Optimal estimates and asymptotics for the stress concentration between closely located stiff inclusions*

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Abstract

If stiff inclusions are closely located, then the stress, which is the gradient of the solution, may become arbitrarily large as the distance between two inclusions tends to zero. In this paper we investigate the asymptotic behavior of the stress concentration factor, which is the normalized magnitude of the stress concentration, as the distance between two inclusions tends to zero. For that purpose we show that the gradient of the solution to the case when two inclusions are touching decays exponentially fast near the touching point. We also prove a similar result when two inclusions are closely located and there is no potential difference on boundaries of two inclusions. We then use these facts to show that the stress concentration factor converges to a certain integral of the solution to the touching case as the distance between two inclusions tends to zero. We then present an efficient way to compute this integral.

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Key words. conductivity equation; anti-plane elasticity; stress; gradient blow-up; extreme conductivity; stress concentration factor

1 Introduction and statement of results

In presence of closely located stiff inclusions embedded in the relatively weak matrix, high stress concentration occurs in the narrow region between two inclusions. Such a phenomenon typically occurs in fiber-reinforced materials and the stiff inclusions represent the cross section of fibers. Recently, much effort has been devoted to quantitative understanding of this stress concentration. In this paper we continue our investigation on this and establish an efficient method to compute the magnitude of the stress concentration that immediately yields an asymptotic formula for the stress distribution and an optimal estimate for the concentration.

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To describe the problem and results in a precise manner, let D_1^0 and D_2^0 be a pair of (touching) bounded domains with $\mathcal{C}^{2,\gamma}$ ($\gamma > 0$) boundaries such that

$$D_1^0 \subset \{(x,y) \in \mathbb{R}^2 \mid x < 0\}, \quad D_2^0 \subset \{(x,y) \in \mathbb{R}^2 \mid x > 0\}, \tag{1.1}$$

$$\partial D_1^0 \cap \partial D_2^0 = \{(0,0)\}, \tag{1.2}$$

and D_1^0 and D_2^0 are convex at $(0,0)$. The domains D_1^0 and D_2^0 are strongly convex at $(0,0)$ if both D_1^0 and D_2^0 have positive curvatures there. By translating D_2^0 by a positive number ϵ along x-axis, while D_1^0 is fixed, we obtain D_2^ϵ, *i.e.*,

$$D_2^\epsilon := D_2^0 + (\epsilon, 0)\,. \tag{1.3}$$

When there is no possibility of confusion, we drop superscripts and denote

$$D_1 := D_1^0, \quad D_2 := D_2^\epsilon. \tag{1.4}$$

For a given harmonic function h in $\mathbb{R}^2$, let u_ϵ be the solution to the problem

$$\begin{cases} \Delta u_\epsilon = 0 & \text{in } \mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)}, \\ u_\epsilon = \lambda_j \text{ (constant)} & \text{on } \partial D_j,\ j = 1,2, \\ u_\epsilon(\mathbf{x}) - h(\mathbf{x}) = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty, \end{cases} \tag{1.5}$$

where the constants λ_i are determined by the conditions

$$\int_{\partial D_1} \partial_\nu u_\epsilon ds = \int_{\partial D_2} \partial_\nu u_\epsilon ds = 0. \tag{1.6}$$

Here and throughout this paper $\partial_\nu u_\epsilon$ denotes the outward normal derivative of u_ϵ on ∂D_j ($j = 1, 2$). It is worth emphasizing that the constants λ_1 and λ_2 may or may not be different depending on the given h.

As mentioned before, inclusions D_1 and D_2 represent the two dimensional cross-sections of two parallel elastic fibers embedded in an infinite elastic matrix and ϵ is the distance between them. The solution u_ϵ represents the out-of-plane elastic displacement, and ∇u_ϵ is proportional to the shear stress. The problem (1.5) may also be regarded as two dimensional conductivity equation in which case D_1 and D_2 represent perfect conductors of infinite conductivity. It is worth mentioning that we consider the situation where there are only two inclusions since our interest lies in estimating local high concentration of stress in the narrow region between two inclusions. There is a study to estimate global stress in a composite (with many inclusions) using a network approximation. We refer to [8] and references therein for that. We also mention that the problem under consideration in this paper has some connection with effective properties of composites with highly conducting inclusion. See [25, Section 10.10] for this connection.

In general, ∇u_ϵ becomes arbitrarily large as the distance ϵ between two inclusions tends to zero, and the problem is to derive pointwise estimates of ∇u_ϵ in terms of ϵ. This problem was raised in [5] and there has been significant progress on it. It has been proved that the generic blow-up rate of ∇u_ϵ is $1/\sqrt{\epsilon}$ in two dimensions [2, 3, 4, 6, 10, 14, 22, 26, 27], and $|\epsilon \ln \epsilon|^{-1}$ in three dimensions [6, 7, 13, 15, 16, 17, 21]. We emphasize that the gradient may or may not blow up depending on the given background harmonic function h. For example, in the configuration of this paper the gradient blows up if $h(x,y) = x$ and it does not if $h(x,y) = y$ for circular inclusions D_1 and D_2. It is worth while to mention that the insulating case in two dimensions can be treated by duality as done in [4] for example. But the insulating case in three dimensions is an open problem: it is not even clear if the gradient actually blows up in three dimensions. It is also worth while to mention that if the conductivity of the inclusions is finite (away from ∞ and 0), ∇u_ϵ is bounded regardless of ϵ [9, 19, 20].

Recently a better understanding of the stress concentration has been obtained: an asymptotic behavior of ∇u_ϵ has been characterized by the singular function associated with D_1 and D_2, as ϵ tends to 0. The singular function, denoted by q_ϵ, is the solution to the following problem:

$$\begin{cases} \Delta q_\epsilon = 0 & \text{in } \mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)}, \\ q_\epsilon = \text{constant} & \text{on } \partial D_j, \ j = 1,2, \\ q_\epsilon(\mathbf{x}) = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty, \\ \displaystyle\int_{\partial D_1} \partial_\nu q_\epsilon ds = -1, & \displaystyle\int_{\partial D_2} \partial_\nu q_\epsilon ds = 1. \end{cases} \tag{1.7}$$

We emphasize that the constant values of q_ϵ on ∂D_1 and ∂D_2 are different, so that ∇q_ϵ blows up as $\epsilon \to 0$.

Let us recall some important facts about q_ϵ: If D_1^0 and D_2^0 are disks, then q_ϵ is given explicitly by

$$q_\epsilon(\mathbf{x}) := \frac{1}{2\pi} \left(\ln |\mathbf{x} - \mathbf{p}_1| - \ln |\mathbf{x} - \mathbf{p}_2| \right), \tag{1.8}$$

where $\mathbf{p}_1 \in D_1$ is the fixed point of the mixed reflection $R_1 R_2$ where R_j is the reflection with respect to ∂D_j, $j = 1,2$, and $\mathbf{p}_2 \in D_2$ is that of $R_2 R_1$. We emphasize that these points can be computed easily (see (4.11)). More generally, if D_1^0 and D_2^0 are strongly convex at $(0,0)$, then let B_1 and B_2 be disks osculating to D_1 and D_2 at $(0,0)$ and $(\epsilon, 0)$, respectively, and let $q_{B,\epsilon}$ be the singular function associated with B_1 and B_2 as given in (1.8). Then, it is proved in [1] that the behavior of ∇q_ϵ is almost explicitly described as

$$\nabla q_\epsilon = \nabla q_{B,\epsilon} (1 + O(\epsilon^{\gamma/2})) + O(1) \tag{1.9}$$

when ∂D_j is $\mathcal{C}^{2,\gamma}$ and $\gamma \in (0,1)$. Thus it follows that

$$\|\nabla q_\epsilon\|_\infty = \frac{\sqrt{\kappa_1 + \kappa_2}}{\sqrt{2}\pi} \frac{1}{\sqrt{\epsilon}} + O(1), \tag{1.10}$$

where κ_2 and κ_2 are the curvatures of D_1^0 and D_2^0 at $(0,0)$, respectively.

Using the singular function q_ϵ, the solution u_ϵ to (1.5) can be decomposed as

$$u_\epsilon = \alpha_\epsilon q_\epsilon + r_\epsilon \tag{1.11}$$

where

$$\alpha_\epsilon = \frac{u_\epsilon|_{\partial D_2} - u_\epsilon|_{\partial D_1}}{q_\epsilon|_{\partial D_2} - q_\epsilon|_{\partial D_1}}. \tag{1.12}$$

Observe that r_ϵ is also constant on ∂D_1 and ∂D_2, and $r_\epsilon|_{\partial D_1} = r_\epsilon|_{\partial D_2}$, so that ∇r_ϵ is bounded on bounded subsets of $\mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)}$ (see [12]). It means that the term $\alpha_\epsilon \nabla q_\epsilon$ is responsible for the blow-up of ∇u_ϵ, or more precisely,

$$\nabla u_\epsilon = \alpha_\epsilon \nabla q_\epsilon + O(1) \quad \text{as } \epsilon \to 0. \tag{1.13}$$

In particular, α_ϵ represents the magnitude (normalized by $|\nabla q_\epsilon|$) of the blow-up. So, it is appropriate to call the constant α_ϵ the *stress concentration factor*.

The purpose of this paper is to analyze the stress concentration factor α_ϵ when D_1^0 and D_2^0 are convex at $(0,0)$. We are particularly interested in finding $\lim_{\epsilon\to 0} \alpha_\epsilon$ (existence of the limit is a part of the study).

There have been some work on the stress concentration factor. It is proved in [12] that if D_1^0 and D_2^0 are disks, then

$$\alpha_\epsilon = \frac{2r_1 r_2}{r_1 + r_2} (\mathbf{n} \cdot \nabla h)(0,0) + O(\sqrt{\epsilon}) \quad \text{as } \epsilon \to 0, \tag{1.14}$$

where r_j is the radius of D_j, $j = 1, 2$, and $\mathbf{n}$ is the outward unit normal vector to ∂D_1 at $(0,0)$. An estimate for α_ϵ in terms of curvatures, size and ϵ was established in [22] under the assumption that an inclusion has a much higher curvature than its size. It is also proved in [1] that if D_1^0 and D_2^0 are strongly convex at $(0,0)$, then

$$\alpha_\epsilon = \frac{\sqrt{2}\pi}{\sqrt{\kappa_1 + \kappa_2}} \frac{1}{\sqrt{\epsilon}} \int_{\partial D_1 \cup \partial D_2} h \partial_\nu q_\epsilon ds \left(1 + O(\epsilon^{\gamma/2})\right), \tag{1.15}$$

and, as a consequence, that α_ϵ is bounded regardless of ϵ.

Observe that even if (1.9) yields a good information of q_ϵ on the narrow region in between two inclusions, it is still difficult to evaluate the integral on the righthand side of (1.15) since it requires global information of $\partial_\nu q_\epsilon$. In this paper, we present a new efficient method for finding $\lim_{\epsilon \to 0} \alpha_\epsilon$. It turns out that the limit is given as a certain integral of the solution u_0 for the touching case, namely,

$$\begin{cases} \Delta u_0 = 0 & \text{in } \Omega, \\ u_0 = \lambda_0 & \text{on } \partial\Omega, \\ u_0(\mathbf{x}) - h(\mathbf{x}) = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty, \end{cases} \tag{1.16}$$

where $\Omega := \mathbb{R}^2 \setminus \overline{(D_1^0 \cup D_2^0)}$ and λ_0 is a constant determined by the additional condition

$$\int_\Omega |\nabla(u_0 - h)|^2 dA < \infty. \tag{1.17}$$

To obtain the main result of this paper (Theorem 1.1), we first consider the touching case problem (1.16). We show that there exists a unique solution u_0 to (1.16) and $\nabla u_0(x,y)$ decays exponentially fast as (x,y) approaches to $(0,0)$ in Ω (Theorem 2.1). It is worth mentioning that Ω has cusps at the origin. We also prove a similar theorem for the residual part r_ϵ in (1.11) (Theorem 3.1). This result was also obtained in [18] in a more general context. However, we include a proof in this paper since it is completely different from that in the paper mentioned above.

We prove these results in somewhat more general setting: We assume that the domains D_1^0 and D_2^0 are convex at $(0,0)$ and their order of contact at the point is $2m$ for some positive integer m. Thus, if ∂D_j^0 near $(0,0)$ is given as the graph of $x = x_j(y)$ $(j = 1,2)$, then there are constants $\delta_0 > 0$ and $c_j > 0$, $j = 1, \ldots, 4$, such that

$$-c_1 y^{2m} \le x_1(y) \le -c_2 y^{2m} \quad \text{and} \quad c_3 y^{2m} \le x_2(y) \le c_4 y^{2m}, \tag{1.18}$$

for $|y| < \delta_0$. If D_1^0 and D_2^0 are strongly convex at $(0,0)$, then $m = 1$.

In terms of the solution u_0 to (1.16) we obtain the following theorem regarding the asymptotic behavior of the stress concentration factor.

Theorem 1.1 *Suppose that* ∂D_j^0 $(j = 1,2)$ *are* $\mathcal{C}^{2,\gamma}$ *for some* $\gamma > 0$*,* D_1^0 *and* D_2^0 *are convex at* $(0,0)$*, and their order of contact at* $(0,0)$ *is* $2m$*. Let* u_0 *be the solution to* (1.16) *and let*

$$\alpha_0 := \int_{\partial D_1^0} \partial_\nu u_0 \, ds. \tag{1.19}$$

Then,

$$\alpha_\epsilon = \alpha_0 + O\left(\epsilon |\log \epsilon|^{2m-1}\right) \quad \text{as } \epsilon \to 0. \tag{1.20}$$

As an immediate consequence of Theorem 1.1 and (1.13), we obtain

$$\nabla u_\epsilon = \alpha_0 \nabla q_\epsilon + O(1) \quad \text{as } \epsilon \to 0 \tag{1.21}$$

in any bounded subset of $\mathbb{R}^2 \setminus (D_1 \cup D_2)$. Thus, the limit α_0 can be regarded as an alternative concentration factor. Moreover, if D_1^0 and D_2^0 are strongly convex at $(0,0)$, then we have from (1.8) and (1.9) that

$$\nabla u_\epsilon(\mathbf{x}) = \frac{\alpha_0}{2\pi} \left(\frac{\mathbf{x} - \mathbf{p}_1}{|\mathbf{x} - \mathbf{p}_1|^2} - \frac{\mathbf{x} - \mathbf{p}_2}{|\mathbf{x} - \mathbf{p}_2|^2} \right) (1 + O(\epsilon^{\gamma/2})) + O(1). \tag{1.22}$$

These formulas have some important consequences. As a first consequence, we have the following identity (see section 4 for a proof):

$$\lim_{\epsilon \to 0} \sqrt{\epsilon} |\nabla u_\epsilon(\epsilon/2, 0)| = \frac{\alpha_0 \sqrt{\kappa_1 + \kappa_2}}{\sqrt{2}\pi}, \tag{1.23}$$

where κ_i is the curvature of ∂D_i^0 at $(0,0)$ for $i = 1,2$. Note that $(\epsilon/2, 0)$ is a point where $|\nabla u_\epsilon|$ has a value close to the maximal concentration.

Another consequence of (1.21) and (1.22) is related to numerical computation of ∇u_ϵ. Since high concentration of the gradient occurs in the narrow region, fine meshes may be required to compute ∇u_ϵ. However, since (1.21) and (1.22) extract the major singular term in an explicit way, it suffices to compute the residual term ∇r_ϵ for which only regular meshes are required. This idea was exploited in [12] in the special case when D_j's are disks using (1.8), (1.11) and (1.14). Implementation of this idea for the general case of strongly convex domains will be the subject of the forthcoming work. It is worth mentioning that there are some other methods to compute the solution when D_1 and D_2 are disks. See, for examples, [11, 23].

The last subject of this paper is regarding computation of α_0. It turns out that, thanks to exponentially decaying property of the solution to the touching case, α_0 can be computed numerically only using regular meshes by truncating the narrow region near $(0,0)$.

This paper is organized as follows. We investigate the touching case in section 2. In section 3 we obtain an estimate for the gradient of the residual term r_ϵ. Section 4 is to prove Theorem 1.1. In the last section we present a way to compute good approximations of α_0.

2 The solution for the touching case

In this section we prove the following theorem regarding the problem (1.16).

Theorem 2.1 *Suppose that ∂D_j^0 $(j=1,2)$ are $\mathcal{C}^{2,\gamma}$ for some $\gamma>0$, D_1^0 and D_2^0 are convex at $(0,0)$, and their order of contact at $(0,0)$ is finite. Then, there is a unique solution u_0 to (1.16), and there are positive constants A, C and δ such that*

$$|\nabla u_0(x,y)| \le C\exp\left(-\frac{A}{|y|}\right) \tag{2.1}$$

for $|y|\le\delta$ and $x_1(y)<x<x_2(y)$, where x_1 and x_2 are the defining functions of ∂D_1^0 and ∂D_2^0 near $(0,0)$.

The estimate (2.1) follows from

$$|u_0(x,y)-\lambda_0| \le C\exp\left(-\frac{A}{|y|}\right) \tag{2.2}$$

by a standard estimate for harmonic functions. Here the constants A and C may differ at each occurrence. In fact, since ∂D_j^0 are $\mathcal{C}^{2,\gamma}$ and $u_0-\lambda_0=0$ on ∂D_j^0, one can show that $u_0(x,y)-\lambda_0$ can be extended by reflection (after the conformal transformations to outside a disk) as harmonic functions for (x,y) satisfying

$$x_1(y)-sy^{2m}<x<x_2(y)$$

and

$$x_1(y)<x<x_2(y)+sy^{2m}$$

for some $s>0$, and the same estimate (2.2) holds for the extended functions. Here $2m$ is the order of contact at $(0,0)$. So, for each $(x,y)\in\overline{\mathbb{R}^2\setminus(D_1^0\cup D_2^0)}$, there is $r>0$ such that $r>ty^{2m}$ for some $t>0$ and u_0 is harmonic in $B_r(x,y)$. ($B_r(c)$ denotes the disk of radius r with the center at c.) So, we have

$$|\nabla u_0(x,y)| \le \frac{C}{r^3}\int_{B_r(x,y)}|u_0-\lambda_0|\,dA \le \frac{C}{y^{6m}}\exp\left(-\frac{A}{|y|}\right) \le C'\exp\left(-\frac{A'}{|y|}\right) \tag{2.3}$$

for some constant A' and C', which is the desired estimate.

The rest of this section is devoted to proving existence and uniqueness of u_0, and (2.2). To construct the solution to (1.16) we use the transformation $1/z$, following [24]. Identify $\mathbf{x}=(x,y)$ in the plane with $z=x+iy$ and let

$$\Phi(z)=\frac{1}{z}.$$

Define

$$\widetilde{\Omega} := \Phi(\Omega), \quad \Gamma_1 := \Phi(\partial D_1^0), \quad \Gamma_2 := \Phi(\partial D_2^0).$$

Note that Γ_1 and Γ_2 are simple curves lying in the left and right half spaces, respectively, and $\widetilde{\Omega}$ is the region enclosed by Γ_1 and Γ_2. Since ∂D_j^0 is $\mathcal{C}^{2,\gamma}(\gamma > 0)$ and D_j^0 is convex at $(0,0)$ for $j = 1, 2$, one can easily see that there are constant $a < b$ such that

$$\widetilde{\Omega} \subset \{w = \xi + i\eta \mid a < \xi < b\}. \tag{2.4}$$

Moreover, Γ_1 near ∞ is given by $\xi = \psi_1(\eta)$ for some function ψ_1 satisfying

$$\psi_1(\eta) \le -C_1|\eta|^{2-2m} \tag{2.5}$$

for some constant $C_1 > 0$, and Γ_2 near ∞ is given by $\xi = \psi_2(\eta)$ for some function ψ_2 satisfying

$$\psi_2(\eta) \ge C_2|\eta|^{2-2m} \tag{2.6}$$

for some constant $C_2 > 0$. In fact, we have

$$\psi_1(\eta) = \frac{x_1(y)}{y^2 + x_1(y)^2} \quad \text{with} \quad \eta = \frac{-y}{y^2 + x_1(y)^2} \tag{2.7}$$

on ∂D_1^0 near $(0,0)$. Thanks to (1.18), we have

$$a < \psi_1(\eta) \le \frac{-c_1 y^{2m}}{y^2 + x_1(y)^2} \le -C_1|\eta|^{2-2m}.$$

Thus we have (2.5). (2.6) can be proved similarly.

We need the following lemma whose proof will be given after completing the proof of Theorem 2.1.

Lemma 2.2 *Let ψ_j $(j = 1, 2)$ be as defined by* (2.7)*, and let a and b be the constants such that*

$$a < \psi_1(\eta) < \psi_2(\eta) < b \tag{2.8}$$

for all $\eta > L$, where L is a large number. Let R be a domain given by

$$R := \{(\xi, \eta) \mid \eta > L, \ \psi_1(\eta) < \xi < \psi_2(\eta)\}, \tag{2.9}$$

and let U be the solution in $H^1(R)$ to the problem

$$\begin{cases} \Delta U &= 0 & \text{in } R, \\ U &= 0 & \text{on } \xi = \psi_j(\eta), \ j = 1, 2, \\ U &= \varphi & \text{on } \Gamma := \{(\xi, L) \mid \psi_1(L) < \xi < \psi_2(L)\}, \end{cases} \tag{2.10}$$

where φ is a bounded function. Then there are positive constants A and C such that

$$|U(\xi, \eta)| \le Ce^{-A\eta} \tag{2.11}$$

for all $\eta > L$.

Because of (2.4), the Poincaré inequality holds in $\widetilde{\Omega}$: for all $\tilde{u} \in H_0^1(\widetilde{\Omega})$ (the standard Sobolev space with the zero trace)

$$\|\tilde{u}\|_{L^2(\widetilde{\Omega})}^2 \le C\|\nabla \tilde{u}\|_{L^2(\widetilde{\Omega})}^2 \tag{2.12}$$

for some constant C. So, one can apply the Lax-Milgram Theorem to show that for $f \in H^{-1}(\widetilde{\Omega})$ there exists a unique solution $\tilde{v} \in H_0^1(\widetilde{\Omega})$ to

$$\begin{cases} \Delta \tilde{v} &= f \quad \text{in } \widetilde{\Omega}, \\ \tilde{v} &= 0 \quad \text{on } \partial\widetilde{\Omega}. \end{cases} \tag{2.13}$$

We choose $r_0 > 0$ such that

$$D_1^0 \cup D_2^0 \subset B_{r_0/2}(0).$$

Let χ be a smooth function such that $\chi(z) = 1$ if $z \in B_{r_0}(0)$ and $\chi(z) = 0$ if $z \notin B_{2r_0}(0)$. For h given in (1.16), let

$$f(w) = \Delta_w \left(\chi\left(\frac{1}{w}\right) h\left(\frac{1}{w}\right)\right),$$

and let $\tilde{v}$ be the solution to (2.13) with this f. Then one can check that u_0 given by

$$u_0(z) = h(z) + \left(\tilde{v}\left(\frac{1}{z}\right) - \chi(z)h(z) - \tilde{v}(0)\right) \tag{2.14}$$

is the solution to (1.16) and the constant value λ_0 is given by $-\tilde{v}(0)$. The uniqueness of the solution follows easily from the maximum principle.

Now, we show (2.2). If $z \in B_{r_0}(0) \setminus \overline{D_1^0 \cup D_2^0}$, then we have

$$u_0(z) - \lambda_0 = \tilde{v}\left(\frac{1}{z}\right). \tag{2.15}$$

Choose L so large that the support of f lies in between two lines $\eta = \pm L$. Let $\widetilde{\Omega}_{\pm L} := \widetilde{\Omega} \cap \{\pm\eta > L\}$, respectively. The boundedness of $\tilde{v}(\xi \pm (L+1)i)$ can be shown easily by a standard estimate for harmonic functions similarly to (2.3), since $\tilde{v} = 0$ on $\partial\widetilde{\Omega}_{\pm L} \cap \partial\widetilde{\Omega}$ and $\tilde{v} \in L^2(\widetilde{\Omega})$. We thus apply Lemma 2.2 to obtain

$$|\tilde{v}(\xi + i\eta)| \le C_1 e^{-A_1|\eta|} \quad \text{for } |\eta| > L+1 \tag{2.16}$$

for some positive constant A_1 and C_1. We may choose a small positive number δ_1 so that $\Phi(x,y) \in \widetilde{\Omega}_{+(L+1)} \cup \widetilde{\Omega}_{-(L+1)}$ for all $x+iy$ satisfying $|y| < \delta_1$ and $x_1(y) < x < x_2(y)$. Then, by (2.15) and (2.16), we have

$$|u_0(x,y) + \tilde{v}(0)| = |\tilde{v}(\xi + \eta i)| \le C_2 e^{-A_1|\eta|} \le C_3 e^{-\frac{A_2}{|y|}}, \tag{2.17}$$

for $|y| < \delta_1$. The last inequality follows from (2.7), since $|x_1(y)| \simeq |y|^{2m}$. Here and throughout this paper, $a \simeq b$ stands for $\frac{1}{C}a \le b \le Ca$ for some constant C independent of ϵ. This completes the proof of Theorem 2.1. □

Proof of Lemma 2.2. By translating and scaling if necessary, we may assume $a = 0$, $b = \pi$ and $L = 0$. Let

$$\tilde{R} := \{(\xi, \eta) \mid \eta > 0,\ 0 < \xi < \pi\}.$$

Decompose φ as $\varphi = \varphi_+ - \varphi_-$ where $\varphi_\pm$ are nonnegative and bounded, and then extend $\varphi_\pm$ to $[0, \pi] \times \{0\}$ by assigning 0 outside Γ, and denote them by $\tilde{\varphi}_\pm$. Let $V_\pm$ be a solution in $H^1(\tilde{R})$ to

$$\begin{cases} \Delta V_\pm = 0 & \text{in } \tilde{R}, \\ V_\pm(0, \eta) = V_\pm(\pi, \eta) = 0, & \eta > 0, \\ V_\pm(\xi, 0) = \tilde{\varphi}_\pm(\xi), & 0 \le \xi \le \pi. \end{cases}$$

Since $\tilde{\varphi}_\pm \ge 0$, we have $V_\pm \ge 0$, and by the maximum principle, we have

$$-V_- \le U \le V_+ \quad \text{in } \tilde{R}. \tag{2.18}$$

One can find the solutions $V_\pm$ by separation of variables. In fact, we have

$$V_\pm(\xi, \eta) = \sum_{n=1}^{\infty} a_n^\pm \sin n\xi \, e^{-n\eta},$$

where $a_n^\pm$ is the Fourier coefficients of $\varphi_\pm$. In particular, we have

$$|V_\pm(\xi, \eta)| \le \left(\sum_{n=1}^{\infty} |a_n^\pm|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} e^{-2n\eta} \right)^{1/2} \le Ce^{-\eta}. \tag{2.19}$$

for $\eta \ge 1$. Even for $0 < \eta < 1$, this inequality holds with another constant C since $\tilde{\varphi}_\pm$ are bounded. Thus, (2.11) follows from (2.18). This completes the proof. □

3 The behavior of ∇r_ϵ in the narrow region

In this section, we consider the behavior of the gradient of r_ϵ given in (1.11) in the narrow region between D_1 and D_2 which we denote by N_δ for $\delta > 0$, namely,

$$N_\delta := \{(x, y) \mid x_1(y) < x < x_2(y) + \epsilon,\ |y| < \delta\}. \tag{3.1}$$

Recall that r_ϵ satisfies

$$\begin{cases} \Delta r_\epsilon = 0 \quad \text{in } \Omega := \mathbb{R}^2 \setminus \overline{D_1 \cup D_2}, \\ r_\epsilon|_{\partial D_1} = r_\epsilon|_{\partial D_2} = \text{constant}, \\ r_\epsilon(\mathbf{x}) - h(\mathbf{x}) = O(|\mathbf{x}|^{-1}) \quad \text{as } |\mathbf{x}| \to \infty. \end{cases} \tag{3.2}$$

In the previous section, it has been shown that ∇u_0 is decreasing exponentially near origin. The following theorem shows that ∇r_ϵ has such a decay property. As mentioned in Introduction, this result was also obtained in [18] in a more general setting. But two proofs are completely different.

Theorem 3.1 *Suppose that ϵ is sufficiently small. There are positive constants A, C and δ independent of $\epsilon > 0$ such that*

$$|\nabla r_\epsilon(x,y)| \le C \exp\left(-\frac{A}{\sqrt{\epsilon}+|y|}\right) \tag{3.3}$$

for any $(x,y) \in N_\delta$.

We prove the following lemma from which Theorem 3.1 follows by a standard elliptic estimate as explained briefly in the previous section.

Lemma 3.2 *Suppose that ϵ is sufficiently small. There are positive constants A, C and δ independent of ϵ such that*

$$|r_\epsilon(x,y) - r_\epsilon|_{\partial D_1}| \le C \exp\left(-\frac{A}{\sqrt{\epsilon}+|y|}\right) \tag{3.4}$$

for any $(x,y) \in N_\delta$.

Proof. As before we identify points (x,y) in $\mathbb{R}^2$ with $z = x+iy$ in $\mathbb{C}$. Choose two disks B_1 and B_2 whose centers are on the real axis such that

$$B_j \subset D_j, \ j=1,2, \ \ \partial B_1 \cap \partial D_1 = \{0\}, \ \text{ and } \ \partial B_2 \cap \partial D_2 = \{\epsilon\}.$$

Let c_j and ρ_j be the center and radius, respectively, of B_j for $j=1,2$. It is convenient to assume that $c_2 = 1+\epsilon$ so that $\rho_2 = 1$. Let

$$\Phi_1(z) = \frac{1}{z-(1+\epsilon)} \tag{3.5}$$

which is the reflection with respect to ∂B_2 (and translation), and

$$\Omega_1 := \Phi_1(\Omega), \quad B_3 := \Phi_1(B_1), \quad B_4 = \Phi_1(B_2), \tag{3.6}$$

and let c_j and ρ_j be the center and radius of B_j. Then, $c_4 = 0$, $\rho_4 = 1$, and $\rho_3 = c_3 - \Phi_1(0)$. Observe that

$$\Omega_1 \subset B_4 \setminus \overline{B_3}, \tag{3.7}$$

the reflected domain Ω_1 touches ∂B_3 and ∂B_4 at $\Phi_1(0)$ and $\Phi_1(\epsilon)$, respectively, and

$$\text{dist}\,(\partial B_3, \partial B_4) = \Phi_1(0) - \Phi_1(\epsilon) = -\frac{1}{1+\epsilon} + 1 = \epsilon + O(\epsilon^2). \tag{3.8}$$

Let

$$S := \left\{ w \mid w = \xi + i\eta \in B_4 \setminus \overline{B_3}, \ \xi < c_3, \ |\eta| < \rho_3/2 \right\}. \tag{3.9}$$

Then one can choose δ independently of ϵ so that $\Phi_1(N_\delta) \subset S$. Let

$$\tilde{r}_\epsilon(w) := r_\epsilon \circ \Phi_1^{-1}(w) - r_\epsilon|_{\partial D_1}, \quad w \in \Omega_1. \tag{3.10}$$

If $z = x+iy \in N_\delta$ and $w = \xi + i\eta = \Phi_1(z)$, then $\eta \simeq y$. Thus in order to prove (3.4), it suffices to show

$$|\tilde{r}_\epsilon(w)| \le C \exp\left(-\frac{A}{\sqrt{\epsilon}+|\eta|}\right) \tag{3.11}$$

for any $w \in \Phi_1(N_\delta)$.

We now transform B_3 so that the transformed disk becomes concentric to B_4 (B_4 is the unit disc). For that purpose let us write a lemma which can be easily verified.

Lemma 3.3 *Let $B_\rho(c)$ be a disk such that $\overline{B_\rho(c)} \subset B_1(0)$. Then there is α with $|\alpha| < 1$ and $\rho_* > 0$ such that the Möbius transform φ_α defined by*

$$\varphi_\alpha(w) = \frac{w - \alpha}{1 - \bar{\alpha} w} \tag{3.12}$$

maps $B_\rho(c)$ onto $B_{\rho_*}(0)$. In fact, α is given by

$$\alpha = \left[(|c|^2 - \rho^2 + 1) - \sqrt{(|c|^2 - \rho^2 + 1)^2 - 4|c|^2} \right] \frac{c}{2|c|^2}. \tag{3.13}$$

It is worth mentioning that Möbius transforms are automorphisms on $B_1(0)$.

Let Φ_2 be the Möbius transform defined by (3.12) and (3.13) with $c = c_3$ and $\rho = \rho_3$, and let $B_5 = B_{\rho_5}(c_5) := \Phi_2(B_3)$. Then $c_5 = 0$. Since $\rho_3 = c_3 + \frac{1}{1+\epsilon}$, one can see from (3.13) that α is real and satisfies

$$\alpha = -1 + \beta\sqrt{\epsilon} + (1 + \frac{1}{c})\epsilon + O(\epsilon\sqrt{\epsilon}), \tag{3.14}$$

where

$$\beta = \sqrt{\frac{2(c_3 + 1)}{|c_3|}}.$$

To compute ρ_5, we observe that $\Phi_2(-\frac{1}{1+\epsilon}) \in \partial B_5$, and from (3.14) that

$$\Phi_2(-\frac{1}{1+\epsilon}) = \frac{-\frac{1}{1+\epsilon} - \alpha}{1 + \frac{\overline{\alpha}}{1+\epsilon}} = -1 + \gamma\sqrt{\epsilon} + O(\epsilon),$$

where $\gamma = 2/\beta$. So, we have

$$\rho_5 = 1 - \gamma\sqrt{\epsilon} + O(\epsilon). \tag{3.15}$$

We emphasize that (3.15) implies in particular that

$$\text{dist}\,(\partial B_5, \partial B_4) = \gamma\sqrt{\epsilon} + O(\epsilon), \tag{3.16}$$

since $B_4 = B_1(0)$.

The proof of the following lemma will be given later in this section. Here $\arg(z)$ for $z \neq 0$ is supposed to take a value in $[0, 2\pi)$.

Lemma 3.4 *Suppose that ϵ is sufficiently small. There exists a constant $C > 0$ independent of ϵ such that*

$$arg\,(\Phi_2\,(w)) \geq \frac{C\sqrt{\epsilon}}{|\eta| + \sqrt{\epsilon}} \tag{3.17}$$

for $w = \xi + \eta i \in S$ with $\eta \geq 0$, and

$$2\pi - arg\,(\Phi_2\,(w)) \geq \frac{C\sqrt{\epsilon}}{|\eta| + \sqrt{\epsilon}} \tag{3.18}$$

for $\eta \leq 0$.

Let us introduce one more transformation Φ_3:

$$\Phi_3(\zeta) = \log \zeta \tag{3.19}$$

with the branch cut on the positive real axis. Then Φ_3 maps $(B_4 \setminus \overline{B_5}) \setminus \{\text{positive real axis}\}$ onto the rectangle $(a_0, 0) \times (0, 2\pi)$ where $a_0 = \log \rho_5 < 0$. We emphasize that

$$a_0 = -\gamma\sqrt{\epsilon} + O(\epsilon), \tag{3.20}$$

which is a consequence of (3.15).

Let θ_0 be the constant on the righthand side of (3.17) with $\eta = \rho_3/2$, *i.e.*,

$$\theta_0 := \frac{C\sqrt{\epsilon}}{\frac{\rho_3}{2} + \sqrt{\epsilon}}. \tag{3.21}$$

Define $\Phi := \Phi_3 \circ \Phi_2$, and $R_{\theta_0} := (a_0, 0) \times (\theta_0, 2\pi - \theta_0)$. Then $\Phi^{-1}(R_{\theta_0}) \cap \Omega_1$ is a bounded subset of Ω_1. Define

$$\Omega_{\theta_0} := \Phi(\Phi^{-1}(R_{\theta_0}) \cap \Omega_1). \tag{3.22}$$

Then Ω_{θ_0} is a connected subset of R_{θ_0} and has two lateral boundaries denoted by l_1 and l_2. Let

$$\check{r}_\epsilon(r, \theta) := (\tilde{r}_\epsilon \circ \Phi^{-1})(r, \theta), \quad (r, \theta) \in \Omega_{\theta_0}, \tag{3.23}$$

where $\tilde{r}_\epsilon$ is given in (3.10). Then, $\check{r}_\epsilon$ satisfies

$$\begin{cases} \Delta \check{r}_\epsilon = 0 & \text{in } \Omega_{\theta_0}, \\ \check{r}_\epsilon = 0 & \text{on } l_1 \cup l_2. \end{cases} \tag{3.24}$$

We have the following lemma whose proof will be given at the end of this section.

Lemma 3.5 *There is a constant C such that for $(r, \theta) \in \Omega_{\theta_0}$*

$$|\check{r}_\epsilon(r, \theta)| \le C \exp\left(-\frac{\pi}{|a_0|}(\theta - \theta_0)\right) \quad \text{if } \theta \le \pi \tag{3.25}$$

and

$$|\check{r}_\epsilon(r, \theta)| \le C \exp\left(-\frac{\pi}{|a_0|}(2\pi - \theta_0 - \theta)\right) \quad \text{if } \theta \ge \pi. \tag{3.26}$$

The desired inequality (3.11) follows from (3.25) and (3.26). To see this, we first observe that if $r + i\theta = \Phi_3 \circ \Phi_2(w)$, then $e^{r+i\theta} = \Phi_2(w)$, in other words, $\theta = \arg\Phi_2(w)$. Because of (3.20) and (3.21), we have $\theta_0/|a_0| \le C$ for some constant C independent of ϵ provided that ϵ is sufficiently small. Observe that if $w = \xi + i\eta \in S$ and $\eta > 0$, then $\theta = \arg\Phi_2(w) < \pi$. So it follows from Lemma 3.4 and (3.25) that

$$|\tilde{r}_\epsilon(w)| \le C \exp\left(-\frac{\pi}{|a_0|}\theta\right) \le C_1 \exp\left(-\frac{A}{\sqrt{\epsilon} + |\eta|}\right).$$

For $w = u + iv \in S$ with $\eta \le 0$, $\theta = \arg\Phi_2(w) \ge \pi$. Lemmas 3.4 and 3.5 also yield

$$|\tilde{r}_\epsilon(w)| \le C \exp\left(-\frac{\pi}{|a_0|}(2\pi - \theta)\right) \le C_2 \exp\left(-\frac{A_1}{\sqrt{\epsilon} + |\eta|}\right).$$

So we have (3.11) and the proof of Lemma 3.2 is completed. $\square$

Let us now prove Lemma 3.4 and Lemma 3.5.

Proof of Lemma 3.4. In this proof, we shall consider the case when $w = \xi + \eta i \in S$ with $\eta \geq 0$ only. We first note that

$$\operatorname{Im}\Phi_2(w) = \frac{\eta(1-\alpha^2)}{(1-\alpha\xi)^2 + \alpha^2\eta^2}.$$

Using (3.14) one can see that

$$1 - \alpha^2 \geq C\sqrt{\epsilon}$$

if ϵ is sufficiently small, since $|\xi| \leq \frac{1}{2}\rho_3 \leq \frac{1}{2}$. We observe that for $w = \xi + i\eta \in S$,

$$1 + \xi \leq 1 + c_3 - \sqrt{\rho_3^2 - \eta^2} = 1 + c_3 - \rho_3 + \frac{\eta^2}{\rho_3 + \sqrt{\rho_3^2 - \eta^2}} \leq \epsilon + \frac{\eta^2}{\rho_3},$$

since $w \in B_4 \setminus B_3$ and $\operatorname{dist}(\partial B_3, \partial B_4) = -\frac{1}{1+\epsilon} + 1 \leq \epsilon$ by (3.8).

If $\eta \geq \sqrt{\epsilon}$, then

$$\operatorname{Im}\Phi_2(w) \geq C_1 \frac{\eta\sqrt{\epsilon}}{\epsilon + \eta^2} \geq C_2 \frac{\sqrt{\epsilon}}{\eta}$$

by (3.14) and the property that $\frac{\eta}{\rho_3} \leq 1$. Since $|\Phi_2(w)| \geq \frac{1}{2}$ by (3.15), we have

$$\sin(\arg\Phi_2(w)) = \frac{\operatorname{Im}\Phi_2(w)}{|\Phi_2(w)|} \geq 2C_2 \frac{\sqrt{\epsilon}}{\eta}.$$

Thus we have

$$\arg\Phi_2(w) \geq C_3 \frac{\sqrt{\epsilon}}{\eta}. \tag{3.27}$$

If $0 \leq \eta < \sqrt{\epsilon}$, then there exists $w_0 = \xi_0 + i\eta_0$ with $|\eta_0| = \sqrt{\epsilon}$ so that

$$\arg\Phi_2(w) \geq \arg\Phi_2(w_0),$$

so it follows from (3.27) that

$$\arg\Phi_2(w) \geq C_3.$$

This proves (3.17). $\square$

Proof of Lemma 3.5. By definition, Ω_{θ_0} is a subset of R_{θ_0}, and $\partial\Omega_{\theta_0} \cap \partial R_{\theta_0}$ belongs to $\theta = \theta_0$ or $2\pi - \theta_0$. We define functions $\psi_\pm$ in R_{θ_0} as the solutions to

$$\begin{cases} \Delta\psi_\pm = 0 & \text{in } R_{\theta_0}, \\ \psi_\pm(r,\theta) = 0 & \text{on } \partial R_{\theta_0} \setminus \partial\Omega_{\theta_0}, \\ \psi_\pm(r,\theta) = \max\{\pm\check{r}_\epsilon(r,\theta), 0\} & \text{on } \partial R_{\theta_0} \cap \partial\Omega_{\theta_0}. \end{cases} \tag{3.28}$$

It was shown in [1] that $\|r_\epsilon - h\|_{L^\infty(\Omega)}$ is bounded independently of ϵ. So, there is a constant M independent of $\epsilon > 0$ such that

$$|\psi_\pm(r,\theta)| \leq M \quad \text{for all } (r,\theta) \in R_{\theta_0}, \tag{3.29}$$

and it can be shown in the same way as (2.18) in the previous section that

$$-\psi_- \le \check{r}_\epsilon \le \psi_+ \quad \text{in } \Omega_{\theta_0}. \tag{3.30}$$

So to prove (3.26) it suffices to show

$$|\psi_\pm(r,\theta)| \le C \exp\left(-\frac{\pi}{|a_0|}(2\pi - \theta_0 - \theta)\right) \text{ for } \theta \in [\pi, 2\pi - \theta_0]. \tag{3.31}$$

We prove (3.31) only for ψ_+ since the proof for ψ_- is identical. The solution ψ_+ can be found explicitly:

$$\psi_+ = \psi_+^e + \psi_+^o$$

where ψ_+^e and ψ_+^o are the even and odd parts about $\theta = \pi$ given by

$$\begin{aligned}
\psi_+^e(r,\theta) &= \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi}{|a_0|} r\right) \cosh\left(\frac{n\pi}{|a_0|}(\theta - \pi)\right) \\
\psi_+^o(r,\theta) &= \sum_{n=1}^{\infty} \beta_n \sin\left(\frac{n\pi}{|a_0|} r\right) \sinh\left(\frac{n\pi}{|a_0|}(\theta - \pi)\right)
\end{aligned}$$

for some constants α_n and β_n.

Suppose that $\theta \ge \pi$. Then we have

$$\begin{aligned}
\left|\psi_+^e(r,\theta)\right| &\le \sum_{n=1}^{\infty} |\alpha_n| \exp\left(\frac{n\pi}{|a_0|}(\theta - \pi)\right) \\
&\le 2\sum_{n=1}^{\infty} |\alpha_n| \cosh\left(\frac{n\pi}{|a_0|}(\pi - \theta_0)\right) \exp\left(-\frac{n\pi}{|a_0|}(2\pi - \theta_0 - \theta)\right)
\end{aligned}$$

Note that

$$\left(\sum_{n=1}^{\infty} |\alpha_n|^2 \cosh^2\left(\frac{n\pi}{|a_0|}(\pi - \theta_0)\right)\right)^{1/2} = \left(\frac{2}{|a_0|}\right)^{1/2} \|\psi_+^e(\cdot, 2\pi - \theta_0)\|_{L^2([a_0,0])} \le \sqrt{2}M,$$

since $\|\sin\left(\frac{n\pi}{|a_0|} r\right)\|_{L^2([a_0,0])} = \left(\frac{1}{2}|a_0|\right)^{1/2}$. So it follows from the Cauchy-Schwarz inequality that

$$\left|\psi_+^e(r,\theta)\right| \le 2\sqrt{2}M \left(\sum_{n=1}^{\infty} \exp\left(-\frac{2n\pi}{|a_0|}(2\pi - \theta_0 - \theta)\right)\right)^{1/2},$$

and hence

$$\left|\psi_+^e(r,\theta)\right| \le C \exp\left(-\frac{\pi}{|a_0|}(2\pi - \theta_0 - \theta)\right) \tag{3.32}$$

for some constant C. Since $\psi_+^e(r,\theta) = \psi_+^e(r, 2\pi - \theta)$,

$$\left|\psi_+^e(r,\theta)\right| \le C \exp\left(-\frac{\pi}{|a_0|}(\theta - \theta_0)\right)$$

when $\theta < \pi$ as well.

Since $\sinh B \le \sinh A\, e^{B-A}$ if $0 < B < A$, we obtain, for $\theta \ge \pi$,

$$\begin{aligned}\left|\psi_+^o(r,\theta)\right| &\le \sum_{n=1}^{\infty} |\beta_n| \sinh\left(\frac{n\pi}{|a_0|}(\theta-\pi)\right) \\ &\le \sum_{n=1}^{\infty} |\beta_n| \sinh\left(\frac{n\pi}{|a_0|}(\pi-\theta_0)\right) \exp\left(-\frac{n\pi}{|a_0|}(2\pi-\theta_0-\theta)\right),\end{aligned}$$

and hence

$$\left|\psi_+^o(r,\theta)\right| \le C \exp\left(-\frac{\pi}{|a_0|}(2\pi-\theta_0-\theta)\right). \tag{3.33}$$

Because of symmetry of ψ_+^o, we have

$$\left|\psi_+^o(r,\theta)\right| \le C \exp\left(-\frac{\pi}{|a_0|}(\theta-\theta_0)\right),$$

when $\theta < \pi$ as well. So we have (3.31) and the proof is complete. □

4 Proofs of Theorem 1.1

In this section we prove Theorem 1.1 and (1.23).

Proof of Theorem 1.1. One can see from (1.6), (1.7) and (1.11) that

$$\alpha_\epsilon = \int_{\partial D_1^0} \partial_\nu r_\epsilon \, ds. \tag{4.1}$$

So, it is enough to prove

$$\left| \int_{\partial D_1^0} \partial_\nu (r_\epsilon - u_0) ds \right| \le C\epsilon |\log \epsilon|^{2m-1} \tag{4.2}$$

for some constant C independent of ϵ.

Let $V := \mathbb{R}^2 \setminus \overline{(D_1^0 \cup D_2^0 \cup D_2^\epsilon)}$, and let $\Gamma_1 := \partial D_2^0 \setminus D_2^\epsilon$ and $\Gamma_2 := \partial D_2^\epsilon \setminus D_2^0$. Then, ∂D_1, Γ_1, and Γ_2 constitute the boundary of V. Let

$$\varphi_\epsilon(\mathbf{x}) := r_\epsilon(\mathbf{x}) - u_0(\mathbf{x}) - (r_\epsilon(0,0) - u_0(0,0)). \tag{4.3}$$

Then, φ_ϵ is a bounded harmonic function in V and $\varphi_\epsilon \equiv 0$ on ∂D_1. We claim that

$$|\varphi_\epsilon(\mathbf{x})| \le C\epsilon, \quad \mathbf{x} \in V. \tag{4.4}$$

In fact, if $\mathbf{x} \in \Gamma_1$, then $u_0(\mathbf{x}) - u_0(0,0) = 0$ and $r_\epsilon(\mathbf{x}+\epsilon) - r_\epsilon(0,0) = 0$. Therefore, since ∇r_ϵ is bounded on any bounded subset of $\mathbb{R}^2 \setminus \overline{D_1 \cup D_2}$ (refer to [1]), we have

$$|\varphi_\epsilon(\mathbf{x})| = |r_\epsilon(\mathbf{x}) - r_\epsilon(0,0)| = |r_\epsilon(\mathbf{x}) - r_\epsilon(\mathbf{x}+\epsilon)| \le C\epsilon. \tag{4.5}$$

Likewise we have for $\mathbf{x} \in \Gamma_2$

$$|\varphi_\epsilon(\mathbf{x})| \le C\epsilon. \tag{4.6}$$

Since $\varphi_\epsilon \equiv 0$ on ∂D_1 and $\varphi_\epsilon(\mathbf{x})$ is bounded, we obtain (4.4) by the maximum principle.

Choose M so large that

$$D_1^0 \subset \left(-\frac{M}{2}, 0\right) \times \left(-\frac{M}{2}, \frac{M}{2}\right),$$

and let $\omega = (-M, 0) \times (-M, M)$. Since φ_ϵ is harmonic in V, we have

$$\int_{\partial D_1^0} \partial_\nu (r_\epsilon - u_0) ds = \int_{\partial D_1^0} \partial_\nu \varphi_\epsilon ds = \int_{\partial \omega} \partial_\nu \varphi_\epsilon ds.$$

Divide $\partial\omega$ into three pieces: $\partial\omega = \gamma_1 \cup \gamma_2 \cup \gamma_3$ where

$$\gamma_1 := \left\{(0, y) \mid |y| \leq \frac{A_0}{|\log \epsilon|}\right\}, \quad \gamma_2 := \left\{(0, y) \mid \frac{A_0}{|\log \epsilon|} < |y| \leq M\right\}, \quad \gamma_3 := \partial\omega \setminus (\gamma_1 \cup \gamma_2),$$

and write

$$\int_{\partial\omega} \partial_\nu \varphi_\epsilon ds = \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} \partial_\nu \varphi_\epsilon ds := I + II + III.$$

Here, the constant A_0 is given by Theorems 2.1 and 3.1 so that

$$|\nabla u_0(x, y)| + |\nabla r_0(x, y)| \leq C \exp\left(-\frac{A_0}{|y| + \sqrt{\epsilon}}\right) \tag{4.7}$$

for $|y| < \delta$ and $x \in (x_1(y), x_2(y))$.

If $-\frac{A_0}{|\log \epsilon|} \leq y \leq \frac{A_0}{|\log \epsilon|}$, then (4.7) implies that

$$|\nabla \varphi_\epsilon(0, y)| \leq C \exp\left(-\frac{A_0}{|y| + \sqrt{\epsilon}}\right).$$

Thus we have

$$|I| \leq C\epsilon. \tag{4.8}$$

If $\frac{A_0}{|\log \epsilon|} < |y| \leq M$, there is $r > Cy^{2m}$ for some C such that $B_r(0, y) \subset V$. It then follows from a gradient estimate for harmonic functions and (4.4) that

$$|\nabla \varphi_\epsilon(0, y)| \leq C \frac{\epsilon}{y^{2m}},$$

and

$$|II| \leq C\epsilon \int_{\frac{A_0}{|\log \epsilon|} < |y| \leq M} \frac{1}{y^{2m}} dy \leq C\epsilon |\log \epsilon|^{2m-1}. \tag{4.9}$$

There is a constant $r > 0$ such that $B_r(\mathbf{x}) \subset V$ for all $\mathbf{x} \in \gamma_3$. So, we have from (4.4) that for any $\mathbf{x} \in \gamma_3$,

$$|\nabla \varphi_\epsilon(\mathbf{x})| \leq C \frac{\epsilon}{r} \leq C\epsilon,$$

and

$$|III| \leq C\epsilon. \tag{4.10}$$

Now, (4.2) follows from (4.8), (4.9), and (4.10), and the proof is complete. □

The formula (1.23) is an immediate consequence of (1.22). In fact, if r_1 and r_2 are radii of circles osculating to ∂D_1 and ∂D_2 at $(0,0)$ and $(\epsilon, 0)$, respectively, then it is proved in [21] that $\mathbf{p}_1$ and $\mathbf{p}_2$ which are fixed points of mixed reflections are given by

$$\mathbf{p}_i = \left((-1)^i \sqrt{2} \sqrt{\frac{r_1 r_2}{r_1 + r_2}} \sqrt{\epsilon} + O(\epsilon), 0 \right) \quad \text{as } \epsilon \to 0. \tag{4.11}$$

So we obtain (1.23) from (1.22).

5 Approximations of α_0

The region outside $D_1^0 \cup D_2^0$ has cusps at $(0,0)$, and it may cause some problem in computing α_0. To avoid this trouble, we show that by replacing the cusp with a neck a good approximation of α_0 can be obtained.

For $\rho > 0$ let

$$D_{(\rho)} = \left(D_1^0 \cup D_2^0\right) \cup ([-\rho, \rho] \times [-\rho, \rho]) \tag{5.1}$$

which is of dumbbell shape, and let $u_{(\rho)}$ be the solution to

$$\begin{cases} \Delta u_{(\rho)} = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D_{(\rho)}}, \\ u_{(\rho)} = \lambda_{(\rho)} \text{ (constant)} \quad \text{on } \partial D_{(\rho)}, \\ u_{(\rho)}(\mathbf{x}) - h(\mathbf{x}) = O(|\mathbf{x}|^{-1}) \quad \text{as } |\mathbf{x}| \to \infty, \end{cases} \tag{5.2}$$

where the constant $\lambda_{(\rho)}$ is determined by the additional condition

$$\int_{\partial D_{(\rho)}} \frac{\partial u_{(\rho)}}{\partial \nu}\bigg|_+ ds = 0. \tag{5.3}$$

We have the following theorem.

Theorem 5.1 *Let δ be the number appearing in Theorem 2.1. For $\rho \in (0, \delta/2)$, let*

$$\alpha_{(\rho)} = \int_{\partial D_1^0 \setminus [-2\rho, 2\rho] \times [-2\rho, 2\rho]} \partial_\nu u_{(\rho)} \, ds. \tag{5.4}$$

Then, there are constants C and A such that

$$\left|\alpha_0 - \alpha_{(\rho)}\right| \le C \exp\left(-\frac{A}{\rho}\right). \tag{5.5}$$

Proof. Choose a point z_0 on the common boundary of $D_{(\rho)}$ and $D_1^0 \cup D_2^0$ and let

$$\varphi(z) := u_{(\rho)}(z) - u_0(z) - (u_{(\rho)}(z_0) - u_0(z_0)).$$

Since $u_{(\rho)}(z) - u_{(\rho)}(z_0) = 0$ for all $z \in \partial D_{(\rho)}$ and $u_0(z) - u_0(z_0) = 0$ on $\partial D_1^0 \cup \partial D_2^0$, we have

$$\varphi(z) = 0, \quad z \in \partial D_{(\rho)} \setminus ([-\rho, \rho] \times [-\rho, \rho]). \tag{5.6}$$

On the other hand, if $x_1(\rho) \le x \le x_2(\rho)$, then we have from (2.2)

$$|u_0(x + i\rho) - u_0(z_0)| \le C e^{-\frac{A}{\rho}},$$

and hence

$$|\varphi(x+i\rho)| = |u_0(x,\rho) - u_0(z_0)| \le Ce^{-\frac{A}{\rho}}. \tag{5.7}$$

Similarly one can see that if $x_1(-\rho) \le x \le x_2(-\rho)$, then

$$|\varphi(x-i\rho)| \le Ce^{-\frac{A}{\rho}}. \tag{5.8}$$

It follows from (5.6), (5.7), and (5.8) that

$$|\varphi(z)| \le Ce^{-\frac{A}{\rho}} \tag{5.9}$$

for all $z \in \partial D_{(\rho)}$, and hence for all $z \in \mathbb{R}^2 \setminus \overline{D_{(\rho)}}$ by the maximum principle. Note that we may apply the maximum principle since $u_{(\rho)}(z) - u_0(z) \to 0$ as $|z| \to \infty$.

We now estimate $\nabla(u_{(\rho)}(z) - u_0(z)) = \nabla\varphi(z)$ on $\partial D_1^0 \setminus [-2\rho, 2\rho] \times [-2\rho, 2\rho]$. Because of (5.6), one can apply the argument used right after of Theorem 2.1 to see that $\varphi(z)$ can be extended across $\partial D_1^0 \setminus [-2\rho, 2\rho] \times [-2\rho, 2\rho]$ so that the extended function is harmonic in $\overline{B_r(z)}$ for all $z \in \partial D_1^0 \setminus [-2\rho, 2\rho] \times [-2\rho, 2\rho]$ where $r = s\rho^{2m}$ for some $s > 0$ (independent of ρ). Then by the gradient estimate for harmonic functions we have

$$|\nabla(u_{(\rho)} - u_0)(z)| \le C_2 e^{-\frac{A_2}{\rho}}, \quad z \in \partial D_1^0 \setminus [-2\rho, 2\rho] \times [-2\rho, 2\rho]. \tag{5.10}$$

It then follows from (2.1) and (5.10) that

$$\begin{aligned} |\alpha_0 - \alpha_{(\rho)}| &\le \int_{(\partial D_1^0)\setminus[-2\rho,2\rho]\times[-2\rho,2\rho]} \left|\partial_\nu(u_{(\rho)} - u_0)\right| \, ds + \int_{(\partial D_1^0)\cap([-2\rho,2\rho]\times[-2\rho,2\rho])} |\partial_\nu u_0| \, ds \\ &\le Ce^{-\frac{A_3}{\rho}}. \end{aligned}$$

This completes the proof. □

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