Spectral Geometry and Analysis of the Neumann-Poincaré operator, A Review

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Abstract The Neumann-Poincaré operator is an integral operator defined on the boundary of a bounded domain. The history of research on it goes back to the era of the mathematicians whose names appear on the name of the operator. The spectral theory of the Neumann-Poincaré operator attracts much attention lately mainly due to its connection to plasmon resonance and cloaking by anomalous localized resonance. There are rapidly growing literature of research results on its spectral geometry and analysis, and the purpose of this paper is to review some of them. Topics of review in this paper include cloaking by anomalous localized resonance and analysis of surface localization of plasmon, negative eigenvalues and spectrum on tori, spectrum on polygonal domains, spectral structure of thin domains, and analysis of stress in terms of spectral theory. These topics are chosen not to overlap those in another review paper on the same subject [**22**]. We also discuss some related problems to be considered for further development.

1. Introduction

This paper is a survey on some of recent development on the spectral theory of the Neumann-Poincaré (abbreviated by NP) operator. The NP operator is an integral operator defined on the boundary of a bounded domains which appears naturally when solving classical Dirichlet or Neumann boundary value problems using layer potentials. The history of research

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on the NP operator goes back to the era of C. Neumann and Poincaré as the name of the operator suggests. It gave birth to the Fredholm theory of integral equations. If the domain on which the NP operator is defined has a corner, then it is a singular integral operator the theory of which has been one of central themes of research of the last century. For example, its L^2-boundedness was proved [**29**]. Lately, the spectral theory of the NP operator attracts much attention in connection with plasmon resonance and cloaking by anomalous localized resonance.

The author has written another survey paper on the spectral theory of the NP operator with his coauthors [**22**]. In this paper we omit introductory remarks and more historical accounts leaving them to that paper. The topics of review in this paper are chosen so that they don't overlap those in that paper. Topics in [**22**] include

- essential spectrum,
- decay estimates of NP eigenvalues in two dimensions,
- Weyl-type asymptotic formula for eigenvalues in three dimensions,
- the elastic NP operator,
- spectral analysis in a space with two norms.

Topics in this paper are

- NP operator and plasmon resonance,
- cloaking by anomalous localized resonance and analysis of surface localization of plasmon,
- concavity and negative eigenvalues including NP spectral structure on tori,
- spectrum on polygonal domains (with emphasis on pure point spectrum),
- spectral structure of thin domains,
- analysis of stress in terms of the spectral theory.

These topics are complementary to each other. Be aware that the NP operator in [**22**] is 2 times the NP operator of this paper.

The plan of this paper is as follows. Recent rapid growth of interest in the NP spectrum is mainly due to its connection to plasmon resonance and cloaking by anomalous localized resonance (abbreviated by CALR). In section 2 we explain plasmon resonance in quasi-static limit in terms of a transmission problem. The NP operator is naturally introduced in the course of explanation. In section 3, we discuss the spectral nature of CALR in terms of surface localization of plasmon and the decay rate of NP eigenvalues (eigenvalues of the NP operator). We review CALR on ellipses and a recent proof of non-occurrence of CALR on strictly convex three-dimensional domains, and peculiar spectral properties on tori in relation with CALR. In section 4 we review recent results on negative NP eigenvalues and concavity in three dimensions. We include a discussion on possible advantage of having negative eigenvalues. In section 5 we review results on the NP spectrum on planar domains with corners and discuss existence of eigenvalues in addition to continuous spectrum. In section 6 we review the spectral structure of thin domains in two- and three-dimensions which is related to negative eigenvalues and polygonal domains. Quantitative analysis of the stress or field concentration in between two closely located inclusions has been an active area research for last thirty years or so. In section 7 we review some of important results on this subject and discuss a spectral nature of stress concentration, especially when one inclusion is an insulator and the other is a perfect conductor. In the course of review, proper references will be given in each corresponding section and some related open problems are discussed. This paper ends with a conclusion.

2. NP operator and plasmon resonance

In this section we discuss plasmon resonance, which occurs on meta-materials of negative dielectric constants, as a motivation to study the NP operator and its spectrum. In the course of discussion, the single layer potential and the NP operator appear naturally.

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$. The boundary $\partial\Omega$ is allowed to have several connected components and its connected component is assumed to be Lipschitz continuous. Suppose that Ω is immersed in the free space $\mathbb{R}^d$ and the dielectric constant of Ω is $\epsilon_c = k + i\delta$ and that of the background is 1 after normalization $(k \neq 1)$; $(k, \delta$ are real constants and δ is the lossy parameter tending to 0). The constant k can be negative and a material whose dielectric constant has the negative real part is called a meta-material. So, the distribution of the conductivity is given by

$$\epsilon = \epsilon_c \chi(\Omega) + \chi(\mathbb{R}^d \setminus \overline{\Omega}), \tag{0.1}$$

where χ denotes the indicator function of the corresponding set. We consider the following transmission problem: for a given harmonic function h in $\mathbb{R}^d$

$$\begin{cases} \nabla \cdot \epsilon \nabla u = 0 & \text{in } \mathbb{R}^d, \\ u(x) - h(x) = O(|x|^{1-d}) & \text{as } |x| \to \infty. \end{cases} \tag{0.2}$$

The solution to the problem, denoted by u_δ, satisfies the transmission conditions along $\partial\Omega$:

$$u_\delta|_- = u_\delta|_+, \quad \epsilon_c \partial_\nu u_\delta|_- = \partial_\nu u_\delta|_+ \quad \text{on } \partial\Omega, \tag{0.3}$$

which are continuity of the potential and the flux. Here and afterwards, subscripts $+$ and $-$ indicate the limits (to $\partial\Omega$) from outside and inside of Ω, respectively, and ∂_ν the outward normal derivative on $\partial\Omega$.

The solution to (0.2) can be represented using the single layer potential which is defined, for $\varphi \in H^{-1/2}(\partial\Omega)$ ($H^{-1/2}(\partial\Omega)$ is the usual Sobolev space on $\partial\Omega$), by

$$\mathcal{S}_{\partial\Omega}[\varphi](x) := \int_{\partial\Omega} \Gamma(x-y)\varphi(y)\, d\sigma(y) , \quad x \in \mathbb{R}^d, \tag{0.4}$$

where $\Gamma(x)$ is the fundamental solution to the Laplacian, *i.e.*,

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi} \ln |x| , & d = 2 , \\ -\dfrac{1}{4\pi} |x|^{-1} , & d = 3 . \end{cases} \tag{0.5}$$

It turns out (see, for example, [**10, 50**]) that the solution u_δ to (0.2) can be represented as

$$u_\delta(x) = h(x) + \mathcal{S}_{\partial\Omega}[\varphi_\delta](x), \quad x \in \mathbb{R}^d, \tag{0.6}$$

for some $\varphi_\delta \in H_0^{-1/2}(\partial\Omega)$ (the subscript 0 indicates that its elements are of zero mean value). The potential function φ_δ is determined by the transmission condition (0.3). The first condition (continuity of the potential) is automatically fulfilled since $\mathcal{S}_{\partial\Omega}[\varphi_\delta]$ is continuous across $\partial\Omega$. The second condition (continuity of the flux) takes the following form:

$$\epsilon_c \partial_\nu \mathcal{S}_{\partial\Omega}[\varphi_\delta]|_- - \partial_\nu \mathcal{S}_{\partial\Omega}[\varphi_\delta]|_+ = (1-\epsilon_c)\partial_\nu h. \tag{0.7}$$

For any $\varphi \in H^{-1/2}(\partial\Omega)$, $\partial_\nu \mathcal{S}_{\partial\Omega}[\varphi]$ satisfies the following well-known jump relation (see, for example, [**10**]):

$$\partial_\nu \mathcal{S}_{\partial\Omega}[\varphi]\big|_\pm(x) = \Big(\pm\frac{1}{2}I + \mathcal{K}_{\partial\Omega}^*\Big)[\varphi](x), \quad x \in \partial\Omega\ , \tag{0.8}$$

where the operator $\mathcal{K}_{\partial\Omega}$ is defined by

$$\mathcal{K}_{\partial\Omega}[\varphi](x) = \frac{1}{\omega_d}\int_{\partial\Omega} \frac{\langle y-x, \nu_y\rangle}{|x-y|^d}\varphi(y)\, d\sigma(y)\ , \quad x \in \partial\Omega, \tag{0.9}$$

and $\mathcal{K}_{\partial\Omega}^*$ is its L^2-adjoint, that is,

$$\mathcal{K}_{\partial\Omega}^*[\varphi](x) = \frac{1}{\omega_d}\int_{\partial\Omega} \frac{\langle x-y, \nu_x\rangle}{|x-y|^d}\varphi(y)\, d\sigma(y)\ . \tag{0.10}$$

Here, $\omega_2 = 2\pi$ and $\omega_3 = 4\pi$. The operator $\mathcal{K}_{\partial\Omega}$ (or $\mathcal{K}_{\partial\Omega}^*$) is called the NP operator on $\partial\Omega$. The operator $\mathcal{K}_{\partial\Omega}$ is also commonly called the double layer potential.

In view of (0.8), the relation (0.7) can be written as the integral equation

$$(\mu_\delta I - \mathcal{K}_{\partial\Omega}^*)\,[\varphi_\delta] = \partial_\nu h \quad \text{on } \partial\Omega, \tag{0.11}$$

where

$$\mu_\delta := \frac{\epsilon_c + 1}{2(\epsilon_c - 1)} = \frac{k+1+i\delta}{2(k-1)+2i\delta}. \tag{0.12}$$

We will come back to the integral equation (0.11) after recalling some important spectral properties of the NP operator.

The most important spectral property of the operator $\mathcal{K}_{\partial\Omega}^*$ is that it can be realized as a self-adjoint operator on the space $H^{-1/2}(\partial\Omega)$ by introducing a new inner product on it. Let $\langle\ ,\ \rangle$ be the usual $H^{-1/2}$–$H^{1/2}$ duality pairing. We define $\langle\varphi,\psi\rangle_*$ for $\varphi,\psi \in H^{-1/2}(\partial\Omega)$ by

$$\langle\varphi,\psi\rangle_* := -\langle\varphi, \mathcal{S}_{\partial\Omega}[\psi]\rangle. \tag{0.13}$$

Note that $-\mathcal{S}_{\partial\Omega}$ is a non-negative operator. Since $\mathcal{S}_{\partial\Omega}$ maps $H^{-1/2}(\partial\Omega)$ into $H^{1/2}(\partial\Omega)$, $\langle\varphi, \mathcal{S}_{\partial\Omega}[\psi]\rangle$ is well-defined. The bilinear form $\langle\ ,\ \rangle_*$ is actually an inner product on $H_0^{-1/2}(\partial\Omega)$ in two dimensions, and on $H^{-1/2}(\partial\Omega)$ in three dimensions. The NP operator $\mathcal{K}_{\partial\Omega}^*$ is self-adjoint on $H^{-1/2}(\partial\Omega)$ with respect to this inner product. In fact, thanks to the relation

$$\mathcal{S}_{\partial\Omega}\mathcal{K}_{\partial\Omega}^* = \mathcal{K}_{\partial\Omega}\mathcal{S}_{\partial\Omega},$$

which is known as the Plemelj's symmetrization principle (also known as Calderón's identity), we have

$$\langle\varphi, \mathcal{K}_{\partial\Omega}^*[\psi]\rangle_* = -\langle\varphi, \mathcal{S}_{\partial\Omega}\mathcal{K}_{\partial\Omega}^*[\psi]\rangle = -\langle\varphi, \mathcal{K}_{\partial\Omega}\mathcal{S}_{\partial\Omega}[\psi]\rangle = \langle\mathcal{K}_{\partial\Omega}^*[\varphi], \psi\rangle_*.$$

For proofs of properties of the NP operator reviewed so far we refer to the survey paper [**22**] and references therein.

Spectrum of $\mathcal{K}_{\partial\Omega}^*$ on $H^{-1/2}(\partial\Omega)$, which is denoted by $\sigma(\mathcal{K}_{\partial\Omega}^*)$, lies in the interval $(-\frac{1}{2}, \frac{1}{2}]$. Since $\mathcal{K}_{\partial\Omega}^*$ is self-adjoint, $\sigma(\mathcal{K}_{\partial\Omega}^*)$ consists of essential spectrum and pure point spectrum, and essential spectrum consists of absolutely continuous spectrum, singularly continuous spectrum, and limit points of eigenvalues (some of them can be void), namely,

$$\sigma(\mathcal{K}_{\partial\Omega}^*) = \sigma_{ess} \cup \sigma_{pp} = \sigma_{ac} \cup \sigma_{sc} \cup \overline{\sigma_{pp}}. \tag{0.14}$$

(see [**66**]). If $\partial\Omega$ is $\mathcal{C}^{1,\alpha}$ for some $\alpha > 0$, then $\mathcal{K}_{\partial\Omega}^*$ is a compact operator on $H^{-1/2}(\partial\Omega)$. In fact, because of orthogonality of the normal vector and the surface or the curve where the operator is defined, we have

$$\frac{|\langle x - y, \nu_x \rangle|}{|x - y|^d} \leqslant \frac{C}{|x - y|^{d-1-\alpha}}, \quad x, y \in \partial\Omega. \tag{0.15}$$

So the singularity of the integral kernel of $\mathcal{K}_{\partial\Omega}^*$ is weaker than the critical singularity whose order is the same as the dimension of $\partial\Omega$ where the integral is defined. Because of this, $\mathcal{K}_{\partial\Omega}^*$ becomes a compact operator and $\sigma(\mathcal{K}_{\partial\Omega}^*)$ consists of eigenvalues of finite multiplicities (except 0 which can have an infinite multiplicity if it is an eigenvalue) accumulating to 0.

Since $\mathcal{K}_{\partial\Omega}^*$ is self-adjoint, the spectral resolution theorem holds [**70**], namely, there is a family of projection operators $\mathcal{E}(t)$ on $H^{-1/2}(\partial\Omega)$, called the resolution of identity, such that

$$\mathcal{K}_{\partial\Omega}^* = \int_{-1/2}^{1/2} t \, d\mathcal{E}(t). \tag{0.16}$$

This formula implies, in particular,

$$(\lambda I - \mathcal{K}_{\partial\Omega}^*)^{-1} = \int_{-1/2}^{1/2} \frac{1}{\lambda - t} \, d\mathcal{E}(t). \tag{0.17}$$

If $\partial\Omega$ is $\mathcal{C}^{1,\alpha}$, (0.16) takes the form

$$\mathcal{K}_{\partial\Omega}^* = \sum_{j=1}^{\infty} \lambda_j \psi_j \otimes \psi_j, \tag{0.18}$$

where $\lambda_1, \lambda_2, \ldots$ ($|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots$) are eigenvalues of $\mathcal{K}_{\partial\Omega}^*$ counting multiplicities, and $\psi_1, \psi_2, \ldots$ are the corresponding (normalized) eigenfunctions.

The following addition formula is proved in [**16**]: if $\partial\Omega$ is $\mathcal{C}^{1,\alpha}$, then for $x \in \overline{\Omega}$ and $z \in \mathbb{R}^d \backslash \overline{\Omega}$

$$\Gamma(x - z) = -\sum_{j=1}^{\infty} \mathcal{S}_{\partial\Omega}[\psi_j](z) \mathcal{S}_{\partial\Omega}[\psi_j](x) + \mathcal{S}_{\partial\Omega}[\varphi_0](z). \tag{0.19}$$

If Ω is a ball, then $\mathcal{S}_{\partial\Omega}[\psi_j]$ is a spherical harmonics. Therefore (0.19) is the expansion of $\Gamma(x - z)$ in terms of the spherical harmonics, which is well-known (see, for example, [**51**]). If Ω is an ellipsoid, then $\mathcal{S}_{\partial\Omega}[\psi_j]$ is an ellipsoidal harmonic (see [**30**, section 7.2]) and the formula (0.19) is due to Heine [**35**] (it is called the Heine expansion formula in [**30**]). It is interesting to generalize, if possible, the addition formula (0.19) to domains with corners. For that (0.16) may be useful.

We now move back to and look into the integral equation (0.11). Note that μ_δ converges to $\mu_0 = \frac{k+1}{2(k-1)}$ as $\delta \to 0$. If k is positive, then μ_0 does not belong to $[-1/2, 1/2]$, the interval where $\sigma(\mathcal{K}_{\partial\Omega}^*)$ is contained, and hence the problem (0.11) is uniquely solvable. Note that the problem (0.2) is elliptic if k is positive. However, if Ω is a meta-material with the negative k, then $\mu_0 \in (-1/2, 1/2)$, so it is possible (depending on k) for μ_0 to belong to $\sigma(\mathcal{K}_{\partial\Omega}^*)$ and for the solution φ_δ to (0.11) to blow up as $\delta \to 0$. In fact, according to (0.18), the solution φ_δ to (0.11) (assuming that $\partial\Omega$ is $\mathcal{C}^{1,\alpha}$) is given by

$$\varphi_\delta = \sum_{j=1}^{\infty} \frac{\langle \partial_\nu h, \psi_j \rangle_*}{\mu_\delta - \lambda_j} \psi_j. \tag{0.20}$$

If $\mu_0 = \lambda_i$ for some i and $\langle \partial_\nu h, \psi_i \rangle_* \neq 0$, namely $\partial_\nu h$ has the eigen-mode of ψ_i, then φ_δ blow up as $\delta \to 0$, and so does the solution to (0.2). This is the plasmon resonance in the quasi-static limit which is one of major reasons for renewed interest in the spectral properties of the NP operator in recent years.

We refer to [**13, 14**] for the connection of NP spectrum with the sub-wavelength imaging. There asymptotic formulas of resonance frequencies near NP eigenvalues for the Helmholtz equation and the Maxwell system are derived. Convergence of a resonance frequency to a NP eigenvalue for the Helmholtz equation is also proved in [**18**]. If μ_0 is an eigenvalue of $\mathcal{K}_{\partial\Omega}^*$, then the corresponding k is called a plasmonic eigenvalue and the single layer potential of the corresponding eigenfunction is called a surface localized plasmon [**33**]. The formula (0.19) shows that $\mathcal{S}_{\partial\Omega}[\psi_j](z) \to 0$ as $j \to \infty$ for all $z \notin \partial\Omega$ (see also section 3). So, if j is large, then $\mathcal{S}_{\partial\Omega}[\psi_j]$ is localized near $\partial\Omega$. It explains why $\mathcal{S}_{\partial\Omega}[\psi_j]$ is called a *surface localized* plasmon.

3. Analysis of cloaking by anomalous localized resonance

Suppose that $\partial\Omega$ is $\mathcal{C}^{1,\alpha}$ for some $\alpha > 0$ so that $\sigma(\mathcal{K}_{\partial\Omega}^*)$ consists of eigenvalues accumulating to 0. Consider the operator-valued function $\lambda \mapsto (\lambda I - \mathcal{K}_{\partial\Omega}^*)^{-1}$. It is a meromorphic function in λ except at $\lambda = 0$. Each NP eigenvalue (other than 0 if 0 is an eigenvalue) is a pole of the function, and 0 is an essential singularity as the limit point of pole. If the dielectric constant k of Ω is -1, then μ_δ in (0.12) tends to 0, namely, the essential singularity. Near an essential singularity of a meromorphic function, many strange phenomena may occur. It is no exception here. If $k = -1$, then CALR (cloaking by anomalous localized resonance) occurs. We review some recent results on CALR in this section.

Let ϵ be the coefficient given by (0.1). We consider the following inhomogeneous problem: for a given function f compactly supported in $\mathbb{R}^d \setminus \overline{\Omega}$

$$\begin{cases} \nabla \cdot \epsilon \nabla u = f & \text{in } \mathbb{R}^d, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{0.21}$$

The source function f satisfies the condition

$$\int_{\mathbb{R}^d} f dx = 0,$$

which is the compatibility condition for existence of the solution to (0.21). Typically, a dipole is chosen a source function, that is, $f(x) = a \cdot \nabla \delta_z(x)$, where a is a constant vector, δ_z is the Dirac delta function at z which lies outside Ω.

Let u_δ be the solution to this problem and let

$$E_\delta := \Im \int_{\mathbb{R}^d} \epsilon_\delta |\nabla u_\delta|^2 \, dx = \int_{\Omega\setminus D} \delta |\nabla u_\delta|^2 \, dx \tag{0.22}$$

($\Im$ for the imaginary part). The problem of CALR is formulated as that of identifying the sources f such that

$$E_\delta \to \infty \quad \text{as } \delta \to 0, \tag{0.23}$$

and u_δ is bounded outside some radius a.

The quantity E_δ approximately represents the time averaged electromagnetic power produced by the source dissipated into heat. So, (0.23) implies an infinite amount of energy dissipated per unit time in the limit $\delta \to 0$ which is unphysical. If we scale the source f by a factor of $1/\sqrt{E_\delta}$, then $u_\delta/\sqrt{E_\delta}$ approaches zero outside the radius a. Hence, CALR occurs: the normalized source is essentially invisible from the outside.

The phenomena of anomalous resonance was first discovered in [**62**] and is related to invisibility cloaking in [**59**] where Ω is an annulus and $f = a \cdot \nabla \delta_z$. We refer to the recent survey paper [**58**] for physics related to CALR including superlensing and for a comprehensive list of relevant references. In this paper we discuss the NP spectral nature of CALR.

Let F be the Newtonian potential of f, *i.e.*,

$$F(x) = \int_{\mathbb{R}^d} \Gamma(x-y) f(y) dy, \quad x \in \mathbb{R}^d.$$

Then, the solution u_δ to (0.21) takes the form (0.6) with h replaced with F. There, the potential φ_δ is the solution to the integral equation (0.11) with $\partial_\nu h$ replaced with $\partial_\nu F$, and hence it is given analogously to (0.20) by

$$\varphi_\delta = \sum_{j=1}^{\infty} \frac{\langle \partial_\nu F, \psi_j \rangle_*}{\mu_\delta - \lambda_j} \psi_j.$$

One can see that $\mu_\delta \sim \delta$ since $k = -1$. Here and throughout this paper, we write $A \lesssim B$ to imply that there is a constant C independent of the parameter (in this case it is δ). The meaning of $A \gtrsim B$ is analogous, and $A \sim B$ means both $A \lesssim B$ and $A \gtrsim B$ hold. Suppose that 0 is not an NP eigenvalue on Ω. The CALR condition (0.23) is equivalent to

$$\widetilde{E}_\delta := \delta \sum_{n=1}^{\infty} \frac{\langle \partial_\nu F, \psi_n \rangle_*^2}{\delta^2 + \lambda_n^2} \to \infty \quad \text{as } \delta \to 0. \tag{0.24}$$

In this way, CALR is related to the spectral theory of the NP operator and the result of [**59**] on CALR with the dipole source on an annulus is extended to more general source f [**5**] using the spectral approach. Note that the circle has 0 as its NP eigenvalue. By adding another circle the NP eigenvalues are perturbed and 0 is not an NP eigenvalue but the limit point of eigenvalues on an annulus. Furthermore, NP eigenvalues and corresponding eigenfunctions on annuli are explicitly known so that $\widetilde{E}_\delta$ can be computed. This is an advantage of working with annuli. Actually 0 is the only NP eigenvalue, other than 1/2 which is of multiplicity 1, on a circle. There is no known example of a planar domain other than disks where 0 is an NP eigenvalue (of finite or infinite multiplicity). It is known that if the NP operator on a planar domain is of finite rank so that the eigenvalue 0 has the finite co-multiplicity, then the domain must be a disk [**68**].

When $f = a \cdot \nabla \delta_z$ for some $z \in \mathbb{R}^d \backslash \overline{\Omega}$ and Ω is an annulus, it is proved in [**59**] (see also [**5**]) that there is a virtual radius r_* such that if $|z| < r_*$, then (0.24) holds and CALR takes place; if $|z| > r_*$, then E_δ is bounded regardless of δ. This result has been extended to confocal ellipses in [**28**]. Since CALR is a phenomenon occurring at the limit point of NP eigenvalues, the structure does not have to be doubly connected. In fact, it is proved in [**16**] that if Ω is an ellipse, then there is an ellipse Ω_* confocal to Ω such that if $z \in \overline{\Omega_*} \backslash \overline{\Omega}$, then CALR takes place, and it does not if $z \notin \overline{\Omega_*}$ (see subsection 3.1). On three dimensional ball or concentric balls CALR does not occur [**7, 16**]; on concentric balls with folded geometry CALR may occur [**6**].

Let us look closely the quantity $\widetilde{E}_\delta$ in (0.24). It is proved in [**16**] that if $f = a \cdot \nabla \delta_z$, then

$$|\langle \partial_\nu F, \psi_n \rangle_*| \sim |a \cdot \nabla \mathcal{S}_{\partial\Omega}[\psi_n](z)|.$$

As a consequence, we have

$$\widetilde{E}_\delta \sim \delta \sum_{n=1}^{\infty} \frac{|a \cdot \nabla \mathcal{S}_{\partial\Omega}[\psi_n](z)|^2}{\delta^2 + \lambda_n^2}. \tag{0.25}$$

Therefore, to investigate the property $\widetilde{E}_\delta \to \infty$, we need to look into the following two questions:

(i) how fast λ_n tends to 0,
(ii) how fast $\nabla \mathcal{S}_{\partial\Omega}[\psi_n](z)$ tends to 0,

as $n \to \infty$. The question (i) is about the convergence rate of NP eigenvalues, and the question (ii) is regarding the surface localization of the plasmon $\mathcal{S}_{\partial\Omega}[\psi_n]$. Results on the convergence rate of NP eigenvalues are reviewed in [**22**]. We recall some of them in the following subsection for smooth discussion.

3.1. CALR in two dimensions. Let Ω be an ellipse in $\mathbb{R}^2$. The elliptic coordinates $x = (x_1, x_2) = (x_1(\rho, \omega), x_2(\rho, \omega))$, $\rho > 0$ and $0 \leqslant \omega < 2\pi$, is given by

$$x_1(\rho, \omega) = R \cos\omega \cosh\rho, \quad x_2(\rho, \omega) = R \sin\omega \sinh\rho.$$

We denote the elliptic coordinates of x by $\rho = \rho_x$ and $\omega = \omega_x$. Then $\partial\Omega$ is represented by

$$\partial\Omega = \{x \in \mathbb{R}^2 \ : \ \rho_x = \rho_0\} \tag{0.26}$$

for some $\rho_0 > 0$. The number ρ_0 is called the elliptic radius of Ω.

The NP eigenvalues on $\partial\Omega$ are

$$\pm\lambda_n = \pm \frac{1}{2e^{2n\rho_0}}, \quad n = 1, 2, \cdots,$$

and eigenfunctions corresponding $+\lambda_n$ and $-\lambda_n$ are respectively given by

$$\phi_n^c(\omega) := \Xi(\rho_0, \omega)^{-1} \cos n\omega, \quad \phi_n^s(\omega) := \Xi(\rho_0, \omega)^{-1} \sin n\omega,$$

where

$$\Xi = \Xi(\rho_0, \omega) := R\sqrt{\sinh^2 \rho_0 + \sin^2 \omega}$$

(see, for example, [**28**]). Using these facts, it is proved in [**16**] that

$$\widetilde{E}_\delta \sim \begin{cases} \delta^{-2+\rho_z/\rho_0} |\log \delta| & \text{if } \rho_0 < \rho_z < 3\rho_0, \\ \delta |\log \delta|^2 & \text{if } \rho_z = 3\rho_0, \\ \delta & \text{if } \rho_z > 3\rho_0, \end{cases}$$

as $\delta \to 0$, where z is the position of the source. As a consequence, the following theorem is obtained.

THEOREM 3.1. *Let $\partial\Omega$ be the ellipse given by (0.26). CALR takes place if $\rho_0 < \rho_z \leqslant 2\rho_0$ and does not take place if $\rho_z > 2\rho_0$, namely, the critical (elliptic) radius for CALR is $2\rho_0$.*

If Ω is a planar domain with the real analytic boundary, then it is not known yet if CALR takes place. One may attempt to prove the following problem:

Problem 1. Is it true that if the location z of the source is close enough to $\partial\Omega$, then CALR takes place, and if z is sufficiently away from $\partial\Omega$, it does not take place?

In relation to this, the question (i) above has been answered in [**19**] (see also [**22**] and references therein for related work): λ_n converges to 0 exponentially fast. In fact, if λ_j is the positive NP eigenvalues on $\partial\Omega$ enumerated as in descending order, then for any $\epsilon < \epsilon_{\partial\Omega}$ there exists a constant C such that

$$\lambda_n \leqslant C e^{-\epsilon n}$$

for all n, where $\epsilon_{\partial\Omega}$ is the modified maximal Grauert radius of $\partial\Omega$ which is basically the radius up to which the real analytic defining function of $\partial\Omega$ is extended as a complex analytic function (see [**19**] for a precise definition). Therefore, we obtain from (0.25)

$$\widetilde{E}_\delta \gtrsim \delta \sum_{n=1}^{\infty} \frac{|a \cdot \nabla \mathcal{S}_{\partial\Omega}[\psi_n](z)|^2}{\delta^2 + e^{-2\epsilon n}}$$

for any $\epsilon < \epsilon_{\partial\Omega}$. However, it is not known whether CALR occurs because of lack of knowledge on the localization of plasmon in the question (ii). In fact, to the best of author's knowledge, there is no result on (ii) in two dimensions except on disks, annuli, and ellipses where NP eigenvalues and eigenfunctions are known explicitly. This brings the following problem

Problem 2. How fast does $\mathcal{S}_{\partial\Omega}[\psi_n](z)$ tend to 0 as $n \to \infty$ when $\partial\Omega$ is real analytic and of general shape?

3.2. CALR in three dimensions. Let Ω be a bounded domain in $\mathbb{R}^3$ with the smooth boundary. It is proved in [**60**] (see also [**22**]) that $\{\lambda_n\}_{n\in\mathbb{N}}$ asymptotically behaves like

$$\lambda_n^2 \sim C_{\partial\Omega} n^{-1} \quad \text{as } n \to \infty, \tag{0.27}$$

in the sense that $\lambda_n^2 n \to C_{\partial\Omega}$ as $n \to \infty$. The constant $C_{\partial\Omega}$ is given by

$$C_{\partial\Omega} = \frac{3W(\partial\Omega) - 2\pi\chi(\partial\Omega)}{128\pi},$$

where $W(\partial\Omega)$ and $\chi(\partial\Omega)$, respectively, are the Willmore energy and the Euler characteristic of the boundary surface $\partial\Omega$. The Willmore energy is defined by

$$W(\partial\Omega) = \int_{\partial\Omega} H(x)^2 dS,$$

where $H(x)$ is the mean curvature and it is known that $W(\partial\Omega) \geqslant 4\pi$ [**56**] (see also [**22**]). Thus we have $C_{\partial\Omega} > 0$, and hence

$$\widetilde{E}_\delta \sim \delta \sum_{n=1}^{\infty} \frac{|a \cdot \nabla \mathcal{S}_{\partial\Omega}[\psi_n](z)|^2}{\delta^2 + n^{-1}}. \tag{0.28}$$

The following theorem is proved in [**21**].

THEOREM 3.2. *Let Ω be a strictly convex bounded domain in $\mathbb{R}^3$ with the C^∞-smooth boundary. For any compact set K in $\mathbb{R}^3 \backslash \overline{\Omega}$ and for each integers k and s there is a constant $C_{k,s}$ such that*

$$\|\mathcal{S}_{\partial\Omega}[\varphi_n]\|_{C^k(K)} \leqslant C_{k,s} n^{-s}$$

for all sufficiently large n.

The main ingredient in proving this theorem is the fact that since $\partial\Omega$ is C^∞-smooth, the NP operator $\mathcal{K}^*_{\partial\Omega}$ is known to be a strictly homogeneous pseudo-differential operator of order -1 [**32**], and its principal symbol is positive definite if Ω is strictly convex as proved in [**60, 61**]. Thus all eigenvalues of $\mathcal{K}^*_{\partial\Omega}$, except possibly finitely many, are positive [**61**]. By altering $\mathcal{K}^*_{\partial\Omega}$ on a finite dimensional subspace if necessary, we can realize $\mathcal{K}^*_{\partial\Omega}$ as a positive definite pseudo-differential operator of order -1. This fact together with (0.27) leads us to Theorem 3.2.

We infer from Theorem 3.2 and (0.28) that $\widetilde{E}_\delta < +\infty$, and hence CALR does not take place on three-dimensional strictly convex bounded domains.

In order to see what happens if $\partial\Omega$ is not convex, the first 450 plasmons, namely, $\mathcal{S}_{\partial\Omega}[\varphi_n]$ ($1 \leqslant n \leqslant 450$), corresponding to largest eigenvalues in descending order, are computed numerically on a cross section close to $\partial\Omega$ when Ω is the Clifford torus in [**21**]. To our surprise, there are four out of 450 plasmons which do not decay fast enough (see Fig 1). It is utterly interesting to investigate it rigorously and explore the possible connection to CALR. We include in this paper the figures from [**21**] to compare $\mathcal{S}_{\partial\Omega}[\varphi_n]$ of fast decay and slow decay (see Fig. 2). We also include figures of two exceptional eigenfunctions together with one non-exceptional one for comparison. The exceptional eigenfunctions have a very interesting feature which we discuss in the next section (see Fig. 3).

For general domains the following theorem is proved using the addition formula (0.19) (we only state a special case of the theorem obtained in [**21**]).

THEOREM 3.3. *Suppose that Ω is a bounded domain in $\mathbb{R}^3$ with the $C^{1,\alpha}$ smooth boundary for some $\alpha > 0$. For any compact set K in $\mathbb{R}^3 \backslash \overline{\Omega}$,*

$$\|\mathcal{S}_{\partial\Omega}[\varphi_n]\|_{C^1(K)} = o(n^{-1/2}) \quad \textit{almost surely as } n \to \infty.$$

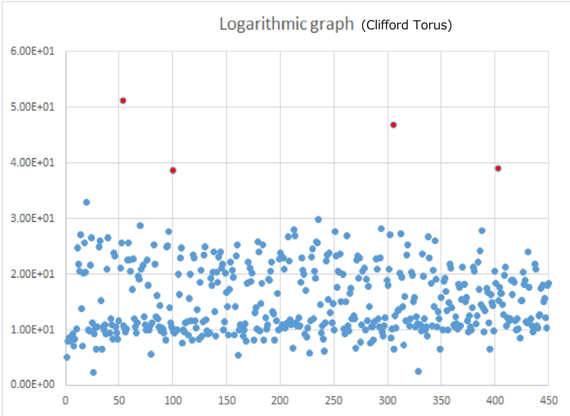

FIGURE 1. The graph of $\|\mathcal{S}_{\partial\Omega}[\varphi_n]\|_{L^2(X)}$ $(1 \leqslant n \leqslant 450)$ after some normalization on the Clifford torus. The horizontal axis represents positive eigenvalues of the NP operator enumerated in decreasing order up to 450. The red dots indicate values drastically larger than neighboring points. They occur at 53rd, 100th, 305th and 402nd eigenvalues. (The figure is from [**21**]. The region X where $\mathcal{S}_{\partial\Omega}[\varphi_n]$ is computed is the rectangle-shaped cross section shown in Fig. 2 and $\mathcal{S}_{\partial\Omega}[\varphi_n]$ is normalized.)

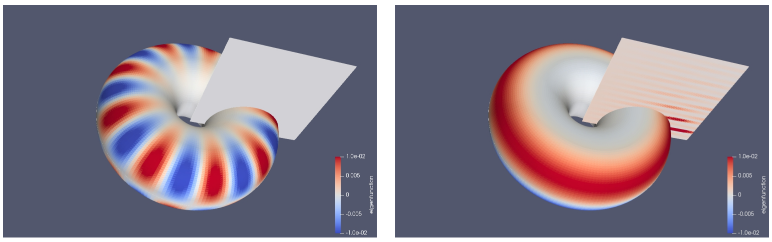

FIGURE 2. $\mathcal{S}_{\partial\Omega}[\varphi_n]$ of fast decay (left) and of slow decay (right). The rectangular cross section represents the region where $\mathcal{S}_{\partial\Omega}[\varphi_n]$ is evaluated, and the color on it represents its value. (Figures are from [**21**].)

For a sequence $\{a_n\}$ of numbers and a non-negative number s, we say $a_n = o(n^{-s})$ almost surely as $n \to \infty$ if

$$\lim_{\delta \downarrow 0} \limsup_{N \to \infty} \frac{\sharp\{n < N \ : \ |a_n| > \delta n^{-s}\}}{N} = 0.$$

It is equivalent to existence of a subsequence $\{n_k\}$ such that $a_{n_k} = o(n_k^{-s})$ as $k \to \infty$ and $\lim_{k\to\infty} n_k/k = 1$.

4. Concavity and negative eigenvalues

In two dimensions the NP spectrum always appears in pairs, namely, if $\lambda \in \sigma(\mathcal{K}_{\partial\Omega}^*)$, then $-\lambda \in \sigma(\mathcal{K}_{\partial\Omega}^*)$. This can be proved using existence of harmonic conjugates. However, there are domains in three dimensions where the NP operators have only positive eigenvalues: the NP eigenvalues

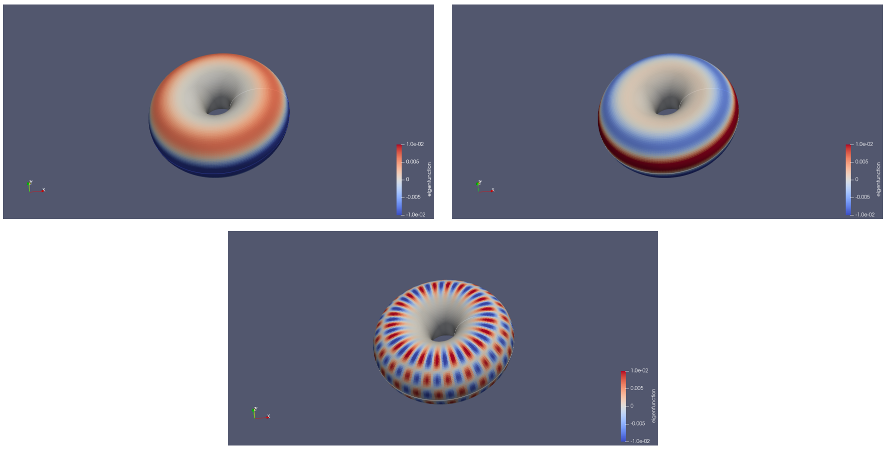

FIGURE 3. Top: exceptional eigenfunctions whose single layer potentials do not decay fast. They do not oscillate in the toroidal direction. Bottom: a non-exceptional eigenfunction whose single layer potential decays fast. It oscillates in the toroidal direction. (Figures are from [**21**].)

on a sphere are $1/(4n+2)$ for $n = 0, 1, 2 \ldots$, and they are all positive on prolate spheroids [**3**]. The first example of three-dimensional domains with a negative NP eigenvalue was found in [**2**]: it is an oblate spheroid. We now know that concavity allows negative NP eigenvalues which we review in this section.

Let us first recall the discussion in [**17**] on possible advantage of having negative eigenvalues. Suppose that $\epsilon_c = k + i\delta$ is the dielectric constant of Ω as before and assume $\delta = 0$ so that $\epsilon_c = k$. Then plasmon resonance in the quasi-static limit occurs if

$$\frac{k+1}{2(k-1)} = \lambda, \tag{0.29}$$

where λ is an eigenvalue of the NP operator on $\partial\Omega$ as mentioned at the end of section 2. The relation (0.29) can be achieved by a larger k (a smaller $|k|$) if λ is negative (see Fig. 4). This may yield an advantage in practice.

4.1. A concavity condition for negative eigenvalues. Results of this subsection are from [**38**]. For a fixed $r > 0$ and $p \in \mathbb{R}^3$, let $T_p : \mathbb{R}^3 \backslash \{p\} \to \mathbb{R}^3 \backslash \{p\}$ be the inversion in a sphere, namely,

$$T_p x := \frac{r^2}{|x-p|^2}(x-p) + p. \tag{0.30}$$

For a given bounded domain Ω in $\mathbb{R}^3$, let $\partial\Omega_p^*$ be the inversion of $\partial\Omega$, *i.e.*, $\partial\Omega_p^* = T_p(\partial\Omega)$.

The following theorem is obtained in [**38**].

THEOREM 4.1. *Let Ω be a bounded domain in $\mathbb{R}^3$ whose boundary is Lipschitz continuous. If there are $p \in \Omega$ and $x \in \partial\Omega$ such that*

$$(x-p) \cdot \nu_x < 0 \tag{0.31}$$

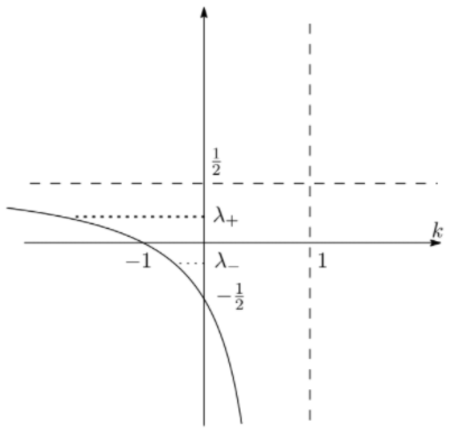

FIGURE 4. The curve is the graph of $\frac{k+1}{2(k-1)}$ and λ_+, λ_- are positive and negative NP eigenvalues, respectively. λ_- is attained by k whose absolute value is smaller. (The figure is from [**17**]).

and $\partial\Omega$ is C^1 near x, then either $\sigma(\mathcal{K}^*_{\partial\Omega})$ or $\sigma(\mathcal{K}^*_{\partial\Omega^*_p})$ has a negative value.

The condition (0.31) indicates that $\partial\Omega$ is concave with respect to $p \in \Omega$. For example, this condition is fulfilled if there is a point on $\partial\Omega$ where the Gaussian curvature is negative. Thus the following corollary is obtained.

COROLLARY 4.2. *Suppose $\partial\Omega$ is C^2 smooth. If there is a point on $\partial\Omega$ where the Gaussian curvature is negative, then either $\mathcal{K}^*_{\partial\Omega}$ or $\mathcal{K}^*_{\partial\Omega^*_p}$ for some $p \in \Omega$ has a negative eigenvalue.*

Let us briefly see how Theorem 4.1 is proved. Just for simplicity we assume $p = 0$ and denote $\partial\Omega^*_p$ by $\partial\Omega^*$. For a function φ defined on $\partial\Omega$, define φ^* on $\partial\Omega^*$ by

$$\varphi^*(x^*) := \varphi(x)\frac{|x|^d}{r^d}, \quad x^* = T_p x.$$

The following identity holds for all φ:

$$\langle \mathcal{K}^*_{\partial\Omega^*}[\varphi^*], \varphi^* \rangle_{\partial\Omega^*} + \langle \mathcal{K}^*_{\partial\Omega}[\varphi], \varphi \rangle_{\partial\Omega} = \int_{\partial\Omega} \frac{y \cdot \nu_y}{|y|^2} |\mathcal{S}_{\partial\Omega}[\varphi](y)|^2 \, dS. \tag{0.32}$$

Here, $\langle\ ,\ \rangle_{\partial\Omega^*}$ and $\langle\ ,\ \varphi\rangle_{\partial\Omega}$ denote the inner product (0.13) on $\partial\Omega^*$ and $\partial\Omega$, respectively. If (0.31) holds, namely, $x \cdot \nu_x < 0$ for some $x \in \partial\Omega$, then, since $\partial\Omega$ is C^1 near x and $\mathcal{S}_{\partial\Omega} : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$ is invertible, we can choose $\varphi \in H^{-1/2}(\partial\Omega)$ so that $\mathcal{S}_{\partial\Omega}[\varphi]$ is supported in a small neighborhood of x and

$$\int_{\partial\Omega} \frac{y \cdot \nu_y}{|y|^2} |\mathcal{S}_{\partial\Omega}[\varphi](y)|^2 \, dS < 0.$$

Thus we have

$$\langle \mathcal{K}^*_{\partial\Omega^*}[\varphi^*], \varphi^* \rangle_{\partial\Omega^*} + \langle \mathcal{K}^*_{\partial\Omega}[\varphi], \varphi \rangle_{\partial\Omega} < 0.$$

Thus, the numerical range of either $\mathcal{K}^*_{\partial\Omega^*}$ or $\mathcal{K}^*_{\partial\Omega}$ has a negative element. The numerical range of the operator T is defined to be

$$W(T) := \{\langle Tx, x\rangle \ ; \ \|x\| = 1\}.$$

Theorem 4.1 then follows from Hausdorff-Toeplitz theorem (see, for example, [**34**]).

The identity (0.32) is derived using the following three transformation formulas,

$$dS(x^*) = \frac{r^4}{|x|^4} dS(x)$$

for the surface measure dS,

$$\Gamma(x^* - y^*) = \frac{|x||y|}{r^2} \Gamma(x - y)$$

for the fundamental solution, and

$$\nu_{x^*} = (-1)^m \Big(I - 2 \frac{x}{|x|} \frac{x^t}{|x|} \Big) \nu_x$$

for the normal vectors.

4.2. NP spectrum on tori. We now review the results in [**15**] on NP spectrum on tori. We do so in some detail having in mind the possible connection to CALR as mentioned in the previous section.

The toroidal coordinate system (ξ, θ, η) for the Cartesian coordinates (x_1, x_2, x_3) is given by

$$x_1 = \frac{R_0 \sqrt{1-\xi^2} \cos\eta}{1 - \xi\cos\theta}, \quad x_2 = \frac{R_0 \sqrt{1-\xi^2} \sin\eta}{1 - \xi\cos\theta}, \quad x_3 = -\frac{R_0 \xi \sin\theta}{1 - \xi\cos\theta}.$$

Here, $R_0 = \sqrt{r_0^2 - a^2}$ where r_0 and a are the major and minor radii, respectively, of a toroidal system. The variable ξ $(0 < \xi < 1)$ is similar to the minor radius, θ $(0 \leqslant \theta < 2\pi)$ is the poloidal angle, and η $(0 \leqslant \eta < 2\pi)$ is the toroidal angle (see [**25**] for the toroidal coordinate system).

The surface ξ = constant is a torus, on which (θ, η) is the coordinate system. Let, with the fixed ξ,

$$\mu(\eta - \eta') := \frac{1}{\xi^2} + \left(1 - \frac{1}{\xi^2}\right) \cos(\eta - \eta')$$

and

$$\chi(\theta) := 1 - \xi\cos\theta.$$

Then, $\mathcal{K}^*_{\partial\Omega}[\varphi]$ for a function φ on the torus can be written as

$$\mathcal{K}^*_{\partial\Omega}[\varphi](\theta, \eta) = \int_0^{2\pi} \int_0^{2\pi} k(\theta, \theta'; \eta - \eta') \varphi(\theta', \eta') \, d\theta' d\eta',$$

where

$$\begin{aligned} k(\theta, \theta'; \eta - \eta') = & \frac{1-\xi^2}{8\pi\sqrt{2}\xi} \frac{\chi(\theta)^{1/2}}{\chi(\theta')^{3/2}} \frac{1}{(\mu(\eta - \eta') - \cos(\theta - \theta'))^{1/2}} \\ & - \frac{1-\xi^2}{8\pi\sqrt{2}\xi^3} \frac{\chi(\theta)^{3/2}}{\chi(\theta')^{3/2}} \frac{1 - \cos(\eta - \eta')}{(\mu(\eta - \eta') - \cos(\theta - \theta'))^{3/2}}. \end{aligned}$$

Since $0 < \xi < 1$, χ is a positive smooth function. Thus any function φ in $H^{-1/2}$ on the torus admits the Fourier series expansion

$$\varphi(\theta,\eta) = \chi(\theta)^{3/2} \sum_{k=-\infty}^{\infty} \hat{\varphi}_k(\theta) e^{ik\eta}. \tag{0.33}$$

We thus have

$$\mathcal{K}^*_{\partial\Omega}[\varphi](\theta,\eta) = \frac{1-\xi^2}{8\pi\sqrt{2}\xi}\chi(\theta)^{3/2} \sum_{k=-\infty}^{\infty} \mathcal{K}_k[\hat{\varphi}_k](\theta) e^{ik\eta}, \tag{0.34}$$

where the operator $\mathcal{K}_k$ is defined by

$$\mathcal{K}_k[f](\theta) := \int_0^{2\pi} a_k(\theta,\theta') f(\theta')\, d\theta'$$

with

$$\begin{aligned} a_k(\theta,\theta') = \chi(\theta)^{-1} \int_0^{2\pi} \frac{e^{-ik\eta'}}{(\mu(\eta') - \cos(\theta-\theta'))^{1/2}}\, d\eta' \\ - \frac{1}{\xi^2} \int_0^{2\pi} \frac{(1-\cos\eta')e^{-ik\eta'}}{(\mu(\eta') - \cos(\theta-\theta'))^{3/2}}\, d\eta'. \end{aligned}$$

The single layer potential also admits the decomposition

$$\mathcal{S}_{\partial\Omega}[\varphi](\theta,\eta) = \frac{R_0\sqrt{1-\xi^2}}{4\pi\sqrt{2}}\chi(\theta)^{1/2} \sum_{k=-\infty}^{\infty} \mathcal{S}_k[\hat{\varphi}_k](\theta) e^{ik\eta}, \tag{0.35}$$

where

$$\mathcal{S}_k[f](\theta) := \int_0^{2\pi} s_k(\theta-\theta') f(\theta')\, d\theta'$$

with

$$s_k(\theta) := \int_0^{2\pi} \frac{e^{-ik\eta'}}{(\mu(\eta') - \cos\theta)^{1/2}}\, d\eta'.$$

Let $H^{-1/2}(T)$ be the Sobolev space of order $-1/2$ on the unit circle T. If we define $\langle\ ,\ \rangle_k$ by

$$\langle f, g\rangle_k := \int_0^{2\pi} f(\theta)\overline{\mathcal{S}_k[g](\theta)}\, d\theta, \quad f,g \in H^{-1/2}(T), \tag{0.36}$$

then it is an inner product on $H^{-1/2}(T)$ for each k and the following relation holds:

$$\langle \varphi, \psi\rangle_* = \frac{{R_0}^3\xi(1-\xi^2)}{4\pi\sqrt{2}} \sum_{k=-\infty}^{\infty} \langle \hat{\varphi}_k, \hat{\psi}_k\rangle_k, \tag{0.37}$$

where $\hat{\varphi}_k = \hat{\varphi}_k(\theta)$ is the Fourier coefficient as defined in (0.33). Moreover, $\mathcal{K}_k$ is compact and self-adjoint on $H^{-1/2}(T)$, that is,

$$\langle \mathcal{K}_k[g_1], g_2\rangle_k = \langle g_1, \mathcal{K}_k[g_2]\rangle_k.$$

We infer from the relation (0.34) that if λ is an eigenvalue of $\mathcal{K}_k$ and g is the corresponding eigenfunction, then $(1-\xi^2)\lambda/(8\sqrt{2}\pi\xi)$ is an eigenvalue of $\mathcal{K}^*_{\partial\Omega}$ and the corresponding eigenfunction is given by

$$\varphi(\theta,\eta) = \chi(\theta)^{3/2} g(\theta) e^{ik\eta}. \tag{0.38}$$

Note that the function φ oscillates in toroidal direction if $k \neq 0$. As we saw in Fig. 3, the exceptional eigenfunctions, whose single layer potentials (namely, plasmon) do not decay fast, do not oscillate in the toroidal direction. On the other hand, the non-exceptional eigenfunctions oscillate in the toroidal direction.

It is proved using the stationary phase method that for any $0 < \xi < 1$ there exists a positive integer k_0 such that the numerical range of $\mathcal{K}_k$ has both positive and negative values and hence $\mathcal{K}_k$ has both positive and negative eigenvalues for all $k \in \mathbb{Z}$ with $|k| > k_0$. Thus we have the following theorem from Hausdorff-Toeplitz theorem again:

Theorem 4.3. *The NP operator on tori has infinitely many negative eigenvalues.*

Any bounded domain with $\mathcal{C}^{1,\alpha}$ boundary has infinitely many positive NP eigenvalues.

Two questions arise naturally:

Problem 3. If φ is given by (0.38), the does the decay of $\mathcal{S}_{\partial\Omega}[\varphi]$ depend on k and all the exceptional eigenvalues (the single layer potential of the corresponding eigenfunctions as discussed in section 3) are from $k = 0$?

Problem 4. Is it true that all eigenfunctions of $\mathcal{K}_{\partial\Omega}^*$ are of the form (0.38), in particular, any eigenvalue of $\mathcal{K}_{\partial\Omega}^*$ is an eigenvalue of $\mathcal{K}_k$ for some k?

4.3. Further results. We mention general results in [**61**]. For more precise statements of the results and discussions on them, we refer to [**22**].

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^∞ boundary.*
- (i) *If one of the principal curvatures at $x \in \partial\Omega$ is positive, then there are infinitely many negative NP eigenvalues.*
- (ii) *If the principal curvatures are negative at every $x \in \partial\Omega$, then there are at most finitely many negative NP eigenvalues.*

In a recent paper [**37**] the NP spectrum on surfaces of revolution of planar curves with a corner is considered and the continuous spectrum can be both positive or negative depending on the angle of the corner. This paper is the first attempt to investigate continuous spectrum on three-dimensional domain with corners.

5. NP spectrum on polygonal domains

In order to introduce an outstanding problem on essential spectrum, we begin the discussion in this section by briefly reviewing results on essential spectrum. For more extensive review and interesting historical account, we refer to [**22**].

Let $\mathcal{E}(t)$ be the resolution of identity given in (0.16). For each $\varphi \in H^{-1/2}(\partial\Omega)$, the measure $\mu_\varphi := \langle \mathcal{E}(t)\varphi, \varphi \rangle_*$ is called the spectral measure associated with φ. According to Lebesgue decomposition theorem, $H^{-1/2}(\partial\Omega)$ can be decomposed as

$$H^{-1/2}(\partial\Omega) = H_{pp} \oplus H_{ac} \oplus H_{sc},$$

where H_{sc} is the collection of all φ such that μ_φ is singularly continuous, and H_{pp} and H_{ac} are defined likewise. The singularly continuous $\sigma_{sc} = \sigma_{sc}(\mathcal{K}^*_{\partial\Omega})$ is the spectrum of $\mathcal{K}^*_{\partial\Omega}$ when restricted H_{sc}, and σ_{ac} and σ_{pp} are defined likewise. See [**66**].

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$ with finite number of corners whose inner angles are $\alpha_1, \ldots, \alpha_N$. Let

$$b_{ess} = \frac{1}{2} \max_{1 \leqslant j \leqslant N} \left(1 - \frac{\alpha_j}{\pi}\right). \tag{0.39}$$

It is proved in [**64**] that b_{ess} is a bound of $\sigma_{ess}(\mathcal{K}^*_{\partial\Omega})$. In [**47**], a lens domain (an intersection of two disks) is considered and a complete spectral resolution of $\mathcal{K}^*_{\partial\Omega}$ is derived. It enables us to infer that

$$\sigma(\mathcal{K}^*_{\partial\Omega}) = \sigma_{ess}(\mathcal{K}^*_{\partial\Omega}) = \sigma_{ac}(\mathcal{K}^*_{\partial\Omega}) = [-b_{ess}, b_{ess}].$$

In particular, it implies that $\sigma_{pp}(\mathcal{K}^*_{\partial\Omega})$ and $\sigma_{sc}(\mathcal{K}^*_{\partial\Omega})$ are void on a lens domain. If Ω be a bounded Lipschitz domain in $\mathbb{R}^2$ with finite number of corners, it is proved in [**65**] that $\sigma_{ess}(\mathcal{K}^*_{\partial\Omega}) = [-b_{ess}, b_{ess}]$, and in [**63**] that $\sigma_{sc}(\mathcal{K}^*_{\partial\Omega})$ is void.

An interesting problem arises:

Problem 5. Find geometric conditions on $\partial\Omega$ which guarantee $\sigma_{sc}(\mathcal{K}^*_{\partial\Omega}) \neq \emptyset$. (Since no example of domains with nonempty singularly continuous spectrum is known, even a single example of such a domain would be interesting.)

We now review the results of [**36**] where the question whether $\sigma_{pp}(\mathcal{K}^*_{\partial\Omega})$ is void or not. The crux of the matter is to use resonance to distinguish σ_{pp} and σ_{ac}. This idea also appears in [**16, 36**].

Let $f \in H_0^{-1/2}(\partial\Omega)$. For $t \in (-1/2, 1/2)$ and $\delta > 0$, let $\varphi_{t,\delta}$ be the solution of the integral equation

$$\big((t + i\delta)I - \mathcal{K}^*_{\partial\Omega}\big)[\varphi_{t,\delta}] = f \quad \text{on } \partial\Omega.$$

By the spectral resolution (0.17), the solution is given by

$$\varphi_{t,\delta} = \int_{\sigma(\mathcal{K}^*_{\partial\Omega})} \frac{1}{t + i\delta - s} \, d\mathcal{E}(s)[f],$$

and hence

$$\|\varphi_{t,\delta}\|_*^2 = \langle \varphi_{t,\delta}, \varphi_{t,\delta} \rangle_* = \int_{\sigma(\mathcal{K}^*_{\partial\Omega})} \frac{1}{(s-t)^2 + \delta^2} \, d\langle \mathcal{E}(s)[f], f \rangle_*.$$

If $t \notin \sigma(\mathcal{K}^*_{\partial\Omega})$, one can immediately see from (0.20) that $\|\varphi_{t,\delta}\|_* < C$ for some C regardless of δ.

If $t \in \sigma(\mathcal{K}^*_{\partial\Omega})$, then $\|\varphi_{t,\delta}\|_*$ may blow up as $\delta \to 0$. The key idea is that the blow-up rate at the eigenvalue t is different from that at continuous spectrum. To see this, we recall that an eigenvalue t of $\mathcal{K}^*_{\partial\Omega}$ is characterized by discontinuity $\mathcal{E}(t+) - \mathcal{E}(t) \neq 0$ (and t is isolated) (see [**70**]). So, if f satisfies

$$\langle \mathcal{E}(t+)[f], f \rangle_* - \langle \mathcal{E}(t)[f], f \rangle_* > 0, \tag{0.40}$$

then

$$\|\varphi_{t,\delta}\|_*^2 \geqslant \frac{\langle \mathcal{E}(t+)[f], f \rangle_* - \langle \mathcal{E}(t)[f], f \rangle_*}{\delta^2},$$

and hence

$$\|\varphi_{t,\delta}\|_*^2 \approx \delta^{-2}.$$

If $t \in \sigma_{ac}(\mathcal{K}_{\partial\Omega}^*)$, then there is f such that the spectral measure $\langle \mathcal{E}_s[f], f\rangle_*$ is absolutely continuous near t, namely, there is $\epsilon > 0$ and a function $\mu_f(s)$ which is integrable on $[t-\epsilon, t+\epsilon]$ such that

$$d\langle \mathcal{E}(s)[f], f\rangle_* = \mu_f(s)ds, \quad s \in [t-\epsilon, t+\epsilon]. \tag{0.41}$$

Then it is proved that

$$\lim_{\delta\to 0} \delta \|\varphi_{t,\delta}\|_*^2 = \frac{\pi}{2}(\mu_f(t+) + \mu_f(t-)) > 0,$$

and

$$\lim_{\delta\to 0} \delta^2 \|\varphi_{t,\delta}\|_*^2 = 0.$$

Define an indicator function $\alpha_f(t)$ by

$$\alpha_f(t) := \sup\left\{ \alpha \;\middle|\; \limsup_{\delta\to 0} \delta^\alpha \|\varphi_{t,\delta}\|_* = \infty \right\}, \quad t \in (-1/2, 1/2). \tag{0.42}$$

We see that $0 \leqslant \alpha_f(t) \leqslant 1$ for all t. The following theorem for classification of NP spectra of is obtained in [**36**].

Theorem 5.1. *Let $f \in H_0^{-1/2}(\partial\Omega)$.*
(i) *If $\alpha_f(t) > 0$, then $t \in \sigma(\mathcal{K}_{\partial\Omega}^*)$.*
(ii) *If $\alpha_f(t) = 1$ and t is isolated, then $t \in \sigma_{pp}(\mathcal{K}_{\partial\Omega}^*)$.*
(iii) *If $1/2 \leqslant \alpha_f(t) < 1$, then $t \in \sigma_{ess}(\mathcal{K}_{\partial\Omega}^*)$.*

The indicator function $\alpha_f(t)$ can be computed numerically using the following identity:

$$\alpha_f(t) = -\lim_{\delta\to 0} \frac{\log \|\varphi_{t,\delta}\|_*}{\log \delta}$$

if the limit exists.

The source function f should satisfy (0.40) and (0.41) with $\mu_f(t+) + \mu_f(t-) > 0$. In [**36**], $f_z(x) = \nu(x) \cdot \nabla(a \cdot \nabla_x \Gamma(x-z))$ (a is a constant vector and z lies outside Ω) is chosen as a source function because for any $\varphi \in H_0^{-1/2}(\partial\Omega)$, $\langle f_z, \varphi\rangle_* \neq 0$ for almost all z among several advantages. Then the indicator function is modified as

$$\alpha_\sharp(t) := \max_{1\leqslant j\leqslant N} \{\alpha_{f_{z_j}}(t)\}.$$

for some $z_1, \ldots, z_N$.

This method of characterizing NP spectra has been tested for various polygonal domains, ellipses perturbed by a corner, and so on. The computational results clearly show effectiveness of the method in distinguishing eigenvalues from continuous spectrum. Some results show that eigenvalues can be embedded inside the continuous spectrum (see Fig. 5). Lately, it is rigorously proved that it is possible that infinitely many eigenvalues are embedded in the continuous spectrum [**52, 53**].

The computational results on rectangles are particularly interesting. We know by (0.39) that if Ω is rectangle, then $\sigma_{ess}(\mathcal{K}_{\partial\Omega}^*) = [-1/4, 1/4]$. However, as the aspect ratio of the rectangle gets larger, more and more eigenvalues show up (see Fig. 6). We can formulate an interesting problem out of these numerical experiments.

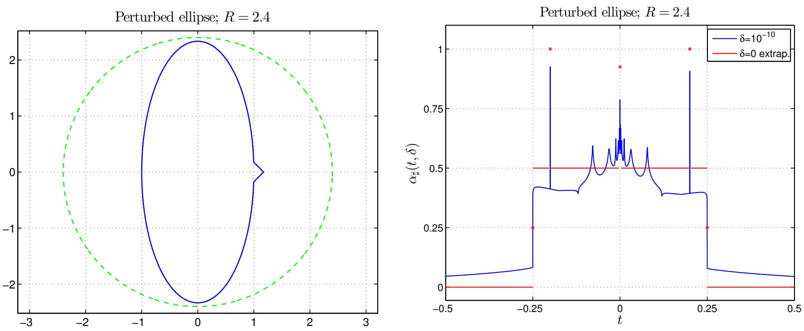

FIGURE 5. An ellipse perturbed by a corner. Two eigenvalues are embedded in the continuous spectrum. (Figures from [**36**]).

Problem 6. Is it true that there is a sequence of numbers $1 = r_0 < r_1 < r_2 < \cdots$ such that if the aspect ratio lies in (r_{k-1}, r_k), then the number of positive NP eigenvalues is k? The first number r_1 seems around 2.201592 (see the second figure in Fig. 6).

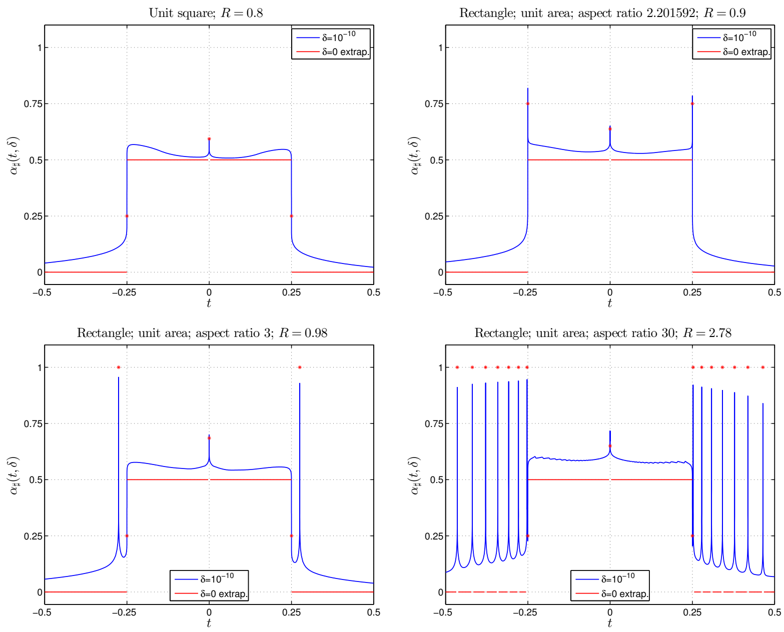

FIGURE 6. Graphs of the indicator function $\alpha_\sharp$ on rectangles of aspect ratios, $r = 1,\ 2.201592,\ 3,\ 30$. When $r = 2.201592$, eigenvalues just about to emerge, and more and more eigenvalues emerge as the aspect ratio increase. (Figures from [**36**]).

Fig. 7 show the NP spectrum of an isosceles triangle. The interval of continuous spectrum is determined by the smallest interior angle as explained earlier. It shows no eigenvalue. It is not clear whether triangles have no NP eigenvalues or not and it would be interesting to clarify this.

Problem 7. Is it true that there is no NP eigenvalues on triangles?

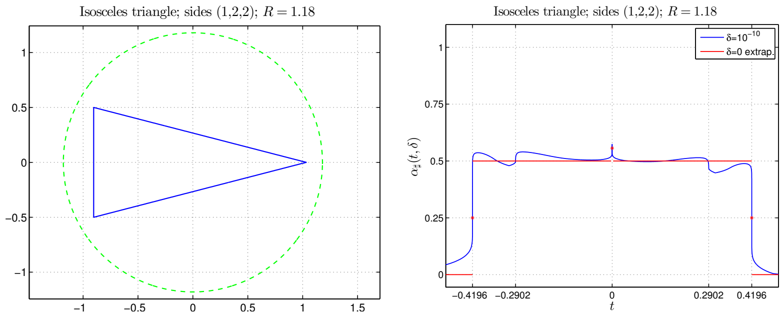

FIGURE 7. NP spectrum of the isosceles triangle with sides 1, 2 and 2. There is no eigenvalue. (Figures from [**36**]).

6. Spectral structure of thin domains

Motivated by the numerical study on the NP spectral structure on rectangles as explained in the previous section, the spectral structure of the NP operators on thin domains has been investigated in two dimensions [**20**] and in three dimensions [**17**]. We review those results in this section.

There is another motivation behind these work. They are motivated by observations that as the boundary $\partial\Omega$ of the domain becomes singular in some sense, the corresponding NP spectrum seems to approach to the bound $\pm 1/2$. For example, as we saw at the beginning of the previous section, if a planar domain has corners and if a corner gets sharper and the domain becomes needle-like around the corner, then the essential spectrum approaches $[-1/2, 1/2]$. If Ω consists of two strictly convex planar domains and boundaries get closer, then more and more eigenvalues of the corresponding NP operator approach $\pm 1/2$ [**26, 27**]. This causes stress concentration in the narrow region between two inclusions (see section 7).

Results to be reviewed in this section show that as the domain gets thinner, NP spectrum approaches $[-1/2, 1/2]$ in two dimensions. In three dimensions, it approaches either $[-1/2, 1/2]$ or $[0, 1/2]$ depending upon the kind of thinness (thin and long, or thin and flat). So, in some case there is no negative eigenvalue.

In this section we work with $\mathcal{K}_{\partial\Omega}$, not $\mathcal{K}^*_{\partial\Omega}$, since it is more convenient. The spectrum of $\mathcal{K}_{\partial\Omega}$ on $H^{1/2}(\partial\Omega)$ is the same as the spectrum of $\mathcal{K}^*_{\partial\Omega}$ on $H^{-1/2}(\partial\Omega)$.

The two-dimensional thin domains considered in [**20**] is given as follows: For $R \geqslant 1$, let Ω_R be a rectangle-shaped domain whose boundary consists of three parts, say

$$\partial\Omega_R = \Gamma_R^+ \cup \Gamma_R^- \cup \Gamma_R^s,$$

where the top and bottom are

$$\Gamma_R^+ = [-R, R] \times \{1\}, \quad \Gamma_R^- = [-R, R] \times \{-1\},$$

and the side Γ_R^s consists of the left and right sides, namely, $\Gamma_R^s = \Gamma_R^l \cup \Gamma_R^r$, where Γ_R^l and Γ_R^r are translates of curves Γ^l and Γ^r connecting points $(0, 1)$

and $(0, -1)$, namely, $\Gamma_R^l = \Gamma^l - (R, 0)$ and $\Gamma_R^r = \Gamma^r + (R, 0)$. If both Γ^l and Γ^r are line segments, Ω_R is a rectangle. The boundary $\partial\Omega_R$ is assumed to be Lipschitz continuous. We say that the domain Ω_R is thin because the scaled domain $R^{-1}\Omega_R$ is thin (like a needle) and its NP spectrum is the same as that of Ω_R (NP spectrum is dilation invariant).

The following theorem is the main result of [**19**].

THEOREM 6.1. *If $\{R_j\}$ be an increasing sequence such that $R_j \to \infty$ as $j \to \infty$, then*

$$\overline{\cup_{j=1}^{\infty} \sigma(\mathcal{K}_{\partial\Omega_{R_j}})} = [-1/2, 1/2]. \tag{0.43}$$

If Ω_R is a rectangle, then $\sigma_{ess}(\mathcal{K}_{\partial\Omega_R}) = [-1/4, 1/4]$ as we discussed in section 5. Theorem 6.1 says that as R increase to ∞, more and more eigenvalues appear outside $[-1/4, 1/4]$ and their totality densely fills up intervals $[-1/2, -1/4] \cup [1/4, 1/2]$. This is in accordance with the computational result in [**36**]. What is surprising is that (0.43) holds regardless of the choice of the sequence R_j.

Theorem 6.1 is proved as follows. The NP operator on $\partial\Omega_R$ behaves like $1/2$ times the one-dimensional Poisson integral. Since the Fourier transform of the Poisson kernel is $e^{-2\pi t|\xi|}$, the Poisson integral has $[0, 1]$ as its essential spectrum. Using this fact, one can construct a function $\varphi_R \in H^{-1/2}(\partial\Omega_R)$ such that

$$\lim_{R\to\infty} \frac{\|(\lambda I - \mathcal{K}_{\partial\Omega_R})[\varphi_R]\|_*}{\|\varphi_R\|_*} = 0 \tag{0.44}$$

for each $\lambda \in (0, 1/2]$. It implies that $[0, 1/2] \subset \overline{\cup_{j=1}^{\infty} \sigma(\mathcal{K}_{\partial\Omega_{R_j}})}$. In fact, if $\lambda \notin \overline{\cup_{j=1}^{\infty} \sigma(\mathcal{K}_{\partial\Omega_{R_j}})}$, there is $\delta > 0$ such that $[\lambda-\delta, \lambda+\delta] \cap \overline{\cup_{j=1}^{\infty} \sigma(\mathcal{K}_{\partial\Omega_{R_j}})} = \emptyset$. Thus,

$$\|(\lambda I - \mathcal{K}_{\partial\Omega_R})[\varphi]\|_* \lesssim \|\varphi\|_*$$

for all φ, contradicting (0.44). Since NP spectrum on planar domain is symmetric with respect to 0, we have (0.43).

The property (0.43) seems a generic property of thin planar domains. For example, thin ellipses enjoy it. In fact, if E_j, $j = 1, 2, \ldots$, is the ellipse defined by $x_1^2/a_j^2 + x_2^2/b_j^2 < 1$ and $\mathcal{K}_{\partial E_j}$ is the corresponding NP operator, where a_j and b_j are positive numbers such that $b_j < a_j$ for all j and $b_j/a_j \to 0$ as $j \to \infty$, then

$$\overline{\cup_{j=1}^{\infty} \sigma(\mathcal{K}_{\partial E_j})} = [-1/2, 1/2]. \tag{0.45}$$

Since eigenvalues of $\mathcal{K}_{\partial E_j}$ are explicitly given by $\pm 1/2(a_j - b_j)^n/(a_j + b_j)^n$, $n = 1, 2, \ldots$, (see [**1**]), the proof of (0.45) is not difficult, and a short proof can be found in [**19**].

Let us now move to the three-dimensional thin domains. There are two different kinds of thinness in three dimensions: thin and long (like prolate spheroids), thin and flat (like oblate ellipsoids). Their NP spectral structures are different as we see below.

Let Π_R be the prolate spheroid defined by, for $R \geqslant 1$,

$$\Pi_R := \Big\{(x_1, x_2, x_3) : x_1^2 + x_2^2 + \frac{x_3^2}{R^2} < 1\Big\}.$$

Let a_j ($j = 1, 2$) be positive numbers. For a positive number R, let Ω_R be the oblate ellipsoid defined by

$$\Omega_R := \left\{ (x_1, x_2, x_3) : \frac{x_1^2}{(a_1 R)^2} + \frac{x_2^2}{(a_2 R)^2} + x_3^2 < 1 \right\}.$$

If $a_1 = a_2$, then Ω_R is an oblate spheroid.

The following theorems are obtained in [**17**].

THEOREM 6.2. *If R_j is a sequence of numbers such that $R_j \geqslant 1$ for all j and $R_j \to \infty$ as $j \to \infty$, then*

$$\overline{\cup_{j=1}^{\infty} \sigma(\mathcal{K}_{\partial \Pi_{R_j}})} = [0, 1/2]. \tag{0.46}$$

THEOREM 6.3. *If R_j is a sequence of positive numbers such that $R_j \to \infty$ as $j \to \infty$, then*

$$\overline{\cup_{j=1}^{\infty} \sigma(\mathcal{K}_{\partial \Omega_{R_j}})} = [-1/2, 1/2]. \tag{0.47}$$

Theorem 6.2 shows that totality of eigenvalues of $\mathcal{K}_{\partial \Pi_{R_j}}$ is dense in $[0, 1/2]$ regardless of choice of the sequence R_j as long as $1 \leqslant R_j \to \infty$. As mentioned before, it is proved in [**3**] that there is no negative eigenvalues on prolate spheroids. Theorem 6.3 shows that totality of eigenvalues of $\mathcal{K}_{\partial \Omega_{R_j}}$ is dense in $[-1/2, 1/2]$. This is rather surprising since, as mentioned in Theorem 4.4, $\mathcal{K}_{\partial \Omega_{R_j}}$ admits at most finitely many negative eigenvalues (since Ω_R is strictly convex). However, (0.47) says that negative eigenvalues in $\cup_{j=1}^{\infty} \sigma(\mathcal{K}_{\partial \Omega_{R_j}})$ are dense in $[-1/2, 0]$.

There are many significant works on the NP spectrum on ellipsoids, [**2, 3, 4, 57, 67**] to name a few. However, it is unlikely that Theorems 6.2 and 6.3 can be proved using those results.

Like two-dimensional case, Theorems 6.2 and 6.3 are proved by investigating the limiting behaviour of the NP operators as $R \to \infty$. It is proved that the NP operator on Π_R converges (on some test functions) to the one-dimensional convolution operator $L * f$ as $R \to \infty$, where

$$L(t) := \frac{1}{2\pi} \int_0^{\pi} \frac{1 - \cos\theta}{[(2 - 2\cos\theta) + t^2]^{3/2}} d\theta.$$

It is then proved that the Fourier transform of L has values in $(0, 1/2]$ and hence the convolution operator has continuous spectrum $[0, 1/2]$. Using this fact, a sequence of functions satisfying (0.44) is constructed. Then the inclusion

$$[0, 1/2] \subset \overline{\cup_{j=1}^{\infty} \sigma(\mathcal{K}_{\partial \Pi_{R_j}})} \tag{0.48}$$

follows. The opposite inclusion is proved in [**3**].

The NP operator on oblate ellipsoids has two pieces defined on the upper and lower parts of ellipsoids. It is proved that each piece converges to 1/2 times the two-dimensional Poisson integral. The Poisson integral operator has continuous spectrum $[0, 1]$. By choosing proper signs on the upper and lower parts, a sequence of functions satisfying (0.44) is constructed for $\lambda \in [-1/2, 1/2]$ ($\lambda \neq 0$) which yields Theorem 6.3.

A natural question arises: whether Theorem 6.2 holds for cylinder-like convex domains or even prolate ellipsoids. One can show that (0.48) holds for such domains. But we do not know if the reverse inclusion is true. We do not know negative NP eigenvalues, if any, on prolate spheroids, disappear eventually if they become thinner.

The property (0.47) seems to be a generic property of thin, flat domains. To demonstrate it, a typical thin, flat domain is considered. To define such a domain, let U be a bounded planar domain with the Lipschitz continuous boundary ∂U. Let Φ be the domain in $\mathbb{R}^3$ whose boundary consists of three pieces, namely,

$$\partial\Phi = \Sigma^+ \cup \Sigma^- \cup \Sigma^s$$

where the top and bottom are given by $\Sigma^\pm = U \times \{\pm 1\}$ and Σ^s is a surface connecting $\partial U \times \{+1\}$ and $\partial U \times \{-1\}$. We assume that $\partial\Phi$ is Lipschitz continuous. For $R > 0$ let

$$\Phi_R := \{(Rx_1, Rx_2, x_3) : (x_1, x_2, x_3) \in \Phi\}. \tag{0.49}$$

The following theorem is proved in [**17**].

THEOREM 6.4. *If R_j is a sequence of positive numbers such that $R_j \to \infty$ as $j \to \infty$, then*

$$\overline{\cup_{j=1}^{\infty} \sigma(\mathcal{K}_{\partial\Phi_{R_j}})} = [-1/2, 1/2].$$

7. Analysis of field concentration

In a composite which consists of inclusions of different material properties and matrix, some inclusions are located close to each other, and a strong stress may occur in the region between closely located inclusions. During last three decades or so, there has been significant progress in quantitative analysis of stress or field concentration about which an extensive survey has been made and several open problems are discussed in [**42**]. In this section, we briefly review some of them.

Let us mention what the problem is with a brief review of results and how it is related to the spectral theory of the NP operator. Let the domain Ω consist of two closely located but disjoint domains D_1 and D_2, namely, $\Omega = D_1 \cup D_2$. Let ϵ be the distance between D_1 and D_2. Let k_j be the conductivity of D_j for $j = 1, 2$, while that of $\mathbb{R}^d \backslash (D_1 \cup D_2)$ is assumed to be 1. So the conductivity distribution for this section is given by

$$\epsilon = k_1 \chi_{D_1} + k_2 \chi_{D_2} + \chi_{\mathbb{R}^d \backslash (D_1 \cup D_2)}.$$

The conductivities k_1 and k_2 are different from 1 and allowed to be 0 or ∞. The conductivity being ∞ means that the inclusion is perfectly conducting, and 0 means insulating.

We consider the problem (0.2). The problem is to derive estimates for ∇u (and higher order derivatives) in terms of ϵ (and k_1, k_2, if possible) as ϵ tends to 0. Another problem is to characterize asymptotically the singularity of ∇u. The asymptotic characterization, as ϵ tends to 0, means a decomposition of the form

$$u = s + r, \tag{0.50}$$

where s is the singular part, namely, ∇s carries the full information of the singularity of ∇u, while r is a regular part, namely, ∇r is bounded.

When D_1 and D_2 are disks, an optimal estimate for the gradient has been derived in [**11, 12**]. It is extended to higher order derivatives in [**31**] (see also [**39**]):

$$\|u\|_{n,U} \lesssim \left(4\lambda_1\lambda_2 - 1 + \sqrt{\epsilon}\right)^{-n} \tag{0.51}$$

provided that $(k_1 - 1)(k_2 - 1) > 0$. Here U be a bounded set containing $\overline{D_1 \cup D_2}$, $\|u\|_{n,U}$ denotes the piecewise C^n norm on U, namely,

$$\|u\|_{n,U} := \|u\|_{C^n(\overline{D_1})} + \|u\|_{C^n(\overline{D_2})} + \|u\|_{C^n(U\setminus\Omega)},$$

and

$$\lambda_j := \frac{k_j + 1}{2(k_j - 1)}, \quad j = 1, 2. \tag{0.52}$$

If $k_1 = k_2 = \infty$ (or $k_1 = k_2 = 0$), then the estimate (0.51) yields

$$|\nabla u(z)| \lesssim \epsilon^{-1/2}. \tag{0.53}$$

This estimate has been extended to strictly convex inclusions in two dimensions (more precisely, strictly convex near the unique points on ∂D_1 and ∂D_2 of the shortest distance) [**71**]. In three dimensions, the optimal estimate for ∇u has been obtained in [**23**]: If $k_1 = k_2 = \infty$ and inclusions are strictly convex inclusions, then

$$|\nabla u(z)| \lesssim \frac{1}{\epsilon|\ln\epsilon|}.$$

However, despite important progress made in [**24, 55, 69, 72**], the insulating case ($k_1 = k_2 = 0$) in three dimensions remains unsolved.

The asymptotic characterization of the form (0.50) for planar strictly convex domains has been obtained in [**8, 48, 43**]. Moreover, it is proved in [**43**] that the regular part r converges to the touching case solution as $\epsilon \to 0$ and the singular part disappear as soon as the inclusions are touching, namely, $\epsilon = 0$. The asymptotic characterization for three-dimensional balls has been derived in [**49, 54**].

If $(k_1-1)(k_2-1) < 0$ and D_1, D_2 are disks, then the following unexpected estimate is obtained in [**39**]:

$$\|u\|_{n,\Omega} \lesssim \left(4|\lambda_1\lambda_2| - 1 + \sqrt{\epsilon}\right)^{-n+1}. \tag{0.54}$$

If $k_1 = 0$ and $k_2 = \infty$ (or the other way around), namely, D_1 is an insulator and D_2 is a perfect conductor, then

$$\lambda_1 = -\frac{1}{2}, \quad \lambda_2 = \frac{1}{2}. \tag{0.55}$$

Thus (0.54) yields that

$$\|u\|_{n,\Omega} \lesssim \epsilon^{\frac{-n+1}{2}}.$$

In particular, it implies that ∇u is bounded and

$$\|\nabla^2 u\|_{L^\infty(U)} \lesssim \epsilon^{-1/2}.$$

This estimate is optimal in the sense that there are harmonic functions h such that the opposite inequality holds.

It is an intriguing problem is to extend the results for circular inclusions to inclusions of general shape (or to prove they do not hold on inclusions of general shape). If D_1 is an insulator and D_2 is a perfect conductor, then the corresponding conductivity problem can be expressed as follows:

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^d \backslash \overline{\Omega}, \\ \partial_\nu u = 0 & \text{on } \partial D_1, \\ u = c & \text{on } \partial D_2, \\ v(x) - h(x) = O(|x|^{-d+1}) & \text{as } |x| \to \infty, \end{cases}$$

where c is a constant to be determined by the additional condition

$$\int_{\partial D_2} \partial_\nu u|_+ = 0.$$

The results in [**39**] are obtained using the spectral theory of NP operator on two circles. It was possible since NP eigenvalues and eigenfunctions on two circles can be computed explicitly. But it may not be possible to apply the NP spectral theory to inclusions of general shape. However, NP spectral theory may provide some insight to the problem.

The representation (0.6) of the solution takes the following form if there are two inclusions, namely, $\Omega = D_1 \cup D_2$:

$$u(x) = h(x) + \mathcal{S}_{\partial D_1}[\varphi_1](x) + \mathcal{S}_{\partial D_2}[\varphi_2](x), \quad x \in \mathbb{R}^d.$$

The continuity of the flux along ∂D_j takes the following form:

$$k_j \partial_\nu (\mathcal{S}_{\partial D_1}[\varphi_1] + \mathcal{S}_{\partial D_2}[\varphi_2])|_- - \partial_\nu (\mathcal{S}_{\partial D_1}[\varphi_1] + \mathcal{S}_{\partial D_2}[\varphi_2])|_+ = (1 - k_j)\partial_\nu h,$$

which can be written in short as

$$(\Lambda - \mathcal{K}^*_{\partial\Omega})[\varphi] = \begin{bmatrix} \partial_\nu h|_{\partial D_1} \\ \partial_\nu h|_{\partial D_2} \end{bmatrix}, \tag{0.56}$$

where

$$\Lambda := \begin{bmatrix} \lambda_1 I & 0 \\ 0 & \lambda_2 I \end{bmatrix}$$

(λ_j is defined by (0.52)). Here, I is the identity operator. The NP operator $\mathcal{K}^*_{\partial\Omega}$ on $\partial\Omega = \partial D_1 \cup \partial D_2$ is given by

$$\mathcal{K}^*_{\partial\Omega} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} \mathcal{K}^*_{\partial D_1}[\varphi_1] & \partial_\nu \mathcal{S}_{\partial D_2}[\varphi_2]|_{\partial D_1} \\ \partial_\nu \mathcal{S}_{\partial D_1}[\varphi_1]|_{\partial D_2} & \mathcal{K}^*_{\partial D_2}[\varphi_2] \end{bmatrix}$$

for $\varphi = (\varphi_1, \varphi_2) \in H^{-1/2}(\partial\Omega) = H^{-1/2}(\partial D_1) \times H^{-1/2}(\partial D_2)$.

If $k_1 = 0$ and $k_2 = \infty$, then

$$\Lambda := \frac{1}{2} \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$$

by (0.55). The question is if the integral equation and spectral properties of $\mathcal{K}^*_{\partial\Omega}$ can provide some insight why ∇u is bounded and the second derivative (and higher order derivatives) blows up.

We may be able to grasp the question better by comparing the insulator-conductor case with the conductor-conductor case ($k_1 = k_2 = \infty$) where the conductivity of both inclusions is ∞. In that case, Λ in (0.56) is given by

$$\Lambda := \frac{1}{2} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Since more and more eigenvalues of $\mathcal{K}^*_{\partial\Omega}$ approach $1/2$ as ϵ (the distance between inclusions) tends to 0 as proved in [**26, 27**], the solution φ to (0.56) blows up and so does ∇u as ϵ tends to 0.

A special spectral feature of circular inclusions which enables us to solve the integral equation (0.56) is that $\varphi = (\varphi_1, \varphi_2)$ and $\psi = (\psi_1, \psi_2)$ are orthogonal to each other if and only of φ_j and ψ_j are orthogonal to each other for $j = 1, 2$.

8. Conclusion

We review recent development in spectral geometry and analysis of the NP operator in various topics including plasmon resonance and the NP spectral theory, CALR and analysis of surface localization of plasmon, negative eigenvalues and spectrum on tori, spectrum on polygonal domains, spectral structure of thin domains, and analysis of stress in terms of the NP spectral theory. These topics are complementary to those in another survey paper [**22**].

During the course of review we discuss some problems to be solved. Among them are

- estimates for the surface localization of the plasmon on planar domains with real analytic boundaries and applications to CALR,
- NP spectrum on tori and possible connection to CALR,
- geometric conditions which guarantee existence of singularly continuous spectrum, (An example with non-empty singularly continuous spectrum would already be interesting.)
- the question on the appearance of more and more eigenvalues on rectangles as their aspect ratios tend to ∞,
- non-existence (or existence) of an NP eigenvalue on triangles,
- an optimal gradient estimate for the insulating problem,
- derivative estimates for the insulator-conductor problem: general shape.

These problems are all quite interesting and may be quite challenging as well.

The NP operator is also used effectively for solving inverse problems, especially in detection of small inclusions, via the notion of generalized polarization tensors. For example, one can see from (0.6) and (0.11) that the solution to the problem (0.2) admits the dipole expansion

$$u(x) = h(x) - M\nabla h(0) \cdot \nabla\Gamma(x) + O(|x|^{-d}), \quad |x| \to \infty,$$

where M is a $d \times d$ matrix called the polarization matrix. It is a signature of the inclusion Ω and can be used to reconstruct some information of Ω. We refer to [**9, 10, 41**] for that application and some other applications of the NP operator.

If the polarization tensor of an inclusion is zero, then $u(x) = h(x) + O(|x|^{-d})$ as $|x| \to \infty$, which means that the inclusion is invisible or vaguely visible, and hence hard to be detected. A inclusion whose polarization tensor is made to vanish is called a weakly neutral inclusion (or a polarization tensor vanishing structure). A homogeneous simply connected domain cannot be weakly neutral. But, weakly neutral inclusions may be attained by coating simply connected domains (see [**44, 45, 46**]). We refer to another survey paper [**40**] for the study on weakly neutral inclusions (as well as neutral inclusions) and related over-determined problem for confocal ellipsoids which is a quite challenging mathematical problem.

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