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The Siegel-Jacobi Operator

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1 Introduction

For any positive integer $g \in \mathbb{Z}^+$, we let H_g the Siegel upper half plane of degree g and let $\Gamma_g := \mathrm{Sp}(g, \mathbb{Z})$ the Siegel modular group of degree g. Let ρ be a rational finite dimensional representation of the general linear group $\mathrm{GL}(g, \mathbb{C})$ on V_ρ and let $\mathcal{M}$ be a symmetric half-integral semipositive matrix of degree h. Let $J_{\rho,\mathcal{M}}(\Gamma_g)$ be the vector space of all Jacobi forms on Γ_g of index $\mathcal{M}$ with respect to ρ (see Definition 2.1). For a positive integer r with $r < g$, we let $\rho^{(r)} : \mathrm{GL}(r, \mathbb{C}) \to \mathrm{GL}(V_\rho)$ be a rational representation of $\mathrm{GL}(r, \mathbb{C})$ defined by

$$\rho^{(r)}(a)v := \rho\left(\begin{pmatrix} a & 0 \\ 0 & E_{g-r} \end{pmatrix}\right)v, \quad a \in \mathrm{GL}(r, \mathbb{C}), v \in V_\rho .$$

The *Siegel-Jacobi operator* $\Psi_{g,r} : J_{\rho,\mathcal{M}}(\Gamma_g) \to J_{\rho^{(r)},\mathcal{M}}(\Gamma_r)$ is defined by

$$(\Psi_{g,r} f)(Z, W) := \lim_{t\to\infty} f\left(\begin{pmatrix} Z & 0 \\ 0 & itE_{g-r} \end{pmatrix}, (W, 0)\right),$$

where $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$, $Z \in H_r$ and $W \in \mathbb{C}^{(h,r)}$. We observe that the above limit always exists and the Siegel-Jacobi operator is a linear mapping (cf. [14]).

The aim of this paper is to investigate some properties of the Siegel-Jacobi operator. This article is organized as follows. In section **2**, we establish the notations and give a definition of Jacobi forms. In section **3**, we obtain the Shimura isomorphism based on ZIEGLER's work [14]. Using this isomorphism and the theory of singular modular forms, we obtain an injectivity or a surjectivity of the Siegel-Jacobi operator under certain conditions. In the final section, we define an action of the Hecke operator of Γ_g on $J_{\rho,\mathcal{M}}(\Gamma_g)$ and prove that the action of the Siegel-Jacobi operator on Jacobi forms is compatible with that of the Hecke algebra.

Notations. We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of rational numbers, the field of real numbers, and the field of complex numbers respectively. For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(g, \mathbb{R})$ and $Z \in H_g$, we set $M\langle Z\rangle := (AZ + B)(CZ + D)^{-1}$. $[\Gamma_g, k]$ (resp. $[\Gamma_g, \rho]$) denotes the vector space of all Siegel modular forms of weight k (resp. of type ρ). We denote by

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$\mathbb{Z}^+$ the set of all positive integers. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. E_n denotes the identity matrix of degree n.

2 Jacobi Forms

In this section, we establish the notations and define the concept of Jacobi forms.

Let

$$\mathrm{GSp}(g,\mathbb{R})^+ = \{M \in \mathbb{R}^{(2g,2g)} \mid {}^tMJ_gM = vJ_g \quad \text{for some} \quad v > 0\}$$

be the group of *similitudes* of degree g, where

$$J_g := \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}.$$

Let $M \in \mathrm{GSp}(g,\mathbb{R})^+$. If ${}^tMJ_gM = vJ_g$, we write $v = v(M)$. It is easy to see that $\mathrm{GSp}(g,\mathbb{R})^+$ acts on H_g transitively by

$$M\langle Z\rangle := (AZ+B)(CZ+D)^{-1},$$

where $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GSp}(g,\mathbb{R})^+$ and $Z \in H_g$.

For two positive integers g and h, we consider the *Heisenberg group*

$$H_{\mathbb{R}}^{(g,h)} := \{[(\lambda,\mu),\kappa] \mid \lambda,\mu \in \mathbb{R}^{(h,g)}, \kappa \in \mathbb{R}^{(h,h)}, \kappa + \mu{}^t\lambda \quad \text{symmetric}\}$$

endowed with the following multiplication law

$$[(\lambda,\mu),\kappa] \circ [(\lambda',\mu'),\kappa'] := [(\lambda+\lambda',\mu+\mu'),\kappa+\kappa'+\lambda{}^t\mu' - \mu{}^t\lambda'].$$

We define the semidirect product of $\mathrm{GSp}(g,\mathbb{R})^+$ and $H_{\mathbb{R}}^{(g,h)}$

$$\hat{G}^J := \mathrm{GSp}(g,\mathbb{R})^+ \ltimes H_{\mathbb{R}}^{(g,h)}$$

endowed with the following multiplication law

$$\begin{aligned} &(M,[(\lambda,\mu),\kappa]) \cdot (M',[(\lambda',\mu'),\kappa']) \\ &:= (MM',[(v(M')^{-1}\tilde{\lambda}+\lambda', v(M')^{-1}\tilde{\mu}+\mu'), v(M')^{-1}\kappa+\kappa'+v(M')^{-1}(\tilde{\lambda}{}^t\mu'-\tilde{\mu}{}^t\lambda')]),\end{aligned}$$

with M, $M' \in \mathrm{GSp}\,(g,\mathbb{R})^+$ and $(\tilde{\lambda},\tilde{\mu}) := (\lambda,\mu)M'$. Clearly the *Jacobi group* $G^J := \mathrm{Sp}(g,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$ is a normal subgroup of $\hat{G}^J$. It is easy to see that $\hat{G}^J$ acts on $H_g \times \mathbb{C}^{(h,g)}$ transitively by

(2.1) $$(M,[(\lambda,\mu),\kappa]) \cdot (Z,W) := (M\langle Z\rangle, v(W+\lambda Z+\mu)(CZ+D)^{-1}),$$

where $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GSp}(g,\mathbb{R})^+$, $v = v(M)$, $(Z,W) \in H_g \times \mathbb{C}^{(h,g)}$.

Let ρ be a rational representation of $\mathrm{GL}(g,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbb{R}^{(h,h)}$ be a symmetric half integral semi-positive matrix of degree h. Let $C^\infty(H_g \times \mathbb{C}^{(h,g)}, V_\rho)$ be the algebra of all C^∞ functions on $H_g \times \mathbb{C}^{(h,g)}$ with values in V_ρ. For $f \in C^\infty(H_g \times \mathbb{C}^{(h,g)}, V_\rho)$, we define

$$(2.2)\qquad \begin{aligned} &(f|_{\rho,\mathcal{M}}[(M,[(\lambda,\mu),\kappa])])(Z,W) \\ :=\ & e^{-2\pi\nu i\sigma(\mathcal{M}[W+\lambda Z+\mu](CZ+D)^{-1}C)}e^{2\pi\nu i\sigma(\mathcal{M}(\lambda Z^t\lambda+2\lambda^t W+(\kappa+\mu^t\lambda)))} \\ & \times \rho(CZ+D)^{-1}f(M\langle Z\rangle, \nu(W+\lambda Z+\mu)(CZ+D)^{-1}), \end{aligned}$$

where $\nu = \nu(M)$.

Definition 2.1. Let ρ and $\mathcal{M}$ be as above. Let

$$H_{\mathbb{Z}}^{(g,h)} := \{[(\lambda,\mu),\kappa] \in H_{\mathbb{R}}^{(g,h)} \mid \lambda,\mu \in \mathbb{Z}^{(h,g)}, \kappa \in \mathbb{Z}^{(h,h)}\}.$$

A *Jacobi form* of index $\mathcal{M}$ with respect to ρ on a subgroup $\Gamma \subset \Gamma_g$ of finite index is a holomorphic function $f \in C^\infty(H_g \times \mathbb{C}^{(h,g)}, V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho,\mathcal{M}}[\tilde{\gamma}] = f$ for all $\tilde{\gamma} \in \Gamma^J := \Gamma \ltimes H_{\mathbb{Z}}^{(g,h)}$.

(B) f has a Fourier expansion of the following form:

$$f(Z,W) = \sum_{\substack{T\geq 0\\ \text{half-integral}}} \sum_{R\in\mathbb{Z}^{(g,h)}} c(T,R)\cdot e^{\frac{2\pi i}{\lambda_\Gamma}\sigma(TZ)}\cdot e^{2\pi i\sigma(RW)}$$

with some $\lambda_\Gamma \in \mathbb{Z}$ and $c(T,R) \neq 0$ only if

$$\begin{pmatrix} \frac{1}{\lambda_\Gamma}T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} \geq 0.$$

If $g \geq 2$, the condition (B) is superfluous by Koecher principle (see [14] Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ. In the special case $V_\rho = \mathbb{C}$, $\rho(A) = (\det A)^k$ ($k \in \mathbb{Z}$, $A \in \mathrm{GL}(g,\mathbb{C})$), we write $J_{k,\mathcal{M}}(\Gamma)$ instead of $J_{\rho,\mathcal{M}}(\Gamma)$ and call k the *weight* of a Jacobi form $f \in J_{k,\mathcal{M}}(\Gamma)$.

Ziegler ([14] Theorem 1.8 or [2] Theorem 1.1) proves that the vector space $J_{\rho,\mathcal{M}}(\Gamma)$ is finite dimensional.

3 The Siegel-Jacobi Operator

Let (ρ, V_ρ) be a finite dimensional representation of $\mathrm{GL}(g,\mathbb{C})$. For any positive integer r with $r < g$, we denote by $V_\rho^{(r)}$ the subspace of V_ρ generated by the values $\{\Psi_{g,r}f(Z,W) \mid f \in J_{\rho,\mathcal{M}}(\Gamma_g), (Z,W) \in H_g \times \mathbb{C}^{(h,g)}\}$. According to [10], $V_\rho^{(r)}$ is invariant under

$$\left\{\begin{pmatrix} a & 0 \\ 0 & E_{g-r}\end{pmatrix} : a \in \mathrm{GL}(r,\mathbb{C})\right\}.$$

Then we have a rational representation $\rho^{(r)}$ of $\mathrm{GL}(r,\mathbb{C})$ on $V_\rho^{(r)}$ defined by

$$\rho^{(r)}(a)v := \rho\left(\begin{pmatrix} a & 0 \\ 0 & E_{g-r}\end{pmatrix}\right)v, \quad a \in \mathrm{GL}(r,\mathbb{C}), v \in V_\rho^{(r)}.$$

Following the argument of [10], we obtain

Lemma 3.1. *If (ρ, V_ρ) is irreducible, then $(\rho^{(r)}, V_\rho^{(r)})$ is also irreducible.*

Now we assume that $\mathcal{M}$ is a symmetric positive half-integral matrix of degree h. For any a, $b \in \mathbb{Q}^{(h,g)}$, we consider the theta series

$$\vartheta_{2\mathcal{M},a,b}(Z,W) := \sum_{\lambda\in\mathbb{Z}^{(h,g)}} e^{\pi i\sigma(2\mathcal{M}((\lambda+a)Z{}^t(\lambda+a)+2(\lambda+a){}^t(W+b)))}$$

with characteristic (a,b) converging uniformly on any compact subset of $H_g \times \mathbb{C}^{(h,g)}$.

We fix an element $Z_0 \in H_g$. Let $\mathcal{N}$ be a complete system of representatives of the cosets $(2\mathcal{M})^{-1}\mathbb{Z}^{(h,g)}/\mathbb{Z}^{(h,g)}$. We denote by $T_{\mathcal{M}}(Z_0)$ the vector space of all holomorphic functions $\varphi: \mathbb{C}^{(h,g)} \to \mathbb{C}$ satisfying the condition

$$\varphi(W+\lambda Z_0+\mu) = e^{-2\pi i\sigma(\mathcal{M}(\lambda Z_0{}^t\lambda+2\lambda{}^tW))}\varphi(W) \tag{3.1}$$

for every λ, $\mu \in \mathbb{Z}^{(h,g)}$. The functions $\{\vartheta_{2\mathcal{M},a,0}(Z_0,W) \mid a \in \mathcal{N}\}$ form a basis of $T_{\mathcal{M}}(Z_0)$ and its dimension is clearly $\{\det(2\mathcal{M})\}^g$. If f is a Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma_g)$, it is easy to see that each component of $\phi(W) := f(Z_0,W)$ satisfies the relation (3.1). So we may write

$$f(Z,W) = \sum_{a\in\mathcal{N}} f_a(Z)\cdot\vartheta_{2\mathcal{M},a,0}(Z,W), \quad Z \in H_g, W \in \mathbb{C}^{(h,g)}, \tag{3.2}$$

where $\{f_a: H_g \to V_\rho \mid a \in \mathcal{N}\}$ are uniquely determined holomorphic functions on H_g.

According to [14], we have

$$\begin{aligned} f_a(-Z^{-1}) &= \left\{\det\left(\frac{Z}{i}\right)\right\}^{-\frac{h}{2}}\cdot\{\rho(-Z)\}\cdot\{\det(2\mathcal{M})\}^{-\frac{g}{2}} \\ &\quad \times\sum_{b\in\mathcal{N}} e^{2\pi i\sigma(2\mathcal{M}a{}^tb)}\cdot f_b(Z) \end{aligned} \tag{3.3}$$

and

$$f_a(Z+S) = e^{-2\pi i\sigma(\mathcal{M}aS{}^ta)}\cdot f_a(Z), \quad S = {}^tS \in \mathbb{Z}^{(g,g)}. \tag{3.4}$$

By an easy argument, we see that the functions $\{f_a \mid a \in \mathcal{N}\}$ must have the Fourier expansion of the form

$$f_a(Z) = \sum_{\substack{T={}^tT\geq 0\\ \text{half-integral}}} c(T)\cdot e^{2\pi i\sigma(TZ)}. \tag{3.5}$$

Conversely, suppose there is given a family $\{f_a \mid a \in \mathcal{N}\}$ of holomorphic functions $f_a : H_g \to V_\rho$ satisfying the transformation laws (3.3), (3.4) and the cusp condition (3.5). Then we obtain a Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma_g)$ by defining $f(Z,W)$ via the equation (3.2).

So we obtain the Shimura isomorphism:

Theorem. (SHIMURA) *The equation* (3.2) *gives an isomorphism between* $J_{\rho,\mathcal{M}}(\Gamma_g)$ *and the vector space of* V_ρ*-valued Siegel modular forms of half integral weight satisfying the transformation laws* (3.3), (3.4) *and the cusp condition* (3.5).

Corollary 3.2. *Let* $2\mathcal{M}$ *be unimodular. We assume that* ρ *satisfies the following condition:*

$$(3.6) \qquad \rho(A) = \rho(-A) \quad \textit{for all} \quad A \in GL(g,\mathbb{C}).$$

Then we have

$$(3.7) \qquad J_{\rho,\mathcal{M}}(\Gamma_g) = [\Gamma_g, \tilde{\rho}] \cdot \vartheta_{2\mathcal{M},0,0}(Z,W) \cong [\Gamma_g, \tilde{\rho}],$$

where $\tilde{\rho} = \rho \otimes \det^{-\frac{h}{2}}$. *In particular, if* $k \cdot g$ *is even,*

$$(3.8) \qquad J_{k,\mathcal{M}}(\Gamma_g) = [\Gamma_g, k - \frac{h}{2}] \cdot \vartheta_{2\mathcal{M},0,0}(Z,W) \cong [\Gamma_g, k - \frac{h}{2}].$$

Proof. The proof of (3.7) follows from (3.3), (3.4) and (3.5). The representation $\det^k : \mathrm{GL}(g,\mathbb{C}) \to \mathbb{C}^\times$ defined by $\det^k(A) = (\det(A))^k$ satisfies the condition (3.6). Hence (3.8) follows from (3.7). □

Notations 3.3. In corollary 3.2, we denote the isomorphism of $J_{\rho,\mathcal{M}}(\Gamma_g)$ (resp. $J_{k,\mathcal{M}}(\Gamma_g)$) onto $[\Gamma_g, \tilde{\rho}]$ (resp. $[\Gamma_g, k - \frac{h}{2}]$) by

$$S_\rho : J_{\rho,\mathcal{M}}(\Gamma_g) \to [\Gamma_g, \tilde{\rho}] \quad (\text{resp. } S_{g,k} : J_{k,\mathcal{M}}(\Gamma_g) \to [\Gamma_g, k - \tfrac{h}{2}]).$$

We denote the Siegel operator by $\Phi_{g,r} : [\Gamma_g, \rho] \to [\Gamma_r, \rho^{(r)}]$, $0 < r < g$.

Definition 3.4. An irreducible finite dimensional representation ρ of $\mathrm{GL}(g,\mathbb{C})$ is determined by its highest weight $(\lambda_1, \lambda_2, \cdots, \lambda_g) \in \mathbb{Z}^g$ with $\lambda_1 \geq \cdots \geq \lambda_g$. We denote this representation by $\rho = (\lambda_1, \cdots, \lambda_g)$. The number $k(\rho) := \lambda_g$ is called the *weight* of ρ.

Theorem 3.5. *Let* $2\mathcal{M}$ *be a positive unimodular symmetric even matrix of degree* h. *We assume that* ρ *is irreducible and satisfies the condition* (3.6). *If* $2k(\rho) < g + \mathrm{rank}(\mathcal{M})$, *then the Siegel-Jacobi operator* $\Psi_{g,g-1}$ *is injective.*

Proof. By corollary 3.2, we have

$$(3.9) \qquad J_{\rho,\mathcal{M}}(\Gamma_g) = [\Gamma_g, \rho \otimes \det^{-\frac{h}{2}}] \cdot \vartheta_{2\mathcal{M},0,0}(Z,W).$$

By an easy computation, we have

$$S_{\rho^{(g-1)}} \circ \Psi_{g,g-1} = \Phi_{g,g-1} \circ S_\rho . \tag{3.10}$$

According to the assumption, the irreducible representation $\rho \otimes \det^{-\frac{h}{2}}$ of $GL(g, \mathbb{C})$ is *singular*, that is, $2k(\rho \otimes \det^{-\frac{h}{2}}) < g$. According to the well-known theory of singular modular forms ([10] Satz 4), every $f \in [\Gamma_g, \rho \otimes \det^{-\frac{h}{2}}]$ is a singular modular form. Thus the Siegel operator $\Phi_{g,g-1}$ is injective (see [11] for the proof of the injectivity of $\Phi_{g,g-1}$). Since S_ρ and $S_{\rho^{(g-1)}}$ are isomorphisms, the Siegel-Jacobi operator $\Psi_{g,g-1}$ is injective by (3.10). This completes the proof of Theorem 3.5. □

Theorem 3.6. *Let* $2\mathcal{M}$ *be as above in Theorem* 3.5. *Assume that* ρ *is irreducible and satisfies the condition* (3.6). *If* $2k(\rho) + 1 < g + \text{rank}(\mathcal{M})$, *then the Siegel-Jacobi operator* $\Psi_{g,g-1}$ *is an isomorphism.*

Proof. By corollary 3.2, we have the relation (3.9). Similarly, we have the commutation relation (3.10). Since $2k(\rho \otimes \det^{-\frac{h}{2}}) + 1 < g$ by the assumption, according to the theory of singular modular forms (cf. [3] and [11]), the Siegel operator $\Phi_{g,g-1}$ is an isomorphism. Since S_ρ, $S_{\rho^{(g-1)}}$ and $\Phi_{g,g-1}$ are all isomorphisms, $\Psi_{g,g-1}$ is an isomorphism. □

Theorem 3.7. *Let* $2\mathcal{M}$ *be as above in Theorem* 3.5. *Assume that* $2k(\rho) > 4g + \text{rank}(\mathcal{M})$ *and* $k \equiv 0 \pmod 2$. *Then the Siegel-Jacobi operator* $\Psi_{g,g-1} : J_{k,\mathcal{M}}(\Gamma_g) \to J_{k,\mathcal{M}}(\Gamma_{g-1})$ *is surjective.*

Proof. By corollary 3.2, we have

$$J_{k,\mathcal{M}}(\Gamma_g) = [\Gamma_g, k - \frac{h}{2}] \cdot \vartheta_{2\mathcal{M},0,0}(Z, W) \cong [\Gamma_g, k - \frac{h}{2}] .$$

By the assumption, $2(k - \frac{h}{2}) > g$ and $k - \frac{h}{2} \equiv 0 \pmod 2$. According to Maass [6], the Siegel operator

$$\Phi_{g,g-1} : [\Gamma_g, k - \frac{h}{2}] \to [\Gamma_{g-1}, k - \frac{h}{2}]$$

is surjective. Consequently the surjectivity of the Siegel-Jacobi operator $\Psi_{g,g-1}$ follows immediately from the commutation relation

$$S_{g-1,k} \circ \Psi_{g,g-1} = \Phi_{g,g-1} \circ S_{g,k} .$$ □

4 Hecke Operator

In this section, we give the action of Hecke operators on Jacobi forms and prove that this action is compatible with that of the Siegel-Jacobi operator.

For a positive integer l, we define

$$O_g(l) := \{M \in \mathbb{Z}^{(2g,2g)} \mid {}^tMJ_gM = lJ_g\},$$

where

$$J_g := \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}.$$

$O_g(l)$ is decomposed into finitely many double cosets *mod* Γ_g, i.e.,

$$O_g(l) = \bigcup_{j=1}^{m} \Gamma_g g_j \Gamma_g \quad \text{(disjoint union)}.$$

We define

$$T(l) := \sum_{j=1}^{m} \Gamma_g g_j \Gamma_g \in \mathscr{H}^{(g)}, \quad \text{the Hecke algebra.}$$

Let $M \in O_g(l)$. For a Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$, we define

$$f|_{\rho,\mathcal{M}}(\Gamma_g M \Gamma_g) := l^{gk(\rho)-\frac{g(g+1)}{2}} \sum_{i=1}^{m} f|_{\rho,\mathcal{M}}[(M_i,[(0,0),0])],$$

where $\Gamma_g M \Gamma_g = \bigcup_i^m \Gamma_g M_i$ (finite disjoint union) and $k(\rho)$ denotes the weight of ρ. See (2.2) in section **2** for the definition of $f|_{\rho,\mathcal{M}}[(M_i,[(0,0),0])]$.

Proposition 4.1. *Let l be a positive integer. Let $M \in O_g(l)$ and $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$. Then*

$$f|_{\rho,\mathcal{M}}(\Gamma_g M \Gamma_g) \in J_{\rho,l\mathcal{M}}(\Gamma_g).$$

Proof. It is easy to compute it and so we omit the proof. □

For a prime p, we define

$$O_{g,p} := \bigcup_{l=0}^{\infty} O_g(p^l).$$

Let $\check{\mathscr{L}}_{g,p}$ be the $\mathbb{C}$-module generated by all left cosets $\Gamma_g M$, $M \in O_{g,p}$ and $\check{\mathscr{H}}_{g,p}$ the $\mathbb{C}$-module generated by all double cosets $\Gamma_g M \Gamma_g$, $M \in O_{g,p}$. Then $\check{\mathscr{H}}_{g,p}$ is a commutative associative algebra. We associate to a double coset

$$\Gamma_g M \Gamma_g = \bigcup_{i=1}^{m} \Gamma_g M_i, \quad M, M_i \in O_{g,p} \quad \text{(disjoint union)}$$

the element

$$j(\Gamma_g M \Gamma_g) = \sum_{i=1}^{m} \Gamma_g M_i \in \check{\mathscr{L}}_{g,p}.$$

We extend j linearly to the Hecke algebra $\check{\mathscr{H}}_{g,p}$ and then we have a monomorphism $j:\check{\mathscr{H}}_{g,p} \to \check{\mathscr{L}}_{g,p}$. We now define a bilinear mapping

$$\check{\mathscr{H}}_{g,p} \times \check{\mathscr{L}}_{g,p} \to \check{\mathscr{L}}_{g,p}$$

by

$$(\Gamma_g M \Gamma_g)\cdot(\Gamma_g M_0) = \sum_{i=1}^{m} \Gamma_g M_i M_0\,, \quad \text{where} \quad \Gamma_g M \Gamma_g = \bigcup_{i=1}^{m} \Gamma_g M_i\,.$$

This mapping is well defined because the definition does not depend on the choice of representatives.

Let $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ be a Jacobi form. For a left coset $L := \Gamma_g N$ with $N \in O_{g,p}$, we put

$$f|L := f|_{\rho,\mathcal{M}}[(N,[(0,0),0])]\,. \tag{4.1}$$

We extend this operator (4.1) linearly to $\check{\mathscr{L}}_{g,p}$. If $T \in \check{\mathscr{H}}_{g,p}$, we write

$$f|T := f|j(T)\,.$$

Obviously we have

$$(f|T)|L = f|(TL)\,, \quad f \in J_{\rho,\mathcal{M}}(\Gamma_g)\,.$$

In a left coset $\Gamma_g M$, $M \in O_{g,p}$, we can choose a representative M of the form

$$M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \quad {}^tAD = p^{k_0}E_g, {}^tBD = {}^tDB\,, \tag{4.2}$$

$$A = \begin{pmatrix} a & {}^t\alpha \\ 0 & A^* \end{pmatrix}, \quad B = \begin{pmatrix} b & {}^t\beta_1 \\ \beta_2 & B^* \end{pmatrix}, \quad D = \begin{pmatrix} d & 0 \\ \delta & D^* \end{pmatrix}, \tag{4.3}$$

where α, β_1, β_2, $\delta \in \mathbb{Z}^{g-1}$. Then we have

$$M^* := \begin{pmatrix} A^* & B^* \\ 0 & D^* \end{pmatrix} \in O_{g-1,p}\,. \tag{4.4}$$

For an integer $r \in \mathbb{Z}$, we define

$$(\Gamma_g M)^* := \frac{1}{d^r}\Gamma_{g-1}M^*\,. \tag{4.5}$$

If $\Gamma_g M \Gamma_g = \bigcup_{j=1}^{m} \Gamma_g M_j$ (disjoint union), M, $M_j \in O_{g,p}$, then we define in a natural way

$$(\Gamma_g M \Gamma_g)^* = \frac{1}{d^r}\sum_{j=1}^{m} \Gamma_{g-1}M_j^*\,. \tag{4.6}$$

We extend the above map (4.6) linearly on $\check{\mathscr{H}}_{g,p}$ and then we obtain an algebra homomorphism

$$(4.7) \qquad \begin{array}{ccc} \check{\mathscr{H}}_{g,p} & \longrightarrow & \check{\mathscr{H}}_{g-1,p} \\ T & \longmapsto & T^* . \end{array}$$

It is known that the above map is a surjective map ([13] Theorem 2).

Let $\Psi^0_{g,r} : J_{\rho,\mathcal{M}}(\Gamma_g) \to J_{\rho_0^{(r)},\mathcal{M}}(\Gamma_r)$ be the *modified Siegel-Jacobi operator* defined by

$$(\Psi^0_{g,r} f)(Z,W) := \lim_{t\to\infty} f\left(\begin{pmatrix} itE_{g-r} & 0 \\ 0 & Z \end{pmatrix}, (0,W)\right), \quad (Z,W) \in H_r \times \mathbb{C}^{(h,r)},$$

where $\rho_0^{(r)} : \mathrm{GL}(r,\mathbb{C}) \to \mathrm{GL}(V_\rho)$ is a finite dimensional representation of $\mathrm{GL}(r,\mathbb{C})$ defined by

$$\rho_0^{(r)}(A) = \rho\begin{pmatrix} E_{g-r} & 0 \\ 0 & A \end{pmatrix}, \quad A \in \mathrm{GL}(r,\mathbb{C}).$$

The following theorem is a variant of the Siegel version [4].

Theorem 4.2. *Suppose we have*

(a) *a rational finite dimensional representation*

$$\rho : \mathrm{GL}(g,\mathbb{C}) \to \mathrm{GL}(V_\rho),$$

(b) *a rational finite dimensional representation*

$$\rho_0 : \mathrm{GL}(g-1,\mathbb{C}) \to \mathrm{GL}(V_{\rho_0}),$$

(c) *a linear map* $R : V_\rho \to V_{\rho_0}$,

satisfying the following properties (1) *and* (2):

(1) $R \circ \rho\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} = \rho_0(A) \circ R$ *for all* $A \in \mathrm{GL}(g-1,\mathbb{C})$.

(2) $R \circ \rho\begin{pmatrix} a & 0 \\ 0 & E_{g-1} \end{pmatrix} = a^r R$ *for some* $r \in \mathbb{Z}$.

Then for any $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ *and* $T \in \check{\mathscr{H}}_{g,p}$, *we have*

$$(R \circ \Psi^0_{g,g-1})(f|T) = R(\Psi^0_{g,g-1} f)|T^* .$$

Proof. Let $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ be a Jacobi form. Then we have the Fourier expansion

$$f(Z,W) = \sum_{T,R} c(T,R) e^{2\pi i\sigma(TZ)} \cdot e^{2\pi i\sigma(RW)} .$$

By an easy computation, we have

$$(\Psi^0_{g,g-1}f)(Z,W)=\sum_{T,R} c\left(\begin{pmatrix}0&0\\0&T\end{pmatrix},\begin{pmatrix}0\\R\end{pmatrix}\right)\cdot e^{2\pi i\sigma(TZ+RW)},$$

where $(Z,W)\in H_{g-1}\times\mathbb{C}^{(h,g-1)}$, $T\in\mathbb{Q}^{(g-1,g-1)}$ runs over the set of all half integral matrices of degree $g-1$ and R runs over the set of all $(g-1)\times h$ integral matrices.

Lemma 4.3. *Let $f\in J_{\rho,\mathcal{M}}(\Gamma_g)$ be a Jacobi form. Then for any $\xi\in\mathbb{C}^{g-1}$,*

$$\Psi^0_{g,g-1}\left(\rho\begin{pmatrix}1&0\\\xi&E_{g-1}\end{pmatrix}f\right)=\Psi^0_{g,g-1}f.$$

Proof. Since ρ is rational, it suffices to show the above formula for *integral* $\xi\in\mathbb{Z}^{g-1}$. For convenience, we put

$$U=\begin{pmatrix}1&0\\\xi&E_{g-1}\end{pmatrix},\quad \xi\in\mathbb{Z}^{g-1}.$$

Then $M_U:=\begin{pmatrix}{}^tU^{-1}&0\\\xi&E_{g-1}\end{pmatrix}$ is an element in Γ_g. Since $f\in J_{\rho,\mathcal{M}}(\Gamma_g)$, we have $f|_{\rho,\mathcal{M}}[M_U]=f$ and hence

$$f(Z[U^{-1}],WU^{-1})=\rho(U)f(Z,W).$$

Thus we have

$$\begin{aligned}(\Psi^0_{g,g-1}(\rho(U)f))(Z,W)&=\lim_{t\to\infty}f\left(\begin{pmatrix}it+Z[\xi]&-{}^t\xi Z\\-Z\xi&Z\end{pmatrix},(-W\xi,W)\right)\\&=(\Psi^0_{g,g-1}f)(Z,W).\end{aligned}$$

Hence this completes the proof of the above lemma. □

Let $L:=\Gamma_gM\in\check{\mathcal{L}}_{g,p}$ $(M\in O_{g,p})$ be fixed, where M is of the form (4.2). We write $v:=v(M)=p^{k_0}$. Then we have

$$(f|L)(\tilde{Z},\tilde{W})=\rho(D)^{-1}f(\frac{1}{v}(\tilde{Z}[{}^tA]+A{}^tB),\tilde{W}{}^tA),$$

where $(\tilde{Z},\tilde{W})\in H_g\times\mathbb{C}^{(h,g)}$.

Therefore we have

$$\begin{aligned}&(\Psi^0_{g,g-1}(f|L))(Z,W)\\=\;&\rho(D)^{-1}\lim_{t\to\infty}f\left(\frac{1}{v}\begin{pmatrix}ita^2+Z[\alpha]&{}^t\alpha Z{}^tA^*\\A^*Z\alpha&Z[{}^tA^*]\end{pmatrix}+BD^{-1},(W\alpha,W{}^tA^*)\right)\\=\;&\rho(D)^{-1}(\Psi^0_{g,g-1}f)(\frac{1}{v}(Z[{}^tA^*]+B^{*t}A^*),W{}^tA^*).\end{aligned}$$

And we have

$$\begin{aligned} & d^r((\Psi^0_{g,g-1}f)|(\Gamma_g M)^*)(Z,W) \\ = \ & \rho\begin{pmatrix} 1 & 0 \\ 0 & D^* \end{pmatrix}(\Psi^0_{g,g-1}f)(\frac{1}{v}(Z[{}^tA^*]+B^{*t}A^*),W{}^tA^*)\,. \end{aligned}$$

According to Lemma 4.3, we may take

$$D=\begin{pmatrix} d & 0 \\ 0 & D^* \end{pmatrix}.$$

Thus we have

$$(\Psi^0_{g,g-1}(f|L))(Z,W)=\rho\begin{pmatrix} d & 0 \\ 0 & D^* \end{pmatrix}\cdot\rho\begin{pmatrix} 1 & 0 \\ 0 & D^* \end{pmatrix}((\Psi^0_{g,g-1}f)|(\Gamma_{g-1}M^*))(Z,W)\,.$$

Finally according to the assumption (c) in Theorem 4.2, we obtain

$$R(\Psi^0_{g,g-1}(f|(\Gamma_g M)))=R(\Psi^0_{g,g-1})|(\Gamma_g M)^*\,.$$

Hence for any $T\in\check{\mathscr{H}}_{g,p}$, we have

$$R(\Psi^0_{g,g-1}(f|T))=R(\Psi^0_{g,g-1}f)|T^*\,.$$

This completes the proof of Theorem 4.2. □

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