Asymptotic Expansion of Solutions to the Lamé System in the Presence of Inclusions and Applications

Hyeonbae Kang
Seoul National University

ICSSD2002, Dec. 16-Dec. 18, 2002

Joint Works with

Habib Ammari (Ecole Polytechnique)
Gen Nakamura (Hokkaido University)
Kazumi Tanuma (Gunma University)

and

Eunjoo Kim (Seoul National University)
June-Yub Lee (Ewha Womans University)

- Ω : elastic body $\mathbb{R}^d$ (with a connected Lipschitz boundary),
- (λ, μ) : Lamé constants of Ω,
- Elastic inhomogeneity D in Ω :

$$D = \cup_{j=1}^{m} D_j = \cup_{j=1}^{m} (\epsilon B_j + z_j)$$

  where B_j is a bounded Lipschitz domain in $\mathbb{R}^d$ and z_j represents the location of D_j, and ϵ is the common order of magnitude.

- $(\widetilde{\lambda}_j, \widetilde{\mu}_j)$: Lamé constants of D_j
- Assume

$$\widetilde{\mu}_j > 0, \quad d\widetilde{\lambda}_j + 2\widetilde{\mu}_j > 0, \quad (\lambda - \widetilde{\lambda}_j)(\mu - \widetilde{\mu}_j) > 0.$$

- D_j are well-separated: there exists $d_0 > 0$ such that

$$\inf_{x \in D} \operatorname{dist}(x, \partial\Omega) > d_0, \quad |z_i - z_j| > d_0.$$

Inverse Problem

Determine the location and size (or other geometry) of inclusions by means of traction-displacement relation measured on $\partial\Omega$.

We only use linear and quadratic tractions.

Transmission Problem

$$\begin{cases} \displaystyle\sum_{j,k,l=1}^{3} \frac{\partial}{\partial x_j}\left(C_{ijkl}\frac{\partial u_k}{\partial x_l}\right) = 0 \quad \text{in } \Omega, \quad i = 1,2,3, \\ \dfrac{\partial \vec{u}}{\partial \nu}|_{\partial\Omega} = \vec{g}, \end{cases}$$

where

$$\begin{aligned} C_{ijkl} := \quad & \Big(\lambda\chi(\Omega \setminus D) + \sum_{s=1}^{m} \widetilde{\lambda}_s \chi(D_s)\Big)\delta_{ij}\delta_{kl} \\ & + \Big(\mu\chi(\Omega \setminus D) + \sum_{s=1}^{m} \widetilde{\mu}_s \chi(D_s)\Big)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \end{aligned}$$

($\chi(D)$ is the characteristic function of D),

$\frac{\partial}{\partial \nu}$ denotes the conormal derivative:

$$\frac{\partial \vec{u}}{\partial \nu} := \lambda(\text{div } \vec{u})N + \mu(\nabla \vec{u} + \nabla \vec{u}^T)N \quad \text{on } \partial D,$$

(N: outward unit normal to ∂D, T: the transpose),

Theorem 1 (Asymptotic Formula).

$$\vec{u}(x) = \vec{U}(x)$$
$$+\sum_{s=1}^{m}\sum_{j=1}^{d}\sum_{|\alpha|=1}^{d}\sum_{|\beta|=1}^{d+1-|\alpha|}\frac{\epsilon^{|\alpha|+|\beta|+1}}{\alpha!\beta!}(\partial^{\alpha}U_j)(z)\partial_z^{\beta}N(x,z)M_{\alpha\beta}^{(s)j}$$
$$+O(\epsilon^{2d}), \quad \textit{uniformly } x \in \partial\Omega.$$

where $N(x,y)$ be the Neumann function (matrix) for $\mathcal{L}_{\lambda,\mu}$ in Ω:

$$\begin{cases} \mathcal{L}_{\lambda,\mu}N(x,y) = -\delta_y(x)I \quad \textit{in } \Omega, \\ \dfrac{\partial N}{\partial \nu}\Big|_{\partial\Omega} = -\dfrac{1}{|\partial\Omega|}I, \\ N(\cdot,y) \perp \Psi \quad \textit{for each } y \in \Omega, \end{cases}$$

where the differentiations act on the x-variables, and $M_{\alpha\beta}^{(s)j}$ is the (generalized) Elastic Moment Tensor (Pólya-Szegö tensor), and U is the

solution in the absence of the inclusions.

Related works :

- Conductivity : Cedio.Fenya-Moskow-Vogelius, Ammari-Kang, Kang-Kim-Kim
- Maxwell System : Ammari-Vogelius-Volkov, Ammari-Kang
- Elasticity : Alves-Ammari, Maz'ya-Nazarov

Layer Potentials

The Kelvin matrix of fundamental solutions $\Gamma = (\Gamma_{ij})$ for the Lamé system :

$$\Gamma_{ij}(x) := \begin{cases} \dfrac{A}{4\pi}\dfrac{\delta_{ij}}{|x|} + \dfrac{B}{4\pi}\dfrac{x_i x_j}{|x|^3}, & \text{if } d = 3, \\ -\dfrac{A}{2\pi}\delta_{ij}\log|x| + \dfrac{B}{2\pi}\dfrac{x_i x_j}{|x|^2}, & x \neq 0, \text{ if } d = 2, \end{cases} \qquad i, j = 1, ..., d,$$

where

$$A = \frac{1}{2}\left(\frac{1}{\mu} + \frac{1}{(d-1)\mu + \lambda}\right) \quad \text{and} \quad B = \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{(d-1)\mu + \lambda}\right).$$

The single and double layer potentials

$$\mathcal{S}_D\vec{\phi}(x) := \int_{\partial D} \Gamma(x-y)\vec{\phi}(y)d\sigma(y), \quad x \in \mathbb{R}^d,$$

$$\mathcal{D}_D\vec{\phi}(x) := \int_{\partial D} \frac{\partial}{\partial \nu_y}\Gamma(x-y)\vec{\phi}(y)d\sigma(y), \quad x \in \mathbb{R}^d \setminus \partial D,$$

Definition 2. *(Elastic Moment Tensors). For multi-index $\alpha \in \mathbb{N}^3$ and $j = 1, 2, 3$, let $\vec{f}^j_\alpha$ and $\vec{g}^j_\alpha$ in $L^2(\partial B)$ be the solution of*

$$\begin{cases} \widetilde{\mathcal{S}}_B \vec{f}^j_\alpha|_+ - \mathcal{S}_B \vec{g}^j_\alpha|_- = x^\alpha e_j|_{\partial B}, \\ \dfrac{\partial}{\partial \widetilde{\nu}} \widetilde{\mathcal{S}}_B \vec{f}^j_\alpha|_+ - \dfrac{\partial}{\partial \nu} \mathcal{S}_B \vec{g}^j_\alpha|_- = \dfrac{\partial (x^\alpha e_j)}{\partial \nu}|_{\partial B}. \end{cases}$$

For $\beta \in \mathbb{N}^3$, the elastic moment tensor (EMT) $M^j_{\alpha\beta}$ associated with the domain B and Lamé parameters (λ, μ) for the background and $(\widetilde{\lambda}, \widetilde{\mu})$ for B is defined by

$$M^j_{\alpha\beta} = (m^j_{\alpha\beta 1}, m^j_{\alpha\beta 2}, m^j_{\alpha\beta 3}) = \int_{\partial B} y^\beta \vec{g}^j_\alpha(y) d\sigma(y).$$

Remark.

- The first order EMT is the elastic version of the polarization tensor in electro-magnetism introduced by Pólya-Schiffer-Szegö
- In the case of cavities and hard inclusions, the first order EMT was introduced by Maz'ya-Nazarov, and studied by Lewiński-Sokolowski, Movchan-Serkov, and a lot more.
- Our definition includes non-cavity cases and higher order tensors.

When $\alpha = e_i$ and $\beta = e_p$ $(i, p = 1, 2, 3)$, put

$$m^{ij}_{pq} := m^{j}_{\alpha\beta q}, \quad p, j = 1, 2, 3.$$

Lemma 3. Properties of EMT

- *EMT is symmetric:* $m^{ij}_{pq} = m^{ij}_{qp}$, $m^{ij}_{pq} = m^{ji}_{pq}$, *and* $m^{ij}_{pq} = m^{pq}_{ij}$, $p, q, i, j = 1, 2, 3$.
- *EMT is positive definite on the space of symmetric matices.*
- *Suppose* $i \neq j$ *and that* B *satisfies* $diam(B)|\partial B| \leq C_0|B|$ *for some* C_0. *Then there exists* $C = C(\lambda, \mu, \widetilde{\lambda}, \widetilde{\mu}, C_0)$ *such that*

$$\mu \left| \frac{\mu - \widetilde{\mu}}{\mu + \widetilde{\mu}} \right| |B| \leq |m^{ij}_{ij}| \leq C|B|.$$

Detection of Inclusions

For a given traction boundary $\vec{g}$,

$$\vec{H}[\vec{g}](x) := -\mathcal{S}_\Omega(\vec{g})(x) + \mathcal{D}_\Omega(\vec{f})(x), \quad x \in \mathbb{R}^d \setminus \partial\Omega, \quad \vec{f} := \vec{u}|_{\partial\Omega}.$$

Theorem 4. *For $x \in \mathbb{R}^d \setminus \overline{\Omega}$,*

$$\vec{H}[\vec{g}](x) = \sum_{j=1}^{d} \sum_{|\alpha|=1}^{d} \sum_{|\beta|=1}^{d+1-|\alpha|} \frac{\epsilon^{|\alpha|+|\beta|+d-2}}{\alpha!\beta!} (\partial^\alpha U_j)(z) \partial^\beta \Gamma(x-z) M^j_{\alpha\beta} + O\left(\frac{\epsilon^{2d}}{|x|^{d-1}}\right),$$

where $M^j_{\alpha\beta}$ are the elastic moment tensors defined by (2) and Γ is the Kelvin matrix of fundamental solutions corresponding to the Lamé parameters (λ, μ).

Disk Reconstruction Procedure

$$E_{uv} = (\delta_{iu}\delta_{jv})_{i,j=1}^{d} \quad \text{and} \quad \vec{g}_{uv} := \frac{\partial(E_{uv}\vec{x})}{\partial\nu}|_{\partial\Omega}.$$

Step R Compute $\epsilon^2 m_{kl}^{uv}$ using the formulae

$$h_{kl}^{uv} := \lim_{t\to\infty} tH_k[\vec{g}_{uv}](te_l)$$

and

$$\epsilon^2 m_{kl}^{uv} = \begin{cases} -\pi\mu\left[\dfrac{\lambda}{2\mu}\displaystyle\sum_{j=1}^{d} h_{jj}^{uv} + h_{kk}^{uv}\right], & k = l, \\ -\pi(\lambda+\mu)h_{kl}^{uv}, & k \neq l. \end{cases}$$

Then computed radius r_c is $\sqrt{\frac{|m_{12}^{12}|}{\pi}}$.

Step C1 Compute the matrix $(n_{ij}^{pq})_{i,j,p,q=1,2}$:

$$\left(\sum_{p,q=1} a_{pq} n_{pq}^{ij}\right) := P^{-1}((a_{pq})) \quad \text{where} \quad P((a_{pq})) = \left(\sum_{p,q=1} a_{pq}\epsilon^2 m_{pq}^{uv}\right).$$

Then find the unique zero $z_k^*, k = 1, \cdots, d$, of

$$\Phi_k(x_k) = -2\pi(\mu+\lambda)|x| \sum_{i,j=1}^{d} H_k[\vec{g}_{ji}](x) \sum_{p=1}^{2} n_{ij}^{pp}$$

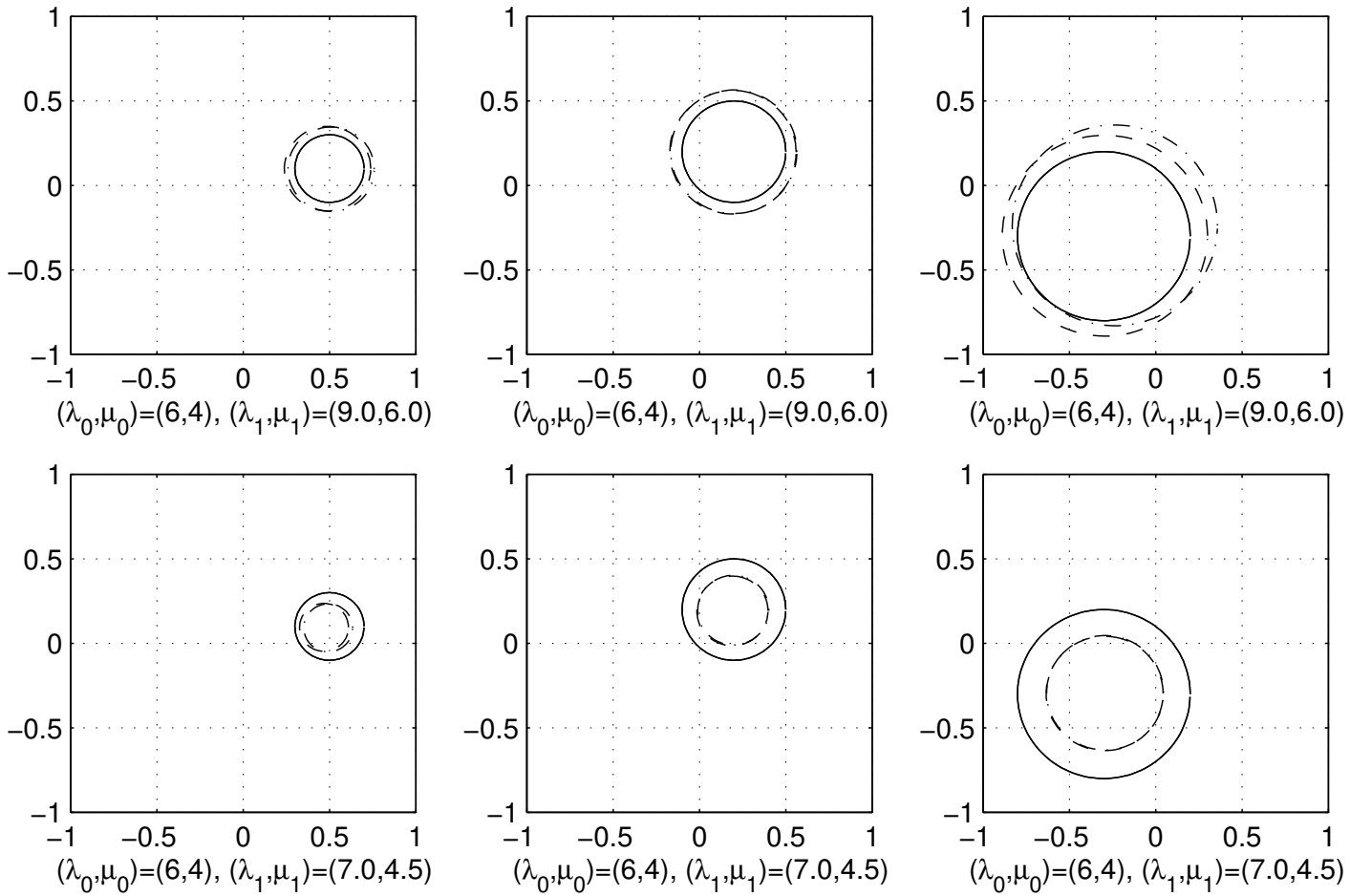
by Newton's method with $H_k[\vec{g}_{ji}](x)$ and $\frac{\partial}{\partial x_k} H_k[\vec{g}_{ji}](x)$.

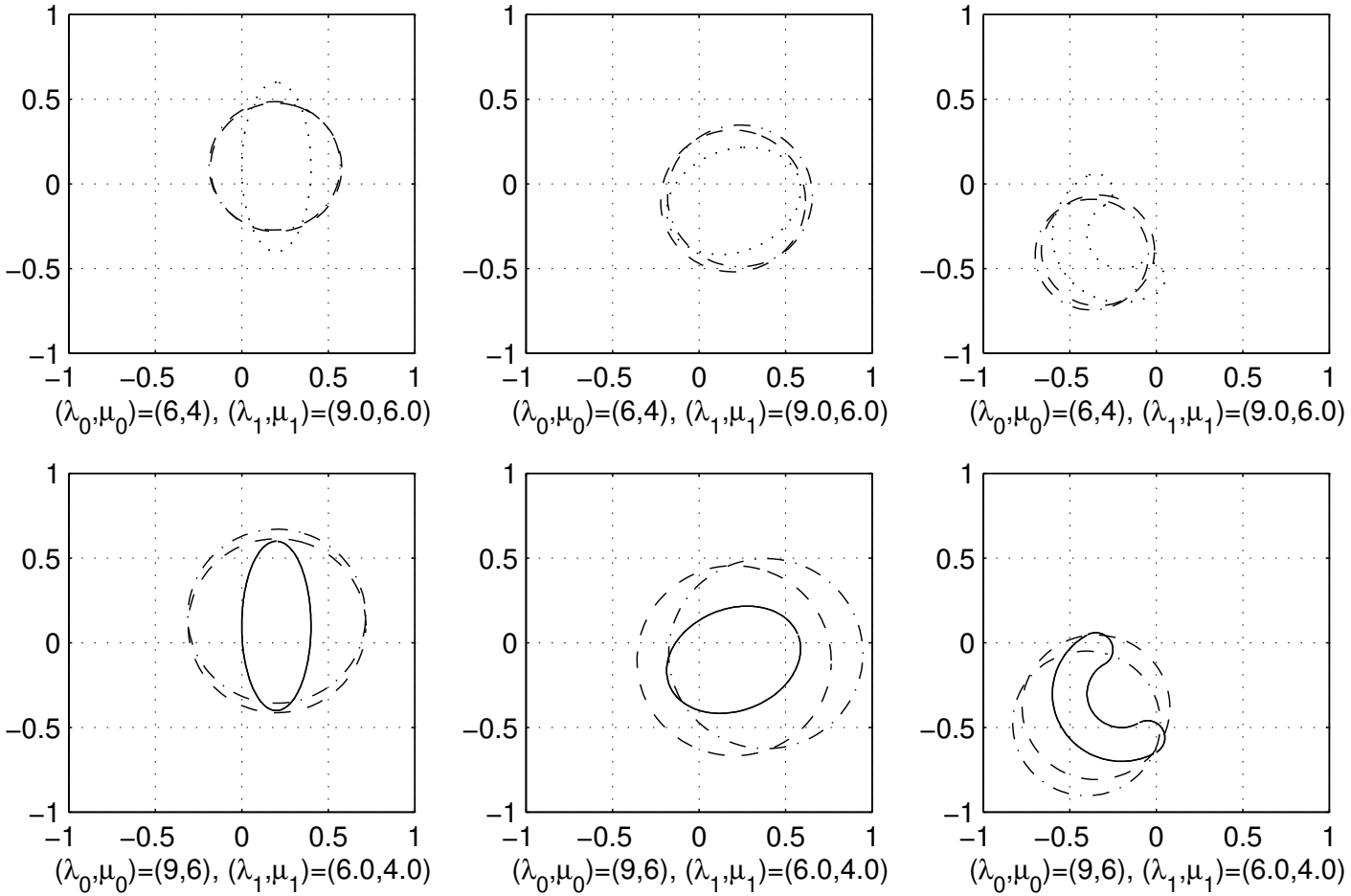
Step C2 Compute center point z by

$$\sum_{i,j=1}^{2} (\partial_i U_j)(z)\epsilon^d m_{pq}^{ij} = 2\pi T^{-1}(tH_k[\vec{g}](te_l))_{pq}.$$

where

$$T(a_{pq}) := \frac{1}{2}\sum_{p,q=1}^{2}(e_{klpq} + e_{klqp})a_{pq} \quad , \quad \vec{U}(x) = (2x_1x_2, x_1^2 - x_2^2).$$


1
0.5
0
−0.5
−1
−1 −0.5 0 0.5 1
$(\lambda_0,\mu_0)=(6,4)$, $(\lambda_1,\mu_1)=(9.0,6.0)$
$(\lambda_0,\mu_0)=(6,4)$, $(\lambda_1,\mu_1)=(9.0,6.0)$
$(\lambda_0,\mu_0)=(6,4)$, $(\lambda_1,\mu_1)=(9.0,6.0)$
$(\lambda_0,\mu_0)=(6,4)$, $(\lambda_1,\mu_1)=(7.0,4.5)$
$(\lambda_0,\mu_0)=(6,4)$, $(\lambda_1,\mu_1)=(7.0,4.5)$
$(\lambda_0,\mu_0)=(6,4)$, $(\lambda_1,\mu_1)=(7.0,4.5)$


1
0.5
0
−0.5
−1
−1 −0.5 0 0.5 1
(λ0,μ0)=(6,4), (λ1,μ1)=(9.0,6.0)
(λ0,μ0)=(6,4), (λ1,μ1)=(9.0,6.0)
(λ0,μ0)=(6,4), (λ1,μ1)=(9.0,6.0)
(λ0,μ0)=(9,6), (λ1,μ1)=(6.0,4.0)
(λ0,μ0)=(9,6), (λ1,μ1)=(6.0,4.0)
(λ0,μ0)=(9,6), (λ1,μ1)=(6.0,4.0)

Ellipse Reconstruction Procedure

Given a tolerance τ, if both $|m_{12}^{11} + m_{22}^{12}|$ and $|m_{11}^{11} - m_{22}^{22}|$ are smaller than τ, then find the disk of the size. If either $|m_{12}^{11} + m_{22}^{12}|$ or $|m_{11}^{11} - m_{22}^{22}|$ is larger than τ, then

(1) To find the angle of orientation, Solve

$$\frac{m_{12}^{11} + m_{22}^{12}}{m_{11}^{11} - m_{22}^{22}} = \frac{1}{2} \tan 2\theta.$$

(2) Using the angle found in (1), solve

$$\begin{aligned}
\hat{M}_{11}^{11} &= \cos^4\theta M_{11}^{11} + \tfrac{1}{2}\sin^2(2\theta)M_{22}^{11} + 4\sin\theta\cos^3\theta M_{12}^{11} + \sin^2(2\theta)M_{12}^{12} \\
&\quad +4\sin^3\theta\cos\theta M_{22}^{12} + \sin^4\theta M_{22}^{22}, \\
\hat{M}_{22}^{11} &= \tfrac{1}{2}\sin^2(2\theta)M_{11}^{11} + (1 - \tfrac{1}{2}\sin^2(2\theta))M_{22}^{11} - \tfrac{1}{2}\sin 4\theta M_{12}^{11} \\
&\quad - \sin^2(2\theta)M_{12}^{12} + \frac{1}{2}\sin 4\theta M_{22}^{12} + \frac{1}{2}\sin^2(2\theta)M_{22}^{22}, \\
\hat{M}_{12}^{12} &= \tfrac{1}{2}\sin^2(2\theta)M_{11}^{11} - \tfrac{1}{2}\sin^2(2\theta)M_{22}^{11} - \tfrac{1}{2}\sin 4\theta M_{12}^{11} + \cos^2(2\theta)M_{12}^{12} \\
&\quad +\frac{1}{2}\sin 4\theta M_{22}^{12} + \frac{1}{2}\sin^2(2\theta)M_{22}^{22}, \\
\hat{M}_{22}^{22} &= \sin^4\theta M_{11}^{11} + \tfrac{1}{2}\sin^2(2\theta)M_{22}^{11} - 4\sin^3\theta\cos\theta M_{12}^{11} + \sin^2(2\theta)M_{12}^{12} \\
&\quad -4\sin\theta\cos^3\theta M_{22}^{12} + \cos^4\theta M_{22}^{22}.
\end{aligned}$$

(3) Solve

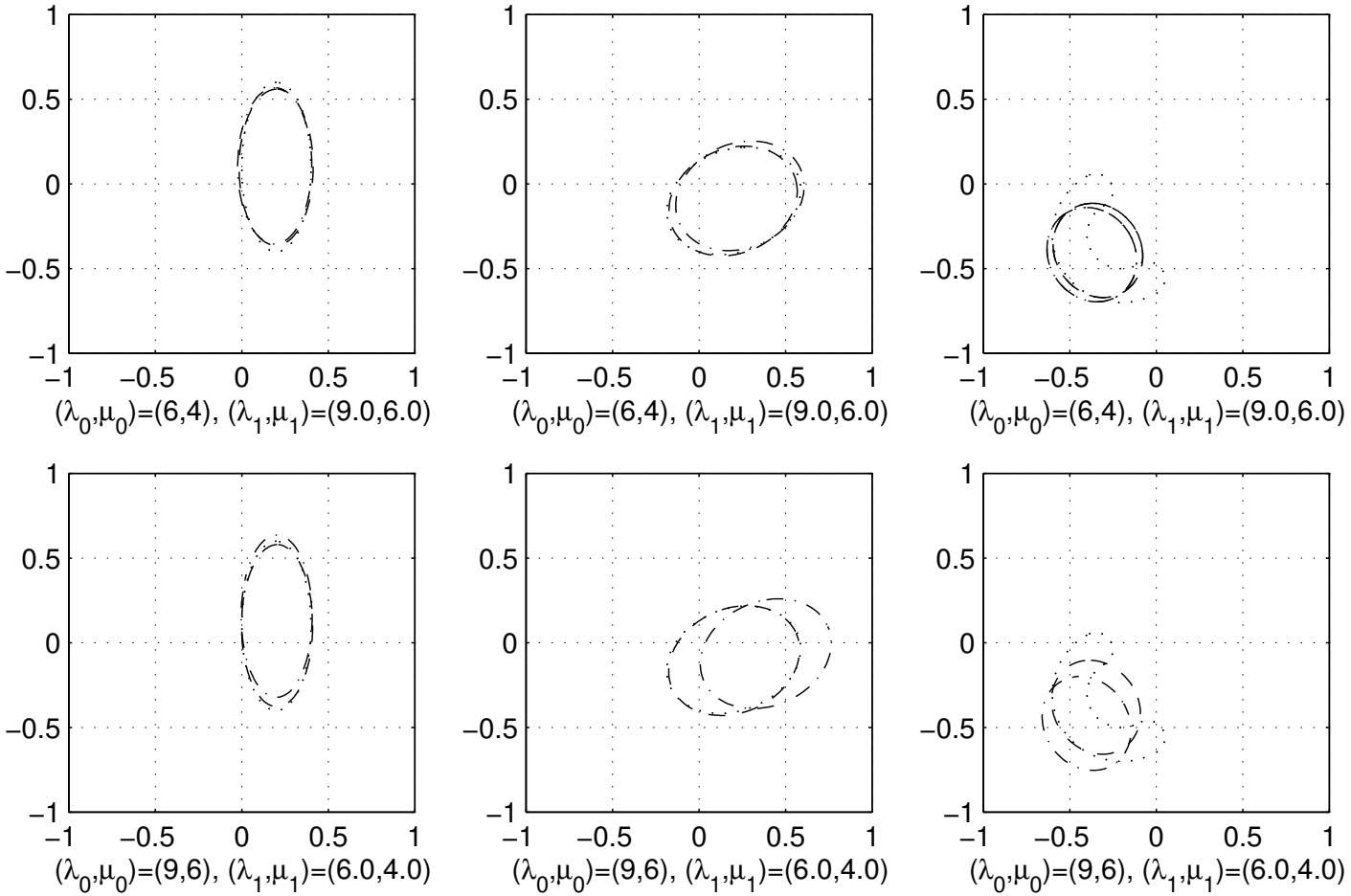
$$\begin{aligned} 2(\hat{m}_{22}^{11} + 2\hat{m}_{12}^{12}) - (\hat{m}_{11}^{11} + \hat{m}_{22}^{22}) &= 2(\hat{M}_{22}^{11} + 2\hat{M}_{12}^{12}) - (\hat{M}_{11}^{11} + \hat{M}_{22}^{22}) \\ \hat{m}_{12}^{12} &= \hat{M}_{12}^{12}. \end{aligned} \tag{1}$$

to find long and short axes.

(4) Among the solutions in (2-3), choose the one minimizing

$$\left|\hat{m}_{11}^{11} - \hat{M}_{11}^{11}\right| + \left|\hat{m}_{22}^{22} - \hat{M}_{22}^{22}\right| + \left|\hat{m}_{22}^{11} - \hat{M}_{22}^{11}\right| + \left|\hat{m}_{12}^{12} - \hat{M}_{12}^{12}\right|. \tag{2}$$

.


1
0.5
0
−0.5
−1
−1 −0.5 0 0.5 1
(λ0,μ0)=(6,4), (λ1,μ1)=(9.0,6.0)
(λ0,μ0)=(6,4), (λ1,μ1)=(9.0,6.0)
(λ0,μ0)=(6,4), (λ1,μ1)=(9.0,6.0)
(λ0,μ0)=(9,6), (λ1,μ1)=(6.0,4.0)
(λ0,μ0)=(9,6), (λ1,μ1)=(6.0,4.0)
(λ0,μ0)=(9,6), (λ1,μ1)=(6.0,4.0)