Conjectures of Pólya-Szegö and Eshelby, and the Newtonian Potential Problem: A Review

Hyeonbae Kang*

February 22, 2008

Abstract

In this paper we review recent solutions to conjectures of Pólya-Szegö and Eshelby, and their relation to the classical Newtonian potential problem. We also review some recent progress on construction of extremal structures with multiple components.

1 Introduction

The Pólya-Szegö conjecture asserts that the inclusion whose electrical polarization tensor has the minimal trace takes the shape of a disk or a ball, while the Eshelby conjecture does that if the field inside an inclusion is uniform for all uniform loadings, then the inclusion is of elliptic or ellipsoidal shape. Recently Kang & Milton found a strong connection between these two seemingly unrelated conjectures [15] and proved a stronger version of the Pólya-Szegö conjecture, which implies the original Pólya-Szegö conjecture, and the Eshelby conjecture by reducing them to the Newton potential problem [16]. Following earlier work of Cherepanov they (with Kim) also constructed extremal structures with two components inside which the fields are uniform for all uniform loadings, and equivalently the polarization tensor associated with the structure satisfies the minimal equality [14]. In this paper we review these results for the purpose of conveying basic ideas of proofs. In the course of review, we will get a glimpse of history of the problems and some of their applications.

Quite recently, Liu [22] also proved the Eshelby conjecture by transforming the problem to an obstacle problem and solving it by a variational method. By similar methods he was able to show that the stronger version of the Eshelby conjecture is false in three dimensions and to construct extremal structures with arbitrary number of components in which the fields are uniform for each uniform loading. We will briefly discuss the connection between methods of Kang-Milton and Liu.

2 The Eshelby conjecture

Consider in $\mathbb{R}^d$, $d = 2, 3$, an inclusion Ω, which is a bounded Lipschitz domain being inserted into a homogeneous medium of conductivity 1 in which there existed a uniform electric field

*Department of Mathematical Sciences and RIM, Seoul National University, Seoul 151-747, Korea (hbkang@snu.ac.kr). Partially supported by the grant KOSEF R01-2006-000-10002-0.

$E = -a$. We assume that the conductivity of Ω is $k \neq 1$. The insertion of the inclusion perturbs the uniform electric field and the perturbed electric field is given by $E = -\nabla u$ where the potential u is the solution to the electric polarization problem:

$$\begin{cases} \nabla \cdot \big(1 + (k-1)\chi(\Omega)\big)\nabla u = 0 & \text{in } \mathbb{R}^d, \\ u(x) - a \cdot x = O(|x|^{1-d}) & \text{as } |x| \to \infty, \end{cases} \tag{2.1}$$

where a is a constant vector in $\mathbb{R}^d$ indicating the direction of the uniform field and $\chi(\Omega)$ denotes the indicator function of Ω. Fig. 1 shows the equipotential lines of the solution to (2.1). As one can see from Fig. 1, the field inside and outside the inclusion is perturbed and it seems unlikely for the field inside the inclusion to be uniform. However, if the inclusion takes an elliptic shape, it turns out that the field inside the inclusion is uniform as one can see from Fig. 2.

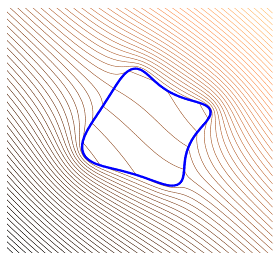

Figure 1: Equipotential lines of the solution

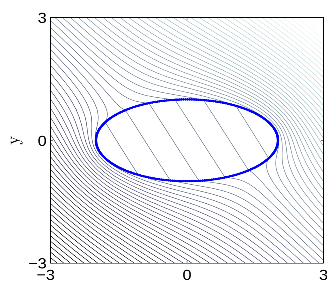

Figure 2: Elliptic inclusion

Apparently the fact that the electric field inside elliptic or ellipsoidal inclusions is uniform has been known for long time and its proofs go back to Poisson (1826) and Maxwell (1873) (see [22, 23]). The simpler version of Eshelby's conjecture for conductivity is the converse: if the electric field inside is uniform for any uniform field $-a$, then the inclusion is of an elliptic or ellipsoidal shape.

In [16] Kang & Milton showed that the Eshelby conjecture is equivalent to the following problem which is called the *Newtonian Potential Problem*: Let Ω be a simply connected domain and $\Gamma(x)$ be the fundamental solution (the Green function) for the Laplacian, namely,

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi} \ln |x| \ , & d = 2 \ , \\ -\dfrac{1}{4\pi |x|} \ , & d = 3 \ . \end{cases} \tag{2.2}$$

If

$$-\int_{\Omega} \Gamma(x-y) dy = \frac{1}{2} \sum_{j=1}^{d} a_j x_j^2 + C, \quad x \in \Omega \tag{2.3}$$

with $a_j > 0$, then Ω must be an ellipse or an ellipsoid. In other words, if the Newtonian potential of the constant function 1 is quadratic in Ω, then Ω must be an ellipse or an ellipsoid.

Surprisingly, the Newtonian potential problem was solved by Dive in 1931 [8] and by Nikliborc in 1932 [32] (see also [7]). The reason why Dive and Nikliborc considered this problem was to prove the converse of a theorem of Newton: Let Ω be a simply connected domain whose center of mass is $0 \in \Omega$ and let $\lambda\Omega$ be a dilation of Ω by $\lambda > 1$, *i.e.*, $\lambda\Omega = \{\lambda x : x \in \Omega\}$. If the gravitational force induced by the uniform mass on $\lambda\Omega \setminus \Omega$ is equal to zero in Ω, then Ω must be an ellipsoid. A theorem due to Newton states that if Ω is an ellipsoid, then the gravitational force in Ω is zero [19].

Let us now see why the Eshelby conjecture and the Newtonian potential problem are equivalent. The solution u to (2.1) satisfies the harmonic equation in Ω and $\mathbb{R}^d \setminus \overline{\Omega}$, and the following boundary conditions along the interface $\partial\Omega$:

$$\begin{cases} u|_+ = u|_- & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial n}\Big|_+ = k \dfrac{\partial u}{\partial n}\Big|_- & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where the subscripts $\pm$ indicate the limit from outside and inside of Ω, respectively, and $\frac{\partial}{\partial n}$ denotes the derivative in the direction of the outward normal on $\partial\Omega$. The condition (2.4) means that the matrix and the inclusion are perfectly bonded.

In view of the condition (2.4), it is quite natural to represent the solution u to (2.1) in terms of the single layer potential. The single layer potential for the harmonic equation on $\partial\Omega$ is defined by

$$\mathcal{S}_\Omega[\phi](x) := \int_{\partial\Omega} \Gamma(x-y)\phi(y) d\sigma(y), \quad x \in \mathbb{R}^d, \tag{2.5}$$

for functions ϕ defined on $\partial\Omega$. Then $\mathcal{S}_\Omega[\phi]$ is continuous across $\partial\Omega$ and its normal derivative enjoys the following jump relation:

$$\frac{\partial}{\partial n} \mathcal{S}_\Omega[\phi]\Big|_+ (x) - \frac{\partial}{\partial n} \mathcal{S}_\Omega[\phi]\Big|_- (x) = \phi(x) \quad \text{a.e. } x \in \partial\Omega. \tag{2.6}$$

For the theory of layer potentials we refer to [2].

Indeed, it is known (see, for example, [17]) that the solution u to (2.1) is given by

$$u(x) = a \cdot x + (k-1)\mathcal{S}_\Omega \left[\frac{\partial u}{\partial n}\Big|_- \right](x), \quad x \in \mathbb{R}^d. \tag{2.7}$$

Therefore, if u is linear in Ω, it follows from (2.7) that

$$\mathcal{S}_\Omega[n_j](x) = \text{linear in } \Omega, \quad j = 1, \cdots, d. \tag{2.8}$$

Then by the divergence theorem, we end up with the formula (2.3).

The connection between the Eshelby conjecture and the Newtonian potential problem and the solution of the latter problem by Dive and Nikliborc yields the proof of the Eshelby's conjecture. The precise statement of the theorem obtained in [16] is as follows.

Theorem 2.1 *Let Ω be a simply connected bounded Lipschitz domain in $\mathbb{R}^d$, $d = 2, 3$. The solution u to (2.1) is linear in Ω for any vector a if and only if Ω is an ellipse or ellipsoid.*

We now describe the original Eshelby's conjecture in the context of the linear isotropic elasticity. Consider an elastic inclusion Ω, whose Lamé parameters are $\widetilde{\lambda}, \widetilde{\mu}$, embedded in a medium in $\mathbb{R}^d$ with Lamé parameters λ, μ. The elasticity tensor of the inclusion-matrix composite $C = (C_{ijkl})$ is given by

$$C_{ijkl} := \Big(\lambda\, \chi(\mathbb{R}^d \setminus \overline{\Omega}) + \widetilde{\lambda}\, \chi(\Omega)\Big)\delta_{ij}\delta_{kl} + \Big(\mu\, \chi(\mathbb{R}^d \setminus \overline{\Omega}) + \widetilde{\mu}\, \chi(\Omega)\Big)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,. \tag{2.9}$$

For given constants $d \times d$ matrix A, consider the following problem for the Lamé system of the linear elasticity:

$$\begin{cases} \nabla \cdot \big(C(\nabla \mathbf{u} + \nabla \mathbf{u}^T))\big) = 0 & \text{in } \mathbb{R}^d, \\ \mathbf{u}(x) - Ax = O(|x|^{1-d}) & \text{as } |x| \to \infty. \end{cases} \tag{2.10}$$

If $\mathbf{u}$ is the solution to (2.10), then $\nabla \mathbf{u}$ represents the field perturbed due to the presence of the inclusion Ω under the uniform loading given by $\nabla(Ax)$. The conductivity model (2.1) in two dimensions can be regarded as the anti-plane elasticity model of (2.10). In [9], Eshelby showed that if Ω is an ellipse or an ellipsoid, then for any given uniform loading the elastic field inside Ω is uniform, and in [10] he conjectured that ellipses and ellipsoids are the only domains with this property.

The following theorem was obtained in [16].

Theorem 2.2 *Let Ω be a simply connected bounded Lipschitz domain in $\mathbb{R}^d$, and suppose that $\lambda - \widetilde{\lambda}$ and $\mu - \widetilde{\mu}$ have the same signs. The solution $\mathbf{u}$ to (2.10) is linear in Ω for all A if and only if Ω is an ellipse or an ellipsoid.*

Theorem 2.2 was proved using the single layer potential for the Lamé system. The special structure of the Kelvin matrix of the fundamental solutions to the Lamé system made it possible to reduce the proof to the Newtonian potential problem.

Eshelby's conjecture was proved by making use of its connection to the Newtonian potential problem. Yet there is another equivalence to be exploited. Let

$$w_\Omega(x) := -\int_\Omega \Gamma(x - y)dy, \quad x \in \mathbb{R}^d. \tag{2.11}$$

When (2.3) holds, w_Ω is the solution to

$$\begin{cases} \Delta w = \chi(\Omega), \\ w(x) = \dfrac{1}{2}\displaystyle\sum_{j=1}^d a_j x_j^2 + C \quad x \in \Omega, \end{cases} \tag{2.12}$$

with an appropriate condition at infinity which is determined by the total of the inclusion and the behavior of $\Gamma(x)$ when $|x| \to \infty$. The problem (2.12) is an obstacle problem and Liu showed in [22] that ellipses or ellipsoids are the only free boundary solutions to the problem to prove Theorem 2.1 and Theorem 2.2.

It should be noted that only the three dimensional case in Theorem 2.1 and Theorem 2.2 is new. In two dimensions the strong Eshelby conjecture was proved by Sendeckyj for elasticity [40]. What we call the strong Eshelby conjecture asserts that if the solution $\mathbf{u}$ to (2.10) for a single nonzero A is linear inside Ω, then Ω is an ellipse or an ellipsoid. The strong Eshelby conjecture for antiplane elasticity was proved by Ru & Schiavone [39] by the same method as that of Sendeckyj, which uses the conformal mapping. Alternative proofs for the strong Eshelby conjecture in two dimensional elasticity are given by Kang & Milton [15, 16] and Liu [22].

With help of the Newtonian potential problem (2.3), one can see rather clearly why the strong Eshelby conjecture should be true in two dimensions. To see that, let w_Ω be as in (2.11). The solution u being linear in Ω for a certain direction a is equivalent to $b \cdot \nabla w_\Omega$ being linear in Ω for some nonzero vector b. But since $\Delta w_\Omega(x) = 1$ in Ω, we immediately conclude that $b^\perp \cdot \nabla w_\Omega$ is linear where $b^\perp$ is the orthogonal vector to b, and hence $\xi \cdot \nabla w_\Omega$ is linear in Ω for any vector ξ in two dimensions. However in three dimensions, even if $b \cdot \nabla w_\Omega$ is linear for some b, there are two other linearly independent directions to be determined, and there is Ω such that ∇w_Ω is *not* linear in Ω even if $b \cdot \nabla w_\Omega$ is linear for some nonzero b. Such an inclusion Ω was constructed by Liu [22] to show that the strong Eshelby conjecture is not true in three dimensions.

Since Sendeckyj's work in two dimensions, there have been some partial solutions of the Eshelby conjecture in three dimensions, mainly dealing with polyhedra and domains whose boundaries have flat parts. Rodin [38] proved that the field cannot be uniform inside polygons or polyhedra, and exact expressions for these non-uniform fields were later obtained [33, 18, 34]. Markenscoff showed that the field cannot be uniform if any portion of the boundary was planar [27] and that the only small perturbations of any ellipsoid boundary that preserve field uniformity in the interior are those which perturb the ellipsoid into another ellipsoid [28]. Lubarda and Markenscoff [24] showed that the field cannot be uniform for inclusions bounded by polynomial surfaces of higher than second degree, nor for inclusions bounded by segments of two or more different surfaces, and argued that non-convex inclusions are also excluded. It is worth emphasizing that Theorem 2.1 and Theorem 2.2 hold for domains with Lipschitz boundaries which include polygons or polyhedra.

It was also proved in [16] that even when the conductivity of the matrix and inclusions are anisotropic, the same result as that of Theorem 2.1 holds. However, if the elasticity tensor of either the matrix or the inclusion is anisotropic (or transversally isotropic), it does not seem to be known whether the Eshelby conjecture is true or not.

3 The Pólya-Szegö conjecture

In Fig. 1, not only the field inside the inclusion but also the one outside undergoes perturbation and we now turn our attention to the field outside the inclusion. The solution u to (2.1) has a dipole asymptotic expansion at infinity:

$$u(x) = a \cdot x + \frac{1}{\omega_d} \frac{\langle a, Mx \rangle}{|x|^d} + O(|x|^{-d}), \quad \text{as } |x| \to \infty. \tag{3.1}$$

Here ω_d is the area of the $d-1$ dimensional unit sphere and M is a constant $d \times d$ symmetric matrix independent of a and x. The matrix $M = M(\Omega, k) := (M_{ij})$ is called the polarization tensor (PT) associated with the inclusion Ω and the conductivity contrast k. See [1, 29]. It is worthwhile noting that the concept of PT can be defined even when Ω consists of multiple components and depends only on Ω and the conductivity contrast k.

The concept of the PT appears in various contexts such as the theory of composites as the low volume faction limit of the effective conductivity (see [29, 2] and references therein) and the study of potential flow [37]. Another important recent usage of the concept is for the electrical impedance tomography problem to detect diametrically small inclusions by means of boundary measurements. In fact, the leading order approximation of the boundary voltage induced by the injected current is expressed in terms of the location and the PT of the inclusions, and hence one can approximately detect, by boundary measurements, the location and the PT of the inclusion. Since the PT carries important geometric information, such as the volume of the inclusion, we are able to recover that information from boundary measurements. It was Friedman & Vogelius [11] who first used the PT for the detection of small inclusions. We refer to [1, 2] and references therein for recent developments of this theory.

In their book [37] Pólya and Szegö conjectured that the inclusion whose PT has the minimal trace takes the shape of a disk or a ball. In connection with this conjecture various kinds of isoperimetric inequalities for the PT have been obtained. See, for example, [35, 36, 41]. After about 40 years since Pólya and Szegö wrote their conjecture, the optimal isoperimetric inequalities for the PT have been obtained by Lipton [21], and later by Capdeboscq & Vogelius [5] using the variational argument similar to that of Kohn & Milton [20]. The bounds are called the Hashin-Shtrikman (HS) bounds after names of the scientists who first found the optimal bounds on the effective conductivity of isotropic two-phase composites [13]. These bounds can be derived as the low volume fraction limit of the bounds of the effective conductivity [25, 26, 30, 31]. The HS bounds for PT are given as follows: Let $|\Omega|$ denote the volume of Ω. Then

$$\mathrm{Tr}(M) \le |\Omega|(k-1)(d-1+\frac{1}{k}), \tag{3.2}$$

and

$$|\Omega|\mathrm{Tr}(M^{-1}) \le \frac{d-1+k}{k-1}, \tag{3.3}$$

where Tr denotes the trace and $|\Omega|$ is the volume of Ω. These bounds are optimal in the sense that every point inside the bounds (except the upper bound) is the pair (or triple) of eigenvalues of the PT associated with a certain domain [3, 5]. We note that the PTs for the ellipses and ellipsoids can be computed explicitly and they satisfy the lower HS-bound (3.3). For example, if Ω is an ellipse whose semi-axes are on the x_1- and x_2-axes and of length a and b, respectively, then its polarization tensor M takes the form

$$M = (k-1)|\Omega| \begin{pmatrix} \dfrac{a+b}{a+kb} & 0 \\ 0 & \dfrac{a+b}{b+ka} \end{pmatrix}, \tag{3.4}$$

and hence it satisfies the equality in (3.3) [4]. Thus, from the viewpoint of the isoperimetric inequality, we ask ourselves the following stronger version of the Pólya-Szegö conjecture: If the PT satisfies the equality in (3.3), then the domain must be an ellipse or an ellipsoid. This stronger version of the Pólya-Szegö conjecture was proved by Kang & Milton [16].

Theorem 3.1 *Let Ω be a simply connected bounded Lipschitz domain in $\mathbb{R}^d$, $d = 2, 3$. If the polarization tensor $M(\Omega)$ of Ω satisfies the equality in (3.3), then Ω must be an ellipse or an ellipsoid.*

The original Pólya-Szegö conjecture asserts that if in addition to (3.3) the eigenvalues of the PT are all the same, then the inclusion must be a disk or a ball. This conjecture immediately follows from Theorem 3.1. We also note that Theorem 3.1 is not true if the domain is not simply connected as we will see in the next section.

In proving Theorem 3.1 we relate the stronger version of the Pólya-Szegö conjecture with the Eshelby conjecture. The following theorem was proved in [15].

Theorem 3.2 *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d = 2, 3$. If the polarization tensor $M(\Omega)$ of Ω satisfies the equality in (3.3), then for any vector $a \in \mathbb{R}^d$ the solution u to (2.1) is linear in Ω.*

The stronger version of the Pólya-Szegö conjecture follows from Theorem 2.1 and the above theorem.

The finding of Theorem 3.2 is rather surprising in some sense. In view of the dipole expansion (3.1), the PT M provides only the first order approximation the perturbation of the solution, namely, $u(x) - a \cdot x$, at infinity. Therefore it is not so easy to imagine that the PT determines the behavior of the field inside the inclusion, and it is not the case in general. However, if the PT satisfies the lower HS-bound, then the field inside the inclusion must be uniform.

In order to prove Theorem 3.2, we follow the variational argument to derive HS-bounds in [5]. One can show that if the equality holds in (3.3), then for any direction a the maximizer of the functional

$$G(v) = -\frac{1}{k-1}\int_\Omega |v|^2 \, dx + 2a \cdot \int_\Omega v \, dx - \int_{\mathbb{R}^d} \nabla\Delta^{-1}(\nabla \cdot v) \cdot v \, dx \tag{3.5}$$

is given by $b\chi(\Omega)$ where b is a constant vector in $\mathbb{R}^d$, from which Theorem 3.2 follows.

4 Extremal structures with many components

Theorems in previous sections show that ellipses and ellipsoids are unique structures, among simply connected ones, which are extremal in two equivalent senses: in the sense of Eshelby (the field inside is uniform) and that of Pólya-Szegö (the PT satisfies the lower HS-bound). There are other extremal structures which have multiple components. These structures are important as counter examples to the conjectures of Eshelby and Pólya-Szegö. Even more important aspect of these structures is that these extremal structures have the minimal internal energy [12].

We are constructing inclusions $\Omega_1, \ldots, \Omega_m$ in $\mathbb{R}^d$, with the same conductivities k such that for any uniform loading (or uniform electric field) $-a$, the field inside Ω_j is uniform for $j = 1, \ldots, m$. In two dimensions, it amounts to constructing bounded domains $\Omega_1, \ldots, \Omega_m$ in the complex plane such that for a given complex number α they admit a function f holomorphic in $\mathbb{C} \setminus \cup_{s=1}^m \overline{\Omega}_s$ satisfying

$$f(z) - \alpha z = O(1) \quad \text{as } |z| \to \infty, \tag{4.1}$$

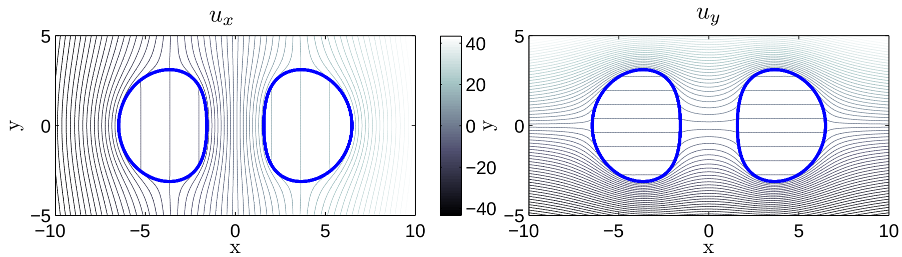

Figure 3: Equipotential lines of u_x and u_y where u_x and u_y are solutions corresponding to the $\mathbf{e}_1$ and $\mathbf{e}_2$ fields.

and

$$f(z) = \Re(\beta z) + q_s \quad \text{on } \partial\Omega_s \tag{4.2}$$

for some complex number β and q_s, $s = 1, \ldots, m$. Note that we require the number β to be the same for all $s = 1, \ldots, m$, which is equivalent to saying that the uniform fields inside the inclusions have the same direction and magnitude. It is not clear whether we can have structures with many components such that uniform fields inside inclusions have different directions. Constructing Ω_s, $s = 1, \ldots, m$, and f satisfying (4.1) and (4.2) is a free boundary problem.

Cherepanov [6] solved the free boundary problem to construct the extremal structures with arbitrary number of components. His method uses a powerful theorem in complex analysis: the region outside of multiple simply connected domains in the complex plane is conformally equivalent to a complex plane with the same number of slits. In recent work [14] (without knowledge of Cherepanov's results), Kang-Kim-Milton explicitly solved the free boundary problem to construct a two parameter family of structures with two components using the Schwarz-Christoffel formula and the Weierstrass ζ-function, and exploited their various connections with the conjectures of Eshelby and Pólya-Szegö. Figure 4 shows a typical shape of the structure and the fields inside the inclusion: they are uniform!

Another way of constructing the extremal structures is to use the Newtonian potential. Similarly to the Newtonian potential problem, inclusions $\Omega_1, \ldots, \Omega_m$ should satisfy

$$-\sum_{s=1}^{m} \int_{\Omega_s} \Gamma(x-y)dy = x \cdot Qx + b_s \cdot x + c_s \quad \text{in } \Omega_s, \tag{4.3}$$

where Q is a constant $d \times d$ matrix, b_s is a constant vector and c_s is a constant for $s = 1, \ldots, m$. Note that Q is independent of s since we imposed the condition that the direction of fields are the same inside all inclusions. If we put

$$w(x) = -\sum_{s=1}^{m} \int_{\Omega_s} \Gamma(x-y)dy, \tag{4.4}$$

then w satisfies

$$\begin{cases} \Delta w = \sum_{s=1}^{m} \chi(\Omega_s), \\ w(x) = x \cdot Qx + b_s \cdot x + c_s \quad x \in \Omega_s, \quad s = 1, \ldots, m, \end{cases} \tag{4.5}$$

with an appropriate condition at infinity which is determined by the total volume of the inclusions and the behavior of $\Gamma(x)$ when $|x| \to \infty$.

In [22], Liu used variational methods to solve (4.5) and numerically constructed structures with arbitrary number of components. His construction is valid in three dimensions as well as in two dimensions.

For the application to composites, construction of such extremal structures in periodic setting are important. The periodic extremal structures with a single component was constructed by Vigdergauz [42, 43]. Gravobsky and Kohn reconstructed the Vigdergauz structure in two dimensions in mathematically rigorous way [12]. They also showed in the same paper that the low volume limit of the Vigdergauz structure is an ellipse. Recently Liu *et al* constructed the periodic extremal structures in two and three dimensions using the variational approach described above [23].

Acknowledgement: I would like to take this opportunity to thank Graeme Milton for wonderful collaborations. I also thank him and Habib Ammari for helpful comments on this article.

References

[1] H. Ammari and H. Kang, *Reconstruction of small inhomogeneities from boundary measurements*, Lecture Notes in Math. 1846, Springer, 2004.

[2] H. Ammari and H. Kang, *Polarization and Moment Tensors with Applications to Inverse Problems and Effective Medium Theory*, Applied Mathematical Sciences Series, Vol 162, Springer, 2007.

[3] H. Ammari, Y. Capdeboscq, H. Kang, E. Kim, and M. Lim, Attainability by Simply Connected Domains of Optimal Bounds for Polarization Tensors, European Jour. of Applied Math, 17 (2) (2006), 201-219.

[4] M. Brühl, M. Hanke, and M.S. Vogelius, A direct impedance tomography algorithm for locating small inhomogeneities, Numer. Math., 93 (2003), 635–654.

[5] Y. Capdeboscq and M.S. Vogelius, A review of some recent work on impedance imaging for inhomogeneities of low volume fraction. Partial differential equations and inverse problems, 69–87, Contemp. Math., 362, Amer. Math. Soc., Providence, RI, 2004.

[6] G.P. Cherepanov, Inverse problems of the plane theory of elasticity, PMM Vol. 38 (6) (1974), 963-979.

[7] E. DiBenedetto and A. Friedman, Bubble growth in Porous Media, Indiana Univ. Math. J., 35 (2) (1986), 573-606.

[8] P. Dive, Attraction des ellipsoides homogènes et réciproques d'un théorème de Newton, Bull. Soc. Math. France, 59 (1931), 128–140.

[9] J.D. Eshelby, The determination of the elastic field of an ellipsoidal inclusion, and related problems. Proc. Roy. Soc. London. Ser. A. 241 (1957), 376–396.

[10] J.D. Eshelby, Elastic inclusions and inhomogeneities. In *Progress in Solid Mechanics*, ed. by I.N. Sneddon and R. Hill, Vol. II (1961) pp. 87–140 North-Holland, Amsterdam.

[11] A. Friedman and M. Vogelius, Identification of small inhomogeneities of extreme conductivity by boundary measurements: a theorem on continuous dependence, Arch. Rat. Mech. Anal. 105 (1989), 299–326.

[12] Y. Grabovsky and R. Kohn, Microstructures minimizing the energy of a two phase elastic composite in two space dimensions. II. The Vigdergauz microstructure, J. Mech. Phys. Solids 43 (1995), no. 6, 949–972.

[13] Z. Hashin and S. Shtrikman, A variational approach to the theory of the elastic behavior of multiphase materials, J. Mech. Phys. Solids 11 (1963), 127–140.

[14] H. Kang, E. Kim, and G.W. Milton, Inclusion pairs satisfying Eshelby's uniformity property, preprint.

[15] H. Kang and G.W. Milton, On Conjectures of Polya-Szego and Eshelby, in *lnverse Problems, Multi-scale Analysis and Effective Medium Theory* (H. Kang and H. Ammari Eds), Contemporary Math. 408 (2006), 75-80.

[16] H. Kang and G.W. Milton, Solutions to the Pólya-Szegö Conjecture and the Weak Eshelby Conjecture, Arch. Rational Mech. Anal. 188 (2008), 93-116.

[17] H. Kang and J.K. Seo, Recent progress in the inverse conductivity problem with single measurement, in *Inverse Problems and Related Fields*, CRC Press, Boca Raton, FL, 69–80, 2000

[18] M. Kawashita and H. Nozaki, Eshelby tensor of a polygonal inclusion and its special properties, J. Elasticity, 64 (2001), no. 1, 71–84.

[19] O.D. Kellog, *Foundation of Potential Theory*, Dover, New York, 1953.

[20] R.V. Kohn and G.W. Milton, On bounding the effective conductivity of anisotropic composites, in *Homogenization and Effective Moduli of Materials and Media*, eds. J.L. Ericksen, D. Kinderlehrer, R.V. Kohn, and J.L. Lions, IMA Volumes in Mathematics and its Applications, 1, 97–125, Springer-Verlag, Berlin, 1986

[21] R. Lipton, Inequalities for electric and elastic polarization tensors with applications to random composites, J. Mech. Phys. Solids 41 (1993), no. 5, 809–833.

[22] L.P. Liu, Solutions to the Eshelby conjectures, Proc. R. Soc. A. 464 (2008), 573-594.

[23] L.P. Liu, R. James, and P. Leo, New extremal inclusions and their applications to two-phase composites, preprint.

[24] V.A. Lubarda and X. Markenscoff, On the absence of Eshelby property for non-ellipsoidal inclusions, Int. J. Solids Structures 35 (1998), no. 25, 3405–3411.

[25] K.A. Lurie and A.V. Cherkaev, G-closure of a set of anisotropic conducting media in the case of two-dimensions, Dokl. Akad. Nauk SSSR 259 (1981) 328–331, translated in Soviet Phys. Dokl. 26 (1981) 657–659.

[26] K.A. Lurie and A.V. Cherkaev, Exact estimates of the conductivity of a binary mixture of isotropic materials, Proc. Roy. Soc. Edinburgh A 104 (1986), 21–38.

[27] X. Markenscoff, On the shape of Eshelby inclusions, J. Elasticity 49 (1997), 163–166.

[28] X. Markenscoff, Inclusions with constant eigenstress, J. Mech. Phys. Solids 46 (1998), no. 12, 2297–2301.

[29] G. W. Milton, *The Theory of Composites*, Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, 2002.

[30] G.W. Milton, R.C. McPhedran, and D.R. McKenzie, Transport properties of arrays of intersecting cylinders, Appl. Phys. 25 (1981), 23–30.

[31] F. Murat and L. Tartar, Calcul des variations et homogénísation. In *Les méthodes de l'homogénéisation: théorie et applications en physique* Collection de la Direction des études et recherches d'Électricité de France, Eyrolles, Paris, 57 (1985) 319–370. [Translated in *Topics in the Mathematical Modeling of Composite Materials*, ed. by A. Cherkaev and R. Kohn., *Progress in Nonlinear Differential Equations and Their Applications* Volume 31, pp. 139–173, Birkhäuser. ISBN 0-8176-3662-5.]

[32] W. Nikliborc, Eine Bemerkung über die Volumpotentiale, Math. Zeit. 35 (1932), 625–631.

[33] H. Nozaki and M. Taya, Elastic fields in a polygon-shaped inclusion with uniform eigenstrains, ASME J. Appl. Mech. 64 (1997), 495–502.

[34] H. Nozaki and M. Taya, Elastic fields in a polyhedral inclusion with uniform eigenstrains and related problems, ASME J. Appl. Mech. 68 (2001), 441–452.

[35] L. Payne, Isoperimetric inequalities and their applications, SIAM Rev. 9 (1967), 453–488.

[36] L. Payne and G. Philippin, Isoperimetric inequalities for polarization and virtual mass, J. Anal. Math. 47 (1986), 255–267.

[37] G. Pólya and G. Szegö, *Isoperimetric Inequalities in Mathematical Physics*, Annals of Mathematical Studies, Number 27, Princeton University Press, Princeton, 1951.

[38] G.J. Rodin, Eshelby's inclusion problem for polygons and polyhedra, J. Mech. Phys. Solids, 44 (1996), no. 12, 1977–1995.

[39] C.-Q. Ru and P. Schiavone, On the elliptic inclusion in anti-plane shear. Math. Mech. Solids 1 (1996), no. 3, 327–333.

[40] G.P. Sendeckyj, Elastic inclusion problems in plane elastostatics, Int. J. Solids Structures 6 (1970), 1535–1543.

[41] M. Schiffer and G. Szegö, Virtual mass and polarization, Trans. Amer. Math. Soc. 67 (1949), 130–205.

[42] S.B. Vigdergauz, Effective elastic parameters of a plate with a regular system of equal-strength holes, Inzhenernyi Zhurnal. Mekhnika Tverdogo Tela 21 (1986), 165–169.

[43] S.B. Vigdergauz, Two dimensional grained composites of extreme rigidity, Journal of Applied Mechanics 61 (2) (1994), 390-394.