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A Note on Maass-Jacobi Forms II

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Abstract. This article is a continuation of the paper [21]. In this paper we deal with Maass-Jacobi forms on the Siegel-Jacobi space $\mathbb{H} \times \mathbb{C}^m$, where $\mathbb{H}$ denotes the Poincaré upper half plane and m is any positive integer.

1. Introduction

This article is a continuation of the paper [21]. Recently A. Pitale [14], K. Bringmann and O. Richter [4], and C. Conley and M. Raum [5] defined another notion of Maass-Jacobi forms and studied some properties of Maass-Jacobi forms. In [4], [14] and [21], the authors considered the case $n = m = 1$ and in [5], the authors dealt with the case $n = 1$ and m is arbitrary. In this paper, we consider mainly the case $n = 1$ and m is an arbitrary positive integer.

This paper is organized as follows. In Section 2, we give some useful geometric properties of the Siegel-Jacobi space $\mathbb{H} \times \mathbb{C}^m$. We study the invariant metrics, their Laplacians, a fundamental domain, geodesics, the scalar curvature and invariant differential forms on $\mathbb{H} \times \mathbb{C}^m$. In Section 3 we describe the center of the universal enveloping algebra of the complexfied Jacobi Lie algebra. This work is due to Conley and Raum [5]. In Section 4, we present some interesting and important results on invariant differential operators on the Siegel-Jacobi space $\mathbb{H} \times \mathbb{C}^m$. In Section 5, we discuss the notion of Maass-Jacobi forms introduced by J.-H. Yang [21]. Maass-Jacobi forms play an important role in the spectral theory of the Laplace operator on a fundamental domain for the Siegel-Jacobi space $\mathbb{H} \times \mathbb{C}^m$. In Section 6, we discuss the notion of Maass-Jacobi forms introduced by A. Pitale [14], Bringman-Richter [4] and Conley-Raum [5]. We describe the results obtained in [4] and [5]. More precisely the authors of [4] and [5] obtained an explicit Fourier expansion of

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the Poincaré series that is an example of harmonic Maass-Jacobi form. In Section 7, we discuss skew-holomorphic Jacobi forms introduced by N.-P. Skoruppa [18]. We describe the relation between cuspidal harmonic Maass-Jacobi forms and cuspidal skew-holomorphic Jacobi forms via the lowering operator $D_-^{(\mathcal{M})}$ (cf. (7.3)) In Section 8, we briefly review some results on covariant differential operators on the Siegel-Jacobi space $\mathbb{H}\times\mathbb{C}^m$ obtained by Conley and Raum [5]. In the final section we briefly mention two notions of Maass-Jacobi forms on the Siegel-Jacobi space $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ for the general case $n > 1$ and $m > 1$. Here $\mathbb{H}_n$ denotes the Siegel upper half plane of degree n. We present some natural problems related to the study of Maass-Jacobi forms.

Notations: We denote by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers respectively. $\mathbb{R}^\times$ denotes the set of all nonzero real numbers. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\mathrm{tr}(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. For a complex matrix A, $\overline{A}$ denotes the complex *conjugate* of A. For $A \in \mathbb{C}^{(k,l)}$ and $B \in \mathbb{C}^{(k,k)}$, we use the abbreviation $B\{A\} = {}^t\overline{A}BA$. For a positive integer n, I_n denotes the identity matrix of degree n. For a positive integer m and a commutative ring F, we denote by $S(m,F)$ the space of all $m \times m$ symmetric matrices with entries in F. For a complex number z, $|z|$ denotes the absolute value of z. For a complex number z, $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ denote the real part of z and the imaginary part of z respectively.

2. Geometric properties of the Siegel-Jacobi space $\mathbb{H} \times \mathbb{C}^m$

We fix a positive integer m throughout this paper and let

$$\mathbb{H} = \{\, \tau \in \mathbb{C} \mid \mathrm{Im}\, \tau > 0 \,\}$$

be the Poincaré upper half plane. Let $G = SL_2(\mathbb{R})$ be the special linear group of degree 2 and let

$$H_{\mathbb{R}}^{(m)} = \big\{\, (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^m,\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\, {}^t\lambda \text{ symmetric } \big\}$$

be the Heisenberg group endowed with the following multiplication law

$$\big(\lambda, \mu; \kappa\big) \circ \big(\lambda', \mu'; \kappa'\big) = \big(\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda'\big)$$

with $\big(\lambda, \mu; \kappa\big), \big(\lambda', \mu'; \kappa'\big) \in H_{\mathbb{R}}^{(m)}$. We define the semidirect product of $SL_2(\mathbb{R})$ and $H_{\mathbb{R}}^{(m)}$

$$G^J = SL_2(\mathbb{R}) \ltimes H_{\mathbb{R}}^{(m)}$$

endowed with the following multiplication law

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot\big(M',(\lambda',\mu';\kappa'\,)\big) = \big(MM',(\tilde{\lambda}+\lambda',\tilde{\mu}+\mu';\kappa+\kappa'+\tilde{\lambda}\,{}^t\mu'-\tilde{\mu}\,{}^t\lambda'\,)\big)$$

with $M,M'\in SL_2(\mathbb{R}),(\lambda,\mu;\kappa),\,(\lambda',\mu';\kappa')\in H_{\mathbb{R}}^{(m)}$ and $(\tilde{\lambda},\tilde{\mu})=(\lambda,\mu)M'$. Then G^J acts on the Siegel-Jacobi space $\mathbb{H}\times\mathbb{C}^m$ of degree 1 and index m transitively by

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot(\tau,z)=\Big((a\tau+b)(c\tau+d)^{-1},(z+\lambda\tau+\mu)(c\tau+d)^{-1}\Big), \tag{2.1}$$

where $M=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{R}),\ (\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(m)}$, $\tau\in\mathbb{H}$ and $z={}^t(z_1,z_2,\cdots,z_m)\in\mathbb{C}^m$ with $z_i\in\mathbb{C}\,(1\le i\le m)$. We note that the Jacobi group G^J is *not* a reductive Lie group and that the homogeneous space $\mathbb{H}\times\mathbb{C}^m$ is not a symmetric space.

For a coordinate $(\tau,z)\in\mathbb{H}\times\mathbb{C}^n$, we write $\tau=x+i\,y$ with x real and $y>0$, and

$$z={}^t(z_1,z_2,\cdots,z_m),\quad z_j=u_j+i\,v_j,\quad u_j,\,v_j\ \text{real},\quad i=1,2,\cdots,m.$$

According to [23], for any two positive real numbers A and B, the following metric given by

$$\begin{aligned} ds^2_{m;A,B} &= \frac{1}{y^3}\left(A\,y\,+\,B\sum_{j=1}^m v_j^2\right)d\tau d\overline{\tau}\\ &\quad+\frac{B}{y^2}\left\{y\sum_{j=1}^m dz_j d\overline{z}_j-\sum_{j=1}^m v_j(d\tau d\overline{z}_j+d\overline{\tau}d\overline{z}_j)\right\}\\ &= \frac{1}{y^3}\left(A\,y\,+\,B\sum_{j=1}^m v_j^2\right)(dx^2+dy^2)\\ &\quad+\frac{B}{y^2}\left\{y\sum_{j=1}^m(du_j^2+dv_j^2)-2\sum_{j=1}^m v_j(dx du_j+dy dv_j)\right\} \end{aligned} \tag{2.2}$$

is a Kähler metric on $\mathbb{H}\times\mathbb{C}^m$ invariant under the action (2.1) of G^J.

We put

$$M_1:=\,\mathrm{tr}\left(y\frac{\partial}{\partial z}\,{}^t\!\left(\frac{\partial}{\partial\overline{z}}\right)\right)=y\sum_{j=1}^m\frac{\partial^2}{\partial z_j\partial\overline{z}_j}=\frac{y}{4}\left(\frac{\partial}{\partial u_j^2}+\frac{\partial}{\partial v_j^2}\right) \tag{2.3}$$

and

$$\begin{aligned}(2.4)\quad M_2 : &= y^2\frac{\partial^2}{\partial\tau\partial\overline{\tau}} + \sum_{a,b=1}^m v_a v_b \frac{\partial^2}{\partial z_a\partial\overline{z}_b} + y\sum_{j=1}^m v_j\left(\frac{\partial^2}{\partial\tau\partial\overline{z}_j} + \frac{\partial^2}{\partial\overline{\tau}\partial z_j}\right)\\ &= \frac{1}{4}\left\{y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + \sum_{a=1}^m v_a^2\left(\frac{\partial^2}{\partial u_a^2}+\frac{\partial^2}{\partial v_a^2}\right)\right\}\\ &\quad +\frac{1}{2}\sum_{1\le a<b\le m} v_a v_b\left(\frac{\partial^2}{\partial u_a\partial u_b}+\frac{\partial^2}{\partial v_a\partial v_b}\right).\\ &\quad +\frac{y}{2}\sum_{j=1}^m v_j\left(\frac{\partial^2}{\partial x\partial u_j}+\frac{\partial^2}{\partial y\partial v_j}\right).\end{aligned}$$

Then M_1 and M_2 are differential operators on $\mathbb{H}\times\mathbb{C}^m$ invariant under the action (2.1). The author [23] proved that

$$(2.5)\qquad \Delta_{m;A,B} := \frac{4}{B}\,M_1 + \frac{4}{A}\,M_2$$

is the Laplacian of $(\mathbb{H}\times\mathbb{C}^m, ds^2_{m;A,B})$. Furthermore the following $2(m+1)$-differential form

$$(2.6)\qquad dv = dx\wedge dy\wedge du_1\wedge\cdots\wedge du_m\wedge dv_1\wedge\cdots\wedge dv_m$$

is a G^J-invariant volume element on the Siegel-Jacobi space $\mathbb{H}\times\mathbb{C}^m$.

Let K^J be the stabilizer of G^J at $(i,0)$. Then

$$K^J = \left\{\left(\begin{pmatrix} a & -b\\ b & a\end{pmatrix}, (0,0,R)\right)\ \Big|\ a^2+b^2=1,\ a,b\in\mathbb{R},\ R={}^tR\in\mathbb{R}^{(m,m)}\right\}.$$

Thus G^J/K^J is diffeomorphic to $\mathbb{H}\times\mathbb{C}^m$ via

$$gK^J \longmapsto g\cdot(i,0) = \left(\frac{a\,i+b}{c\,i+d}, \frac{\lambda\,i+\mu}{c\,i+d}\right),$$

where $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{R})$ and $(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(m)}$. The Siegel-Jacobi space $\mathbb{H}\times\mathbb{C}^m$ is a homogeneous space which is not symmetric. Let $\mathfrak{k}^J$ be the Lie algebra of K^J. Then the Lie algebra $\mathfrak{g}^J$ of G^J has the Cartan decomposition

$$(2.7)\qquad \mathfrak{g}^J = \mathfrak{k}^J + \mathfrak{p}^J,$$

where

$$\begin{aligned}\mathfrak{g}^J &= \left\{\left(\begin{pmatrix} x & y\\ z & -x\end{pmatrix}, (P,Q,R)\right)\ \Big|\ x,y,z\in\mathbb{R},\ P,Q\in\mathbb{R}^m,\ R={}^tR\in\mathbb{R}^{(m,m)}\right\},\\ \mathfrak{k}^J &= \left\{\left(\begin{pmatrix} 0 & x\\ -x & 0\end{pmatrix}, (0,0,R)\right)\ \Big|\ x\in\mathbb{R},\ R={}^tR\in\mathbb{R}^{(m,m)}\right\},\\ \mathfrak{p}^J &= \left\{\left(\begin{pmatrix} x & y\\ y & -x\end{pmatrix}, (P,Q,0)\right)\ \Big|\ x,y\in\mathbb{R},\ P,Q\in\mathbb{R}^m\right\}.\end{aligned}$$

Lemma 2.1. *We have the relations*

$$[\mathfrak{k}^J,\mathfrak{k}^J]\subset\mathfrak{k}^J \quad \textit{and} \quad [\mathfrak{k}^J,\mathfrak{p}^J]\subset\mathfrak{p}^J. \tag{2.8}$$

Proof. The Lie bracket operation on $\mathfrak{g}^J$ is given by

$$[(X_1,(P_1,Q_1,R_1)),(X_2,(P_2,Q_2,R_2))]=(X^*,(P^*,Q^*,R^*)), \tag{2.9}$$

where $X_1,X_2\in\mathfrak{sl}_2(\mathbb{R}),\ P_1,Q_1,P_2,Q_2\in\mathbb{R}^m,\ R_1={}^tR_1,\ R_2={}^tR_2\in\mathbb{R}^{(m,m)}$,

$$\begin{aligned} X^* &= [X_1,X_2]=X_1X_2-X_2X_1,\\ (P^*,Q^*) &= (P_1,Q_1)X_2-(P_2,Q_2)X_1,\\ R^* &= P_1\,{}^tQ_2-P_2\,{}^tQ_1+Q_2\,{}^tP_1-Q_1\,{}^tP_2. \end{aligned}$$

The relations (2.8) follow immediately from Formula (2.9). □

Remark 2.1. The relation

$$[\mathfrak{p}^J,\mathfrak{p}^J]\subset\mathfrak{k}^J$$

does not hold.

The vector space $\mathfrak{p}^J$ can be regarded as the tangent space of the Siegel-Jacobi space $\mathbb{H}\times\mathbb{C}^m\cong G^J/K^J$ at $(i,0)$. We define a complex structure I^J on the tangent space $\mathfrak{p}^J$ of $\mathbb{H}\times\mathbb{C}^m\cong G^J/K^J$ at $(i,0)$ by

$$I^J\left(\begin{pmatrix} x & y\\ y & -x\end{pmatrix},(P,Q,0)\right)=\left(\begin{pmatrix} y & -x\\ -x & -y\end{pmatrix},(Q,-P,0)\right). \tag{2.10}$$

Let

$$\mathfrak{p}=\left\{\begin{pmatrix} x & y\\ y & -x\end{pmatrix}\in\mathbb{R}^{(2,2)}\ \Big|\ x,y\in\mathbb{R}\right\}$$

be the real vector space of dimension 2. Identifying $\mathfrak{p}$ with $\mathbb{C}$ via

$$\begin{pmatrix} x & y\\ y & -x\end{pmatrix}\longmapsto x+i\,y\in\mathbb{C}$$

and identifying $\mathbb{R}^m\times\mathbb{R}^m$ with $\mathbb{C}^m$ via

$$(P,Q)\longmapsto Q+i\,P,\quad P,Q\in\mathbb{R}^m,$$

we may regard the complex structure I^J as a real linear map on $\mathbb{C}\times\mathbb{C}^m$ defined by

$$I^J(x+i\,y,Q+i\,P)=(-y+i\,x,-P+i\,Q),\quad x+i\,y\in\mathbb{C},\ Q+i\,P\in\mathbb{C}^m. \tag{2.11}$$

Clearly I^J extends complex linearly on the complexification $\mathfrak{p}^J_{\mathbb{C}}=\mathfrak{p}^J\otimes_{\mathbb{R}}\mathbb{C}$ of $\mathfrak{p}^J$. Then $\mathfrak{p}^J_{\mathbb{C}}$ has a decomposition

$$\mathfrak{p}^J_{\mathbb{C}}=\mathfrak{p}^J_+\oplus\mathfrak{p}^J_-, \tag{2.12}$$

where $\mathfrak{p}_+^J$ (resp. $\mathfrak{p}_-^J$) denotes the $(+i)$-eigenspace (resp. $(-i)$-eigenspace) of I^J. Precisely, both $\mathfrak{p}_+^J$ and $\mathfrak{p}_-^J$ are given by

$$\mathfrak{p}_+^J = \left\{ \left(\begin{pmatrix} x & i\,x \\ i\,x & -x \end{pmatrix}, (P, iP, 0) \right) \,\Big|\, x \in \mathbb{C},\ P \in \mathbb{C}^m \right\}$$

and

$$\mathfrak{p}_-^J = \left\{ \left(\begin{pmatrix} x & -i\,x \\ -i\,x & -x \end{pmatrix}, (P, -iP, 0) \right) \,\Big|\, x \in \mathbb{C},\ P \in \mathbb{C}^m \right\}.$$

Proposition 2.1. *Fix an element* $g = \big(M, (\lambda, \mu; \kappa)\big) \in G^J$ *with* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ *and* $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(m)}$. *We let* $(\tau_*, z_*) = g \cdot (\tau, z)$. *Let*

$$\mathbb{F}_g : \mathbb{H} \times \mathbb{C}^m \longrightarrow \mathbb{H} \times \mathbb{C}^m$$

be the biholomorphic mapping defined by the action (2.1) of g. *Then the differential mapping*

$$d\mathbb{F}_g : T_{(\tau,z)}\big(\mathbb{H} \times \mathbb{C}^m\big) \longrightarrow T_{(\tau_*,z_*)}\big(\mathbb{H} \times \mathbb{C}^m\big)$$

is given by

$$(w, \xi) \longmapsto \big(w(g), \xi(g)\big), \quad w \in \mathbb{C},\ \xi \in \mathbb{C}^m \tag{2.13}$$

with

$$w(g) = \frac{w}{(c\,\tau + d)^2} \qquad \text{and} \qquad \xi(g) = \frac{\xi}{c\,\tau + d} + \frac{w(d\,\lambda - c\,\mu - c\,z)}{(c\,\tau + d)^2}.$$

Here we identified $\mathfrak{p}^J$ *with* $\mathbb{C} \times \mathbb{C}^m$.

Proof. Let $\alpha(t) = \big(\tau(t), z(t)\big)$ $(-\epsilon < t < \epsilon,\ \epsilon > 0)$ be a smooth curve in $\mathbb{H} \times \mathbb{C}^m$ passing through $\alpha(0) = (\tau, z)$ with $\alpha'(0) = (w, \xi) \in T_{(\tau,z)}\big(\mathbb{H} \times \mathbb{C}^m\big)$. Then

$$\begin{aligned} \chi(t) :=\ & g \cdot \alpha(t) = \big(\tau(g;t), z(g;t)\big) \\ =\ & \left(\frac{a\,\tau(t) + b}{c\,\tau(t) + d}, \frac{z(t) + \lambda\,\tau(t) + \mu}{c\,\tau(t) + d} \right) \end{aligned}$$

is a smooth curve in $\mathbb{H} \times \mathbb{C}^m$ passing through $\chi(0) = (\tau_*, z_*)$. Then by an easy computation, we see that

$$\tau'(g;0) = \frac{\partial}{\partial t}\Big|_{t=0} \tau(g;t) = \frac{\tau'(0)}{(c\,\tau + d)^2} = \frac{w}{(c\,\tau + d)^2}$$

and

$$z'(g;0) = \frac{\partial}{\partial t}\Big|_{t=0} z(g;t) = \frac{\xi}{c\,\tau + d} + \frac{w(d\,\lambda - c\,\mu - c\,z)}{(c\,\tau + d)^2}.$$

□

Let $\Gamma_1 := SL_2(\mathbb{Z})$ be the elliptic modular group. We let

$$\Gamma_{1,m} := \Gamma_1 \ltimes H_{\mathbb{Z}}^{(m)}$$

be the arithmetic subgroup of G^J, where

$$H_{\mathbb{Z}}^{(m)} := \left\{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(m)} \mid \lambda, \mu, \kappa \textit{ are integral} \right\}$$

is a discrete subgroup of $H_{\mathbb{R}}^{(m)}$. Let $E_k := {}^t(0, \cdots, 1, 0, \cdots, 0)$ $(1 \le k \le m)$ be the $m \times 1$ matrix with the $(k,1)$-th entry 1 and other entries 0. For an element $\tau \in \mathbb{H}$, we set for brevity

$$F_k(\tau) := \tau\, E_k, \quad 1 \le k \le m.$$

Let

$$\mathcal{F} := \left\{ \tau \in \mathbb{H} \,\middle|\, |\tau| \ge 1, \ \ |\mathrm{Re}\,\tau| \le 1/2 \right\}$$

be a fundamental domain for $\Gamma_1 \backslash \mathbb{H}$. We refer to [16], pp. 78-79 for more detail. For each $\tau \in \mathcal{F}$, we define the subset P_τ of $\mathbb{C}^m$ by

$$P_\tau := \left\{ \sum_{k=1}^{m} \lambda_k E_k + \sum_{k=1}^{m} \mu_k F_k(\tau) \,\middle|\, 0 \le \lambda_k, \mu_k \le 1 \right\}.$$

For each $\tau \in \mathcal{F}$, we define the subset $\mathcal{D}_\tau$ of $\mathbb{H} \times \mathbb{C}^m$ by

$$\mathcal{D}_\tau := \{ (\tau, z) \in \mathbb{H} \times \mathbb{C}^m \mid z \in P_\tau \}.$$

Theorem 2.1. *The following subset*

$$\mathcal{F}_{[m]} := \bigcup_{\tau \in \mathcal{F}} \mathcal{D}_\tau \tag{2.14}$$

is a fundamental domain for $\Gamma_{1,m} \backslash (\mathbb{H} \times \mathbb{C}^m)$ *with respect to the action (2.1).*

Proof. Let (τ_*, z_*) be an arbitrary element of $\mathbb{H} \times \mathbb{C}^m$. We must find an element (τ, z) of $\mathcal{F}_{[m]}$ and $\gamma_* = (\gamma, (\lambda, \mu; \kappa)) \in \Gamma_{1,m}$ with $\gamma \in \Gamma_1 = SL_2(\mathbb{Z})$ such that $\gamma_* \cdot (\tau, z) = (\tau_*, z_*)$. Since $\mathcal{F}$ is a fundamental domain for $\Gamma_1 \backslash \mathbb{H}$, there is an element γ of Γ_1 and an element $\tau \in \mathcal{F}$ such that $\tau_* = \gamma \cdot \tau$. Here τ is unique up to the boundary of $\mathcal{F}$. We write

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 = SL_2(\mathbb{Z}).$$

We can find $\lambda, \mu \in \mathbb{Z}^m$ and $z \in P_\tau$ satisfying the equation

$$z + \lambda\, \tau + \mu = z_*(x\, \tau + d).$$

If we take $\gamma_* = (\gamma, (\lambda, \mu; 0)) \in \Gamma_{1,m}$, we see that $\gamma_* \cdot (\tau, z) = (\tau_*, z_*)$. Therefore

$$\mathbb{H} \times \mathbb{C}^m = \bigcup_{\gamma_* \in \Gamma_{1,m}} \gamma_* \cdot \mathcal{F}_{[m]}.$$

Let (τ, z) and $\gamma_* \cdot (\tau, z)$ be two elements of $\mathcal{F}_{[m]}$ with $\gamma_* = (\gamma, (\lambda, \mu; \kappa)) \in \Gamma_{1,m}$ with $\gamma \in \Gamma_1$. Then both τ and $\gamma \cdot \tau$ lie in $\mathcal{F}$. Therefore both of them either lie in the boundary of $\mathcal{F}$ or $\gamma = \pm I_2$. In the case that both τ and $\gamma \cdot \tau$ lie in the boundary of $\mathcal{F}$, both (τ, z) and $\gamma_* \cdot (\tau, z)$ lie in the boundary of $\mathcal{F}_{[m]}$. If $\gamma = \pm I_2$, we get

$$z \in P_\tau \qquad \text{and} \qquad \pm\,(z + \lambda\,\tau + \mu) \in P_\tau. \tag{2.15}$$

From the definition of P_τ and (2.16), we see that either $\lambda = \mu = 0,\ \gamma \neq -I_2$ or both z and $\pm(z + \lambda\,\tau + \mu)$ lie on the boundary of the parallelepiped P_τ. Hence either both (τ, z) and $\gamma_* \cdot (\tau, z)$ lie in the boundary of $\mathcal{F}_{[m]}$ or $\gamma_* = (I_2, (0, 0; \kappa)) \in \Gamma_{1,m}$. Consequently $\mathcal{F}_{[m]}$ is a fundamental domain for $\Gamma_{1,m}\backslash(\mathbb{H} \times \mathbb{C}^m)$ with respect to the action (2.1). □

Now we consider the Siegel-Jacobi space $\mathbb{H}_{1,1} := \mathbb{H} \times \mathbb{C}$ endowed with the Riemannian metric (cf. (2.2))

$$ds^2_{1;1,1} = \frac{y + v^2}{y^3}\,(dx^2 + dy^2)\, +\, \frac{1}{y}\,(du^2 + dv^2) - \frac{2v}{y^2}\,(dx\,du\, +\, dy\,dv),$$

where $\tau = x + i\,y$ with $x,\ y > 0$ real and $z = u + i\,v$ with u, v real are coordinates in $\mathbb{H}_{1,1}$. Then

$$E_1 := \frac{\partial}{\partial x}, \quad E_2 := \frac{\partial}{\partial y}, \quad E_3 := \frac{\partial}{\partial u}, \quad E_4 := \frac{\partial}{\partial v}$$

form a local frame field on $\mathbb{H}_{1,1}$. Let Γ^k_{ij} $(i, j, k = 1, 2, 3, 4)$ be the Christoffel symbols for the Riemannian connection ∇ determined uniquely by the Riemannian metric $ds^2_{1;1,1}$. That is,

$$\nabla_{E_i} E_j = \sum_{k=1}^{4} \Gamma^k_{ij}\, E_k, \qquad i, j = 1, 2, 3, 4.$$

Lemma 2.2. *For all $i, j, k = 1, 2, 3, 4,\ \Gamma^k_{ij} = \Gamma^k_{ji}$. The Christoffel symbols Γ^k_{ij}'s $(1 \leq i, j, k \leq 4)$ are given by*

$$\begin{aligned}
\Gamma^2_{11} &= \frac{2\,y + y^2}{2\,y^2}, \qquad \Gamma^1_{12} = \Gamma^2_{22} = -\,\frac{2\,y + v^2}{2\,y^2},\\
\Gamma^4_{11} &= \frac{v^3}{2\,y^3}, \qquad \Gamma^3_{12} = \Gamma^4_{22} = -\,\frac{v^3}{2\,y^3},\\
\Gamma^1_{14} &= \Gamma^1_{23} = \Gamma^2_{24} = \Gamma^4_{33} = \frac{v}{2\,y},\\
\Gamma^2_{13} &= \Gamma^3_{34} = \Gamma^4_{44} = -\,\frac{v}{2\,y}, \qquad \Gamma^4_{13} = \frac{y - v^2}{2\,y^2},\\
\Gamma^3_{14} &= \Gamma^3_{23} = \Gamma^4_{24} = -\,\frac{y - v^2}{2\,y^2}, \quad \Gamma^2_{33} = \frac{1}{2}, \quad \Gamma^1_{34} = \Gamma^2_{44} = -\,\frac{1}{2}
\end{aligned}$$

and all other $\Gamma_{ij}^k = 0$.

Proof. It is easy to prove the above lemma. We leave the proof to the reader. □

Proposition 2.2. *Let* $\gamma(t) = \big(x(t) + i\,y(t), u(t) + i\,v(t)\big)$ *be a smooth curve in* $\mathbb{H}_{1,1}$*. For brevity we write*

$$\begin{aligned} \ddot{x} &= \frac{d^2x}{dt^2}, \quad \ddot{y} = \frac{d^2y}{dt^2}, \quad \ddot{u} = \frac{d^2u}{dt^2}, \quad \ddot{v} = \frac{d^2v}{dt^2}, \\ \dot{x} &= \frac{dx}{dt}, \quad \dot{y} = \frac{dy}{dt}, \quad \dot{u} = \frac{du}{dt}, \quad \dot{v} = \frac{dv}{dt}. \end{aligned}$$

Then the curve $\gamma(t)$ *is a geodesic in* $\mathbb{H}_{1,1}$ *with respect to the metric* $ds^2_{1;1,1}$ *if and only if it satisfies the following four differential equations*

$$\ddot{x} - \frac{2\,y + y^2}{2\,y^2}\,\dot{x}\,\dot{y} + \frac{v}{y}\,\dot{x}\,\dot{v} + \frac{v}{y}\,\dot{y}\,\dot{u} - \dot{u}\,\dot{v} = 0 \tag{2.16}$$

$$\ddot{y} + \frac{2\,y + y^2}{2\,y^2}\,\dot{x}^2 - \frac{2\,y + y^2}{2\,y^2}\,\dot{y}^2 + \frac{1}{2}\,\dot{u}^2 - \frac{1}{2}\,\dot{v}^2 - \frac{v}{y}\,\dot{x}\,\dot{u} + \frac{v}{y}\,\dot{y}\,\dot{v} = 0 \tag{2.17}$$

$$\ddot{u} - \frac{v^3}{y^3}\,\dot{x}\,\dot{y} - \frac{y - v^2}{y^2}\,\dot{x}\,\dot{v} - \frac{y - v^2}{y^2}\,\dot{y}\,\dot{u} - \frac{v}{y}\,\dot{u}\,\dot{v} = 0 \tag{2.18}$$

$$\ddot{v} + \frac{v^3}{2y^3}\,\dot{x}^2 - \frac{v^3}{2y^3}\,\dot{y}^2 + \frac{v}{2y}\,\dot{u}^2 - \frac{v}{2y}\,\dot{v}^2 + \frac{y - v^2}{y^2}\,\dot{x}\,\dot{u} - \frac{y - v^2}{y^2}\,\dot{y}\,\dot{v} = 0 \tag{2.19}$$

Proof. Using Lemma 2.2 and the geodesic equations, we obtain the above equations. □

Remark 2.2. If $u = v = 0$, the equations (2.16)-(2.19) reduce to the following two equations

$$\ddot{x} - \frac{2}{y}\,\dot{x}\,\dot{y} = 0 \tag{2.20}$$

and

$$\ddot{y} + \frac{1}{y}\,\dot{x}^2 - \frac{1}{y}\,\dot{y}^2 = 0. \tag{2.21}$$

Thus these two equations (2.20) and (2.21) give geodesics in the Poincaré upper half plane $\mathbb{H}$ which are circles perpendicular to the x-axis or straight lines perpendicular to the x-axis. Therefore the curve $\gamma(t) = \big(x(t) + i\,y(t), 0\big)$ $(-\infty < t < \infty)$ such that $\alpha(t) = x(t) + i\,y(t)$ is a geodesic in $\mathbb{H}$ is a geodesic in $\mathbb{H}_{1,1}$ with respect to the metric $ds^2_{1;1,1}$.

Proposition 2.3. *Let $\gamma(t)$ be a geodesic in $\mathbb{H}_{1,1}$ joining two points $\gamma(0) = (\tau_1, 0)$ and $\gamma(1) = (\tau_2, 0)$ such that $\gamma(t)$ is contained in the subset $\{(\tau, 0) \in \mathbb{H}_{1,1} \mid \tau \in \mathbb{H}\}$. Then the length ρ of the geodesic segment between $\gamma(0) = (\tau_1, 0)$ and $\gamma(1) = (\tau_2, 0)$ is given by*

$$\rho = \log \frac{1 + R^{1/2}}{1 - R^{1/2}}, \tag{2.22}$$

where $R := R(\tau_1, \tau_2)$ is the cross-ratio of τ_1 and τ_2 defined by

$$R(\tau_1, \tau_2) := \frac{\tau_1 - \tau_2}{\tau_1 - \overline{\tau}_2} \cdot \frac{\overline{\tau}_1 - \overline{\tau}_2}{\overline{\tau}_1 - \tau_2}.$$

Proof. By remark 2.2, the length ρ is equal to the length ρ_0 of the geodesic in $\mathbb{H}$ joining τ_1 and τ_2 with respect to the Poincaré metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

It is well known that ρ_0 is given by the formula (2.22). We refer to [17] for the general case. □

Proposition 2.4. *Let (τ_1, z_1) and (τ_2, z_2) be two points in the Siegel-Jacobi space $\mathbb{H} \times \mathbb{C}^m$. Then there exists an element $g = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu; \kappa) \right) \in G^J$ such that*

$$g \cdot (\tau_1, z_1) = (i, 0) \qquad \text{and} \qquad g \cdot (\tau_2, z_2) = \left(i\,\delta, \frac{z_2 + \lambda\,\tau_2 + \mu}{c\,\tau_2 + d} \right)$$

with $\delta > 0$. Therefore the length of the geodesic joining (τ_1, z_1) to (τ_2, z_2) with respect to the Riemannian metric $ds^2_{m;A,B}$ is equal to that of the geodesic joining $(i, 0)$ to $\left(i\,\delta, \frac{z_2 + \lambda\,\tau_2 + \mu}{c\,\tau_2 + d} \right)$ with respect to the metric $ds^2_{m;A,B}$.

Proof. We see that there is an element $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ such that

$$h \cdot \tau_1 = \frac{a\,\tau_1 + b}{c\,\tau_1 + d} = i \qquad \text{and} \qquad h \cdot \tau_2 = \frac{a\,\tau_2 + b}{c\,\tau_2 + d} = i\,\delta$$

with $\delta > 0$. We take

$$\lambda = -\frac{\operatorname{Im} z_1}{\operatorname{Im} \tau_1} \qquad \text{and} \qquad \mu = -\operatorname{Re} z_1 + \frac{\operatorname{Re} \tau_1 \cdot \operatorname{Im} z_1}{\operatorname{Im} \tau_1}$$

We easily see that the element

$$g = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu; \kappa) \right) \in G^J$$

satisfies the condition

$$g \cdot (\tau_1, z_1) = (i, 0) \qquad \text{and} \qquad g \cdot (\tau_2, z_2) = \left(i\,\delta, \frac{z_2 + \lambda\,\tau_2 + \mu}{c\,\tau_2 + d} \right)$$

with $\delta > 0$.

For each fixed element $g \in G^J$, according to the G^J-invariance of the metric $ds^2_{m;A,B}$, the map $\mathbb{F}_g$ of $\mathbb{H} \times \mathbb{C}^m$ defined by the action (2.1) of g is an isometry of $\mathbb{H} \times \mathbb{C}^m$ with respect to the metric $ds^2_{m;A,B}$. Consequently we obtain the second statement. □

Proposition 2.5. *The scalar curvature* $r(p)$ *of the Siegel-Jacobi space* $\left(\mathbb{H}_{1,1},\, ds^2_{1;1,1}\right)$ *is* -3 *for each point* p *of* $\mathbb{H}_{1,1}$.

Proof. Using Lemma 2.2, we obtain the scalar curvature $r(p) = -3$ for each point p of $\mathbb{H}_{1,1}$ by a tedious computation. □

Now we study differential forms on $\mathbb{H} \times \mathbb{C}^m$ invariant under the action (2.1) of $\Gamma_{1,m}$.

Proposition 2.6. *(a) Assume that*

$$\alpha = f(\tau, z)\, d\tau \, + \sum_{k=1}^{m} \phi_k(\tau, z)\, dz_k$$

is a differential 1*-form on* $\mathbb{H} \times \mathbb{C}^m$ *invariant under the action (2.1) of* $\Gamma_{1,m}$*. Then the functions* f *and* ϕ_k $(k = 1, 2, \cdots, m)$ *satisfy the following conditions*

$$f(\gamma \cdot (\tau, z)) = (c\,\tau + d)^2 f(\tau, z) \, + \, (c\,\tau + d) \sum_{k=1}^{m} (c\,z_k + c\,\mu_k - d\,\lambda_k)\, \phi_k(\tau, z) \tag{2.23}$$

and

$$\phi_k(\gamma \cdot (\tau, z)) = (c\,\tau + d)\, \phi_k(\tau, z), \qquad k = 1, 2, \cdots, m \tag{2.24}$$

for all $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu; \kappa) \right) \in \Gamma_{1,m}$ *with* $\lambda = {}^t(\lambda_1, \cdots, \lambda_m) \in \mathbb{Z}^m$ *and* $\mu = {}^t(\mu_1, \cdots, \mu_m) \in \mathbb{Z}^m$.

(b) Let

$$\eta = \, d\tau \wedge dz_1 \wedge dz_2 \wedge \cdots \wedge dz_m$$

be a differential $(m+1)$*-form on* $\mathbb{H} \times \mathbb{C}^m$*. Assume that*

$$\theta = g(\tau, z)\, \eta^{\otimes \ell}, \qquad \ell = 1, 2, 3, \cdots,$$

is a differential $\ell(m+1)$*-form on* $\mathbb{H} \times \mathbb{C}^m$ *invariant under the action (2.1) of* $\Gamma_{1,m}$*. Then the function* g *satisfies the following condition*

$$g(\gamma \cdot (\tau, z)) = (c\,\tau + d)^{\ell(m+2)} g(\tau, z) \tag{2.25}$$

for all $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu; \kappa) \right) \in \Gamma_{1,m}$.

(c) For $k = 1, 2, \cdots, m$, *we let*

$$\widetilde{\omega}_k = (-1)^{m-k}\, d\tau \wedge dz_1 \wedge \cdots \wedge dz_{k-1} \wedge \widehat{dz_k} \wedge dz_{k+1} \wedge \cdots \wedge dz_m$$

be a differential m*-form on* $\mathbb{H} \times \mathbb{C}^m$. *Assume that*

$$\beta = \sum_{k=1}^{m} a_k(\tau, z)\, \widetilde{\omega}_k \; + \; (-1)^m\, b(\tau, z)\, dz_1 \wedge \cdots \wedge dz_m$$

is a differential m*-form on* $\mathbb{H} \times \mathbb{C}^m$ *invariant under the action (2.1) of* $\Gamma_{1,m}$. *Then the functions* $a(\tau, z)$ *and* b_k $(k = 1, 2, \cdots, m)$ *satisfy the following conditions*

$$a_k(\gamma \cdot (\tau, z)) = (c\,\tau + d)^{m+1}\, a_k(\tau, z) - (c\,\tau + d)^m (c\, z_k + c\,\mu_k - d\,\lambda_k)\, b(\tau, z) \tag{2.26}$$

for $k - 1, 2, \cdots, m$ *and*

$$b(\gamma \cdot (\tau, z)) = (c\,\tau + d)^m\, b(\tau, z) \tag{2.27}$$

for all $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu; \kappa) \right) \in \Gamma_{1,m}$ *with* $\lambda = {}^t(\lambda_1, \cdots, \lambda_m) \in \mathbb{Z}^m$ *and* $\mu = {}^t(\mu_1, \cdots, \mu_m) \in \mathbb{Z}^m$.

Proof. For $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu; \kappa) \right) \in \Gamma_{1,m}$ with $\lambda = {}^t(\lambda_1, \cdots, \lambda_m) \in \mathbb{Z}^m$ and $\mu = {}^t(\mu_1, \cdots, \mu_m) \in \mathbb{Z}^m$ and $(\tau, z) \in \mathbb{H} \times \mathbb{C}^m$ with $z = {}^t(z_1, \cdots, z_m) \in \mathbb{C}^m$, we set $(\tau^*, z^*) = \gamma \cdot (\tau, z)$. In other words,

$$\tau^* = \frac{a\,\tau + b}{c\,\tau + d}, \qquad z_k^* = \frac{z_k + \lambda_k\,\tau + \mu_k}{c\,\tau + d}, \qquad k = 1, 2, \cdots, m.$$

Then we have

$$d\tau^* = \frac{d\tau}{(c\,\tau + d)^2} \tag{2.28}$$

and

$$dz_k^* = \left\{ \frac{\lambda_k}{c\,\tau + d} - \frac{c\,(z_k + \lambda_k\,\tau + \mu_k)}{(c\,\tau + d)^2} \right\} d\tau \; + \; \frac{dz_k}{c\,\tau + d}, \qquad k = 1, 2, \cdots, m. \tag{2.29}$$

Using the formulas (2.28) and (2.29), we obtain the desired results (a), (b) and (c). □

3. The center of the universal enveloping algebra of $\mathfrak{g}^J$

In this section we describe the center of the universal enveloping algebra of the complexication of the Jacobi Lie algebra $\mathfrak{g}^J$ explicitly.

Let $\mathfrak{g}^J_{\mathbb{C}}$ be the complexification of the Jacobi Lie algebra $\mathfrak{g}^J$. We put the 2×2 matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $\{H, E, F\}$ is a basis of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Let $\epsilon_{ij}\,(1 \le i \le m,\ j = 1, 2)$ be the $m \times 2$ matrices whose (i, j)-th entry is 1 and whose other entries are zero, and let E_{kl} be the $m \times m$ elementary matrix whose (k, l)-th entry is 1 and whose other entries are zero. We set $e_i := \epsilon_{i1},\ f_i := \epsilon_{i2}\ (1 \le i \le m)$ and

$$R_{kl} := \frac{1}{2}\,(E_{kl} + E_{ji}), \quad R_{kl} = R_{lk}, \quad 1 \le k, l \le m.$$

Then $\{H, E, F, e_i, f_i, R_{kl} \mid 1 \le i \le m,\ 1 \le k \le l \le m\}$ is a basis for $\mathfrak{g}^J_{\mathbb{C}}$. It is easily seen that

$$\mathcal{Z}_m := \left\{ (0, (0, 0, R)) \in \mathfrak{g}^J_{\mathbb{C}} \mid R = {}^tR \in \mathbb{C}^{(m,m)} \right\}$$

is the center of $\mathfrak{g}^J_{\mathbb{C}}$.

Lemma 3.1. *We have the following.*

(1) $[H, E] = 2\,E, \quad [H, F] = -2\,F, \quad [E, F] = H.$

(2) $[H, e_i] = -e_i, \quad [H, f_i] = f_i, \quad 1 \le i \le m.$

(3) $[E, e_i] = f_i, \quad [E, f_i] = 0, \quad 1 \le i \le m.$

(4) $[F, e_i] = 0, \quad [F, f_i] = -e_i, \quad 1 \le i \le m.$

(5) $[e_i, f_j] = 2\,R_{ij}, \quad 1 \le i, j \le m.$

Proof. The proof follows immediately from the fact that

$$\begin{aligned} &(3.1) \quad [(X_1, (P_1, Q_1, R_1)), (X_2, (P_2, Q_2, R_2))] \\ &\quad = \Big([X_1, X_2], \big((P_1, Q_1)X_2 - (P_2, Q_2)X_1, P_1\,{}^tQ_2 - P_2\,{}^tQ_1 + Q_2\,{}^tP_1 - Q_1\,{}^tP_2\big)\Big), \end{aligned}$$

where $X_1, X_2 \in \mathfrak{sl}_2(\mathbb{C})$, $[X_1, X_2] = X_1X_2 - X_2X_1$, $P_i, Q_i \in \mathbb{C}^{(m,1)}\,(i = 1, 2)$, $R_1, R_2 \in \mathbb{C}^{(m,m)}$ with $R_1 = {}^tR_1$ and $R_2 = {}^tR_2$. □

Formally we put

$$e := \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{pmatrix}, \qquad f := \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix},$$

and

$$R := \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1m} \\ R_{21} & R_{22} & \cdots & R_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ R_{m1} & R_{m2} & \cdots & R_{mm} \end{pmatrix}, \quad R_{kl} = R_{lk}, \quad 1 \le k, l \le m.$$

Theorem 3.1. *The center* $\mathcal{Z}_m\big(\mathfrak{g}^J_{\mathbb{C}}\big)$ *of the universal enveloping algebra* $\mathcal{U}(\mathfrak{g}^J_{\mathbb{C}})$ *of* $\mathfrak{g}^J_{\mathbb{C}}$ *is given by*

$$\mathcal{Z}_m\big(\mathfrak{g}^J_{\mathbb{C}}\big) = \mathbb{C}[\Omega_m, R_{kl} \mid 1 \le k \le l \le m].$$

That is, $\mathcal{Z}_m\big(\mathfrak{g}^J_{\mathbb{C}}\big)$ *is a polynomial algebra on* $1 + \frac{m(m+1)}{2}$ *generators* Ω_m, R_{kl} $(1 \le k \le l \le m)$. *Here*

$$\begin{aligned} \Omega_m : \;=\; & \det R \left\{ H^2 - (m+2)H \; + 4\,EF \right\} \\ & + \det R \left\{ E\, {}^t e R^{-1} e - {}^t f R^{-1} f F - \left(H - \frac{m+3}{2} \right) {}^t f R^{-1} e \right\} \\ & + \det R \left\{ \frac{1}{4}\, {}^t f \big({}^t f R^{-1} e \big) R^{-1} e - \frac{1}{4} \big({}^t e R^{-1} f \big) \big({}^t e R^{-1} e \big) \right\} \end{aligned}$$

is a Casimir operator of $\mathcal{U}(\mathfrak{g}^J\mathbb{C})$ *of degree* $m+2$.

Proof. Using the method computing the center of the universal enveloping algebra of a certain class of semidirect sum Lie algebras invented by Campoamer-Stursburg and Low [6] (cf. [2], [15]), Conley and Raum [5] proved the above theorem. We refer to [5] for the detail. □

Let $\gamma : G^J \times \big(\mathbb{H} \times \mathbb{C}^m\big) \longrightarrow \mathbb{C}^\times$ be a scalar cocycle with respect to the action (2.1). This means that γ is a smooth function satisfying the cocycle condition

$$\gamma(g_1 g_2, (\tau, z)) = \gamma(g_1, g_2 \cdot (\tau, z))\, \gamma(g_2, (\tau, z)) \tag{3.2}$$

for all $g_1, g_2 \in G^J$ and $(\tau, z) \in \mathbb{H} \times \mathbb{C}^m$. Then we get the map

$$\widehat{\gamma}(g) : G^J \longrightarrow C^\infty(\mathbb{H} \times \mathbb{C}^m)$$

defined by

$$\widehat{\gamma}(g)(\tau, z) := \gamma(g, (\tau, z)), \quad g \in G^J,\ (\tau, z) \in \mathbb{H} \times \mathbb{C}^m.$$

Then we obtain the right action $|_\gamma$ of G^J on $C^\infty(\mathbb{H} \times \mathbb{C}^m)$ defined by

$$(g \cdot f)(\tau, z) := \big(f|_\gamma [g^{-1}] \big)(\tau, z) := \gamma(g^{-1}, (\tau, z)) f(g^{-1} \cdot (\tau, z)), \tag{3.3}$$

where $g \in G^J$, $f \in C^\infty(\mathbb{H} \times \mathbb{C}^m)$ and $(\tau, z) \in \mathbb{H} \times \mathbb{C}^m$.

We note that the differential $d\widehat{\gamma}$ of $\widehat{\gamma}$ at the identity is given by

$$d\widehat{\gamma}(Y)(\tau, z) = \frac{d}{dt}\Big|_{t=0} \gamma(\exp(tY), (\tau, z)).$$

Therefore we have the differential right action $|_\gamma$ of $\mathfrak{g}^J_{\mathbb{C}}$ on $C^\infty(\mathbb{H}\times\mathbb{C}^m)$ defined by

$$(3.4)\qquad (\phi|_\gamma[Y])(\tau,z): \;=\; \frac{d}{dt}\Big|_{t=0}(\gamma(\exp(tY),(\tau,z))\phi(\exp(tY)\cdot(\tau,z)))$$

$$(3.5)\qquad \;=\; \gamma(Y,(\tau,z))\phi(\tau,z) + \frac{d}{dt}\Big|_{t=0}\phi(\exp(tY),(\tau,z)),$$

where $Y\in\mathfrak{g}^J_{\mathbb{C}}$ and $\phi\in C^\infty(\mathbb{H}\times\mathbb{C}^m)$. The action (3.4) extends to $\mathcal{U}(\mathfrak{g}^J_{\mathbb{C}})$ as usual, and elements of $\mathcal{U}(\mathfrak{g}^J_{\mathbb{C}})$ of order r act by differential operators of order $\le r$.

Let $\mathbb{D}_\gamma$ be the algebra of all differential operators D on $\mathbb{H}\times\mathbb{C}^m$ satisfying the following condition

$$(3.6)\qquad (D\phi)|_\gamma[g] = D\big(\phi|_\gamma[g]\big)$$

for all $g\in G^J$ and for all $\phi\in C^\infty(\mathbb{H}\times\mathbb{C}^m)$. Since G^J is connected, $\mathbb{D}_\gamma$ is the algebra of all differential operators $\mathbb{D}_\gamma$on $\mathbb{H}\times\mathbb{C}^m$ commuting with the $|_\gamma$-action of $\mathfrak{g}^J_{\mathbb{C}}$. In particular, the action $|_\gamma$ maps the center $\mathcal{Z}_m(\mathfrak{g}^J_{\mathbb{C}})$ of $\mathcal{U}(\mathfrak{g}^J_{\mathbb{C}})$ into the center $\mathcal{Z}_m(\mathbb{D}_\gamma)$ of $\mathbb{D}_\gamma$.

Throughout this section we let $\mathcal{M}$ be a positive definite half-integral symmetric matrix of degree m and let $k\in\mathbb{Z}^+$. We let $\gamma_{k,\mathcal{M}}: G^J\times(\mathbb{H}\times\mathbb{C}^m)\longrightarrow\mathbb{C}^\times$ be the canonical automorphic factor for G^J on $\mathbb{H}\times\mathbb{C}^m$ defined by

$$\gamma_{k,\,\mathcal{M}}\big((M,(\lambda,\mu;\kappa)),(\tau,z)\big):$$

$$(3.7)\qquad =\; (c\tau+d)^k\, e^{2\pi i\,\mathcal{M}[z+\lambda\tau+\mu]\,c\,(c\tau+d)^{-1}}\, e^{-2\pi i\,\mathrm{tr}\left(\mathcal{M}(\tau\lambda\,{}^t\lambda+2\,\lambda\,{}^tz+\kappa+\mu\,{}^t\lambda)\right)},$$

where $(M,(\lambda,\mu;\kappa))\in G^J$ with $M=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{R})$, $(\lambda,\mu;\kappa)\in H^{(m)}_{\mathbb{R}}$ and $(\tau,z)\in\mathbb{H}\times\mathbb{C}^m$.

For brevity we write

$$\partial_\tau: \;=\; \frac{\partial}{\partial\tau}=\frac12\left(\frac{\partial}{\partial x}-i\,\frac{\partial}{\partial y}\right),\qquad \partial_{\overline\tau}:=\frac{\partial}{\partial\overline\tau}=\frac12\left(\frac{\partial}{\partial x}+i\,\frac{\partial}{\partial y}\right),$$

$$\partial_{z_j}: \;=\; \frac{\partial}{\partial z_j}=\frac12\left(\frac{\partial}{\partial u_j}-i\,\frac{\partial}{\partial v_j}\right),\quad 1\le j\le m,$$

$$\partial_{\overline z_j}: \;=\; \frac{\partial}{\partial \overline z_j}=\frac12\left(\frac{\partial}{\partial u_j}+i\,\frac{\partial}{\partial v_j}\right),\quad 1\le j\le m,$$

$$\partial_z: \;=\; {}^t\big(\partial_{z_1},\partial_{z_2},\cdots,\partial_{z_m}\big),\quad \partial_{\overline z}:={}^t\big(\partial_{\overline z_1},\partial_{\overline z_2},\cdots,\partial_{\overline z_m}\big).$$

Lemma 3.2. *Let* $\mathcal{M}$ *and* k *be as above. We set* $\widetilde{\mathcal{M}}:=2\pi i\mathcal{M}$. *Then we have the following:*

$$(3.8)\qquad |_{\gamma_{k,\,\mathcal{M}}}[E] \;=\; 2\,\mathrm{Re}\,(\partial_\tau),$$

$$(3.9)\qquad |_{\gamma_{k,\,\mathcal{M}}}[F] \;=\; -2\,\mathrm{Re}\big(\tau\,(\tau\,\partial_\tau+{}^tz\,\partial_z)\big)-k\,\tau-\widetilde{\mathcal{M}}[z],$$

$$(3.10)\qquad |_{\gamma_{k,\,\mathcal{M}}}[H] \;=\; 2\,\mathrm{Re}\big(2\,\tau\,\partial_\tau+{}^tz\,\partial_z\big)+k,$$

$$(3.11)\qquad |_{\gamma_{k,\,\mathcal{M}}}[(0,(P,Q,R))] \;=\; 2\,\mathrm{Re}\big({}^t(P\,\tau+Q)\,\partial_z\big)+2\,{}^tP\widetilde{\mathcal{M}}\,z+\mathrm{tr}\big(R\widetilde{\mathcal{M}}\big).$$

Proof. We observe that if $(X,(P,Q,R))\in\mathfrak{g}^J_{\mathbb{C}}$ with $X\in\mathfrak{sl}_2(\mathbb{C}),\ P,Q\in\mathbb{C}^{(m,1)}$ and $R={}^tR\in\mathbb{C}^{(m,m)}$, then

$$\exp\big((X,(P,Q,R))\big)=\Big(\exp(X),\big((P,Q)\,g(X),R-(P,Q)\,h(X)\,{}^t(-Q,P)\big)\Big), \tag{3.12}$$

where

$$\exp(t):=\sum_{n=0}^{\infty}\frac{t^n}{n!},\quad g(t):=\frac{e^t-1}{t}\quad\text{and}\quad h(t):=\frac{e^t-1-t}{t}.$$

Using the formula (3.12) we easily obtain the formulas (3.8)-(3.11). □.

Theorem 3.2.

$$|_{\gamma_{k,\,\mathcal{M}}}[\Omega_m]=\det\big(\widetilde{\mathcal{M}}\big)\ \big\{k(k-m-2)\,-\,2\,\mathcal{C}^{k,\,\mathcal{M}}\big\}, \tag{3.13}$$

where

$$\begin{aligned}\mathcal{C}^{k,\,\mathcal{M}}:\ =\ & -8\,y^2\partial_\tau\partial_{\overline{\tau}}\,+\,4\,i\Big(k-\frac{m}{2}\Big)\,y\,\partial_{\overline{\tau}}\\ & +2\,y^2\Big(\partial_{\overline{\tau}}\,\widetilde{\mathcal{M}}^{-1}[\partial_z]\,+\,\partial_\tau\,\widetilde{\mathcal{M}}^{-1}[\partial_{\overline{z}}]\Big)\,-\,8\,y\,\partial_\tau\,{}^tv\,\partial_{\overline{z}}\\ & -\frac{1}{2}\,y^2\Big\{\widetilde{\mathcal{M}}^{-1}[\partial_{\overline{z}}]\,\widetilde{\mathcal{M}}^{-1}[\partial_z]-\,{}^t\big(\partial_{\overline{z}}\widetilde{\mathcal{M}}^{-1}\partial_z\big)^2\Big\}\,+\,2\,y\,\big({}^tv\,\partial_{\overline{z}}\big)\,{}^t\partial_z\widetilde{\mathcal{M}}^{-1}\partial_u\\ & -\frac{i}{2}\,(2k-m+1)\,y\,{}^t\partial_{\overline{z}}\,\widetilde{\mathcal{M}}^{-1}\partial_u\,+\,2\,{}^tv\,\big({}^tv\,\partial_{\overline{z}}\big)\partial_{\overline{z}}\,+\,i\,(2k-m-1)\,{}^tv\,\partial_{\overline{z}}.\end{aligned}$$

The operator $\mathcal{C}^{k,\,\mathcal{M}}$ *generates the image of the* $|_{\gamma_{k,\mathcal{M}}}$*-action of the center* $\mathcal{Z}_m(\mathfrak{g}^J_{\mathbb{C}})$*. In particular,* $\mathcal{C}^{k,\,\mathcal{M}}$ *is an element of the center of* $\mathbb{D}_{\gamma_{k,\mathcal{M}}}$*.*

Proof. We write $\widetilde{\mathcal{M}}=\big(\widetilde{\mathcal{M}}_{pq}\big)$. According to (3.11), we have the relation $|_{\gamma_{k,\mathcal{M}}}[R_{pq}]=\widetilde{\mathcal{M}}_{pq}$ for all $1\le p\le q\le m$. The proof follows from Theorem 3.1. and Lemma 3.2. □

4. Invariant differential operators on $\mathbb{H}\times\mathbb{C}^m$

For brevity we put

$$T_{1,m}:=\mathbb{C}\times\mathbb{C}^m.$$

We define the real linear map $\Phi_m:\mathfrak{p}^J\longrightarrow T_{1,m}$ by

$$\Phi_m\left(\begin{pmatrix}x & y\\ y & -x\end{pmatrix},(P,Q,0)\right)=(x+i\,y,P+i\,Q), \tag{4.1}$$

where $\left(\begin{pmatrix}x & y\\ y & -x\end{pmatrix},(P,Q,0)\right)\in\mathfrak{p}^J$. Obviously Φ_m is a real linear isomorphism of $\mathfrak{p}^J$ onto $T_{1,m}$.

Let $S(m,\mathbb{R})$ denote the additive group consisting of all $m \times m$ real symmetric matrices. We define the group isomorphism $\theta_m : K^J \longrightarrow U(1) \times S(m,\mathbb{R})$ by

$$\theta_m \left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, (0,0;\kappa) \right) = (a + i\,b, \kappa), \tag{4.2}$$

where $\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, (0,0;\kappa) \right) \in K^J$.

Theorem 4.1. *The adjoint representation Ad of K^J on $\mathfrak{p}^J$ is compatible with the natural action of $U(1) \times S(m,\mathbb{R})$ on $T_{1,m} = \mathbb{C} \times \mathbb{C}^m$ defined by*

$$(h,\kappa)\cdot(w,\xi) := (h^2 w, h\,\xi), \quad h \in U(1),\ \kappa \in S(m,\mathbb{R}),\ w \in \mathbb{C},\ \xi \in \mathbb{C}^m \tag{4.3}$$

through the map Φ_m and θ_m. Precisely if $k^J \in K^J$ and $\alpha \in \mathfrak{p}^J$, then we have the following equality

$$\Phi_m\big(Ad(k^J)\alpha\big) = \theta_m(k^J)\cdot\Phi_m(\alpha). \tag{4.4}$$

We recall that we identified $\mathfrak{p}^J$ with $\mathbb{C} \times \mathbb{C}^m$.

Proof. We refer to [26] for the proof. □

The action (4.3) induces the action of $U(1)$ on the polynomial algebra $\mathrm{Pol}_{[m]} := \mathrm{Pol}(T_{1,m})$. We denote by $\mathrm{Pol}_{[m]}^{U(1)}$ the subalgebra of $\mathrm{Pol}_{[m]}$ consisting of $U(1)$-invariants. We let $\mathbb{D}\big(\mathbb{H} \times \mathbb{C}^m\big)$ be the algebra of all differential operators invariant under the action (2.1) of G^J. According to [7], one gets a canonical linear bijection

$$\Theta_{[m]} : \mathrm{Pol}_{[m]}^{U(1)} \longrightarrow \mathbb{D}\big(\mathbb{H} \times \mathbb{C}^m\big) \tag{4.5}$$

of $\mathrm{Pol}_{[m]}^{U(1)}$ onto $\mathbb{D}\big(\mathbb{H} \times \mathbb{C}^m\big)$. But $\Theta_{[m]}$ is not multiplicative. The map $\Theta_{[m]}$ is described explicitly as follows. Let $\big\{\eta_\alpha \mid 1 \le \alpha \le 2(m+1)\big\}$ be a basis of $\mathfrak{p}^J$. If $P \in \mathrm{Pol}_{[m]}^{U(1)}$, then

$$\Big(\Theta_{[m]}(P)f\Big)(gK^J) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g \exp\left(\sum_{\alpha=1}^{2(m+1)} t_\alpha \eta_\alpha \right) K^J \right) \right]_{(t_\alpha)=0}, \tag{4.6}$$

where $g \in G^J$ and $f \in C^\infty(\mathbb{H} \times \mathbb{C}^m)$.

Theorem 4.2. $\mathrm{Pol}_{[m]}^{U(1)}$ *is generated by*

$$q(w,\xi) = \mathrm{tr}\big(w\,\overline{w}\big), \tag{4.7}$$

$$\alpha_{kp}(w,\xi) = \mathrm{Re}\,\big(\xi\,{}^t\overline{\xi}\big)_{kp}, \quad 1 \le k \le p \le m, \tag{4.8}$$

$$\beta_{lq}(w,\xi) = \mathrm{Im}\,\big(\xi\,{}^t\overline{\xi}\big)_{lq}, \quad 1 \le l < q \le m, \tag{4.9}$$

$$f_{kp}(w,\xi) = \mathrm{Re}\,(\overline{w}\,\xi\,{}^t\xi)_{kp}, \quad 1 \le k \le p \le m, \tag{4.10}$$

$$g_{kp}(w,\xi) = \mathrm{Im}\,(\overline{w}\,\xi\,{}^t\xi)_{kp}, \quad 1 \le k \le p \le m, \tag{4.11}$$

where $w \in \mathbb{C}$ *and* $\xi \in \mathbb{C}^m$.

Proof. We refer to [9] or [26] for the general case. □

We let

$$w = r + i\,s \in \mathbb{C} \quad \text{and} \quad \xi = {}^t(\xi_1, \cdots, \xi_m) \in \mathbb{C}^m \text{ with } \xi_k = \zeta_k + i\,\eta_k,\ 1 \leq k \leq m,$$

where $r, s, \zeta_1, \eta_1, \cdots, \zeta_m, \eta_m$ are real. The invariants q, α_{kp}, β_{lq}, f_{kp} and g_{kp} are expressed in terms of $r, s, \zeta_k, \eta_l\ (1 \leq k, l \leq m)$ as follows:

$$\begin{aligned}
q(w,\xi) &= r^2 + s^2, \\
\alpha_{kp}(w,\xi) &= \zeta_k\zeta_p + \eta_k\eta_p, \quad 1 \leq k \leq p \leq m, \\
\beta_{lq}(w,\xi) &= \zeta_q\eta_l - \zeta_l\eta_q, \quad 1 \leq l < q \leq m, \\
f_{kp}(w,\xi) &= r(\zeta_k\zeta_p - \eta_k\eta_p) + s(\zeta_k\eta_p + \eta_k\zeta_p), \quad 1 \leq k \leq p \leq m, \\
g_{kp}(w,\xi) &= r(\zeta_k\eta_p + \eta_k\zeta_p) - s(\zeta_k\zeta_p - \eta_k\eta_p), \quad 1 \leq k \leq p \leq m.
\end{aligned}$$

Theorem 4.3. *The* $\frac{m(m+1)}{2}$ *relations*

$$f_{kp}^2 + g_{kp}^2 = q\,\alpha_{kk}\,\alpha_{pp}, \quad 1 \leq k \leq p \leq m \tag{4.12}$$

exhaust all the relations among a complete set of generators q, α_{kp}, β_{lq}, f_{kp} *and* g_{kp} *of* $\mathrm{Pol}_{[m]}^{U(1)}$ *with* $1 \leq k \leq p \leq m$ *and* $1 \leq l < q \leq m$.

Theorem 4.4. *The action of* $U(1)$ *on* $Pol_{1,m}$ *is not multiplicity-free. In fact, if*

$$\mathrm{Pol}_{[m]} = \sum_{\sigma \in \widehat{U(1)}} m_\sigma\, \sigma,$$

then $m_\sigma = \infty$.

For the proofs of the above theorems we refer to [26].

We consider the case $m = 1$. For a coordinate (w, ξ) in $T_{1,1}$, we write $w = r + i\,s,\ \xi = \zeta + i\,\eta,\ r, s, \zeta, \eta$ real. The author [21] proved that the algebra $\mathrm{Pol}_{[1]}^{U(1)}$ is generated by

$$\begin{aligned}
q(w,\xi) &= \frac{1}{4}\, w\,\overline{w} = \frac{1}{4}\big(r^2 + s^2\big), \\
\alpha(w,\xi) &= \xi\,\overline{\xi} = \zeta^2 + \eta^2, \\
\phi(w,\xi) &= \frac{1}{2}\,\mathrm{Re}\,\big(\xi^2\overline{w}\big) = \frac{1}{2}\, r\big(\zeta^2 - \eta^2\big) + s\,\zeta\eta, \\
\psi(w,\xi) &= \frac{1}{2}\,\mathrm{Im}\,\big(\xi^2\overline{w}\big) = \frac{1}{2}\, s\big(\eta^2 - \zeta^2\big) + r\,\zeta\eta.
\end{aligned}$$

In [21], using Formula (3.6) the author calculated explicitly the images

$$D_1 = \Theta_{[1]}(q), \quad D_2 = \Theta_{[1]}(\alpha), \quad D_3 = \Theta_{[1]}(\phi) \quad \text{and} \quad D_4 = \Theta_{[1]}(\psi)$$

of q, α, ϕ and ψ under the Halgason map $\Theta_{[1]}$. We can show that the algebra $\mathbb{D}(\mathbb{H}\times\mathbb{C})$ is generated by the following differential operators

$$\begin{aligned} D_1 =& y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \\ & + 2\, y\, v \left(\frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v} \right), \end{aligned}$$

$$D_2 = y \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right),$$

$$\begin{aligned} D_3 =& y^2 \frac{\partial}{\partial y} \left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) - 2y^2 \frac{\partial^3}{\partial x \partial u \partial v} \\ & - \left(v \frac{\partial}{\partial v} + 1 \right) D_2 \end{aligned}$$

and

$$\begin{aligned} D_4 =& y^2 \frac{\partial}{\partial x} \left(\frac{\partial^2}{\partial v^2} - \frac{\partial^2}{\partial u^2} \right) - 2\, y^2 \frac{\partial^3}{\partial y \partial u \partial v} \\ & - v \frac{\partial}{\partial u} D_2, \end{aligned}$$

where $\tau = x + iy$ and $z = u + iv$ with real variables x, y, u, v. Moreover, we have

$$\begin{aligned} D_1 D_2 - D_2 D_1 =& 2\, y^2 \frac{\partial}{\partial y} \left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) \\ & - 4\, y^2 \frac{\partial^3}{\partial x \partial u \partial v} - 2 \left(v \frac{\partial}{\partial v} D_2 + D_2 \right). \end{aligned}$$

In particular, the algebra $\mathbb{D}(\mathbb{H}\times\mathbb{C})$ is not commutative. We refer to [1, 21] for more detail.

Recently Hiroyuki Ochiai [13] (see also [1]) proved the following result.

Theorem 4.5. *We have the following relations*

(*a*) $[D_1, D_2] = 2D_3$

(*b*) $[D_1, D_3] = 2D_1 D_2 - 2D_3$

(*c*) $[D_2, D_3] = -D_2^2$

(*d*) $[D_4, D_1] = 0$

(*e*) $[D_4, D_2] = 0$

(*f*) $[D_4, D_3] = 0$

(g) $D_3^2 + D_4^2 = D_2 D_1 D_2$

These seven relations exhaust all the relations among the generators D_1, D_2, D_3 *and* D_4 *of* $\mathbb{D}(\mathbb{H}\times\mathbb{C})$.

Remark 4.1. According to Theorem 4.5, we see that D_4 is a generator of the center of $\mathbb{D}(\mathbb{H}\times\mathbb{C})$. We observe that the Lapalcian

$$\Delta_{1;A,B} = \frac{4}{A}\,D_1 + \frac{4}{B}\,D_2 \qquad \text{(see (2.5))}$$

of $(\mathbb{H}\times\mathbb{C}, ds^2_{1;A,B})$ does not belong to the center of $\mathbb{D}(\mathbb{H}\times\mathbb{C})$.

5. Maass-Jacobi Forms due to Yang

Using G^J-invariant differential operators on the Siegel-Jacobi space, we introduce a notion of Maass-Jacobi forms.

Definition 5.1. Let

$$\Gamma_{1,m} := SL_2(\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(m)}$$

be the discrete subgroup of G^J, where

$$H_{\mathbb{Z}}^{(m)} = \Big\{ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(m)} \mid \lambda,\mu,\kappa \text{ are integral } \Big\}.$$

A smooth function $f : \mathbb{H}\times\mathbb{C}^m \longrightarrow \mathbb{C}$ is called a Maass-Jacobi form on $\mathbb{H}\times\mathbb{C}^m$ if f satisfies the following conditions (MJ1)-(MJ3) :

(MJ1) f is invariant under $\Gamma_{1,m}$.
(MJ2) f is an eigenfunction of the Laplacian $\Delta_{m;A,B}$ (cf. Formula (2.5)).
(MJ3) f has a polynomial growth, that is, there exist a constant $C>0$ and a positive integer N such that

$$|f(x+iy,z)| \le C\,|p(y)|^N \quad \text{as } y \longrightarrow \infty,$$

where $p(y)$ is a polynomial in y.

Remark 5.1. Let $\mathbb{D}_*$ be a commutative subalgebra of $\mathbb{D}(\mathbb{H}\times\mathbb{C}^m)$ containing the Laplacian $\Delta_{m;A,B}$. We say that a smooth function $f : \mathbb{H}\times\mathbb{C}^m \longrightarrow \mathbb{C}$ is a Maass-Jacobi form with respect to $\mathbb{D}_*$ if f satisfies the conditions $(MJ1)$, $(MJ2)_*$ and $(MJ3)$: the condition $(MJ2)_*$ is given by

$(MJ2)_*$ f is an eigenfunction of any invariant differential operator in $\mathbb{D}_*$.

It is natural to propose the following problems.

Problem A : Find all the eigenfunctions of $\Delta_{m;A,B}$.

Problem B : Construct Maass-Jacobi forms.

Problem C : Develop the spectral theory of the Laplacian $\Delta_{m;A,B}$ on a fundamental domain for the Siegel-Jacobi space $\mathbb{H}\times\mathbb{C}^m$ with respect to $\Gamma_{1,m}$.

If we find a *nice* eigenfunction ϕ of the Laplacian $\Delta_{m;A,B}$, we can construct a Maass-Jacobi form f_ϕ on $\mathbb{H}\times\mathbb{C}^m$ in the usual way defined by

(5.1)
$$f_\phi(\tau,z) := \sum_{\gamma\in\Gamma^\infty_{1,m}\backslash\Gamma_{1,m}} \phi\big(\gamma\cdot(\tau,z)\big),$$

where

$$\Gamma^\infty_{1,m} = \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda,\mu;\kappa) \right) \in \Gamma_{1,m} \,\Big|\, c=0 \right\}$$

is a subgroup of $\Gamma_{1,m}$.

We consider the simple case $m=1$ and $A=B=1$. We take a coordinate $(\tau,z)\in\mathbb{H}\times\mathbb{C}$ with $\tau=x+iy,\ x\in\mathbb{R},\ y>0$ and $z=u+iv,\ u,v$ real. A metric $ds^2_{1;1,1}$ on $\mathbb{H}\times\mathbb{C}$ given by

$$\begin{aligned} ds^2_{1;1,1} &= \frac{y+v^2}{y^3}\,(\,dx^2+dy^2\,) + \frac{1}{y}\,(\,du^2+dv^2\,) \\ &\quad - \frac{2v}{y^2}\,(\,dx\,du + dy\,dv\,) \end{aligned}$$

is a G^J-invariant Kähler metric on $\mathbb{H}\times\mathbb{C}$. Its Laplacian $\Delta_{1;1,1}$ is given by

$$\begin{aligned} \Delta_{1;1,1} &= y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) \\ &\quad + (\,y+v^2\,)\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right) \\ &\quad + 2\,y\,v\left(\frac{\partial^2}{\partial x\partial u}+\frac{\partial^2}{\partial y\partial v}\right). \end{aligned}$$

We provide some examples of eigenfunctions of $\Delta_{1;1,1}$.

(1) $h(x,y)=y^{\frac12}K_{s-\frac12}(2\pi|a|y)\,e^{2\pi iax}$ $(s\in\mathbb{C},\ a\neq 0)$ with eigenvalue $s(s-1)$. Here

(5.2)
$$K_s(z) := \frac12\int_0^\infty \exp\left\{-\frac{z}{2}(t+t^{-1})\right\} t^{s-1}\,dt,$$

where $\mathrm{Re}\,z>0$.

(2) $y^s,\ y^sx,\ y^su$ $(s\in\mathbb{C})$ with eigenvalue $s(s-1)$.

(3) $y^sv,\ y^suv,\ y^sxv$ with eigenvalue $s(s+1)$.

(4) $x,\ y,\ u,\ v,\ xv,\ uv$ with eigenvalue 0.

(5) All Maass wave forms.

We let $f;\mathbb{H}\times\mathbb{C}\longrightarrow\mathbb{C}$ be a Maass-Jacobi form with $\Delta_{1;1,1}f=\Lambda f$. Then f satisfies the following invariance relations

$$f(\tau+n,z)=f(\tau,z)\qquad\text{for all } n\in\mathbb{Z}$$

and

$$f(\tau,z+n_1\tau+n_2)=f(\tau,z)\qquad\text{for all } n_1,n_2\in\mathbb{Z}.$$

Therefore f is a smooth function on $\mathbb{H}\times\mathbb{C}$ which is periodic in x and u with period 1. So f has the following Fourier series

$$f(\tau,z)=\sum_{n\in\mathbb{Z}}\sum_{r\in\mathbb{Z}}c_{n,r}(y,v)\,e^{2\pi i\,(nx+ru)}. \tag{5.3}$$

For two fixed integers n and r, for brevity, we set $\varphi(y,v)=c_{n,r}(y,v)$. Then φ satisfies the following differential equation

$$\left[y^2\frac{\partial^2}{\partial y^2}+(y+v^2)\frac{\partial^2}{\partial v^2}+2yv\,\frac{\partial^2}{\partial y\partial v}-\left\{(A\,y+B\,v)^2+B^2\,y+\Lambda\right\}\right]\varphi=0, \tag{5.4}$$

where $A=2\pi n$ and $B=2\pi r$ are constants. We note that the function $\phi(y)=y^{\frac12}\,K_{s-\frac12}(2\pi|n|y)$ satisfies the the differential equation (5.4) with $\Lambda=s(s-1)$. Here $K_s(z)$ is the K-Bessel function defined by (5.2) (cf. [10], [19]).

6. Maass-Jacobi forms due to Pitale, Bringmann et al

We fix a positive integer m. Let $\mathcal{M}$ be a symmetric half-integral semi-positive definite matrix of degree m. Let $C^\infty(\mathbb{H}\times\mathbb{C}^m)$ be the algebra of all C^∞-functions on $\mathbb{H}\times\mathbb{C}^m$. For any nonnegative integer $k\in\mathbb{Z}$, we define the $|_{k,\mathcal{M}}$-slash action of G^J on $C^\infty(\mathbb{H}\times\mathbb{C}^m)$ as follows: If $f\in C^\infty(\mathbb{H}\times\mathbb{C}^m)$, and $(M,(\lambda,\mu;\kappa))\in G^J$ with $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{R})$ and $(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(m)}$,

$$\begin{aligned}&\big(f|_{k,\mathcal{M}}[(M,(\lambda,\mu;\kappa))]\big)(\tau,z):\\ =\;&(c\tau+d)^{-k}\,e^{-2\pi i\,\mathcal{M}[z+\lambda\tau+\mu]\,c\,(c\tau+d)^{-1}}\\ &\times e^{2\pi i\,\mathrm{tr}\left(\mathcal{M}(\tau\lambda\,{}^t\lambda+2\lambda\,{}^tz+\kappa+\mu\,{}^t\lambda)\right)}\,f\left(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\mu}{c\tau+d}\right),\end{aligned} \tag{6.1}$$

where $\tau\in\mathbb{H}$ and $z\in\mathbb{C}^m$. We recall the Siegel's notation $\alpha[\beta]={}^t\beta\alpha\beta$ for suitable matrices α and β. Let $\mathbb{D}_{k,\mathcal{M}}$ be the algebra of all differential operators D on $\mathbb{H}\times\mathbb{C}^m$ satisfying the following condition

$$(Df)|_{k,\mathcal{M}}[g]=D\big(f|_{k,\mathcal{M}}[g]\big) \tag{6.2}$$

for all $f\in C^\infty(\mathbb{H}\times\mathbb{C}^m)$ and for all $g\in G^J$. We recall the arithmetic subgroup $\Gamma_{1,m}$ of G^J defined by

$$\Gamma_{1,m}:=SL_2(\mathbb{Z})\ltimes H_{\mathbb{Z}}^{(m)}.$$

Definition 6.1. Let $\mathcal{C}^{k,\mathcal{M}}$ be the Casimir operator defined in Theorem 3.2. A smooth function $\phi : \mathbb{H} \times \mathbb{C}^m \longrightarrow \mathbb{C}$ is called a Maass-Jacobi form of weight k and index $\mathcal{M}$ if it satisfies the following conditions:

(MJ1*) $\phi|_{k,\mathcal{M}}[\gamma] = \phi$ for all $\gamma \in \Gamma_{1,m}$.

(MJ2*) ϕ is an eigenfunction of the Casimir operator $\mathcal{C}^{k,\mathcal{M}}$.

(MJ3*) For some $a > 0$,

$$\phi(\tau, z) = O\big(e^{ay}\, e^{2\,\pi\, i\, \mathcal{M}[v]/y}\big) \quad \text{as } y \longrightarrow \infty.$$

Furthermore if $\mathcal{C}^{k,\mathcal{M}}\phi = 0$, it is said to be a *harmonic* Maass-Jacobi form of weight k and index $\mathcal{M}$. We denote by $\mathbb{J}_{k,\mathcal{M}}$ the space of all harmonic Maass-Jacobi forms of weight k and index $\mathcal{M}$.

For the present being we let $\mathcal{M}$ be a positive definite integral even lattice of rank m and k an integer. We identify $\mathcal{M}$ with its Gram matrix with respect to a fixed basis, that is, a positive definite half-integral symmetric matrix of degree m. We write $|\mathcal{M}|$ for the determinant of the Gram matrix of $\mathcal{M}$. Throughout this section n will be an integer and r will be in $\mathbb{Z}^m$. For $r = {}^t(r_1, \cdots, r_m) \in \mathbb{Z}^m$ and $z = {}^t(z_1, \cdots, z_m) \in \mathbb{C}^m$, we put

$$\zeta^r := \prod_{j=1}^{m} e^{2\,\pi\, i\, r_j z_j},$$

where $\zeta = (\zeta_1, \cdots, \zeta_m)$ with $\zeta_j = e^{2\pi i z_j}$ $(1 \le j \le m)$. For $a \in \mathbb{C}$, we write $e(a) := e^{2\,\pi\, i\, a}$. For two vectors $\xi = {}^t(\xi_1, \cdots, \xi_m)$ and $\eta = {}^t(\eta_1, \cdots, \eta_m)$ in $\mathbb{C}^m$, we let

$$\langle \xi, \eta \rangle := \sum_{j=1}^{m} \xi_j\, \eta_j$$

be the standard scalar product.

We set

$$D = D_{\mathcal{M}}(n, r) := |\mathcal{M}|\big(4n - \mathcal{M}^{-1}[r]\big) \quad \text{and} \quad h = h_{\mathcal{M}}(r) := |\mathcal{M}|\, \mathcal{M}^{-1}[r]. \tag{6.3}$$

Let $M_{\nu,\mu}(w)$ be the usual M-Whittaker function, which is a solution to the following differential equation

$$\frac{\partial^2}{\partial w^2} f(w) + \left(-\frac{1}{4} + \frac{\nu}{w} + \frac{\frac{1}{4} - \mu^2}{w^2} \right) f(w) = 0. \tag{6.4}$$

For $s \in \mathbb{C},\ \kappa \in \frac{1}{2}\mathbb{Z}$ and $t \in \mathbb{R}^{\times}$, we define the function

$$\mathcal{M}_{s,\kappa}(t) := |t|^{-\frac{\kappa}{2}} M_{\mathrm{sgn}(t)\frac{\kappa}{2},\, s - \frac{1}{2}}(|t|) \tag{6.5}$$

and

$$\phi^{(n,r)}_{k,\mathcal{M},s}(\tau,z) := \mathcal{M}_{s,k-\frac{m}{2}}\left(\frac{\pi\,Dy}{|\mathcal{M}|}\right)\, e^{2\,\pi\, i\,\left(\langle r,z\rangle + \frac{i}{4}\,\mathcal{M}^{-1}[r]y + nx\right)}. \tag{6.6}$$

We define the Poincaré series

$$P^{(n,r)}_{k,\mathcal{M},s}(\tau,z) := \sum_{\gamma\in\Gamma^{\infty}_{1,m}\backslash\Gamma_{1,m}} \left(\phi^{(n,r)}_{s,\mathcal{M},s}\Big|_{k,\mathcal{M}}[r]\right)(\tau,z). \tag{6.7}$$

Obviously $P^{(n,r)}_{k,\mathcal{M},s}$ is holomorphic in $\mathbb{C}^m$. It is easily seen that $P^{(n,r)}_{k,\mathcal{M},s}$ is an eigenfunction of the Casimir operator $\mathcal{C}^{k,\mathcal{M}}$ with eigenvalue

$$-2\,s(1-s) - \frac{1}{2}\left\{k^2 - k(m+2) + \frac{1}{4}\,m(m+4)\right\}.$$

For $s\in\mathbb{C},\ \kappa\in\frac{1}{2}\mathbb{Z}$ and $t\in\mathbb{R}^{\times}$, we set

$$\mathcal{W}_{s,\kappa}(t) := |t|^{-\frac{\kappa}{2}}\, W_{\mathrm{sgn}(t)\,\frac{\kappa}{2},\, s-\frac{1}{2}}(|t|), \tag{6.8}$$

where $W_{\nu,\mu}$ denotes the usual W-Whittaker function which is also a solution to the differential equation (6.4).

For $r\in\mathbb{Z}^m$, we define the theta series

$$\theta^{(r)}_{k,\mathcal{M}}(\tau,z) := \sum_{\lambda\in\mathbb{Z}^m} e^{2\,\pi\, i\,\mathcal{M}[\lambda]}\,\zeta^{2\mathcal{M}\lambda}\left\{e^{2\,\pi\, i\,\langle r,\lambda\rangle}\zeta^{r} \,+\, (-1)^k\, e^{-2\,\pi\, i\,\langle r,\lambda\rangle}\zeta^{r}\right\}. \tag{6.9}$$

Theorem 6.1(Bringmann-Richter [4] and Conley-Raum [5])**.** *The Poincaré series $P^{(n,r)}_{s,\mathcal{M},s}(\tau,z)$ has the Fourier expansion*

$$\begin{aligned} P^{(n,r)}_{k,\mathcal{M},s}(\tau,z) \;&=\; \mathcal{M}_{s,k-\frac{m}{2}}\left(\frac{\pi Dy}{|\mathcal{M}|}\right)\, e\left(\frac{-iDy}{4|\mathcal{M}|}\right)\,\theta^{(r)}_{k,\mathcal{M}}(\tau,z)\, e^{2\pi\, i\, n\,\tau} \\ &\quad+ \sum_{n'\in\mathbb{Z},\, r'\in\mathbb{Z}^m} c_{y,s}(n',r')\, e^{2\,\pi\, i\, n'\tau}\,\zeta^{r'}. \end{aligned} \tag{6.10}$$

Here the coefficients $c_{y,s}(n',r')$ are

$$c_{y,s}(n',r') := b_{y,s}(n',r') \,+\, (-1)^k\, b_{y,s}(n',-r')$$

with $b_{y,s}$ depending on D and $D' = |\mathcal{M}|\,(4\,n' \,-\, \mathcal{M}^{-1}[r'])$ and $b_{y,s}(n',r')$ is given as follows:

(1) If $D'=0$, there is a constant $a_s(n',r')$ such that

$$b_{y,s}(n',r') = a_s(n',r')\,\frac{y^{1+\frac{m}{4}-\frac{k}{2}-s}}{\Gamma\left(s+\frac{k}{2}-\frac{m}{4}\right)\Gamma\left(s-\frac{k}{2}+\frac{m}{4}\right)}.$$

(2) If $DD' > 0$,

$$\begin{aligned} b_{y,s}(n',r') &= 2^{1-\frac{m}{2}}\,\pi\, i^{-k}\,|\mathcal{M}|^{-\frac{1}{2}}\,\frac{\Gamma(2s)}{\Gamma\left(s-\operatorname{sgn}(D')\left(\frac{k}{2}-\frac{m}{4}\right)\right)} \\ &\times \left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{m+2}{4}} e\left(-\frac{i\,D'y}{4|\mathcal{M}|}\right) \mathcal{W}_{s,k-\frac{m}{2}}\left(\frac{\pi D'y}{|\mathcal{M}|}\right) \\ &\times \sum_{c\in\mathbb{Z}^+} c^{-\frac{m+2}{2}}\,K_{c,\mathcal{M}}(n,r,n',r')\,J_{2s-1}\left(\frac{\pi\sqrt{DD'}}{c|\mathcal{M}|}\right), \end{aligned}$$

where Γ *is the usual Gamma function,* J_s *is the usual* J*-Bessel function and* $K_{c,\mathcal{M}}(n,r,n',r')$ *is the Kloosterman sum defined by*

$$(6.11)\quad \begin{aligned} K_{c,\mathcal{M}}(n,r,n',r') &:= e^{-\pi\, i\, c^{-1}\langle r,\,\mathcal{M}^{-1}r'\rangle} \\ &\times \sum_{\substack{d\in(\mathbb{Z}/c\mathbb{Z})^\times,\\ \lambda\in\mathbb{Z}^m/c\mathbb{Z}^m}} e^{2\,\pi\, i\left(c^{-1}\bar{d}\mathcal{M}[\lambda]\,+\,n'd-\langle r',\lambda\rangle\,+\,\bar{d}\,n\,+\,\bar{d}\,\langle r,\lambda\rangle\right)}, \end{aligned}$$

where $\bar{d}$ *is an integer inverse of* d *modulo* c*.*

(3) If $DD' < 0$,

$$\begin{aligned} b_{y,s}(n',r') &= 2^{1-\frac{m}{2}}\,\pi\, i^{-k}\,|\mathcal{M}|^{-\frac{1}{2}}\,\frac{\Gamma(2s)}{\Gamma\left(s-\operatorname{sgn}(D')\left(\frac{k}{2}-\frac{m}{4}\right)\right)} \\ &\times \left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{m+2}{4}} e\left(-\frac{i\,D'y}{4|\mathcal{M}|}\right) \mathcal{W}_{s,k-\frac{m}{2}}\left(\frac{\pi D'y}{|\mathcal{M}|}\right) \\ &\times \sum_{c\in\mathbb{Z}^+} c^{-\frac{m+2}{2}}\,K_{c,\mathcal{M}}(n,r,n',r')\,I_{2s-1}\left(\frac{\pi\sqrt{DD'}}{c|\mathcal{M}|}\right), \end{aligned}$$

where I_s *is the usual* I*-Bessel function.*

Proof. We refer to [4] for the proof in the case $n = m = 1$ and to [5] in the case $n = 1$, m is arbitrary. □

Remark 6.1. If $s = \frac{k}{2} - \frac{m}{4}$ (resp. $s = 1 + \frac{m}{4} - \frac{k}{2}$) , then the Poincaré series $P^{(n,r)}_{k,\mathcal{M},s}(\tau,z)$ converges for $k > m+2$ (resp. $k < 0$). In both cases Poincaré series $P^{(n,r)}_{k,\mathcal{M},s}(\tau,z)$ is a *harmonic* Maass-Jacobi form of weight k and index $\mathcal{M}$ which is holomorphic in $\mathbb{C}^m$.

Remark 6.2. The Fourier coefficients $c^{(n,r)}_{y,s} = c^{(n,r)}_{k,\mathcal{M},s}$ of the Poincaré series $P^{(n,r)}_{k,\mathcal{M},s}(\tau,z)$ satisfy the the so-called *Zagier-type duality* with dual weights k and $m+2-k$. More precisely, if $D < 0$ and $D' < 0$, there is a constant $h_{k,s}$ depending only on k and s such that

$$c^{(n,r)}_{k,\mathcal{M},s}(n',r') = h_{k,s}\, c^{(n',r')}_{m+2-k,\mathcal{M},s}(n,r) \tag{6.12}$$

while if $D<0$ and $D'>0$, there is a constant $\hat{h}_{k,s}$ depending only on k and s such that

$$c_{k,\mathcal{M},s}^{(n,r)}(n',r') = \hat{h}_{k,s}\, c_{m+2-k,\mathcal{M},s}^{(n',r')}(n,r). \tag{6.13}$$

7. Skew-Holomorphic Jacobi Forms

We define the *skew-slash action* of G^J on $C^\infty(\mathbb{H}\times\mathbb{C}^m)$ as follows: If $f\in C^\infty(\mathbb{H}\times\mathbb{C}^m)$, and $(M,(\lambda,\mu;\kappa))\in G^J$ with $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{R})$ and $(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(m)}$,

$$\begin{aligned}&\big(f|_{k,\mathcal{M}}^{sk}[(M,(\lambda,\mu;\kappa))]\big)(\tau,z):\\ =\ &(c\overline{\tau}+d)^{1-k}\,|c\tau+d|^{-1}\,e^{-2\pi i\,\mathcal{M}[z+\lambda\tau+\mu]\,c\,(c\tau+d)^{-1}}\\ &\times e^{2\pi i\,\mathrm{tr}\left(\mathcal{M}(\tau\lambda\,{}^t\lambda+2\,\lambda\,{}^tz+\kappa+\mu\,{}^t\lambda)\right)}\,f\left(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\mu}{c\tau+d}\right),\end{aligned}\tag{7.1}$$

where $\tau\in\mathbb{H}$ and $z\in\mathbb{C}^m$.

Definition 7.1. A smooth $f:\mathbb{H}\times\mathbb{C}^m\longrightarrow\mathbb{C}$ is said to be a *skew-holomorphic* Jacobi form of weight k and index $\mathcal{M}$ if it is real analytic in τ and is holomorphic in $z\in\mathbb{C}^m$ and satisfies the following conditions:

(SK1) $f|_{k,\mathcal{M}}^{sk}[\gamma]=f$ for all $\gamma\in\Gamma^J$.

(SK2) The Fourier expansion of f is of the form

$$f(\tau,z)=\sum_{\substack{n\in\mathbb{Z},\,r\in\mathbb{Z}^m\\ D\gg-\infty}} c(n,r)\,e^{\pi\,Dy/|\mathcal{M}|}\,e^{2\pi i n\tau}\,\zeta^r.$$

We denote by $\mathbb{J}_{k,\mathcal{M}}^{sk}$ the space of all skew-holomorphic Jacobi forms of weight k and index $\mathcal{M}$.

Remark 7.1. The notion of skew-holomorphic Jacobi forms was introduced by N.-P. Skoruppa [18].

Let

$$e_{n,r,\mathcal{M}}(\tau,z):=e^{2\pi i\,(n\tau+\langle r,z\rangle)}\,e^{\pi Dy/|\mathcal{M}|}.$$

We define the Poincaré series

$$P_{k,\mathcal{M}}^{(n,r),sk}(\tau,z):=\sum_{\gamma\in\Gamma_{1,m}^\infty\backslash\Gamma_{1,m}}\big(e_{n,r,\mathcal{M}}|_{k,\mathcal{M}}^{sk}[\gamma]\big)(\tau,z).\tag{7.2}$$

Theorem 7.1. *The Poincaré series $P_{k,\mathcal{M}}^{(n,r),sk}(\tau,z)$ defined in (7.2) is a cuspidal skew-holomorphic Jacobi form of weight k and index $\mathcal{M}$. And it has the Fourier* *expansion*

$$\begin{aligned} P_{k,\mathcal{M}}^{(n,r),sk}(\tau,z) &= e^{\pi D y/|\mathcal{M}|}\,\theta_{k-1,\mathcal{M}}^{(r)}(\tau,z)\,e^{2\pi i n\tau} \\ &\quad + \sum_{\substack{n'\in\mathbb{Z},\, r'\in\mathbb{Z}^m \\ D'>0}} c(n',r')\,e^{\pi D' y/|\mathcal{M}|}\,e^{2\pi i n'\tau}\,\zeta^{r'}, \end{aligned}$$

where $\theta_{k,\mathcal{M}}^{(r)}(\tau,z)$ *is defined in Formula (6.9) and the coefficients* $c(n',r')$ *are*

$$c(n',r') = b(n',r') + (-1)^k b(n',-r').$$

Here

$$\begin{aligned} b(n',r') : &= 2^{1-\frac{m}{2}}\,\pi\,i^{1-k}\left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{m+2}{4}} \\ &\quad \times \sum_{c\in\mathbb{Z}^+} c^{-\frac{m+2}{2}}\,K_{c,\mathcal{M}}(n,r,n',-r')\,J_{k-\frac{m+2}{2}}\left(\frac{\pi\sqrt{DD'}}{c|\mathcal{M}|}\right). \end{aligned}$$

Proof. The proof is analogous to that of Theorem 6.1. □

We define the following lowering operator

$$\begin{aligned} (7.3)\quad D_-^{(\mathcal{M})} &= \left(\frac{\tau-\overline{\tau}}{2\,i}\right)\left\{-(\tau-\overline{\tau})\,\partial_{\overline{\tau}} - {}^t(z-\overline{z})\partial_{\overline{z}} + \frac{\tau-\overline{\tau}}{8\,\pi\,i}\,\mathcal{M}^{-1}[\partial_{\overline{z}}]\right\} \\ &= -2\,i\,y\left(y\,\partial_{\overline{\tau}} + {}^t v\,\partial_{\overline{z}} - \frac{y}{8\,\pi\,i}\,\mathcal{M}^{-1}[\partial_{\overline{z}}]\right). \end{aligned}$$

We note that $D_-^{(\mathcal{M})}$ satisfies the following relation

$$(7.4)\qquad \left(D_-^{(\mathcal{M})}\phi\right)\Big|_{k-2,\mathcal{M}}[\gamma] = D_-^{(\mathcal{M})}\left(\phi|_{k,\mathcal{M}}[\gamma]\right)$$

for all $\phi \in C^\infty(\mathbb{H}\times\mathbb{C}^m)$ and for all $\gamma\in\Gamma_{1,m}$.

Now we define the differential operator

$$(7.5)\qquad \xi_{k,\mathcal{M}} := \left(\frac{\tau-\overline{\tau}}{2\,i}\right)^{k-\frac{5}{2}} D_-^{(\mathcal{M})} = y^{k-\frac{5}{2}}\,D_-^{(\mathcal{M})}.$$

It is easily seen that if f is a harmonic Maass-Jacobi form of weight k and index $\mathcal{M}$ which is holomorphic in $\mathbb{C}^m$, then the image $\xi_{k,\mathcal{M}}f$ of f under $\xi_{k,\mathcal{M}}$ is a skew-holomorphic Jacobi form of weight $3-k$ and index $\mathcal{M}$.

Theorem 7.2. *The Poincaré series* $P_{k,\mathcal{M}}^{(n,r),sk}(\tau,z)$ *span the space* $\mathbb{J}_{k,\mathcal{M}}^{sk,cusp}$ *of all cuspidal skew-holomorphic Jacobi forms of weight* k *and index* $\mathcal{M}$.

Proof. The proof can be found in [18]. □

Now we consider the special case $s = \frac{k}{2} - \frac{m}{4}$ and $s = 1 + \frac{m}{4} - \frac{k}{2}$.

Proposition 7.1. *The Poincaré series* $P^{(n,r)}_{k,\,\mathcal{M},\,\frac{k}{2}-\frac{m}{4}}$ *with* $k > 2+m$ *is meromorphic. If* $k < 0$,

$$\xi_{k,\mathcal{M}}\left(P^{(n,r)}_{k,\,\mathcal{M},1+\frac{m}{4}-\frac{k}{2}}\right) = c_{k,\mathcal{M}}\,P^{(n,r),sk}_{3-k,\,\mathcal{M}},$$

where $c_{k,\mathcal{M}}$ *is a constant depending on* k *and* $\mathcal{M}$.

Proof. We refer to [5], p. 18 for the proof. □

Proposition 7.2. *Let* $\mathbb{J}^{cusp,*}_{k,\,\mathcal{M}}$ *be the space of all cuspidal harmonic Maass-Jacobi forms of weight* k *and index* $\mathcal{M}$ *which are holomorphic in* $\mathbb{C}^m$. *Then we have the relation*

$$\xi_{k,\,\mathcal{M}}\left(\mathbb{J}^{cusp,*}_{k,\,\mathcal{M}}\right) = \mathbb{J}^{sk,cusp}_{k,\,\mathcal{M}}.$$

Proof. We refer to [5], p. 18 for the proof. □

8. Covariant differential operators on $\mathbb{H}\times\mathbb{C}^m$

Let G be a real Lie group, H a closed subgroup and V a finite dimensional complex vector space. For an element $x \in G$ we denote the coset xH by $\overline{x}$. A 1-cocycle of G on G/H with values in V is a smooth function $\alpha : G\times G/H \longrightarrow GL(V)$ satisfying the following condition

$$\alpha(g_1g_2,\overline{x}) = \alpha(g_2,\overline{x})\,\alpha(g_1,g_2\overline{x})$$

for all $g_1, g_2, x \in G$. The associated right action of G on $C^\infty\big(G/H\big)\otimes V$ is

$$f|_\alpha[g](\overline{x}) := \alpha(g,\overline{x})f(g\overline{x}), \qquad g, x \in G$$

and the associated representation of H on V is

$$\pi_\alpha(h) := \alpha(h,\overline{x}),$$

where $h \in H$ and e is the identity element of G.

Definition 8.1. Let V and V' be two finite dimensional complex vector spaces. Let α and α' be two 1-cocycles of G on G/H with values in V and V' respectively. A differential operator $D : C^\infty(G/H)\otimes V \longrightarrow C^\infty(G/H)\otimes V'$ is *covariant* from $|_\alpha$ to $|_{\alpha'}$ if for all $g \in G$ and $f \in C^\infty(G/H)\otimes V$, we have

$$D\big(f|_\alpha[g]\big) = (Df)|_{\alpha'}[g].$$

Let $\mathbb{D}_{\alpha,\alpha'}(G/H)$ be the space of all covariant differential operators from $|_\alpha$ to $|_{\alpha'}$ and $\mathbb{D}^q_{\alpha,\alpha'}(G/H)$ be the space of those of order $\le q$. When $\alpha = \alpha'$, we refer to such operators as $|_\alpha$-invariant, and we write simply $\mathbb{D}_\alpha(G/H)$ and $\mathbb{D}^q_\alpha(G/H)$

We consider our case

$$G^J = SL_2(\mathbb{R}) \ltimes H_{\mathbb{R}}^{(m)} \quad \text{and} \quad K^J = SO(2) \ltimes S(m,\mathbb{R}).$$

We observe that K^J is an abelian closed subgroup of G^J. We define the linear map $\xi : \mathfrak{g}_{\mathbb{C}}^J \longrightarrow \mathfrak{g}_{\mathbb{C}}^J$ by $\xi(X) = \widehat{X}$ with $X \in \mathfrak{g}_{\mathbb{C}}^J$, where

$$\begin{aligned} \widehat{H} : &= i\,(F-E), \qquad \widehat{E} := \frac{1}{2}\,\{H + i\,(E+F)\}\,, \qquad \widehat{F} := \frac{1}{2}\,\{H - i\,(E+F)\}\,,\\ \widehat{R}_{kl} : &= \frac{1}{2}\,R_{kl}, \qquad \widehat{e}_j := \frac{1}{2}\,(e_j - i\,f_j), \qquad \widehat{f}_j := \frac{1}{2}\,(e_j + i\,f_j). \end{aligned}$$

It is easy to see that there is a unique K^J-splitting

$$\mathfrak{g}_{\mathbb{C}}^J = \mathfrak{k}_*^J \oplus \mathfrak{p}_*^J, \tag{8.1}$$

where

$$\mathfrak{k}_*^J = \text{span}\{\widehat{H},\, \widehat{R}_{kl} \mid 1 \le k \le l \le m\,\}$$

and

$$\mathfrak{p}_*^J = \text{span}\{\widehat{E},\, \widehat{F},\, \widehat{e}_j,\, \widehat{f}_j \mid 1 \le j \le m\,\}.$$

We note that ξ is an automorphism of Lie algebras and so the given basis of $\mathfrak{p}_*^J$ is a K^J-eigenbasis : the $\widehat{H}$=weights of $\widehat{E}$, $\widehat{F}$, $\widehat{e}_j$ and $\widehat{f}_j$ are 1, -2, -1 and 1 respectively. We take the scalar valued 1-cocycle $\alpha := \gamma_{k,\mathcal{M}}$ defined by (3.7). We set $\mathcal{M} = (\mathcal{M}_{kl})$. We let $\pi_{k,\mathcal{M}} : K^J \longrightarrow GL_1(\mathbb{C})$ be the one-dimensional representation of K^J defined by

$$\pi_{k,\mathcal{M}}(h) := \gamma_{k,\mathcal{M}}(h,\overline{e})^{-1},$$

where $h \in K^J$ and $\overline{e} = (i,0) = eK^J$ with the identity element e in G^J. We remark that ξ maps the Casimir operator Ω_m to $\left(\frac{i}{2}\right)^m \Omega_m$.

Definition 8.2. Let $k \in \mathbb{Z}$ and $\mathcal{M} \in S(m,\mathbb{C})$. We define the raising operators X_+, Y_+ and the lowering operators X_- and Y_-:

$$\begin{aligned} X_+^{k,\mathcal{M}} : &= 2\,i\left(\partial_\tau + y^{-1}\,{}^t v \partial_z + y^{-2}\widetilde{\mathcal{M}}[v]\right), \qquad X_-^{k,\mathcal{M}} := -2\,i\,y\big(y\,\partial_{\overline{\tau}} + {}^t v\,\partial_{\overline{z}}\big),\\ Y_+^{k,\mathcal{M}} : &= i\,\partial_z + 2\,i\,y^{-1}\widetilde{\mathcal{M}} v, \qquad Y_-^{k,\mathcal{M}} := -\,i\,y\,\partial_{\overline{z}}, \qquad \widetilde{\mathcal{M}} := 2\,\pi\,i\,\mathcal{M}. \end{aligned}$$

We write $Y_{\pm,j}^{k,\mathcal{M}}$ for the j-th entry of $Y_\pm^{k,\mathcal{M}}$ $(1 \le j \le m)$.

For brevity, we write

$$\mathbb{D}(k,\mathcal{M};k',\mathcal{M}') := \mathbb{D}_{\gamma_{k,\mathcal{M}},\gamma_{k',\mathcal{M}'}}\big(G^J/K^J\big)$$

and

$$\mathbb{D}^q(k,\mathcal{M};k',\mathcal{M}') := \mathbb{D}^q_{\gamma_{k,\mathcal{M}},\gamma_{k',\mathcal{M}'}}\big(G^J/K^J\big),$$

where $k, k' \in \mathbb{Z},\ \mathcal{M}, \mathcal{M}' \in S(m, \mathbb{C})$, $q \in \mathbb{Z} \cup \{0\}$ and $G^J/K^J = \mathbb{H} \times \mathbb{C}^m$. We also write

$$\mathbb{D}_{k,\mathcal{M}} := \mathbb{D}(k, \mathcal{M}; k, \mathcal{M}) \qquad \text{and} \qquad \text{and} \qquad \mathbb{D}^q_{k,\mathcal{M}} := \mathbb{D}^q(k, \mathcal{M}; k, \mathcal{M}).$$

Conley and Raum [5] obtained the following three results.

Proposition 8.1. *(1) The spaces* $\mathbb{D}^1(k, \mathcal{M}; k \pm 2, \mathcal{M})$ *are one-dimensional. In fact* $\mathbb{D}^1(k, \mathcal{M}; k \pm 2, \mathcal{M}) = \mathbb{C} X_{\pm}^{k,\mathcal{M}}$.

(2) $\mathbb{D}^1(k, \mathcal{M}; k \pm 1, \mathcal{M}) = \mathrm{Span}\{ Y_{\pm,j}^{k,\mathcal{M}} \mid 1 \le j \le m \}$ *are m-dimensional.*

(3) $\mathbb{D}^0_{k,\mathcal{M}} = \mathbb{D}^1_{k,\mathcal{M}} = \mathbb{C}$.

(4) All other $\mathbb{D}^1(k, \mathcal{M}; k', \mathcal{M}')$ *are zero.*

(5) We have the following commutation relations

$$[X_-, X_+] = -k, \quad [Y_{-,j}, Y_{+,j'}] = i\, \widetilde{\mathcal{M}}_{jj'}, \quad [X_-, Y_+] = -Y_-,$$
$$[Y_-, X_+] = Y_+, \quad [X_+, Y_+] = [X_-, Y_-] = 0.$$

Proposition 8.2. *Any covariant differential operator of order q may be expressed as a linear combination of products up to q raising and lowering operators. There is a unique such expression in which the raising operators are all to the left of the lowering operators. The expression of this form for the Casimir operator* $\mathcal{C}^{k,\mathcal{M}}$ *is*

$$\begin{aligned} \mathcal{C}^{k,\mathcal{M}} \;=\; & -2\, X_+ X_- + i \left(X_+ \widetilde{\mathcal{M}}^{-1}[Y_-] - \widetilde{\mathcal{M}}^{-1}[Y_+] X_- \right) \\ & - \frac{1}{2} \left\{ \widetilde{\mathcal{M}}^{-1}[Y_+]\, \widetilde{\mathcal{M}}^{-1}[Y_-] - {}^tY_+ \left({}^tY_+ \widetilde{\mathcal{M}}^{-1} Y_- \right) \widetilde{\mathcal{M}}^{-1} Y_- \right\} \\ & - \frac{i}{2} \left(2\,k - m - 3 \right) {}^tY_+ \widetilde{\mathcal{M}}^{-1} Y_-. \end{aligned} \tag{8.2}$$

Proposition 8.3. *The algebra* $\mathbb{D}_{k,\mathcal{M}}$ *is generated by* $\mathbb{D}^3_{k,\mathcal{M}}$. *Bases for* $\mathbb{D}^2_{k,\mathcal{M}}$ *and* $\mathbb{D}^3_{k,\mathcal{M}}$ *are given by*

$$\begin{aligned} \mathbb{D}^2_{k,\mathcal{M}} &= \mathrm{Span}\{ 1,\ X_+ X_-,\ Y_{+,i}\, Y_{-,j} \mid 1 \le i, j \le m \}, \\ \mathbb{D}^3_{k,\mathcal{M}} &= \mathrm{Span}\{ X_+ Y_{-,i}\, Y_{-,j},\ Y_{+,i}\, Y_{+,j}\, X_- \mid 1 \le i \le j \le m \} \oplus \mathbb{D}^2_{k,\mathcal{M}}. \end{aligned}$$

Therefore we have

$$\dim_{\mathbb{C}} \mathbb{D}^2_{k,\mathcal{M}} = m^2 + 2 \qquad \text{and} \qquad \dim_{\mathbb{C}} \mathbb{D}^3_{k,\mathcal{M}} = 2\,m^2 + m + 2.$$

9. Final remarks

In this final section we briefly remark the general case $n > 1$ and $m > 1$.

We let

$$\mathbb{H}_n = \{\, \Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega, \quad \operatorname{Im}\Omega > 0 \,\}$$

be the Siegel upper half plane of degree n and let

$$Sp(n,\mathbb{R}) = \{ M \in \mathbb{R}^{(2n,2n)} \mid {}^tM J_n M = J_n \, \}$$

be the symplectic group of degree n, where

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

$Sp(n,\mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{9.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $\Omega \in \mathbb{H}_n$.

For brevity, we write $G_n = Sp(n,\mathbb{R})$. The isotropy subgroup K_n at iI_n for the action (9.1) is a maximal compact subgroup given by

$$K_n = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \,\Big|\, A\,{}^tA + B\,{}^tB = I_n,\ A\,{}^tB = B\,{}^tA,\ A, B \in \mathbb{R}^{(n,n)} \right\}.$$

Let $\mathfrak{k}_n$ be the Lie algebra of K_n. Then the Lie algebra $\mathfrak{g}_n$ of G_n has a Cartan decomposition $\mathfrak{g}_n = \mathfrak{k}_n \oplus \mathfrak{p}_n$, where

$$\mathfrak{g}_n = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & -\,{}^tX_1 \end{pmatrix} \,\Big|\, X_1, X_2, X_3 \in \mathbb{R}^{(n,n)},\ X_2 = {}^tX_2,\ X_3 = {}^tX_3 \right\},$$

$$\mathfrak{k}_n = \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \,\Big|\, {}^tX + X = 0,\ Y = {}^tY \right\},$$

$$\mathfrak{p}_n = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \,\Big|\, X = {}^tX,\ Y = {}^tY,\ X, Y \in \mathbb{R}^{(n,n)} \right\}.$$

The subspace $\mathfrak{p}_n$ of $\mathfrak{g}_n$ may be regarded as the tangent space of $\mathbb{H}_n$ at iI_n.

We consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \{\, (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\,{}^t\lambda \text{ symmetric} \,\}$$

endowed with the following multiplication law

$$(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') = (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda\,{}^t\mu' - \mu\,{}^t\lambda')$$

with $\big(\lambda,\mu;\kappa\big),\big(\lambda',\mu';\kappa'\big)\in H_{\mathbb R}^{(n,m)}$. We define the semidirect product of $Sp(n,\mathbb R)$ and $H_{\mathbb R}^{(n,m)}$

$$G^J_{n,m}=Sp(n,\mathbb R)\ltimes H_{\mathbb R}^{(n,m)}$$

endowed with the following multiplication law

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot\big(M',(\lambda',\mu';\kappa')\big)=\big(MM',(\tilde\lambda+\lambda',\tilde\mu+\mu';\kappa+\kappa'+\tilde\lambda\,{}^t\mu'-\tilde\mu\,{}^t\lambda')\big)$$

with $M,M'\in Sp(n,\mathbb R),(\lambda,\mu;\kappa),\ (\lambda',\mu';\kappa')\in H_{\mathbb R}^{(n,m)}$ and $(\tilde\lambda,\tilde\mu)=(\lambda,\mu)M'$. Then $G^J_{n,m}$ acts on $\mathbb H_n\times\mathbb C^{(m,n)}$ transitively by

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot(\Omega,Z)=\Big(M\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}\Big), \tag{9.2}$$

where $M=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in Sp(n,\mathbb R),\ (\lambda,\mu;\kappa)\in H_{\mathbb R}^{(n,m)}$ and $(\Omega,Z)\in\mathbb H_n\times\mathbb C^{(m,n)}$.

The stabilizer $K^J_{n,m}$ of $G^J_{n,m}$ at $(iI_n,0)$ for the action (9.2) is given by

$$K^J_{n,m}=\Big\{\big(k,(0,0;\kappa)\big)\ \big|\ k\in K_n,\ \kappa={}^t\kappa\in\mathbb R^{(m,m)}\Big\}.$$

Therefore $\mathbb H_n\times\mathbb C^{(m,n)}\cong G^J_{n,m}/K^J_{n,m}$ is a homogeneous space of *non-reductive type*. The Lie algebra $\mathfrak g^J_{n,m}$ of $G^J_{n,m}$ has a decomposition

$$\mathfrak g^J_{n,m}=\mathfrak k^J_{n,m}+\mathfrak p^J_{n,m},$$

where

$$\mathfrak g^J_{n,m}=\Big\{\big(Z,(P,Q,R)\big)\ \big|\ Z\in\mathfrak g_n,\ P,Q\in\mathbb R^{(m,n)},\ R={}^tR\in\mathbb R^{(m,m)}\Big\},$$

$$\mathfrak k^J_{n,m}=\Big\{\big(X,(0,0,R)\big)\ \big|\ X\in\mathfrak k_n,\ R={}^tR\in\mathbb R^{(m,m)}\Big\},$$

$$\mathfrak p^J_{n,m}=\Big\{\big(Y,(P,Q,0)\big)\ \big|\ Y\in\mathfrak p_n,\ P,Q\in\mathbb R^{(m,n)}\Big\}.$$

Thus the tangent space of the homogeneous space $\mathbb H_{n,m}$ at $(iI_n,0)$ is identified with $\mathfrak p^J_{n,m}$. We note that the Jacobi group $G^J_{n,m}$ is *not* a reductive Lie group and that the homogeneous space $\mathbb H_n\times\mathbb C^{(m,n)}$ is not a symmetric space. From now on, for brevity we write $\mathbb H_{n,m}=\mathbb H_n\times\mathbb C^{(m,n)}$, called the Siegel-Jacobi space of degree n and index m.

For a coordinate $(\Omega,Z)\in\mathbb H_{n,m}$ with $\Omega=(\omega_{\mu\nu})\in\mathbb H_n$ and $Z=(z_{kl})\in\mathbb C^{(m,n)}$, we put

$$\begin{aligned}\Omega&=X+iY,\quad X=(x_{\mu\nu}),\quad Y=(y_{\mu\nu})\ \text{ real},\\ Z&=U+iV,\quad U=(u_{kl}),\quad V=(v_{kl})\ \text{ real},\\ d\Omega&=(d\omega_{\mu\nu}),\quad d\overline{\Omega}=(d\overline{\omega}_{\mu\nu}),\\ dZ&=(dz_{kl}),\quad d\overline{Z}=(d\overline{z}_{kl}),\end{aligned}$$

$$\frac{\partial}{\partial\Omega} = \left(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial\omega_{\mu\nu}}\right), \quad \frac{\partial}{\partial\overline{\Omega}} = \left(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial\overline{\omega}_{\mu\nu}}\right),$$

$$\frac{\partial}{\partial Z} = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \cdots & \frac{\partial}{\partial z_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_{1n}} & \cdots & \frac{\partial}{\partial z_{mn}} \end{pmatrix}, \quad \frac{\partial}{\partial\overline{Z}} = \begin{pmatrix} \frac{\partial}{\partial\overline{z}_{11}} & \cdots & \frac{\partial}{\partial\overline{z}_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial\overline{z}_{1n}} & \cdots & \frac{\partial}{\partial\overline{z}_{mn}} \end{pmatrix},$$

where δ_{ij} denotes the Kronecker delta symbol.

C. L. Siegel [17] introduced the symplectic metric ds_n^2 on $\mathbb{H}_n$ invariant under the action (9.1) of $Sp(n,\mathbb{R})$ given by

$$ds_n^2 = \sigma\Big(Y^{-1}d\Omega\, Y^{-1}d\overline{\Omega}\Big) \tag{9.3}$$

and H. Maass [11] proved that the differential operator

$$\Delta_n = 4\,\sigma\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right) \tag{9.4}$$

is the Laplacian of $\mathbb{H}_n$ for the symplectic metric ds_n^2. Here $\sigma(A)$ denotes the trace of a square matrix A. In [23], the author proved that for any two positive real numbers A and B, the following metric

$$\begin{aligned} ds^2_{n,m;A,B} &= A\,\sigma\Big(Y^{-1}d\Omega\,Y^{-1}d\overline{\Omega}\Big) \\ &\quad + B\left\{\sigma\Big(Y^{-1}\,{}^tV\,V\,Y^{-1}d\Omega\,Y^{-1}d\overline{\Omega}\Big) + \sigma\Big(Y^{-1}\,{}^t(dZ)\,d\overline{Z}\Big)\right. \\ &\quad \left. -\sigma\Big(V\,Y^{-1}d\Omega\,Y^{-1}\,{}^t(d\overline{Z})\Big) - \sigma\Big(V\,Y^{-1}d\overline{\Omega}\,Y^{-1}\,{}^t(dZ)\Big)\right\} \end{aligned} \tag{9.5}$$

is a Riemannian metric on $\mathbb{H}_{n,m}$ which is invariant under the action (9.2) of the Jacobi group $G^J_{n,m}$.

The author [23] proved that for any two positive real numbers A and B, the Laplacian $\Delta_{n,m;A,B}$ of $(\mathbb{H}_{n,m}, ds^2_{n,m;A,B})$ is given by

$$\begin{aligned} \Delta_{n,m;A,B} &= \frac{4}{A}\left\{\sigma\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right) + \sigma\left(VY^{-1}\,{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial Z}\right)\right. \\ &\quad \left. +\sigma\left(V\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial Z}\right) + \sigma\left({}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial\Omega}\right)\right\} \\ &\quad + \frac{4}{B}\,\sigma\left(Y\frac{\partial}{\partial Z}\,{}^t\!\left(\frac{\partial}{\partial\overline{Z}}\right)\right). \end{aligned} \tag{9.6}$$

Using $G^J_{n,m}$-invariant differential operators on the Siegel-Jacobi space $\mathbb{H}_{n,m}$, we introduce a notion of Maass-Jacobi forms.

Definition 9.1. Let

$$\Gamma_{n,m} := Sp(n,\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(n,m)}$$

be the discrete subgroup of G^J, where

$$H_{\mathbb{Z}}^{(n,m)} = \Big\{ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)} \mid \lambda,\mu,\kappa \text{ are integral } \Big\}.$$

A smooth function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is called a Maass-Jacobi form on $\mathbb{H}_{n,m}$ if f satisfies the following conditions (MJ1)-(MJ3) :

(MJ1) f is invariant under $\Gamma_{n,m}$.
(MJ2) f is an eigenfunction of the Laplacian $\Delta_{n,m;A,B}$ (cf. (9.6)).
(MJ3) f has a polynomial growth, that is, there exist a constant $C>0$ and a positive integer N such that

$$|f(X+iY,Z)| \leq C\,|p(Y)|^N \quad \text{as } \det Y \longrightarrow \infty,$$

where $p(Y)$ is a polynomial in $Y=(y_{ij})$.

Remark 9.1. Let $\mathbb{D}_*$ be a commutative subalgebra of $\mathbb{D}(\mathbb{H}_{n,m})$ containing the Laplacian $\Delta_{n,m;A,B}$. We say that a smooth function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is a Maass-Jacobi form with respect to $\mathbb{D}_*$ if f satisfies the conditions $(MJ1)$, $(MJ2)_*$ and $(MJ3)$: the condition $(MJ2)_*$ is given by

$(MJ2)_*$ f is an eigenfunction of any invariant differential operator in $\mathbb{D}_*$.

Let ρ be a rational representation of $GL(n,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbb{R}^{(m,m)}$ be a symmetric half-integral semi-positive definite matrix of degree m. Let $C^\infty(\mathbb{H}_{n,m},V_\rho)$ be the algebra of all C^∞ functions on $\mathbb{H}_{n,m}$ with values in V_ρ. Let $J_{\rho,\mathcal{M}} : G^J_{n,m} \times \mathbb{H}_{n,m} \longrightarrow GL(V_\rho)$ be the canonical automorphic factor for $G^J_{n,m}$ on $\mathbb{H}_{n,m}$ given by

$$\begin{aligned} J_{\rho,\mathcal{M}}(g,(\Omega,Z)) \;&=\; e^{2\pi i\,\mathrm{tr}(\mathcal{M}[Z+\lambda\Omega+\mu](C\Omega+D)^{-1}C)} \\ &\quad\times e^{-2\pi i\,\mathrm{tr}(\mathcal{M}(\lambda\Omega\,{}^t\lambda+2\lambda\,{}^tZ+\kappa+\mu\,{}^t\lambda))}\rho(C\Omega+D), \end{aligned} \tag{9.7}$$

where $g=(M,(\lambda,\mu;\kappa)) \in G^J_{n.m}$ with $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $(\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$. We recall the Siegel's notation $\alpha[\beta]={}^t\beta\alpha\beta$ for suitable matrices α and β.

We define the $|_{\rho,\mathcal{M}}$-slash action of $G^J_{n,m}$ on $C^\infty(\mathbb{H}_{n,m},V_\rho)$ as follows: If $f \in C^\infty(\mathbb{H}_{n,m},V_\rho)$ and $g \in G^J_{n,m}$,

$$\Big(f|_{\rho,\mathcal{M}}[g]\Big)(\Omega,Z) := J_{\rho,\mathcal{M}}(g,(\Omega,Z))^{-1} f(g\cdot(\Omega,Z)). \tag{9.8}$$

We define $\mathbb{D}_{\rho,\mathcal{M}}$ to be the algebra of all differential operators D on $\mathbb{H}_{n,m}$ satisfying the following condition

$$(9.9) \qquad (Df)|_{\rho,\mathcal{M}}[g] = D(f|_{\rho,\mathcal{M}}[g])$$

for all $f \in C^\infty(\mathbb{H}_{n,m}, V_\rho)$ and for all $g \in G^J_{n,m}$. We denote by $\mathcal{Z}_{\rho,\mathcal{M}}$ the center of $\mathbb{D}_{\rho,\mathcal{M}}$.

We define an another notion of Maass-Jacobi forms as follows.

Definition 9.2. A vector-valued smooth function $\phi : \mathbb{H}_{n,m} \longrightarrow V_\rho$ is called a Maass-Jacobi form on $\mathbb{H}_{n,m}$ of type ρ and index $\mathcal{M}$ if it satisfies the following conditions $(MJ1)_{\rho,\mathcal{M}}$, $(MJ2)_{\rho,\mathcal{M}}$ and $(MJ3)_{\rho,\mathcal{M}}$:

$(MJ1)_{\rho,\mathcal{M}}$ $\phi|_{\rho,\mathcal{M}}[\gamma] = \phi$ for all $\gamma \in \Gamma_{n,m}$.

$(MJ2)_{\rho,\mathcal{M}}$ f is an eigenfunction of all differential operators in the center $\mathcal{Z}_{\rho,\mathcal{M}}$ of $\mathbb{D}_{\rho,\mathcal{M}}$.

$(MJ3)_{\rho,\mathcal{M}}$ f has a growth condition

$$\phi(\Omega, Z) = O\Big(e^{a \det Y} \cdot e^{2\pi \operatorname{tr}(\mathcal{M}[V]Y^{-1})}\Big)$$

as $\det Y \longrightarrow \infty$ for some $a > 0$.

The case $n = 1$, $m = 1$ and $\rho = \det^k (k = 0, 1, 2, \cdots)$ was studied by R. Bendt and R. Schmidt [1], A. Pitale [14] and K. Bringmann and O. Richter [4]. The case $n = 1$, $m =$arbitrary and $\rho = \det^k (k = 1, 2, \cdots)$ was dealt with by C. Conley and M. Raum [5]. In [5] the authors proved that the center $\mathcal{Z}_{\det^k,\mathcal{M}}$ of $\mathbb{D}_{\det^k,\mathcal{M}}$ is the polynomial algebra with one generator $\mathcal{C}^{k,\mathcal{M}}$ (cf. Theorem 3.2), the so-called *Casimir* operator which is a $|_{\det^k,\mathcal{M}}$-slash invariant differential operator of degree three for the case $n = m = 1$ or of degree four for the case $n = 1$, $m \geq 2$. As described in Section 6, Bringmann and Richter [4] considered the Poincaré series $\mathcal{P}^{(n,r)}_{k,\mathcal{M},s}$ (the case $n = m = 1$) (cf. (6.7)) that is a *harmonic* Maass-Jacobi form in the sense of Definition 9.2 and investigated its Fourier expansion and its Fourier coefficients. Here the *harmonicity* of $\mathcal{P}^{(n,r)}_{k,\mathcal{M},s}$ means that $\mathcal{C}^{k,\mathcal{M}}\mathcal{P}^{(n,r)}_{k,\mathcal{M},s} = 0$, i.e., $\mathcal{P}^{(n,r)}_{k,\mathcal{M},s}$ is an eigenfunction of $\mathcal{C}^{k,\mathcal{M}}$ with zero eigenvalue. Conley and Raum [5] generalized the results in [14] and [4] to the case $n = 1$ and m is an arbitrary positive integer.

Remark 9.2. In [3], Bringmann, Conley and Richter proved that the center of the algebra of differential operators invariant under the action of the Jacobi group over a complex quadratic field is generated by two Casimir operators of degree three. They also introduce an analogue of Kohnen's plus space for modular forms of half-integral weight over $K = \mathbb{Q}(i)$, and provide a lift from it to the space of Jacobi forms over K.

Definition 9.3. Let ρ and ρ' be two rational representations of $GL(n, \mathbb{C})$ on finite dimensional complex vector spaces V_ρ and V'_ρ respectively. Let $\mathcal{M}$ and $\mathcal{M}'$ be two symmetric half-integral semi-positive matrices of degree m. A differential operator

$T : C^\infty(\mathbb{H}_{n,m}) \otimes V_\rho \longrightarrow C^\infty(\mathbb{H}_{n,m}) \otimes V_{\rho'}$ is covariant from $|_{\rho,\,\mathcal{M}}$ to $|_{\rho',\,\mathcal{M}'}$ if T satisfies the following condition

$$T\Big(f|_{\rho,\mathcal{M}}[g]\Big) = (Tf)|_{\rho',\mathcal{M}'}[g]) \tag{9.10}$$

for all $f \in C^\infty(\mathbb{H}_{n,m}) \otimes V_\rho$ and for all $g \in G^J_{n,m}$.

Let $\mathbb{D}(\rho,\mathcal{M};\rho',\mathcal{M}')$ be the space of all covariant differential operators on $\mathbb{H}_{n,m}$ from $|_{\rho,\,\mathcal{M}}$ to $|_{\rho',\,\mathcal{M}'}$, and let $\mathbb{D}^q(\rho,\mathcal{M};\rho',\mathcal{M}')$ be the space of all covariant differential operators of order $\leq q$ on $\mathbb{H}_{n,m}$ from $|_{\rho,\,\mathcal{M}}$ to $|_{\rho',\,\mathcal{M}'}$. When $\rho = \rho'$ and $\mathcal{M} = \mathcal{M}'$, we refer to such differential operators as $|_{\rho,\,\mathcal{M}}$-invariant, and we write simply $\mathbb{D}_{\rho,\,\mathcal{M}}$ and $\mathbb{D}^q_{\rho,\,\mathcal{M}}$ instead of $\mathbb{D}(\rho,\mathcal{M};\rho,\mathcal{M})$ and $\mathbb{D}^q(\rho,\mathcal{M};\rho,\mathcal{M})$ respectively.

We present the natural problems.

Problem 1. Find the generators of the algebra $\mathbb{D}_{\rho,\,\mathcal{M}}$.

Problem 2. Find all the relations among a complete list of generators of $\mathbb{D}_{\rho,\,\mathcal{M}}$.

Finally we consider the special case that $\rho = \mathbf{1}$ is a trivial representation of $GL(n,\mathbb{C})$ and $\mathcal{M} = 0$. Let

$$T_{n,m} := S(m,\mathbb{C}) \times \mathbb{C}^{(m,n)}$$

be the complex vector space of dimension $\frac{n(n+1)}{2} + mn$. We obtain the natural action of $U(n)$ on $T_{n,m}$ given by

$$h \cdot (\omega,\zeta) := (h\,\omega\,{}^t h, \zeta\,{}^t h), \quad h \in U(n),\ \omega \in S(m,\mathbb{C}),\ \zeta \in \mathbb{C}^{(m,n)}. \tag{9.11}$$

We refer to [26] for a precise detail. Then the action (9.11) induces the action $\tau_{n,m}$ of $U(n)$ on the polynomial algebra $\mathrm{Pol}(T_{n,m})$ consisting of all polynomial functions on $T_{n,m}$. We denote by $\mathrm{Pol}(T_{n,m})^{U(n)}$ the subalgebra of $\mathrm{Pol}(T_{n,m})$ invariant under the action $\tau_{n,m}$ of $U(n)$. The we have the so-called Helgason map

$$\Theta_{n,m} : \mathrm{Pol}(T_{n,m})^{U(n)} \longrightarrow \mathbb{D}_{\mathbf{1},0} = \mathbb{D}(\mathbf{1},0;\mathbf{1},0)$$

defined by

$$\Big(\Theta_{n,m}(P)f\Big)(gK^J) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g \exp\left(\sum_{\alpha=1}^{N_\star} t_\alpha \eta_\alpha \right) K^J \right) \right]_{(t_\alpha)=0}, \tag{9.12}$$

where $N_\star = n(n+1) + 2mn$, $\big\{\eta_\alpha \,|\, 1 \leq \alpha \leq N_\star\big\}$ is a basis of $\mathfrak{p}^J_{n,m}$ and $P \in \mathrm{Pol}(T_{n,m})^{U(n)}$. The map $\Theta_{n,m}$ is a linear bijection but is not multiplicative.

The following natural problems arise.

Problem 3. Find a complete list of explicit generators of $\mathrm{Pol}(T_{n,m})^{U(n)}$.

Problem 4. Find all the relations among a complete list of generators of $\mathrm{Pol}(T_{n,m})^{U(n)}$.

Problem 5. Find an easy or effective way to express the images of the above invariant polynomials or generators of $\mathrm{Pol}(T_{n,m})^{U(n)}$ under the Helgason map $\Theta_{n,m}$ explicitly.

Recently Problem 3 was solved completely in [9].

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