LAYER POTENTIAL APPROACHES TO INTERFACE PROBLEMS

by

Hyeonbae Kang

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1. Introduction

This paper reviews recent progress on imaging by generalized polarization tensors (GPTs), enhancement of near-cloaking by GPT-vanishing structures, cloaking by anomalous localized resonance, and analysis of stress concentration. These seemingly unrelated problems are all interface problems, and an integral operator called the Neumann-Poincaré operator arises naturally from them. We discuss about boundedness and invertibility properties, and spectral property of this operator, and then relate these properties with above mentioned problems.

2. Neumann-Poincaré operator

We begin our investigation by looking into the classical Neumann boundary value problem. Let Ω be a bounded domain in $\mathbb{R}^d$ (smoothness of the boundary $\partial\Omega$ will be specified later) and consider for a given Neumann data g the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here and throughout this paper ν denotes the unit outward normal vector to $\partial\Omega$. We emphasize that g satisfies $\int_{\partial\Omega} g = 0$ for compatibility and the solution u is assumed to satisfy $\int_{\partial\Omega} u = 0$ to guarantee uniqueness of the solution.

A classical way of solving (1) is to use layer potentials. Let $\Gamma(x)$ be the fundamental solution to the Laplacian, *i.e.*,

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi} \ln |x| \, , & d = 2 \, , \\ \dfrac{1}{(2-d)\omega_d} |x|^{2-d} \, , & d \geq 3 \, , \end{cases} \tag{2}$$

where ω_d denotes the area of the unit sphere in $\mathbb{R}^d$. The single layer potential $\mathcal{S}_{\partial\Omega}[\varphi]$ of a density function $\varphi \in L^2(\partial\Omega)$ is defined by

$$\mathcal{S}_{\partial\Omega}[\varphi](x) := \int_{\partial\Omega} \Gamma(x-y)\varphi(y)\, d\sigma(y) \, , \quad x \in \mathbb{R}^d. \tag{3}$$

If we set $u(x) = \mathcal{S}_{\partial\Omega}[\varphi](x)$ for some function φ, then u is harmonic in Ω. So in order for u to be the solution to (1), it suffices to choose φ so that the boundary condition is fulfilled.

The single layer potential $\mathcal{S}_{\partial\Omega}[\varphi]$ satisfies the jump relation

$$\frac{\partial}{\partial \nu} \mathcal{S}_{\partial\Omega}[\varphi]\big|_{\pm}(x) = \left(\pm \frac{1}{2} I + \mathcal{K}_{\partial\Omega}^* \right)[\varphi](x), \quad x \in \partial\Omega \, , \tag{4}$$

where the operator $\mathcal{K}_{\partial\Omega}$ is defined by

$$\mathcal{K}_{\partial\Omega}[\varphi](x) = \frac{1}{\omega_d} \int_{\partial\Omega} \frac{\langle y-x, \nu_y \rangle}{|x-y|^d} \varphi(y)\, d\sigma(y) \, , \quad x \in \partial\Omega, \tag{5}$$

and $\mathcal{K}_{\partial\Omega}^*$ is its L^2-adjoint, *i.e.*,

$$\mathcal{K}_{\partial\Omega}^*[\varphi](x) = \frac{1}{\omega_d} \int_{\partial\Omega} \frac{\langle x-y, \nu_x \rangle}{|x-y|^d} \varphi(y)\, d\sigma(y) \, . \tag{6}$$

Here $\pm$ indicates the limits (to $\partial\Omega$) from outside and inside of Ω, respectively. So in order to fulfill the boundary condition in (1), we need to solve the integral equation

$$\left(-\frac{1}{2} I + \mathcal{K}_{\partial\Omega}^* \right)[\varphi] = g \quad \text{on } \Omega. \tag{7}$$

The operator $\mathcal{K}_{\partial\Omega}$ (or $\mathcal{K}^*_{\partial\Omega}$) is called the Neumann-Poincaré (NP) operator associated with the domain Ω. It is well known (see for example [**21, 53, 76**]) that if $\partial\Omega$ is smooth ($\mathcal{C}^{1,\alpha}$ for some $\alpha > 0$), then

(i) $\mathcal{K}^*_{\partial\Omega}$ is a compact operator on $L^2(\partial\Omega)$,
(ii) the spectrum of $\mathcal{K}^*_{\partial\Omega}$ lies in $(-\frac{1}{2}, \frac{1}{2}]$,
(iii) $\frac{1}{2}I + \mathcal{K}^*_{\partial\Omega}$ is invertible on $L^2(\partial\Omega)$ and $-\frac{1}{2}I + \mathcal{K}^*_{\partial\Omega}$ is invertible on $L^2_0(\partial\Omega)$.

Here, $L^2_0(\partial\Omega)$ is the collection of all L^2 functions with the mean zero. If $g \in L^2_0(\partial\Omega)$ (to satisfy the compatibility condition), the solution to (7) is given by

$$\varphi = \left(-\frac{1}{2}I + \mathcal{K}^*_{\partial\Omega}\right)^{-1}[g],$$

and the solution to (1) by

$$u(x) = \mathcal{S}_{\partial\Omega}\left(-\frac{1}{2}I + \mathcal{K}^*_{\partial\Omega}\right)^{-1}[g](x), \quad x \in \Omega.$$

A few remarks on the above-mentioned properties of $\mathcal{K}^*_{\partial\Omega}$ are in order. If $\partial\Omega$ is $\mathcal{C}^{1,\alpha}$, then because of orthogonality of the normal vector and the tangential vector, we have

$$\frac{|\langle x-y, \nu_x\rangle|}{|x-y|^d} \le \frac{C}{|x-y|^{d-1-\alpha}}, \quad x, y \in \partial\Omega, \tag{8}$$

which makes $\mathcal{K}^*_{\partial\Omega}$ compact. Property (iii) can be proved using the Fredholm alternative. We emphasize that property (ii) holds not only for smooth domains but also for domains with Lipschitz boundaries. Even if we restrict our investigation here mostly to the domains with smooth boundaries, it is worth while to review two important results on the properties of the NP operators associated with Lipschitz domains. If $\partial\Omega$ is Lipschitz, then $\mathcal{K}^*_{\partial\Omega}$ is a singular integral operator, and L^2-boundedness of $\mathcal{K}^*_{\partial\Omega}$ was proved by Calderón [**44**] when the Lipschitz constant of $\partial\Omega$ is small, and by Coifman, McIntosh, and Meyer [**50**] for the general case. In this regards, it is worth mentioning $T[1]$ Theorem of David and Journé [**52**] which states that a singular integral operator T is bounded on L^2 if and only if $T[1]$ is a function of bounded mean oscillation. Invertibility as stated in (iii) for Lipschitz domains was proved by Verchota [**109**].

To motivate our discussion on the spectrum of the NP operator, we consider another problem: a transmission problem. Suppose that an inclusion Ω is immersed in the free space $\mathbb{R}^d$. Suppose that the conductivity (or the dielectric constant) of Ω is ϵ_c and that of the background is ϵ_m ($\epsilon_c \neq \epsilon_m$). So, the distribution of the conductivity is given by

$$\sigma_\Omega = \epsilon_c \chi(\Omega) + \epsilon_m \chi(\mathbb{R}^d \setminus \overline{\Omega}),$$

where χ denotes the indicator function. The problem we consider is

$$\begin{cases} \nabla \cdot \sigma_\Omega \nabla u = 0 & \text{in } \mathbb{R}^d, \\ u(x) - h(x) = O(|x|^{1-d}) & \text{as } |x| \to \infty, \end{cases} \tag{9}$$

for a given harmonic function h in $\mathbb{R}^d$. Note that without the inclusion Ω the solution to (9) is nothing but $u(x) = h(x)$. In the presence of the inclusion, the solution takes the form $u = h +$ something, and this something is generated by the discontinuity of the conductivity along $\partial\Omega$. It turns out (see for example [**21, 74**]) that there is a potential φ on $\partial\Omega$ such that the solution is given by

$$u(x) = h(x) + \mathcal{S}_{\partial\Omega}[\varphi](x), \quad x \in \mathbb{R}^d. \tag{10}$$

Since u satisfies the transmission conditions, $u|_- = u|_+$ (continuity of potential) and $\epsilon_c \frac{\partial u}{\partial \nu}|_- = \epsilon_m \frac{\partial u}{\partial \nu}|_+$ (continuity of flux), one can see from the jump relation (4) that the following relation holds:

$$\left(\frac{\epsilon_c + \epsilon_m}{2(\epsilon_c - \epsilon_m)} I - \mathcal{K}^*_{\partial\Omega} \right) [\varphi] = \frac{\partial h}{\partial \nu} \quad \text{on } \partial\Omega. \tag{11}$$

We emphasize that the problem (9) is elliptic if (and only if) ϵ_c and ϵ_m are positive, and in this case the number $\frac{\epsilon_c+\epsilon_m}{2(\epsilon_c-\epsilon_m)}$ does not belong to $[-1/2, 1/2]$, where the spectrum of $\mathcal{K}^*_{\partial\Omega}$ lies. So, as long as we are interested in elliptic problems, there is no need to look into the spectrum of $\mathcal{K}^*_{\partial\Omega}$. The spectrum of the NP operator is a classical subject of research since Poincaré. See a recent paper [**105**] and references therein for a brief history of this. Recently there has been renewed interest in the spectrum of the NP operator in relation to the plasmonic structures consisting of inclusions with negative dielectric constants, *i.e.*, with $\epsilon_c < 0$ (while ϵ_m stays positive). In this case, $\frac{\epsilon_c+\epsilon_m}{2(\epsilon_c-\epsilon_m)}$ may lie in the spectrum of $\mathcal{K}^*_{\partial\Omega}$. As we will see in the next section, if $\partial\Omega$ is $\mathcal{C}^{1,\alpha}$ (so that $\mathcal{K}^*_{\partial\Omega}$ is compact), then the spectrum of $\mathcal{K}^*_{\partial\Omega}$ is discrete and accumulating to 0. The number $\frac{\epsilon_c}{\epsilon_m}$ such that $\frac{\epsilon_c+\epsilon_m}{2(\epsilon_c-\epsilon_m)}$ is an eigenvalue of $\mathcal{K}^*_{\partial\Omega}$ is called a plasmonic eigenvalue and the single layer potential of the corresponding eigenfunction is called a localized plasmon [**59**].

2.1. Spectrum of the NP operator. — We first emphasize that $\mathcal{K}^*_{\partial\Omega}$ is not self-adjoint on the usual L^2-space. In fact, it is self-adjoint on $L^2(\partial\Omega)$ only if Ω is a disk or a ball [**89**]. However, we may realize $\mathcal{K}^*_{\partial\Omega}$ as a self-adjoint operator by using a different inner product.

Let $\langle \, , \, \rangle$ be the usual inner product on $L^2(\partial\Omega)$. It is easy to see that $\mathcal{S}_{\partial\Omega}$ is self-adjoint on $L^2(\partial\Omega)$, which is nothing but saying $\Gamma(x-y) = \Gamma(y-x)$. Let $\varphi \in L^2_0(\partial\Omega)$ and define

$$u(x) = \mathcal{S}_{\partial\Omega}[\varphi](x), \quad x \in \mathbb{R}^d. \tag{12}$$

Then $u(x) = O(|x|^{1-d})$ as $|x| \to \infty$, and we have

$$\int_\Omega |\nabla u|^2 dx = \int_{\partial\Omega} u\Big(-\frac{1}{2}\varphi + \mathcal{K}^*_{\partial\Omega}[\varphi]\Big) d\sigma,$$

$$\int_{\mathbb{R}^d \setminus \overline{\Omega}} |\nabla u|^2 dx = -\int_{\partial\Omega} u\Big(\frac{1}{2}\varphi + \mathcal{K}^*_{\partial\Omega}[\varphi]\Big) d\sigma.$$

Summing up these two identities we find

$$\int_{\mathbb{R}^d} |\nabla u|^2 dx = -\langle \varphi, \mathcal{S}_{\partial\Omega}[\varphi] \rangle. \tag{13}$$

Thus $-\langle \varphi, \mathcal{S}_{\partial\Omega}[\varphi] \rangle \geq 0$. On the other hand, if $\langle \varphi, \mathcal{S}_{\partial\Omega}[\varphi] \rangle = 0$, then (13) and the decay condition of u at infinity imply $u \equiv 0$, and hence $\varphi = 0$. So, $-\langle \varphi, \mathcal{S}_{\partial\Omega}[\varphi] \rangle$ is an inner product on $L^2_0(\partial\Omega)$[(1)]. Let $\mathcal{H}$ be the Hilbert space $L^2_0(\partial\Omega)$ equipped with this inner product, and define

$$\langle \varphi, \psi \rangle_{\mathcal{H}} := -\langle \varphi, \mathcal{S}_{\partial\Omega}[\psi] \rangle, \quad \varphi, \psi \in \mathcal{H}. \tag{14}$$

That $\mathcal{K}^*_{\partial\Omega}$ is self-adjoint on $\mathcal{H}$ follows from the well-known Calderón's identity (also known as Plemeljs symmetrization principle):

$$\mathcal{S}_{\partial\Omega}\mathcal{K}^*_{\partial\Omega} = \mathcal{K}_{\partial\Omega}\mathcal{S}_{\partial\Omega}. \tag{15}$$

In fact,

$$\langle \varphi, \mathcal{K}^*_{\partial\Omega}[\psi] \rangle_{\mathcal{H}} = -\langle \varphi, \mathcal{S}_{\partial\Omega}\mathcal{K}^*_{\partial\Omega}[\psi] \rangle = -\langle \varphi, \mathcal{K}_{\partial\Omega}\mathcal{S}_{\partial\Omega}[\psi] \rangle = \langle \mathcal{K}^*_{\partial\Omega}[\varphi], \psi \rangle_{\mathcal{H}}.$$

We refer to [**7**, Lemma 3.3] for a proof of the Calderón's identity. We emphasize that the Calderón's identity holds, and hence $\mathcal{K}^*_{\partial\Omega}$ is self-adjoint on $\mathcal{H}$, even if $\partial\Omega$ is only Lipschitz. It is worth mentioning that when $\partial\Omega$ is $\mathcal{C}^{1,\alpha}$, self-adjointness of $\mathcal{K}^*_{\partial\Omega}$ also follows from the Calderón's identity and a symmetrizability result in [**77**].

If $\partial\Omega$ is $\mathcal{C}^{1,\alpha}$, then $\mathcal{K}^*_{\partial\Omega}$ is compact on $\mathcal{H}$. So, $\mathcal{K}^*_{\partial\Omega}$ has eigenvalues accumulating to 0. Let $\lambda_1, \lambda_2, \ldots$ ($|\lambda_1| \geq |\lambda_2| \geq \ldots$) be eigenvalues of $\mathcal{K}^*_{\partial\Omega}$ on $\mathcal{H}$ counting multiplicities, and $\varphi_1, \varphi_2, \ldots$ be the corresponding (normalized) eigenfunctions. Then $\mathcal{K}^*_{\partial\Omega}$ admits the spectral resolution

$$\mathcal{K}^*_{\partial\Omega} = \sum_{j=1}^{\infty} \lambda_j \varphi_j \otimes \varphi_j. \tag{16}$$

More generally, if $\partial\Omega$ is merely Lipschitz, then by the spectral resolution theorem [**110**] there is a family of projection operators $\mathcal{E}(t)$ on $\mathcal{H}$ (called a resolution of identity) such that

$$\mathcal{K}^*_{\partial\Omega} = \int_{-1/2}^{1/2} t \, d\mathcal{E}(t). \tag{17}$$

In some special cases, we can find those eigenvalues explicitly:

(i) If Ω is a disk, then 0 is the only eigenvalue of $\mathcal{K}^*_{\partial\Omega}$ on $\mathcal{H}$ [**72**]. ($\frac{1}{2}$ is an eigenvalue of $\mathcal{K}^*_{\partial\Omega}$ on $L^2(\partial\Omega)$ as we will see later.)

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(1) Since $\mathcal{S}_{\partial\Omega}$ maps $H^{-1/2}(\partial\Omega)$ into $H^{1/2}(\partial\Omega)$, $\langle \, , \, \rangle_{\mathcal{S}}$ may be considered as an inner product on $H_0^{-1/2}(\partial\Omega)$.

(ii) If Ω is a ball, then the eigenvalues of $\mathcal{K}^*_{\partial\Omega}$ are

$$\frac{1}{2(2n+1)}, \quad n = 1, 2, \dots, \tag{18}$$

and the corresponding eigenfunctions are the spherical harmonics of order n [**73**] (see also [**9**]).

(iii) If Ω is an ellipse of the long axis a and short axis b, then they are

$$\frac{1}{2}\Big(\frac{a-b}{a+b}\Big)^n, \quad n = 1, 2, \dots, \tag{19}$$

and the corresponding eigenfunctions are elliptic harmonics [**24**].

(iv) If Ω is an ellipsoid, the first few eigenvalues were computed in [**24**] and the same method can be applied to compute all the eigenvalues.

We now show that $\frac{1}{2}$ is an eigenvalue of $\mathcal{K}^*_{\partial\Omega}$ on $L^2(\partial\Omega)$ (not on $\mathcal{H}$). If Ω is a disk or a ball, then

$$\mathcal{K}^*_{\partial\Omega}[1] = \frac{1}{2}. \tag{20}$$

There are many ways to see this and one of them is to use the double layer potential. The double layer potential is defined by

$$\mathcal{D}_{\partial\Omega}[\varphi](x) := \int_{\partial\Omega} \frac{\partial}{\partial\nu_y}\Gamma(x-y)\varphi(y)\, d\sigma(y)\,, \quad x \in \mathbb{R}^d \setminus \partial\Omega. \tag{21}$$

Then $\mathcal{D}_{\partial\Omega}[\varphi]$ enjoys the jump relation

$$\mathcal{D}_{\partial\Omega}[\varphi]\big|_{\pm}(x) = \Big(\mp\frac{1}{2}I + \mathcal{K}_{\partial\Omega}\Big)[\varphi](x), \quad x \in \partial\Omega\,. \tag{22}$$

Since $\mathcal{D}_{\partial\Omega}[1] = 1$ which can be proved using Green's identity, we have

$$\mathcal{K}_{\partial\Omega}[1] = \frac{1}{2} \tag{23}$$

for any domain Ω. Since $\mathcal{K}_{\partial\Omega}$ is self-adjoint if Ω is a disk or a ball, we have (20)[(2)].

Suppose that $\mathcal{K}^*_{\partial\Omega}[1] \neq \frac{1}{2}$. Since

$$\int_{\partial\Omega} (\frac{1}{2}I - \mathcal{K}^*_{\partial\Omega})[1]\, d\sigma = \int_{\partial\Omega} (\frac{1}{2} - \mathcal{K}_{\partial\Omega}[1])\, d\sigma = 0$$

by (23), and $\frac{1}{2}I - \mathcal{K}^*_{\partial\Omega}$ is invertible on $L^2_0(\partial\Omega)$, there is a unique $\psi \in L^2_0(\partial\Omega)$ such that

$$(\frac{1}{2}I - \mathcal{K}^*_{\partial\Omega})[\psi] = (\frac{1}{2}I - \mathcal{K}^*_{\partial\Omega})[1]$$

Let $\varphi_0 := \psi - 1$. Then $\varphi_0 \neq 0$ (because $\int_{\partial\Omega}\varphi_0 = -|\partial\Omega|$) and

$$\mathcal{K}^*_{\partial\Omega}[\varphi_0] = \frac{1}{2}\varphi_0. \tag{24}$$

(2) Greuber conjectured that a disk or a ball is the only domain such that $\mathcal{K}^*_{\partial\Omega}[1] = \frac{1}{2}$. This conjecture has been proved to be true for bounded Lipschitz domains in $\mathbb{R}^2$ and for bounded Lipschitz convex domains in $\mathbb{R}^3$ by Mendez and Reichel [**94**].

So, $1/2$ is an eigenvalue of $\mathcal{K}^*_{\partial\Omega}$ on $L^2(\partial\Omega)$.

It is worth mentioning that the eigenfunction φ_0 in three dimensions is the equilibrium distribution of charge which minimizes the energy of the electrical field, namely,

$$E(\varphi) = \int_{\mathbb{R}^3} |\nabla \mathcal{S}_{\partial\Omega}[\varphi](x)|^2 dx$$

subject to $\int_{\partial\Omega} \varphi d\sigma$ = constant and $\varphi \geq 0$. See [**94**] in this connection. We also mention that the function u defined by $u(x) = \mathcal{S}_{\partial\Omega}[\varphi_0](x)$ satisfies $\frac{\partial u}{\partial \nu}|_- = (-\frac{1}{2}I + \mathcal{K}^*_{\partial\Omega})[\varphi_0] = 0$ and so u is constant in Ω. So u is harmonic in $\mathbb{R}^d \setminus \overline{\Omega}$ and u = const. on $\partial\Omega$ such that $u(x) = O(\ln|x|)$ in two dimensions and $u(x) = O(|x|^{-1})$ in three dimensions as $|x| \to \infty$.

Further discussion. — As discussed in the text, if $\partial\Omega$ is $\mathcal{C}^{1,\alpha}$, then $\mathcal{K}^*_{\partial\Omega}$ is compact on $\mathcal{H}$ and $\mathcal{K}^*_{\partial\Omega}$ has discrete eigenvalues. An interesting question is what if $\partial\Omega$ is merely Lipschitz, or what the spectrum of the NP operator looks like in this case. I am not aware of any example of domains with Lipschitz boundary whose NP operator has spectrum other than eigenvalues. The spectrum of the NP operator on simple domains with corners like the square in two dimensions may already have a quite interesting structure. (See also Further discussion at the end of Section 3.) In this respect, we refer to [**105**] and references therein. There the upper bounds of the essential spectrum of the NP operator are obtained when the boundary has corners. However, it is not clear if the essential spectrum does exists when the domain is not a two dimensional disc.

As we will see in Section 5 the slower convergence (to 0) of the eigenvalues of the NP operator on the ball is responsible for non-occurrence of the cloaking by anomalous localized resonance. In relation to this, we conjecture that eigenvalues of the NP operator on the ball has the fastest convergence rate in some sense among simply connected bounded domain in three dimensions. Regarding the convergence rate of eigenvalues, it would be interesting to relate the convergence rate with analyticity of the boundary in two dimensions.

3. Generalized polarization tensors and applications to imaging

Let u be the solution to (9). Then it admits the representation (10) and (11). Suppose that $0 \in \Omega$. Since the harmonic function h admits the expansion

$$h(x) = \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!} (\partial^\alpha h)(0) x^\alpha,$$

if we define $\varphi_\alpha \in L^2_0(\partial\Omega)$ by

$$\varphi_\alpha(x) := (\lambda I - \mathcal{K}^*_{\partial\Omega})^{-1}[\nu \cdot \nabla y^\alpha](x), \quad x \in \partial\Omega \,, \tag{25}$$

then by linearity φ, the solution to (11), is given by

$$\varphi = \sum_{n=1}^{\infty} \sum_{|\alpha|=n} \frac{1}{\alpha!} (\partial^\alpha h)(0) \varphi_\alpha.$$

Here we use the multi-index notation $\alpha = (\alpha_1, \ldots, \alpha_d)$, $|\alpha| = \alpha_1 + \ldots + \alpha_d$, and $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. And the solution u is given by

$$u(x) = h(x) + \sum_{n=1}^{\infty} \sum_{|\alpha|=n} \frac{1}{\alpha!} (\partial^\alpha h)(0) \mathcal{S}_{\partial\Omega}[\varphi_\alpha](x), \quad |x| \to \infty.$$

Recall that

$$\mathcal{S}_{\partial\Omega}[\varphi_\alpha](x) = \int_{\partial\Omega} \Gamma(x-y) \varphi_\alpha(y)\, d\sigma(y).$$

Since $\Gamma(x-y)$ admits the expansion

$$\Gamma(x-y) = \sum_{m=0}^{\infty} \sum_{|\beta|=m} \frac{(-1)^{|\beta|}}{\beta!} \partial^\beta \Gamma(x) y^\beta$$

for $|x|$ large and y in a bounded set and $\varphi_\alpha \in L^2_0(\partial\Omega)$, we have

$$u(x) = h(x) + \sum_{n,m=1}^{\infty} \sum_{|\alpha|=n} \sum_{|\beta|=m} \frac{(-1)^{|\beta|}}{\alpha!\beta!} \partial^\alpha h(0) m_{\alpha\beta} \partial^\beta \Gamma(x), \quad |x| \to \infty, \tag{26}$$

where

$$m_{\alpha\beta} = \int_{\partial\Omega} y^\beta \varphi_\alpha(y)\, d\sigma(y). \tag{27}$$

We emphasize that the expansion (26) uses polynomials as a basis. We will rewrite this expansion in a different form using the spherical harmonics as a basis (see (47)).

The quantities $m_{\alpha\beta}$ is called the generalized polarization tensors (GPT). Note that $m_{\alpha\beta}$ depends on the domain Ω and the constant $\lambda = \frac{\epsilon_c + \epsilon_m}{2(\epsilon_c - \epsilon_m)}$. So we denote $m_{\alpha\beta}$ as $m_{\alpha\beta}(\lambda, \Omega)$ to indicate its dependence on the arguments, or $m_{\alpha\beta}(\Omega)$ when λ is fixed, or $m_{\alpha\beta}(\lambda)$ when Ω is fixed. We emphasize that $m_{\alpha\beta}(\lambda)$ is a holomorphic function of λ in $\mathbb{C} \setminus [-\frac{1}{2}, \frac{1}{2}]$. The asymptotic expansion (26) shows that the GPTs are building blocks of the 'far field' expansion of u in the presence of the inclusion Ω. As we will see later, GPTs carry rich information on the shape of the inclusion, so they can be used for imaging and shape description. For example, the full set of GPTs determines the domain uniquely [**18**]. Conversely, if we design a structure so that first a few terms of its GPTs vanish, then the structure is vaguely seen by the far field measurements. In this way, the notion of GPTs has an important connection with the invisibility cloaking. It is worth mentioning here that asymptotic expansion (26) is valid in some cases when Ω is not homogeneous, namely, its dielectric property ϵ_c is not constant (see [**11**]). It holds for example when the Ω has multi-layered radial structure which we will deal with later in the note.

When $|\alpha| = |\beta| = 1$, we may write $m_{\alpha\beta}$ as m_{ij}, $i, j = 1, \dots, d$, so m_{ij} is defined by

$$m_{ij} = \int_{\partial\Omega} y^j (\lambda I - \mathcal{K}^*_{\partial\Omega})^{-1}[\nu_i](y)\, d\sigma(y). \tag{28}$$

The $d \times d$ matrix $M = (m_{ij})$ is called the polarization (or polarizability) tensor and appeared in many context such as low frequency asymptotic of wave [**51**], the study of potential flow [**106**], and theory of composites (see [**95**] and references therein). The leading order term in (26) may be written as

$$u(x) = h(x) - M\nabla h(0) \cdot \nabla\Gamma(x) + O(|x|^{-d}), \quad |x| \to \infty. \tag{29}$$

3.1. Properties of GPTs. — We now collect some important properties of GPTs.

Symmetry. If $\{a_\alpha\}$ and $\{b_\beta\}$ are such that $\sum a_\alpha x^\alpha$ and $\sum b_\beta x^\beta$ are harmonic polynomials (such coefficients are called harmonic coefficients), then

$$\sum a_\alpha b_\beta m_{\alpha\beta} = \sum a_\alpha b_\beta m_{\beta\alpha}.$$

In particular, the PT M is a symmetric matrix.

Bounds and positivity. If $f(x) = \sum_{\alpha\in I} a_\alpha x^\alpha$ is a harmonic polynomial and $|\lambda| \geq \frac{1}{2}$ (and real), then

$$\frac{2}{2\lambda+1}\int_\Omega |\nabla f|^2\, dx \leq \sum_{\alpha,\beta\in I} a_\alpha a_\beta m_{\alpha\beta}(\lambda,\Omega) \leq \frac{2}{2\lambda-1}\int_\Omega |\nabla f|^2\, dx\ . \tag{30}$$

These bounds imply that if λ is real and $\lambda \geq \frac{1}{2}$, then $\sum a_\alpha a_\beta m_{\alpha\beta}(\lambda,\Omega) > 0$. In particular, the M is positive-definite. If $\lambda \leq -\frac{1}{2}$, then it is negative-definite. Moreover, if κ is an eigenvalue of M, then

$$\frac{2}{2\lambda+1}|\Omega| \leq \kappa \leq \frac{2}{2\lambda-1}|\Omega|. \tag{31}$$

Here, $|\Omega|$ denote the volume of Ω.

Optimal bounds for PT(3). The bounds (31) can be improved to

$$\frac{1}{k-1}\mathrm{Tr}(M) \leq |\Omega|(d-1+\frac{1}{k}), \tag{32}$$

and

$$(k-1)\mathrm{Tr}(M^{-1}) \leq \frac{d-1+k}{|\Omega|}, \tag{33}$$

where Tr denotes the trace and $k = \frac{2\lambda+1}{2\lambda-1}$. These bounds have been obtained by Lipton [**93**], and later by Capdeboscq-Vogelius [**48**] based on the variational argument in [**79**]. The bounds are optimal in the sense that every matrix satisfying bounds is realized as the PT of an inclusion [**5, 47**]. It is worth mentioning that these bounds are geometry independent and can be improved for domains with some thickness [**46**]. It is proved

(3) The bounds are called the Hashin-Shtrikman bounds after names of the scientists who first found the optimal bounds on the effective conductivity of isotropic two-phase composites [**61**]

in [**71**] that if the lower bound (33) is attained, then Ω is an ellipse or an ellipsoid, and as a consequence the Pólya-Szegö conjecture[(4)] is proved.

Unique determination of the domain by GPT. If bounded Lipschitz domains Ω_1 and Ω_2 satisfy

$$\sum a_\alpha b_\beta m_{\alpha\beta}(\lambda_1, \Omega_1) = \sum a_\alpha b_\beta m_{\alpha\beta}(\lambda_2, \Omega_2)$$

for all harmonic coefficients a_α and b_β, then

$$\lambda_1 = \lambda_2 \quad \text{and} \quad \Omega_1 = \Omega_2.$$

Transformation formula for PT. The following relations under scaling, shifting, and rotation hold:

(i) Scaling:

$$M(\lambda, s\Omega) = s^d M(\lambda, \Omega). \tag{34}$$

(ii) Shifting. PT is invariant under translation, *i.e.*,

$$M(\lambda, \Omega + z) = M(\lambda, \Omega). \tag{35}$$

(iii) Rotation. Let $\mathcal{R}$ be an orthogonal transformation. Then the following relation holds:

$$M(\lambda, \mathcal{R}\Omega) = \mathcal{R} M(\lambda, \Omega) \mathcal{R}^T. \tag{36}$$

The proofs of all above mentioned properties can be found in [**21**].

Transformation formula for (higher order) GPTs are also important. To derive those formula, it is more convenient to use complex harmonic combinations of GPTs. In two dimensions, let a^m_α and b^m_β be coefficients such that

$$\sum_{|\alpha|=m} a^m_\alpha x^\alpha = r^m \cos m\theta, \text{ and } \sum_{|\alpha|=m} b^m_\alpha x^\beta = r^m \sin m\theta. \tag{37}$$

We then define contracted GPTs (CGPT) as follows:

$$M^{cc}_{mn} = \sum_{|\alpha|=m} \sum_{|\beta|=n} a^m_\alpha a^n_\beta m_{\alpha\beta}, \quad M^{cs}_{mn} = \sum_{|\alpha|=m} \sum_{|\beta|=n} a^m_\alpha b^n_\beta m_{\alpha\beta},$$
$$M^{sc}_{mn} = \sum_{|\alpha|=m} \sum_{|\beta|=n} b^m_\alpha a^n_\beta m_{\alpha\beta}, \quad M^{ss}_{mn} = \sum_{|\alpha|=m} \sum_{|\beta|=n} b^m_\alpha b^n_\beta m_{\alpha\beta}.$$

It is worth mentioning that M^{cc}_{mn} is defined by

$$M^{cc}_{mn} = \int_{\partial\Omega} (r^n \cos n\theta)(\lambda I - \mathcal{K}^*_\Omega)^{-1}[\nu \cdot \nabla(r^m \cos m\theta)]\, d\sigma, \tag{38}$$

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(4) The Pólya-Szegö conjecture asserts that the inclusion whose electrical polarization tensor has the minimal trace takes the shape of a disk or a ball [**106**].

and M^{cs}_{mn} is defined by switching the first $\cos n\theta$ by $\sin n\theta$, and the others are defined similarly. We also define complex CGPTs as

$$\mathbf{N}^{(1)}_{mn}(\lambda,\Omega) = \int_{\partial\Omega} P_n(y)(\lambda I - \mathcal{K}^*_\Omega)^{-1}[\nu\cdot\nabla P_m](y)\,d\sigma(y), \tag{39}$$
$$\mathbf{N}^{(2)}_{mn}(\lambda,\Omega) = \int_{\partial\Omega} P_n(y)(\lambda I - \mathcal{K}^*_\Omega)^{-1}[\nu\cdot\nabla \overline{P_m}](y)\,d\sigma(y),$$

where $P_n(x) = (x_1 + ix_2)^n$. Then one can see immediately that

$$\mathbf{N}^{(1)}_{mn}(\lambda,\Omega) = (M^{cc}_{mn} - M^{ss}_{mn}) + i(M^{cs}_{mn} + M^{sc}_{mn}),$$
$$\mathbf{N}^{(2)}_{mn}(\lambda,\Omega) = (M^{cc}_{mn} + M^{ss}_{mn}) + i(M^{cs}_{mn} - M^{sc}_{mn}).$$

Then the following transformation formula hold [**3**]: Let $R_\theta\Omega$, $s\Omega$, and $T_z\Omega$ be rotation by θ, scaling, and shifting by z of Ω, respectively. For all integers m,n, we have

$$\mathbf{N}^{(1)}_{mn}(R_\theta\Omega) = e^{i(m+n)\theta}\mathbf{N}^{(1)}_{mn}(\Omega), \quad \mathbf{N}^{(2)}_{mn}(R_\theta\Omega) = e^{i(n-m)\theta}\mathbf{N}^{(2)}_{mn}(\Omega), \tag{40}$$

$$\mathbf{N}^{(1)}_{mn}(s\Omega) = s^{m+n}\mathbf{N}^{(1)}_{mn}(\Omega), \quad \mathbf{N}^{(2)}_{mn}(s\Omega) = s^{m+n}\mathbf{N}^{(2)}_{mn}(\Omega), \tag{41}$$

(42)
$$\mathbf{N}^{(1)}_{mn}(T_z\Omega) = \sum_{l=1}^{m}\sum_{k=1}^{n}\mathbf{C}^z_{ml}\mathbf{N}^{(1)}_{lk}(\Omega)\mathbf{C}^z_{nk}, \quad \mathbf{N}^{(2)}_{mn}(T_z\Omega) = \sum_{l=1}^{m}\sum_{k=1}^{n}\overline{\mathbf{C}^z_{ml}}\mathbf{N}^{(2)}_{lk}(\Omega)\mathbf{C}^z_{nk},$$

where $\mathbf{C}^z$ is the lower triangle matrix with the m,n-th entry given by

$$\mathbf{C}^z_{mn} = \binom{m}{n} z^{m-n}, \tag{43}$$

and $\overline{\mathbf{C}^z}$ denotes its conjugate. Here, we identify $z = (z_1, z_2)$ with $z = z_1 + iz_2$.

In three dimensions, the complex CGPTs are defined as follows: let Y^m_n, $-n \le m \le n$, be the (complex) spherical harmonic of homogeneous degree n and order m, *i.e.*,

$$Y^m_n(\theta,\varphi) = (-1)^m\left[\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}\right]^{1/2} e^{im\varphi}\mathcal{P}^m_n(\cos\theta)\,, \quad -n\le m\le n\,,$$

where $\mathcal{P}^m_n$ are the associated Legendre polynomials of degree n and order m. If

$$r^n Y^m_n(\theta,\varphi) = \sum_{|\alpha|=n} a^{mn}_\alpha x^\alpha,$$

then CGPT M_{mnkl} is defined by

$$M_{nmlk} = \sum_{|\alpha|=n,|\beta|=l} \overline{a^{mn}_\alpha} a^{kl}_\beta M_{\alpha\beta}, \quad m,n,k,l = 1,2,\cdots. \tag{44}$$

In other words, we have

$$M_{nmlk} = \int_{\partial D} r^l_y \overline{Y^k_l(\theta_y,\varphi_y)}(\lambda I - \mathcal{K}^*_D)^{-1}\left[\frac{\partial}{\partial\nu} r^n_y Y^m_n(\theta_y,\varphi_y)\Big|_{\partial D}\right](y)d\sigma(y), \tag{45}$$

where $y = r_y(\cos\varphi_y \sin\theta_y, \sin\varphi_y \sin\theta_y, \cos\theta_y)$. Then similar transformation formula for M_{nmlk} can be obtained. See [**6**] for details.

Explicit formula for PT and GPTs can be found for shapes like disks, balls, ellipses, and ellipsoids. For example, if Ω is an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$, then

$$M(\lambda, \Omega) = 2|\Omega| \begin{bmatrix} \dfrac{a+b}{(2\lambda-1)a + (2\lambda+1)b} & 0 \\ 0 & \dfrac{a+b}{(2\lambda-1)b + (2\lambda+1)a} \end{bmatrix}. \tag{46}$$

We may derive an asymptotic expansion of the solution to (9) which is different from (26) using the contracted GPTs. If h admits the Fourier expansion

$$h(x) = a_0 + \sum_{n=1}^{\infty} r^n (a_n^c \cos n\theta + a_n^s \sin n\theta),$$

then, as $|x| \to \infty$, we have

(47)

$$u(x) = h(x) - \sum_{m,n=1}^{\infty} \left[\frac{\cos m\theta}{2\pi m r^m} (M_{mn}^{cc} a_n^c + M_{mn}^{cs} a_n^s) + \frac{\sin m\theta}{2\pi m r^m} (M_{mn}^{sc} a_n^c + M_{mn}^{ss} a_n^s) \right].$$

3.2. Shape description and imaging by GPTs. — There are many geometric quantities intrinsically associated with domains (or shapes) such as eigenvalues, moments, and capacities. The sequence of GPTs is one of them. In fact, GPTs contain richer information than eigenvalues since the full sequence of GPTs determines the domain uniquely as we discussed before while the full sequence of eigenvalues does not [**56**]. We now show that first few terms of GPTs, not the full sequence, can recover a good approximation of the shape.

Note that there is one-to-one correspondence between the class of PTs and that of ellipses. In fact, suppose $M = M(\Omega)$ is the PT of the domain Ω. Since M is a positive (or negative) definite symmetric matrix, M can be diagonalized as $M = \mathcal{R}\Lambda\mathcal{R}^T$ where Λ is a diagonal matrix whose entries have the same sign and $\mathcal{R}$ is an orthogonal matrix. Then using (46) one can determine the ellipse E such that $M(\lambda, E) = \Lambda$. By (36), $M(\lambda, \mathcal{R}E) = M$ (this argument works for the equivalent ellipsoid as well[(5)]). The ellipse $\mathcal{R}E$ is called the equivalent ellipse of Ω. In other words, given a domain Ω, the ellipse (or ellipsoid) whose PT is the same as that of Ω is called the equivalent ellipse or ellipsoid of Ω. The equivalent ellipse reveals an overall property of the shape. Figure 1 shows the equivalent ellipse of a kite-shaped domain. It is worthwhile to mention that since an ellipse which is not a disk has an orientation (the direction of the long axis), we may define the direction of the given domain by that of the equivalent ellipse.

(5) One can find the formula for the PT of ellipsoids in [**95**]

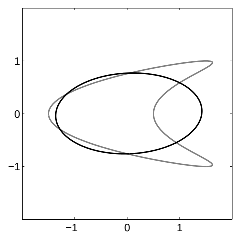

FIGURE 1. Equivalent Ellipse. A figure from [**31**]

Since the full set of GPTs determines the shape completely and the PT represents the overall property, one may guess that the higher order GPTs carry information on finer details of the shape. However, it is not clear how to represent geometric figures contained in higher order GPTs, like equivalent ellipses in PTs. So, in [**31**] an optimal control method is used to recover geometric features from first few terms of GPTs.

The aim is to make use of $\sum_{|\alpha|+|\beta|\le K} a_\alpha b_\beta m_{\alpha\beta}$ for a fixed $K \ge 2$, where a_α and b_β are harmonic coefficients, to image finer details of the shape of the inclusion. The optimization problem to recover the shape of the given target domain Ω is to minimize over domains D

$$J[D] := \frac{1}{2} \sum_{|\alpha|+|\beta|\le K} w_{|\alpha|+|\beta|} \left| \sum_{\alpha,\beta} a_\alpha b_\beta m_{\alpha\beta}(k, D) - \sum_{\alpha,\beta} a_\alpha b_\beta m_{\alpha\beta}(k, \Omega) \right|^2 .$$

Here $w_{|\alpha|+|\beta|}$ is a binary weight: 0 for 'off' and 1 for 'on'. For example $w_2 = 1$ and others are zero means that only the PT is used.

To minimize $J[D]$ we use a iterative scheme: we modify the initial shape D^n to obtain D^{n+1} by applying the gradient descent method:

$$\partial D^{n+1} = \partial D^n - \left(\frac{J[D^n]}{\sum_j \left(\langle d_S J[D^n], \psi_j \rangle\right)^2} \sum_j \langle d_S J[D^n], \psi_j \rangle \psi_j \right) \nu,$$

where ν is the outward unit normal to D^n, $\{\psi_j\}$ is a basis of $L^2(\partial D^n)$, and $d_S J[D^n]$ is the shape derivative. We have a good initial guess for the iteration: the equivalent ellipse! Given the PT of the inclusion Ω, we can find an ellipse with the same PT but not its location since the PT is invariant under translation. We can locate the inclusion provided that its GPTs with $|\alpha| + |\beta| = 3$ are known. Suppose that $B = B_r(x^*)$ is a ball in $\mathbb{R}^d$, $d = 2, 3$. Let $\alpha_l := \mathbf{e}_l$ and $\beta_l := 2\mathbf{e}_l$, $j = 1, \ldots, d$, where $\mathbf{e}_l$ is the standard basis of $\mathbb{R}^d$. Then it is known, see [**20**], that

$$(m_{\alpha_1\beta_1}, \ldots, m_{\alpha_d\beta_d}) = \frac{2d(k-1)}{k+d-1} |B| x^*.$$

(Here and throughout this section we assume that $\epsilon_m = 1$ and $\epsilon_c = k$ so that $\lambda = \frac{k+1}{2(k-1)}$.) Temporarily assuming that B is a ball, we get the center for the initial guess from $m_{\alpha_l \beta_l}$, $l = 1, \ldots, d$. We emphasize that in each step of iteration we may use a different binary weights w_k.

The shape derivative $d_S J[D^n]$ can be computed as follows: Let D_ϵ be an ϵ-perturbation of D, *i.e.*,

$$\partial D_\epsilon := \{\tilde{x} = x + \epsilon h(x)\nu(x) \mid x \in \partial D\}.$$

Let $H = \sum_\alpha a_\alpha x^\alpha$ and $F = \sum_\beta b_\beta x^\beta$ be harmonic polynomials. Then

$$\begin{aligned} &\sum_{\alpha,\beta} a_\alpha b_\beta m_{\alpha\beta}(k, D_\epsilon) - \sum_{\alpha,\beta} a_\alpha b_\beta m_{\alpha\beta}(k, D) \\ &= \epsilon(k-1) \int_{\partial D} h(x) \left[\frac{\partial v}{\partial \nu}\Big|_- \frac{\partial u}{\partial \nu}\Big|_- + \frac{1}{k} \frac{\partial u}{\partial T}\Big|_- \frac{\partial v}{\partial T}\Big|_- \right](x)\, d\sigma(x) + O(\epsilon^2), \end{aligned}$$

where

$$\begin{cases} \Delta u = 0 & \text{in } D \cup (\mathbb{R}^2 \setminus \overline{D}), \\ u|_+ - u|_- = 0 & \text{on } \partial D, \\ \dfrac{\partial u}{\partial \nu}\Big|_+ - k \dfrac{\partial u}{\partial \nu}\Big|_- = 0 & \text{on } \partial D, \\ (u - H)(x) = O(|x|^{-1}) & \text{as } |x| \to \infty, \end{cases}$$

and

$$\begin{cases} \Delta v = 0 & \text{in } D \cup (\mathbb{R}^2 \setminus \overline{D}), \\ kv|_+ - v|_- = 0 & \text{on } \partial D, \\ \dfrac{\partial v}{\partial \nu}\Big|_+ - \dfrac{\partial v}{\partial \nu}\Big|_- = 0 & \text{on } \partial D, \\ (v - F)(x) = O(|x|^{-1}) & \text{as } |x| \to \infty. \end{cases}$$

Then the shape derivative of $J[D]$ in the direction of h is given by

$$\langle d_S J[D], h \rangle_{L^2(\partial D)} = \sum_{|\alpha|+|\beta| \le K} w_{|\alpha|+|\beta|} \delta_D^{HF} \langle \phi_D^{HF}, h \rangle_{L^2(\partial D)},$$

where

$$\phi_D^{HF}(x) = (k-1) \left[\frac{\partial v}{\partial \nu}\Big|_- \frac{\partial u}{\partial \nu}\Big|_- + \frac{1}{k} \frac{\partial u}{\partial T}\Big|_- \frac{\partial v}{\partial T}\Big|_- \right],$$

and

$$\delta_D^{HF} = \sum_{\alpha,\beta} a_\alpha b_\beta m_{\alpha\beta}(k, D_\epsilon) - \sum_{\alpha,\beta} a_\alpha b_\beta m_{\alpha\beta}(k, D).$$

Figure 2 shows the result of shape reconstruction using $m_{\alpha\beta}$, $|\alpha| + |\beta| \le 6$, after 6 iterations. The kite shape is recovered well. However, the optimization algorithm used here has a limitation. As Figure 3 shows, the iteration can not change the topology. In [**16**] this limitation has been overcome using the level set method. Figure 4 shows the result of reconstruction by the level-set method using $m_{\alpha\beta}$, $|\alpha| + |\beta| \le 6$, after 100 iterations. It is remarkable that first few GPTs carry information on the topology. GPTs can be used to recover inhomogeneous conductivity as Figure 5 shows.

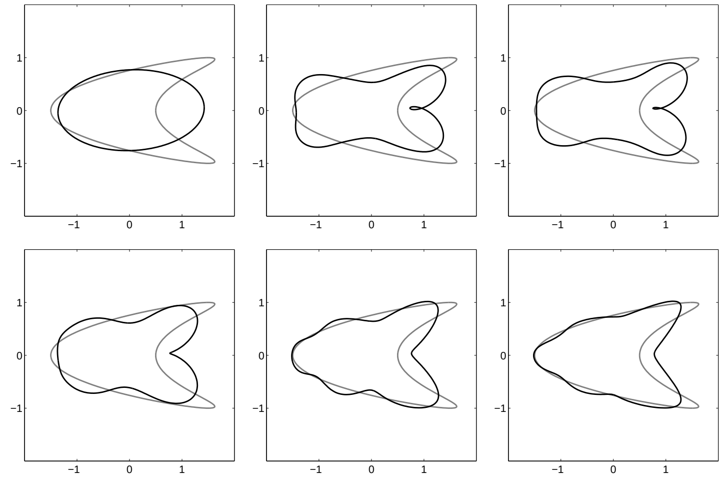

FIGURE 2. Shape reconstruction using $m_{\alpha\beta}$, $|\alpha|+|\beta| \leq 6$, after 6 iterations. Figures from [**31**].

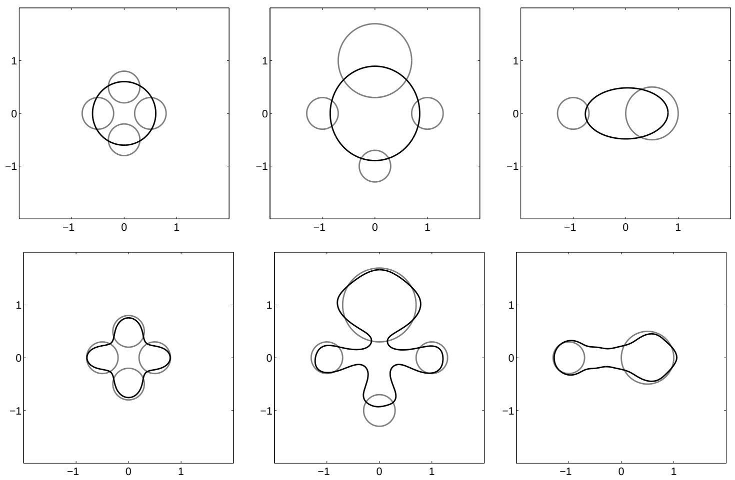

FIGURE 3. Reconstruction of clusters of inclusions using $m_{\alpha\beta}$, $|\alpha|+|\beta| \leq 6$. The upper images: the equivalent ellipses, and the lower ones: results after 6 iterations. Figures from [**31**]

GPTs as shape descriptors. So far we have seen that GPTs carry rich information of the shape of the inclusion. In previous subsection we showed that GPTs (or

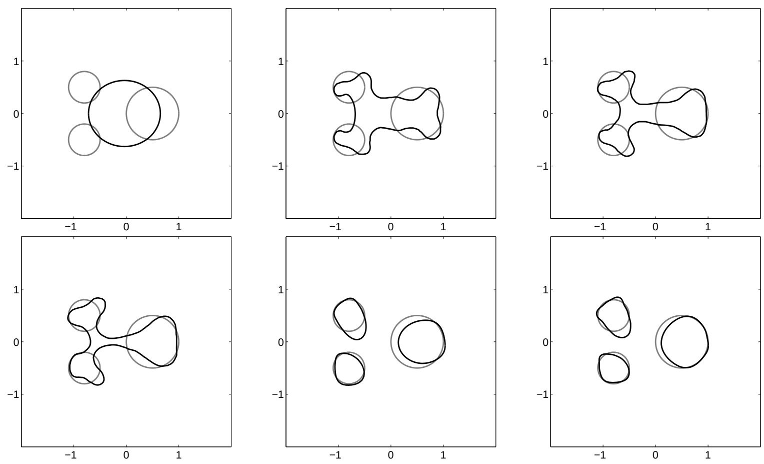

FIGURE 4. Reconstruction of clusters of inclusions by the level-set method using $m_{\alpha\beta}$, $|\alpha| + |\beta| \leq 6$, after 100 iterations. Figures from [**16**].

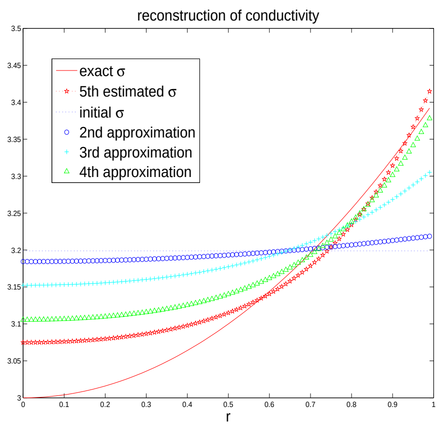

FIGURE 5. Recovery of radial conductivities using GPTs. $r = 0$ is the center of the circle and $r = 1$ is the boundary. A figure from [**11**].

CGPTs) obey certain transformation laws under scaling, rotation, and shifting. This property makes GPTs suitable for the dictionary matching problem (or pattern recognition). The problem is to identify the object in the dictionary when the target object is identical to one of the objects in the dictionary up to shifting, rotation, and scaling. The standard method of dictionary matching is to construct invariants, called shape descriptors, under rigid motions and scaling, and to compare those invariants, and a common way to construct such invariants uses the moments [**63, 113**]. In the recent paper [**3, 6**] new invariants are constructed using GPTs in two and three dimensions using translation formula and viability of the method is demonstrated by numerical experiments. We also mention a recent work of Ammari *et al* [**4**] where GPT based invariants are used for shape recognition and classification in electro-sensing.

3.3. Expansion method of imaging. — Let σ be the conductivity profile of the domain Ω. For a given $g \in L^2_0(\partial\Omega)$ we consider the Neumann boundary value problem

$$\begin{cases} \nabla \cdot (\sigma(x)\nabla u) = 0 \quad \text{in } \Omega, \\ \sigma \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = g, \\ \displaystyle\int_{\partial\Omega} u \, d\sigma = 0. \end{cases} \tag{48}$$

If u is the solution to this problem, the map Λ defined by

$$\Lambda : g \mapsto u|_{\partial\Omega}$$

is called the Neumann-to-Dirichlet (NtD) map. One of main problems in inverse problems is to determine the conductivity σ from the NtD map. This problem is called the Calderón problem (or Electrical Impedance Tomography), and since the work of Calderón [**45**], Kohn-Vogelius [**83**] and Sylvester-Uhlmann [**107**] huge literature has been devoted to this problem. We refer interested readers to [**32, 65, 108**].

Suppose that the conductivity distribution is given by

$$\sigma = \chi(\Omega \setminus D) + k\chi(D),$$

meaning that the inclusion D of conductivity $k \neq 1$ is buried in Ω of conductivity 1. The problem here is to reconstruct the inclusion D from the NtD map Λ. Even though uniqueness of the reconstruction of the inclusion via the NtD map holds as was proved by Isakov [**64**], this reconstruction problem is ill-posed and it is hard to have stable reconstruction. For example, if the boundary of the inclusion has high oscillation, the reconstruction becomes unstable (see [**17**]). It is even more so if we consider a finite measurements problem. Reconstruction of D by the NtD map is an infinite measurements problem, namely, the data set is $\{\Lambda[g] : g \in L^2_0(\partial\Omega)\}$. One may consider finite measurements problems to reconstruct D from finite set of data $\Lambda[g_j]$, $j = 1, \ldots, N$, for some N.

To overcome ill-posedness and instability of the reconstruction many regularization methods were invented. One way of regularization is to either restrict the class of domains or to restrict the geometric features of the inclusion to be reconstructed (e.g. giving up detecting high oscillation). It is worth mentioning that uniqueness of reconstruction within the class of polygons, disks, and balls by one or two measurements was proved (see [**65**] and references therein). We also mention that uniqueness within the class of ellipses (using only a finite number of measurements) is still an open problem.

Suppose now that the inclusion to be reconstructed is diametrically small. In this case the leading order terms of the asymptotic expansion of the solution to (48) as the diameter tends to 0 provides a good approximation of the Dirichlet data, and reconstruction (of the location and some geometric information) becomes stable. To be more precise, suppose that the inclusion D is represented as

$$D = \delta B + z$$

where B is a reference domain containing 0, δ is a small parameter of diameter, and z indicates the location of the inclusion. If B has several components, then D is a cluster of small inclusions. We further assume that D is at some distance from the boundary $\partial\Omega$ of Ω, *i.e.*,

$$\text{dist}(D, \partial\Omega) \geq C_0 \tag{49}$$

for some C_0. Let u_δ be the solution to (48). The condition (49) means that the boundary value $u_\delta|_{\partial\Omega}$ is a kind of 'far-field' pattern. So, in view of (29), one may guess that

$$u_\delta(x) \approx u_0(x) - M(\lambda, D)\nabla u_0(z) \cdot \nabla\Gamma(x - z),$$

where u_0 is the solution in absence of D, *i.e.*,

$$\begin{cases} \Delta u_0 = 0 \quad \text{in } \Omega, \\ \dfrac{\partial u_0}{\partial \nu}\Big|_{\partial\Omega} = g, \\ \displaystyle\int_{\partial\Omega} u_0 \, d\sigma = 0, \end{cases}$$

and $M(\lambda, D)$ is the PT of D with $\lambda = \frac{k+1}{2(k-1)}$. The guess is almost right except that since we are dealing with the Neumann boundary value problem, Γ, which is the Green function in the free space, should be replaced by the Neumann function. The Neuman function on Ω is defined by

$$\begin{cases} -\Delta_x N(x, z) = \delta_z(x) \quad \text{in } \Omega \, , \\ \dfrac{\partial N}{\partial \nu_x}\Big|_{\partial\Omega} = -\dfrac{1}{|\partial\Omega|} \, , \displaystyle\int_{\partial\Omega} N(x, z) \, d\sigma(x) = 0 \quad \text{for } z \in \Omega \, . \end{cases}$$

Since $M(D) = \delta^d M(B)$ by (34), the correct asymptotic formula is

$$u_\delta(x) = u_0(x) - \delta^d M(B)\nabla u_0(z) \cdot \nabla_z N(x, z) + O(\delta^{d+1}). \tag{50}$$

This formula was first derived in [**54**]. The formula (50) reveals that one can recover the location z and the PT $M(D)$ of D, and hence the equivalent ellipse of D. This idea appeared and was implemented in [**42**]. The same expansion and reconstruction work for a cluster of inclusions [**23**]. Figure 6 is from the same paper. In the figure, resolved imaging would be reconstructing three inclusions separately. Instead we reconstruct the location and an overall (averaged) feature of the inclusions, and the reconstruction is stable! As we saw in previous subsections, we may recover a resolved image by using higher order GPTs, but the reconstruction becomes unstable.

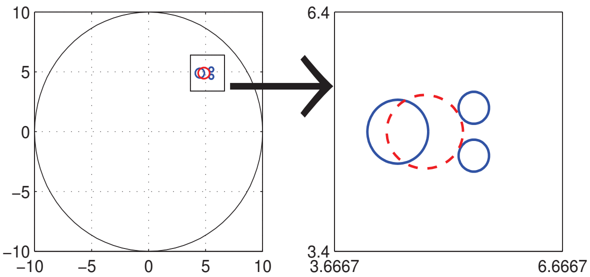

FIGURE 6. Reconstruction of a cluster of inclusions by the equivalent ellipse. A figure from [**23**]

We emphasize here that the expansion formula may not be directly used for reconstruction of the location and the PT since the Neumann function depends on Ω. This difficulty can be removed by using the formula

$$\left(-\frac{1}{2}I + \mathcal{K}_\Omega\right)[N(\cdot, z)](x) = \Gamma(x - z) \text{ modulo constant}, \quad x \in \partial\Omega, \ z \in \Omega, \tag{51}$$

and the new expansion

$$\left(-\frac{1}{2}I + \mathcal{K}_\Omega\right)[u_\delta](x) = u_0(x) - \delta^d M(B)\nabla u_0(z) \cdot \nabla_z \Gamma(x - z) + O(\delta^{d+1}). \tag{52}$$

This is a kind of pre-conditioning. See [**21**, Lemma 2.28] for a proof of (51).

The expansion (50) is extended to include higher order terms [**19**]:

$$u_\delta(x) = u_0(x) + \sum_{n=2}^{2d-1} \frac{(-1)^{|\beta|}}{\alpha!\beta!} \sum_{|\alpha|+|\beta|=n} \partial^\alpha u_0(z) m_{\alpha\beta} \partial_z^\beta N(x, z) + O(\delta^{2d}). \tag{53}$$

It is possible to derive terms even higher than δ^{2d}. But those terms involve not only GPTs (which are intrinsic quantities associated only with the inclusion) but also interaction between D and $\partial\Omega$. The expansion (53) shows that it is possible to recover $m_{\alpha\beta}$, $|\alpha| + |\beta| \leq 2d - 1$, from the boundary measurements.

The expansion method for reconstruction of small inclusions has been applied in various contexts such as wave imaging, elasticity imaging, etc. We refer readers to [**22**] and references therein for development in this direction. We also mention modeling of the weakly electric fish by Ammari, Boulier, and Garnier [**2**].

Yet there is another important way of imaging: in terms of multi-static measurements. There is an array of transducers and receivers. They emit waves and receive responses, and from this data one can construct the multi-static response matrix. Since the multi-static response matrix admits an asymptotic expansion with respect to the diameter of the inclusion, one can recover GPTs of higher orders. Then one can recover fine details of the shape of the inclusion. Here we emphasize that if we use higher order GPTs, then we recover resolved image. On the other hand, recovery of higher order GPTs is unstable while the lower order ones can be recovered in a stable way. So there is a trade-off between resolution and stability. We refer to the book [**14**] and references therein for recent development on multi-static imaging and statistical analysis of resolution and stability.

Further discussion. — Suppose that $\partial\Omega$ is Lipschitz and that $\mathcal{K}^*_{\partial\Omega}$ admits the spectral resolution (17). Then, we have

$$(\lambda I - \mathcal{K}^*_{\partial\Omega})^{-1}[\nu_i] = \int_{-1/2}^{1/2} \frac{1}{\lambda - t}\, d\mathcal{E}(t)[\nu_i],$$

and hence

$$m_{ij} = \langle y_j, (\lambda I - \mathcal{K}^*_{\partial\Omega})^{-1}[\nu_i]\rangle = \int_{-1/2}^{1/2} \frac{1}{\lambda - t}\, d\langle \mathcal{E}(t)[\nu_i], y_j\rangle.$$

We emphasize that $\langle \mathcal{E}(t)[\nu_i], y_j\rangle$ is the usual inner product. Let

$$d\mu_{ij}^{\Omega} := d\langle \mathcal{E}(t)[\nu_i], y_j\rangle.$$

Then, we have

$$m_{ij}(\lambda, \Omega) = \int_{-1/2}^{1/2} \frac{d\mu_{ij}^{\Omega}(t)}{\lambda - t}, \quad \lambda \in \mathbb{C} \setminus [-\frac{1}{2}, \frac{1}{2}]. \tag{54}$$

If $\partial\Omega$ is $\mathcal{C}^{1,\alpha}$ and hence $\mathcal{K}^*_{\partial\Omega}$ admits the spectral resolution (16), then

$$d\mu_{ij}^{\Omega} = \sum_{k=1}^{\infty} \langle \varphi_k, \nu_i\rangle_{\mathcal{H}} \langle \varphi_k, y_j\rangle \delta(t - \lambda_k) dt.$$

Since $\langle \varphi_k, \nu_i\rangle_{\mathcal{H}} = -\langle \varphi_k, \mathcal{S}_{\partial\Omega}[\nu_i]\rangle$, we have

$$m_{ij}(\lambda, \Omega) = -\sum_{k=1}^{\infty} \frac{\langle \varphi_k, \mathcal{S}_{\partial\Omega}[\nu_i]\rangle \langle \varphi_k, y_j\rangle}{\lambda - \lambda_k}. \tag{55}$$

In particular, $m_{ij}(\lambda, \Omega)$ is a meromorphic function on $\mathbb{C}$ except 0 where it has an essential singularity.

One interesting question here is whether $m_{ij}(\lambda, \Omega)$ determines Ω uniquely (up to shifting). It is reminiscent of the inverse spectral problem to determine the domain Ω from its eigenvalues. Here we have extra data, namely,

$$\langle \varphi_k, \mathcal{S}_{\partial\Omega}[\nu_i]\rangle \langle \varphi_k, y_j\rangle, \quad k = 1, 2, \dots,$$

in addition to eigenvalues of the NP operator.

It is also quite interesting to reconstruct finer details of the shape using $M(\lambda,\Omega)$ for finite number of λ's. This is different from imaging using higher order GPTs. It is worth mentioning that $M(\lambda,\Omega)$ for several λ can be actually obtained by multi-frequency measurements. For this we refer to recent work of Ammari *et al* [**4**]

As we discussed in Section 2, it is interesting to investigate the property of the measure $d\mu_{ij}^{\Omega}$ when $\partial\Omega$ has corners. If $\partial\Omega$ is $\mathcal{C}^{1,\alpha}$, then this measure is singular. It would be interesting to see if it has an absolutely continuous part (or an essential singularity) if $\partial\Omega$ has corners. In this regard, we mention the paper [**62**] where $m_{ij}(\lambda,\Omega)$ is numerically computed when Ω is a cube. If Ω is a cube, then $(m_{ij}(\lambda,\Omega))_{i,j=1}^3$ is isotropic, *i.e.*, $(m_{ij}(\lambda,\Omega)) = m(\lambda,\Omega)I$ for some scalar function m.

4. Enhancement of cloaking using GPT-vanishing structure

So far we discussed how one can use GPTs to see (imaging). In this section we discuss how one can use the notion of GPTs to hide (invisibility cloaking). We discuss results of [**26**] in the quasi-static (zero frequency) case. The results have been extended to the Helmholtz equation [**15, 27**] and the full Maxwell equation [**29**].

4.1. Invisibility by far-field measurements. — We first emphasize that the expansion (47) holds even if the conductivity (or dielectric constant) is not constant. For clarity suppose that the background function $h(x)$ is $x = r\cos\theta$. We then obtain from (47) that

$$u(x) = h(x) - \sum_{m=1}^{\infty} \left[\frac{\cos m\theta}{2\pi m r^m} M_{m1}^{cc} + \frac{\sin m\theta}{2\pi m r^m} M_{m1}^{sc} \right]$$

as $|x| \to \infty$. Suppose that the inclusion Ω has the property that $M_{m1} = 0$ for all m, then $u(x) = h(x)$, as if there is no inclusion. In other words, we can not see the inclusion from the far-field measurement of u. Suppose that Ω has the property that $M_{m1} = 0$ for all $m \le N$, then $u(x) - h(x) = O(|x|^{-N-1})$. So as N becomes larger, Ω is seen vaguely. In summary, we can make the inclusion vaguely visible from the far-field measurements by making a first few terms of its GPTs vanish.

To construct a structure whose a first few GPTs vanish, let $1 = r_{N+1} < r_N < \ldots < r_1 = 2$ and define

$$A_j := \{r_{j+1} < r \le r_j\}, \quad j = 1, 2, \ldots, N,$$

and $A_0 = \{r_1 < r\}$, $A_{N+1} = \{r \le r_{N+1}\}$. We choose σ_j to be the conductivity of A_j for $j = 1, 2, \ldots, N+1$, and $\sigma_0 = 1$. Let

$$\sigma = \chi(A_0) + \sum_{j=1}^{N} \sigma_j \chi(A_j) + \sigma_{N+1}\chi(A_{N+1}). \tag{56}$$

We emphasize that σ_{N+1} may or may not be fixed. If σ_{N+1} is fixed to be 0, it means that the core is insulated. Because of the symmetry of the structure, we have

$$M^{cs}_{mn} = M^{sc}_{mn} = 0 \quad \text{for all } m, n,$$
$$M^{cc}_{mn} = M^{ss}_{mn} = 0 \quad \text{if } m \neq n,$$

and

$$M^{cc}_{nn} = M^{ss}_{nn} \quad \text{for all } n.$$

Let $M_n = M^{cc}_{nn}$, $n = 1, 2, \ldots$, then the expansion (47) becomes

$$(u-h)(x) = -\sum_{n=1}^{\infty} \left[\frac{M_n}{2\pi n r^n} (a^c_n \cos n\theta + a^s_n \sin n\theta) \right]. \tag{57}$$

If $M_n = 0$ for all $n \leq N$, then $u(x) - h(x) = O(|x|^{-N-1})$. In fact, more than this is true: $u(x) = h(x)$ outside the inclusion if $h(x) = r^n \cos n\theta$ or $r^n \sin n\theta$. We call such structure Ω (and γ) a GPT-vanishing structure of order N. If $N = 1$, we call it a PT-vanishing structure.

In order for σ to be a GPT-vanishing structure of order N, for given $h(x) = r^k \cos k\theta$, $k = 1, \ldots, N$, the solution u to (9) should satisfy $u(x) = h(x)$ in the matrix. If $h(x) = r^k \cos k\theta$, then the solution u takes the form

$$u(x) = a_j r^k \cos k\theta + \frac{b_j}{r^k} \cos k\theta \quad \text{in } A_j, \quad j = 0, 1, \ldots, N+1,$$

with $a_0 = 1$ and $b_{N+1} = 0$. Since u satisfies

$$(u-h)(x) = \frac{b_0}{r^k} \cos k\theta \quad \text{as } |x| \to \infty,$$

we have

$$M_k = -2\pi k b_0. \tag{58}$$

The transmission conditions (continuity of the potential and the flux) on the interface $\{r = r_j\}$ yield

$$\begin{bmatrix} a_j \\ b_j \end{bmatrix} = \frac{1}{2\sigma_j} \begin{bmatrix} \sigma_j + \sigma_{j-1} & (\sigma_j - \sigma_{j-1}) r_j^{-2k} \\ (\sigma_j - \sigma_{j-1}) r_j^{2k} & \sigma_j + \sigma_{j-1} \end{bmatrix} \begin{bmatrix} a_{j-1} \\ b_{j-1} \end{bmatrix},$$

and hence

$$\begin{bmatrix} a_{N+1} \\ 0 \end{bmatrix} = \prod_{j=1}^{N+1} \frac{1}{2\sigma_j} \begin{bmatrix} \sigma_j + \sigma_{j-1} & (\sigma_j - \sigma_{j-1}) r_j^{-2k} \\ (\sigma_j - \sigma_{j-1}) r_j^{2k} & \sigma_j + \sigma_{j-1} \end{bmatrix} \begin{bmatrix} 1 \\ b_0 \end{bmatrix}.$$

Let

$$P^{(k)} = \begin{bmatrix} p^{(k)}_{11} & p^{(k)}_{12} \\ p^{(k)}_{21} & p^{(k)}_{22} \end{bmatrix} := \prod_{j=1}^{N+1} \frac{1}{2\sigma_j} \begin{bmatrix} \sigma_j + \sigma_{j-1} & (\sigma_j - \sigma_{j-1}) r_j^{-2k} \\ (\sigma_j - \sigma_{j-1}) r_j^{2k} & \sigma_j + \sigma_{j-1} \end{bmatrix}.$$

Then,

$$b^{(k)}_0 = -\frac{p^{(k)}_{21}}{p^{(k)}_{22}}.$$

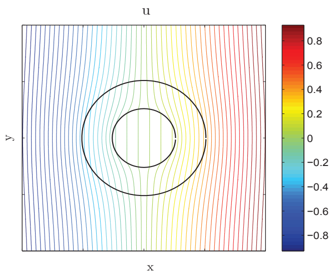

FIGURE 7. A neutral inclusion (=PT-vanishing structure). The uniform field is not perturbed in the presence of the structure. A figure prepared by E. Kim. (It looks like the outside field is perturbed. But it is a numerical error.)

So, we achieve $M_k = 0$ for $k = 1, \dots, N$ by choosing σ_j, $j = 1, \dots, N$ (or $N+1$) so that

$$p_{21}^{(k)} = 0, \quad k = 1, \dots, N, \tag{59}$$

in other words,

$$\prod_{j=1}^{N+1} \frac{1}{2\sigma_j} \begin{bmatrix} \sigma_j + \sigma_{j-1} & (\sigma_j - \sigma_{j-1}) r_j^{-2k} \\ (\sigma_j - \sigma_{j-1}) r_j^{2k} & \sigma_j + \sigma_{j-1} \end{bmatrix} \text{ is a upper triangular matrix.}$$

For arbitrary N, the equation is a non-linear algebraic equation of $\sigma_1, \dots, \sigma_N$ and existence of solutions is not proved. Of course, for small N, it can be solved easily. In fact, for the circular multi-layered structure as we constructed, a GPT-vanishing structure (or a PT-vanishing structure) is known as a neutral inclusion. A neutral inclusion is an inclusion such that even if we insert it into the homogeneous space, the uniform field is not perturbed. Figure 7 shows a neutral inclusion. We refer to [**95**] for neutral inclusions and their connection to the theory of composites. The neutral inclusion here consists of the core of radius r_2 and conductivity σ_2, the shell of radius r_1 and conductivity σ_1, and the matrix of conductivity 1 which satisfy

$$(\sigma_2 - \sigma_1)(\sigma_1 + 1) r_2^2 + (\sigma_2 + \sigma_1)(\sigma_1 - 1) r_1^2 = 0. \tag{60}$$

Instead, the equation (59) is solved numerically in [**26**]. Figure 8 shows the conductivity profile of a GPT-vanishing structure of order 6 when the conductivity of the core is 0. It is interesting to observe that the conductivity in the figure fluctuates below and above the conductivity of the matrix.

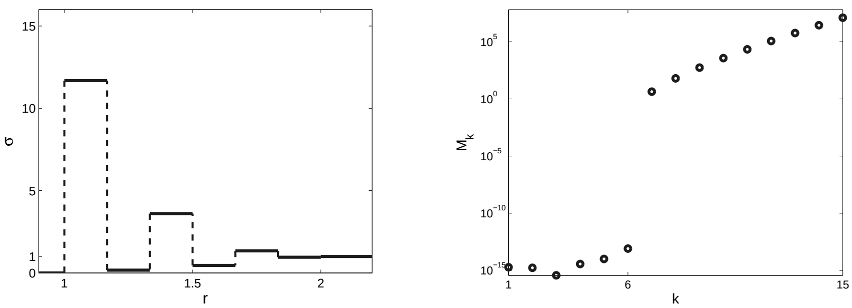

FIGURE 8. Conductivity profile of a GPT-vanishing structure of order 6 with the conductivity of core 0. The right-hand side is the values of GPTs. A figure from [**26**].

4.2. Invisibility by DtN map. — Let σ be the conductivity distribution of Ω. The the Drichlet-to-Neumann (DtN) map $\Lambda[\sigma]$ corresponding to σ is defined by

$$\Lambda[\sigma](\phi) = \sigma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega},$$

where u is the solution to

$$\begin{cases} \nabla \cdot \sigma \nabla u = 0, & \text{in } \Omega, \\ u = \phi, & \text{on } \partial\Omega. \end{cases}$$

As mentioned in the previous subsection, the Calderón problem[(6)] is to reconstruct σ in terms of $\Lambda[\sigma]$. In this sense the DtN map is the tool we look inside Ω.

There is an obstacle in reconstructing σ from $\Lambda[\sigma]$. If F is a diffeomorphism of Ω which is the identity on $\partial\Omega$, then

$$\Lambda[\sigma] = \Lambda[F_*\sigma]$$

where $F_*\sigma$ is the push-forward of σ by F:

$$F_*\sigma(y) = \frac{DF(x)\sigma(x)DF(x)^T}{\det(DF(x))}, \quad x = F^{-1}(y).$$

See [**82**]. In [**58**] Greenleaf *et al* use this idea to show non-uniqueness of the Calderón problem. Let

$$F(x) := \left(1 + \frac{|x|}{2}\right) \frac{x}{|x|} \tag{61}$$

which is a diffeomorphism from the punctured disk $\{x : 0 < |x| < 2\}$ onto the annulus $\{x : 1 < |x| < 2\}$ and is the identity on $|x| = 2$. Thus $\Lambda[1] = \Lambda[F_* 1]$.

Pendry et al [**104**] used the exactly same transformation to show that the structure after the transformation can bend the electro-magnetic waves so that things inside

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(6) We used NtD map before. But the DtN map is more convenient when comparing with other work.

$|x| < 1$ are cloaked (see also [**85**]). Since then huge literature has been devoted to the study of cloaking by transformation optics. It is not possible to include relevant references here. Instead we refer to an excellent survey [**57**] and references therein. We mention that cloaking by transformation optics is a very active area of research and related literature is ever growing.

The structure of the transformation optics proposed in [**58**] has demerits. For example, F_*1 is singular on the inner boundary $|x| = 1$ (0 in the normal direction, ∞ in tangential direction in two dimensions). In order to avoid the singularity of the conductivity, Kohn *et al* [**81**] (see also [**80**]) came up with the idea of blowing up a small disk instead of a single point. For a small number δ, let

$$\sigma_\delta = \begin{cases} \gamma & \text{if } |x| < \delta, \\ 1 & \text{if } \delta \le |x| \le 2. \end{cases}$$

Here, the conductivity γ of the core is constant, and it can be 0 (the core is insulated) or ∞ (perfect conductor). Let

$$F(x) = \begin{cases} \left(\dfrac{2-2\delta}{2-\delta} + \dfrac{1}{2-\delta}|x| \right) \dfrac{x}{|x|} & \text{if } \delta \le |x| \le 2, \\ \dfrac{x}{\delta} & \text{if } |x| \le \delta. \end{cases}$$

Then F maps B_2 (the disk of radius 2) onto B_2 and blows up B_δ onto B_1. Then it is shown that

$$\|\Lambda[F_*\sigma] - \Lambda[1]\| \le C\delta^2 \tag{62}$$

for some constant C independent of δ and γ. Here the norm is the operator norm of the DtN map as a map from $H^{1/2}(\partial B_2)$ into $H^{-1/2}(\partial B_2)$. If the core is insulated ($\gamma = 0$), then B_1 after the transformation is also insulated. So, (62) shows that things in B_1 is almost cloaked (up to the order of δ^2).

Let us discuss briefly why (62) holds. Let u_δ be the solution to

$$\begin{cases} \nabla \cdot \sigma_\delta \nabla u_\delta = 0 & \text{in } B_2, \\ u_\delta = \phi & \text{on } \partial B_2, \end{cases}$$

and u_0 be the solution to

$$\begin{cases} \Delta u_0 = 0 & \text{in } B_2, \\ u_0 = \phi & \text{on } \partial B_2. \end{cases}$$

Like (50) one can show that

$$\frac{\partial u_\delta}{\partial \nu}(x) = \frac{\partial u_0}{\partial \nu}(x) + \nabla u_0(0) \cdot M \frac{\partial}{\partial \nu_x} \nabla_y G(x,0) + O(\delta^3), \quad x \in \partial B_2,$$

where M is the polarization tensor of B_δ and $G(x,y)$ is the Green function for Δ on B_2 (see [**21**] for a proof). We emphasize that the expansion holds uniformly for γ (see

[**101**]). This expansion can be rewritten as

$$\Lambda[\sigma](\phi)(x) = \Lambda[1](\phi)(x) + \nabla u_0(0) \cdot M \frac{\partial}{\partial \nu_x} \nabla_y G(x, 0) + O(\delta^3).$$

Since the PT for B_δ (with conductivity γ) is $M = \frac{2(\gamma-1)}{\gamma+1}|B_\delta|I$ (see (46)), we have

$$\|\Lambda[\sigma] - \Lambda[1]\| \le C\delta^2.$$

Since $\Lambda[F_*\sigma] = \Lambda[\sigma]$, (62) follows.

The main result of [**26**] is that if we coat the small disk of radius δ with the GPT-vanishing structure of order N, then the cloaking effect is enhanced to the order $(2\rho)^{2N+2}$. To show this, let σ be the GPT-vanishing structure of order N as constructed in the previous section with $r_1 = 2$, $r_{N+1} = 1$, and the conductivity of the core $\sigma_{N+1} = \gamma$. Let $M_n[\sigma]$ be the CGPTs associated with the structure σ. Then, $M_n = 0$ for $n = 1, \ldots, N$. Moreover, it is proved that

$$|M_n[\sigma]| \le 2\pi n 2^{2n}, \quad n = 1, 2, \ldots. \tag{63}$$

Let

$$\sigma_\delta^N(x) = \sigma(\frac{1}{\delta}x). \tag{64}$$

Then, σ_δ^N has conductivity γ inside the radius δ, multi-layered structure in between the radius δ and 2δ with fluctuating conductivities, and conductivity 1 outside 2δ. We also have

$$M_n[\sigma_\delta^N] = 0 \quad \text{for } n = 1, \ldots, N, \tag{65}$$

and from the scaling property (41) of CGPTs and (63) that

$$|M_n[\sigma_\delta^N]| = \delta^{2n}|M_n[\sigma]| \le 2\pi n(2\delta)^{2n}, \quad n = 1, 2, \ldots. \tag{66}$$

Using (57) with $h(x) = r^k e^{ik\theta}$, it is proved that

$$\Big(\Lambda[\sigma_\delta^N] - \Lambda[1]\Big)(f) = \sum_{k=-\infty}^{\infty} \frac{2|k| M_{|k|}[\sigma_\delta^N]}{2\pi|k| - M_{|k|}[\sigma_\delta^N]} f_k e^{ik\theta},$$

where f_k is the Fourier coefficients of f on ∂B_2. It then follows from (65) and (66) that

$$\| \Lambda[\sigma_\delta^N] - \Lambda[1] \| \le C(2\delta)^{2N+2}.$$

Since $\Lambda[F_*\sigma_\delta^N] = \Lambda[\sigma_\delta^N]$, we finally have an estimate for enhancement of the near-cloaking:

$$\| \Lambda[F_*\sigma_\delta^N] - \Lambda[1] \| \le C(2\delta)^{2N+2}. \tag{67}$$

Further discussion. — The GPT-vanishing structure about which we discussed in this section is circular. It is a challenging problem (with possibility of various applications) to construct GPT-vanishing structure of order 1 (or higher order) of arbitrary shape. The problem is that for the given core D of any shape and conductivity σ_c find Ω containing D and the conductivity σ_s of the shell $\Omega \setminus D$ so that the PT of the structure is 0. Recently Jarczyk and Mityushev [**66**] constructed a coating on the core so that the structure is neutral to a given field. We also mention a paper [**99**] of Milton and Serkov where the neutral coating is constructed when $\sigma_c = 0$ or ∞. There they proved that it is only confocal ellipses (with $\sigma_c = 0$ or ∞) which is neutral to multiple fields. We emphasize that a PT-vanishing structure is the same as a neutral inclusion if the structure is radial as we explained, but nor for general shapes. If a structure is neutral to two fields in two dimensions, then it is PT-vanishing, but, not vice versa. If the structure is neutral to the uniform field in the direction a, then the solution u satisfies $u(x) = a \cdot x$ outside the structure. But for the PT-vanishing structure only satisfies $u(x) - a \cdot x = O(|x|^{-d})$ as $|x| \to \infty$. So the PT-vanishing structure may be regarded as a weakly neutral structure.

The method for enhancement of approximate cloaking discussed in this section uses multi-coated structures with vanishing GPTs. Recently a different method using a transformation (a change of variables) is proposed in [**60**].

5. Analysis of cloaking by anomalous localized resonance

We now discuss another kind of invisibility cloaking: cloaking due to anomalous localized resonance (CALR). If a body of dielectric material (core) is coated by a plasmonic structure of negative dielectric constant with nonzero loss parameter (shell), then anomalous localized resonance may occur and the source outside the structure may be cloaked as the loss parameter tends to zero. We note that unlike the cloaking by transformation optics, the cloaking due to anomalous localized resonance is an exterior cloaking.

Let Ω be a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, and D be a domain whose closure is contained in Ω. In other words, D is the core and $\Omega \setminus \overline{D}$ is the shell. For a given loss parameter $\delta > 0$, the permittivity distribution in $\mathbb{R}^d$ is given by

$$\epsilon_\delta = \begin{cases} 1 & \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ -1 + i\delta & \text{in } \Omega \setminus \overline{D}, \\ 1 & \text{in } D. \end{cases} \tag{68}$$

Here $-1 + i\delta$ represents the negative dielectric constant of the shell with the lossy parameter δ (plasmonic structure). See Figure 9. For a given function f compactly supported in $\mathbb{R}^d \setminus \overline{\Omega}$ satisfying

$$\int_{\mathbb{R}^d} f \, d\mathbf{x} = 0 \tag{69}$$

(which is required by conservation of charge), we consider the following dielectric problem:

$$\nabla \cdot \epsilon_\delta \nabla u_\delta = f \quad \text{in } \mathbb{R}^d, \tag{70}$$

with the decay condition $u_\delta(x) \to 0$ as $|x| \to \infty$. Let

$$E_\delta := \Im \int_{\mathbb{R}^d} \epsilon_\delta |\nabla u_\delta|^2 \, d\mathbf{x} = \int_{\Omega \setminus D} \delta |\nabla u_\delta|^2 \, d\mathbf{x} \tag{71}$$

($\Im$ for the imaginary part). The problem of cloaking by anomalous localized resonance (CALR) can be formulated as the problem of identifying the sources f such that first

$$E_\delta \to \infty \quad \text{as } \delta \to 0, \tag{72}$$

and secondly, $u_\delta/\sqrt{E_\delta}$ goes to zero outside some radius a, as $\delta \to 0$:

$$|u_\delta(x)/\sqrt{E_\delta}| \to 0 \quad \text{as } \delta \to 0 \quad \text{when } |x| > a. \tag{73}$$

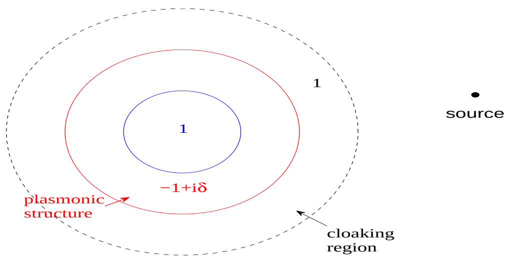

FIGURE 9. Configuration for cloaking due to anomalous localized resonance

The equation (70) is known as the quasistatic equation and the real part of $-\nabla u_\delta(x)e^{-i\omega t}$, where ω is the frequency and t is the time, represents an approximation for the electric field in the vicinity of Ω, when the wavelength of the electromagnetic radiation is large compared to the size of Ω. The quantity E_δ approximately represents the time averaged electromagnetic power produced by the source dissipated into heat. So, (72) implies an infinite amount of energy dissipated per unit time in the limit $\delta \to 0$ which is unphysical. If we rescale the source f by a factor of $1/\sqrt{E_\delta}$ then the source will produce the same power independently of δ and the new associated potential $u_\delta/\sqrt{E_\delta}$ will, by (73), approach zero outside the radius a. Hence, cloaking due to anomalous localized resonance (CALR) occurs. The normalized source is essentially invisible from the outside, yet the fields inside are very large.

This phenomena of anomalous resonance was first discovered by Nicorovici, McPhedran and Milton [**100**] and is related to invisibility cloaking [**96**]. It is also related to superlenses [**102, 103**] since, as shown in [**100**], the anomalous resonance can create apparent point sources. For these connections and further developments tied to this form of invisibility cloaking, we refer to [**7, 9, 8, 40, 41, 98**] and references therein. In this section we review results of first three papers which rely on spectral theory of corresponding NP operator. The results of [**7**] for circular structure was extended in [**78**] by a different method-variational approach for sources supported on a circle.

5.1. Neumann-Poincaré type operator. — Let u_δ be the solution to (70). The transmission conditions along the interfaces ∂D and $\partial\Omega$ are given by

$$\begin{aligned} (-1+i\delta)\frac{\partial u_\delta}{\partial \nu}\Big|_+ &= \frac{\partial u_\delta}{\partial \nu}\Big|_- \quad \text{on } \partial D \\ \frac{\partial u_\delta}{\partial \nu}\Big|_+ &= (-1+i\delta)\frac{\partial u_\delta}{\partial \nu}\Big|_- \quad \text{on } \partial\Omega. \end{aligned} \tag{74}$$

So it is natural to represent the solution in terms of single layer potentials. Let F be the Newtonian potential of f, *i.e.*,

$$F(x) = \int_{\mathbb{R}^2} \Gamma(x-y) f(y)dy, \quad x \in \mathbb{R}^d. \tag{75}$$

Then F satisfies $\Delta F = f$ in $\mathbb{R}^d$, and the solution u_δ may be represented as

$$u_\delta(x) = F(x) + \mathcal{S}_{\partial D}[\varphi_i](x) + \mathcal{S}_{\partial\Omega}[\varphi_e](x), \quad x \in \mathbb{R}^d \tag{76}$$

for some functions $\varphi_i \in L^2_0(\partial D)$ and $\varphi_e \in L^2_0(\partial\Omega)$. By (74), the pair of potentials (φ_i, φ_e) is the solution to

$$\begin{bmatrix} z_\delta I - \mathcal{K}^*_{\partial D} & -\dfrac{\partial}{\partial \nu_i}\mathcal{S}_{\partial\Omega} \\ \dfrac{\partial}{\partial \nu_e}\mathcal{S}_{\partial D} & z_\delta I + \mathcal{K}^*_{\partial\Omega} \end{bmatrix} \begin{bmatrix} \varphi_i \\ \varphi_e \end{bmatrix} = \begin{bmatrix} \dfrac{\partial F}{\partial \nu_i} \\ -\dfrac{\partial F}{\partial \nu_e} \end{bmatrix} \tag{77}$$

on $L^2_0(\partial D) \times L^2_0(\partial\Omega)$, where ν_i and ν_e denote the (outward) normal vectors to ∂D and $\partial\Omega$, respectively, and

$$z_\delta = \frac{i\delta}{2(2-i\delta)}. \tag{78}$$

It is worth mentioning about the off-diagonal entries of the above matrix. For example, $\frac{\partial}{\partial\nu_i}\mathcal{S}_{\partial\Omega}$ is an operator from $L^2(\partial\Omega)$ into $L^2(\partial D)$ defined by

$$\frac{\partial}{\partial \nu_i}\mathcal{S}_{\partial\Omega}[\varphi](x) = \frac{\partial}{\partial \nu_i}\int_{\partial\Omega} \Gamma(x-y)\varphi(y)d\sigma(y), \quad x \in \partial D.$$

Let

$$\mathbb{K}^* := \begin{bmatrix} -\mathcal{K}^*_{\partial D} & -\dfrac{\partial}{\partial \nu_i}\mathcal{S}_{\partial\Omega} \\ \dfrac{\partial}{\partial \nu_e}\mathcal{S}_{\partial D} & \mathcal{K}^*_{\partial\Omega} \end{bmatrix}. \tag{79}$$

This is the Neumann-Poincaré operator for the problem under consideration defined on $L^2(\partial D) \times L^2(\partial\Omega)$. Then the integral equation (77) can be written as

$$(z_\delta \mathbb{I} + \mathbb{K}^*)[\Phi] = \mathbf{g} \tag{80}$$

where

$$\mathbb{I} := \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad \Phi := \begin{bmatrix} \varphi_i \\ \varphi_e \end{bmatrix}, \quad \mathbf{g} := \begin{bmatrix} \dfrac{\partial F}{\partial \nu_i} \\ -\dfrac{\partial F}{\partial \nu_e} \end{bmatrix}.$$

Note that as $\delta \to 0$, $z_\delta \to 0$. Since eigenvalues of $\mathbb{K}^*$ accumulate at 0 and hence 0 is an essential singularity of the operator-valued meromorphic function

$$\lambda \in \mathbb{C} \mapsto (\lambda \mathbb{I} + \mathbb{K}^*)^{-1}.$$

This causes a serious difficulty in solving (80). We overcome this difficulty using the spectrum of $\mathbb{K}^*$.

The following properties of $\mathbb{K}^*$ have been proved in [**7**]:

(i) The spectrum of $\mathbb{K}^*$ lies in the interval $(-1/2, 1/2]$.

(ii) The L^2-adjoint of $\mathbb{K}^*$, $\mathbb{K}$, is given by

$$\mathbb{K} = \begin{bmatrix} -\mathcal{K}_{\partial D} & \mathcal{D}_{\partial\Omega} \\ -\mathcal{D}_{\partial D} & \mathcal{K}_{\partial\Omega} \end{bmatrix} \tag{81}$$

where $\mathcal{D}$ is the double layer potential defined in (21).

(iii) The operator $\mathbb{S}$ defined by

$$\mathbb{S} = \begin{bmatrix} \mathcal{S}_{\partial D} & \mathcal{S}_{\partial\Omega} \\ \mathcal{S}_{\partial D} & \mathcal{S}_{\partial\Omega} \end{bmatrix}$$

is self-adjoint, and $-\mathbb{S} > 0$ on $L_0^2(\partial D) \times L_0^2(\partial\Omega)$.

(iv) The Calderón's identity holds

$$\mathbb{S}\mathbb{K}^* = \mathbb{K}\mathbb{S},$$

i.e., $\mathbb{S}\mathbb{K}^*$ is self-adjoint.

To see the second assertion in (iii), let $\Phi = (\varphi_i, \varphi_e)^T \in L_0^2(\partial D) \times L_0^2(\partial\Omega)$ and define

$$u(x) = \mathcal{S}_{\partial D}[\varphi_i](x) + \mathcal{S}_{\partial\Omega}[\varphi_e](x).$$

Then we have

$$\int_{\mathbb{R}^d} |\nabla u|^2 dx = -\int_{\partial D} \bar{u}\varphi_i d\sigma - \int_{\partial\Omega} \bar{u}\varphi_e d\sigma = -\langle \Phi, \mathbb{S}[\Phi] \rangle$$

where the inner product is the standard one on $L^2(\partial D) \times L^2(\partial\Omega)$.

These properties make the map

$$(\Phi, \Psi) \mapsto -\langle \Phi, \mathbb{S}[\Psi] \rangle = -\langle \varphi_i, \mathcal{S}_{\partial D}[\psi_i] \rangle - \langle \varphi_e, \mathcal{S}_{\partial\Omega}[\psi_e] \rangle$$

an inner product on $L^2_0(\partial D) \times L^2_0(\partial\Omega)$. As in section 2, we let $\mathcal{H}^2$ be the Hilbert space $L^2_0(\partial D) \times L^2_0(\partial\Omega)$ with this inner product and let

$$\langle \Phi, \Psi \rangle_{\mathcal{H}^2} := -\langle \Phi, \mathbb{S}[\Psi] \rangle. \tag{82}$$

If we denote $\mathcal{H}$ defined in section 2 by $\mathcal{H}(\partial\Omega)$, then $\mathcal{H}^2$ is nothing but $\mathcal{H}(\partial D)\times\mathcal{H}(\partial\Omega)$. The Calderón identity implies that $\mathbb{K}^*$ is self-adjoint on $\mathcal{H}^2$.

Let $\lambda_1, \lambda_2, \dots$ with $|\lambda_1| \geq |\lambda_2| \geq \dots$ be the nonzero eigenvalues (counting multipl-cities) of $\mathbb{K}^*$ and Ψ_n be the corresponding eigenfunctions normalized by $\|\Psi_n\|_{\mathcal{H}^2} = 1$. Then we have

$$\langle \Psi_i, \Psi_j \rangle_{\mathcal{H}^2} = \delta_{ij} \tag{83}$$

where δ_{ij} is the Kronecker's delta, and $\mathbb{K}^*$ admits the spectral decomposition

$$\mathbb{K}^*[\Phi] = \sum_{n=1}^{\infty} \lambda_n \langle \Phi, \Psi_n \rangle_{\mathcal{H}^2} \Psi_n, \quad \Phi \in \mathcal{H}^2. \tag{84}$$

We mention that since $\mathbb{K}^*$ is a Hilbert-Schmidt operator (see [**53**]), we have

$$\sum_{n=1}^{\infty} \lambda_n^2 < \infty. \tag{85}$$

Since F is harmonic in Ω, one can see from the divergence theorem that $\mathbf{g} \in \mathcal{H}^2$. So, the solution $\Phi_\delta = (\varphi_i^\delta, \varphi_e^\delta)^T$ to (80) is given by

$$\Phi_\delta = \sum_{n=1}^{\infty} \frac{\langle \mathbf{g}, \Psi_n \rangle_{\mathcal{H}^2}}{\lambda_n + z_\delta} \Psi_n. \tag{86}$$

Note that CALR occurs, *i.e.*, $E_\delta \to \infty$ if and only if

$$\delta \int_{\Omega\setminus\overline{D}} \left| \nabla(\mathcal{S}_{\partial D}[\varphi_i^\delta] + \mathcal{S}_{\partial\Omega}[\varphi_e^\delta]) \right|^2 dx \to \infty \quad \text{as } \delta \to \infty. \tag{87}$$

On the other hand, one can see that

$$\begin{aligned}\int_{\Omega\setminus\overline{D}} \left| \nabla(\mathcal{S}_{\partial D}[\varphi_i^\delta] + \mathcal{S}_{\partial\Omega}[\varphi_e^\delta]) \right|^2 dx &= -\frac{1}{2}\langle \Phi_\delta, \mathbb{S}\Phi_\delta \rangle + \langle \mathbb{K}^*\Phi_\delta, \mathbb{S}\Phi_\delta \rangle \\ &= \frac{1}{2}\langle \Phi_\delta, \Phi_\delta \rangle_{\mathcal{H}^2} - \langle \mathbb{K}^*\Phi_\delta, \Phi_\delta \rangle_{\mathcal{H}^2} \\ &= \sum_{n=1}^{\infty} \frac{(\frac{1}{2} - \lambda_n)\langle \mathbf{g}, \Psi_n \rangle_{\mathcal{H}^2}^2}{|\lambda_n + z_\delta|^2}.\end{aligned}$$

Since $\lambda_n \to 0$ as $n \to \infty$, we conclude that CALR takes place if and only if

$$\delta \sum_{n=1}^{\infty} \frac{\langle \mathbf{g}, \Psi_n \rangle_{\mathcal{H}^2}^2}{\lambda_n^2 + \delta^2} \to \infty \quad \text{as } \delta \to \infty. \tag{88}$$

This characterization gives a necessary and sufficient condition on the source term f for the blow up of the electromagnetic energy in $\Omega \setminus \overline{D}$. This condition is given in terms of the Newton potential of f.

CALR in an annulus in two dimensions. — If $\Omega = B_e = \{|x| < r_e\}$ and $D = B_i = \{|x| < r_i\}$ in $\mathbb{R}^d$, $d = 2, 3$, where $r_e > r_i$, so that the shell is an annulus, then we can compute the eigenvalues and eigenfunctions of $\mathbb{K}^*$ and the CALR can be studied using (88). In this subsection we review results in two dimensions of [**7**].

If $B = \{|x| < r_0\}$ in $\mathbb{R}^2$, then we have for each integer n

$$\mathcal{S}_{\partial B}[e^{in\theta}](x) = \begin{cases} -\dfrac{r_0}{2|n|}\left(\dfrac{r}{r_0}\right)^{|n|} e^{in\theta} & \text{if } |x| = r < r_0, \\ -\dfrac{r_0}{2|n|}\left(\dfrac{r_0}{r}\right)^{|n|} e^{in\theta} & \text{if } |x| = r > r_0. \end{cases} \tag{89}$$

Using this formula, one can show that the eigenvalues of $\mathbb{K}^*$ on $\mathcal{H}^2$ are

$$-\frac{1}{2}\rho^n, \ \frac{1}{2}\rho^n, \quad n = 1, 2, \dots, \tag{90}$$

and corresponding eigenfunctions are

$$\begin{bmatrix} e^{\pm in\theta} \\ \rho e^{\pm in\theta} \end{bmatrix}, \ \begin{bmatrix} e^{\pm in\theta} \\ -\rho e^{\pm in\theta} \end{bmatrix}, \quad n = 1, 2, \dots, \tag{91}$$

where $\rho = r_i/r_e$. Then using (88) we obtain the following results: let

$$r_* = \sqrt{\frac{r_e^3}{r_i}}. \tag{92}$$

Non-occurrence of CALR. If the source function f is supported in $|x| > r_*$, then CALR does not occur, *i.e.*,

$$E_\delta < C \tag{93}$$

for some C independent of δ. Moreover, we have

$$\sup_{|x| \geq r_*} |u_\delta(x) - F(x)| \to 0 \quad \text{as} \quad \delta \to 0, \tag{94}$$

where F be the Newtonian potential of f.

This result shows that if the source function f is supported outside the critical radius r_*, then by observing u_δ in $|x| \geq r_*$ one can recover F and hence the source f approximately.

Occurrence of CALR. Let f be a source function supported in $r_e < |x| < r_*$ and F be the Newtonian potential of f. Let g_e^n be the Fourier coefficient of $-\frac{\partial F}{\partial \nu_e}$ on $\partial\Omega$, *i.e.*,

$$-\frac{\partial F}{\partial \nu_e} = \sum_{n=-\infty}^{\infty} g_e^n e^{in\theta}.$$

(i) If F is not identically zero, then weak CALR occurs, *i.e.*,

$$\limsup_{\delta \to 0} E_\delta = \infty. \tag{95}$$

(ii) If the Fourier coefficients of F satisfy the gap condition

[GP]: there exists a sequence $\{n_k\}$ with $|n_1| < |n_2| < \cdots$ such that

$$\lim_{k\to\infty} \rho^{|n_{k+1}|-|n_k|} \frac{|g_e^{n_k}|^2}{|n_k|\rho^{|n_k|}} = \infty,$$

then CALR occurs, *i.e.*,

$$\lim_{\delta\to 0} E_\delta = \infty. \tag{96}$$

The gap condition [GP] is mild and the Newtonian potentials of many source functions satisfy it. For examples, dipole and quadrapole sources satisfy [GP]. A dipole source in $B_{r_*} \setminus \overline{B}_e$ is $f(x) = a \cdot \nabla\delta_y(x)$ for a vector a and $y \in B_{r_*} \setminus \overline{B}_e$ where δ_y is the Dirac delta function at y, and a quadrapole source is

$$f(x) = A : \nabla\nabla\delta_y(x) = \sum_{i,j=1}^{2} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \delta_y(x)$$

for a 2×2 matrix $A = (a_{ij})$ and $y \in B_{r_*} \setminus \overline{B}_e$.

These results were extended in [**9**] to the case when

$$\epsilon_\delta = \begin{cases} 1 & \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \\ \epsilon_s + i\delta & \text{in } \Omega \setminus \overline{D}, \\ \epsilon_c & \text{in } D. \end{cases} \tag{97}$$

Here ϵ_c is a positive constant, but ϵ_s is a negative constant representing the negative dielectric constant of the shell. In this case, it is proved that if $\epsilon_s = -1$ and $\epsilon_c \neq 1$, then CALR occurs and the critical radius is $r_* = r_e^2/r_i$[(7)] (see [**9**] for precise statements). If $\epsilon_s \neq -1$, then CALR does not occur for any source.

CALR in an annulus in three dimensions. — Let $\Omega = \{|x| < r_e\}$ and $D = \{|x| < r_i\}$ in $\mathbb{R}^3$. If $B = \{|x| < r_0\}$ in $\mathbb{R}^3$, then we have for each integer n

$$\mathcal{S}_{\partial B}[Y_n^m](x) = \begin{cases} -\dfrac{1}{2n+1} \dfrac{r^n}{r_0^{n-1}} Y_n^m(\hat{x}) & \text{if } |x| = r \leq r_0, \\ -\dfrac{1}{2n+1} \dfrac{r_0^{n+2}}{r^{n+1}} Y_n^m(\hat{x}) & \text{if } |x| = r \geq r_0, \end{cases} \tag{98}$$

where Y_n^m is the spherical harmonic. So, one can show that the eigenvalues of $\mathbb{K}^*$ on $\mathcal{H}^2$ are

$$\pm \frac{1}{2(2n+1)} \sqrt{1 + 4n(n+1)\rho^{2n+1}}, \quad n = 1, 2, \ldots, \tag{99}$$

and corresponding eigenfunctions are

$$\begin{bmatrix} (\sqrt{1 + 4n(n+1)\rho^{2n+1}} - 1)Y_n^m \\ 2(n+1)\rho^{n+2} Y_n^m \end{bmatrix}, \quad \begin{bmatrix} (-\sqrt{1 + 4n(n+1)\rho^{2n+1}} - 1)Y_n^m \\ 2(n+1)\rho^{n+2} Y_n^m \end{bmatrix} \tag{100}$$

(7) This result was obtained in [**96**] when the source is a dipole.

for $m = -n, \dots, n$, respectively, where $\rho = r_i/r_e$. Using this formula, it is proved in [**9**] that ACLR does not occur in three dimensions if the dielectric constant ϵ_s is -1 (or any other constant).

We emphasize that in this case the eigenvalues behave asymptotically as $\pm\frac{1}{2(2n+1)}$ as $n \to \infty$. This slow convergence is responsible for the non-occurrence of CALR in three dimensional annulus.

However, we are able to make CALR occur in three dimensions by using a shell with a specially designed anisotropic dielectric constant. In fact, let D and Ω be concentric balls in $\mathbb{R}^3$ of radii r_i and r_e, and choose r_0 so that $r_0 > r_e$. For a given loss parameter $\delta > 0$, define the dielectric constant ϵ_δ by

$$\epsilon_\delta(x) = \begin{cases} I, & |x| > r_e, \\ (\epsilon_s + i\delta)a^{-1}\left(I + \dfrac{b(b-2|x|)}{|x|^2}\widehat{x}\otimes\widehat{x}\right), & r_i < |x| < r_e, \\ \epsilon_c\sqrt{\dfrac{r_0}{r_i}}I, & |x| < r_i, \end{cases} \tag{101}$$

where I is the 3×3 identity matrix, ϵ_s and ϵ_c constants, $\widehat{x} = \frac{x}{|x|}$, and

$$a := \frac{r_e - r_i}{r_0 - r_e} > 0, \quad b := (1+a)r_e. \tag{102}$$

It is proved that if $\epsilon_s = -1$, then CALR occurs, and the critical radius is $\sqrt{r_e r_0}$ if $\epsilon_c = 1$ and r_0 if $\epsilon_c \neq 1$ (see [**8**] for precise statements). If $\epsilon_s \neq -1$, then CALR does not occur.

Note that ϵ_δ is anisotropic and variable in the shell. This dielectric constant is obtained by push-forwarding (unfolding) that of a folded geometry as in Figure 10. The source to make CALR occur is located in between r_1 and r_e, so it behaves as if it is located inside the outer boundary before unfolding. The idea of a folded geometry were first introduced in [**86**] to explain the properties of superlenses, and has been used in [**97**] to prove CALR in the analogous two-dimensional cylinder structure for a finite set of dipolar sources.

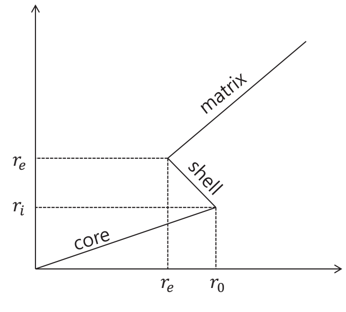

FIGURE 10. Unfolding map. A figure from [**8**]

Further discussion. — It would be interesting to see if the condition [GP] for CALR can be removed or not, or in particular to find a source supported inside the critical radius for which only the weak CALR occurs. An interesting example should be such that the support does not enclose the structure.

Rigorous analysis for CALR for non-circular (or spherical) structures seems quite hard since we need to find eigenvalues and eigenfunctions of the NP-operator. However, when Ω is a scaling of D, then $\mathcal{K}_{\partial\Omega}^*$ and $\mathcal{K}_{\partial D}^*$ have the same spectrums since the NP-operator is invariant under scaling. In this case, it may be possible to study CALR using the spectrum of $\mathcal{K}_{\partial D}^*$. When D is an ellipse, the work is in progress [**49**].

6. Analysis of stress concentration

In composites which consists of inclusions and the matrix (the background medium), some inclusions are located very close or even touching to each other. If the conductivity of inclusions stays away from 0 and ∞ (bounded below and above by some constants), then the stress is bounded regardless of the distance between inclusions as proved in [**39, 88**] (see also a recent paper [**1**] for a different proof using layer potentials). More precisely, it is proved that the $\mathcal{C}^{1,\alpha}$ norm of the solution u is bounded when boundaries of the inclusions are $\mathcal{C}^{2,\alpha}$ smooth.

However, if the conductivity of inclusions degenerates to either ∞ or 0, the ellipticity holds only outside the inclusions and completely different phenomena occur: ∇u may blow up as the distance tends to zero [**43, 75**].

Since the stress concentration occurs in between two close inclusions, we consider the case when there are two inclusions. Let D_1 and D_2 be bounded simply connected domains in $\mathbb{R}^d$, $d = 2, 3$. Suppose that they are conductors, whose conductivity is k, $0 < k \neq 1 < \infty$, embedded in the background with conductivity 1. Let σ denote the conductivity distribution, *i.e.*,

$$\sigma = k\chi(D_1 \cup D_2) + \chi(\mathbb{R}^d \setminus (D_1 \cup D_2)). \tag{103}$$

We consider the elliptic problem (9) when the inclusions are arbitrarily close to each other. The problem may be considered as a conductivity problem or anti-plane elasticity in two dimensions.

Let

$$\delta := \mathrm{dist}(D_1, D_2), \tag{104}$$

and assume that δ is small. We emphasize that the shapes of D_1 and D_2 do not depend on δ. More precisely, there are fixed domains $\widetilde{D}_1$ and $\widetilde{D}_2$ such that D_j is a translate of $\widetilde{D}_j$, namely, there are vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ such that

$$D_j = \widetilde{D}_j + \mathbf{a}_j, \quad j = 1, 2. \tag{105}$$

The problem is to estimate $|\nabla u|$ (u is the solution) in terms of δ when δ tends to 0, or to characterize the asymptotic singular behavior of ∇u as $\delta \to 0$.

When $k=\infty$, the problem becomes

$$\begin{cases} \Delta u = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{(D_1 \cup D_2)}, \\ u = \lambda_i \text{ (constant)} \quad \text{on } \partial D_i,\ i=1,2, \\ u(\mathbf{x}) - h(\mathbf{x}) = O(|\mathbf{x}|^{1-d}) \quad \text{as } |\mathbf{x}| \to \infty, \end{cases} \tag{106}$$

where the constants λ_i are determined by the conditions

$$\int_{\partial D_1} \frac{\partial u}{\partial \nu^{(1)}}\Big|_+ = \int_{\partial D_2} \frac{\partial u}{\partial \nu^{(2)}}\Big|_+ = 0, \tag{107}$$

with $\nu^{(j)}$ being the outward unit normal to ∂D_j, $j=1,2$. If $k=0$, the problem is

$$\begin{cases} \Delta u = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{(D_1 \cup D_2)}, \\ \dfrac{\partial u}{\partial \nu^{(i)}}\Big|_+ = 0 \quad \text{on } \partial D_i,\ i=1,2, \\ u(\mathbf{x}) - h(\mathbf{x}) = O(|\mathbf{x}|^{1-d}) \quad \text{as } |\mathbf{x}| \to \infty. \end{cases} \tag{108}$$

Inclusions with $k=\infty$ are hard inclusions in anti-plane elasticity or perfectly conducting ones in electrostatics, and those with $k=0$ are holes or insulating inclusions. In these cases, ∇u may blow up as δ tends to 0.

As shown in [**25, 30, 35, 111, 112**], in two dimensions the generic rate of gradient blow-up is $\delta^{-1/2}$, while it is $|\delta \log \delta|^{-1}$ in three dimensions [**35, 36, 84, 92**]. The blow-up of the gradient may or may not occur depending on the background potential (the harmonic function h in (9)) and those background potentials which actually make the gradient blow up are characterized in [**28**] when D_1 and D_2 are disks. In two dimensions, the hole case or the perfectly insulating case, where $k=0$, can be dealt with using the conjugate relation (see [**30, 75**]) and in this case the blow-up rate is also $\delta^{-1/2}$.

In recent papers [**10, 69**] the singular behavior of ∇u is completely characterized, namely, the solution u is decomposed as

$$u = cq + b \tag{109}$$

where c is a constant, q is a singular function representing the singular behavior of ∇u, and b is a good function such that ∇b is bounded regardless of δ. In this section we review these results.

Analysis of stress concentration can be applied for solving two longstanding problems. The first one is the study of material failure. In fact, the problem of estimation of the gradient blow-up was raised by Babuška in relation to the study of material failure of composites [**33**]. In composites which consist of inclusions and the matrix, some inclusions may be closely located and stress occurs in between them. The problems (9), (106) and (108) are anti-plane elasticity equations, and ∇u represents the shear stress tensor. So results like (109) provide clear quantitative understanding of the stress concentration, which would be a fundamental ingredient in the study of material failure.

The second application is computation of the electrical field in the presence of closely located inclusions with extreme conductivities (0 or ∞) which is known to be a hard problem. Because $|\nabla u|$, the intensity of the electric field, becomes arbitrarily large, we need fine meshes to compute ∇u numerically. Since (109) for example provides complete characterization of the singular behavior of ∇u, the complexity of computation can be greatly reduced by removing the singular term there. In fact, effectiveness of this scheme is already demonstrated in [**69**] when inclusions are disks.

Eigenfunctions of the NP-operator. — Suppose that D_1 and D_2 are strictly convex and consider the problem (106) when $k = \infty$. As before we represent the solution u to (106) as

$$u(x) = h(x) + \mathcal{S}_{\partial D_1}[\varphi^{(1)}](x) + \mathcal{S}_{\partial D_2}[\varphi^{(2)}](x), \quad x \in \mathbb{R}^d \setminus (D_1 \cup D_2) \tag{110}$$

for a pair of functions $(\varphi^{(1)}, \varphi^{(2)}) \in L^2_0(\partial D_1) \times L^2_0(\partial D_2)$. Since u is constant on ∂D_1 and ∂D_2, we have

$$\frac{\partial}{\partial \nu^{(j)}} \big(\mathcal{S}_{\partial D_1}[\varphi^{(1)}] + \mathcal{S}_{\partial D_2}[\varphi^{(2)}] \big)\Big|_- = -\frac{\partial h}{\partial \nu^{(j)}} \quad \text{on } \partial D_j, \ j = 1, 2,$$

which can be written as

$$\begin{aligned} \left(\frac{1}{2} I - \mathcal{K}^*_{\partial D_1}\right)[\varphi^{(1)}] - \frac{\partial}{\partial \nu^{(1)}} \mathcal{S}_{\partial D_2}[\varphi^{(2)}] &= \frac{\partial h}{\partial \nu^{(1)}} \quad \text{on } \partial D_1, \\ -\frac{\partial}{\partial \nu^{(2)}} \mathcal{S}_{\partial D_2}[\varphi^{(1)}] + \left(\frac{1}{2} I - \mathcal{K}^*_{\partial D_2}\right)[\varphi^{(2)}] &= \frac{\partial h}{\partial \nu^{(2)}} \quad \text{on } \partial D_2. \end{aligned}$$

So, the Neumann-Poincaré operator here is

$$\mathbb{K}^* := \begin{bmatrix} \mathcal{K}^*_{\partial D_1} & \dfrac{\partial}{\partial \nu^{(1)}} \mathcal{S}_{\partial D_2} \\ \dfrac{\partial}{\partial \nu^{(2)}} \mathcal{S}_{\partial D_1} & \mathcal{K}^*_{\partial D_2} \end{bmatrix}. \tag{111}$$

The system of integral equations can be written in a condensed form as

$$\big(\frac{1}{2}\mathbb{I} - \mathbb{K}^*\big)[\Phi] = \partial h, \tag{112}$$

where

$$\Phi := \begin{bmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{bmatrix}, \quad \partial h := \begin{bmatrix} \frac{\partial h}{\partial \nu^{(1)}} \\ \frac{\partial h}{\partial \nu^{(2)}} \end{bmatrix}.$$

The difficulty in solving (112) lies in the distribution of the eigenvalues of $\mathbb{K}^*$. In fact, Bonnetier and Triki [**37, 38**] showed that $\mathbb{K}^*$ in two dimensions has eigenvalues λ_n of the form $\lambda_n \sim \frac{1}{2} - c_n\sqrt{\delta}$, where c_n is a constant, for $n = 1, 2, \ldots$. So, the operator $\frac{1}{2}\mathbb{I} - \mathbb{K}^*$ has many small eigenvalues and its condition number is pretty bad. On the other hand, it is proved in [**10**] that the multiplicity of the eigenvalue 1/2 of $\mathbb{K}^*$ is 2[8]. Since $\frac{1}{2}\mathbb{I} - \mathbb{K}^*$ is invertible $L^2_0(\partial D_1) \times L^2_0(\partial D_2)$, we may choose two eigenfunctions corresponding to 1/2, $\Phi_j = (\varphi_j^{(1)}, \varphi_j^{(2)})^T$, $j = 1, 2$, in such a way that

$$\int_{\partial D_1} \varphi_1^{(1)} d\sigma \neq 0, \quad \int_{\partial D_2} \varphi_1^{(2)} d\sigma = 0$$

and

$$\int_{\partial D_1} \varphi_2^{(1)} d\sigma = 0, \quad \int_{\partial D_2} \varphi_2^{(2)} d\sigma \neq 0.$$

Let $\Phi = (\varphi^{(1)}, \varphi^{(2)})^T$ be an eigenfunction corresponding to 1/2, and let

$$u(x) = \mathcal{S}_{\partial D_1}[\varphi^{(1)}](x) + \mathcal{S}_{\partial D_2}[\varphi^{(2)}](x), \quad x \in \mathbb{R}^d \setminus (D_1 \cup D_2).$$

One can see using (4) that u is constant on ∂D_1 and ∂D_2 (the constants may be different), and

$$\int_{\partial D_1} \frac{\partial u}{\partial \nu^{(1)}}\Big|_+ d\sigma = \int_{\partial D_1} \varphi^{(1)} d\sigma, \quad \int_{\partial D_2} \frac{\partial u}{\partial \nu^{(2)}}\Big|_+ d\sigma = \int_{\partial D_2} \varphi^{(2)} d\sigma.$$

We now choose an eigenfunction $\mathbf{g} = (g_1, g_2)^T$ as a linear combination of Φ_1 and Φ_2 so that

$$\int_{\partial D_1} g_1 d\sigma = 1, \quad \int_{\partial D_2} g_2 d\sigma = -1. \tag{113}$$

Let

$$q(x) = \mathcal{S}_{\partial D_1}[g_1](x) + \mathcal{S}_{\partial D_2}[g_2](x), \quad x \in \mathbb{R}^d \setminus (D_1 \cup D_2). \tag{114}$$

Then q is the solution to

$$\begin{cases} \Delta q = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{D_1 \cup D_2}, \\ q(x) = O(|x|^{1-d}) \quad \text{as } |x| \to \infty, \\ q = \text{constant} \quad \text{on } \partial D_j, \ j = 1, 2, \\ \displaystyle\int_{\partial D_1} \frac{\partial q}{\partial \nu} d\sigma = 1, \quad \int_{\partial D_1} \frac{\partial q}{\partial \nu} d\sigma = -1. \end{cases} \tag{115}$$

This is the function which determines the singular behavior of ∇u as we will see in the next subsection.

In some special cases, the singular function q can be found explicitly. Suppose that D_1 and D_2 are disks, and let R_j be the reflection with respect to ∂D_j, $j = 1, 2$. Then repeated reflections $R_1 R_2$ and $R_2 R_1$ have unique fixed points, which we denote by $p_1 \in D_1$ $p_2 \in D_2$. Then the singular function q is given by

$$q(x) := \frac{1}{2\pi} \left(\ln |x - p_1| - \ln |x - p_2| \right). \tag{116}$$

This function is constant on ∂D_1 and ∂D_2 since these circles are Apollonius circles of p_1 and p_2. See Figure 11. This function was found by Yun [**111**].

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[8] If there are N simply connected inclusions, then the multiplicity is N.

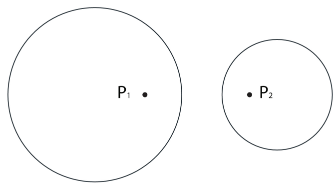

FIGURE 11. Apollonius circles of p_1 and p_2

If D_1 and D_2 have general shape, there is no such a nice formula for q like (116). Recently, it is proved in [**70**] that if D_1 and D_2 are spheres of the same radii, say 1, then q is approximately given the integral

$$q(x) := \int_p^1 \left(\frac{1}{|x-(c,0,0)|} - \frac{1}{|x+(c,0,0)|} \right) \frac{1}{\sqrt{c^2-p^2}}\, dc, \tag{117}$$

where p is a specially chosen number less than 1.

Characterization of the gradient blow-up. — Let u be the solution to (106) and q be the singular function defined by (115). Define

$$r(x) := u(x) - c_\delta q(x), \quad x \in \mathbb{R}^d \setminus (D_1 \cup D_2),$$

where the constant c_δ is given by

$$c_\delta := \frac{u|_{\partial D_1} - u|_{\partial D_2}}{q|_{\partial D_1} - q|_{\partial D_2}}.$$

Then, r is constant on ∂D_1 and ∂D_2, and one can easily see that $r|_{\partial D_1} = r|_{\partial D_2}$. Since the gradient blow-up is caused by the potential difference on ∂D_1 and ∂D_2 and there is no potential difference for r, one can prove that

$$\|\nabla r\|_{L^\infty(\Omega)} \le C$$

for some bounded set Ω containing D_1 and D_2. Let $z_1 \in \partial D_1$ and $z_2 \in \partial D_2$ be two closest points, n be the unit vector in the direction of $z_2 - z_1$, and p be the middle point of z_1 and z_2. Then we have

$$c_\delta \approx \frac{4\pi r_1 r_2}{r_1 + r_2} (n \cdot \nabla h)(p).$$

Here r_j is the radius of D_j. So, we have the following characterization of the stress concentration [**69**]:

$$u(x) = \frac{a}{2\pi} \left(\ln|x-p_1| - \ln|x-p_2| \right) + b(x) \tag{118}$$

where the stress concentration factor a is given by

$$a = \frac{4\pi r_1 r_2}{r_1 + r_2} (n \cdot \nabla h)(p) \tag{119}$$

and ∇b is bounded in a bounded set.

The singular behavior of the solution to the insulating case in two dimensions can be characterized using conjugation. Let $\arg : \mathbb{R}^2 \setminus \{(0,0)\} \to [-\pi, \pi)$ be the argument function with a branch cut along the negative real axis. The a harmonic conjugate $q_\perp$ of q is given by

$$q_\perp(x) = \frac{1}{2\pi}\Big(\arg(x-p_1) - \arg(x-p_2) - \arg(x-c_1) + \arg(x-c_2) \Big), \tag{120}$$

where c_j is the center of D_j, $j = 1, 2$. Note that $q_\perp$ is well-defined outside D_1 and D_2. The following characterization is obtained in the same paper: the solution u to (108) can be expressed as follows:

$$u(x) = a_\perp q_\perp(x) + b_\perp(x), \quad x \text{ outside } D_1 \cup D_2 \tag{121}$$

where

$$a_\perp = \frac{4\pi r_1 r_2}{r_1 + r_2}(t \cdot \nabla h)(p) \tag{122}$$

where t is the vector perpendicular to n, and

$$\|\nabla b_\perp\|_{L^\infty(\Omega \setminus (D_1 \cup D_2))} \le C$$

for a constant C independent of δ.

As mentioned before, it is necessary to use fine meshes to solve (112) numerically. However, if we use (118) or (121), then one can solve it using regular meshes by removing the singular term which is explicit. This was done in [**69**] and Figures 12 and 13 show the results of computation.

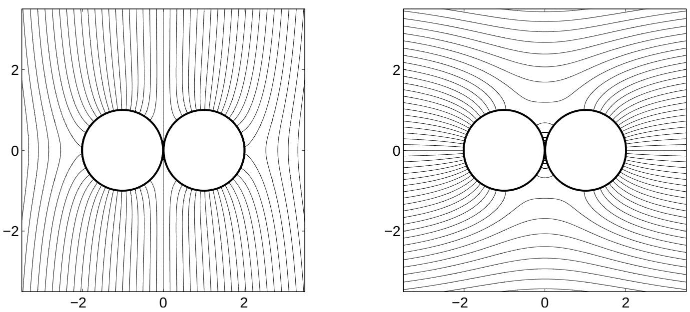

FIGURE 12. Level curve of the solution in insulating case computed using regular mesh. A figure from [**69**].

If D_1 and D_2 take general shapes other than disks, it is unlikely to find the corresponding singular function q. However, we were able to show that q can be approximated by the singular function, denoted by q_B, corresponding to the osculating disks. To be precise, let z_1 and z_2 be the closest points on strictly convex domains D_1 and

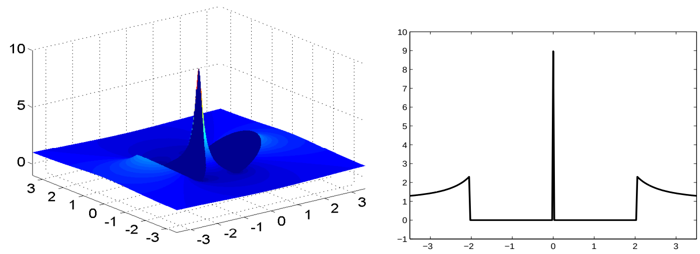

FIGURE 13. Profiles of the solution. Computation by M. Lim.

D_2. Let B_j be the osculating disk to D_j at z_j, $j = 1, 2$. Let p_1 and p_2 be the fixed points of the repeated reflections with respect to ∂B_1 and ∂B_2, and let

$$q_B(x) = \frac{1}{2\pi}\left(\ln|x - p_1| - \ln|x - p_2|\right).$$

It is proved in [**10**] that u can be expressed as

$$u(x) = a_\delta q_B(x) + b(x) \tag{123}$$

where ∇b is bounded regardless of δ in a bounded set containing D_1 and D_2, and the stress concentration factor a_δ is bounded regardless of δ (not explicit) and satisfies

$$a_\delta \approx -\frac{\sqrt{2}\pi\langle \mathbf{h}, \mathbf{g}\rangle}{\sqrt{\epsilon(\kappa_1 + \kappa_2)}}.$$

Here $\mathbf{g}$ is the eigenfunction of $\mathbb{K}^*$ satisfying (113), $\mathbf{h} = (h|_{\partial D_1}, h|_{\partial D_2})^T$, and κ_j is the curvature of ∂D_j at z_j. We emphasize that similar characterizations are obtained for the insulating case and boundary value problems.

Unlike the disk case, a_δ is not explicit since it involves the eigenfunction $\mathbf{g}$. However, (123) can be used to to solve (112) numerically without using fine meshes by treating a_δ as one of unknowns. This numerical computation is in progress.

Further discussion. — There are many problems regarding the gradient blow-up. It is a challenging open problem to clarify whether $|\nabla u|$ may blow up or not in the insulating case in three dimensions and to find the blow-up rate if the blow-up occurs. It is also quite interesting to clarify the dependence of $|\nabla u|$ on k as $k \to \infty$ or $k \to 0$. In this relation we mention that a precise dependence on k when D_1 and D_2 are disks was shown in [**25, 30**].

In these papers and in [**92**] dependence of the ∇u when inclusions are disks or balls is explicit. The estimates show that if the magnitude of inclusions is in the same order as the distance, then the blow-up does not occur. It is interesting to prove this when inclusions are not disks or balls. This result will be useful in studying nano composites.

I am not aware of any result on the stress estimation for the isotropic elasticity system when the shear modulus is ∞ (hard inclusions) or the shear and compressional moduli are 0 (holes). For the hard inclusions we need to assume that the compressional modulus is bounded since otherwise the elasticity equation converges to a modified Stokes' system as proved in [**12, 13**]. We expect the same blow-up rate is valid as in the conductivity case, as shown numerically [**67**]. In this regard, we mention that if the shear and bulk moduli are bounded, then the blow-up of the gradient does not occur as proved by Li and Nirenberg [**87**].

The problems for the heat equation and the acoustic equation are important in analysis of heat and noise concentration. We expect that these problems are quite challenging. It is worth mentioning that a similar blow-up phenomenon for the p-Laplacian equation was investigated by Gorb and Novikov [**55**].

After submission of this paper much progress has been made on the problems mentioned above. Refined asymptotic formula describing the high concentration in between two dimensional convex domains and three dimensional spheres have been obtained in [**68**] and [**91**], respectively; precise dependence on the conductivity k has been characterized when domains are discs [**90**]. There also has been a progress on the problem of two dimensional linear elasticity: an upper bound on the gradient has been obtained when inclusions are strictly convex [**34**].

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Hyeonbae Kang, Department of Mathematics, Inha University, Incheon 402-751, S. Korea.
E-mail : hbkang@inha.ac.kr