Remark on Harmonic Analysis on Siegel-Jacobi Space

Jae-Hyun Yang

Inha University

jhyang@inha.ac.kr

Department of Mathematics
Kyoto University
Kyoto, Japan
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My work was inspired by the spirit of the great number theorists of the 20th century

Carl Ludiwig Siegel (1896-1981)

André Weil (1906-1998)

Hans Maass (1911-1992)

Atle Selberg (1917-2007)

Robert P. Langlands (1936-)

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[A] **C. L. Siegel**, *Symplectic Geometry,* Amer. J. Math. **65** (1943), 1-86; Academic Press, New York and London (1964); Gesammelte Abhandlungen, no. **41, vol. II**, Springer-Verlag (1966), 274-359.

[B] **H. Maass**, *Über eine neue Art von nicht-analytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funtionalgleichungen,* Math. Ann. **121** (1949), 141-183.

[C] **A. Selberg**, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series,* J. Indian Math. Soc. B. **20** (1956), 47-87.

[D] **A. Weil**, *Sur certains groupes d'operateurs unitaires (French)*, Acta Math. **111** (1964), 143–211.

Introduction

Let

$$\mathbf{H}_n = \left\{ \Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega,\ \operatorname{Im}\Omega > 0 \right\}$$

be the Siegel upper half plane and let

$$\mathbf{H}_{n,m} = \mathbf{H}_n \times \mathbb{C}^{(m,n)}$$

be the Siegel-Jacobi space.

Notations : Here $F^{(m,n)}$ denotes the set of all $m \times n$ matrices with entries in a commutative ring F and ${}^tA$ denotes the transpose of a matrix A. For an $n \times m$ matrix B and an $n \times n$ matrix A, we write $A[B] = {}^tBAB$.

Let

$$Sp(n,\mathbb{R}) = \left\{ M \in \mathbb{R}^{(2n,2n)} \mid {}^tMJ_nM = J_n \right\}$$

be the symplectic group of degree n, where

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Then $Sp(n,\mathbb{R})$ acts on $\mathbf{H}_n$ transitively by

$$M \circ \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \qquad (1)$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $\Omega \in \mathbf{H}_n$. Therefore

$$Sp(n,\mathbb{R})/U(n) \cong \mathbf{H}_n$$

is a (Hermitian) symmetric space.

Let

$$H_{\mathbb{R}}^{(n,m)} = \left\{ (\lambda,\mu,\kappa) \,\middle|\, \lambda,\mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)} \right\}$$

be the Heisenberg group. Let

$$G^J = Sp(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

be the **Jacobi group** with the multiplication law

$$\begin{aligned} &(M_0, (\lambda_0,\mu_0,\kappa_0)) \cdot (M, (\lambda,\mu,\kappa)) \\ &= \Big(M_0 M, \big(\tilde{\lambda}_0 + \lambda, \tilde{\mu}_0 + \mu, \kappa_0 + \kappa + \tilde{\lambda}_0 {}^t\mu - \tilde{\mu}_0 {}^t\lambda \big) \Big), \end{aligned}$$

where $(\tilde{\lambda}_0, \tilde{\mu}_0) = (\lambda_0, \mu_0)M$. Then G^J acts on the **Siegel-Jacobi space** $\mathbf{H}_{n,m}$ transitively by

$$\Big(M, (\lambda, \mu, \kappa)\Big) \cdot (\Omega, Z) \qquad (2)$$
$$= \Big(M \circ \Omega, (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}\Big),$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}),\ (\lambda, \mu, \kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega, Z) \in \mathbf{H}_{n,m}$. Thus

$$G^J/K^J \cong \mathbf{H}_{n,m}$$

is a **non-reductive** complex manifold, where

$$K^J = U(n) \times \mathrm{Sym}(n, \mathbb{R}).$$

Let Γ_* be an arithmetic subgroup of $Sp(n, \mathbb{R})$ and $\Gamma_*^J = \Gamma_* \ltimes H_{\mathbb{Z}}^{(n,m)}$. For instance, $\Gamma_* = Sp(n, \mathbb{Z})$. Here

$$H_{\mathbb{Z}}^{(n,m)} = \Big\{ (\lambda, \mu, \kappa) \in H_{\mathbb{R}}^{(n,m)} \,\Big|\, \lambda, \mu, \kappa \text{ integral } \Big\}.$$

We have the following **natural problems** :

Problem 1 : Find the spectral decomposition of

$$L^2(\Gamma_*^J \backslash \mathbf{H}_{n,m})$$

for the Laplacian $\Delta_{n,m}$ on $\mathbf{H}_{n,m}$ or a commuting set $\mathbb{D}_*$ of G^J-invariant differential operators on $\mathbf{H}_{n,m}$.

Problem 2 : Decompose the regular representation $R_{\Gamma_*^J}$ of G^J on $L^2(\Gamma_*^J \backslash G^J)$ into irreducibles.

The above problems are very important arithmetically and geometrically. However the above problems are very **difficult** to solve at this moment. One of the reason is that it is difficult to deal with Γ_*. Unfortunately the unitary dual of $Sp(n,\mathbb{R})$ is not known yet for $n \geq 3$.

For a coordinate $(\Omega, Z) \in \mathbf{H}_{n,m}$ with $\Omega = (\omega_{\mu\nu}) \in \mathbb{H}_n$ and $Z = (z_{kl}) \in \mathbb{C}^{(m,n)}$, we put

$$\begin{aligned}
\Omega =& X + iY, \quad X = (x_{\mu\nu}), \quad Y = (y_{\mu\nu}) \text{ real}, \\
Z =& U + iV, \quad U = (u_{kl}), \quad V = (v_{kl}) \text{ real}, \\
d\Omega =& (d\omega_{\mu\nu}), \quad d\overline{\Omega} = (d\overline{\omega}_{\mu\nu}), \\
dZ =& (dz_{kl}), \quad d\overline{Z} = (d\overline{z}_{kl}),
\end{aligned}$$

$$\frac{\partial}{\partial \Omega} = \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial \omega_{\mu\nu}} \right), \quad \frac{\partial}{\partial \overline{\Omega}} = \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial \overline{\omega}_{\mu\nu}} \right),$$
$$\frac{\partial}{\partial X} = \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial x_{\mu\nu}} \right), \quad \frac{\partial}{\partial Y} = \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial y_{\mu\nu}} \right),$$

$$\frac{\partial}{\partial Z} = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \cdots & \frac{\partial}{\partial z_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_{1n}} & \cdots & \frac{\partial}{\partial z_{mn}} \end{pmatrix},$$

$$\frac{\partial}{\partial \overline{Z}} = \begin{pmatrix} \frac{\partial}{\partial \overline{z}_{11}} & \cdots & \frac{\partial}{\partial \overline{z}_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \overline{z}_{1n}} & \cdots & \frac{\partial}{\partial \overline{z}_{mn}} \end{pmatrix},$$

$$\frac{\partial}{\partial U} = \begin{pmatrix} \frac{\partial}{\partial u_{11}} & \cdots & \frac{\partial}{\partial u_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial u_{1n}} & \cdots & \frac{\partial}{\partial u_{mn}} \end{pmatrix},$$

$$\frac{\partial}{\partial V} = \begin{pmatrix} \frac{\partial}{\partial v_{11}} & \cdots & \frac{\partial}{\partial v_{m1}} \\ \vdots & \ddots & \cdots \\ \frac{\partial}{\partial v_{1n}} & \cdots & \frac{\partial}{\partial v_{mn}} \end{pmatrix}.$$

1. Invariant metrics on $\mathbf{H}_{n,m}$

We recall that for a positive real number A, the metric

$$\boxed{ds^2_{n;A} = A \cdot \mathrm{tr}\Big(Y^{-1} d\Omega Y^{-1} d\overline{\Omega}\Big)}$$

is a $Sp(n,\mathbb{R})$-invariant Kähler metric on $\mathbf{H}_n$ introduced by C. L. Siegel (cf. [A] or [8], 1943).

Theorem 1 (J.-H. Yang [16], 2005). For any two positive real numbers A and B, the following metric

$$
\begin{aligned}
& ds^2_{n,m;A,B} \\
= \;& A \cdot \mathrm{tr}\Big(Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega}\Big) \\
& + B \cdot \Big\{ \mathrm{tr}\Big(Y^{-1}\, {}^t V\, V\, Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega}\Big) \\
& \qquad + \mathrm{tr}\Big(Y^{-1}\, {}^t(dZ)\, d\overline{Z}\Big) \\
& \qquad - \mathrm{tr}\Big(V\, Y^{-1} d\Omega\, Y^{-1}\, {}^t(d\overline{Z})\Big) \\
& \qquad - \mathrm{tr}\Big(V\, Y^{-1} d\overline{\Omega}\, Y^{-1}\, {}^t(dZ)\Big) \Big\}
\end{aligned}
$$

is a Riemannian metric on $\mathbf{H}_{n,m}$ which is invariant under the action (2) of G^J.

For the case $n = m = A = B = 1$, we get

$$
\begin{aligned}
& ds^2_{1,1;1,1} \\
= & \ \frac{y + v^2}{y^3}\Big(dx^2 + dy^2 \Big) + \frac{1}{y}\Big(du^2 + dv^2 \Big) \\
& - \frac{2v}{y^2}\Big(dx\, du + dy\, dv \Big).
\end{aligned}
$$

Lemma A. The following differential form

$$dv_{n,m} = \frac{[dX] \wedge [dY] \wedge [dU] \wedge [dV]}{(\det Y)^{n+m+1}}$$

is a G^J-invariant volume element on $\mathbf{H}_{n,m}$, where

$$
\begin{aligned}
[dX] &= \wedge_{\mu \le \nu}\, dx_{\mu\nu}, \quad [dY] = \wedge_{\mu \le \nu}\, dy_{\mu\nu}, \\
[dU] &= \wedge_{k,l}\, du_{kl}, \quad [dV] = \wedge_{k,l}\, dv_{kl}.
\end{aligned}
$$

Proof. The proof follows from the fact that

$$(\det Y)^{-(n+1)}[dX] \wedge [dY]$$

is a $Sp(n, \mathbb{R})$-invariant volume element on $\mathbf{H}_n$. (cf. [9]) $\square$

2. Laplacians on $\mathbb{H}_{n,m}$

Hans Maass(cf. [3], 1953) proved that for a positive real number A, the differential operator

$$\boxed{\Delta_n = \frac{4}{A} \cdot \mathrm{tr}\left(Y\, {}^t\!\left(Y \frac{\partial}{\partial \overline{\Omega}}\right) \frac{\partial}{\partial \Omega} \right)}$$

is the Laplacian of $\mathbb{H}_n$ for the metric $ds^2_{n;A}$.

[3] H. Maass, *Die Differentialgleichungen in der Theorie der Siegelschen Modulfunktionen*, Math. Ann. **26** (1953), 44–68.

Theorem 2 (J.-H. Yang [16], 2005). For any two positive real numbers A and B, the Laplacian $\Delta_{n,m;A,B}$ of $ds^2_{n,m;A,B}$ is given by

$$\begin{aligned} &\Delta_{n,m;A,B} \\ = \;& \frac{4}{A}\left\{ \mathrm{tr}\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right)\right. \\ &\qquad + \mathrm{tr}\left(VY^{-1}\,{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\,\frac{\partial}{\partial Z}\right) \\ &\qquad + \mathrm{tr}\left(V\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial Z}\right) \\ &\qquad \left. + \mathrm{tr}\left({}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial\Omega}\right)\right\} \\ & + \frac{4}{B}\,\mathrm{tr}\left(Y\,\frac{\partial}{\partial Z}\,{}^t\!\left(\frac{\partial}{\partial\overline{Z}}\right)\right). \end{aligned}$$

For the case $n = m = A = B = 1$, we get

$$\begin{aligned}\Delta_{1,1;1,1} \;&=\; y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \\ &\quad + (\,y + v^2\,)\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) \\ &\quad + 2\,y\,v\left(\frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v}\right).\end{aligned}$$

Remark : $ds^2_{n,m;A,B}$ and $\Delta_{n,m;A,B}$ are expressed in terms of the **trace form**. !!!

3. Invariant differential operators on $\mathbf{H}_{n,m}$

Let $\mathbb{D}(\mathbf{H}_n)$ be the algebra of all $Sp(n,\mathbb{R})$-invariant differential operators on $\mathbf{H}_n$. For brevity, we set $K = U(n)$. Then K acts on the vector space

$$T_n = \left\{ \omega \in \mathbb{C}^{(n,n)} \mid \omega = {}^t\omega \right\}$$

by

$$\boxed{k \cdot \omega = k\,\omega\,{}^t k, \quad h \in K,\ \omega \in T_n.} \qquad (3)$$

The action (3) induces naturally the representation τ_K of K on the polynomial algebra $\text{Pol}(T_n)$ of T_n. Let

$$\text{Pol}(T_n)^K = \Big\{ p \in \text{Pol}(T_n) \,\Big|\, k \cdot p = p,\ \forall\, k \in K \Big\}$$

be the subalgebra of $\text{Pol}(T_n)$ consisting of all K-invariant polynomials on T_n. Then we get a canonical linear bijection (not an algebra isomorphism)

$$\mathfrak{S}_n : \text{Pol}(T_n)^K \longrightarrow \mathbb{D}(\mathbf{H}_n). \qquad (4)$$

Theorem 3. $\text{Pol}(T_n)^K$ is generated by algebraically independent polynomials

$$q_i(\omega) = \text{tr}\Big((\omega\overline{\omega})^i \Big), \quad i = 1, 2, \cdots, n.$$

Proof. The proof follows from the classical invariant theory or the work of Harish-Chandra (1923-1983). $\square$

Remark. Let $D_i = \mathfrak{S}_n(q_i),\ 1 \leq i \leq n.$ According to the work of Harish-Chandra,

$$\mathbb{D}(\mathbf{H}_n) \cong \mathbb{C}[D_1, \cdots, D_n]$$

is a polynomial ring of degree n, where n is the split real rank of $Sp(n, \mathbb{R})$.

Remark. $\mathfrak{S}_n(q_1) = \Delta_{n;1}$ is the Laplacian of $ds^2_{n;1}$ on $\mathbf{H}_n$. So far $\mathfrak{S}_n(q_i)\,(i = 2, \cdots, n)$ were not written explicitly.

Remark. Maass [3] found explicit algebraically independent generators $H_1, H_2, \cdots, H_n$ of $\mathbb{D}(\mathbb{H}_n)$. We will describe $H_1, H_2, \cdots, H_n$ explicitly. For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $\Omega = X + iY \in \mathbb{H}_n$ with real X, Y, we set

$$\Omega_* = M \cdot \Omega = X_* + iY_* \quad \text{with } X_*, Y_* \text{ real.}$$

We set

$$\begin{aligned} K &= \left(\Omega - \overline{\Omega}\right)\frac{\partial}{\partial \Omega} = 2\,i\,Y\frac{\partial}{\partial \Omega}, \\ \Lambda &= \left(\Omega - \overline{\Omega}\right)\frac{\partial}{\partial \overline{\Omega}} = 2\,i\,Y\frac{\partial}{\partial \overline{\Omega}}, \\ K_* &= \left(\Omega_* - \overline{\Omega}_*\right)\frac{\partial}{\partial \Omega_*} = 2\,i\,Y_*\frac{\partial}{\partial \Omega_*}, \\ \Lambda_* &= \left(\Omega_* - \overline{\Omega}_*\right)\frac{\partial}{\partial \overline{\Omega}_*} = 2\,i\,Y_*\frac{\partial}{\partial \overline{\Omega}_*}. \end{aligned}$$

Then it is easily seen that

$$K_* = {}^t(C\overline{\Omega} + D)^{-1}\,{}^t\Big\{(C\Omega + D)\,{}^tK\Big\}, \qquad (5)$$

$$\Lambda_* = {}^t(C\Omega + D)^{-1}\,{}^t\Big\{(C\overline{\Omega} + D)\,{}^t\Lambda\Big\} \qquad (6)$$

and

$${}^t\Big\{(C\overline{\Omega} + D)\,{}^t\Lambda\Big\} = \Lambda\,{}^t(C\overline{\Omega}+D) - \frac{n+1}{2}\left(\Omega - \overline{\Omega}\right){}^tC. \qquad (7)$$

Using Formulas (5), (6) and (7), we can show that

$$
\begin{aligned}
& \Lambda_* K_* + \frac{n+1}{2} K_* \\
= & \; {}^t(C\Omega + D)^{-1} \, {}^t\!\left\{ (C\Omega + D) \, {}^t\!\left(\Lambda K + \frac{n+1}{2} K \right) \right\}.
\end{aligned}
$$

Therefore we get

$$
\mathrm{tr}\left(\Lambda_* K_* + \frac{n+1}{2} K_* \right) = \mathrm{tr}\left(\Lambda K + \frac{n+1}{2} K \right).
$$

We set

$$
A^{(1)} = \Lambda K + \frac{n+1}{2} K.
$$

We define $A^{(j)}\,(j = 2, 3, \cdots, n)$ recursively by

$$
\begin{aligned}
A^{(j)} = \; & A^{(1)} A^{(j-1)} - \frac{n+1}{2} \Lambda \, A^{(j-1)} \\
& + \frac{1}{2} \Lambda \, \sigma\left(A^{(j-1)} \right) \qquad (8) \\
& + \frac{1}{2} \left(\Omega - \overline{\Omega} \right) {}^t\!\left\{ \left(\Omega - \overline{\Omega} \right)^{-1} \, {}^t\!\left({}^t\Lambda \, {}^tA^{(j-1)} \right) \right\}.
\end{aligned}
$$

We set

$$H_j = \mathrm{tr}\Big(A^{(j)}\Big), \quad j = 1, 2, \cdots, n. \qquad (9)$$

As mentioned before, Maass proved that $H_1, H_2, \cdots, H_n$ are algebraically independent generators $H_1, H_2, \cdots, H_n$ of $\mathbb{D}(\mathbb{H}_n)$.

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Let $T_{n,m} = T_n \times \mathbb{C}^{(m,n)}$. Then K acts on $T_{n,m}$ by

$$\boxed{h \cdot (\omega, z) = \Big(h\, \omega\, {}^t h,\ z\, {}^t h\Big),} \qquad (10)$$

where $h \in K,\ \omega \in T_n,\ z \in \mathbb{C}^{(m,n)}$. Then this action induces naturally the action ρ of K on the polynomial algebra

$$\mathrm{Pol}_{m,n} = \mathrm{Pol}\,(T_{n,m}).$$

We denote by $\mathrm{Pol}_{m,n}^K$ the subalgebra of $\mathrm{Pol}_{m,n}$ consisting of all K-invariants of the action ρ of K. We also denote by

$$\mathbb{D}(\mathbf{H}_{n,m})$$

the algebra of all differential operators on $\mathbf{H}_{n,m}$ which is invariant under the action (2) of the Jacobi group G^J. Then we can show that there exists a natural linear bijection

$$\mathfrak{S}_{n,m} : \mathrm{Pol}_{m,n}^K \longrightarrow \mathbb{D}(\mathbf{H}_{n,m})$$

of $\mathrm{Pol}_{m,n}^K$ onto $\mathbb{D}(\mathbf{H}_{n,m})$.

The map $\mathfrak{S}_{n,m}$ is described explicitly as follows.

We put $N_\star = n(n+1) + 2mn$. Let $\Big\{ \eta_\alpha \,|\; 1 \leq \alpha \leq N_\star \Big\}$ be a basis of $T_{n,m}$. If $P \in \mathrm{Pol}_{m,n}^K$, then

$$\Big(\mathfrak{S}_{n,m}(P) f\Big)(gK)$$
$$= \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g \exp\left(\sum_{\alpha=1}^{N_\star} t_\alpha \eta_\alpha \right) K \right) \right]_{(t_\alpha)=0},$$

where $f \in C^\infty(\mathbb{H}_{n,m})$. In general, it is hard to express $\mathfrak{S}_{n,m}(P)$ explicitly for a polynomial $P \in \mathrm{Pol}_{m,n}^K$.

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We present the following **basic** K-invariant polynomials in $\mathrm{Pol}_{m,n}^K$.

$$\begin{aligned}
p_j(\omega,z) &= \mathrm{tr}((\omega\overline{\omega})^j), \quad 1 \leq j \leq n,\\
\psi_k^{(1)}(\omega,z) &= (z\,{}^t\overline{z})_{kk}, \quad 1 \leq k \leq m,\\
\psi_{kp}^{(2)}(\omega,z) &= \mathrm{Re}\,(z\,{}^t\overline{z})_{kp}, \quad 1 \leq k < p \leq m,\\
\psi_{kp}^{(3)}(\omega,z) &= \mathrm{Im}\,(z\,{}^t\overline{z})_{kp}, \quad 1 \leq k < p \leq m,\\
f_{kp}^{(1)}(\omega,z) &= \mathrm{Re}\,(z\overline{\omega}\,{}^t z)_{kp}, \quad 1 \leq k \leq p \leq m,\\
f_{kp}^{(2)}(\omega,z) &= \mathrm{Im}\,(z\overline{\omega}\,{}^t z\,)_{kp}, \quad 1 \leq k \leq p \leq m,
\end{aligned}$$

where $\omega \in T_n$ and $z \in \mathbb{C}^{(m,n)}$.

For an $m \times m$ matrix S, we define the following invariant polynomials in $\mathrm{Pol}_{m,n}^K$.

$$m_{j;S}^{(1)}(\omega, z) = \mathrm{Re}\left(\mathrm{tr}\left(\omega\overline{\omega} + {}^t z S \overline{z}\right)^j\right),$$

$$m_{j;S}^{(2)}(\omega, z) = \mathrm{Im}\left(\mathrm{tr}\left(\omega\overline{\omega} + {}^t z S \overline{z}\right)^j\right),$$

$$q_{k;S}^{(1)}(\omega, z) = \mathrm{Re}\left(\mathrm{tr}\left(({}^t z\, S\, \overline{z})^k\right)\right),$$

$$q_{k;S}^{(2)}(\omega, z) = \mathrm{Im}\left(\mathrm{tr}\left(({}^t z\, S\, \overline{z})^k\right)\right),$$

$$\theta_{i,k,j;S}^{(1)}(\omega, z) = \mathrm{Re}\left(\mathrm{tr}\left((\omega\overline{\omega})^i\, ({}^t z\, S\, \overline{z})^k\, (\omega\overline{\omega} + {}^t z\, S\, \overline{z}\,)^j\right)\right),$$

$$\theta_{i,k,j;S}^{(2)}(\omega, z) = \mathrm{Im}\left(\mathrm{tr}\left((\omega\overline{\omega})^i\, ({}^t z\, S\, \overline{z})^k\, (\omega\overline{\omega} + {}^t z\, S\, \overline{z}\,)^j\right)\right),$$

where $1 \leq i, j \leq n$ and $1 \leq k \leq m$.

We define the following K-invariant polynomials in $\mathrm{Pol}_{m,n}^K$.

$$r_{jk}^{(1)}(\omega, z) = \mathrm{Re}\Big(\mathrm{tr}\Big((\omega\overline{\omega})^j\,({}^t z\overline{z})^k\Big)\Big),$$
$$r_{jk}^{(2)}(\omega, z) = \mathrm{Im}\Big(\mathrm{tr}\Big((\omega\overline{\omega})^j\,({}^t z\overline{z})^k\Big)\Big),$$

where $1 \le j \le n$ and $1 \le k \le m$.

There may be possible other new invariants. We think that at this moment it may be complicated and difficult to find the generators of $\mathrm{Pol}_{m,n}^K$.

We propose the following problems.

Problem A. Find the generators of $\mathrm{Pol}_{m,n}^K$.

Problem B. Find an easy way to express the images of the above invariant polynomials under the map $\mathfrak{S}_{n,m}$ explicitly.

Theorem 4. The algebra $\mathbb{D}(\mathbf{H}_1 \times \mathbb{C})$ is generated by the following differential operators

$$D = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + 2\, y\, v \left(\frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v} \right),$$

$$\Psi = y \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right),$$

$$D_1 = 2y^2 \frac{\partial^3}{\partial x \partial u \partial v} - y^2 \frac{\partial}{\partial y} \left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) + \left(v \frac{\partial}{\partial v} + 1 \right) \Psi$$

and

$$D_2 = y^2 \frac{\partial}{\partial x} \left(\frac{\partial^2}{\partial v^2} - \frac{\partial^2}{\partial u^2} \right) - 2\, y^2 \frac{\partial^3}{\partial y \partial u \partial v} - v \frac{\partial}{\partial u} \Psi,$$

where $\tau = x + iy$ and $z = u + iv$ with real variables x, y, u, v. Moreover, we have

$$D\Psi - \Psi D = 2\, y^2 \frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) - 4\, y^2 \frac{\partial^3}{\partial x \partial u \partial v} - 2\left(v\, \frac{\partial}{\partial v}\Psi + \Psi \right).$$

Remark. We observe that $\Delta_{n,m;A,B} \in \mathbb{D}(\mathbf{H}_{n,m})$. We can show that

$$D = \mathrm{tr}\left(Y\, \frac{\partial}{\partial Z} {}^t\!\left(\frac{\partial}{\partial \overline{Z}}\right) \right)$$

is an element of $\mathbb{D}(\mathbf{H}_{n,m})$. Therefore

$$\Delta_{n,m;A,B} - \frac{4}{B}\, D \in \mathbb{D}(\mathbf{H}_{n,m}).$$

The following differential operator $\mathbb{K}$ on $\mathbb{H}_{n,m}$ of degree $2n$ defined by

$$\mathbb{K} = \det(Y)\, \det\left(\frac{\partial}{\partial Z} {}^t\!\left(\frac{\partial}{\partial \overline{Z}}\right) \right)$$

is invariant under the action (2) of G^J.

The following matrix-valued differential operator $\mathbb{T}$ on $\mathbb{H}_{n,m}$ defined by

$$\mathbb{T} = {}^t\Big(\frac{\partial}{\partial \overline{Z}}\Big)\,Y\frac{\partial}{\partial Z}$$

is invariant under the action (2) of G^J. Therefore each (k,l)-entry $\mathbb{T}_{kl}$ of $\mathbb{T}$ given by

$$\mathbb{T}_{kl} = \sum_{i,j=1}^{n} y_{ij}\,\frac{\partial^2}{\partial \overline{z}_{ki}\partial z_{lj}}, \qquad 1 \leq k, l \leq m$$

is an element of $\mathbb{D}\Big(\mathbb{H}_{n,m}\Big)$.

Indeed it is very complicated and difficult at this moment to express the generators of the algebra of all G^J-invariant differential operators on $\mathbb{D}_{n,m}$ explicitly. In particular, it is extremely difficult to find explicit G^J-invariant differential operators on $\mathbb{H}_{n,m}$ of *odd* degree. We propose an open problem to find other explicit G^J-invariant differential operators on $\mathbb{H}_{n,m}$.

4. Partial Cayley transform

Let

$$\mathbf{D}_n = \left\{ W \in \mathbb{C}^{(n,n)} \mid W = {}^tW,\ I_n - W\overline{W} > 0 \right\}$$

be the generalized unit disk of degree n. We let

$$\mathbf{D}_{n,m} = \mathbf{D}_n \times \mathbb{C}^{(m,n)}$$

be the Siegel-Jacobi disk.

We define the **partial Cayley transform**

$$\Phi_* : \mathbf{D}_{n,m} \longrightarrow \mathbf{H}_{n,m}$$

by

$$\Phi_*(W,\eta) = \tag{11}$$

$$\boxed{\Big(i(I_n + W)(I_n - W)^{-1},\, 2\, i\, \eta\, (I_n - W)^{-1} \Big),}$$

where $W \in \mathbf{D}_n$ and $\eta \in \mathbb{C}^{(m,n)}$. It is easy to see that Φ_* is a biholomorphic mapping.

We set

$$T_* = \frac{1}{\sqrt{2}} \begin{pmatrix} I_{m+n} & I_{m+n} \\ iI_{m+n} & -iI_{m+n} \end{pmatrix}.$$

We now consider the group G_*^J defined by

$$G_*^J = T_*^{-1} G^J T_*.$$

Then G_*^J acts on $\mathbf{D}_{n,m}$ transitively by

$$\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\lambda, \mu, \kappa) \right) \cdot (W, \eta) = \tag{12}$$

$$\Big((PW+Q)(\overline{Q}W+\overline{P})^{-1}, (\eta+\lambda W+\mu)(\overline{Q}W+\overline{P})^{-1} \Big).$$

Theorem 5 (J.-H. Yang [17], 2005). The action (2) of G^J on $\mathbf{H}_{n,m}$ is compatible with the action (12) of G_*^J on $\mathbf{D}_{n,m}$ through the partial Cayley transform Φ_*. More precisely, if $g_0 \in G^J$ and $(W, \eta) \in \mathbf{D}_{n,m}$,

$$g_0 \cdot \Phi_*(W, \eta) = \Phi_* \Big(g_* \cdot (W, \eta) \Big),$$

where $g_* = T_*^{-1} g_0 T_*$.

5. Invariant Differential Operators on $\mathbb{D}_{n,m}$

For a coordinate $(W,\eta) \in \mathbb{D}_{n,m}$ with $W = (w_{\mu\nu}) \in \mathbb{D}_n$ and $\eta = (\eta_{kl}) \in \mathbb{C}^{(m,n)}$, we put

$$\begin{aligned} dW &= (dw_{\mu\nu}), \qquad d\overline{W} = (d\overline{w}_{\mu\nu}), \\ d\eta &= (d\eta_{kl}), \qquad d\overline{\eta} = (d\overline{\eta}_{kl}), \end{aligned}$$

$$\frac{\partial}{\partial W} = \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial w_{\mu\nu}} \right),$$

$$\frac{\partial}{\partial \overline{W}} = \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial \overline{w}_{\mu\nu}} \right),$$

$$\frac{\partial}{\partial \eta} = \begin{pmatrix} \frac{\partial}{\partial \eta_{11}} & \cdots & \frac{\partial}{\partial \eta_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \eta_{1n}} & \cdots & \frac{\partial}{\partial \eta_{mn}} \end{pmatrix},$$

$$\frac{\partial}{\partial \overline{\eta}} = \left(\frac{\partial}{\partial \overline{\eta}_{kl}} \right).$$

Theorem 6 (J.-H. Yang [18], 2005). The following metric $d\tilde{s}^2_{n,m;A,B}$ defined by

$$
\begin{aligned}
& \frac{1}{4}\, d\tilde{s}^2_{n,m;A,B} = \\
& A\, \mathrm{tr}\Big((I_n - W\overline{W})^{-1} dW\, (I_n - \overline{W}W)^{-1} d\overline{W} \Big) \\
+\ & B \Big\{ \mathrm{tr}\Big((I_n - W\overline{W})^{-1}\, {}^t(d\eta)\, d\overline{\eta}\, \Big) \\
+\ & \mathrm{tr}\Big((\eta\overline{W} - \overline{\eta})(I_n - W\overline{W})^{-1} dW \\
& \qquad (I_n - \overline{W}W)^{-1}\, {}^t(d\overline{\eta}) \Big) \\
+\ & \mathrm{tr}\Big((\overline{\eta}W - \eta)(I_n - \overline{W}W)^{-1} d\overline{W} \\
& \qquad (I_n - W\overline{W})^{-1}\, {}^t(d\eta)\, \Big) \\
-\ & \mathrm{tr}\Big((I_n - W\overline{W})^{-1}\, {}^t\eta\, \eta\, (I_n - \overline{W}W)^{-1} \\
& \qquad \overline{W} dW\, (I_n - \overline{W}W)^{-1} d\overline{W}\, \Big) \\
-\ & \mathrm{tr}\Big(W\, (I_n - \overline{W}W)^{-1}\, {}^t\overline{\eta}\, \overline{\eta}\, (I_n - W\overline{W})^{-1} \\
& \qquad dW\, (I_n - \overline{W}W)^{-1} d\overline{W}\, \Big) \\
+\ & \mathrm{tr}\Big((I_n - W\overline{W})^{-1}\, {}^t\eta\, \overline{\eta}\, (I_n - W\overline{W})^{-1} \\
& \qquad dW\, (I_n - \overline{W}W)^{-1} d\overline{W}\, \Big) \\
+\ & \mathrm{tr}\Big((I_n - \overline{W})^{-1}\, {}^t\overline{\eta}\, \eta\, \overline{W}\, (I_n - W\overline{W})^{-1} \\
& \qquad dW\, (I_n - \overline{W}W)^{-1} d\overline{W}\, \Big) \\
+\ & \mathrm{tr}\Big((I_n - \overline{W})^{-1} (I_n - W)(I_n - \overline{W}W)^{-1} \\
& \qquad {}^t\overline{\eta}\, \eta\, (I_n - \overline{W}W)^{-1}\, (I_n - \overline{W})(I_n - W)^{-1} \\
& \qquad dW\, (I_n - \overline{W}W)^{-1} d\overline{W}\, \Big) \Big\}
\end{aligned}
$$

$$- \quad B\,\mathrm{tr}\Big((I_n - W\overline{W})^{-1}(I_n - W)(I_n - \overline{W})^{-1} \\ {}^t\overline{\eta}\,\eta\,(I_n - W)^{-1}\,dW(I_n - \overline{W}W)^{-1}d\overline{W}\,\Big)$$

is a Riemannian metric on $\mathbf{D}_{n,m}$ which is invariant under the action (12) of G^J_*.

If $n = m = A = B = 1$, then $d\tilde{s}^2 = d\tilde{s}^2_{1,1;1,1}$ is given by

$$\begin{aligned} \frac{1}{4}\,d\tilde{s}^2 \;&=\; \frac{dW\,d\overline{W}}{(1-|W|^2)^2} + \frac{1}{(1-|W|^2)}\,d\eta\,d\overline{\eta} \\ &+\; \frac{(1+|W|^2)|\eta|^2 - \overline{W}\eta^2 - W\overline{\eta}^2}{(1-|W|^2)^3}\,dW\,d\overline{W} \\ &+\; \frac{\eta\overline{W} - \overline{\eta}}{(1-|W|^2)^2}\,dW\,d\overline{\eta} \\ &+\; \frac{\overline{\eta}W - \eta}{(1-|W|^2)^2}\,d\overline{W}\,d\eta. \end{aligned}$$

Theorem 7 (J.-H. Yang [18], 2005). The Laplacian $\tilde{\Delta} = \tilde{\Delta}_{n,m;A,B}$ of the above metric $d\tilde{s}^2_{n,m;A,B}$ is given by

$$
\begin{aligned}
\tilde{\Delta} \; = \; & A\Bigg\{\mathrm{tr}\left[(I_n - W\overline{W})^{t}\Big((I_n - W\overline{W})\frac{\partial}{\partial\overline{W}}\Big)\frac{\partial}{\partial W}\right] \\
+ \; & \mathrm{tr}\left[{}^{t}(\eta - \overline{\eta}\,W)^{t}\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial W}\right] \\
+ \; & \mathrm{tr}\left[(\overline{\eta} - \eta\,\overline{W})^{t}\Big((I_n - W\overline{W})\frac{\partial}{\partial\overline{W}}\Big)\frac{\partial}{\partial\eta}\right] \\
- \; & \mathrm{tr}\left[\eta\overline{W}(I_n - W\overline{W})^{-1\,t}\eta^{t}\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}\right] \\
- \; & \mathrm{tr}\left[\overline{\eta}W(I_n - \overline{W}W)^{-1\,t}\overline{\eta}^{t}\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}\right] \\
+ \; & \mathrm{tr}\left[\overline{\eta}(I_n - W\overline{W})^{-1\,t}\eta^{t}\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}\right] \\
+ \; & \mathrm{tr}\left[\eta\overline{W}W(I_n - \overline{W}W)^{-1\,t}\overline{\eta}^{t}\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}\right]\Bigg\} \\
+ \; & B\cdot\mathrm{tr}\left[(I_n - \overline{W}W)\frac{\partial}{\partial\eta}{}^{t}\left(\frac{\partial}{\partial\overline{\eta}}\right)\right].
\end{aligned}
$$

If $n = m = A = B = 1$, we get

$$\begin{aligned} \tilde{\Delta}_{1,1;1,1} \;=\; & (1-|W|^2)^2 \frac{\partial^2}{\partial W \partial \overline{W}} \\ & + (1-|W|^2)\, \frac{\partial^2}{\partial \eta \partial \overline{\eta}} \\ & + (1-|W|^2)(\eta - \overline{\eta}\, W)\, \frac{\partial^2}{\partial W \partial \overline{\eta}} \\ & + (1-|W|^2)(\overline{\eta} - \eta\, \overline{W})\, \frac{\partial^2}{\partial \overline{W} \partial \eta} \\ & -(\overline{W}\, \eta^2 + W \overline{\eta}^2)\, \frac{\partial^2}{\partial \eta \partial \overline{\eta}} \\ & + (1+|W|^2)|\eta|^2\, \frac{\partial^2}{\partial \eta \partial \overline{\eta}}. \end{aligned}$$

The main ingredients for the proof of Theorem 6 and Theorem 7 are the partial Cayley transform (Theorem 5), Theorem 1 and Theorem 2.

Let $\mathbb{D}(\mathbf{D}_{n,m})$ be the algebra of all differential operators $\mathbf{D}_{n,m}$ invariant under the action (12) of G^J_*. By Theorem 5, we have the algebra isomorphism

$$\mathbb{D}(\mathbf{D}_{n,m}) \cong \mathbb{D}(\mathbf{H}_{n,m}).$$

6. A fundamental domain for $\Gamma_{n,m}\backslash\mathbf{H}_{n,m}$

Before we describe a fundamental domain for the Siegel-Jacobi space, we review the Siegel's fundamental domain for the Siegel upper half plane.

We let

$$\mathcal{P}_n = \left\{ Y \in \mathbb{R}^{(n,n)} \mid Y = {}^tY > 0 \right\}$$

be an open cone in $\mathbb{R}^{n(n+1)/2}$. The general linear group $GL(n,\mathbb{R})$ acts on $\mathcal{P}_n$ transitively by

$$h \circ Y = h\,Y\,{}^th, \quad h \in GL(n,\mathbb{R}),\ Y \in \mathcal{P}_n.$$

Thus $\mathcal{P}_n$ is a symmetric space diffeomorphic to $GL(n,\mathbb{R})/O(n)$. We let

$$GL(n,\mathbb{Z}) = \left\{ h \in GL(n,\mathbb{R}) \,\middle|\, h \text{ is integral } \right\}$$

be the discrete subgroup of $GL(n,\mathbb{R})$.

The fundamental domain $\mathcal{R}_n$ for $GL(n,\mathbb{Z})\backslash\mathcal{P}_n$ which was found by H. Minkowski [5] is defined as a subset of $\mathcal{P}_n$ consisting of $Y = (y_{ij}) \in \mathcal{P}_n$ satisfying the following conditions (M.1)-(M.2) (cf. [4] p. 123):

(M.1) $\quad aY\,{}^t a \geq y_{kk}$ for every $a = (a_i) \in \mathbb{Z}^n$ in which $a_k, \cdots, a_n$ are relatively prime for $k = 1, 2, \cdots, n$.

(M.2) $\quad y_{k,k+1} \geq 0$ for $k = 1, \cdots, n-1$.

We say that a point of $\mathcal{R}_n$ is *Minkowski reduced* or simply *M-reduced*.

Siegel [8] determined a fundamental domain $\mathcal{F}_n$ for $\Gamma_n \backslash \mathbf{H}_n$, where $\Gamma_n = Sp(n, \mathbb{Z})$ is the Siegel modular group of degree n. We say that $\Omega = X + iY \in \mathbf{H}_n$ with X, Y real is *Siegel reduced* or *S-reduced* if it has the following three properties:

(S.1) $\det(\mathrm{Im}\,(\gamma\cdot\Omega)) \leq \det(\mathrm{Im}\,(\Omega)) \quad$ for all $\gamma \in \Gamma_n$;

(S.2) $Y = \mathrm{Im}\,\Omega$ is M-reduced, that is, $Y \in \mathcal{R}_n$;

(S.3) $|x_{ij}| \leq \frac{1}{2} \quad$ for $1 \leq i, j \leq n$, where $X = (x_{ij})$.

$\mathcal{F}_n$ is defined as the set of all Siegel reduced points in $\mathbf{H}_n$. Using the highest point method, Siegel [8] proved the following (F1)-(F3) (cf. [4], p. 169):

(F1) $\Gamma_n \cdot \mathcal{F}_n = \mathbf{H}_n$, i.e., $\mathbf{H}_n = \cup_{\gamma \in \Gamma_n} \gamma \cdot \mathcal{F}_n$.

(F2) $\mathcal{F}_n$ is closed in $\mathbf{H}_n$.

(F3) $\mathcal{F}_n$ is connected and the boundary of $\mathcal{F}_n$ consists of a finite number of hyperplanes.

The metric $ds^2_{n;1}$ induces a metric $ds^2_{\mathcal{F}_n}$ on $\mathcal{F}_n$. Siegel [8] computed the volume of $\mathcal{F}_n$

$$\mathrm{vol}\,(\mathcal{F}_n) = 2 \prod_{k=1}^{n} \pi^{-k} \Gamma(k) \zeta(2k),$$

where $\Gamma(s)$ denotes the Gamma function and $\zeta(s)$ denotes the Riemann zeta function. For instance,

$$\mathrm{vol}\,(\mathcal{F}_1) = \frac{\pi}{3}, \quad \mathrm{vol}\,(\mathcal{F}_2) = \frac{\pi^3}{270},$$
$$\mathrm{vol}\,(\mathcal{F}_3) = \frac{\pi^6}{127575}, \quad \mathrm{vol}\,(\mathcal{F}_4) = \frac{\pi^{10}}{200930625}.$$

Let $f_{kl}\,(1 \leq k \leq m,\ 1 \leq l \leq n)$ be the $m \times n$ matrix with entry 1 where the k-th row and the l-th column meet, and all other entries 0. For an element $\Omega \in \mathbb{H}_n$, we set for brevity

$$h_{kl}(\Omega) = f_{kl}\Omega, \quad 1 \leq k \leq m,\ 1 \leq l \leq n.$$

For each $\Omega \in \mathcal{F}_n$, we define a subset P_Ω of $\mathbb{C}^{(m,n)}$ by

$$P_\Omega = \left\{ \sum_{k=1}^{m} \sum_{j=1}^{n} \lambda_{kl} f_{kl} + \textstyle\sum_{k=1}^{m} \sum_{j=1}^{n} \mu_{kl} h_{kl}(\Omega) \,\right| \\ \left. 0 \leq \lambda_{kl}, \mu_{kl} \leq 1 \right\}.$$

For each $\Omega \in \mathcal{F}_n$, we define the subset D_Ω of $\mathbf{H}_{n,m}$ by

$$D_\Omega = \left\{ (\Omega, Z) \in \mathbf{H}_{n,m} \,|\ Z \in P_\Omega \right\}.$$

We define

$$\mathcal{F}_{n,m} = \cup_{\Omega \in \mathcal{F}_n} D_\Omega.$$

Theorem 8 (J.-H. Yang [19], 2005). Let

$$\Gamma_{n,m} = \Gamma_n \ltimes H_{\mathbb{Z}}^{(n,m)}$$

be the discrete subgroup of G^J. Then $\mathcal{F}_{n,m}$ is a fundamental domain for $\Gamma_{n,m} \backslash \mathbf{H}_{n,m}$.

Proof. The proof can be found in [19]. $\square$

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7. Maass-Jacobi forms

Definition. For brevity, we set $\Delta_{n,m} = \Delta_{n,m;1,1}$ (cf. Theorem 2). Let

$$\Gamma_{n,m} = \Gamma_n \ltimes H_{\mathbb{Z}}^{(n,m)}$$

be the discrete subgroup of G^J. A smooth function $f : \mathbf{H}_{n,m} \longrightarrow \mathbb{C}$ is called a **Maass-Jacobi form** on $\mathbf{H}_{n,m}$ if f satisfies the following conditions (MJ1)-(MJ3) :

(MJ1) f is invariant under $\Gamma_{n,m}$.

(MJ2) f is an eigenfunction of $\Delta_{n,m}$.

(MJ3) f has a polynomial growth, that is, there exist a constant $C > 0$ and a positive integer N such that

$$|f(X + iY, Z)| \leq C\,|p(Y)|^N \quad \text{as } \det Y \longrightarrow \infty,$$

where $p(Y)$ is a polynomial in $Y = (y_{ij})$. (cf. See Section 6)

It is natural to propose the following problems.

Problem C. Construct Maass-Jacobi forms.

Problem D. Find all the eigenfunctions of $\Delta_{n,m}$.

We consider the simple case $n = m = A = B = 1$. A metric $ds^2_{1,1} = ds^2_{1,1;1,1}$ on $\mathbf{H}_1 \times \mathbb{C}$ given by

$$ds^2_{1,1} = \frac{y + v^2}{y^3}(\,dx^2 + dy^2\,) + \frac{1}{y}(\,du^2 + dv^2\,) - \frac{2v}{y^2}(\,dx\,du + dy\,dv\,)$$

is a G^J-invariant Kähler metric on $\mathbf{H}_1 \times \mathbb{C}$. Its Laplacian $\Delta_{1,1}$ is given by

$$\begin{aligned}\Delta_{1,1} = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) &+ (\,y + v^2\,)\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) \\ &+ 2\,y\,v\left(\frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v}\right).\end{aligned}$$

We provide some examples of eigenfunctions of $\Delta_{1,1}$.

(1) $h(x,y) = y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|a|y)\, e^{2\pi i a x}$ ($s \in \mathbb{C}$, $a \neq 0$) with eigenvalue $s(s-1)$. Here

$$K_s(z) := \frac{1}{2}\int_0^\infty \exp\left\{-\frac{z}{2}(t+t^{-1})\right\} t^{s-1}\, dt,$$

where $\mathrm{Re}\, z > 0$.

(2) $y^s,\ y^s x,\ y^s u$ ($s \in \mathbb{C}$) with eigenvalue $s(s-1)$.

(3) $y^s v,\ y^s uv,\ y^s xv$ with eigenvalue $s(s+1)$.

(4) x, y, u, v, xv, uv with eigenvalue 0.

(5) All Maass wave forms.

7.1. Eisenstein Series

Let

$$\Gamma_{1,1}^{\infty} = \left\{ \left(\begin{pmatrix} \pm 1 & m \\ 0 & \pm 1 \end{pmatrix}, (0, n, \kappa) \right) \,\middle|\, m, n, \kappa \in \mathbb{Z} \right\}$$

be the subgroup of $\Gamma_{1,1} = SL_2(\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(1,1)}$.

For $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu, \kappa) \right) \in \Gamma_{1,1}$, we put $(\tau_\gamma, z_\gamma) = \gamma \cdot (\tau, z)$. That is,

$$\tau_\gamma = (a\tau + b)(c\tau + d)^{-1},$$
$$z_\gamma = (z + \lambda\tau + \nu)(c\tau + d)^{-1}.$$

We note that if $\gamma \in \Gamma_{1,1}$,

$$\operatorname{Im} \tau_\gamma = \operatorname{Im} \tau, \quad \operatorname{Im} z_\gamma = \operatorname{Im} z$$

if and only if $\gamma \in \Gamma_{1,1}^\infty$. For $s \in \mathbb{C}$, we define an Eisenstein series formally by

$$E_s(\tau, z) = \sum_{\gamma \in \Gamma_{1,1}^\infty \backslash \Gamma_{1,1}} (\operatorname{Im} \tau_\gamma)^s \cdot \operatorname{Im} z_\gamma.$$

Then $E_s(\tau, z)$ satisfies formally

$$E_s(\gamma \cdot (\tau, z)) = E_s(\tau, z), \quad \gamma \in \Gamma_{1,1}$$

and

$$\Delta E_s(\tau, z) = s(s+1) E_s(\tau, z).$$

7.2. Fourier Expansion of Maass-Jacobi Form

We let $f : \mathbf{H}_1 \times \mathbb{C} \longrightarrow \mathbb{C}$ be a Maass-Jacobi form with $\Delta f = \lambda f$. Then f satisfies the following invariance relations

$$f(\tau + n, z) = f(\tau, z) \quad \text{for all } n \in \mathbb{Z}$$

and

$$f(\tau, z + n_1\tau + n_2) = f(\tau, z)$$

for all $n_1, n_2 \in \mathbb{Z}$. Therefore f is a smooth function on $\mathbf{H}_1 \times \mathbb{C}$ which is periodic in x and u with period 1. So f has the following Fourier series

$$f(\tau, z) = \sum_{n\in\mathbb{Z}} \sum_{r\in\mathbb{Z}} c_{n,r}(y, v)\, e^{2\pi i(nx+ru)}.$$

For two fixed integers n and r, we have to calculate the function $c_{n,r}(y, v)$. For brevity, we put $F(y,v) = c_{n,r}(y,v)$. Then F satisfies the following differential equation

$$\left[y^2 \frac{\partial^2}{\partial y^2} + (y+v^2)\frac{\partial^2}{\partial v^2} + 2yv\frac{\partial^2}{\partial y \partial v} \right] F$$
$$= \left\{ (ay+bv)^2 + b^2 y + \lambda \right\} F.$$

Here $a = 2\pi n$ and $b = 2\pi r$ are constant. We note that the function $u(y) = y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|y)$ satisfies the above differential equation with $\lambda = s(s-1)$. Here $K_s(z)$ is the K-Bessel function before.

<u>Problem</u> : Find the solutions of the above differential equations explicitly.

<u>Problem</u> : Develop a Fourier expansion of a Maass-Jacobi form in terms of the Whittaker functions.

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8. Spectral theory of $\Delta_{n,m;A,B}$

Problem : Develop the spectral theory of $\Delta_{n,m}$ on $\mathcal{F}_{n,m}$.

Step I. Spectral Theory of Δ_Ω on A_Ω

Step II. Spectral Theory of Δ_n on $\mathcal{F}_n$ (Hard at this moment)

Step III. Mixed Spectral Theory

Step IV. Combine Step I-III and more advanced works to develop the Spectral Theory of $\Delta_{n,m}$ on $\mathcal{F}_{n,m}$.
[Very Complicated and Hard at this moment]

I will explain Step I-IV in more detail.

[Step I] For a fixed element $\Omega \in \mathbf{H}_n$, we set

$$L_\Omega = \mathbb{Z}^{(m,n)} + \mathbb{Z}^{(m,n)}\Omega$$

Then L_Ω is a lattice in $\mathbb{C}^{(m,n)}$ and the period matrix $\Omega_* = (I_n, \Omega)$ satisfies the Riemann conditions (RC.1) and (RC.2) :

(RC.1) $\quad \Omega_* J_n \, \Omega_*^T = 0\,;$

(RC.2) $\quad -\frac{1}{i}\Omega_* J_n \, \overline{\Omega}_*^T > 0.$

Thus the complex torus $A_\Omega = \mathbb{C}^{(m,n)}/L_\Omega$ is an abelian variety. For more details on A_Ω, we refer to [6].

We write $\Omega = X + iY$ of $\mathbf{H}_n$ with $X = \mathrm{Re}\,\Omega$ and $Y = \mathrm{Im}\,\Omega$. For a pair (A, B) with $A, B \in \mathbb{Z}^{(m,n)}$, we define the function $E_{\Omega;A,B} : \mathbb{C}^{(m,n)} \longrightarrow \mathbb{C}$ by

$$E_{\Omega;A,B}(Z) = e^{2\pi i\left(\text{tr}\,(A^T U\,) + \text{tr}\left((B - AX)Y^{-1}V^T\right)\right)},$$

where $Z = U + iV$ is a variable in $\mathbb{C}^{(m,n)}$ with real U, V.

Theorem: The set $\left\{ E_{\Omega;A,B} \,|\, A, B \in \mathbb{Z}^{(m,n)} \right\}$ is a complete orthonormal basis for $L^2(A_\Omega)$. Moreover we have the following spectral decomposition of Δ_Ω:

$$L^2(A_\Omega) = \oplus_{A,B \in \mathbb{Z}^{(m,n)}} \mathbb{C} \cdot E_{\Omega;A,B}.$$

[Step II] The inner product $(\ ,\ )$ on $L^2(\mathcal{F}_n)$ is defined by

$$(f, g) = \int_{\mathcal{F}_n} f(\Omega)\,\overline{g(\Omega)}\ \frac{[dX] \wedge [dY]}{(\det Y)^{n+1}}.$$

$L^2(\mathcal{F}_n)$ is decomposed as follows:

$$L^2(\mathcal{F}_n) = L^2_{\texttt{cusp}}(\mathcal{F}_n) \oplus L^2_{\texttt{res}}(\mathcal{F}_n) \oplus L^2_{\texttt{cont}}(\mathcal{F}_n)$$

The continuous part $L^2_{\texttt{cont}}(\mathcal{F}_n)$ can be understood by the theory of **Eisenstein series** developed by Alte Selberg and Robert Langlands. Also the residual part $L^2_{\texttt{res}}(\mathcal{F}_n)$ can be understood. But the cuspidal part $L^2_{\texttt{cusp}}(\mathcal{F}_n)$ has not been well developed yet. We have little knowledge of **cusp forms**.

For instance, if $n = 1$, then every element f in $L^2(\mathcal{F}_1)$ is decomposed into

$$f = \sum_{n=0}^{\infty} (f, g_n) g_n + \frac{1}{4\pi i} \int_{\mathrm{Re}\, s = \frac{1}{2}} (f, E_s)\, E_s \, ds$$

Here $g_0 = \sqrt{\frac{3}{\pi}}$, $\{ g_n \,|\, n \geq 1 \}$ is an orthonormal basis consisting of **cusp Maass forms**. The Eisenstein series E_s $(s \in \mathbb{C})$ is defined by

$$E_s(\Omega) = \sum_{\gamma \in \Gamma_1(\infty) \backslash \Gamma_1} (\mathrm{Im}\,(\gamma \cdot \Omega))^s , \quad \Omega \in \mathbb{H}_1$$

Here $\Gamma_1 = Sp(1, \mathbb{Z}) = SL(2, \mathbb{Z})$ and

$$\Gamma_1(\infty) = \{ \gamma \in \Gamma_1 \,|\, \gamma \cdot \infty = \infty \} .$$

[Step III-IV] The inner product $(\ ,\ )_{n,m}$ on $L^2(\mathcal{F}_{n,m})$ is defined by

$$\begin{aligned}&(f,g)_{n,m}\\ =&\int_{\mathcal{F}_{n,m}} f(\Omega,Z)\,\overline{g(\Omega,Z)}\ \frac{[dX][dY][dU][dV]}{(\det Y)^{n+m+1}}.\end{aligned}$$

$L^2(\mathcal{F}_{n,m})$ is decomposed into

$$L^2(\mathcal{F}_{n,m})=L^2_{\mathtt{cusp}}\oplus L^2_{\mathtt{res}}\oplus L^2_{\mathtt{cont}}$$

The continuous part $L^2_{\mathtt{cont}}$ can be understood by the theory of Eisenstein series with some more work. But the cuspidal part $L^2_{\mathtt{cusp}}$ has not been developed yet. This part is closely related to the theory of **Maass-Jacobi cusp forms**.

We have the following natural question :

Problem. Develop the theory of Maass-Jacobi forms (e.g., Hecke theory of Maass-Jacobi forms, Whittaker functions etc).

9. Decomposition of the regular representation of G^J

It is very important to decompose the **regular** representation of G^J on $L^2\Big(\Gamma_{n,m}\backslash G^J\Big)$ into irreducible (unitary) representations. Here

$$\Gamma_{n,m} = Sp(n,\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(n,m)}.$$

For brevity, we put

$$L^2 = L^2\Big(\Gamma_{n,m}\backslash G^J\Big).$$

Then the regular representation of G^J is decomposed into

$$L^2 = L^2_d \oplus L^2_c,$$

where L^2_d is the discrete part of L^2 and L^2_c is the continuous part of L^2. The continuous part of L^2 can be understood by the Langlands' theory of Eisenstein series with some more work. We decompose L^2_d as

$$L^2_d = \sum_{\pi} m_\pi \pi.$$

10. Open Problems

We list the problems to be investigated in the future.

Problem 1. Find explicit algebraically independent generators of $\mathbb{D}(\mathbf{H}_{n,m})$.

Problem 2. Find explicit algebraically independent generators of $\text{Pol}_{m,n}^{K} = \text{Pol}(T_{n,m})^{K}$. Here $K = U(n)$. Decompose the representation ρ of K or $K_{\mathbb{C}} = GL(n,\mathbb{C})$ on $\text{Pol}(T_{n,m})$ explicitly. More precisely if

$$\rho = \sum_{\sigma\in\widehat{K}} m_\sigma\,\sigma$$

we want to know the multiplicity m_σ. I think that the representation is not multiplicity free.

[Remark]: For a positive integer r, we let $\text{Pol}_{[r]}(T_n)$ denote the subspace of $\text{Pol}(T_n)$ consisting of homogeneous polynomial functions on T_n of degree r. The action of K or $K_{\mathbb{C}}$ on $\text{Pol}_{[r]}(T_n)$ is multiplicity-free (cf. L. Hua, W. Schmid, G. Shimura et al).

Problem 3. Let (Ω_1, Z_1) and (Ω_2, Z_2) be two given points in $\mathbf{H}_{n,m}$. Express the distance between (Ω_1, Z_1) and (Ω_2, Z_2) for the metric $ds^2_{n,m;A,B}$ explicitly.

Problem 4. Compute the multiplicity m_π in $L^2_d = \sum_\pi m_\pi \pi$ in Section 9. Investigate the unitary dual of G^J.

[**Remark]:** The unitary dual of $Sp(n, \mathbb{R})$ is not known for $n \geq 3$.

Problem 5. Investigate the Schrödinger-Weil representations of G^J in detail.

Problem 6. Develop the theory of the orbit method for G^J.

Problem 7. Find the trace formula for G^J with respect to $\Gamma_{n,m}$.

Problem 8. Find Weyl's law for G^J. Discuss the existence of nonzero Maass-Jacobi cusp forms.

Problem 9. Describe the Fourier transform, the inversion formula, the Plancherel formula and the spherical transform explicitly.

Problem 10. Discuss the existence and uniqueness of the Whittaker model (e.g., via an integral transform). In the case $n = m = 1$, R. Berndt and R. Schmidt gave two methods to obtain the Whittaker models (1) by the infinitesimal method and the the method of differential operators, and (2) via an integral transform [cf. Progress in Math. Vol. 163 (1998], pp. 63-73].

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Thank You Very Much !!!

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