SPECTRAL PERMANENCE IN A SPACE WITH TWO NORMS

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Abstract. A generalization of a classical argument of Mark G. Krein leads us to the conclusion that the Neumann-Poincaré operator associated to the Lamé system of linear elastostatics equations in two dimensions has the same spectrum on Lebesgue space of the boundary as well the more natural energy space. A similar result for the Neumann-Poincaré operator associated to the Laplace equation was stated by Poincaré's and was proved rigorously a century ago my means of a symmetrization principle for non-selfadjoint operators. We develop the necessary theoretical framework underlying the spectral analysis of the Neumann-Poincaré operator, including also a discussion of spectral asymptotics of a Galerkin type approximation. Several examples from function theory of a complex variables and harmonic analysis are included.

1. Introduction

The present note is motivated by some recently accumulated spectral analysis facts referring to the Neumann-Poincaré operator (henceforth denoted by the NP operator). This is a boundary integral operator acting on the Lebesgue spaces of the boundary of a domain in Euclidean space, explicitly appearing in the double layer potential of a charge with some prescribed regularity. It is a compact operator if the boundary of the domain is smooth enough, and a singular integral operator if the boundary has corners. It is not our aim to present the full theory of the NP operator, but merely to extract from its (sometimes ad-hoc) spectral analysis a general framework which might be of interest for wider classes of problems.

A basic observation, going back more than a century ago, and attributed to Poincaré, is that a natural space to consider the NP operator is an energy space, rather than any natural functional space on the boundary of the underlying domain. This was one of the first occurrences of the necessity to study a linear transformation in more than a single normed space. In modern terminology, the energy space isolated by Poincaré is a negative fractional Sobolev space, a concept which took shape a few good decades later. The article [13] contains details about Poincaré's variational problem, its modern interpretation and his amazing intuition on what we call today the completeness of the root vectors of the non-symmetric NP operator.

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It was Mark G. Krein who found in 1947 (and probably much earlier) a theoretical explanation for the spectral behavior of “symmetrisable compact linear transformations”, a term already used in potential theory by a good dozen of authors. For an authoritative account of these early efforts see the chapter with the same title in Zaanen’s book [21]. Krein’s arguments are simple, irreducible in their beauty and applicability, and have been rediscovered by P. Lax, J. Dieudonné, and possibly others, see for references [13].

We adapt Krein’s proofs to the case on non-compact and not necessarily symmetrisable operators (in a Hilbert space endowed with a weaker inner product norm), and as an application we prove that the NP operators associated to the Laplace equation and the Lamé system of linear elastostatics have the same spectra in Lebesgue L^2 space and the energy space. This is the first proof of spectral permanence for the Lamé system of equations.

In practice, linear operators are approximated by finite rank truncations, via adapted Galerkin type approximation methods [15, 20]. It is notorious that the spectrum of these truncations may not converge to the spectrum of the whole operator. Think for instance to the unilateral shift. Moreover, the choice of the increasing sequence of finite rank projections may significantly alter the spectral asymptotics. A second theme of our note adds a few observations on these topic and has a numerical analysis flavor. Specifically, we study the simultaneous finite rank approximation of an operator acting on a Hilbert space endowed with a weaker norm. We analyze the norm gap between the two sequences of finite central truncations of a linear bounded operator with respect to an ascending nest of finite dimensional subspaces and the orthogonal projections onto them in the two different metrics. Here the examples from complex or harmonic analysis abound, and we only discuss a perturbation argument (inspired by the last part of Krein’s article [14]). A partial conclusion is that performing the finite central truncation with respect to the subspaces spanned by the simultaneous orthogonal system of vectors (in the two metrics) will not distort their limiting spectra. A second conclusion is that a small perturbation (in a precise multiplicative sense) of the weaker inner product will also leave invariant the spectral asymptotics of the finite truncations. References on finite central truncations, known also under the name of the moment method, are [4, 15, 20].

2. Preliminaries

Let H be a complex separable Hilbert space and $T \in \mathcal{L}(H)$ a linear bounded operator acting on H. The spectrum of T is denoted by $\sigma(T)$ and the numerical range of T is by $W(T) = \{\langle T\varphi, \varphi\rangle,\ \|\varphi\| = 1\}$, where $\langle\ ,\ \rangle$ denotes the inner product on H. By a theorem of Hausdorff and Toeplitz we know that the closure of $W(T)$ is a compact set, containing $\sigma(T)$. In many applications, notably in the stability analysis of semigroups, it turns out that locating the numerical range of an operator (bounded or not) is much easier than computing the spectrum. For a proof of Hausdorff-Toeplitz theorem and a variety of insights into numerical range estimates we refer to the monograph [9].

We endow H with a weaker pre-hilbertian space norm:

$$(\varphi, \psi) = \langle A\varphi, \psi\rangle, \quad \varphi, \psi \in H, \tag{2.1}$$

where A is a positive, bounded linear symmetric operator acting on H. Let K denote the Hilbert space completion of H with respect to the new norm. We have $H \subset K$, with dense range inclusion. If the operator A has a bounded inverse, then $H = K$, although not isometrically. This scenario is less interesting for our aims, and we assume henceforth that the operator A is not invertible.

An outstanding example is $H = L^2(\partial\Omega)$, where Ω is a bounded domain in $\mathbb{R}^d$ ($d \geq 2$) with the Lipschitz boundary and A being the single layer potential on $\partial\Omega$, namely,

$$A[\varphi](x) = \mathcal{S}[\varphi](x) := \int_{\partial\Omega} \Gamma(x-y)\varphi(y)\, d\sigma(y) \ , \quad x \in \partial\Omega, \tag{2.2}$$

where $\Gamma(x)$ is the fundamental solution to the Laplacian, *i.e.*,

$$\Gamma(x) = \begin{cases} -\dfrac{1}{2\pi} \ln |x| \ , & d = 2 \ , \\ \dfrac{1}{(d-2)\omega_d} |x|^{2-d} \ , & d \geq 3 \ , \end{cases} \tag{2.3}$$

with ω_d being the area of the unit sphere in $\mathbb{R}^d$. We emphasize that the single layer potential here has the opposite sign to that in [1]. This is to make it a positive operator. The single layer potential $\mathcal{S}$ is symmetric and positive in three dimensions, and the completion of $L^2(\partial\Omega)$ with respect to the inner product $(\ , \ )$ is $K = H^{-1/2}(\partial\Omega)$. In two dimensions, $\mathcal{S}_{\partial\Omega}$ may not be positive. However, we slightly vary the definition of the single layer potential so that the varied one, which we still denote by $\mathcal{S}$, becomes positive (see [2]). It is known (see [11]) that the norm induced by $(\ , \ )$ is equivalent to the usual Sobolev norm on $H^{-1/2}(\partial\Omega)$.

3. Boundedness and spectral permanence

It is known since Krein's landmark article [14] that an operator $T \in \mathcal{L}(H)$ which is symmetric with respect to the weaker norm is automatically bounded with respect to it. Moreover, he proved in the same work that the additional compactness assumption on $T \in \mathcal{L}(H)$ implies the compactness and spectral permanence of T, with respect to the weaker norm. We slightly generalize below these two observations, having as an example the NP operator.

Proposition 3.1. *Assume that two linear bounded operators $S, T \in \mathcal{L}(H)$ satisfy*

$$AT = SA. \tag{3.1}$$

Then T extends to a bounded linear transform of the Hilbert space K, that is there exists a linear bounded map $M \in \mathcal{L}(H)$ satisfying

$$\sqrt{A}T = M\sqrt{A}. \tag{3.2}$$

Proof. Since $T^*A = AS^*$, the operators $X = T + S^*$ and $Y = i(T - S^*)$ satisfy

$$AX = X^*A \quad \text{and} \quad AY = Y^*A,$$

which imply that X and Y are symmetric with respect to $(\ , \ )$. So, according to Krein's observation [14], both X and Y extend continuously to the Hilbert space K. Let us include a simple proof. For every vector $\varphi \in H$, we have

$$\begin{aligned}(X\varphi, X\varphi)^2 &= \langle AX\varphi, X\varphi\rangle^2 = \langle A\varphi, X^2\varphi\rangle^2 \\ &= (\varphi, X^2\varphi)^2 \leq (\varphi, \varphi)(X^2\varphi, X^2\varphi).\end{aligned}$$

Consequently, leaving aside the trivial case of vanishing denominators, we have

$$\frac{(X\varphi, X\varphi)}{(\varphi, \varphi)} \leq \frac{(X^2\varphi, X^2\varphi)}{(X\varphi, X\varphi)} \leq \ldots \leq \frac{(X^n\varphi, X^n\varphi)}{(X^{n-1}\varphi, X^{n-1}\varphi)}.$$

The product of all factors is telescopic, and yields

$$\left[\frac{(X\varphi, X\varphi)}{(\varphi, \varphi)}\right]^n \leq \frac{(X^n\varphi, X^n\varphi)}{(\varphi, \varphi)} \leq \frac{\|A\|\|X\|^{2n}\|\varphi\|}{(\varphi, \varphi)},$$

where $\| \ \|$ is the norm on H. Hence we have

$$\frac{(X\varphi, X\varphi)}{(\varphi, \varphi)} \leq \liminf_{n\to\infty} \left[\frac{\|A\|\|X\|^{2n}\|\varphi\|}{(\varphi, \varphi)}\right]^{1/n} = \|X\|^2.$$

Thus X extends to a linear bounded operator of the space K, with the norm not exceeding the norm on H. Then $T = \frac{X-iY}{2}$ is bounded on K, that is,

$$\langle AT\varphi, T\varphi\rangle \leq C\langle A\varphi, \varphi\rangle, \quad \varphi \in H,$$

for a universal positive constant C. Or equivalently,

$$T^*AT \leq CA$$

in the operator norm. Written on vector, the above inequality becomes

$$\|\sqrt{A}T\varphi\| \leq \|\sqrt{C}\sqrt{A}\varphi\|.$$

Thus, the linear map

$$\sqrt{C}\sqrt{A}\varphi \mapsto \sqrt{A}T\varphi$$

is well-defined and contractive.

Consequently there exists a linear bounded map $M_1 \in \mathcal{L}(H)$ satisfying

$$\sqrt{A}T = M_1\sqrt{C}\sqrt{A},$$

and we can absorb the constant into the intertwiner: $M = M_1\sqrt{C}$. □

Note that T is symmetric with respect to the inner space K if and only if $M = M^*$ as an operator of H.

Corollary 3.2. *The following spectral inclusion relations hold:*

$$\sigma_p(T, H) \subset \sigma_p(M, H), \tag{3.3}$$

and

$$\sigma_{ap}(T, K) = \sigma_{ap}(M, H), \tag{3.4}$$

where σ_p and σ_{ap} denote point spectrum and approximative point spectrum, respectively.

Proof. The first inclusion is obvious from (3.2). For proving the second one, let $\varphi_n \in H$ be a sequence of vectors, normalized in the weaker norm: $(\varphi_n, \varphi_n) = \langle A\varphi_n, \varphi_n\rangle = 1$, $n \geq 1$. If $\lim_{n\to\infty}(T\varphi_n, T\varphi_n) = 0$, then and only then $\lim_{n\to\infty} \|M\sqrt{A}\varphi_n\| = 0$. This proves that the point $\lambda = 0$ belongs to the approximative point spectrum of $T \in \mathcal{L}(K)$ if and only if it belongs to the approximative point spectrum of $M \in \mathcal{L}(H)$. For the case when $\lambda \neq 0$ we simply consider $T - \lambda$. □

Theorem 3.3. *Let $T \in \mathcal{L}(H)$ be a symmetric operator with respect to the second inner product, namely,*

$$AT = T^*A, \tag{3.5}$$

and assume that the point spectrum of T on H is a discrete subset in the complement of the essential spectrum of $T \in \mathcal{L}(H)$. Then the point spectrum of the extension $T \in \mathcal{L}(K)$ is real, and equal to the point spectrum of $T \in \mathcal{L}(H)$, with equal multiplicities, respectively.

Proof. By a *root vector*, corresponding to the eigenvalue λ, we mean a non-zero element of $\ker(T - \lambda)$. Due to the symmetry of T with respect to the inner product $(\cdot, \cdot)$, we have $\ker(T-\lambda) \neq 0$ only for real values of λ. We prove that root subspaces of $T \in \mathcal{L}(H)$ and $T \in \mathcal{L}(K)$ coincide.

By assumption, every element $\lambda \in \sigma_p(T, H)$ does not belong to the essential spectrum of T, and hence $\ker(T - \lambda)$ is finite dimensional and $\operatorname{ran}(T - \lambda)$ is a closed subspaces of H, of finite codimension. Moreover, $T - \lambda$ is of Fredholm index 0. Let

$$V_\lambda = \{\varphi \in H : \ (\varphi, \psi) = 0 \text{ for all } \psi \in \ker(T - \lambda, H)\}.$$

Notice that V_λ is a closed finite codimensional subspace of H, invariant under T. Moreover, $\ker(T - \lambda, V_\lambda) = 0$ and hence $(T - \lambda)V_\lambda = V_\lambda$ by the invariance of the Fredholm index under finite rank perturbations. Then the operator $(T - \lambda, V_\lambda)^{-1}$ is bounded on V_λ and it is also symmetric, hence bounded by Proposition 3.1 in the weak norm of the space $K \ominus \ker(T - \lambda, H)$. In conclusion, we have $\ker(T - \lambda, H) = \ker(T - \lambda, K)$. □

Corollary 3.4. *Assume, in addition to assumptions of Theorem 3.3, that $\sigma_p(T, H)$ is dense in $\sigma(T, H)$. Then $\sigma(T, H) = \sigma(T, K)$.*

Proof. We have from Theorem 3.3 the equality of spectra $\sigma_p(T, H) = \sigma_p(T, K)$, and by density $\sigma(T, H) \subset \sigma(T, K)$. On the other hand, we infer from (3.4) that $\sigma(T, K) \subset \sigma(T, H)$. □

4. Finite section method

In this section we study the asymptotic equivalence of finite central truncations of a linear operator T defined on the stronger space H and the weaker space K whose inner product is defined in terms of A. It is helpful to have in mind the examples $H = L^2$ and $K = H^{-1/2}$ explained in section 2.

Let $H_n \subset H$ be an increasing sequence of finite-dimensional subspaces, whose union is dense in H. In most applications H_n is the *Krylov subspace*, that is the span of

$\{\xi, T\xi, \ldots, T^{n-1}\xi\}$, where ξ is a non-null vector of H. In this scenario, and others, the chain of subspaces (H_n) is related to the operator T by the assumption

$$T(H_n) \subset H_{n+1}, \quad n \geq 0. \tag{4.1}$$

This means that the block matrix decomposition of T with respect to the orthogonal direct sum $H = H_0 \oplus (H_1 \ominus H_0) \oplus (H_2 \ominus H_1) \oplus \ldots$ has only the first sub-diagonal non-zero. This structure is known in numerical analysis as a *block Hessenberg matrix*. Note that we do not ask T to be a symmetric transformation. If it were so, the matrix associated to T would have only the main and adjacent block-diagonals non-zero, a classical structure known under the name of a *block Jacobi matrix*.

We denote by P_n the orthogonal projection of H onto H_n, and by Q_n the orthogonal projection of K onto H_n. Note that $P_n \to I$ in the strong operator topology of $\mathcal{L}(H)$. Two sets of projections satisfy:

$$P_n Q_n = Q_n, \quad Q_n P_n = P_n, \tag{4.2}$$

when regarded as linear endomorphisms of H. The compression $A_n = P_n A P_n$ of the operator A to the subspace H_n is positive, hence invertible.

Lemma 4.1. *For every vector $\varphi \in H$ one has*

$$Q_n \varphi = A_n^{-1} P_n A \varphi. \tag{4.3}$$

Proof. For every vector $\psi \in H_n$ we obtain

$$\begin{aligned}(\varphi - A_n^{-1} P_n A\varphi, \psi) &= \langle A(\varphi - A_n^{-1} P_n A\varphi), \psi\rangle \\ &= \langle P_n A\varphi - P_n A P_n A_n^{-1} P_n A\varphi, \psi\rangle = \langle P_n A\varphi - P_n A\varphi, \psi\rangle = 0.\end{aligned}$$

This completes the proof. □

We are concerned with the asymptotic behavior of the spectra of the finite central truncations. We focus on the distance between the finite central truncations

$$T_n := P_n T P_n \quad \text{and} \quad \tilde{T}_n := Q_n T Q_n. \tag{4.4}$$

The starting point is the following simple identity.

Lemma 4.2. *For every $n \geq 0$ one has*

$$\tilde{T}_n - T_n = A_n^{-1} P_n A (I - P_n) T P_n. \tag{4.5}$$

Moreover, if (4.1) holds, then

$$\tilde{T}_n - T_n = A_n^{-1} P_n A (P_{n+1} - P_n) T P_n. \tag{4.6}$$

Proof. Let $\varphi, \psi \in H_n$. Then we have

$$(\tilde{T}_n \varphi, \psi) = \langle A\tilde{T}_n\varphi, \psi\rangle = \langle A P_n \tilde{T}_n \varphi, P_n\psi\rangle = \langle A_n \tilde{T}_n \varphi, \psi\rangle,$$

and on the other hand

$$(\tilde{T}_n\varphi, \psi) = (Q_n T Q_n \varphi, \psi) = (T Q_n \varphi, Q_n \psi) = (T\varphi, \psi) = \langle AT\varphi, \psi\rangle.$$

Furthermore, we have

$$\begin{aligned}\langle AT\varphi, \psi\rangle = \langle P_n A T P_n \varphi, \psi\rangle &= \langle A_n P_n T P_n \varphi, \psi\rangle + \langle P_n A (I - P_n) T P_n \varphi, \psi\rangle \\ &= \langle A_n T_n \varphi, \psi\rangle + \langle P_n A (I - P_n) T P_n \varphi, \psi\rangle\end{aligned}$$

Therefore, we have

$$A_n\tilde{T}_n = A_nT_n + P_nA(I-P_n)TP_n,$$

which immediately yields (4.5). The identity (4.6) follows from the observation $(I-P_n)TP_n = (P_{n+1}-P_n)TP_n$ which holds because of (4.1). □

The residual term

$$X_n = A_n^{-1}P_nA(P_{n+1}-P_n) \tag{4.7}$$

in (4.6) naturally appears in the Cholesky type decomposition of the one-step extension of the matrix A_n to A_{n+1}. Specifically, the matrix A_{n+1} has the following block structure factorization with respect to the orthogonal decomposition $H_{n+1} = H_n \oplus (H_{n+1}\ominus H_n)$:

$$\begin{aligned} A_{n+1} &= \begin{bmatrix} A_n & P_nA(P_{n+1}-P_n) \\ (P_{n+1}-P_n)AP_n & (P_{n+1}-P_n)A(P_{n+1}-P_n) \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ X_n^* & I \end{bmatrix} \begin{bmatrix} A_n & 0 \\ 0 & D_{n+1} \end{bmatrix} \begin{bmatrix} I & X_n \\ 0 & I \end{bmatrix}, \end{aligned} \tag{4.8}$$

where

$$D_{n+1} = (P_{n+1}-P_n)A(P_{n+1}-P_n) - X_n^*A_nX_n.$$

This matrix factorization, sometimes abridged in numerical analysis by the initials LDU, was instrumental in the classical study of Volterra type operators (see [8]), and later entered in the theory of nest algebras [4].

We are interested in conditions assuring

$$\lim_{n\to\infty} \|\tilde{T}_n - T_n\| = 0. \tag{4.9}$$

A stronger requirement would be

$$\lim_{n\to\infty} \mathrm{trace}|\tilde{T}_n - T_n| = 0, \tag{4.10}$$

which would imply that the counting measures

$$\tilde{\mu}_n = \frac{1}{n}\sum_{\lambda\in\sigma(\tilde{T}_n)} \delta_\lambda$$

and

$$\mu_n = \frac{1}{n}\sum_{\lambda\in\sigma(T_n)} \delta_\lambda$$

have the same cluster points in the weak-* topology, This is a much desired outcome in the theory of orthogonal polynomials and random matrices, see for instance [18, 19].

The weaker condition (4.9) is well suited for numerical range estimates. Indeed, remark first that

$$W(T_n) \subset W(T), \tag{4.11}$$

because in the definition of $W(T_n)$ only a subset of unit vectors in H are occurring. Assume next that $\lambda_n \in \sigma(\tilde{T}_n) \subset W(\tilde{T}_n)$. Pick a unit vector $\varphi \in H_n$ so that $\lambda_n = \langle \tilde{T}_n\varphi, \varphi\rangle$. Since

$$\langle(\tilde{T}_n - T_n)\varphi, \varphi\rangle \le \|\tilde{T}_n - T_n\|,$$

we infer from (4.11) that

$$\mathrm{dist}(\lambda_n, W(T)) \leq \|\tilde{T}_n - T_n\|.$$

Thus, if λ is the limit of a subsequence $\lambda_{n(k)}$ we find

$$\mathrm{dist}(\lambda, W(T)) \leq \limsup \|\tilde{T}_n - T_n\| = 0.$$

One obvious instance for condition (4.9) to hold is when the operator A is block-diagonal with respect to the chain of subspaces H_n:

$$A = \mathrm{diag}\ (D_0, D_1, D_2, \ldots). \tag{4.12}$$

In such a case, $X_n = 0$ for all n, and (4.9) is achieved by (4.6).

A second sufficient condition for the asymptotic equivalence (4.9) is

$$\lim_{n\to\infty} \|A_n^{-1} P_n A(P_{n+1} - P_n)\| = 0, \tag{4.13}$$

and a third

$$\sup_n \|A_n^{-1} P_n A(P_{n+1} - P_n)\| < \infty \quad \text{and} \quad \lim_{n\to\infty} \|(I - P_n)TP_n\| = 0. \tag{4.14}$$

Next we show that the latter two sufficient conditions for the asymptotic equivalence of the two sequences of finite central truncations are not affected by a structured compact perturbation of the operator A. By a *strictly lower triangular operator* with respect to the chain of subspaces $(H_n)_{n=0}^{\infty}$ we mean an element $L \in \mathcal{L}(H)$ satisfying

$$P_n L = P_n L P_{n-1}, \quad n \geq 1$$

with $P_0 \equiv 0$, or more intuitively and equivalently

$$L^* P_n = P_{n-1} L^* P_n, \quad n \geq 1,$$

which in turn implies $L^*(H_n) \subset H_{n-1}$. Note that in this case $\ker(I + L) = 0$.In fact, if $(I + L)\varphi = 0$, then

$$P_n \varphi = -P_n L P_{n-1} \varphi, \quad n \geq 1,$$

and hence

$$P_n\varphi = -P_n L P_{n-1}\varphi = P_n L P_{n-1} L P_{n-2}\varphi = \ldots (-1)^n P_n L P_{n-1} \ldots L P_0 \varphi = 0.$$

If L is in addition compact, then $I + L$ is invertible by the Fredholm alternative.

The next result affects only perturbations of the weaker norm induced by the operator A.

Theorem 4.3. *Let $(x, y) = \langle Ax, y\rangle$ be a second, weaker inner product structure on a Hilbert space H, implemented by the positive operator $A \in \mathcal{L}(H)$. Suppose that one of the following two assumptions holds: either*

$$\lim_{n\to\infty} \|A_n^{-1} P_n A(P_{n+1} - P_n)\| = 0 \tag{4.15}$$

or

$$\sup_n \|A_n^{-1} P_n A(P_{n+1} - P_n)\| < \infty. \tag{4.16}$$

Then for any strictly lower triangular compact operator $L \in \mathcal{L}(H)$, the multiplicative perturbation $B = (I + L)A(I + L^*)$ fulfills the same conditions, respectively:

$$\lim_{n\to\infty} \|B_n^{-1} P_n B(P_{n+1} - P_n)\| = 0, \tag{4.17}$$

or

$$\sup_n \|B_n^{-1}P_nB(P_{n+1}-P_n)\| < \infty. \tag{4.18}$$

Proof. Recall that any compact operator K satisfies

$$\lim_{n\to\infty} \|(I-P_n)K\| = \lim_{n\to\infty} \|K(I-P_n)\| = 0.$$

In particular, the finite central truncations $K_n = P_nKP_n$ converge in operator norm to K. To prove this it is enough to approximate K by finite rank operators.

Assume that either (4.15) or (4.16) holds true. Denote, as customary by now, $L_n = P_nLP_n$ and consider the matrix decomposition

$$I + L_{n+1} = \begin{bmatrix} I+L_n & 0 \\ G_n & F_n \end{bmatrix}$$

where $F_n = (P_{n+1}-P_n)(I+L)(P_{n+1}-P_n)$ and $G_n = (P_{n+1}-P_n)LP_n$. Note that $I+L_n$ is invertible, that $\lim_{n\to\infty} G_n = 0$ by the compactness assumption, and $\sup_n \|F_n\| \le \|I+L\|$.

The LD factorization is in order:

$$I + L_{n+1} = \begin{bmatrix} I+L_n & 0 \\ G_n & F_n \end{bmatrix} = \begin{bmatrix} I & 0 \\ G_n(I+L_n)^{-1} & I \end{bmatrix} \begin{bmatrix} I+L_n & 0 \\ 0 & F_n \end{bmatrix}.$$

Next we multiply $I + L_{n+1}$ by the left factor of A_{n+1} in (4.8):

$$\begin{aligned}(I+L_{n+1}) \begin{bmatrix} I & 0 \\ X_n^* & I \end{bmatrix} &= \begin{bmatrix} I & 0 \\ G_n(I+L_n)^{-1} & I \end{bmatrix} \begin{bmatrix} I+L_n & 0 \\ F_nX_n^* & F_n \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ G_nX_n^*(I+L_n)^{-1} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ F_nX_n^*(I+L_n)^{-1} & I \end{bmatrix} \begin{bmatrix} I+L_n & 0 \\ 0 & F_n \end{bmatrix}.\end{aligned}$$

All in all

$$\begin{aligned}&P_{n+1}BP_{n+1} = P_{n+1}(I+L)P_{n+1}AP_{n+1}(I+L^*)P_{n+1} \\ &= \begin{bmatrix} I & 0 \\ (G_n+F_nX_n^*)(I+L_n)^{-1} & I \end{bmatrix} \begin{bmatrix} B_n & 0 \\ 0 & F_nD_{n+1}F_n^* \end{bmatrix} \begin{bmatrix} I & (I+L_n^*)^{-1}(G_n^*+X_nF_n^*) \\ 0 & I \end{bmatrix}\end{aligned}$$

We obtain this way and from the factorization similar to 4.8 a closed form expression:

$$B_n^{-1}P_nB(P_{n+1}-P_n) = (I+L_n^*)^{-1}(G_n^*+X_nF_n^*). \tag{4.19}$$

It remains to remark that $\sup_n \|(I+L_n^*)^{-1}\| < \infty$. This can be inferred from $\lim_n \|L_n - L\| = 0$ and the factorization

$$(I+L_n) = (I+L+(L_n-L)) = (I+L)(I+(I+L)^{-1}(L_n-L)).$$

This completes the proof. □

5. Examples

5.1. **Spectrum of the Neumann-Poincaré operator.** Let Ω be a bounded domain in $\mathbb{R}^d$, $d \ge 2$, with the Lipschitz boundary. The Neumann-Poincaré (NP) operator on $\partial\Omega$, denoted by $\mathcal{K}$, is defined by

$$\mathcal{K}[\varphi](x) = \text{p.v.}\frac{1}{\omega_d}\int_{\partial D} \frac{(x-y)\cdot\nu(y)}{|x-y|^d}\varphi(y)\,d\sigma(y)\ , \quad x \in \partial\Omega, \tag{5.1}$$

where p.v. stands for the Cauchy principal value, ω_d the area of the unit sphere in $\mathbb{R}^d$, and $\nu(y)$ the outward unit normal vector to $\partial\Omega$ at y. This operator appears naturally when solving Dirichlet or Neumann boundary value problems for Laplacian using layer potentials. In fact, if we take the normal derivative of the single layer potential in (2.2), then the following jump relation holds:

$$\frac{\partial}{\partial \nu}\mathcal{S}[\varphi]\Big|_{\pm}(x) = \left(\mp\frac{1}{2}I - \mathcal{K}^*\right)[\varphi](x) \quad \text{a.e. } x \in \partial\Omega, \tag{5.2}$$

where the subscripts + and − above indicate the limits from outside and inside of Ω, respectively, and $\mathcal{K}^*$ (this is also called the NP operator) is the adjoint operator of $\mathcal{K}$ in L^2-space (see, for example, [1, 7]).

If $\partial\Omega$ is $C^{1,\alpha}$ for some $\alpha > 0$, then the operator $\mathcal{K}^*$ is compact on $L^2(\partial\Omega)$ (and $H^{-1/2}(\partial\Omega)$). It is worth mentioning that if $\partial\Omega$ is merely Lipschitz continuous, then $\mathcal{K}^*$ is a singular integral operator and its boundedness on $L^2(\partial\Omega)$ was proved in [5]. The NP operator is not symmetric with respect to the inner product on $H = L^2(\partial\Omega)$ unless Ω is a disk or a ball [16]. However, Plemelj's symmetrization principle

$$\mathcal{S}\mathcal{K}^* = \mathcal{K}\mathcal{S} \tag{5.3}$$

makes it possible for $\mathcal{K}^*$ to be realized as a symmetric operator on K [13], and K is actually the space $H^{-1/2}(\partial\Omega)$ [11]. So, the two norm scenario shows that $\mathcal{K}^*$ is bounded on $H^{-1/2}(\partial\Omega)$, and

$$\sigma(\mathcal{K}^*, H^{-1/2}(\partial\Omega)) = \sigma(\mathcal{K}^*, L^2(\partial\Omega)), \tag{5.4}$$

provided that $\partial\Omega$ is $C^{1,\alpha}$ smooth (so that $\sigma(\mathcal{K}^*, H^{-1/2}(\partial\Omega))$ is discrete with 0 as accumulation point). We emphasize that if $\partial\Omega$ has a corner, then $\mathcal{K}^*$ may have continuous spectrum as was shown for intersecting disks in [12], and we do not know if (5.4) holds in this case. We also emphasize that $\mathcal{K}^*$ exhibits a completely different spectra on L^p spaces for $p \neq 2$ (see [17]).

Let us now consider the NP operator for the Lamé system of linear elastostatics. Let $\mathbf{\Gamma} = (\Gamma_{ij})_{i,j=1}^d$ is the Kelvin matrix of fundamental solutions to the Lamé operator:

$$\Gamma_{ij}(\mathbf{x}) = \begin{cases} \dfrac{\alpha_1}{4\pi}\dfrac{\delta_{ij}}{|\mathbf{x}|} + \dfrac{\alpha_2}{4\pi}\dfrac{x_i x_j}{|\mathbf{x}|^3}, & \text{if } d = 3, \\ -\dfrac{\alpha_1}{2\pi}\delta_{ij}\ln|\mathbf{x}| + \dfrac{\alpha_2}{2\pi}\dfrac{x_i x_j}{|\mathbf{x}|^2}, & \text{if } d = 2, \end{cases} \tag{5.5}$$

where

$$\alpha_1 = \frac{1}{2}\left(\frac{1}{\mu} + \frac{1}{2\mu+\lambda}\right) \quad \text{and} \quad \alpha_2 = \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{2\mu+\lambda}\right). \tag{5.6}$$

Here λ and μ are Lamé constants. The NP operator for the Lamé system is defined by

$$\mathbf{K}[\mathbf{f}](\mathbf{x}) := \text{p.v.}\int_{\partial\Omega} \partial_{\nu_{\mathbf{y}}}\mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\mathbf{f}(\mathbf{y})d\sigma(\mathbf{y}) \quad \text{a.e. } \mathbf{x} \in \partial\Omega. \tag{5.7}$$

Here, the conormal derivative on $\partial\Omega$ is defined to be

$$\partial_\nu \mathbf{u} := (\mathbb{C}\widehat{\nabla}\mathbf{u})\mathbf{n} = \lambda(\nabla\cdot\mathbf{u})\mathbf{n} + 2\mu(\widehat{\nabla}\mathbf{u})\mathbf{n} \quad \text{on } \partial\Omega, \tag{5.8}$$

with $\mathbf{n}$ being the outward unit normal to $\partial\Omega$, and the conormal derivative $\partial_{\nu_{\mathbf{y}}}\mathbf{\Gamma}(\mathbf{x}-\mathbf{y})$ of the Kelvin matrix with respect to $\mathbf{y}$-variables is defined by

$$\partial_{\nu_{\mathbf{y}}}\mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\mathbf{b} = \partial_{\nu_{\mathbf{y}}}(\mathbf{\Gamma}(\mathbf{x}-\mathbf{y})\mathbf{b}) \tag{5.9}$$

for any constant vector $\mathbf{b}$.

There is a significant difference between NP operators for Laplace and Lamé operators: The one for the Lamé operator is not compact even if the domain has a smooth boundary (see [6]). However, it is proved in [3] that if the domain Ω in two dimensions has $C^{1,\alpha}$ boundary, then $\mathbf{K}^2 - k_0^2$ is compact where $k_0 = \frac{\mu}{2(2\mu+\lambda)}$. As an immediate consequence it is shown that the elasto-static NP operator $\mathbf{K}$ on planar domains with $C^{1,\alpha}$ boundaries has only eigenvalues accumulating at k_0 and $-k_0$. Since the symmetrization principle like (5.3) holds with the single layer potential for the Lamé system, we infer from results of this paper that

$$\sigma(\mathbf{K}^*, H^{-1/2}(\partial\Omega)^2) = \sigma(\mathbf{K}^*, L^2(\partial\Omega)^2), \tag{5.10}$$

if Ω is a planar domain and has $C^{1,\alpha}$ boundary.

5.2. **Restriction operators on spaces of analytic functions.** We illustrate below by means of a simple scenario what can go wrong with the spectral asymptotics of the finite central truncation of a simple operator, in the presence of two non-equivalent Hilbert space norms.

Let Ω be a bounded, simply connected open set of the complex plane and denote by $L^2_a(\Omega)$ the associated Bergman space. That is the space of analytic functions in Ω which are square summable with respect to the area measure dA. Let μ be an arbitrary positive measure, with infinite and closed support K contained in Ω. Then the restriction map, defined by

$$R : L^2_a(\Omega) \longrightarrow L^2(\mu), \quad Rf = f|_K, \tag{5.11}$$

is linear and compact. The operator $A = R^*R$ is positive, compact, injective and non-invertible. The assumption about the cardinality of K implies, via the uniqueness principle for analytic functions, that the linear operator A is injective.

For a detailed potential theoretic study of the restriction map we refer to [10, 18]. The scenario of two norms is now evident:

$$\langle Af, g\rangle_{2,\Omega} = \langle R^*Rf, g\rangle_{2,\Omega} = \langle f, g\rangle_{2,\mu}, \quad f, g \in L^2_a(\Omega).$$

Let $T = M_z$ denote the multiplication by the complex variable operator (sometimes called the *Bergman shift*) and let H_n be the Krylov subspaces generated by the constant function and T, that is the spaces of polynomials of degree less than n:

$$H_n = \{p \in \mathbf{C}[z],\ \deg p < n\}.$$

The finite central truncations of T with respect to the two norms are classical. First the spectra of T_n, the compressions of T to the spaces H_n, coincide with the zeros of the complex orthogonal polynomials. Their asymptotic behavior is quite involved, and not fully understood, reflecting the geometry of the boundary of Ω, the equipotential lines of the complement and the Schwarz reflection map, see for details [19]. On the other hand, finite central truncations $\tilde{T}_n$ of T in the norm of the space $L^2(\mu)$ are governed by the normality of the operator $T \in L^2(\mu)$. In particular, the spectra of $\tilde{T}_n$ cluster in the convex hull of the support of the measure μ. As the choice of the measure μ was arbitrary, the two finite central truncations are far from being asymptotically equivalent.

On the other hand, letting aside the Krylov subspace method, let ϕ_n denote the eigenfunctions of the positive compact operator A. Since A is injective, the system $(\phi_n)_{n=0}^{\infty}$ spans the Bergman space $L^2_a(\Omega)$ and at the same time it is dense in the closure of the range of the restriction operator R in $L^2(\mu)$. Choose now

$$H_n = \text{span}\{\phi_0, \phi_1, \ldots, \phi_{n-1}\}.$$

Since the operator A is diagonal with respect to the chain of subspaces H_n, Lemma 4.2 implies $T_n = \tilde{T}_n$. That is the two finite central truncations of the operator T are in this case identical.

The **ellipse** offers the optimal scenario for both cases analyzed above. Indeed, denoting E a solid ellipse with foci at ± 1, Chebyshev polynomials of the second kind are orthogonal and span the Bergman space $L^2_a(E)$, and in the same time they are orthogonal and span $L^2(\mu)$, where $d\mu = \frac{dx}{\sqrt{1-x^2}}$, see [10] for details.

5.3. **Pseudodifferential operators of order zero.** Let $T = P(x, D)$ denote a pseudo-differential operator of order zero, acting on a torus $\mathbb{T}^n$. The basis formed by the characters $e^{i\alpha\cdot x}$, $\alpha \in \mathbb{Z}^n$, diagonalizes all partial derivative operators. Hence, the finite central truncations of T in any Sobolev space $W^{s,2}(\mathbb{T}^n)$ are, according to Lemma 4.2, asymptotically equivalent. This simple observation implies that the asymptotics of the finite central truncations of $P(x, D)$, or any operator T which is bounded on every $W^{s,2}(\mathbb{T}^n)$, is not sensitive to the Sobolev scale, as soon as one works with increasing sequences of finite projections on the Fourier modes.

When dealing with the NP or Lamé operator in two real dimensions, a parametrization of the (assumed) smooth Jordan boundary by the unit circle would put these operators in the framework of this subsection. Indeed, it is well known that the NP operators for the Laplace equation or Lamé system is bounded on $H^s(\partial\Omega)$ and has values in the same space for any s if $\partial\Omega$ is C^∞. In this way the finite central truncations along the pull-back of the Fourier modes on $\partial\Omega$ will have the same spectral asymptotics, regardless of the underlying Sobolev space.

Finally, with respect to the same nest of finite dimensional Fourier modes subspaces, we note that a perturbation of the norm of the energy space of the form $\langle (I+L)S(I+L^*)f, f\rangle_{2,\partial\Omega}$, where L is *any* lower triangular and compact operator, will not alter the spectral asymptotics of the two sequences of truncated NP operators.

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