Mathematical analysis of the stationary motion of an incompressible viscous fluid

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Abstract

The motion of an incompressible viscous fluid is governed by a nonlinear system of partial differential equations, called the Navier-Stokes equations. The purpose of this talk is to review some recent results on existence and uniqueness of stationary solutions of the Navier-Stokes equations in a bounded or exterior domain Ω in $\mathbb{R}^3$. The first existence result without any smallness condition on the external data $\mathbf{f}$ was established in 1933 by a famous French mathematician, Jean Leray(1906-1998). He proved that for each $\mathbf{f} = \operatorname{div} \mathbf{F}$ with $\mathbf{F} \in \mathbf{L}_2(\Omega)$, there exists at least one weak solution satisfying the energy inequality. Such a solution will be called a *Leray weak solution*. But uniqueness of Leray weak solutions can be guaranteed only when $\mathbf{F}$ is suitably small; nonuniqueness may occur for large $\mathbf{F}$.

We first consider the case when Ω is a bounded domain. Then uniqueness of Leray weak solutions can be easily proved if $\|\mathbf{F}\|_2$ is sufficiently small. More general uniqueness and existence results have been obtained by Galdi-Simader-Sohr(2005), Kim(2009) and Choe-Kim(preprint), with $\mathbf{F}$ belonging to $\mathbf{L}_{3/2}(\Omega)$.

We next consider the more difficult case when Ω is an exterior domain. Then uniqueness of Leray weak solutions is not trivial at all, due to the lack of the Poincaré inequality. Nevertheless Galdi(1992) and Galdi-Simader(1994) were able to prove the uniqueness for small $\mathbf{F}$ in some suitable norms. Further improvements have been made by Kozono-Yamazaki(1998, 1999), Kim-Kozono(preprint) and Heck-Kim-Kozono(preprint) using the full L_q-theory of linear Stokes and Oseen equations.