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Abstract. Let $\mathbb{H}_g$ and $\mathbb{D}_g$ be the Siegel upper half plane and the generalized unit disk of degree g respectively. Let $\mathbb{C}^{(h,g)}$ be the Euclidean space of all $h \times g$ complex matrices. We present a partial Cayley transform of the Siegel–Jacobi disk $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ onto the Siegel–Jacobi space $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ which gives a partial bounded realization of $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ by $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$. We prove that the natural actions of the Jacobi group on $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ and $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ are compatible via a partial Cayley transform. A partial Cayley transform plays an important role in computing differential operators on the Siegel–Jacobi disk $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ invariant under the natural action of the Jacobi group on $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ explicitly.

1. Introduction

For a given fixed positive integer g, we let

$$\mathbb{H}_g = \left\{ \Omega \in \mathbb{C}^{(g,g)} \mid \Omega = {}^t\Omega, \quad \operatorname{Im} \Omega > 0 \right\}$$

be the Siegel upper half plane of degree g and let

$$Sp(g,\mathbb{R}) = \left\{ M \in \mathbb{R}^{(2g,2g)} \mid {}^tM J_g M = J_g \right\}$$

be the symplectic group of degree g, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l, ${}^tM$ denotes the transpose matrix of a matrix M and

$$J_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

We see that $Sp(g,\mathbb{R})$ acts on $\mathbb{H}_g$ transitively by

$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{1.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R})$ and $\Omega \in \mathbb{H}_g$.

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Let

$$\mathbb{D}_g = \Big\{ W \in \mathbb{C}^{(g,g)} \mid W = {}^tW,\ I_g - W\overline{W} > 0 \Big\}$$

be the generalized unit disk of degree g. The Cayley transform $\Phi : \mathbb{D}_g \longrightarrow \mathbb{H}_g$ defined by

$$\Phi(W) = i\,(I_g + W)(I_g - W)^{-1}, \quad W \in \mathbb{D}_g \tag{1.2}$$

is a biholomorphic mapping of $\mathbb{D}_g$ onto $\mathbb{H}_g$ which gives the bounded realization of $\mathbb{H}_g$ by $\mathbb{D}_g$ (cf. [8, pp. 281–283]). And the action (2.8) of the symplectic group on $\mathbb{D}_g$ is compatible with the action (1.1) via the Cayley transform Φ. A. Korányi and J. Wolf [4] gave a realization of a bounded symmetric domain as a Siegel domain of the third kind investigating a generalized Cayley transform of a bounded symmetric domain that generalizes the Cayley transform Φ of $\mathbb{D}_g$.

For two positive integers g and h, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(g,h)} = \Big\{ (\lambda,\mu;\kappa) \,\Big|\, \lambda,\mu \in \mathbb{R}^{(h,g)},\ \kappa \in \mathbb{R}^{(h,h)},\ \ \kappa + \mu\,{}^t\lambda \text{ symmetric } \Big\}$$

endowed with the following multiplication law

$$(\lambda,\mu;\kappa) \circ (\lambda',\mu';\kappa') = (\lambda+\lambda', \mu+\mu'; \kappa+\kappa' + \lambda\,{}^t\mu' - \mu\,{}^t\lambda').$$

The Jacobi group G^J is defined as the semidirect product of $Sp(g,\mathbb{R})$ and $H_{\mathbb{R}}^{(g,h)}$

$$G^J = Sp(g,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$$

endowed with the following multiplication law

$$\begin{aligned} &\Big(M, (\lambda,\mu;\kappa)\Big) \cdot \Big(M', (\lambda',\mu';\kappa')\Big) \\ &= \Big(MM', (\tilde{\lambda}+\lambda', \tilde{\mu}+\mu'; \kappa+\kappa' + \tilde{\lambda}\,{}^t\mu' - \tilde{\mu}\,{}^t\lambda')\Big) \end{aligned}$$

with $M, M' \in Sp(g,\mathbb{R}), (\lambda,\mu;\kappa),\ (\lambda',\mu';\kappa') \in H_{\mathbb{R}}^{(g,h)}$, and $(\tilde{\lambda},\tilde{\mu}) = (\lambda,\mu)M'$. Then G^J acts on $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ transitively by

$$\Big(M, (\lambda,\mu;\kappa)\Big) \cdot (\Omega, Z) = \Big(M\cdot\Omega, (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}\Big), \tag{1.3}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R}),\ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(g,h)}$, and $(\Omega, Z) \in \mathbb{H}_g \times \mathbb{C}^{(h,g)}$. In [11, p. 1331], the author presented the natural construction of the action (1.3).

We mention that studying the Siegel–Jacobi space or the Siegel–Jacobi disk associated with the Jacobi group is useful to the study of the universal family of polarized abelian varieties (cf. [12], [14]). The aim of this paper is to present a partial Cayley transform of the Siegel–Jacobi disk $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ onto the Siegel–Jacobi space $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ which gives a partially bounded realization of $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ by $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ and to prove that the natural action of the Jacobi group on $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ and $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ is compatible via a partial Cayley transform. The main reason that we study a partial Cayley transform is that this transform is usefully applied to computing differential operators on the Siegel–Jacobi disk $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ invariant under the action (3.5) of the Jacobi group G_*^J (cf. (3.2)) explicitly.

This paper is organized as follows. In Section 2, we review the Cayley transform of the generalized unit disk $\mathbb{D}_g$ onto the Siegel upper half plane $\mathbb{H}_g$ which gives a bounded realization of $\mathbb{H}_g$ by $\mathbb{D}_g$. In Section 3, we construct a partial Cayley transform of the Siegel–Jacobi disk $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ onto the Siegel–Jacobi space $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ which gives a partially bounded realization of $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ by $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ (cf. (3.6)). We prove that the action (1.3) of the Jacobi group G^J is compatible with the action (3.5) of the Jacobi group G_*^J through a partial Cayley transform (cf. Theorem 3.1). In the final section, we present the canonical automorphic factors of the Jacobi group G_*^J.

NOTATIONS: We denote by $\mathbb{R}$ and $\mathbb{C}$ the field of real numbers, and the field of complex numbers respectively. For a square matrix $A \in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. For $\Omega \in \mathbb{H}_g$, $\operatorname{Re}\Omega$ (*resp.* $\operatorname{Im}\Omega$) denotes the real (*resp.* imaginary) part of Ω. For a matrix $A \in F^{(k,k)}$ and $B \in F^{(k,l)}$, we write $A[B] = {}^tBAB$. I_n denotes the identity matrix of degree n.

2. The Cayley transform

Let

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} I_g & I_g \\ iI_g & -iI_g \end{pmatrix} \tag{2.1}$$

be the $2g \times 2g$ matrix represented by Φ. Then

$$T^{-1} Sp(g,\mathbb{R})\, T = \left\{ \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \;\Big|\; {}^tP\overline{P} - {}^t\overline{Q}Q = I_g,\ {}^tP\overline{Q} = {}^t\overline{Q}P \right\}. \tag{2.2}$$

Indeed, if $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R})$, then

$$T^{-1} M T = \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \tag{2.3}$$

where

$$P = \frac{1}{2} \Big\{ (A+D) + i\,(B-C) \Big\} \tag{2.4}$$

and

$$Q = \frac{1}{2} \Big\{ (A-D) - i\,(B+C) \Big\}. \tag{2.5}$$

For brevity, we set

$$G_* = T^{-1} Sp(g,\mathbb{R}) T.$$

Then G_* is a subgroup of $SU(g,g)$, where

$$SU(g,g) = \left\{ h \in \mathbb{C}^{(g,g)} \;\Big|\; {}^th I_{g,g} \overline{h} = I_{g,g} \right\}, \quad I_{g,g} = \begin{pmatrix} I_g & 0 \\ 0 & -I_g \end{pmatrix}.$$

In the case $g = 1$, we observe that

$$T^{-1}Sp(1,\mathbb{R})T = T^{-1}SL_2(\mathbb{R})T = SU(1,1).$$

If $g > 1$, then G_* is a *proper* subgroup of $SU(g,g)$. In fact, since ${}^tTJ_gT = -i\,J_g$, we get

$$G_* = \left\{ h \in SU(g,g) \,\middle|\, {}^thJ_gh = J_g \right\} = SU(g,g) \cap Sp(g,\mathbb{C}), \tag{2.6}$$

where

$$Sp(g,\mathbb{C}) = \left\{ \alpha \in \mathbb{C}^{(2g,2g)} \,\middle|\, {}^t\alpha\, J_g\, \alpha = J_g \right\}.$$

Let

$$P^+ = \left\{ \begin{pmatrix} I_g & Z \\ 0 & I_g \end{pmatrix} \,\middle|\, Z = {}^tZ \in \mathbb{C}^{(g,g)} \right\}$$

be the P^+-part of the complexification of $G_* \subset SU(g,g)$. We note that the Harish–Chandra decomposition of an element $\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}$ in G_* is

$$\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} = \begin{pmatrix} I_g & Q\overline{P}^{-1} \\ 0 & I_g \end{pmatrix} \begin{pmatrix} P - Q\overline{P}^{-1}\overline{Q} & 0 \\ 0 & \overline{P} \end{pmatrix} \begin{pmatrix} I_g & 0 \\ \overline{P}^{-1}\overline{Q} & I_g \end{pmatrix}.$$

For more detail, we refer to [3, p. 155]. Thus the P^+-component of the following element

$$\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \cdot \begin{pmatrix} I_g & W \\ 0 & I_g \end{pmatrix}, \quad W \in \mathbb{D}_g$$

of the complexification of G_*^J is given by

$$\begin{pmatrix} I_g & (PW+Q)(\overline{Q}W+\overline{P})^{-1} \\ 0 & I_g \end{pmatrix}. \tag{2.7}$$

We note that $Q\overline{P}^{-1} \in \mathbb{D}_g$. We get the Harish-Chandra embedding of $\mathbb{D}_g$ into P^+ (cf. [3, p. 155] or [7, pp. 58–59]). Therefore we see that G_* acts on $\mathbb{D}_g$ transitively by

$$\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \cdot W = (PW+Q)(\overline{Q}W+\overline{P})^{-1}, \quad \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*,\ W \in \mathbb{D}_g. \tag{2.8}$$

The isotropy subgroup at the origin o is given by

$$K = \left\{ \begin{pmatrix} P & 0 \\ 0 & \overline{P} \end{pmatrix} \,\middle|\, P \in U(g) \right\}.$$

Thus G_*/K is biholomorphic to $\mathbb{D}_g$. It is known that the action (1.1) is compatible with the action (2.8) via the Cayley transform Φ (cf. (1.2)). In other words, if $M \in Sp(g,\mathbb{R})$ and $W \in \mathbb{D}_g$, then

$$M \cdot \Phi(W) = \Phi(M_* \cdot W), \tag{2.9}$$

where $M_* = T^{-1}MT \in G_*$. For a proof of Formula (2.9), we refer to the proof of Theorem 3.1.

For $\Omega = (\omega_{ij}) \in \mathbb{H}_g$, we write $\Omega = X + iY$ with $X = (x_{ij}),\ Y = (y_{ij})$ real and $d\Omega = (d\omega_{ij})$. We also put

$$\frac{\partial}{\partial \Omega} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial \omega_{ij}} \right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{\Omega}} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial \overline{\omega}_{ij}} \right).$$

Then

$$ds^2 = \sigma\Big(Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega}\Big) \tag{2.10}$$

is a $Sp(g,\mathbb{R})$-invariant metric on $\mathbb{H}_g$ (cf. [8]). H. Maass [5] proved that its Laplacian is given by

$$\Delta = 4\,\sigma\left(Y\ {}^t\!\left(Y \frac{\partial}{\partial \overline{\Omega}} \right) \frac{\partial}{\partial \Omega} \right). \tag{2.11}$$

For $W = (w_{ij}) \in \mathbb{D}_g$, we write $dW = (dw_{ij})$ and $d\overline{W} = (d\overline{w}_{ij})$. We put

$$\frac{\partial}{\partial W} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial w_{ij}} \right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{W}} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial \overline{w}_{ij}} \right).$$

Using the Cayley transform $\Phi : \mathbb{D}_g \longrightarrow \mathbb{H}_g$, H. Maass proved (cf. [5]) that

$$ds_*^2 = 4\,\sigma\Big((I_g - W\overline{W})^{-1} dW\, (I_g - \overline{W}W)^{-1} d\overline{W} \Big) \tag{2.12}$$

is a G_*-invariant Riemannian metric on $\mathbb{D}_g$ and its Laplacian is given by

$$\Delta_* = \sigma\left((I_g - W\overline{W})\ {}^t\!\left((I_g - W\overline{W}) \frac{\partial}{\partial \overline{W}} \right) \frac{\partial}{\partial W} \right). \tag{2.13}$$

3. A partial Cayley transform

In this section, we present a partial Cayley transform of $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ onto $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ and prove that the action (1.3) of G^J on $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ is compatible with the *natural action* (cf. (3.5)) of the Jacobi group G_*^J on $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ via a partial Cayley transform.

From now on, for brevity we write $\mathbb{H}_{g,h} = \mathbb{H}_g \times \mathbb{C}^{(h,g)}$. We can identify an element $g = \Big(M, (\lambda, \mu; \kappa)\Big)$ of $G^J,\ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R})$ with the element

$$\begin{pmatrix} A & 0 & B & A\,{}^t\mu - B\,{}^t\lambda \\ \lambda & I_h & \mu & \kappa \\ C & 0 & D & C\,{}^t\mu - D\,{}^t\lambda \\ 0 & 0 & 0 & I_h \end{pmatrix}$$

of $Sp(g+h,\mathbb{R})$. This subgroup plays an important role in investigating the Fourier–Jacobi expansion of a Siegel modular form for $Sp(g+h,\mathbb{R})$ (cf. [6]) and studying the theory of Jacobi forms (cf. [1], [2], [9], [10], [11], [17]).

We set

$$T_* = \frac{1}{\sqrt{2}} \begin{pmatrix} I_{g+h} & I_{g+h} \\ iI_{g+h} & -iI_{g+h} \end{pmatrix}.$$

We now consider the group G^J_* defined by

$$G^J_* = T_*^{-1} G^J T_*.$$

If $g = (M, (\lambda, \mu; \kappa)) \in G^J$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbb{R})$, then $T_*^{-1} g T_*$ is given by

$$T_*^{-1} g T_* = \begin{pmatrix} P_* & Q_* \\ \overline{Q}_* & \overline{P}_* \end{pmatrix}, \tag{3.1}$$

where

$$P_* = \begin{pmatrix} P & \frac{1}{2}\Big\{ Q\ {}^t(\lambda + i\mu) - P\ {}^t(\lambda - i\mu) \Big\} \\ \frac{1}{2}(\lambda + i\mu) & I_h + i\frac{\kappa}{2} \end{pmatrix},$$

$$Q_* = \begin{pmatrix} Q & \frac{1}{2}\Big\{ P\ {}^t(\lambda - i\mu) - Q\ {}^t(\lambda + i\mu) \Big\} \\ \frac{1}{2}(\lambda - i\mu) & -i\frac{\kappa}{2} \end{pmatrix},$$

and P, Q are given by Formulas (2.4) and (2.5). From now on, we write

$$\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \left(\frac{1}{2}(\lambda + i\mu),\ \frac{1}{2}(\lambda - i\mu);\ -i\frac{\kappa}{2} \right) \right) = \begin{pmatrix} P_* & Q_* \\ \overline{Q}_* & \overline{P}_* \end{pmatrix}.$$

In other words, we have the relation

$$T_*^{-1} \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu; \kappa) \right) T_* = \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \left(\frac{1}{2}(\lambda + i\mu),\ \frac{1}{2}(\lambda - i\mu);\ -i\frac{\kappa}{2} \right) \right).$$

Let

$$H_{\mathbb{C}}^{(g,h)} = \Big\{ (\xi, \eta\,; \zeta) \mid \xi, \eta \in \mathbb{C}^{(h,g)},\ \zeta \in \mathbb{C}^{(h,h)},\ \zeta + \eta\,{}^t\xi \text{ symmetric} \Big\}$$

be the Heisenberg group endowed with the following multiplication

$$(\xi, \eta\,; \zeta) \circ (\xi', \eta'; \zeta') = (\xi + \xi', \eta + \eta'\,; \zeta + \zeta' + \xi\,{}^t\eta' - \eta\,{}^t\xi')).$$

We define the semidirect product

$$SL(2g, \mathbb{C}) \ltimes H_{\mathbb{C}}^{(g,h)}$$

endowed with the following multiplication

$$\begin{aligned} &\left(\begin{pmatrix} P & Q \\ R & S \end{pmatrix}, (\xi, \eta\,; \zeta) \right) \cdot \left(\begin{pmatrix} P' & Q' \\ R' & S' \end{pmatrix}, (\xi', \eta'; \zeta') \right) \\ = &\left(\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} P' & Q' \\ R' & S' \end{pmatrix}, (\tilde{\xi} + \xi', \tilde{\eta} + \eta'; \zeta + \zeta' + \tilde{\xi}\,{}^t\eta' - \tilde{\eta}\,{}^t\xi') \right), \end{aligned}$$

where $\tilde{\xi} = \xi P' + \eta R'$ and $\tilde{\eta} = \xi Q' + \eta S'$.

If we identify $H_{\mathbb{R}}^{(g,h)}$ with the subgroup

$$\Big\{ (\xi, \overline{\xi}; i\kappa) \mid \xi \in \mathbb{C}^{(h,g)},\ \kappa \in \mathbb{R}^{(h,h)} \Big\}$$

of $H_{\mathbb{C}}^{(g,h)}$, we have the following inclusion

$$G^J_* \subset SU(g,g) \ltimes H_{\mathbb{R}}^{(g,h)} \subset SL(2g, \mathbb{C}) \ltimes H_{\mathbb{C}}^{(g,h)}.$$

More precisely, if we recall $G_* = SU(g,g)\cap Sp(g,\mathbb{C})$ (cf. (2.6)), we see that the Jacobi group G_*^J is given by

(3.2)
$$G_*^J = \left\{ \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\xi, \bar{\xi}\,; i\kappa) \right) \;\Big|\; \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*,\ \xi \in \mathbb{C}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)} \right\}.$$

We define the mapping $\Theta : G^J \longrightarrow G_*^J$ by

(3.3)
$$\Theta\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda,\mu;\kappa) \right) = \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \left(\frac{1}{2}(\lambda + i\mu),\ \frac{1}{2}(\lambda - i\mu)\,;\, -i\frac{\kappa}{2} \right) \right),$$

where P and Q are given by Formulas (2.4) and (2.5). We can see that if $g_1, g_2 \in G^J$, then $\Theta(g_1 g_2) = \Theta(g_1)\Theta(g_2)$.

According to [13, p. 250], G_*^J is of the Harish-Chandra type (cf. [7, p. 118]). Let

$$g_* = \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\lambda, \mu;\, \kappa) \right)$$

be an element of G_*^J. Since the Harish-Chandra decomposition of an element $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ in $SU(g,g)$ is given by

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} I_g & QS^{-1} \\ 0 & I_g \end{pmatrix} \begin{pmatrix} P - QS^{-1}R & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} I_g & 0 \\ S^{-1}R & I_g \end{pmatrix},$$

the P_*^+-component of the following element

$$g_* \cdot \left(\begin{pmatrix} I_g & W \\ 0 & I_g \end{pmatrix}, (0,\eta;0) \right), \quad W \in \mathbb{D}_g$$

of $SL(2g,\mathbb{C}) \ltimes H_{\mathbb{C}}^{(g,h)}$ is given by

(3.4)
$$\left(\begin{pmatrix} I_g & (PW+Q)(\overline{Q}W+\overline{P})^{-1} \\ 0 & I_g \end{pmatrix},\ \left(0,\ (\eta + \lambda W + \mu)(\overline{Q}W + \overline{P})^{-1}\,; 0\right) \right).$$

We can identify $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ with the subset

$$\left\{ \left(\begin{pmatrix} I_g & W \\ 0 & I_g \end{pmatrix}, (0,\eta;0) \right) \;\Big|\; W \in \mathbb{D}_g,\ \eta \in \mathbb{C}^{(h,g)} \right\}$$

of the complexification of G_*^J. Indeed, $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ is embedded into P_*^+ given by

$$P_*^+ = \left\{ \left(\begin{pmatrix} I_g & W \\ 0 & I_g \end{pmatrix}, (0,\eta;0) \right) \;\Big|\; W = {}^tW \in \mathbb{C}^{(g,g)},\ \eta \in \mathbb{C}^{(h,g)} \right\}.$$

This is a generalization of the Harish-Chandra embedding (cf. [7, p. 119]). Hence G_*^J acts on $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ transitively by

$$\begin{aligned}&\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\lambda, \mu;\, \kappa) \right) \cdot (W, \eta) \\ =&\Big((PW + Q)(\overline{Q}W + \overline{P})^{-1}, (\eta + \lambda W + \mu)(\overline{Q}W + \overline{P})^{-1} \Big).\end{aligned} \tag{3.5}$$

From now on, for brevity we write $\mathbb{D}_{g,h} = \mathbb{D}_g \times \mathbb{C}^{(h,g)}$. We define the map Φ_* of $\mathbb{D}_{g,h}$ into $\mathbb{H}_{g,h}$ by

$$\Phi_*(W,\eta) = \Big(i(I_g + W)(I_g - W)^{-1},\, 2\,i\,\eta\,(I_g - W)^{-1} \Big), \quad (W,\eta) \in \mathbb{D}_{g,h}. \tag{3.6}$$

We can show that Φ_* is a biholomorphic map of $\mathbb{D}_{g,h}$ onto $\mathbb{H}_{g,h}$ which gives a partial bounded realization of $\mathbb{H}_{g,h}$ by the Siegel–Jacobi disk $\mathbb{D}_{g,h}$. We call this map Φ_* the *partial Cayley transform* of the Siegel–Jacobi disk $\mathbb{D}_{g,h}$.

Theorem 3.1. *The action* (1.3) *of* G^J *on* $\mathbb{H}_{g,h}$ *is compatible with the action* (3.5) *of* G_*^J *on* $\mathbb{D}_{g,h}$ *through the partial Cayley transform* Φ_*. *In other words, if* $g_0 \in G^J$ *and* $(W,\eta) \in \mathbb{D}_{g,h}$,

$$g_0 \cdot \Phi_*(W,\eta) = \Phi_*(g_* \cdot (W,\eta)), \tag{3.7}$$

where $g_* = T_*^{-1} g_0 T_*$. *We observe that Formula* (3.7) *generalizes Formula* (2.9). *The inverse of* Φ_* *is*

$$\Phi_*^{-1}(\Omega, Z) = \Big((\Omega - iI_g)(\Omega + iI_g)^{-1},\, Z(\Omega + iI_g)^{-1} \Big). \tag{3.8}$$

Proof. Let

$$g_0 = \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu; \kappa) \right)$$

be an element of G^J and let $g_* = T_*^{-1} g_0 T_*$. Then

$$g_* = \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \left(\frac{1}{2}(\lambda + i\mu),\, \frac{1}{2}(\lambda - i\mu)\,;\, -i\frac{\kappa}{2} \right) \right),$$

where P and Q are given by Formulas (2.4) and (2.5).

For brevity, we write

$$(\Omega, Z) = \Phi_*(W,\eta) \quad \text{and} \quad (\Omega_*, Z_*) = g_0 \cdot (\Omega, Z).$$

That is,

$$\Omega = i(I_g + W)(I_g - W)^{-1} \quad \text{and} \quad Z = 2\,i\,\eta(I_g - W)^{-1}.$$

Then we get

$$\begin{aligned}
\Omega_* &= (A\Omega+B)(C\Omega+D)^{-1}\\
&= \Big\{i\,A(I_g+W)(I_g-W)^{-1}+B\Big\}\Big\{i\,C(I_g+W)(I_g-W)^{-1}+D\Big\}^{-1}\\
&= \Big\{i\,A(I_g+W)+B(I_g-W)\Big\}(I_g-W)^{-1}\\
&\quad\times\Big[\Big\{i\,C(I_g+W)+D(I_g-W)\Big\}(I_g-W)^{-1}\Big]^{-1}\\
&= \Big\{(i\,A-B)W+(i\,A+B)\Big\}\Big\{(i\,C-D)W+(i\,C+D)\Big\}^{-1}
\end{aligned}$$

and

$$\begin{aligned}
Z_* &= (Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}\\
&= \Big\{2\,i\,\eta\,(I_g-W)^{-1}+i\,\lambda(I_g+W)(I_g-W)^{-1}+\mu\Big\}\\
&\quad\times\Big\{i\,C(I_g+W)(I_g-W)^{-1}+D\Big\}^{-1}\\
&= \Big\{2\,i\,\eta+i\,\lambda\,(I_g+W)+\mu\,(I_g-W)\Big\}(I_g-W)^{-1}\\
&\quad\times\Big[\Big\{i\,C(I_g+W)+D(I_g-W)\Big\}(I_g-W)^{-1}\Big]^{-1}\\
&= \Big\{2\,i\,\eta+(\lambda\,i-\mu)W+\lambda\,i+\mu\Big\}\Big\{(i\,C-D)W+(i\,C+D)\Big\}^{-1}.
\end{aligned}$$

On the other hand, we set

$$(W_*,\eta_*)=g_*\cdot(W,\eta)\quad\text{and}\quad(\widehat{\Omega},\widehat{Z})=\Phi_*(W_*,\eta_*).$$

Then

$$W_*=(PW+Q)(\overline{Q}W+\overline{P})^{-1}\quad\text{and}\quad\eta_*=(\eta+\lambda_*W+\mu_*)(\overline{Q}W+\overline{P})^{-1},$$

where $\lambda_*=\frac{1}{2}(\lambda+\,i\,\mu)$ and $\mu_*=\frac{1}{2}(\lambda-\,i\,\mu)$.

According to Formulas (2.4) and (2.5), we get

$$\begin{aligned}
\widehat{\Omega} &= i\,(I_g+W_*)(I_g-W_*)^{-1}\\
&= i\Big\{I_g+(PW+Q)(\overline{Q}W+\overline{P})^{-1}\Big\}\Big\{I_g-(PW+Q)(\overline{Q}W+\overline{P})^{-1}\Big\}^{-1}\\
&= i\,(\overline{Q}W+\overline{P}+PW+Q)(\overline{Q}W+\overline{P})^{-1}\\
&\quad\times\Big\{(\overline{Q}W+\overline{P}-PW-Q)(\overline{Q}W+\overline{P})^{-1}\Big\}^{-1}\\
&= i\Big\{(P+\overline{Q}\,)W+\overline{P}+Q\Big\}\Big\{(\overline{Q}-P)W+\overline{P}-Q\Big\}^{-1}\\
&= \Big\{(i\,A-B)W+(i\,A+B)\Big\}\Big\{(i\,C-D)W+(i\,C+D)\Big\}^{-1}.
\end{aligned}$$

Therefore $\widehat{\Omega}=\Omega_*$. In fact, this result is the known fact (cf. Formula (2.9)) that the action (1.1) is compatible with the action (2.8) via the Cayley transform

Φ.

$$\begin{aligned}\widehat{Z} &= 2\,i\,\eta_*(I_g - W_*)^{-1}\\ &= 2\,i\,(\eta+\lambda_* W+\mu_*)(\overline{Q}W+\overline{P})^{-1}\\ &\quad\times\Big\{I_g-(PW+Q)(\overline{Q}W+\overline{P}\,)^{-1}\Big\}^{-1}\\ &= 2\,i\,(\eta+\lambda_* W+\mu_*)(\overline{Q}W+\overline{P}\,)^{-1}\\ &\quad\times\Big\{(\overline{Q}W+\overline{P}-PW-Q)(\overline{Q}W+\overline{P}\,)^{-1}\Big\}^{-1}\\ &= 2\,i\,(\eta+\lambda_* W+\mu_*)\Big\{(\overline{Q}-P)W+\overline{P}-Q\Big\}^{-1}.\end{aligned}$$

Using Formulas (2.4) and (2.5), we obtain

$$\widehat{Z}=\Big\{2\,i\,\eta+(\lambda\,i-\mu)W+\lambda\,i+\mu\Big\}\Big\{(i\,C-D)W+i\,C+D\Big\}^{-1}.$$

Hence $\widehat{Z}=Z_*$. Consequently we get Formula (3.7). Formula (3.8) follows immediately from a direct computation. □

Remark 3.1. R. Berndt and R. Schmidts (cf. [1, pp. 52–53]) investigated a partial Cayley transform in the case $g=h=1$.

For a coordinate $(\Omega,Z)\in\mathbb{H}_{g,h}$ with $\Omega=(\omega_{\mu\nu})\in\mathbb{H}_g$ and $Z=(z_{kl})\in\mathbb{C}^{(h,g)}$, we put

$$\begin{aligned}\Omega &= X\,+\,iY, && X\,=\,(x_{\mu\nu}), && Y\,=\,(y_{\mu\nu})\ \text{ real},\\ Z &= U\,+\,iV, && U\,=\,(u_{kl}), && V\,=\,(v_{kl})\ \text{ real},\\ d\Omega &= (d\omega_{\mu\nu}), && dX\,=\,(dx_{\mu\nu}), && dY\,=\,(dy_{\mu\nu}),\\ dZ &= (dz_{kl}), && dU\,=\,(du_{kl}), && dV\,=\,(dv_{kl}),\\ d\overline{\Omega} &= (d\overline{\omega}_{\mu\nu}), && d\overline{Z}=(d\bar{z}_{kl}),\end{aligned}$$

$$\frac{\partial}{\partial\Omega}=\left(\frac{1+\delta_{\mu\nu}}{2}\,\frac{\partial}{\partial\omega_{\mu\nu}}\right),\quad \frac{\partial}{\partial\overline{\Omega}}=\left(\frac{1+\delta_{\mu\nu}}{2}\,\frac{\partial}{\partial\overline{\omega}_{\mu\nu}}\right),$$

$$\frac{\partial}{\partial Z}=\begin{pmatrix}\frac{\partial}{\partial z_{11}} & \cdots & \frac{\partial}{\partial z_{h1}}\\ \vdots & \ddots & \vdots\\ \frac{\partial}{\partial z_{1g}} & \cdots & \frac{\partial}{\partial z_{hg}}\end{pmatrix},\quad \frac{\partial}{\partial\overline{Z}}=\begin{pmatrix}\frac{\partial}{\partial \bar{z}_{11}} & \cdots & \frac{\partial}{\partial \bar{z}_{h1}}\\ \vdots & \ddots & \vdots\\ \frac{\partial}{\partial \bar{z}_{1g}} & \cdots & \frac{\partial}{\partial \bar{z}_{hg}}\end{pmatrix}.$$

Remark 3.2. The author proved in [15] that for any two positive real numbers A and B, the following metric

$$\begin{aligned}ds^2_{g,h;A,B} &= A\,\sigma\Big(Y^{-1}d\Omega\,Y^{-1}d\overline{\Omega}\Big)\\ &\quad+B\,\Big\{\sigma\Big(Y^{-1}\,{}^tV\,V\,Y^{-1}d\Omega\,Y^{-1}d\overline{\Omega}\Big)+\sigma\Big(Y^{-1}\,{}^t(dZ)\,d\overline{Z}\Big)\\ &\quad-\sigma\Big(V\,Y^{-1}d\Omega\,Y^{-1}\,{}^t(d\overline{Z})\Big)-\sigma\Big(V\,Y^{-1}d\overline{\Omega}\,Y^{-1}\,{}^t(dZ)\,\Big)\Big\}\end{aligned}\tag{3.9}$$

is a Riemannian metric on $\mathbb{H}_{g,h}$ which is invariant under the action (1.3) of the Jacobi group G^J and its Laplacian is given by

$$\begin{aligned}\Delta_{n,m;A,B} \quad = \quad & \frac{4}{A}\left\{\sigma\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right)+\sigma\left(VY^{-1}\,{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial Z}\right)\right.\\ & \left. +\sigma\left(V\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial Z}\right)+\sigma\left({}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial\Omega}\right)\right\} \qquad (3.10)\\ & +\frac{4}{B}\,\sigma\left(Y\,\frac{\partial}{\partial Z}\,{}^t\!\left(\frac{\partial}{\partial\overline{Z}}\right)\right).\end{aligned}$$

We observe that Formulas (3.9) and (3.10) generalize Formulas (2.10) and (2.11). The following differential form

$$dv_{g,h} = (\det Y)^{-(g+h+1)}\,[dX]\wedge[dY]\wedge[dU]\wedge[dV]$$

is a G^J-invariant volume element on $\mathbb{H}_{g,h}$, where

$$[dX]=\wedge_{\mu\le\nu}dx_{\mu\nu},\ [dY]=\wedge_{\mu\le\nu}dy_{\mu\nu},\ [dU]=\wedge_{k,l}du_{kl}\ \text{ and }\ [dV]=\wedge_{k,l}dv_{kl}.$$

Using the partial Cayley transform Φ_* and Theorem 3.1, we can find a G^J_*-invariant Riemannian metric on the Siegel–Jacobi disk $\mathbb{D}_{g,h}$ and its Laplacian explicitly which generalize Formulas (2.12) and (2.13). For more detail, we refer to [16].

4. The canonical automorphic factors

The isotropy subgroup K^J_* at $(0,0)$ under the action (3.5) is

$$K^J_* = \left\{\left(\begin{pmatrix}P & 0\\ 0 & \overline{P}\end{pmatrix},\,(0,0\,;\kappa)\right)\ \Big|\ P\in U(g),\ \kappa\in\mathbb{R}^{(h,h)}\right\}. \qquad (4.1)$$

The complexification of K^J_* is given by

$$K^J_{*,\mathbb{C}} = \left\{\left(\begin{pmatrix}P & 0\\ 0 & {}^tP^{-1}\end{pmatrix},\,(0,0\,;\zeta)\right)\ \Big|\ P\in GL(g,\mathbb{C}),\ \zeta\in\mathbb{C}^{(h,h)}\right\}. \qquad (4.2)$$

By a complicated computation, we can show that if

$$g_* = \left(\begin{pmatrix}P & Q\\ \overline{Q} & \overline{P}\end{pmatrix},(\lambda,\mu;\,\kappa)\right) \qquad (4.3)$$

is an element of G^J_*, then the $K^J_{*,\mathbb{C}}$-component of

$$g_*\cdot\left(\begin{pmatrix}I_g & W\\ 0 & I_g\end{pmatrix},(0,\eta;0)\right)$$

is given by

$$\left(\begin{pmatrix}P-(PW+Q)(\overline{Q}W+\overline{P})^{-1}\overline{Q} & 0\\ 0 & \overline{Q}W+\overline{P}\end{pmatrix},(0,0\,;\kappa_*)\right), \qquad (4.4)$$

where

$$\begin{aligned}\kappa_* &= \kappa + \lambda\, {}^t\eta + (\eta + \lambda W + \mu)\, {}^t\lambda \\ &\qquad - (\eta + \lambda W + \mu)\, {}^t\overline{Q}\, {}^t(\overline{Q}W + \overline{P})^{-1}\, {}^t(\eta + \lambda W + \mu) \\ &= \kappa + \lambda\, {}^t\eta + (\eta + \lambda W + \mu)\, {}^t\lambda \\ &\qquad - (\eta + \lambda W + \mu)(\overline{Q}W + \overline{P})^{-1}\overline{Q}\, {}^t(\eta + \lambda W + \mu).\end{aligned}$$

Here we used the fact that $(\overline{Q}W + \overline{P})^{-1}\overline{Q}$ is symmetric.

For $g_* \in G_*^J$ given by (4.3) with $g_0 = \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*$ and $(W, \eta) \in \mathbb{D}_{g,h}$, we write

$$J(g_*, (W, \eta)) = a(g_*, (W, \eta))\, b(g_0, W), \tag{4.5}$$

where

$$a(g_*, (W, \eta)) = (I_{2g}, (0, 0; \kappa_*)), \quad \text{where } \kappa_* \text{ is given in (4.4)}$$

and

$$b(g_0, W) = \left(\begin{pmatrix} P - (PW + Q)(\overline{Q}W + \overline{P})^{-1}\overline{Q} & 0 \\ 0 & \overline{Q}W + \overline{P} \end{pmatrix}, (0, 0\,; 0) \right).$$

Lemma 4.1. *Let*

$$\rho : GL(g, \mathbb{C}) \longrightarrow GL(V_\rho)$$

be a holomorphic representation of $GL(g, \mathbb{C})$ on a finite dimensional complex vector space V_ρ and $\chi : \mathbb{C}^{(h,h)} \longrightarrow \mathbb{C}^\times$ be a character of the additive group $\mathbb{C}^{(h,h)}$. Then the mapping

$$J_{\chi,\rho} : G_*^J \times \mathbb{D}_{g,h} \longrightarrow GL(V_\rho)$$

defined by

$$J_{\chi,\rho}(g_*, (W, \eta)) = \chi(a(g_*, (W, \eta)))\, \rho(b(g_0, W))$$

is an automorphic factor of G_*^J with respect to χ and ρ.

Proof. We observe that $a(g_*, (W, \eta))$ is a summand of automorphy, i.e.,

$$a(g_1 g_2, (W, \eta)) = a(g_1, g_2 \cdot (W, \eta)) + a(g_2, (W, \eta)),$$

where $g_1, g_2 \in G_*^J$ and $(W, \eta) \in \mathbb{D}_{g,h}$. Together with this fact, the proof follows from the fact that the mapping

$$J_\rho : G_* \times \mathbb{D}_g \longrightarrow GL(V_\rho)$$

defined by

$$J_\rho(g_0, W) := \rho(b(g_0, W)), \quad g_0 \in G_*,\ W \in \mathbb{D}_g$$

is an automorphic factor of G_*. □

Example 4.1. Let $\mathcal{M}$ be a symmetric half-integral semi-positive definite matrix of degree h and let $\rho : GL(g,\mathbb{C}) \longrightarrow GL(V_\rho)$ be a holomorphic representation of $GL(g,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. Then the character

$$\chi_{\mathcal{M}} : \mathbb{C}^{(h,h)} \longrightarrow \mathbb{C}^\times$$

defined by

$$\chi_{\mathcal{M}}(c) = e^{-2\pi i\sigma(\mathcal{M}c)}, \quad c \in \mathbb{C}^{(h,h)}$$

provides the automorphic factor

$$J_{\mathcal{M},\rho} : G_*^J \times \mathbb{D}_{g,h} \longrightarrow GL(V_\rho)$$

defined by

$$J_{\mathcal{M},\rho}(g_*, (W,\eta)) = e^{-2\pi i\sigma(\mathcal{M}\kappa_*)}\rho(\overline{Q}W + \overline{P}),$$

where g_* is an element in G_*^J given by (4.3) and κ_* is given in (4.4). Using $J_{\mathcal{M},\rho}$, we can define the notion of Jacobi forms on $\mathbb{D}_{g,h}$ of index $\mathcal{M}$ with respect to the Siegel modular group $T^{-1}Sp(g,\mathbb{Z})\,T$ (cf. [9], [10], [11]).

Remark 4.1. The P_*^--component of

$$g_* \cdot \left(\begin{pmatrix} I_g & W \\ 0 & I_g \end{pmatrix}, (0,\eta;0) \right)$$

is given by

$$\left(\begin{pmatrix} I_g & 0 \\ (\overline{Q}W + \overline{P})^{-1}\overline{Q} & I_g \end{pmatrix}, \left(\lambda - (\eta + \lambda W + \mu)(\overline{Q}W + \overline{P})^{-1}\overline{Q},\, 0\,;0 \right) \right), \tag{4.6}$$

where

$$P_*^- = \left\{ \left(\begin{pmatrix} I_g & 0 \\ W & I_g \end{pmatrix}, (\xi, 0\,;0) \right) \,\Big|\, W = {}^tW \in \mathbb{C}^{(g,g)},\ \xi \in \mathbb{C}^{(h,g)} \right\}.$$

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