{U}}_0(\epsilon \tilde{y} + z) - \nabla \hat{q}_0(\epsilon \tilde{y} + z)) \, d\tilde{y}, \tag{30}
$$

and

$$
\begin{aligned}
\mathbf{B}(\tilde{x}) = {} & (\tilde{\mu} - \mu) \frac{\partial \hat{\mathbf{U}}_0}{\partial \mathbf{N}}(\epsilon \tilde{x} + z) \\
& + \epsilon (\frac{\tilde{\mu}}{\mu} - 1) \frac{\partial}{\partial \mathbf{N}} \int_B \mathbf{\Gamma}^{\frac{\epsilon \omega}{\sqrt{\tilde{\mu}}}}(\tilde{x} - \tilde{y})(\omega^2 \hat{\mathbf{U}}_0(\epsilon \tilde{y} + z) - \nabla \hat{q}_0(\epsilon \tilde{y} + z)) \, d\tilde{y} \\
& - \epsilon (\frac{\tilde{\mu}}{\mu} - 1) \int_D \mathbf{F}(\tilde{x} - \tilde{y}) \cdot (\omega^2 \hat{\mathbf{U}}_0(\epsilon \tilde{y} + z) - \nabla \hat{q}_0(\epsilon \tilde{y} + z)) \, dy \, \mathbf{N}(\tilde{x}).
\end{aligned} \tag{31}
$$

We emphasize that the normal vector $\mathbf{N}$ above is that on ∂B.

We may rewrite (29) as

$$
\mathcal{T}(\tilde{\varphi}, \tilde{\psi}) = (\mathbf{A}, \mathbf{B}), \tag{32}
$$

where $\mathcal{T}$ is an operator from $L^2(\partial B)^3 \times L^2(\partial B)^3$ into $H^1(\partial B)^3 \times L^2(\partial B)^3$ defined by

$$
\mathcal{T}(\tilde{\varphi}, \tilde{\psi}) = \begin{pmatrix} \mathcal{S}_B^{\frac{\epsilon \omega}{\sqrt{\mu}}} & -\mathcal{S}_B^{\frac{\epsilon \omega}{\sqrt{\mu}}} \\[2mm] \tilde{\mu} \dfrac{\partial}{\partial n} \mathcal{S}_B^{\frac{\epsilon \omega}{\sqrt{\mu}}}|_- & -\mu \dfrac{\partial}{\partial n} \mathcal{S}_B^{\frac{\epsilon \omega}{\sqrt{\mu}}}|_+ \end{pmatrix} \begin{pmatrix} \tilde{\varphi} \\[2mm] \tilde{\psi} \end{pmatrix}.
$$

We then decompose the operator $\mathcal{T}$ as

$$
\mathcal{T} = \mathcal{T}_0 + \mathcal{T}_\epsilon, \tag{33}
$$

where

$$\mathcal{T}_0(\tilde{\varphi}, \tilde{\psi}) := \begin{pmatrix} \mathcal{S}_B^0 & -\mathcal{S}_B^0 \\ \tilde{\mu}\dfrac{\partial}{\partial n}\mathcal{S}_B^0|_- & -\mu\dfrac{\partial}{\partial n}\mathcal{S}_B^0|_+ \end{pmatrix} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix},$$

and $\mathcal{T}_\epsilon = \mathcal{T} - \mathcal{T}_0$. Then by (17) and (18), it follows that

$$\|\mathcal{T}_\epsilon(\tilde{\varphi}, \tilde{\psi})\|_{H^1(\partial B) \times L^2(\partial B)} \leq C\epsilon\omega(\|\tilde{\varphi}\|_{L^2(\partial B)} + \|\tilde{\psi}\|_{L^2(\partial B)}). \tag{34}$$

Note that $\mathcal{S}_B^0$ is invertible, and since $|\frac{\tilde{\mu}+\mu}{2(\tilde{\mu}-\mu)}| > \frac{1}{2}$, the operator $-\frac{(\tilde{\mu}+\mu)}{2(\tilde{\mu}-\mu)}I + (\mathcal{K}_B^0)^*$ is invertible as well (see [4]). Thus one can see that $\mathcal{T}_0$ is also invertible. In fact, one can readily check that the solution is explicit.

Lemma 2.1 *For* $(\mathbf{f}, \mathbf{g}) \in H^1(\partial B)^3 \times L^2(\partial B)^3$ *the solution* $(\tilde{\varphi}, \tilde{\psi}) = \mathcal{T}_0^{-1}(\mathbf{f}, \mathbf{g})$ *is given by*

$$\tilde{\varphi} = \tilde{\psi} + (\mathcal{S}_B^0)^{-1}[\mathbf{f}], \tag{35}$$

$$\tilde{\psi} = \frac{1}{\tilde{\mu} - \mu}\left(-\frac{(\tilde{\mu} + \mu)}{2(\tilde{\mu} - \mu)}I + (\mathcal{K}_B^0)^*\right)^{-1}\left[-\tilde{\mu}(-\frac{1}{2}I + (\mathcal{K}_B^0)^*)(\mathcal{S}_B^0)^{-1}[\mathbf{f}] + \mathbf{g}\right]. \tag{36}$$

In view of (33) and (34), one can see that there is $\epsilon_0 > 0$ such that $\mathcal{T}$ is invertible as long as $\epsilon\omega \leq \epsilon_0$. Moreover $\mathcal{T}^{-1}$ takes the form

$$\mathcal{T}^{-1} = \mathcal{T}_0^{-1} + E, \tag{37}$$

where the operator E satisfies

$$\|E(\mathbf{f}, \mathbf{g})\|_{L^2(\partial B) \times L^2(\partial B)} \leq C\epsilon\omega(\|\mathbf{f}\|_{H^1(\partial B)} + \|\mathbf{g}\|_{L^2(\partial B)}), \tag{38}$$

for some constant C independent of ϵ and ω.

Suppose that $\epsilon\omega \leq \epsilon_0 < 1$. Let $(\widetilde{\varphi}^\omega, \widetilde{\psi}^\omega)$ be the solution to (29). Then by (37) we have

$$(\widetilde{\varphi}^\omega, \widetilde{\psi}^\omega) = \mathcal{T}_0^{-1}(\mathbf{A}, \mathbf{B}) + E(\mathbf{A}, \mathbf{B}).$$

In view of (30) we have

$$\|\mathbf{A}\|_{H^1(\partial B)} \leq C\epsilon(\omega^2 + 1). \tag{39}$$

On the other hand, according to (31), $\mathbf{B}$ can be written as

$$\mathbf{B}(\tilde{x}) = (\tilde{\mu} - \mu)\widehat{\nabla}\hat{\mathbf{U}}_0(z, \omega)\mathbf{N}(\tilde{x}) + \mathbf{B}_1(\tilde{x}),$$

where $\mathbf{B}_1$ satisfies

$$\|\mathbf{B}_1\|_{L^2(\partial B)} \leq C\epsilon(\omega^2 + 1). \tag{40}$$

Therefore, we have

$$(\widetilde{\varphi}^\omega, \widetilde{\psi}^\omega) = (\tilde{\mu} - \mu)\mathcal{T}_0^{-1}\left(0, \widehat{\nabla}\hat{\mathbf{U}}_0(z, \omega)\mathbf{N}\right) + \mathcal{T}_0^{-1}(\mathbf{A}, \mathbf{B}_1) + E(\mathbf{A}, \mathbf{B}). \tag{41}$$

Because of (38), (39), and (40), the last two terms in the above equation are error terms satisfying

$$\|\mathcal{T}_0^{-1}(\mathbf{A}, \mathbf{B}_1) + E(\mathbf{A}, \mathbf{B})\|_{L^2(\partial B) \times L^2(\partial B)} \leq C\epsilon(\omega^2 + 1).$$

We also need to derive asymptotic expansions for $\frac{\partial \widetilde{\varphi}^\omega}{\partial \omega}$ and $\frac{\partial \widetilde{\psi}^\omega}{\partial \omega}$. By differentiating both sides of (29) with respect to ω, we obtain

$$\mathcal{S}_B^{\frac{\epsilon\omega}{\sqrt{\mu}}}\left[\frac{\partial\widetilde{\varphi}^\omega}{\partial\omega}\right](\tilde{x}) - \mathcal{S}_B^{\frac{\epsilon\omega}{\sqrt{\mu}}}\left[\frac{\partial\widetilde{\psi}^\omega}{\partial\omega}\right](\tilde{x}) = \frac{\partial\mathbf{A}(\tilde{x})}{\partial\omega} - \int_{\partial B}\frac{\partial}{\partial\omega}\mathbf{\Gamma}^{\frac{\epsilon\omega}{\sqrt{\mu}}}(\tilde{x}-\tilde{y})\tilde{\varphi}^\omega(\tilde{y})d\sigma(\tilde{y})$$
$$+ \int_{\partial B}\frac{\partial}{\partial\omega}\mathbf{\Gamma}^{\frac{\epsilon\omega}{\sqrt{\mu}}}(\tilde{x}-\tilde{y})\tilde{\psi}^\omega(\tilde{y})d\sigma(\tilde{y}) \qquad (42)$$

and

$$\tilde{\mu}\frac{\partial}{\partial n}\mathcal{S}_B^{\frac{\epsilon\omega}{\sqrt{\mu}}}\left[\frac{\partial\widetilde{\varphi}^\omega}{\partial\omega}\right]\bigg|_-(\tilde{x}) - \mu\frac{\partial}{\partial n}\mathcal{S}_B^{\frac{\epsilon\omega}{\sqrt{\mu}}}\left[\frac{\partial\widetilde{\psi}^\omega}{\partial\omega}\right]\bigg|_+(\tilde{x}) = \frac{\partial\mathbf{B}(\tilde{x})}{\partial\omega}$$
$$- \frac{\partial}{\partial n}\int_{\partial B}\frac{\partial}{\partial\omega}\mathbf{\Gamma}^{\frac{\epsilon\omega}{\sqrt{\mu}}}(\tilde{x}-\tilde{y})\tilde{\varphi}^\omega(\tilde{y})d\sigma(\tilde{y}) + \frac{\partial}{\partial n}\int_{\partial B}\frac{\partial}{\partial\omega}\mathbf{\Gamma}^{\frac{\epsilon\omega}{\sqrt{\mu}}}(\tilde{x}-\tilde{y})\tilde{\psi}^\omega(\tilde{y})d\sigma(\tilde{y}) \qquad (43)$$

on ∂B.

Straightforward computations using (10) and (30) show that the right-hand side of the equality in (42) is of order $\epsilon(\omega+1)$ in the $H^1(\partial B)$-norm. We can also show using (31) that $\frac{\partial G_1}{\partial\omega}$ is also of order $\epsilon(\omega+1)$ in the $L^2(\partial B)$-norm. Thus, using the same argument as before, we readily obtain

$$\left(\frac{\partial\widetilde{\varphi}^\omega}{\partial\omega}, \frac{\partial\widetilde{\psi}^\omega}{\partial\omega}\right) = (\tilde{\mu}-\mu)\mathcal{T}_0^{-1}\left(0, \widehat{\nabla}(\frac{\partial\hat{\mathbf{U}}_0}{\partial\omega})(z,\omega)\mathbf{N}\right) + O(\epsilon(\omega+1)), \qquad (44)$$

where the equality holds in $L^2(\partial B)^3 \times L^2(\partial B)^3$.

In view of (41) and (44), applying Lemma 2.1 (with $\mathbf{f}=0$) yields the following result.

Proposition 2.2 *Let $(\widetilde{\varphi}^\omega, \widetilde{\psi}^\omega)$ be the solution to (29). There exists $\epsilon_0 > 0$ such that if $\epsilon\omega < \epsilon_0$, then the following asymptotic expansions hold:*

$$\widetilde{\varphi}^\omega = \left(\frac{-(\tilde{\mu}+\mu)}{2(\tilde{\mu}-\mu)}I + (\mathcal{K}_B^0)^*\right)^{-1}[\widehat{\nabla}\hat{\mathbf{U}}_0(z,\omega)\mathbf{N}] + O(\epsilon(\omega^2+1)), \qquad (45)$$

$$\widetilde{\psi}^\omega = \left(\frac{-(\tilde{\mu}+\mu)}{2(\tilde{\mu}-\mu)}I + (\mathcal{K}_B^0)^*\right)^{-1}[\widehat{\nabla}\hat{\mathbf{U}}_0(z,\omega)\mathbf{N}] + O(\epsilon(\omega^2+1)), \qquad (46)$$

and

$$\frac{\partial\widetilde{\varphi}^\omega}{\partial\omega} = \left(\frac{-(\tilde{\mu}+\mu)}{2(\tilde{\mu}-\mu)}I + (\mathcal{K}_B^0)^*\right)^{-1}[\widehat{\nabla}\frac{\partial}{\partial\omega}\hat{\mathbf{U}}_0(z,\omega)\mathbf{N}] + O(\epsilon(\omega+1)), \qquad (47)$$

$$\frac{\partial\widetilde{\psi}^\omega}{\partial\omega} = \left(\frac{-(\tilde{\mu}+\mu)}{2(\tilde{\mu}-\mu)}I + (\mathcal{K}_B^0)^*\right)^{-1}[\widehat{\nabla}\frac{\partial}{\partial\omega}\hat{\mathbf{U}}_0(z,\omega)\mathbf{N}] + O(\epsilon(\omega+1)), \qquad (48)$$

where all the equalities hold in $L^2(\partial B)$.

We are now ready to derive the inner expansion for $\mathbf{w}$. Let Ω be a domain containing D and let $\widetilde{\Omega} = \frac{1}{\epsilon}\Omega - z$. After a change of variables, (25) and (26) take the forms:

$$\mathbf{w}(\epsilon\tilde{x}+z,\omega) = \begin{cases} \epsilon^2(\frac{1}{\mu}-\frac{1}{\tilde{\mu}})\displaystyle\int_B\mathbf{\Gamma}^{\frac{\epsilon\omega}{\sqrt{\mu}}}(\tilde{x}-\tilde{y})(\omega^2\hat{\mathbf{U}}_0(\epsilon\tilde{y}+z)-\nabla\hat{q}_0(\epsilon\tilde{y}+z))\,d\tilde{y} \\ \quad + \epsilon\mathcal{S}_B^{\frac{\epsilon\omega}{\sqrt{\mu}}}[\tilde{\varphi}^\omega](\tilde{x}) \quad \text{in } B, \\ \epsilon\mathcal{S}_B^{\frac{\epsilon\omega}{\sqrt{\mu}}}[\tilde{\psi}^\omega](\tilde{x}) \quad \text{in } \mathbb{R}^3\setminus\overline{B}, \end{cases} \qquad (49)$$

and

$$p(\epsilon\tilde{x} + z, \omega) = \begin{cases} \epsilon(\dfrac{1}{\mu} - \dfrac{1}{\tilde{\mu}}) \displaystyle\int_B \mathbf{F}(\tilde{x} - \tilde{y}) \cdot (\omega^2 \hat{\mathbf{U}}_0(\epsilon\tilde{y} + z) - \nabla\hat{q}_0(\epsilon\tilde{y} + z))\, d\tilde{y} \\ \qquad + \epsilon\mathcal{Q}_B[\tilde{\varphi}^\omega](\tilde{x}) \quad \text{in } B, \\ \epsilon\mathcal{Q}_B[\tilde{\psi}^\omega](\tilde{x}) \quad \text{in } \mathbb{R}^3 \setminus \overline{B}. \end{cases} \tag{50}$$

Since

$$\left\| \mathcal{S}_B^{\frac{\epsilon\omega}{\sqrt{\mu}}}[\tilde{\varphi}^\omega] - \mathcal{S}_B^0[\tilde{\varphi}^\omega] \right\|_{H^1(\partial B)} \leq C\epsilon\omega \|\tilde{\varphi}^\omega\|_{L^2(\partial B)},$$

we have

$$\mathbf{w}(\epsilon\tilde{x} + z, \omega) = \begin{cases} \epsilon\mathcal{S}_B^0[\tilde{\varphi}^\omega](\tilde{x}) + O(\epsilon^2(\omega^2 + 1)), & \tilde{x} \in B, \\ \epsilon\mathcal{S}_B^0[\tilde{\psi}^\omega](\tilde{x}) + O(\epsilon^2(\omega + 1)), & \tilde{x} \in \widetilde{\Omega} \setminus \overline{B}. \end{cases}$$

It then follows from (45) and (46) that

$$\mathbf{w}(\epsilon\tilde{x} + z, \omega) = \epsilon\mathcal{S}_B^0 \left(-\frac{(\tilde{\mu} + \mu)}{2(\tilde{\mu} - \mu)} I + (\mathcal{K}_B^0)^* \right)^{-1} [\widehat{\nabla}\hat{\mathbf{U}}_0(z, \omega)\mathbf{N}](\tilde{x}) + O(\epsilon^2(\omega^2 + 1)) \tag{51}$$

for $\tilde{x} \in \widetilde{\Omega}$.

On the other hand, we have

$$\frac{\partial\mathbf{w}}{\partial\omega}(\epsilon\tilde{x} + z, \omega) = \begin{cases} \epsilon\mathcal{S}_B^{\frac{\epsilon\omega}{\sqrt{\mu}}} \left[\dfrac{\partial\tilde{\varphi}^\omega}{\partial\omega} \right](\tilde{x}) + O(\epsilon^2(\omega + 1)), & \tilde{x} \in B, \\ \epsilon\mathcal{S}_B^{\frac{\epsilon\omega}{\sqrt{\mu}}} \left[\dfrac{\partial\tilde{\psi}^\omega}{\partial\omega} \right](\tilde{x}) + O(\epsilon^2), & \tilde{x} \in \widetilde{\Omega} \setminus \overline{B}. \end{cases}$$

Therefore, from (47) and (48) we obtain that

$$\frac{\partial\mathbf{w}}{\partial\omega}(\epsilon\tilde{x} + z, \omega) = \epsilon\mathcal{S}_B^0 \left(-\frac{(\tilde{\mu} + \mu)}{2(\tilde{\mu} - \mu)} I + (\mathcal{K}_B^0)^* \right)^{-1} [\widehat{\nabla}\frac{\partial}{\partial\omega}\hat{\mathbf{U}}_0(z, \omega)\mathbf{N}](\tilde{x}) + O(\epsilon^2(\omega + 1)) \tag{52}$$

for $\tilde{x} \in \widetilde{\Omega}$.

Let

$$\mathbf{v}(\tilde{x}) := \mathcal{S}_B^0 \left(-\frac{(\tilde{\mu} + \mu)}{2(\tilde{\mu} - \mu)} I + (\mathcal{K}_B^0)^* \right)^{-1} [\widehat{\nabla}\hat{\mathbf{U}}_0(z, \omega)\mathbf{N}](\tilde{x}),$$

$$q(\tilde{x}) := \mathcal{Q}_B \left(-\frac{(\tilde{\mu} + \mu)}{2(\tilde{\mu} - \mu)} I + (\mathcal{K}_B^0)^* \right)^{-1} [\widehat{\nabla}\hat{\mathbf{U}}_0(z, \omega)\mathbf{N}](\tilde{x}).$$

It is easy to check that $(\mathbf{v}, q)$ is the solution to

$$\begin{cases} \mu\Delta\mathbf{v} - \nabla q = 0 & \text{in } \mathbb{R}^3 \setminus \overline{B}, \\ \tilde{\mu}\Delta\mathbf{v} - \nabla q = 0 & \text{in } B, \\ \mathbf{v}|_- - \mathbf{v}|_+ = 0 & \text{on } \partial B, \\ (q\mathbf{N} - \tilde{\mu}\dfrac{\partial\mathbf{v}}{\partial\mathbf{N}})\Big|_- - (q\mathbf{N} - \mu\dfrac{\partial\mathbf{v}}{\partial\mathbf{N}})\Big|_+ = (\tilde{\mu} - \mu)\widehat{\nabla}\hat{\mathbf{U}}_0(z, \omega)\mathbf{N} & \text{on } \partial B, \\ \nabla \cdot \mathbf{v} = 0 & \text{in } \mathbb{R}^3, \\ \mathbf{v}(\tilde{x}) \to 0 & \text{as } |\tilde{x}| \to +\infty, \\ q(\tilde{x}) \to 0 & \text{as } |\tilde{x}| \to +\infty. \end{cases} \tag{53}$$

10

We finally obtain the following theorem from (51) and (52).

Theorem 2.3 *Let Ω be a small region containing D and let*

$$R(x,\omega) = \hat{\mathbf{u}}_0(x,\omega) - \hat{\mathbf{U}}_0(x,\omega) - \epsilon\mathbf{v}\left(\frac{x-z}{\epsilon}\right), \quad x \in \Omega. \tag{54}$$

There exists $\epsilon_0 > 0$ such that if $\epsilon\omega < \epsilon_0$, then

$$R(x,\omega) = O(\epsilon^2(\omega^2+1)), \quad \nabla_x R(x,\omega) = O(\epsilon(\omega^2+1)), \quad x \in \Omega. \tag{55}$$

Moreover,

$$\frac{\partial R}{\partial \omega}(x,\omega) = O(\epsilon^2(\omega+1)), \quad \nabla_x\left(\frac{\partial R}{\partial \omega}\right)(x,\omega) = O(\epsilon(\omega+1)), \quad x \in \Omega. \tag{56}$$

Note that the estimates for $\nabla_x R$ in (55) and $\nabla_x(\frac{\partial R}{\partial \omega})$ in (56) can be derived using (49).

We now derive the outer expansion of $\mathbf{u}_0$. To this end, let us first recall the notion of the viscous moment tensor (VMT) from [4]. Let $(\mathbf{v}_{k\ell}, p)$, for $k, \ell = 1, 2, 3$, be the solution to

$$\begin{cases} \mu\Delta\mathbf{v}_{k\ell} - \nabla p = 0 & \text{in } \mathbb{R}^3 \setminus \overline{B}, \\ \tilde{\mu}\Delta\mathbf{v}_{k\ell} - \nabla p = 0 & \text{in } B, \\ \mathbf{v}_{k\ell}|_- - \mathbf{v}_{k\ell}|_+ = 0 & \text{on } \partial B, \\ (p\mathbf{N} - \tilde{\mu}\frac{\partial\mathbf{v}_{k\ell}}{\partial\mathbf{N}})\Big|_- - (p\mathbf{N} - \mu\frac{\partial\mathbf{v}_{k\ell}}{\partial\mathbf{N}})\Big|_+ = 0 & \text{on } \partial B, \\ \nabla\cdot\mathbf{v}_{k\ell} = 0 & \text{in } \mathbb{R}^3, \\ \mathbf{v}_{k\ell}(\tilde{x}) - \tilde{x}_k\mathbf{e}_\ell + \frac{\delta_{k\ell}}{3}\sum_{j=1}^3\tilde{x}_j\mathbf{e}_j = O(|\tilde{x}|^{-2}) & \text{as } |\tilde{x}| \to +\infty, \\ p(\tilde{x}) = O(|\tilde{x}|^{-3}) & \text{as } |\tilde{x}| \to +\infty. \end{cases} \tag{57}$$

Here $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is the standard basis of $\mathbb{R}^3$.

The VMT $V(\tilde{\mu}, \mu, B) = (V_{ijk\ell})_{i,j,k,\ell=1,2,3}$ is defined by

$$V_{ijk\ell}(\tilde{\mu}, \mu, B) := (\tilde{\mu} - \mu)\int_B \nabla\mathbf{v}_{k\ell}(\tilde{x}) : \widehat{\nabla}(\tilde{x}_i\mathbf{e}_j)\,d\tilde{x}, \tag{58}$$

where $:$ denotes the contraction of two matrices, *i.e.*, $A : B = \sum_{ij=1}^3 a_{ij}b_{ij}$.

Since $(\hat{\mathbf{u}}_0 - \hat{\mathbf{U}}_0, \hat{p}_0 - \hat{q}_0)$ satisfies

$$\begin{cases} (\Delta + \frac{\omega^2}{\mu})(\hat{\mathbf{u}}_0 - \hat{\mathbf{U}}_0) - \frac{1}{\mu}\nabla(\hat{p}_0 - \hat{q}_0) = 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}, \\ (\Delta + \frac{\omega^2}{\mu})(\hat{\mathbf{u}}_0 - \hat{\mathbf{U}}_0) - \frac{1}{\mu}\nabla(\hat{p}_0 - \hat{q}_0) = \omega^2\left(\frac{1}{\mu} - \frac{1}{\tilde{\mu}}\right)\hat{\mathbf{u}}_0 - \left(\frac{1}{\mu} - \frac{1}{\tilde{\mu}}\right)\nabla\hat{p}_0 & \text{in } D, \\ (\hat{\mathbf{u}}_0 - \hat{\mathbf{U}}_0)|_+ - (\hat{\mathbf{u}}_0 - \hat{\mathbf{U}}_0)|_- = 0 & \text{on } \partial D, \\ -\frac{1}{\mu}(\hat{p}_0 - \hat{q}_0)|_+\mathbf{N} + \frac{\partial}{\partial\mathbf{N}}(\hat{\mathbf{u}}_0 - \hat{\mathbf{U}}_0)|_+ \\ \quad = -\frac{1}{\mu}(\hat{p}_0 - \hat{q}_0)|_-\mathbf{N} + \frac{\partial}{\partial\mathbf{N}}(\hat{\mathbf{u}}_0 - \hat{\mathbf{U}}_0)|_- + \frac{\tilde{\mu}-\mu}{\mu}\frac{\partial\hat{\mathbf{u}}_0}{\partial\mathbf{N}}\Big|_- & \text{on } \partial D, \\ \nabla\cdot(\hat{\mathbf{u}}_0 - \hat{\mathbf{U}}_0) = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{59}$$

11

together with the radiation condition, the integration of the first equation in (59) against the Green's function $\mathbf{\Gamma}^{\frac{\omega}{\sqrt{\mu}}}(x,y)$ over $y \in \mathbb{R}^3 \setminus \overline{D}$ and the divergence theorem give us the following representation formula:

$$\hat{\mathbf{u}}_0(x) = \hat{\mathbf{U}}_0(x) + (\frac{\tilde{\mu}}{\mu} - 1) \int_{\partial D} \mathbf{\Gamma}^{\frac{\omega}{\sqrt{\mu}}}(x,y) \frac{\partial \hat{\mathbf{u}}_0}{\partial \mathbf{N}}\bigg|_{-}(y) d\sigma(y)$$

$$- (\frac{1}{\mu} - \frac{1}{\tilde{\mu}}) \int_D \mathbf{\Gamma}^{\frac{\omega}{\sqrt{\mu}}}(x,y) \nabla \hat{p}_0(y) \, dy + \omega^2 (\frac{1}{\mu} - \frac{1}{\tilde{\mu}}) \int_D \mathbf{\Gamma}^{\frac{\omega}{\sqrt{\mu}}}(x,y) \hat{\mathbf{u}}_0(y) \, dy. \qquad (60)$$

It follows from the inner expansion in Theorem 2.3 that, for $y \in \partial D$,

$$\frac{\partial \hat{\mathbf{u}}_0}{\partial \mathbf{N}}(y) = \frac{\partial \hat{\mathbf{U}}_0}{\partial \mathbf{N}}(y) + \frac{\partial \mathbf{v}}{\partial \mathbf{N}} \left(\frac{y-z}{\epsilon} \right) + O(\epsilon) \qquad (61)$$

and, for $x \in D$,

$$\nabla \hat{p}_0(x) = \tilde{\mu} \triangle \hat{\mathbf{u}}_0 + \omega^2 \hat{\mathbf{u}}_0 = \frac{\tilde{\mu}}{\epsilon}(\triangle \mathbf{v}) \left(\frac{x-z}{\epsilon} \right) + O(1) = \frac{1}{\epsilon}(\nabla q) \left(\frac{x-z}{\epsilon} \right) + O(1). \qquad (62)$$

Since

$$\tilde{\mu} \int_{\partial D} \frac{\partial \hat{\mathbf{u}}_0}{\partial \mathbf{N}}\bigg|_{-}(y) \, d\sigma(y) - \int_D \nabla \hat{p}_0(y) \, dy = -\omega^2 \int_D \hat{\mathbf{u}}_0(y) \, dy,$$

we obtain that for x far away from z, the following outer expansion holds:

$$\hat{\mathbf{u}}_0(x) \approx \hat{\mathbf{U}}_0(x) - \epsilon^3 \sum_{i,j,\ell=1}^{3} \partial_i \Gamma_{\ell j}^{\frac{\omega}{\sqrt{\mu}}}(x,z) \bigg[(\frac{\tilde{\mu}}{\mu} - 1) \int_{\partial B} \left(\frac{\partial \hat{\mathbf{U}}_0}{\partial \mathbf{N}}(z) + \frac{\partial \mathbf{v}}{\partial \mathbf{N}}\bigg|_{-}(\xi) \right)_j \xi_i \, d\sigma(\xi)$$

$$(\frac{1}{\mu} - \frac{1}{\tilde{\mu}}) \int_B \partial_j q(\xi) \xi_i \, d\xi \bigg] \mathbf{e}_\ell,$$

where $\partial_i \Gamma_{\ell j}^{\frac{\omega}{\sqrt{\mu}}}(x,z)$ is the differentiation with respect to the x variable and $\left(\frac{\partial \mathbf{v}}{\partial \mathbf{N}} \right)_j$ is the j-th component of $\frac{\partial \mathbf{v}}{\partial \mathbf{N}}$, which we may further simplify as follows

$$(\hat{\mathbf{u}}_0 - \hat{\mathbf{U}}_0)(x)$$

$$\approx -\epsilon^3 (\frac{\tilde{\mu}}{\mu} - 1) \sum_{i,j,\ell=1}^{3} \left[\partial_i \Gamma_{\ell j}^{\frac{\omega}{\sqrt{\mu}}}(x,z) \int_B \partial_j \mathbf{v}_i(\xi) + \partial_i \mathbf{v}_j(\xi) + \partial_j \hat{\mathbf{U}}_{0i}(z) + \partial_i \hat{\mathbf{U}}_{0j}(z) \, d\xi \right] \mathbf{e}_\ell.$$

$$(63)$$

Here $\mathbf{v}_j$ denotes the j-th component of $\mathbf{v}$.

Since

$$\mathbf{v}(\xi) = \sum_{p,q=1}^{3} \partial_q \hat{\mathbf{U}}_0(z)_p \mathbf{v}_{pq}(\xi) - \nabla \hat{\mathbf{U}}_0(z)\xi, \qquad (64)$$

we have

$$(\hat{\mathbf{u}}_0 - \hat{\mathbf{U}}_0)(x)$$

$$\approx -\epsilon^3 (\frac{\tilde{\mu}}{\mu} - 1) \sum_{i,j,\ell,p,q=1}^{3} \left[\partial_i \Gamma_{\ell j}^{\frac{\omega}{\sqrt{\mu}}}(x,z) \partial_q \hat{\mathbf{U}}_0(z)_p \int_B \partial_j (\mathbf{v}_{kl})_i(\xi) + \partial_i (\mathbf{v}_{kl})_j(\xi) \, d\xi \right] \mathbf{e}_\ell. \qquad (65)$$

We have the following theorem for the outer expansion.

Theorem 2.4 *Let Ω' be a compact region away from D, namely $\mathrm{dist}(\Omega', D) \geq C > 0$ for some constant C, and let*

$$R(x,\omega) = \hat{\mathbf{u}}_0(x,\omega) - \hat{\mathbf{U}}_0(x,\omega) + \frac{\epsilon^3}{\mu} \sum_{i,j,p,q,\ell=1}^{3} V_{ijkl} \partial_i \Gamma_{\ell j}^{\frac{\omega}{\sqrt{\mu}}}(x,z) \partial_q \hat{\mathbf{U}}_0(z)_p \mathbf{e}_\ell. \qquad (66)$$

There exists $\epsilon_0 > 0$ such that if $\epsilon\omega < \epsilon_0$, then

$$R(x,\omega) = O(\epsilon^4(\omega^3 + 1)), \quad x \in \Omega'. \qquad (67)$$

Moreover,

$$\frac{\partial R}{\partial \omega}(x,\omega) = O(\epsilon^4(\omega^2 + 1)), \quad x \in \Omega'. \qquad (68)$$

3 Far- and near-field asymptotic formulas in the transient regime

Recall that the inverse Fourier transform, $\mathbf{U}_0$, of $\hat{\mathbf{U}}_0$ satisfies

$$\begin{cases} (\partial_t^2 - \mu\Delta)\mathbf{U}_0(x,t) - \nabla F = \delta_{x=\bar{y}}\delta_{t=0}\mathbf{a} & \text{in } \mathbb{R}^3 \times \mathbb{R}, \\ \nabla \cdot \mathbf{U}_0 = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}, \\ \mathbf{U}_0(x,t) = 0 & \text{for } x \in \mathbb{R}^3 \quad \text{and } t \ll 0. \end{cases}$$

For $\rho > 0$, we define the operator P_ρ on tempered distributions by

$$P_\rho[\psi](t) = \int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} \hat{\psi}(\omega) \, d\omega, \qquad (69)$$

where $\hat{\psi}$ denotes the Fourier transform of ψ. The operator P_ρ truncates the high-frequency component of ψ.

One can easily show that $P_\rho[\mathbf{U}_0]$ satisfies

$$\begin{aligned} (\partial_t^2 - \Delta)P_\rho[\mathbf{U}_0](x,t) - \nabla P_\rho[F](x-y) = \delta_{x=\bar{y}}\psi_\rho(t)\mathbf{a} & \quad \text{in } \mathbb{R}^3 \times \mathbb{R}, \\ \nabla \cdot P_\rho[\mathbf{U}_0] = 0 & \quad \text{in } \mathbb{R}^3 \times \mathbb{R}, \end{aligned} \qquad (70)$$

where

$$\psi_\rho(t) := \frac{2\sin\rho t}{t} = \int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} d\omega.$$

The purpose of this section is to derive and asymptotic expansions for $P_\rho[\mathbf{u}_0 - \mathbf{U}_0](x,t)$. For doing so, we observe that

$$P_\rho[\mathbf{u}_0](x,t) = \int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} \hat{\mathbf{u}}_0(x,\omega) d\omega, \qquad (71)$$

where $\hat{\mathbf{u}}_0$ is the solution to (19). Therefore, according to Theorem 2.3, we have

$$P_\rho[\mathbf{u}_0 - \mathbf{U}_0](x,t) - \epsilon \sum_{p,q=1}^{3} \partial_q P_\rho[\mathbf{U}_0](z,t)_p[\mathbf{v}_{pq}(x) - x_p\mathbf{e}_q] = \int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} R(x,\omega) d\omega.$$

13

Suppose that $|t| \geq c_0$ for some positive number c_0 (c_0 is of order the distance between $\bar{y}$ and z). Then, integrating by parts gives

$$\left| \int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} R(x, \omega) d\omega \right| = \left| \frac{1}{t} \int_{|\omega| \leq \rho} \frac{d}{d\omega} e^{-\sqrt{-1}\omega t} R(x, \omega) d\omega \right|$$

$$\leq \frac{1}{|t|} \left(|R(x, \rho)| + |R(x, -\rho)| \right) + \int_{|\omega| \leq \rho} \left| \frac{\partial}{\partial \omega} R(x, \omega) \right| d\omega$$

$$\leq C\epsilon^2 \rho^2.$$

Thus we arrive at the following theorem.

Theorem 3.1 *Suppose that $\rho = O(\epsilon^{-\alpha})$ for some $\alpha < \frac{1}{2}$. Then*

$$P_\rho[\mathbf{u}_0 - \mathbf{U}_0](x, t) = \epsilon \sum_{p,q=1}^{3} \partial_q P_\rho[\mathbf{U}_0](z, t)_p [\mathbf{v}_{pq}(x) - x_p \mathbf{e}_q] + O(\epsilon^{2(1-\alpha)}).$$

We now derive a far-field asymptotic expansion for $P_\rho[\mathbf{u}_0 - \mathbf{U}_0]$. Let $\mathbf{G}_\infty(x, y, t)$ be the inverse Fourier transform of $\mathbf{\Gamma}^{\frac{\omega}{\sqrt{\mu}}}(x, y)$. Note that $\mathbf{G}_\infty$ is the limit of $\mathbf{G}$ given by (5) as $\alpha \to +\infty$. It then follows that

$$P_\rho[\mathbf{G}_\infty](x, y, t) = \int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} \mathbf{\Gamma}^{\frac{\omega}{\sqrt{\mu}}}(x, y) d\omega$$

$$= \frac{1}{4\pi} \frac{3\gamma_i \gamma_j - \delta_{ij}}{r^3} \left[\phi_\rho(t) - \phi_\rho(t - \frac{r}{\sqrt{\mu}}) \right] - \frac{1}{4\pi\mu} \frac{\gamma_i \gamma_j - \delta_{ij}}{r} \psi_\rho(t - \frac{r}{\sqrt{\mu}}), \qquad (72)$$

where $\phi_\rho(t) := \int_0^t \psi_\rho(s) ds$.

From Theorem 2.4, we get

$$\int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} (\hat{\mathbf{u}}_0(x, \omega) - \hat{\mathbf{U}}_0(x, \omega)) \, d\omega$$

$$= -\frac{\epsilon^3}{\mu} \int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} \left(\sum_{i,j,p,q,\ell=1}^{3} V_{ijpq} \partial_i \Gamma_{\ell j}^{\frac{\omega}{\sqrt{\mu}}}(x, z) \partial_q \hat{\mathbf{U}}_0(z)_p \mathbf{e}_\ell \right) d\omega$$

$$+ \int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} R(x, \omega) \, d\omega,$$

where the remainder is estimated by

$$\int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} R(x, \omega) \, d\omega = O(\epsilon^{4(1 - \frac{3}{4}\alpha)}).$$

Since

$$\int_{|\omega|\leq\rho} e^{-\sqrt{-1}\omega t}\left(\sum_{i,j,p,q,\ell=1}^{3} V_{ijpq}\partial_i\Gamma^{\frac{\omega}{\sqrt{\mu}}}_{\ell j}(x,z)\partial_q\hat{\mathbf{U}}_0(z)_p\mathbf{e}_\ell\right)d\omega$$

$$=\mu^{-1}\int_{|\omega|\leq\rho} e^{-\sqrt{-1}\omega t}\left(\sum_{i,j,p,q,k,\ell=1}^{3} V_{ijpq}\partial_i\Gamma^{\frac{\omega}{\sqrt{\mu}}}_{\ell j}(x,z)\partial_q\Gamma^{\frac{\omega}{\sqrt{\mu}}}_{pk}(z,\bar{y})a_k\mathbf{e}_\ell\right)d\omega$$

$$=\mu^{-1}\int_{\mathbb{R}}\left(\sum_{i,j,p,q,k,\ell=1}^{3} V_{ijpq}\partial_i P_\rho[\mathbf{G}_\infty]_{\ell j}(x,z,t-\tau)\partial_q P_\rho[\mathbf{G}_\infty]_{pk}(z,\bar{y},\tau)a_k\mathbf{e}_\ell\right)d\tau,$$

the following theorem holds.

Theorem 3.2 *Let* $\hat{\mathbf{U}}_0(x,\omega):=\frac{1}{\mu}\Gamma^{\frac{\omega}{\sqrt{\mu}}}(x-\bar{y})\mathbf{a}$. *Suppose that* $\rho=O(\epsilon^{-\alpha})$ *for some* $\alpha<\frac{1}{2}$. *Then for* $|x-z|\geq C>0$, *the following far-field expansion holds*

$$P_\rho[\mathbf{u}_0-\mathbf{U}_0](x,t)$$
$$=-\frac{\epsilon^3}{\mu^2}\int_{\mathbb{R}}\left(\sum_{i,j,p,q,k,\ell=1}^{3} V_{ijpq}\partial_i P_\rho[\mathbf{G}_\infty]_{\ell j}(x,z,t-\tau)\partial_q P_\rho[\mathbf{G}_\infty]_{pk}(z,\bar{y},\tau)a_k\mathbf{e}_\ell\right)d\tau \quad (73)$$
$$+O(\epsilon^{4(1-\frac{3}{4}\alpha)}).$$

Note that if we plug (72) in the far-field formula (73) then we can see that, unlike the acoustic case investigated in [3], the perturbation $P_\rho[\mathbf{u}_0-\mathbf{U}_0](x,t)$ can be seen not only as a polarized wave emitted from the anomaly but it contains, because of the term $(1/r^3)\phi_\rho(t)$ in (72), a near fieldlike term which does not propagate.

4 Asymptotic imaging

4.1 Far-field imaging: time-reversal

We present a time-reversal technique for detecting the location z of the anomaly from measurements of the perturbations at x away from the location z. As in the acoustic case, the main idea is to take advantage of the reversibility of the elastic wave equation in a non-viscous medium in order to back-propagate signals to the sources that emitted them [5, 10].

Let S be a sphere englobing the anomaly D. Consider, for simplicity, the harmonic regime, we get

$$\int_S\left[\frac{\partial\overline{\Gamma^{\frac{\omega}{\sqrt{\mu}}}}}{\partial n}(x,z)\Gamma^{\frac{\omega}{\sqrt{\mu}}}(x,y)-\overline{\Gamma^{\frac{\omega}{\sqrt{\mu}}}}(x,z)\frac{\partial\Gamma^{\frac{\omega}{\sqrt{\mu}}}}{\partial n}(x,y)\right]d\sigma(x)=2\sqrt{-1}\Im m\Gamma^{\frac{\omega}{\sqrt{\mu}}}(y,z),$$

for $y\in\Omega$, and therefore, for $\mathbf{w}(x):=\hat{\mathbf{u}}_0(x,\omega)-\hat{\mathbf{U}}_0(x,\omega)$, it follows that

$$\int_S\left[\frac{\partial\overline{\mathbf{w}}}{\partial n}(x,\omega)\Gamma^{\frac{\omega}{\sqrt{\mu}}}(x,z)-\overline{\mathbf{w}}(x,\omega)\frac{\partial\Gamma^{\frac{\omega}{\sqrt{\mu}}}}{\partial n}(x,z)\right]d\sigma(x)$$
$$=2\sqrt{-1}\frac{\epsilon^3}{\mu}\nabla\hat{\mathbf{U}}_0(z,\omega)V(\tilde{\mu},\mu,B)\nabla_z\Im m\Gamma^{\frac{\omega}{\sqrt{\mu}}}(y,z)+O(\epsilon^4\omega^3),$$

15

if $\omega > 1$.

This shows that the anti-derivative of time-reversal perturbation focuses on the location of the anomaly with an anisotropic focal spot. Because of the structure of the Green function $\mathbf{\Gamma}^{\frac{\omega}{\sqrt{\mu}}}(y, z)$, time-reversing the perturbation gives birth to a near fieldlike effect. Moreover, the diffraction limit depends on the direction. It is, unlike the acoustic case, anisotropic. These interesting findings were experimentally observed and first reported in [9]. Our asymptotic formula (73) clearly explains them.

4.2 Near-field imaging: optimization approach

Set Ω to be a window containing the anomaly D. Theorem 3.1 suggests to reconstruct the shape and the shear modulus of the elastic inclusion D by minimizing the following functional:

$$\int_{T-\Delta T}^{T+\Delta T} ||P_\rho[\mathbf{u}_0 - \mathbf{U}_0](x, t) - \epsilon \sum_{p,q=1}^3 \partial_q P_\rho[\mathbf{U}_0](z, t)_p [\mathbf{v}_{pq}(x) - x_p \mathbf{e}_q]||_{L^2(\Omega)}^2,$$

where $T = |\bar{y} - z|/\sqrt{\mu}$ is the arrival time and ΔT is a window time. One can add a total variation regularization term.

The choice of the space and time window sizes are critical. If they are too large, then noisy images are obtained. If they are too small, then resolution is poor. The optimal window sizes are related to the signal-to-noise ratio of the recorded near-field measurements. They express the trade-off between resolution and stability. This will be the subject of a next paper.

5 Conclusion

In this paper we have rigorously establish asymptotic expansions of near- and far-field measurements of the transient elastic wave induced by a small elastic anomaly. We have proved that, after truncation of the high-frequency component, the perturbation due to the anomaly can be seen not only as a polarized wave emitted from the anomaly but it contains unlike the acoustic case a near fieldlike term which does not propagate. We have also shown that time-reversing this perturbation gives birth to a near fieldlike effect. Moreover, the diffraction limit is anisotropic. We have then explained the experimental findings reported in [9].

In this paper we have only considered a purely quasi-incompressible elasticity model. In a forthcoming work, we will consider the problem of reconstructing a small anomaly in a viscoelastic medium from wavefield measurements. Expressing the ideal elastic field without any viscous effect in terms of the measured field in a viscous medium, we will generalize the methods described here to recover the viscoelastic and geometric properties of an anomaly from wavefield measurements.

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