

KAC-MOODY ALGEBRAS, THE MONSTROUS MOONSHINE, JACOBI FORMS AND INFINITE PRODUCTS

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1. Introduction

Recently R. E. Borcherds obtained some quite interesting results in [Bo6-7]. First he solved the Moonshine Conjectures made by Conway and Norton ([C-N]). Secondly he constructed automorphic forms on the orthogonal group $O_{s+2,2}(\mathbb{R})$ which are modular products and then wrote some of the well-known meromorphic modular forms as infinite products. Modular products roughly mean infinite products whose exponents are the coefficients of certain nearly holomorphic modular forms. The theory of Jacobi forms plays an important role in his second work in [Bo7]. More than 10 years ago Feingold and Frenkel ([F-F]) realized the connection between the theory of a special hyperbolic Kac-Moody Lie algebra of the type $HA_1^{(1)}$ and that of Jacobi forms of degree one and then generalized the results of H. Maass to higher levels. So far the relationship between the theory of Jacobi forms of higher degree and that of other hyperbolic Kac-Moody algebras has not been developed yet. The work of Borcherds in [Bo7] gives a light on the possibility for the relationship between them. This fact urged me to write a somewhat supplementary or expository note on Borcherds' recent works which is useful for my research on Jacobi forms although I am not an expert in the theory of Kac-Moody Lie algebras and lattices. I hope that this note will be useful for the readers who are interested in these interesting subjects. I learned a lot about these subjects while I had been preparing this article. I had given a lecture on some of these materials at the 4th Symposium of the Pyungsan Institute for Mathematical Sciences held at the Wonkwang University in September, 1995.

As mentioned above, the purpose of this paper is to give a survey of Borcherds' recent results in [Bo6-7] to the core. This article is organized as follows. In section 2, we collect some of the well-known properties of Kac-Moody Lie algebras, e.g., the Weyl-Kac character formula, the root multiplicity and so on. In the appendix, we discuss the generalized Kac-Moody Lie algebras introduced by Borcherds roughly. In section 3, we give a sketchy survey on the Moonshine Conjectures solved by Borcherds ([Bo6]). We discuss the monster Lie algebra and the no-ghost theorem. This section is completely based on the article [Bo6]. In section 4, we review some of the theory of Jacobi forms and discuss singular Jacobi forms briefly. We present Borcherds' construction of nearly holomorphic Jacobi forms by making use of the concept of affine vector systems. In section 5, we give a brief history of infinite products and present the work of Borcherds that expressed modular forms in the Kohnen "plus" space of weight $1/2$ as infinite products. For instance, we write the well-known modular forms like the discriminant function $\Delta(\tau)$, the modular invariant $j(\tau)$ and the Eisenstein series $E_k(\tau)$ ($k \geq 4$, k : even) as infinite products explicitly. In the final section, we discuss the fake monster Lie algebras and Kac-Moody Lie algebras of the arithmetic hyperbolic type defined by V. V. Nikulin ([N5]). As an example, we explain the generalization of Maass correspon-

dence to higher levels which was done by A. J. Feingold and I. B. Frenkel ([F-F]). Finally we also give some open problems which have to be investigated. In the appendix A, we collect some of the well-known properties of classical modular forms. In the appendix B, we briefly discuss the Kohnen “plus” space and the Maass “Spezialschar” which are essential for the understanding of the works in [Bo7] and [F-F]. In the appendix C, we discuss the geometrical aspect of the orthogonal group $O_{s+2,2}(\mathbb{R})$ briefly. In the final appendix, we collect some of the well-known properties of the Leech lattice Λ . Also we briefly discuss the Jacobi theta functions.

Finally I would like to give my deep thanks to TGRC-KOSEF for its financial support on this work. I also would like to give my hearty thanks to my Korean colleagues for their interest in this work.

Notations: We denote by \mathbb{Z} , \mathbb{Q} , \mathbb{R} and \mathbb{C} the ring of integers, the field of rational numbers, the field of real numbers, and the field of complex numbers respectively. \mathbb{Z}^+ and \mathbb{Z}_+ denote the set of all positive integers and the set of nonnegative integers respectively. For a positive integer n , $\Gamma_n := Sp(n, \mathbb{Z})$ denotes the Siegel modular group of degree n . The symbol “:=” means that the expression on the right is the definition of that on the left. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F . For a square matrix $A \in F^{(k,k)}$ of degree k , $\sigma(A)$ denotes the trace of A . For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. For any $M \in F^{(k,l)}$, tM denotes the transpose matrix of M . E_n denotes the identity matrix of degree n . We denote by Λ the Leech lattice. $\Pi_{1,1}$ denotes the unique unimodular even integral Lorentzian lattice of rank 2. We denote by G the MONSTER group. For $g \in G$, $T_g(q)$ denotes the Thompson series of g . M and V^\sharp denote the monster Lie algebra and the moonshine module respectively. $\eta(\tau)$ denotes the Dedekind eta function. $\tau(n)$ denotes the Ramanujan function. Usually ρ denotes the Weyl vector. We denote by $[\Gamma_n, k]$ (resp. $[\Gamma, k]_0$) the complex vector space of all Siegel modular forms (resp. cusp forms) on H_n of weight k with respect to Γ_n . We denote by $[\Gamma_2, k]^M$ the Maass space or the Maass Spezialschar.

2. Kac-Moody Lie Algebras

In this section, we review the basic definitions and properties of Kac-Moody Lie algebras.

An $n \times n$ matrix $A = (a_{ij})$ is called a *generalized Cartan matrix* if it satisfies the following conditions: (i) $a_{ii} = 2$ for $i = 1, \dots, n$; (ii) a_{ij} are nonpositive integers for $i \neq j$; (iii) $a_{ij} = 0$ implies $a_{ji} = 0$. An indecomposable generalized Cartan matrix is said to be *of finite type* if all its principal minors are positive, *of affine type* if all its proper principal minors are positive and $\det A = 0$, and is said to be *of indefinite type* if A is of neither finite type nor affine type. A is said to be *of*

hyperbolic type if it is of indefinite type and all of its proper principal submatrices are of finite type or affine type, and to be of *almost hyperbolic type* if it is of indefinite type and at least one of its proper principal submatrices is of finite or affine type.

A matrix A is called *symmetrizable* if there exists an invertible diagonal matrix $D = \text{diag}(q_1, \dots, q_n)$ with $q_i > 0$, $q_i \in \mathbb{Q}$ such that DA is symmetric. If A is an $n \times n$ matrix of rank l , then a *realization* of A is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, where \mathfrak{h} is a complex vector space of dimension $2n - l$, $\Pi = \{\alpha_1, \dots, \alpha_n\}$ and $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ are linearly independent subsets of \mathfrak{h}^* and \mathfrak{h} respectively such that $\alpha_j(\alpha_i^\vee) = a_{ij}$ for $1 \leq i, j \leq n$.

Definition 2.1. The *Kac-Moody Lie algebra* $\mathfrak{g}(A)$ associated with the generalized Cartan matrix A is the Lie algebra generated by the elements e_i, f_i ($i = 1, 2, \dots, n$) and \mathfrak{h} with the defining relations

$$\begin{aligned} [h, h'] &= 0 && \text{for all } h, h' \in \mathfrak{h}, \\ [e_i, f_j] &= \delta_{ij} \alpha_i^\vee && \text{for } 1 \leq i, j \leq n, \\ [h, e_i] &= \alpha_i(h) e_i, \quad [h, f_i] = -\alpha_i(h) f_i && \text{for } i = 1, 2, \dots, n, \\ (\text{ad } e_i)^{1-a_{ij}}(e_j) &= (\text{ad } f_i)^{1-a_{ij}}(f_j) = 0 && \text{for } i \neq j. \end{aligned}$$

The elements of Π (resp. Π^\vee) are called *simple roots* (resp. *simple coroots*) of $\mathfrak{g}(A)$.

Let $Q := \sum_{i=1}^n \mathbb{Z} \alpha_i$, $Q_+ := \sum_{i=1}^n \mathbb{Z}_+ \alpha_i$ and $Q_- := -Q_+$. Q is called the *root lattice*. For $\alpha := \sum_{i=1}^n k_i \alpha_i \in Q$ the number $\text{ht}(\alpha) := \sum_{i=1}^n k_i$ is called the *height* of α . We define a *partial ordering* \geq on \mathfrak{h}^* by $\lambda \geq \mu$ if $\lambda - \mu \in Q_+$. For each $\alpha \in \mathfrak{h}^*$, we put

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g}(A) \mid [h, X] = \alpha(h)X \text{ for all } h \in \mathfrak{h}\}.$$

If $\mathfrak{g}_\alpha \neq 0$, α is called a *root* and \mathfrak{g}_α is called the *root space* attached to α . The number $\text{mult } \alpha := \dim \mathfrak{g}_\alpha$ is called the *multiplicity* of α . The Kac-Moody Lie algebra $\mathfrak{g}(A)$ has the following root space decomposition with respect to \mathfrak{h} :

$$(2.1) \quad \mathfrak{g}(A) = \sum_{\alpha \in Q} \mathfrak{g}_\alpha \quad (\text{direct sum}).$$

A root α with $\alpha > 0$ (resp. $\alpha < 0$) is called *positive* (resp. *negative*). All roots are either positive or negative. We denote by Δ, Δ^+ and Δ^- the set of all roots, positive roots and negative roots respectively.

Definition 2.2. Let $\mathfrak{g}(A)$ be a symmetrizable Kac-Moody Lie algebra associated with a symmetrizable generalized Cartan matrix $A = (a_{ij})$. A $\mathfrak{g}(A)$ -module V is

\mathfrak{h} -diagonalizable if $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$, where V_μ is the weight space of weight μ given by

$$V_\mu := \{v \in V \mid hv = \mu(h)v \text{ for all } h \in \mathfrak{h}\} \neq 0.$$

The number $\text{mult}_V \mu := \dim V_\mu$ is called the *multiplicity* of weight μ . When all the weight spaces are finite-dimensional, we define the *character* of V by

$$(2.2) \quad \text{ch } V := \sum_{\mu \in \mathfrak{h}^*} (\dim V_\mu) e^\mu = \sum_{\mu \in \mathfrak{h}^*} (\text{mult}_V \mu) e^\mu.$$

An \mathfrak{h} -diagonalizable $\mathfrak{g}(A)$ -module V is said to be *integrable* if all the Chevalley generators $e_i, f_i (i = 1, 2, \dots, n)$ are locally nilpotent on V . A $\mathfrak{g}(A)$ -module V is called a *highest weight module* with highest weight $\Lambda \in \mathfrak{h}^*$ if there exists a nonzero vector $v \in V$ such that (i) $e_i v = 0$ for all $i = 1, 2, \dots, n$; (ii) $hv = \Lambda(h)v$ for all $h \in \mathfrak{h}$; and (iii) $U(\mathfrak{g}(A))v = V$. A vector v is called a *highest weight vector*. Here $U(\mathfrak{g}(A))$ denotes the universal enveloping algebra of $\mathfrak{g}(A)$.

Let \mathfrak{n}_+ (resp. \mathfrak{n}_-) be the subalgebra of $\mathfrak{g}(A)$ generated by e_1, \dots, e_n (resp. f_1, \dots, f_n). Then we have the *triangular decomposition*

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \quad (\text{direct sum of vector spaces}).$$

Let $\mathfrak{b}_+ := \mathfrak{h} + \mathfrak{n}_+$ be the *Borel subalgebra* of $\mathfrak{g}(A)$. For a fixed $\Lambda \in \mathfrak{h}^*$, we let $\mathbb{C}(\Lambda)$ be the one-dimensional \mathfrak{b}_+ -module with the \mathfrak{b}_+ -action defined by

$$\mathfrak{n}_+ \cdot 1 = 0 \quad \text{and} \quad h \cdot 1 = \Lambda(h)1 \quad \text{for all } h \in \mathfrak{h}.$$

The induced module

$$(2.3) \quad M(\Lambda) := U(\mathfrak{g}(A)) \otimes_{U(\mathfrak{b}_+)} \mathbb{C}(\Lambda)$$

is called the *Verma module* with highest weight Λ . It is known that every $\mathfrak{g}(A)$ -module with highest weight Λ is a quotient of $M(\Lambda)$ and $M(\Lambda)$ contains a unique proper maximal submodule $M'(\Lambda)$.

We put

$$(2.4) \quad L(\Lambda) := M(\Lambda)/M'(\Lambda).$$

Then we can show that $L(\Lambda)$ is an irreducible $\mathfrak{g}(A)$ -module.

We set

$$\begin{aligned} P &:= \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ for } i = 1, \dots, n\}, \\ P_+ &:= \{\lambda \in P \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ for } i = 1, \dots, n\}, \\ P_{++} &:= \{\lambda \in P \mid \langle \lambda, \alpha_i^\vee \rangle > 0 \text{ for } i = 1, \dots, n\}. \end{aligned}$$

The set P is called the *weight lattice*. Elements from P (resp. P_+ or P_{++}) are called *integral weights* (resp. *dominant* or *regular dominant integral weights*). We observe that $Q \subset P$ and $P_{++} \subset P_+ \subset P$. If Λ is an element of P_+ , i.e., Λ is a dominant integral weight, then $L(\Lambda)$ is *integrable* (cf. [K], p. 171) and the *Weyl-Kac character formula* for $L(\Lambda)$ is given by

$$(2.5) \quad \text{ch } L(\Lambda) = \frac{\sum_{w \in W} \epsilon(w) e^{w(\Lambda + \rho) - \rho}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\text{mult } \alpha}}$$

Here $\epsilon(w) := (-1)^{\ell(w)} = \det_{\mathfrak{h}^*} w$ for $w \in W$, W the *Weyl group* of $\mathfrak{g}(A)$ and ρ is an element of \mathfrak{h}^* such that $\langle \rho, \alpha_i^\vee \rangle = 1$ for $i = 1, \dots, n$. We recall that $W \subset \text{Aut}(\mathfrak{h}^*)$ is the subgroup generated by the reflections $\sigma_i(\lambda) := \lambda - \lambda(\alpha_i^\vee)\alpha_i$ ($1 \leq i \leq n$).

We set $\Lambda = 0$ in (2.5). Since $L(0)$ is the trivial module over $\mathfrak{g}(A)$, we obtain the following denominator identity or denominator formula:

$$(2.6) \quad \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\text{mult } \alpha} = \sum_{w \in W} \epsilon(w) e^{w(\rho) - \rho}.$$

Substituting (2.6) into (2.5), we obtain another form of the Weyl-Kac character formula:

$$(2.7) \quad \text{ch } L(\Lambda) = \frac{\sum_{w \in W} \epsilon(w) e^{w(\Lambda + \rho)}}{\sum_{w \in W} \epsilon(w) e^{w(\rho)}}.$$

Of course, in the case when $\mathfrak{g}(A)$ is a finite dimensional semisimple Lie algebra, then (2.7) is the classical Weyl character formula and (2.6) is the Weyl denominator identity. We remark that an integrable highest weight module $L(\Lambda)$ over $\mathfrak{g}(A)$ is unitarizable and conversely if $L(\Lambda)$ is irreducible, then $\Lambda \in P_+$ (cf. [K], p.196).

Let $A = (a_{ij})$ be a generalized Cartan matrix. We associate to A a graph $\mathcal{S}(A)$ called the Dynkin diagram of A as follows. If $a_{ij}a_{ji} \leq 4$ and $|a_{ij}| \geq |a_{ji}|$, the vertices i and j are connected by $|a_{ij}|$ lines, and these lines are equipped with an arrow pointing toward i if $|a_{ij}| > 1$. If $a_{ij}a_{ji} > 4$, the vertices i and j are connected by a bold-faced line equipped with an ordered pair of integers $|a_{ij}|, |a_{ji}|$. We list some of hyperbolic Kac-Moody Lie algebras.

$$HA_1^{(1)} : \quad \begin{array}{c} \circ \text{---} \circ \rightleftharpoons \circ \\ \alpha_{-1} \quad \alpha_0 \quad \alpha_1 \end{array}$$

$$HB_l^{(1)}, l \geq 3 : \quad \begin{array}{c} \alpha_1 \\ \circ \\ | \\ \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \dots \text{---} \circ \rightleftharpoons \circ \\ \alpha_{-1} \quad \alpha_0 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \dots \quad \alpha_{l-1} \quad \alpha_l \end{array}$$

$$HC_l^{(1)}, l \geq 2 : \quad \begin{array}{c} \circ \text{---} \circ \rightleftharpoons \circ \text{---} \circ \text{---} \circ \text{---} \dots \text{---} \circ \leftarrow \circ \\ \alpha_{-1} \quad \alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \dots \quad \alpha_{l-1} \quad \alpha_l \end{array}$$

$$HD_l^{(1)}, l \geq 4 : \quad \begin{array}{c} \alpha_1 \\ \circ \\ | \\ \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \dots \text{---} \circ \text{---} \circ \\ \alpha_{-1} \quad \alpha_0 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \dots \quad \alpha_{l-2} \quad \alpha_{l-1} \end{array}$$

$$HF_4^{(1)} : \quad \begin{array}{c} \circ \text{---} \circ \text{---} \circ \text{---} \circ \rightleftharpoons \circ \text{---} \circ \\ \alpha_{-1} \quad \alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \end{array}$$

$$HG_2^{(1)} : \quad \begin{array}{c} \circ \text{---} \circ \text{---} \circ \rightleftharpoons \circ \\ \alpha_{-1} \quad \alpha_0 \quad \alpha_1 \quad \alpha_2 \end{array}$$

$$HE_7^{(1)} : \quad \begin{array}{c} \alpha_7 \\ \circ \\ | \\ \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \\ \alpha_{-1} \quad \alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \end{array}$$

$$HE_8^{(1)} : \quad \begin{array}{c} \alpha_8 \\ \circ \\ | \\ \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \\ \alpha_{-1} \quad \alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6 \quad \alpha_7 \end{array}$$

$$HA_2^{(2)} : \quad \begin{array}{c} \circ \text{---} \circ \leftarrow \circ \\ \alpha_{-1} \quad \alpha_0 \quad \alpha_1 \end{array}$$

$$HA_{2l}^{(2)}, l \geq 2 : \quad \begin{array}{c} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \cdots \text{---} \circ \text{---} \circ \\ \alpha_{-1} \quad \alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \quad \quad \alpha_{l-1} \quad \alpha_l \end{array}$$

$$HA_{2l-1}^{(2)}, l \geq 3 : \quad \begin{array}{c} \quad \quad \quad \alpha_1 \\ \quad \quad \quad \circ \\ \quad \quad \quad | \\ \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \cdots \text{---} \circ \text{---} \circ \\ \alpha_{-1} \quad \alpha_0 \quad \alpha_2 \quad \alpha_3 \quad \quad \quad \alpha_{l-1} \quad \alpha_l \end{array}$$

$$HD_{l+1}^{(2)}, l \geq 2 : \quad \begin{array}{c} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \cdots \text{---} \circ \text{---} \circ \\ \alpha_{-1} \quad \alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \quad \quad \alpha_{l-1} \quad \alpha_l \end{array}$$

$$HE_6^{(2)} : \quad \begin{array}{c} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \\ \alpha_{-1} \quad \alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \end{array}$$

$$HD_4^{(3)} : \quad \begin{array}{c} \circ \text{---} \circ \text{---} \circ \text{---} \circ \\ \alpha_{-1} \quad \alpha_0 \quad \alpha_1 \quad \alpha_2 \end{array}$$

Let $A = (a_{ij})_{i,j=-1,0,\dots,\ell}$ be a hyperbolic generalized Cartan matrix whose corresponding affine submatrix of A is given by $A_0 = (a_{kl})_{k,l=0,1,\dots,\ell}$. We can realize the hyperbolic Kac-Moody Lie algebra $\mathfrak{g}(A)$ as the minimal graded Lie algebra $L = \bigoplus_{n \in \mathbb{Z}} L_n$ with local part $V + \mathfrak{g}(A_0) + V^*$, where $V = L(-\alpha_{-1})$ is the basic representation of the affine Kac-Moody Lie algebra $\mathfrak{g}(A_0)$ and V^* is the contragredient of V . Thus $L = G/I$, and $L_n = G_n/I_n$, where $G = \bigoplus_{n \in \mathbb{Z}} G_n$ is the maximal graded Lie algebra with local part $V + \mathfrak{g}(A_0) + V^*$ and $I = \bigoplus_{n \in \mathbb{Z}} I_n$ is the maximal graded ideal of G intersecting the local part trivially. Each L_n ($n \in \mathbb{Z}$) is a $\mathfrak{g}(A_0)$ -module. (By definition, $G = \bigoplus_{n \in \mathbb{Z}} G_n$ is called a *graded Lie algebra* if G is a Lie algebra and $[G_i, G_j] \subset G_{i+j}$ for all $i, j \in \mathbb{Z}$.) A graded Lie algebra $G = \bigoplus_{n \in \mathbb{Z}} G_n$ is called *irreducible* if the representation ϕ_{-1} of G_0 on G_{-1} defined by $\phi_{-1}(x_0)x_{-1} = [x_0, x_{-1}]$ for all $x_0 \in G_0$ and $x_{-1} \in G_{-1}$ is irreducible. A graded Lie algebra $G = \bigoplus_{n \in \mathbb{Z}} G_n$ is said to be *maximal* (resp. *minimal*) if for any other graded Lie algebra $G' = \bigoplus_{n \in \mathbb{Z}} G'_n$, every isomorphism of the local parts of G and G' can be extended to an epimorphism of G onto G' (resp. of G' onto G). Kac (cf. [K]) proved that for any local Lie algebra $G_{-1} \oplus G_0 \oplus G_1$, there exist unique up to isomorphism maximal and minimal graded Lie algebras whose local parts are isomorphic to a given Lie algebra $G_{-1} \oplus G_0 \oplus G_1$.

Example 2.3. Let

$$A = (a_{ij})_{i,j=-1,0,1} := \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

be the hyperbolic generalized Cartan matrix. We can realize the corresponding hyperbolic Kac-Moody Lie algebra $\mathfrak{g}(A) := HA_1^{(1)}$ as the minimal graded Lie algebra $L = \bigoplus_{n \in \mathbb{Z}} L_n$ with local part $V + \mathfrak{g}(A_0) + V^*$, where $A_0 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ and $V := L(-\alpha_{-1})$ is the basic representation of the affine Kac-Moody Lie algebra $\mathfrak{g}(A_0) := A_1^{(1)}$. The dimensions $\dim(L_{-n})_\alpha$ for $0 \leq n \leq 5$ were computed by A. J. Feingold, I. B. Frenkel, S.-J. Kang and etc. For instance, $\dim(L_0)_\alpha = 1$ and

$$\dim(L_{-1})_\alpha = p \left(1 - \frac{(\alpha, \alpha)}{2} \right),$$

where p is the partition function defined by

$$(2.8) \quad \sum_{n=0}^{\infty} p(n)q^n = \frac{1}{\phi(q)}, \quad \phi(q) := \prod_{n \geq 1} (1 - q^n).$$

Example 2.4. Let

$$A = (a_{ij})_{i,j=-1,0,1} := \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -4 \\ 0 & -1 & 2 \end{pmatrix}$$

be the generalized Cartan matrix of hyperbolic type. We can realize $\mathfrak{g}(A) := HA_2^{(2)}$ as the minimal graded Lie algebra $L = \bigoplus_{n \in \mathbb{Z}} L_n$. The dimensions $\dim L_{-n}$ ($1 \leq n \leq 3$) were computed by A. J. Feingold and S. J. Kang.

Example 2.5. Kac-Moody-Wakimoto (cf.[KMW]) considered the hyperbolic Kac-Moody Lie algebra $HE_8^{(1)} = E_{10}$. Using the modular invariant property of level 2 string functions, they computed root multiplicities of G_{-2} and I_{-2} . Thus they obtained the formula

$$\dim(L_{-2})_\alpha = \xi \left(3 - \frac{(\alpha, \alpha)}{2} \right),$$

where $\xi(n)$ is defined by the relation

$$(2.9) \quad \sum_{n=0}^{\infty} \xi(n)q^n = \frac{1}{\phi(q)^8} \left(1 - \frac{\phi(q^2)}{\phi(q^4)} \right).$$

REMARK 2.6. In [Fr], Frenkel conjectured that for a hyperbolic Kac-Moody Lie algebra \mathfrak{g} , we have

$$\dim \mathfrak{g}_\alpha \leq p^{(\ell-2)} \left(1 - \frac{(\alpha, \alpha)}{2} \right),$$

where ℓ is the size of the generalized Cartan matrix of \mathfrak{g} and the function $p^{(\ell-2)}(n)$ is defined by

$$(2.10) \quad \sum_{n=0}^{\infty} p^{(\ell-2)}(n)q^n = \frac{1}{\phi(q)^{\ell-2}} = \prod_{n \geq 1} (1 - q^n)^{2-\ell}.$$

But this conjecture does not hold for E_{10} (cf. [KMW]). This conjecture is true for $HA_n^{(1)}$. We observe that $HA_n^{(1)}$ is of hyperbolic type for $n \leq 7$ and that $HA_n^{(1)}$ is of almost hyperbolic type for $n \geq 8$.

Appendix. Generalized Kac-Moody Algebras

Let I be a countable index set. A real matrix $A = (a_{ij})_{i,j \in I}$ is called a *Borcherds-Cartan matrix* if it satisfies the following conditions:

- (BC1) $a_{ii} = 2$ or $a_{ii} \leq 0$ for all $i \in I$;
- (BC2) $a_{ij} \leq 0$ if $i \neq j$ and $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$;
- (BC3) $a_{ij} = 0$ implies $a_{ji} = 0$.

Let $I^{re} := \{i \in I \mid a_{ii} = 2\}$ and $I^{im} := \{i \in I \mid a_{ii} \leq 0\}$. Let $\underline{m} = (m_i \mid i \in I)$ be the *charge* of A , i.e., $m_i = 1$ for all $i \in I^{re}$ and $m_j \in \mathbb{Z}^+$ for all $j \in I^{im}$. A Borcherds-Cartan matrix A is said to be *symmetrizable* if there exists a diagonal matrix $D = \text{diag}(\delta_i \mid i \in I)$ with $\delta_i > 0$ ($i \in I$) such that DA is symmetric.

Definition 2.7. The *generalized Kac-Moody algebra* $\mathfrak{g} = \mathfrak{g}(A, \underline{m})$ with a symmetrizable Borcherds-Cartan matrix A of charge $\underline{m} = (m_i \mid i \in I)$ is the complex Lie algebra generated by the elements h_i, d_i ($i \in I$), e_{ik}, f_{ik} ($i \in I, k = 1, \dots, m_i$) with the defining relations:

$$\begin{aligned} [h_i, h_j] &= [h_i, d_j] = [d_i, d_j] = 0, \\ [h_i, e_{j\ell}] &= a_{ij}e_{j\ell}, \quad [h_i, f_{j\ell}] = -a_{ij}f_{j\ell}, \\ [d_i, e_{j\ell}] &= \delta_{ij}e_{j\ell}, \quad [d_i, f_{j\ell}] = -\delta_{ij}f_{j\ell}, \\ [e_{ik}, f_{j\ell}] &= \delta_{ij}\delta_{k\ell}h_i, \\ (ad e_{ik})^{1-a_{ij}}(e_{j\ell}) &= (ad f_{ik})^{1-a_{ij}}(f_{j\ell}) = 0 \quad \text{if } a_{ii} = 2 \text{ and } i \neq j, \\ [e_{ik}, e_{j\ell}] &= [f_{ik}, f_{j\ell}] = 0 \quad \text{if } a_{ii} = 0 \end{aligned}$$

for all $i, j \in I, k = 1, \dots, m_i, \ell = 1, \dots, m_j$.

The subalgebra $\mathfrak{h} := (\sum_{i \in I} \mathbb{C}h_i) \oplus (\sum_{i \in I} \mathbb{C}d_i)$ is called the *Cartan subalgebra* of \mathfrak{g} . For each $j \in I$, we define a *linear functional* $\alpha_j \in \mathfrak{h}^*$ by

$$\alpha_j(h_i) = a_{ij}, \quad \alpha_j(d_i) = \delta_{ij} \quad \text{for all } i, j \in I.$$

Let $\Pi := \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ and $\Pi^\vee := \{h_i \mid i \in I\} \subset \mathfrak{h}$. The elements of Π (resp. Π^\vee) are called the *simple roots* (resp. *simple coroots*) of \mathfrak{g} . We set

$$Q := \sum_{i \in I} \mathbb{Z}\alpha_i, \quad Q^+ := \sum_{i \in I} \mathbb{Z}_+\alpha_i, \quad Q^- := -Q^+.$$

Q is called the *root lattice* of \mathfrak{g} . We define a partial ordering \leq on \mathfrak{h}^* by $\lambda \leq \mu$ if $\mu - \lambda \in Q^+$. For $\alpha \in \mathfrak{h}^*$, we put

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid [h, X] = \alpha(h)X \text{ for all } h \in \mathfrak{h}\}.$$

If $\mathfrak{g}_\alpha \neq 0$ and $\alpha \neq 0$, α is called *root* of \mathfrak{g} and \mathfrak{g}_α is called the *root space attached to the root* α . The generalized Kac-Moody algebra \mathfrak{g} has the root decomposition

$$(2.11) \quad \mathfrak{g} = \sum_{\alpha \in Q} \mathfrak{g}_\alpha \quad (\text{direct sum}).$$

We observe that $\mathfrak{g}_{\alpha_i} = \sum_{k=1}^{m_i} \mathbb{C}e_{i,k}$ and $\mathfrak{g}_{-\alpha_i} = \sum_{k=1}^{m_i} \mathbb{C}f_{i,k}$. The number $\text{mult } \alpha := \dim \mathfrak{g}_\alpha$ is called the *multiplicity* of α . A root α with $\alpha > 0$ (with $\alpha < 0$) is said to be *positive* (resp. *negative*). We denote by $\Delta, \Delta^+, \Delta^-$ the set of all roots, positive roots, and negative roots respectively. We set

$$(2.12) \quad \mathfrak{n}^+ := \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- := \sum_{\alpha \in \Delta^-} \mathfrak{g}_\alpha.$$

Then we have the triangular decomposition: $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$.

Since A is symmetrizable, there exists a symmetric linear form (\mid) on \mathfrak{h}^* satisfying the condition $(\alpha_i \mid \alpha_j) = \delta_{ij}a_{ij}$ for all $i, j \in I$. We say that a root α is *real* if $(\alpha \mid \alpha) > 0$ and *imaginary* if $(\alpha \mid \alpha) \leq 0$. In particular, the simple root α_i is real if $a_{ii} = 2$ and imaginary if $a_{ii} \leq 0$. We note that the imaginary simple roots may have multiplicity > 1 . For each $i \in I^{re}$, we let $\sigma_i \in \text{Aut}(\mathfrak{h}^*)$ be the *simple reflection* on \mathfrak{h}^* defined by $\sigma_i(\lambda) := \lambda - \lambda(h_i)\alpha_i$ for $\lambda \in \mathfrak{h}^*$. The subgroup W of $\text{Aut}(\mathfrak{h}^*)$ generated by the σ_i 's ($i \in I^{re}$) is called the *Weyl group* of \mathfrak{g} .

Let

$$P_G^+ := \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \geq 0 \text{ for all } i \in I, \lambda(h_i) \in \mathbb{Z}_+ \text{ if } a_{ii} = 2\}.$$

For $\lambda \in P_G^+$, we let $V(\lambda)$ be the irreducible highest weight module over \mathfrak{g} with highest weight λ . We denote by T the set of all imaginary simple roots counted with multiplicities. We choose $\rho \in \mathfrak{h}^*$ such that $\rho(h_i) = \frac{1}{2}a_{ii}$ for all $i \in I$. Then we have the *Weyl-Kac-Borcherds character formula* [Bo1] :

$$(2.13) \quad \text{ch } V(\lambda) = \frac{\sum_{w \in W} \sum_{F \subset T, F \perp \lambda} (-1)^{\ell(w) + |F|} e^{w(\lambda + \rho - s(F)) - \rho}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\text{mult } \alpha}},$$

where F runs over all the finite subsets of T such that any two distinct elements of F are mutually orthogonal, $\ell(w)$ denotes the length of $w \in W$, $|F| := \text{Card}(F)$ and $s(F)$ denotes the sum of elements in F . For $\lambda = 0$, we obtain the *denominator identity*:

$$(2.14) \quad \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\text{mult } \alpha} = \sum_{\substack{w \in W \\ F \subset T}} (-1)^{\ell(w) + |F|} e^{w(\rho - s(F)) - \rho}.$$

REMARK 2.8. The notion of a generalized Kac-Moody algebra was introduced by Borcherds in his study of the vertex algebras and the moonshine conjecture [Bo1-3]. As mentioned above, the structure and the representation theory of generalized Kac-Moody algebras are very similar to those of Kac-Moody algebras. The main difference is that a generalized Kac-Moody algebra may have imaginary *simple* roots.

3. The Moonshine Conjectures and The Monster Lie Algebra

In this section, we give a construction of the *Monster Lie algebra* M and a sketchy proof of the *Moonshine Conjectures* due to R. E. Borcherds [Bo6].

The *Fischer-Griess monster sporadic simple group* G , briefly the MONSTER, is the largest among the 26 sporadic finite simple groups of order

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71.$$

It is known that the dimension of the smallest nontrivial irreducible representation of the MONSTER is 196883 ([FLT]). It was observed by John McKay that $1 + 196883 = 196884$, which is the first nontrivial coefficient of the elliptic modular function $j_*(q) := j(q) - 744$, where $j(q)$ is the *modular invariant*:

$$(3.1) \quad j_*(q) = \sum_{n \geq -1} c(n)q^n = q^{-1} + 196884q + 21493760q^2 + \cdots.$$

Later J.G. Thompson [Th2] found that the early coefficients of the elliptic modular function $j_*(q)$ are simple linear combinations of the irreducible character degrees of

G . Motivated by these observations, J. H. Conway and S. Norton [C-N] conjectured that there is an infinite dimensional graded representation $V^\sharp = \sum_{n \geq -1} V_n^\sharp$ of the MONSTER G with $\dim V_n^\sharp = c(n)$ such that for any element $g \in G$, the Thompson series

$$(3.2) \quad T_g(q) := \sum_{n \geq -1} \operatorname{tr}(g|_{V_n^\sharp}) q^n, \quad c_g(n) := \operatorname{tr}(g|_{V_n^\sharp})$$

is a *Hauptmodul* for a genus 0 discrete subgroup of $SL(2, \mathbb{R})$. It is known that there are 194 conjugacy classes of the MONSTER G . Only 171 of the Thompson series $T_g(q)$, $g \in G$ are distinct. Conway reports on this strange and remarkable phenomenon as follows: “Because these new links are still completely unexplained, we refer to them collectively as the ‘moonshine’ properties of the MONSTER, intending the word to convey our feelings that they are seen in a dim light, and that the whole subject is rather vaguely illicit!”. Therefore the above-mentioned conjectures had been called the *moonshine conjectures*. Recently these conjectures were proved to be true by Borcherds [Bo6] by constructing the so-called *monster Lie algebra*. In his proof, he made use of the *natural* graded representation $V^\sharp := \sum_{n \geq -1} V_n^\sharp$ of the MONSTER G , called the *moonshine module* or the *Monster vertex algebra*, which was constructed by I.B. Frenkel, J. Lepowsky and A. Meurman [FLM]. (The vector space V^\sharp and V_n^\sharp are denoted by V^\natural and V_{-n}^\natural respectively in [FLM].) The *graded dimension* $\dim_* V^\sharp := \sum_{n \geq -1} (\dim V_n^\sharp) q^n$ of the moonshine module V^\sharp is given by $\dim_* V^\sharp = T_1(q) = j(q) - 744$.

Let $\Pi_{1,1} \cong \mathbb{Z}^2$ be the 2-dimensional even Lorentzian lattice with the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. The *Monster Lie algebra* M constructed by Borcherds has the following properties:

(M1) M is a \mathbb{Z}^2 -graded generalized Kac-Moody Lie algebra with Borcherds-Cartan matrix $A = (-(i+j))_{i,j \in I}$ of charge $\underline{m} = ((c(i)| i \in I)$, where $I = \{-1\} \cup \{i | i \geq 1\}$. The root lattice of M is $\Pi_{1,1} \cong \mathbb{Z}^2$.

(M2) M is a \mathbb{Z}^2 -graded representation of the MONSTER G such that $M_{(0,0)} \cong \mathbb{R}^2$ and $M_{(m,n)} \cong V_{mn}^\sharp$ for all $(m,n) \neq (0,0)$. In particular, $\dim M_{(m,n)} = \dim V_{mn}^\sharp = c(mn)$ for all $(m,n) \neq (0,0)$.

(M3) The only *real simple* root of M is $(1, -1)$ and the *imaginary simple* roots of M are of the form $(1, i)$ for $i \geq 1$ with multiplicity $c(i)$.

(M4) $\operatorname{tr}(g|_{M_{(m,n)}}) = \operatorname{tr}(g|_{V_{mn}^\sharp}) = c_g(mn)$ for all $g \in G$ and $(m,n) \neq (0,0)$.

(M5) M has a *contravariant* bilinear form $(,)_0$ which is positive definite on the piece $M_{(m,n)}$ of degree $(m,n) \neq (0,0)$. (By a *contravariant* bilinear form we mean that there is an involution σ on M such that $(u, v) := -(u, \sigma(v))_0$ is invariant and $(u, v) = 0$ if $\deg(u) + \deg(v) \neq 0$.)

We denote by $e_{-1} := e_{1,-1}$, $e_{i,k(i)}$ and $f_{-1} := f_{-1,1}$, $f_{i,k(i)}$ ($i \in I$, $k(i) = 1, 2, \dots, c(i)$) the positive and negative simple root vectors of M respectively. Then we have

$$\begin{aligned} M_{(1,-1)} &= \mathbb{C}e_{-1}, & M_{(-1,1)} &= \mathbb{C}f_{-1}, \\ M_{(1,i)} &= \mathbb{C}e_{i,1} \oplus \mathbb{C}e_{i,2} \oplus \dots \oplus \mathbb{C}e_{i,c(i)}, \\ M_{(-1,-i)} &= \mathbb{C}f_{i,1} \oplus \mathbb{C}f_{i,2} \oplus \dots \oplus \mathbb{C}f_{i,c(i)} \quad (i \geq 1). \end{aligned}$$

Consider a basis of $M_{(1,i)}$ consisting of the eigenvectors $v_{i,k(i)}(g)$ of an element $g \in G$ with eigenvalues $\lambda_{i,k(i)}(g)$, where $k(i) = 1, 2, \dots, c(i)$. Since $M_{(1,i)} \cong V_i$ ($i \geq 1$) as G -modules, we have

$$(3.3) \quad \sum_{k(i)=1}^{c(i)} \lambda_{i,k(i)}(g) = \text{tr}(g|_{M_{(1,i)}}) = \text{tr}(g|_{V_i^\#}) = c_g(i)$$

for all $g \in G$ and $i \geq 1$. In addition, since $M_{(1,-1)} \cong M_{(-1,1)} \cong V_{-1}^\# \cong \mathbb{R}^2$ is the trivial G -module, we have $g \cdot e_{-1} = e_{-1}$, $g \cdot f_{-1} = f_{-1}$ for all $g \in G$.

For small degrees M looks like Fig. 1.

$$\begin{array}{cccccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 & 0 & V_3^\# & V_6^\# & V_9^\# & \cdots \\ \cdots & 0 & 0 & 0 & 0 & V_2^\# & V_4^\# & V_6^\# & \cdots \\ \cdots & 0 & 0 & V_{-1}^\# & 0 & V_1^\# & V_2^\# & V_3^\# & \cdots \\ \cdots & 0 & 0 & 0 & \mathbb{R}^2 & 0 & 0 & 0 & \cdots \\ \cdots & V_3^\# & V_2^\# & V_1^\# & 0 & V_{-1}^\# & 0 & 0 & \cdots \\ \cdots & V_6^\# & V_4^\# & V_2^\# & 0 & 0 & 0 & 0 & \cdots \\ \cdots & V_9^\# & V_6^\# & V_3^\# & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

Fig. 1

Now we give a construction of the Monster Lie algebra. First of all we define the notion of vertex algebras.

Definition 3.1. A *vertex algebra* V over \mathbb{R} is a real vector space with an infinite number of bilinear products, written as $u_n v$ for $u, v \in V$, $n \in \mathbb{Z}$, such that

(V1) $u_n v = 0$ for n sufficiently large (depending on u and v),

(V2) for all $u, v, w \in V$ and $m, n, q \in \mathbb{Z}$, we have

$$\sum_{i \in \mathbb{Z}} \binom{m}{i} (u_{q+i}v)_{m+n-i}w = \sum_{i \in \mathbb{Z}} (-1)^i \binom{q}{i} (u_{m+q-i}(v_{n+i}w) - (-1)^q v_{n+q-i}(u_{m+i}w)),$$

(V3) there is an element $1 \in V$ such that $v_n 1 = 0$ if $n \geq 0$ and $v_{-1} 1 = v$.

Example 3.2. (1) For each even lattice L , there is a vertex algebra V_L associated with L constructed by Borcherds [Bo1]. Let \hat{L} be the central extension of L by \mathbb{Z}_2 , i.e., the double cover of L . The underlying vector space of the vertex algebra V_L is given by $V_L = \mathbb{Q}(\hat{L}) \otimes S(\oplus_{i>0} L_i)$, where $\mathbb{Q}(\hat{L})$ is the twisted group ring of \hat{L} and $S(\oplus_{i>0} L_i)$ is the ring of polynomials over the sum of an infinite number of copies L_i of $L \otimes \mathbb{R}$.

(2) Let V be a commutative algebra over \mathbb{R} with derivation D . Then V becomes a vertex algebra by defining

$$u_n v := \begin{cases} \frac{D^{-n-1}(u)v}{(-n-1)!} & \text{for } n < 0 \\ 0 & \text{for } n \geq 0. \end{cases}$$

Conversely any vertex algebra over \mathbb{R} for which $u_n v = 0$ for $n \geq 0$ arises from a commutative algebra in this way.

(3) Let V and W be two vertex algebras. Then the tensor product $V \otimes W$ as vector spaces becomes a vertex algebra if we define the multiplication by

$$(a \otimes b)_n (c \otimes d) := \sum_{i \in \mathbb{Z}} (a_i c) \otimes (b_{n-1-i} d), \quad n \in \mathbb{Z}.$$

We note that the identity element of $V \otimes W$ is given by $1_V \otimes 1_W$.

(4) The moonshine module V^\sharp is a vertex algebra.

Definition 3.3. Let V be a vertex algebra over \mathbb{R} . A *conformal vector* of *dimension* or *central charge* $c \in \mathbb{R}$ of V is defined to be an element ω of V satisfying the following three conditions:

- (1) $\omega_0 v = D(v)$ for all $v \in V$;
- (2) $\omega_1 \omega = 2\omega$, $\omega_3 \omega = c/2$, $\omega_i \omega = 0$ for $i = 2$ or $i > 3$;
- (3) any element of V is a sum of eigenvectors of the operator $L_0 := \omega_1$ with integral eigenvalues.

Here D is the operator of V defined by $D(v) := v_{-2} 1$ for all $v \in V$. If v is an eigenvector of L_0 , then its eigenvalue $\lambda(v)$ is called the *conformal weight* of v and v is called a *conformal vector* of conformal weight $\lambda(v)$. If vertex algebras V and W have conformal vectors ω_V and ω_W of dimension m and n respectively, then

$\omega_V \otimes \omega_W$ is a conformal vector of the vertex algebra $V \otimes W$ of dimension $m + n$. It is known that the vertex algebra V associated with any c -dimensional even lattice has a *canonical* conformal vector ω of dimension c . We define the operators L_i ($i \in \mathbb{Z}$) of V by

$$(3.4) \quad L_i := \omega_{i+1}, \quad i \in \mathbb{Z}.$$

These operators satisfy the relations

$$(3.5) \quad [L_i, L_j] = (i - j)L_{i+j} + \binom{i+1}{3} \frac{c}{2} \delta_{i+j,0}, \quad i, j \in \mathbb{Z}$$

and so make V into a module over a *Virasoro algebra* spanned by a central element c and L_i ($i \in \mathbb{Z}$). We observe that the operator L_{-1} is equal to the operator D . We define

$$(3.6) \quad P^i = \{v \in V \mid L_0(v) = \omega_1 v = iv, L_k(v) = 0 \text{ for } k > 0\}, \quad i \in \mathbb{Z}.$$

Then the space $P^1/(DV \cap P^1)$ is a subalgebra of the Lie algebra V/DV , which is equal to P^1/DP^0 for the vertex algebra V_L or for the Monster vertex algebra V^\sharp . Here DV denotes the image of V under D . It is known that the algebra P^1/DP^0 is a generalized Kac-Moody algebra. The structure of a Lie algebra on V/DV is given by the bracket: $[u, v] := u_0 v$ ($u, v \in V$).

The vertex algebra V_L associated with an even lattice L has a real valued symmetric bilinear form $(,)$ such that if u has degree k , the adjoint u_n^* of the operator u_n with respect to $(,)$ is given by

$$(3.7) \quad u_n^* = (-1)^k \sum_{j \geq 0} \frac{L_1^j(\sigma(u))_{2k-j-n-2}}{j!},$$

where σ is the automorphism of V_L defined by

$$(3.8) \quad \sigma(e^w) := (-1)^{(w,w)/2} (e^w)^{-1}$$

for e^w an element of the twisted group ring of L corresponding to the vector $w \in L$. If a vertex algebra has a bilinear form with the above properties, we say that *the bilinear form is compatible with the conformal vector*.

Definition 3.4. A *vertex operator algebra* is defined to be a vertex algebra with a conformal vector ω such that the eigenspaces of the operator $L_0 := \omega_1$ are all finite dimensional and their eigenvalues are all nonnegative integers.

For example, the Monster vertex algebra V^\sharp is a vertex operator algebra whose conformal vector spans the subspace V_1^\sharp fixed by the MONSTER G . The vertex algebra $V_{\Pi_{1,1}}$ associated with the 2-dimensional even unimodular Lorentzian lattice $\Pi_{1,1}$ is *not* a vertex operator algebra.

We recall the properties of the Monster vertex algebra V^\sharp .

(V^\sharp 1) V^\sharp is a vertex algebra over \mathbb{R} with conformal vector ω of dimension 24 and a positive definite symmetric bilinear form $(,)$ such that the adjoint of u_n is given by the expression (3.7), where σ is the trivial automorphism of V^\sharp .

(V^\sharp 2) V^\sharp is a sum of eigenspaces V_n^\sharp of the operator $L_0 := \omega_1$, where V_n^\sharp is the eigenspace on which L_0 has eigenvalue $n + 1$ and $\dim V_n^\sharp = c(n)$. Thus V^\sharp is a vertex operator algebra in the sense of Definition 3.4.

(V^\sharp 3) The MONSTER G acts faithfully and homogeneously on V^\sharp preserving the vertex algebra structure, the conformal vector ω and the bilinear form $(,)$. The first few representations V_n^\sharp of the MONSTER G are decomposed as

$$\begin{aligned} V_{-1}^\sharp &= \chi_1, & V_0^\sharp &= 0, \\ V_1^\sharp &= \chi_1 + \chi_2, \\ V_2^\sharp &= \chi_1 + \chi_2 + \chi_3, \\ V_3^\sharp &= 2\chi_1 + 2\chi_2 + \chi_3 + \chi_4, \\ V_5^\sharp &= 4\chi_1 + 5\chi_2 + 3\chi_3 + 2\chi_4 + \chi_5 + \chi_6 + \chi_7, \end{aligned}$$

where χ_n ($1 \leq n \leq 7$) are the first seven irreducible representations of G , indexed in order of increasing dimension.

(V^\sharp 4) For $g \in G$, the Thompson series $T_g(q)$ is a *completely replicable function*, i.e., satisfies the identity

$$(3.9) \quad p^{-1} \exp \left(- \sum_{i>0} \sum_{m \in \mathbb{Z}^+, n \in \mathbb{Z}} \text{tr}(g^i|_{V_{mn}^\sharp}) p^{mi} q^{mi} / i \right) = \sum_{m \in \mathbb{Z}} \text{tr}(g|_{V_m^\sharp}) p^m - \sum_{n \in \mathbb{Z}} \text{tr}(g|_{V_n^\sharp}) q^n.$$

We remark that the properties (V^\sharp 1), (V^\sharp 2) and (V^\sharp 3) characterize V^\sharp completely as a graded representation of G .

CONSTRUCTION OF THE MONSTER LIE ALGEBRA M : The tensor product $V := V^\sharp \otimes V_{\Pi_{1,1}}$ of V^\sharp and $V_{\Pi_{1,1}}$ is also a vertex algebra. Then P^1/DP^0 is a Lie algebra with an *invariant* bilinear form $(,)$ and an involution τ . Here P^1 and $D := L_1$ are defined by (3.4) and (3.6), and τ is the involution on V induced by the trivial automorphism of V^\sharp and the involutuon ω of $V_{\Pi_{1,1}}$. Let $R := \{v \in V \mid (u, v) = 0 \text{ for } u \in V\}$ be the radical of $(,)$. It is easy to see

that DP is a proper subset of R . We define M to be the quotient of the Lie algebra P^1/DP^0 by R . The $\Pi_{1,1}$ -grading of $V_{\Pi_{1,1}}$ induces a $\Pi_{1,1}$ -grading on M . According to the *no-ghost theorem*, $M_{(m,n)}$ is isomorphic to the piece V_{mn}^\sharp of degree $1 - (m, n)^2/2 = 1 + mn$ if $(m, n) \neq (0, 0)$ and $M_{(0,0)} \cong V_0^\sharp \oplus \mathbb{R}^2 \cong \mathbb{R}^2$. And if $v \in M$ is nonzero and homogeneous of nonzero degree in $\Pi_{1,1}$, then $(v, \tau(v)) > 0$. M satisfies the properties (M1)-(M5).

REMARK 3.5. The construction of the Monster Lie algebra M from a vertex algebra can be carried out for any vertex algebra with a conformal vector, but it is only when this vector has *dimension* 24 that we can apply the no-ghost theorem to identify the *homogeneous* pieces of M with those of V^\sharp . The important thing is that the bilinear form $(,)$ on M is positive definite on any piece of nonzero degree, and thus need not be true if the conformal vector has dimension greater than 24, even if the inner product is positive definite.

Problem. Are there some other ways to construct the Monster Lie algebra?

SKETCHY PROOF OF THE MOONSHINE CONJECTURES: The proof is divided into two steps.

STEP I. The Thompson series are determined by the first 5 coefficients $c_g(i)$, $1 \leq i \leq 5$ for all $g \in G$ because of the identities (3.9).

STEP II. The Hauptmoduls listed in Conway and Norton [C-N, Table 2] satisfy the identities (3.9) and have the same first 5 coefficients of the Thompson series.

The proof of step I is done by comparing the coefficients of p^2 and p^4 of both sides of the identities (3.9) and so obtaining the recursion formulas among $c_g(i)$. The proof of step II follows from the result of Norton [No1] and Koike [Koi1] that the modular functions associated with elements of the MONSTER G also satisfy the identities (3.9) and hence satisfy the same recursion formulas. Roughly we explain how Conway and Norton [C-N] associate to an element of G a modular function of genus 0. Let g be an element of G corresponding to an element of odd order in $\text{Aut}(\Lambda)$ with Leech lattice Λ such that g fixes no nonzero vectors. Let $\epsilon_1, \dots, \epsilon_{24}$ be eigenvalues of g on the real vector space $\Lambda \otimes \mathbb{R}$. We define

$$(3.10) \quad \eta_g(q) := \eta_g(\epsilon_1 q) \cdots \eta_g(\epsilon_{24} q), \quad q := e^{2\pi i \tau}, \quad \tau \in H_1,$$

where $\eta(q) := q^{1/24} \prod_{n \geq 1} (1 - q^n)$ is the Dedekind eta function and $H_1 := \{z \in \mathbb{C} \mid \Im z > 0\}$ is the Poincaré upper half plane. We put

$$j_g(q) := \frac{1}{\eta_g(q)} - \frac{1}{\eta_g(0)}.$$

Then $j_g(q)$ is the modular function of genus 0. $j_g(q)$ is the modular function associated with an element g of G by Conway and Norton.

Appendix: The No-Ghost Theorem

Here we discuss the NO-GHOST THEOREM. First of all, we describe the concept of a *Virasoro algebra*.

Let $\mathbb{F} = \mathbb{R}$ or \mathbb{C} . Let $\mathbb{F}[t, t^{-1}]$ be the commutative associative algebra of Laurent polynomials in an indeterminate t , i.e., the algebra of finite linear combinations of integral powers of t . Let $p(t) \in \mathbb{F}[t, t^{-1}]$ and we consider the derivation $D_{p(t)}$ of $\mathbb{F}[t, t^{-1}]$ defined by

$$(1) \quad D_{p(t)} := p(t) \frac{d}{dt}.$$

The vector space \mathfrak{a} spanned by all the derivations of type (1) has the Lie algebra structure with respect to the natural Lie bracket

$$(2) \quad [D_p, D_q] = D_{pq' - p'q} \quad \text{for all } p, q \in \mathbb{F}[t, t^{-1}].$$

We choose the following basis $\{d_n \mid n \in \mathbb{Z}\}$ of \mathfrak{a} defined by

$$(3) \quad d_n := -t^{n+1} \frac{d}{dt}, \quad n \in \mathbb{Z}.$$

By (2), we have the commutation relation

$$(4) \quad [d_m, d_n] = (m - n)d_{m+n}, \quad m, n \in \mathbb{Z}.$$

It is easily seen that \mathfrak{a} is precisely the Lie algebra consisting of all derivations of $\mathbb{F}[t, t^{-1}]$.

Now we consider the one-dimensional central extension \mathfrak{b} of \mathfrak{a} by $\mathbb{F}c$ with basis consisting of a central element c and elements $L_n, n \in \mathbb{Z}$, corresponding to the basis $d_n, n \in \mathbb{Z}$, of \mathfrak{a} . We define the bilinear map $[\cdot, \cdot]_* : \mathfrak{b} \times \mathfrak{b} \longrightarrow \mathfrak{b}$ by

$$[c, \mathfrak{b}]_* = [\mathfrak{b}, c]_* = [c, c]_* = 0$$

and

$$(5) \quad [L_m, L_n]_* = (m - n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c$$

for all $m, n \in \mathbb{Z}$. Then $[\cdot, \cdot]_*$ is anti-symmetric and satisfy the Jacobi identity. Thus $(\mathfrak{b}, [\cdot, \cdot]_*)$ has the Lie algebra structure. The Lie algebra \mathfrak{b} is called a *Virasoro algebra*. It is not difficult to see that the extension \mathfrak{b} of the Lie algebra \mathfrak{a} is the unique nontrivial one-dimensional central extension up to isomorphism.

Now we state the *no-ghost theorem* and give its sketchy proof.

The No-Ghost Theorem. Let V be a vertex algebra with a nondegenerate bilinear form $(,)_V$. Suppose that V is acted on by a Virasoro algebra \mathfrak{b} in such a way that the adjoint of L_k with respect to $(,)_V$ is L_{-k} ($k \in \mathbb{Z}$), the central element of \mathfrak{b} acts as multiplication by 24, any vector of V is a sum of eigenvectors of L_0 with nonnegative integral eigenvalues, and all the eigenspaces of L_0 are finite dimensional. We let $V^k := \{v \in V \mid L_0(v) = kv\}$ ($k \in \mathbb{Z}_+$) be the k -eigenspace of L_0 . Assume that V is acted on by a group G which preserves all this structure. Let $V_{\Pi_{1,1}}$ be the vertex algebra associated with the two-dimensional even unimodular Lorentzian lattice $\Pi_{1,1}$ so that $V_{\Pi_{1,1}}$ is $\Pi_{1,1}$ -graded, has a bilinear form $(,)_{1,1}$ and is acted on by the Virasoro algebra \mathfrak{b} as mentioned in this section. We let

$$P^1 := \{v \in V \otimes V_{\Pi_{1,1}} \mid L_0(v) = v, L_k(v) = 0 \text{ for all } k > 0\}$$

and let P_r^1 be the subspace of P^1 of degree $r \in \Pi_{1,1}$. All these spaces inherit an action of G from the action of G on V and the trivial action of G on $V_{\Pi_{1,1}}$. Let $(,) := (,)_V \otimes (,)_{1,1}$ be the tensor product of $(,)_V$ and $(,)_{1,1}$ and let

$$R := \{v \in V \otimes V_{\Pi_{1,1}} \mid (u, v) = 0 \text{ for all } u \in V \otimes V_{\Pi_{1,1}}\}$$

be the null space of $(,)$. Then as G -modules with an invariant bilinear form,

$$P_r^1/R \cong \begin{cases} V^{1-(r,r)/2}, & \text{for } r \neq 0 \\ V^1 \oplus \mathbb{R}^2, & \text{for } r = 0. \end{cases}$$

REMARK. (1) The name “no-ghost theorem” comes from the fact that in the original statement of the theorem in [G-T], V was a part of the underlying vector space of the vertex algebra associated with a positive definite lattice so that the inner product on V^i was positive definite, and thus P_r^1 had no *ghosts*, i.e., *vectors of negative norm* for $r \neq 0$.

(2) If we take the moonshine module V^\sharp as V , then V_n^\sharp corresponds to V^{n+1} for all $n \in \mathbb{Z}$.

A SKETCHY PROOF: Fix a certain nonzero vector $r \in \Pi_{1,1}$ and some norm 0 vector $w \in \Pi_{1,1}$ with $(r, w) \neq 0$. We have an action of the Virasoro algebra on $V \otimes V_{\Pi_{1,1}}$ generated by its conformal vector. The operators L_m of the Virasoro algebra \mathfrak{b} satisfy the relations

$$(6) \quad [L_m, L_n]_* = (m - n)L_{m+n} + \frac{26}{12}(m^3 - m)\delta_{m+n,0}, \quad m, n \in \mathbb{Z},$$

and the adjoint of L_m is L_{-m} . Here 26 comes from the fact that the central element c acts on V as multiplication by 24 and the dimension of $\Pi_{1,1}$ is two. We define the operators K_m , $m \in \mathbb{Z}$ by

$$(7) \quad K_m := v_{m-1},$$

where $v := e_{-2}^{-w} e^w$ is an element of the vertex algebra $V_{\Pi_{1,1}}$ and e^w is an element of the group ring of the double cover of $\Pi_{1,1}$ corresponding to $w \in \Pi_{1,1}$ and e^{-w} is its inverse. Then these operators satisfy the relations

$$(8) \quad [L_m, K_n]_* = -nK_{m+n}, \quad [K_m, K_n]_* = 0$$

for all $m, n \in \mathbb{Z}$. (8) follows from the fact that the adjoint of K_m is K_{-m} and $(w, w) = 0$.

Now we define the subspaces T^1 and Ve^r of $V \otimes V_{\Pi_{1,1}}$ by

$$T^1 := \{ v \in V \otimes V_{\Pi_{1,1}} \mid \deg(v) = r, L_0(v) = v, L_m(v) = K_m(v) = 0 \text{ for all } m > 0 \}$$

and $Ve^r := V \otimes e^r$. Then we can prove that

$$(9) \quad T^1 \cong V^{1-(r,r)/2} e^r \quad \text{and} \quad T^1 \cong P^1/R.$$

We leave the proof of (9) to the reader. Consequently we have the desired result

$$P^1/R \cong V^{1-(r,r)/2} e^r \cong V^{1-(r,r)/2}.$$

For the case $r = 0$, we leave the detail to the reader.

Finally we remark that in [FI] Frenkel uses the no-ghost theorem to prove some results about Kac-Moody algebras.

4. Jacobi Forms

In this section, we discuss Jacobi forms associated to the symplectic group $Sp(g, \mathbb{R})$ and those associated to the orthogonal group $O_{s+2,2}(\mathbb{R})$ respectively. We also discuss the differences between them.

I. Jacobi forms associated to $Sp(g, \mathbb{R})$.

An exposition of the theory of Jacobi forms associated to the symplectic group $Sp(g, \mathbb{R})$ can be found in [E-Z], [Y1]-[Y4] and [Zi].

In this subsection, we establish the notations and define the concept of Jacobi forms associated to the symplectic group. For any positive integer $g \in \mathbb{Z}^+$, we let

$$Sp(g, \mathbb{R}) = \{ M \in \mathbb{R}^{(2g,2g)} \mid {}^t M J_g M = J_g \}$$

be the symplectic group of degree g , where

$$J_g := \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}.$$

It is easy to see that $Sp(g, \mathbb{R})$ acts on H_g transitively by

$$M < Z > := (AZ + B)(CZ + D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbb{R})$ and $Z \in H_g$. For two positive integers g and h , we consider the *Heisenberg group*

$$H_{\mathbb{R}}^{(g,h)} := \{[(\lambda, \mu), \kappa] \mid \lambda, \mu \in \mathbb{R}^{(h,g)}, \kappa \in \mathbb{R}^{(h,h)}, \kappa + \mu^t \lambda \text{ symmetric}\}$$

endowed with the following multiplication law

$$[(\lambda, \mu), \kappa] \circ [(\lambda', \mu'), \kappa'] := [(\lambda + \lambda', \mu + \mu'), \kappa + \kappa' + \lambda^t \mu' - \mu^t \lambda'].$$

We define the semidirect product of $Sp(g, \mathbb{R})$ and $H_{\mathbb{R}}^{(g,h)}$

$$G^J := Sp(g, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$$

endowed with the following multiplication law

$$\begin{aligned} & (M, [(\lambda, \mu), \kappa]) \cdot (M', [(\lambda', \mu'), \kappa']) \\ & := (MM', [(\tilde{\lambda} + \lambda', \tilde{\mu} + \mu'), \kappa + \kappa' + \tilde{\lambda}^t \mu' - \tilde{\mu}^t \lambda']), \end{aligned}$$

with $M, M' \in Sp(g, \mathbb{R})$ and $(\tilde{\lambda}, \tilde{\mu}) := (\lambda, \mu)M'$. It is easy to see that G^J acts on $H_g \times \mathbb{C}^{(h,g)}$ transitively by

$$(4.1) \quad (M, [(\lambda, \mu), \kappa]) \cdot (Z, W) := (M < Z >, (W + \lambda Z + \mu)(CZ + D)^{-1}),$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbb{R})$, $[(\lambda, \mu), \kappa] \in H_{\mathbb{R}}^{(g,h)}$ and $(Z, W) \in H_g \times \mathbb{C}^{(h,g)}$.

Let ρ be a rational representation of $GL(g, \mathbb{C})$ on a finite dimensional complex vector space V_ρ . Let $\mathcal{M} \in \mathbb{R}^{(h,h)}$ be a symmetric half-integral semipositive definite matrix of degree h . Let $C^\infty(H_g \times \mathbb{C}^{(h,g)}, V_\rho)$ be the algebra of all C^∞ functions on $H_g \times \mathbb{C}^{(h,g)}$ with values in V_ρ . For $f \in C^\infty(H_g \times \mathbb{C}^{(h,g)}, V_\rho)$, we define

$$\begin{aligned} & (f|_{\rho, \mathcal{M}}[(M, [(\lambda, \mu), \kappa])])(Z, W) \\ (4.2) \quad & := e^{-2\pi i \sigma(\mathcal{M}[W + \lambda Z + \mu](CZ + D)^{-1} C)} \cdot e^{2\pi i \sigma(\mathcal{M}(\lambda Z^t \lambda + 2\lambda^t W + (\kappa + \mu^t \lambda)))} \\ & \times \rho(CZ + D)^{-1} f(M < Z >, (W + \lambda Z + \mu)(CZ + D)^{-1}), \end{aligned}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbb{R})$ and $[(\lambda, \mu), \kappa] \in H_{\mathbb{R}}^{(g,h)}$.

Definition 4.1. Let ρ and \mathcal{M} be as above. Let

$$H_{\mathbb{Z}}^{(g,h)} := \{[(\lambda, \mu), \kappa] \in H_{\mathbb{R}}^{(g,h)} \mid \lambda, \mu \in \mathbb{Z}^{(h,g)}, \kappa \in \mathbb{Z}^{(h,h)}\}.$$

A *Jacobi form* of index \mathcal{M} with respect to ρ on Γ_g is a holomorphic function $f \in C^\infty(H_g \times \mathbb{C}^{(h,g)}, V_\rho)$ satisfying the following conditions (A) and (B):

- (A) $f|_{\rho, \mathcal{M}}[\tilde{\gamma}] = f$ for all $\tilde{\gamma} \in \Gamma_g^J := \Gamma_g \ltimes H_{\mathbb{Z}}^{(g,h)}$.
- (B) f has a Fourier expansion of the following form

$$f(Z, W) = \sum_{\substack{T \geq 0 \\ \text{half-integral}}} \sum_{R \in \mathbb{Z}^{(g,h)}} c(T, R) \cdot e^{2\pi i \sigma(TZ)} \cdot e^{2\pi i \sigma(RW)}$$

with $c(T, R) \neq 0$ only if $\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} \geq 0$.

Moreover if $c(T, R) \neq 0$ implies $\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} > 0$, f is called a *cuspidal Jacobi form*.

If $g \geq 2$, the condition (B) is superfluous by the Koecher principle (cf. [Zi], Lemma 1.6). We denote by $J_{\rho, \mathcal{M}}(\Gamma_g)$ the vector space of all Jacobi forms of index \mathcal{M} with respect to ρ on Γ_g . In the special case $V_\rho = \mathbb{C}$, $\rho(A) = (\det A)^k$ ($k \in \mathbb{Z}$, $A \in GL(g, \mathbb{C})$), we write $J_{k, \mathcal{M}}(\Gamma_g)$ instead of $J_{\rho, \mathcal{M}}(\Gamma_g)$ and call k the *weight* of a Jacobi form $f \in J_{k, \mathcal{M}}(\Gamma_g)$.

Ziegler (cf. [Zi], Theorem 1.8 or [E-Z], Theorem 1.1) proves that the vector space $J_{\rho, \mathcal{M}}(\Gamma_g)$ is finite dimensional.

Definition 4.2. A Jacobi form $f \in J_{\rho, \mathcal{M}}(\Gamma_g)$ is said to be *singular* if it admits a Fourier expansion such that a Fourier coefficient $c(T, R)$ vanishes unless $\det \begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} = 0$.

Example 4.3. Let $S \in \mathbb{Z}^{(2k, 2k)}$ be a symmetric, positive definite, unimodular even integral matrix and $c \in \mathbb{Z}^{(2k, h)}$. We define the theta series

$$(4.3) \quad \vartheta_{S, c}^{(g)}(Z, W) := \sum_{\lambda \in \mathbb{Z}^{(2k, g)}} e^{\pi \{ \sigma(S\lambda Z^t \lambda) + 2\sigma({}^t c S \lambda^t W) \}}, \quad Z \in H_g, W \in \mathbb{C}^{(h, g)}.$$

We put $\mathcal{M} := \frac{1}{2}{}^t c S c$. We assume that $2k < g + \text{rank}(\mathcal{M})$. Then it is easy to see that $\vartheta_{S, c}^{(g)}$ is a singular Jacobi form in $J_{k, \mathcal{M}}(\Gamma_g)$ (cf. [Zi], p. 212).

REMARK 4.4. Without loss of generality, we may assume that \mathcal{M} is a *positive definite symmetric*, half-integral matrix of degree h (cf. [Zi], Theorem 2.4).

From now on, throughout this paper \mathcal{M} is assumed to be positive definite.

Definition 4.5. An irreducible finite dimensional representation ρ of $GL(g, \mathbb{C})$ is determined uniquely by its highest weight $(\lambda_1, \lambda_2, \dots, \lambda_g) \in \mathbb{Z}^g$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_g$. We denote this representation by $\rho = (\lambda_1, \lambda_2, \dots, \lambda_g)$. The number $k(\rho) := \lambda_g$ is called the *weight* of ρ .

The author (cf. [Y3]) proved that singular Jacobi forms in $J_{\rho, \mathcal{M}}(\Gamma_g)$ are characterized by their singular weights.

Theorem 4.6 (Yang [Y3]). *Let $2\mathcal{M}$ be a symmetric, positive definite even integral matrix of degree h . Assume that ρ is an irreducible representation of $GL(g, \mathbb{C})$. Then a nonvanishing Jacobi form in $J_{\rho, \mathcal{M}}(\Gamma_g)$ is singular if and only if $2k(\rho) < g + h$. Only the nonnegative integers k with $0 \leq k \leq \frac{g+h}{2}$ can be the weights of singular Jacobi forms in $J_{k, \mathcal{M}}(\Gamma_g)$. These integers are called singular weights in $J_{k, \mathcal{M}}(\Gamma_g)$.*

Proof. The proof can be found in [Y3], Theorem 4.5. \square

II. Jacobi forms associated to $O_{s+2,2}(\mathbb{R})$

An exposition of the theory of Jacobi forms associated to the orthogonal group can be found in [Bo7], [G1] and [G2].

First we fix a positive integer s . We let L_0 be a positive definite even integral lattice with a quadratic form Q_0 and let $\Pi_{1,1}$ be the nonsingular even integral lattice with its associated symmetric matrix $I_2 := \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. We define the following lattices L_1 and M by

$$(4.4) \quad L_1 := L_0 \oplus \Pi_{1,1} \quad \text{and} \quad M := \Pi_{1,1} \oplus L_1.$$

Then L_1 and M are nonsingular even integral lattices of $(s+1, 1)$ and $(s+2, 2)$ respectively. From now on we denote by Q_0, Q_1, Q_M (resp. $(\cdot, \cdot)_0, (\cdot, \cdot)_1, (\cdot, \cdot)_M$) the quadratic forms (resp. the nondegenerate symmetric bilinear forms) associated with the lattices L_0, L_1, M respectively. We also denote by $S_0, S_1,$ and S_M the nonsingular symmetric even integral matrices associated with the lattices L_0, L_1 and M respectively. Thus S_1 and S_M are given by

$$(4.5) \quad S_1 := \begin{pmatrix} 0 & 0 & -1 \\ 0 & S_0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S_M := \begin{pmatrix} 0 & 0 & I_2 \\ 0 & S_0 & 0 \\ I_2 & 0 & 0 \end{pmatrix},$$

where $I_2 := \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. We let $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{C}} := M \otimes_{\mathbb{Z}} \mathbb{C}$ be the quadratic spaces over \mathbb{R} and \mathbb{C} respectively. We let

$$(4.6) \quad O(M_{\mathbb{R}}, S_M) := \{g \in GL(M_{\mathbb{R}}) \mid {}^t g S_M g = S_M\}$$

be the real orthogonal group of the quadratic space $(M_{\mathbb{R}}, Q_M)$. We denote by O_M the isometry group of the lattice (M, Q_M) . Then O_M is an algebraic group defined over \mathbb{Z} . We observe that S_M is congruent to $E_{s+2,2}$ over \mathbb{R} , i.e., $S_M = {}^t a E_{s+2,2} a$ for some $a \in GL(s+4, \mathbb{R})$, where

$$(4.7) \quad E_{s+2,2} := \begin{pmatrix} E_{s+2} & 0 \\ 0 & -E_2 \end{pmatrix}.$$

Then it is easy to see that $O(M_{\mathbb{R}}, S_M) = a^{-1} O(M_{\mathbb{R}}, E_{s+2,2}) a$. Now for brevity we write $O(M_{\mathbb{R}})$ simply instead of $O(M_{\mathbb{R}}, S_M)$. Obviously $O(M_{\mathbb{R}})$ is isomorphic to the real orthogonal group

$$(4.8) \quad O_{s+2,2}(\mathbb{R}) := \{g \in GL(s+4, \mathbb{R}) \mid {}^t g E_{s+2,2} g = E_{s+2,2}\}.$$

$O(M_{\mathbb{R}})$ has four connected components. Let $G_{\mathbb{R}}^0$ be the identity component of $O(M_{\mathbb{R}})$ and let $K_{\mathbb{R}}^0$ be its maximal compact subgroup. Then the pair $(G_{\mathbb{R}}^0, K_{\mathbb{R}}^0)$ of the real semisimple groups isomorphic to the pair $(SO(s+2, 2)^0, SO(s+2, \mathbb{R}) \times SO(2, \mathbb{R}))$ is a symmetric pair of type (BDI) (cf. [H] 445-446). The homogeneous space $X := G_{\mathbb{R}}^0 / K_{\mathbb{R}}^0$ is a Hermitian symmetric space of noncompact type of dimension $s+2$ (cf. see Appendix C). Indeed, X is a bounded symmetric domain of type IV in the Cartan classification. It is known that X is isomorphic to a $G_{\mathbb{R}}^0$ -orbit in the projective space $\mathbb{P}(M_{\mathbb{C}})$. Precisely, if we let $D := \{z \in \mathbb{P}(M_{\mathbb{C}}) \mid (z, z)_M = 0, (z, \bar{z})_M < 0\}$, then

$$(4.9) \quad D \cong G_{\mathbb{R}}^0 x_0 \cup G_{\mathbb{R}}^0 \bar{x}_0 = D^+ \cup \overline{D^+}, \quad D^+ := G_{\mathbb{R}}^0 x_0,$$

where \bar{x}_0 denotes the complex conjugation of x_0 in $\mathbb{P}(M_{\mathbb{C}})$. We shall denote by $G_{\mathbb{R}}$ the subgroup of $O(M_{\mathbb{R}})$ preserving the domain D^+ . It is known that $D^+ \cong G_{\mathbb{R}}^0 / K_{\mathbb{R}}^0$ may be realized as a tube domain in \mathbb{C}^{s+2} given by

$$(4.10) \quad \mathcal{D} := \{{}^t Z = (\omega, z, \tau) \in \mathbb{C}^{s+2} \mid \omega \in H_1, \tau \in H_1, S_1[\text{Im } Z] < 0\},$$

where $\text{Im } Z$ denotes the imaginary part of the column vector Z . An embedding of the tube domain \mathcal{D} into the projective space $\mathbb{P}(M_{\mathbb{C}})$, called the Borel embedding, is of the following form

$$(4.11) \quad p(Z) = p({}^t(\omega, z_1, \dots, z_s, \tau)) = {}^t\left(\frac{1}{2} S_1[Z] : \omega : z_1 : \dots : z_s : \tau : 1\right) \in \mathbb{P}(M_{\mathbb{C}}).$$

$G_{\mathbb{R}}$ acts on \mathcal{D} transitively as follows: if $g = (g_{kl}) \in G_{\mathbb{R}}$ with $1 \leq k, l \leq s+4$ and $Z = {}^t(\omega, z_1, \dots, z_s, \tau) \in \mathcal{D}$, then

$$(4.12) \quad g \langle Z \rangle := (\tilde{\omega}, \tilde{z}_1, \dots, \tilde{z}_s, \tilde{\tau}),$$

where

$$\begin{aligned}\tilde{\omega} &:= \left(\frac{1}{2}g_{2,1}S_1[Z] + g_{2,2}\omega + \sum_{l=3}^{s+2} g_{2,l}z_{l-2} + g_{2,s+3}\tau + g_{2,s+4}\right) J(g, Z)^{-1}, \\ \tilde{z}_k &:= \left(\frac{1}{2}g_{k+2,1}S_1[Z] + g_{k+2,2}\omega + \sum_{l=3}^{s+2} g_{k+2,l}z_{l-2} + g_{k+2,s+3}\tau + g_{k+2,s+4}\right) J(g, Z)^{-1}, \quad 1 \leq k \leq s, \\ \tilde{\tau} &:= \left(\frac{1}{2}g_{s+3,1}S_1[Z] + g_{s+3,2}\omega + \sum_{l=3}^{s+2} g_{s+3,l}z_{l-2} + g_{s+3,s+3}\tau + g_{s+3,s+4}\right) J(g, Z)^{-1}.\end{aligned}$$

Here we put

$$(4.13) \quad J(g, Z) := \frac{1}{2}g_{s+4,1}S_1[Z] + g_{s+4,2}\omega + \sum_{l=3}^{s+2} g_{s+4,l}z_{l-2} + g_{s+4,s+3}\tau + g_{s+4,s+4}.$$

It is easily seen that

$$(4.14) \quad p(g < Z >)J(g, Z) = g \cdot p(Z) \quad (\cdot \text{ is the matrix multiplication})$$

and that $J : G_{\mathbb{R}} \times \mathcal{D} \rightarrow GL(1, \mathbb{C}) = \mathbb{C}^{\times}$ is the automorphic factor, i.e.,

$$J(g_1g_2, Z) = J(g_1, g_2 < Z >)J(g_2, Z)$$

for all $g_1, g_2 \in G_{\mathbb{R}}$ and $Z \in \mathcal{D}$.

Let $O_M(\mathbb{Z})$ be the isometry group of the lattice M . Then $\Gamma_M := G_{\mathbb{R}} \cap O_M(\mathbb{Z})$ is an arithmetic subgroup of $G_{\mathbb{R}}$.

Definition 4.7. Let k be an integer. A holomorphic function f on \mathcal{D} is a modular form of weight k with respect to Γ_M if it satisfies the following transformation behaviour

$$(4.15) \quad (f|_k\gamma)(Z) := J(\gamma, Z)^{-k}f(\gamma < Z >) = f(Z)$$

for all $\gamma \in \Gamma_M$ and $Z \in \mathcal{D}$. For a subgroup Γ of Γ_M of finite index, a modular form with respect to Γ can be defined in the same way.

We denote by $M_k(\Gamma)$ the vector space consisting of all modular forms of weight k with respect to Γ . We now introduce the concept of cusp forms for Γ_M . First of all we note that the realization D^+ of our tube domain \mathcal{D} in the projective space $\mathbb{P}(M_{\mathbb{C}})$ is obtained as a subset of the quadric D in $\mathbb{P}(M_{\mathbb{C}})$ (cf. see (4.10)). A maximal connected complex analytic set X in $\overline{D^+} \setminus D^+$ is called a *boundary component* of D^+ , where $\overline{D^+}$ denotes the closure of D^+ in $\mathbb{P}(M_{\mathbb{C}})$. The normalizer

$N(X) := \{g \in G_{\mathbb{R}} \mid g(X) = X\}$ of a boundary component X of D^+ is a maximal parabolic subgroup of $G_{\mathbb{R}}$. X is called a *rational boundary component* if the normalizer $N(X)$ of X is defined over \mathbb{Q} . A modular form with respect to Γ_M is called a *cuspidal form* if it vanishes on every rational boundary component of D^+ . It is well known that any rational boundary component X of D^+ corresponds to a primitive isotropic sublattice S of M via $X = X_S := \mathbb{P}(S \otimes \mathbb{C}) \cap \overline{D^+}$. Since the lattice M contains only isotropic lines and planes, there exist two types of rational boundary components, which are points and curves.

The orthogonal group $G_{\mathbb{R}}$ has the rank two and so there are two types of maximal parabolic subgroups in Γ_M . Therefore there are two types of Fourier expansions of modular forms. A subgroup of Γ_M fixing a null sublattice of M of rank one is called a *Fourier group*. A subgroup of Γ_M fixing a null sublattice of M of rank two is called a *Jacobi parabolic group**. Both a Fourier group and a Jacobi parabolic group are maximal parabolic subgroups of Γ_M .

Let $f \in M_k(\Gamma_M)$ be a modular form of weight k with respect to Γ_M . Since the following $\gamma_\ell (\ell \in L_1 \cong \mathbb{Z}^{s+2})$ defined by

$$(4.16) \quad \gamma_\ell := \begin{pmatrix} 1 & {}^t\alpha & b \\ 0 & E_{s+2} & \ell \\ 0 & 0 & 1 \end{pmatrix}, \quad b := \frac{1}{2}S_1[\ell], \quad \alpha = S_1\ell$$

are elements of Γ_M , $f(Z + \ell) = f(Z)$ for all $\ell \in L_1$. We note that $\gamma_\ell(Z) = Z + \ell$ for all $Z \in \mathcal{D}$ and $J(\gamma_\ell, Z) = 1$. Hence we have a Fourier expansion

$$(4.17) \quad f(Z) = \sum_{\ell} a(\ell) e^{2\pi i({}^t\ell S_1 Z)},$$

where ℓ runs over the set $\{\ell \in \widetilde{L}_1 \mid i\ell \in \mathcal{D}, S_1[\ell] \geq 0\}$. Here \widetilde{L}_1 denotes the dual lattice of L_1 , that is,

$$\widetilde{L}_1 := \{\ell \in L_1 \otimes_{\mathbb{Z}} \mathbb{Q} \mid {}^t\ell S_1 \alpha \in \mathbb{Z} \text{ for all } \alpha \in L_1\}.$$

We let

$$(4.18) \quad f(Z) = f(\omega, z, \tau) = \sum_{m \geq 0} \phi_m(\tau, z) e^{2\pi i m \omega}$$

be the Fourier-Jacobi expansion of f with respect to the variable w . Obviously the Fourier-Jacobi coefficient

$$(4.19) \quad \phi_0(\tau, z) = \lim_{v \rightarrow \infty} f(iv, z, \tau)$$

*In [Bo7], this group was named just a Jacobi group. The definition of a Jacobi group is different from ours.

depends only on τ . We can show that the Fourier-Jacobi coefficients $\phi_m(\tau, z)$ ($m \geq 0$) satisfies the following functional equations

$$(4.20) \quad \phi_m\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k e^{\pi i m \frac{cS_0[z]}{c\tau + d}} \phi_m(\tau, z)$$

and

$$(4.21) \quad \phi_m(\tau, z + x\tau + y) = e^{-2\pi i m({}^t x S_0 z + \frac{1}{2} S_0[x]\tau)} \phi_m(\tau, z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 = SL(2, \mathbb{Z})$ and all $x, y \in \mathbb{Z}^s$.

Now we define the Jacobi forms associated to the orthogonal group. First we choose the following basis of M such that

$$(4.22) \quad M = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus L_0 \oplus \mathbb{Z}e_{-2} \oplus \mathbb{Z}e_{-1},$$

where e_1, e_2, e_{-1}, e_{-2} are four isotropic vectors with $(e_i, e_j) = \delta_{i, -j}$. Let $P_{\mathbb{R}}$ be the Jacobi parabolic subgroup of $G_{\mathbb{R}}$ preserving the isotropic plane $\mathbb{R}e_1 \oplus \mathbb{R}e_2$. Then it is easily seen that an element g of $P_{\mathbb{R}}$ is given by the following form:

$$(4.23) \quad g = \begin{pmatrix} A^0 & X_1 & Y \\ 0 & U & X \\ 0 & 0 & A \end{pmatrix}, \quad X_1 \in \mathbb{R}^{(2, n)}, Y \in \mathbb{R}^{(2, 2)}, X \in \mathbb{R}^{(n, 2)},$$

$$\begin{aligned} A &\in GL_2(\mathbb{R})^+, \quad S_0[U] = S_0, \quad A^0 = I^t A^{-1} I, \\ X_1 &= I^t A^{-1} {}^t X S_0 U, \quad {}^t Y I A + {}^t A I Y = S_0[X], \end{aligned}$$

where $I := I_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. We denote by GO_L for the general orthogonal group or conformal group of the lattice L consisting of linear transformations multiplying the quadratic form by an invertible element of a lattice L . We let $K := \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ be the 2-dimensional primitive null sublattice of M . Then we have a homomorphism $\pi_P : P_{\mathbb{R}} \rightarrow GO_K(\mathbb{R}) \times GO_{L_0}(\mathbb{R})$ defined by

$$(4.24) \quad \begin{pmatrix} A_0 & X_1 & Y \\ 0 & U & X \\ 0 & 0 & A \end{pmatrix} \longmapsto (A^0, U).$$

The connected component of the kernel of π_P is called a *Heisenberg group*, denoted by $\text{Heis}(M_{\mathbb{R}})$. It is easy to see that $\text{Heis}(M_{\mathbb{R}})$ consists of the following elements

$$(4.25) \quad \{X; r\} = \{x, y; r\} = \begin{pmatrix} 1 & 0 & {}^t y S_0 & {}^t x S_0 y - r & \frac{1}{2} S_0[y] \\ 0 & 1 & {}^t x S_0 & \frac{1}{2} S_0[x] & r \\ 0 & 0 & E_s & x & y \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $X = (x, y)$, with $x, y \in \mathbb{R}^{(s,1)}$ and $r \in \mathbb{R}$. The multiplication on $\text{Heis}(M_{\mathbb{R}})$ is given by

$$(4.26) \quad \{X_1; r_1\}\{X_2; r_2\} := \{X_1 + X_2; r_1 + r_2 + {}^t x_1 S_0 y_2\}, \quad X_1 = (x_1, y_1), \quad X_2 = (x_2, y_2).$$

We let $G_{\mathbb{R}}^J$ be the subgroup of $P_{\mathbb{R}}$ generated by the following elements

$$(4.27) \quad \{A\} := \text{diag}(A^0, E_s, A), \quad A \in SL(2, \mathbb{R}), \quad A^0 = I^t A^{-1} I$$

and $\{X; r\}$ in $\text{Heis}(M_{\mathbb{R}})$. $G_{\mathbb{R}}^J$ is called the (real) *Jacobi group* of the lattice M . We observe that $G_{\mathbb{R}}^J$ is isomorphic to the semidirect product of $SL(2, \mathbb{R})$ and $\text{Heis}(M_{\mathbb{R}})$. It is easy to check that

$$(4.28) \quad \{X; r\}\{A\} = \{A\}\{XA; r + \frac{1}{2}({}^t x_A S_0 y_A - {}^t x S_0 y)\},$$

where x_A and y_A are the columns of the matrix XA . We see easily that $\text{Heis}(M_{\mathbb{R}})$ is a normal subgroup of the Jacobi group $G_{\mathbb{R}}^J$ and the center C^J of $G_{\mathbb{R}}^J$ consists of all elements $\nabla(r) := \{0, 0; r\}$, $r \in \mathbb{R}$. According to (4.12), the actions of $\{A\}$ and $\{x, y; r\}$ on \mathcal{D} are given by as follows:

$$(4.29) \quad \{A\} \langle Z \rangle = {}^t \left(\omega - \frac{cS_0[z]}{2(c\tau + d)}, \frac{{}^t z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right);$$

$$(4.30) \quad \{x, y; r\} \langle Z \rangle = {}^t \left(\omega + r + {}^t S_0 z + \frac{1}{2} S_0[x]\tau, {}^t(z + x\tau + y), \tau \right),$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$ and $Z = {}^t(\omega, z, \tau) \in \mathcal{D}$. From (4.29) and (4.30), we can define the action of the Jacobi group $G_{\mathbb{R}}^J$ on the (τ, z) -domain $H_1 \times \mathbb{C}^s$, which we denote by $g \langle (\tau, z) \rangle$, $g \in G_{\mathbb{R}}^J$.

Let k and m be two integers. For $g \in G_{\mathbb{R}}^J$ and $Z = {}^t(\omega, {}^t z, \tau) \in \mathcal{D}$, we denote by $\omega(g; Z)$ the ω -component of $g \langle Z \rangle$. Now we define the mapping $J_{k,m} : G_{\mathbb{R}}^J \times (H_1 \times \mathbb{C}^s) \rightarrow GL(1, \mathbb{C}) = \mathbb{C}^\times$ by

$$(4.31) \quad J_{k,m}(g, (\tau, z)) := J(g, Z)^k e^{-2\pi i m \omega(g; Z)} \cdot e^{2\pi i m \omega},$$

where $g \in G_{\mathbb{R}}^J$, $Z = {}^t(\omega, {}^t z, \tau) \in \mathcal{D}$ and $J(g, Z)$ is the automorphic factor defined by (4.13). $J_{k,m}$ is well-defined, i.e., it is independent of the choice of $Z = {}^t(\omega, {}^t z, \tau) \in \mathcal{D}$ with given $(\tau, z) \in H_1 \times \mathbb{C}^s$. It is easy to check that $J_{k,m}$ is an automorphic factor for the Jacobi group $G_{\mathbb{R}}^J$. In particular, we have

$$(4.32) \quad J_{k,m}(\{A\}, (\tau, z)) = e^{\pi i \frac{cmS_0[z]}{c\tau+d}} \cdot (c\tau + d)^k, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$$

and

$$(4.33) \quad J_{k,m}(\{x, y; r\}, (\tau, z)) = e^{-2\pi i m(r + {}^t x S_0 z + \frac{1}{2} S_0[x]\tau)}.$$

We have a natural action of $G_{\mathbb{R}}^J$ on the algebra $C^\infty(H_1 \times \mathbb{C}^s)$ of all C^∞ functions on $H_1 \times \mathbb{C}^s$ given by

$$(4.34) \quad (\phi|_{k,m} g)(\tau, z) := J_{k,m}(g, (\tau, z))^{-1} \phi(g < (\tau, z) >),$$

where $\phi \in C^\infty(H_1 \times \mathbb{C}^s)$, $g \in G_{\mathbb{R}}^J$ and $(\tau, z) \in H_1 \times \mathbb{C}^s$. We let $\Gamma_M^J := \Gamma_M \cap G_{\mathbb{R}}^J$ (cf. Definition 4.7). Then Γ_M^J is a discrete subgroup of $G_{\mathbb{R}}^J$ which acts on $H_1 \times \mathbb{C}^s$ properly discontinuously.

Definition 4.8. Let k and m be nonnegative integers. A holomorphic function $\phi : H_1 \times \mathbb{C}^s \rightarrow \mathbb{C}$ is called a Jacobi form of weight k and index m on Γ_M^J if ϕ satisfies the following functional equation

$$(4.35) \quad \phi|_{k,m} \gamma = \phi \quad \text{for all } \gamma \in \Gamma_M^J$$

and $f(\tau, z)$ has a Fourier expansion

$$(4.36) \quad \phi(\tau, z) = \sum_{n \in \mathbb{Z}} \sum_{\ell \in \widehat{L}_0} c(n, \ell) e^{2\pi i(n\tau + {}^t \ell S_0 z)}$$

with $c(n, \ell) \neq 0$ only if $2nm - S_0[\ell] \geq 0$. Here \widehat{L}_0 is the dual lattice of L_0 , i.e.,

$$\widehat{L}_0 := \{\ell \in L_0 \otimes \mathbb{Q} \mid {}^t \ell S_0 \alpha \in \mathbb{Z} \text{ for all } \alpha \in L_0\}.$$

A Jacobi form ϕ of weight k and index m is called a *cuspidal form* if $c(n, \ell) \neq 0$ implies $2nm - S_0[\ell] > 0$. We denote by $J_{k,m}(\Gamma_M^J)$ (resp. $J_{k,m}^{\text{cusp}}(\Gamma_M^J)$) the vector space of all Jacobi forms (resp. cuspidal forms) of weight k and index m on Γ_M^J .

REMARK 4.9. (1) $J_{k,m}(\Gamma_M^J)$ is finite dimensional.

(2) The Fourier-Jacobi coefficients ϕ_m of a modular form f (cf. (4.17)) are Jacobi forms of weight k and index m on Γ_M^J . (cf. (4.20) and (4.21)).

(3) If $\phi \in J_{k,m}(\Gamma_M^J)$, the function $f_\phi(\omega, z, \tau) := \phi(\tau, z) e^{2\pi i m \omega}$ is a modular form with respect to the subgroup of finite index of the integral Jacobi parabolic subgroup $P_{\mathbb{Z}} := P_{\mathbb{R}} \cap \Gamma_M$.

Let m be a nonnegative integer and let $G_m(L_0) := \widehat{L}_0 / mL_0$ be the *discriminant group* of the lattice L_0 . For each $h \in G_m(L_0)$, we define the theta function $\vartheta_{S_0, m, h} := \vartheta_{L_0, m, h}$

$$(4.37) \quad \vartheta_{L_0, m, h}(\tau, z) := \sum_{\ell \in L_0} e^{\pi i m(S_0[\ell + \frac{h}{m}]\tau + 2{}^t(\ell + \frac{h}{m})S_0 z)},$$

where $(\tau, z) \in H_{1,s} := H_1 \times \mathbb{C}^s$. Any Jacobi form $\phi \in J_{k,m}(\Gamma_M^J)$ can be written as

$$(4.38) \quad \phi(\tau, z) = \sum_{h \in G_m(L_0)} \phi_h(\tau) \vartheta_{L_0, m, h}(\tau, z)$$

with

$$\phi_h(\tau) := \sum_{r \geq 0} c((2r + qS_0[h])(2qm)^{-1}, h) e^{2\pi i \frac{r\tau}{qm}},$$

where each $r \geq 0$ satisfies the condition $2r \equiv -qS_0[h] \pmod{2qm}$, $c(n, \ell)$ denotes the Fourier coefficients of $\phi(\tau, z)$ and q is the level of the quadratic form S_0 . We can rewrite (4.38) as follows :

$$(4.39) \quad \phi(\tau, z) = {}^t\Phi(\tau) \cdot \Theta_{L_0, m}(\tau, z),$$

where

$$(4.40) \quad \Phi(\tau) := (\phi_h(\tau))_{h \in G_m(L_0)} \quad \text{and} \quad \Theta_{L_0, m} := (\vartheta_{L_0, m, h})_{h \in G_m(L_0)}.$$

Then we can show that for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, the theta function $\Theta_{L_0, m}$ satisfies the following transformation formula

$$(4.41) \quad \Theta_{L_0, m} \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = e^{\pi i m \frac{cS_0[z]}{c\tau + d}} \cdot (c\tau + d)^{\frac{s}{2}} \cdot \chi(M) \Theta_{L_0, m}(\tau, z),$$

where $\chi(M)$ is a certain unitary matrix of degree $|G_m(L_0)|$ (cf. [G2], p.9 and [O], p.105). And $\Phi(\tau)$ satisfies the following functional equations :

$$(4.42) \quad \Phi(\tau + 1) = e^{-\pi i \frac{S_0[h]}{m}} \Phi(\tau), \quad \Phi(-1/\tau) = \tau^{k - \frac{s}{2}} \overline{U(J)} \Phi(\tau),$$

where $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and

$$(4.43) \quad U(J) := (\det S_0)^{-\frac{1}{2}} \left(\frac{i}{m} \right)^{\frac{s}{2}} \left(e^{-2\pi i \frac{t_g S_0 h}{m}} \right)_{g, h \in G_m(L_0)}.$$

We note that the finite group $G_m(L_0)$ may be regarded as the quadratic space equipped with the quadratic form q_{m, L_0} defined by

$$(4.44) \quad q_{m, L_0}(h + mL_0) := (h + mL_0, h + mL_0) \in (h, h)_0 + 2\mathbb{Z}.$$

From (4.41) and (4.42) it follows that $\Phi(\tau)$ is a vector-valued modular form of a half-integral weight and that the vector space of Jacobi forms of index m depends only on the quadratic space $(G_m(L_0), q_{m, L_0})$.

Lemma 4.10. *Let M_1 and M_2 be two even integral lattices of dimension s_1 and s_2 . We assume that the quadratic spaces $(G_{m_1}(M_1), q_{m_1, M_1})$ and $(G_{m_2}(M_2), q_{m_2, M_2})$ are isomorphic. Then we have the isomorphism*

$$J_{k, m_1}(\Gamma_{M_1}^J) \cong J_{k + \frac{s_2 - s_1}{2}, m_2}(\Gamma_{M_2}^J).$$

Proof. The proof is done if the map

$$(4.45) \quad {}^t\Phi(\tau) \cdot \Theta_{M_1, m_1}(\tau, z) \longmapsto {}^t\Phi(\tau) \cdot \Theta_{M_2, m_2}(\tau, z)$$

is an isomorphism of $J_{k, m_1}(\Gamma_{M_1}^J)$ onto $J_{k + \frac{s_2 - s_1}{2}, m_2}(\Gamma_{M_2}^J)$. The isomorphism can be proved using (4.41) and $s_1 \equiv s_2 \pmod{8}$. \square

Now we discuss the concept of singular modular forms and singular Jacobi forms.

Definition 4.11. A modular form f with respect to Γ_M (or a Jacobi form ϕ of index with respect to Γ_M^J) is said to be *singular* if its Fourier coefficients satisfy the following condition that

$$a(n, \ell, m) \neq 0 \text{ (or } c(n, \ell) \neq 0) \text{ implies } 2nm - S_0[\ell] = 0,$$

where $a(n, \ell, m)$ and $c(n, \ell)$ denote the Fourier coefficients of f and ϕ in their Fourier expansions respectively.

We consider the *differential operators* D and \widehat{D} defined by

$$(4.46) \quad D := \frac{\partial^2}{\partial \omega \partial \tau} - \frac{1}{2} S_0 \left[\frac{\partial}{\partial z} \right]$$

and

$$(4.47) \quad \widehat{D} := \frac{\partial}{\partial z} - \frac{1}{4\pi i m} S_0 \left[\frac{\partial}{\partial z} \right].$$

Then it is easy to see that if f is a singular modular form and if ϕ is a Jacobi form of index m , then $Df = 0$ and $\widehat{D}\phi = 0$. We can also show easily that any Jacobi form with respect to Γ_M^J has its weight $s/2$ and that any Jacobi form of weight $s/2$ with respect to Γ_M^J is singular. From this fact, we see that a weight of a singular modular form with respect to Γ_M is either 0 or $s/2$.

For each positive integer m , we let $M(m) := \Pi_{1,1} \oplus L_0 \oplus \Pi_{1,1}$ be the lattice with its associated symmetric matrix given by

$$(4.48) \quad S(m) := \begin{pmatrix} 0 & 0 & I_2 \\ 0 & mS_0 & 0 \\ I_2 & 0 & 0 \end{pmatrix}, \quad I_2 := \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

We let $\Gamma_m^J := \Gamma_{M(m)}^J$ be the integral Jacobi group of the lattice $M(m)$. It follows immediately from the definitions that if $\phi \in J_{k, m}(\Gamma_M^J)$, then $\phi \in J_{k, 1}(\Gamma_m^J)$. The existence of a nonconstant singular Jacobi form of index 1 with respect to Γ_M^J guarantees the unimodularity of the lattice M . Precisely, we have

Proposition 4.12 ([G2], Lemma 4.5). *Let M be a maximal even integral lattice. This means that M is not a sublattice of any even integral lattice. Then a nonconstant singular Jacobi form of index 1 with respect to Γ_M^J exists if and only if the lattice M is unimodular.*

Proof. The proof can be found in [G2], p.21. But we write his proof here. Let $\phi(\tau, z)$ be a nonconstant singular Jacobi form of index 1 with respect to Γ_M^J . According to (4.39), $\phi(\tau, z) = \Phi(\tau) \cdot \Theta_{L_0,1}(\tau, z_0)$. Thus the components $\phi_h(\tau)$ ($h \in G_1(L_0)$) of Φ are constants because their weights are all zero. By (4.42), we have

$$\phi_h(\tau + 1) = e^{-\pi i S_0[h]} \phi_h(\tau), \quad h \in G_1(L_0).$$

Therefore the components ϕ_h are not zero only for the isotropic vectors h in the group $G_1(L_0) = \widehat{L_0}/L_0$. Since $M = \Pi_{1,1} \oplus L_0 \oplus \Pi_{1,1}$ is maximal, there exists only the trivial isotropic element $h = 0$ in $G_1(L_0)$. Again by (4.42), we obtain that $|G_1(L_0)| = 1$ and so L_0 is unimodular. Hence the lattice M is unimodular. \square

Example 4.13. We assume that M is a *unimodular* even integral lattice of signature $(s + 2, 2)$. Then the theta series

$$(4.49) \quad \vartheta(\tau, z) := \sum_{\lambda \in L} e^{\pi i (S_0[\lambda]\tau + 2^t \lambda S_0 z)}, \quad (\tau, z) \in H_{1,s}$$

is a singular Jacobi form of weight $s/2$ and index 1 with respect to Γ_M^J . The arithmetic lifting f_ϑ of $\vartheta(\tau, z)$ defined by

$$(4.50) \quad f_\vartheta(\omega, z, \tau) := \frac{(s/2 - 1)! \xi(s/2)}{(2\pi i)^{s/2}} + \sum_{\substack{n, m \geq 0, \lambda \in L \\ 2nm = S_0[\lambda] \\ (n, m) \neq (0, 0)}} \sigma_{s/2-1}(n, m; \lambda) e^{2\pi i (n\tau + {}^t \lambda S_0 z + m\omega)}$$

is a singular modular form of weight $s/2$ with respect to Γ_M^J , where $\sigma_{s/2-1}(n, m; \lambda)$ denotes the sum of $(s/2 - 1)$ -powers of all common divisors of the numbers n, m and the vector $\lambda \in L$. For more detail, we refer to Theorem 3.1 and Example 4.4 in [G2].

As we have seen so far, automorphic forms on the real symplectic group and those on the real orthogonal group have different geometric objects, different automorphic factors (cf. (4.12)), and somewhat different properties. For instance, in case of the orthogonal group $O_{s+2,2}(\mathbb{R})$, there is a gap between 0 and $s/2$ such that there exist no modular forms and no Jacobi forms with weights in this gap. By the way, this phenomenon does not happen for automorphic forms and Jacobi forms in the case of the symplectic group $Sp(g, \mathbb{R})$ because all integers less than half the

largest singular weight are also singular weights. For more detail, we refer to [F] for singular modular forms and to [Y3] for singular Jacobi forms. In both cases the number of singular weights is equal to the real rank of the corresponding Lie group. Nonetheless the properties of Jacobi forms for the orthogonal group are similar to those of Jacobi forms for the symplectic group. For example, the Fourier coefficient $c(n, \ell)$ of a Jacobi form of weight k and index m for $O_{s+2,2}(\mathbb{R})$ depends only on the number $2mn - S_0[\ell]$ (which is the norm of the vector (n, ℓ, m) in the lattice \widehat{L}_1) and the equivalence class of ℓ in the discriminant group $G_m(L_0) = \widehat{L}_0/L_0$ (cf. compare Theorem 2.2 in [E-Z] with our case). We observe that the automorphic factors for the Jacobi groups for both cases are quite similar (cf. see (4.2) and (4.31)-(4.33)). The expression of Jacobi forms in terms of (4.38) or (4.39) are similar to that of Jacobi forms for the symplectic group (cf. [E-Z], [Y1], and [Zi]).

REMARK 4.14. In [Bo7], R. Borcherds investigates automorphic forms and Jacobi forms for $O_{s+2,2}(\mathbb{R})$ which are either nearly holomorphic or meromorphic.

Meromorphic functions with all poles at cusps are called *nearly holomorphic ones*.

BORCHERDS' CONSTRUCTION OF JACOBI FORMS : Let K be a positive definite integral lattice of dimension s . A function $c : K \rightarrow \mathbb{Z}^+ \cup \{0\}$ is said to be a *vector system* if it satisfies the following three properties (1)–(3) :

- (1) The set $\{v \in K | c(v) \neq 0\}$ is finite.
- (2) $c(v) = c(-v)$ for all $v \in K$.
- (3) The function taking λ to $\sum_{v \in K} c(v)(\lambda, v)^2$ is constant on the sphere of norm 1 vectors $\lambda \in K \otimes \mathbb{R}$.

We will write V for the *multiset* of vectors in a vector system and so we think of V as containing $c(v)$ copies of each vector $v \in K$. And we write $\sum_{v \in V} f(v)$ instead of $\sum_{v \in K} c(v)f(v)$. The vector system is said to be *trivial* if it only contains vectors of zero norm.

The hyperplanes orthogonal to the vectors of a vector system V divides $K_{\mathbb{R}} := K \otimes \mathbb{R}$ into cones which we call the *Weyl chambers* of V . We note that unlike the case of root systems, the Weyl chamber of V need not be all the same type. If we choose a fixed Weyl chamber W , then we can define the *positive* and *negative* vectors of V by saying that v is positive or negative, denoted by $v > 0$ or $v < 0$ if $(v, \lambda) > 0$ or $(v, \lambda) < 0$ for some vector λ in the interior W^0 of W . It is easy to check that the concept of positivity and negativity does not depend on the choice of a vector λ in W^0 . Obviously every nonzero vector of the vector system V is either positive or negative.

We define the *Weyl vector* $\rho = \rho_W$ of W by

$$\rho := \frac{1}{2} \sum_{\substack{v \in V \\ v > 0}} v.$$

We define d to be the number of vectors in V and define $k := \frac{c(0)}{2}$. The rational number k is called the *weight* of V . We define the *index* m of V by

$$m := (2 \dim K)^{-1} \sum_{v \in V} (v, v)$$

We can show that the index m of V is a nonnegative integer. If V is a vector system in K , we define the (*untwisted*) *affine vector system* of V to be the multiset of vectors $(v, n) \in K \oplus \mathbb{Z}$ with $v \in V$. We say that (v, n) is *positive* if either $n > 0$ or $n = 0, v > 0$. It can be seen that the Weyl vectors for different Weyl chambers differ by elements of K .

Borcherds (cf. [Bo7], p.183) define the function $\psi(\tau, z)$ on $H_1 \times K_{\mathbb{C}}$ with $K_{\mathbb{C}} = K \otimes \mathbb{C} \cong \mathbb{C}^s$ by

$$(4.51) \quad \psi(\tau, z) := q^{\frac{d}{24}} \zeta^{-\rho} \prod_{(v,n) > 0} (1 - q^n \zeta^v), \quad (\tau, z) \in H_1 \times K_{\mathbb{C}},$$

where (v, n) runs over the set of all positive vectors in the affine vector system of V , $q^a := e^{2\pi i a \tau}$ and $\zeta^v = e^{2\pi i (z, v)}$. Then $\psi(\tau, z)$ is a *nearly holomorphic Jacobi form* of weight k and index m . Thus ψ can be written as a finite sum of theta functions times nearly holomorphic modular forms. In fact, ψ satisfies the following transformation formulas:

$$\begin{aligned} \psi(\tau + 1, z) &= e^{\frac{\pi i d}{12}} \psi(\tau, z), \\ \psi(-1/\tau, z/\tau) &= (-i)^{d/2-k} (\tau/i)^k e^{\pi i m(z, z)/\tau} \psi(\tau, z), \\ \psi(\tau, z + \mu) &= (-1)^{2(\rho, \mu)} \psi(\tau, z), \\ \psi(\tau, z + \lambda\tau) &= (-1)^{2(\rho, \lambda)} q^{-m(\lambda, \lambda)/2} \zeta^{-m\lambda} \psi(\tau, z) \end{aligned}$$

for all $\lambda, \mu \in \widehat{K}$ (the dual of K).

5. Infinite Products and Modular Forms

In [Bo7], R. Borcherds constructed automorphic forms on $O_{s+2,2}(\mathbb{R})^0$ which are modular products and using the theory of these automorphic theory expressed some meromorphic modular forms for $SL(2, \mathbb{Z})$ with certain conditions as infinite products. Roughly speaking, a modular product means an infinite product whose exponents are the coefficients of a certain nearly holomorphic modular form. For instance, he wrote modular forms as the modular invariant j and the Eisenstein series E_4 and E_6 as infinite products. These results tell us implicitly that the denominator function of a generalized Kac-Moody algebra is sometimes an automorphic form on $O_{s+2,2}(\mathbb{R})^0$ which is a modular product. In this section we discuss Borcherds' results just mentioned in some detail.

We shall start by giving some well-known classical product identities. First we give some of the product identities of L. Euler (1707-83) which are

$$(5.1) \quad \sum_{n \geq 0} \frac{(-1)^n q^{n(n+1)/2} z^n}{(1-q)(1-q^2) \cdots (1-q^n)} = \prod_{n > 0} (1 - q^n z),$$

$$(5.2) \quad \sum_{n \geq 0} \frac{z^n}{(1-q)(1-q^2) \cdots (1-q^n)} = \prod_{n > 0} (1 - q^n z)^{-1},$$

$$(5.3) \quad \sum_{n \in \mathbb{Z}} (-1)^n q^{3(n+1/6)^2/2} = q^{1/24} \prod_{n > 0} (1 - q^n).$$

A similiar product identity due to C. F. Gauss (1777-1855) is

$$(5.4) \quad \sum_{n \in \mathbb{Z}} q^{n^2} = (1 + q^2)(1 - q^2)(1 + q^3)^2(1 - q^4) \cdots .$$

Both of (5.3) and (5.4) are special cases of the so-called Jacobi's triple product identity [C. G. J. Jacobi (1804-51)]

$$(5.5) \quad \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} z^n = \prod_{n > 0} (1 - q^{2n})(1 - q^{2n-1}z)(1 - q^{2n-1}z^{-1})$$

if we choose z to be some fixed power of q . In fact, if you replace q and z in (5.5) by $q^{3/2}$ and $q^{1/2}$ respectively, you obtain the identity (5.3), and if you replace z in (5.5) by -1 , you get the identity (5.4).

The quintuple product identity derived by G. N. Watson (1886-?) is

$$(5.6) \quad \sum_{n \in \mathbb{Z}} q^{(3n^2+n)/2} (z^{3n} - z^{-3n-1}) \\ = \prod_{n > 0} (1 - q^n)(1 - q^n z)(1 - q^{n-1}z^{-1})(1 - q^{2n-1}z^2)(1 - q^{2n-1}z^{-2}).$$

Historically speaking, in 1929 Watson (cf. [W1]) derived the identity (5.6) in the course of proving some of Ramanujan's theorems on continued fractions. In 1938, Watson (cf. [W2]) proved the following identity:

$$(5.7) \quad \sum_{n \in \mathbb{Z}} q^{n(3n+2)} (z^{-3n} - z^{3n+2}) \\ = \prod_{n > 0} (1 - q^{2n})(1 - q^{2n-2}z^2)(1 - q^{2n}z^{-2})(1 + q^{2n-1}z)^{-1}(1 + q^{2n-1}z^{-1}).$$

Subbarao and Vidyasagar (cf. [S-V]) showed that the identities (5.6) and (5.7) are equivalent. The two identities (5.1) and (5.2) of Euler are easily established (cf. [Be], p. 49). G. E. Andrews showed that the Jacobi's triple product identity (5.5) can be obtained easily from the identities (5.1) and (5.2) in his short paper [A]. Carlitz and Subbarao (cf. [C-S]) gave a simple proof of the quintuple product identity (5.7).

The following denominator formula for a finite dimensional simple Lie algebra \mathfrak{g}

$$(5.8) \quad e^\rho \sum_{w \in W} \det(w) e^{-w(\rho)} = \prod_{\alpha > 0} (1 - e^\alpha)$$

is due to Hermann Weyl (1885-1955), where W is the Weyl group of \mathfrak{g} , ρ is the Weyl vector and the product runs over the set of all positive roots. Macdonald (cf. [Mac]) observed that the Weyl denominator formula is just a statement about finite root systems, and then generalized this formula to *affine* root systems producing the so-called *Macdonald identities*. He noticed that the Jacobi's triple product identity is just the Macdonald identity for the simplest affine root system. Kac observed that the Macdonald identities are just the denominator formulas for the Kac-Moody Lie algebras in the early 1970s. Thereafter he obtained the so-called *Weyl-Kac character formulas* for representations of the affine Kac-Moody algebras generalizing the Weyl character formula (see (2.5)-(2.7) and [K], p. 173). The Weyl-Kac character formula for the affine Kac-Moody algebra is given as follows:

$$(5.9) \quad e^\rho \sum_{w \in W} \det(w) e^{-w(\rho)} = \prod_{\alpha > 0} (1 - e^{-\alpha})^{\text{mult}(\alpha)},$$

where $\text{mult}(\alpha)$ is the multiplicity of the root α . For more detail we refer to (2.6) and [K]. For instance, the Jacobi's triple product identity is just the Weyl-Kac character formula for the affine Kac-Moody algebra $SL_2(\mathbb{R}[z, z^{-1}])$ and the Weyl-Kac character formulas for the affine Kac-Moody algebras $SL_n(\mathbb{R}[z, z^{-1}])$ are just the Macdonald identities. It seems that the Weyl-Kac character formula is true for non-affine Kac-Moody algebras. Borchers obtained the so-called *Weyl-Kac-Borchers character formula* for a generalized Kac-Moody algebra (cf. (2.13)-(2.14)). The Weyl-Kac character formula is proved by the Euler-Poincaré principle applied to the cohomology of the Lie subalgebra E of \mathfrak{g} associated to the positive roots of the Kac-Moody algebra \mathfrak{g} .

It seems to the author that Borchers was the first one that discovered that the denominator functions of the generalized Kac-Moody algebras which could be written as infinite products are often automorphic forms on the orthogonal group $O_{s+2,2}(\mathbb{R})^0$. Moreover he gave a method of constructing automorphic forms on $O_{s+2,2}(\mathbb{R})^0$ through modular forms of weight $-s/2$ with integer coefficients and obtained the connection between the Kohnen's "plus" space of weight $1/2$ and the space of modular forms on Γ_1 satisfying some conditions.

Now we are in a position to describe his works on infinite products related to automorphic forms on the orthogonal group $O_{s+2,2}(\mathbb{R})$.

We let L be the unimodular even integral Lorentzian lattice $\Pi_{s+1,1}$ of dimension $s+2$ and let $M := L \oplus \Pi_{1,1}$, where $\Pi_{1,1}$ is the unique 2-dimensional unimodular even integral Lorentzian lattice with its inner product matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. We choose a negative norm vector α in $L_{\mathbb{R}} := L \otimes \mathbb{R}$. We say that a vector v in $L_{\mathbb{R}}$ is *positive*, denoted by $v > 0$ if $(v, \alpha) > 0$.

Theorem 5.1 (Borcherds [Bo7], Theorem 10.1). Let $f(\tau) = \sum_n c(n)q^n$ be a nearly holomorphic modular form of weight $-s/2$ for Γ_1 with integer coefficients, with $24|c(0)$ if $s = 0$. Then there is a unique vector $\delta \in L$ such that

$$(5.10) \quad \Phi(v) := e^{-2\pi i(\delta, v)} \prod_{r>0, r \in L} \left(1 - e^{-2\pi i(r, v)}\right)^{c(-r^2/2)}, \quad v \in \Omega$$

is a meromorphic automorphic form of weight $c(0)/2$ for $O_M(\mathbb{Z})^0 \cong O_{s+2,2}(\mathbb{Z})^0$, where $r^2 := (r, r)$ and

$$\Omega := \{z \in M \otimes \mathbb{C} \mid (z, z) = 0, (z, \bar{z}) > 0\}.$$

REMARK 5.2. Borcherds showed that all the zeros and poles of Φ lie on the rational quadratic divisors and computed the multiplicities of the zeros of Φ . Roughly speaking a rational quadratic divisor means the zero set of $a(y, y) + (b, y) + c = 0$ with $a, c \in \mathbb{Z}$ and $b \in L$.

Definition 5.3. We define the function $H : \mathbb{Z}_+ \rightarrow \mathbb{Q}$ by

$$H(n) := \begin{cases} \text{the Hurwitz class number of the discriminant } -n & \text{if } n > 0; \\ -1/12 & \text{if } n = 0. \end{cases}$$

We note that

$$\tilde{H}(q) := \sum_{n \geq 0} H(n)q^n = -1/12 + q^3/3 + q^4/2 + q^7 + q^8 + q^{11} + (4/3)q^{12} + \dots$$

Now we state a very interesting result.

Theorem 5.4 (Borcherds [Bo7], Theorem 14.1). Let \mathcal{A} be the additive group consisting of nearly holomorphic modular forms of weight $1/2$ for $\Gamma_0(4)$ whose coefficients are integers and satisfy the Kohnen's "plus space" condition. We also let \mathcal{B} be the multiplicative group consisting of meromorphic modular forms for some characters of Γ_1 of integral weight with leading coefficient 1 whose coefficients are integers and all of whose zeros and poles are either cusps or imaginary quadratic irrationals. To each $f(\tau) = \sum_n c(n)q^n$ in \mathcal{A} we associate the function $\Psi_f : H_1 \rightarrow \mathbb{C}$ defined by

$$(5.11) \quad \Psi_f(\tau) := q^{-h} \prod_{n>0} (1 - q^n)^{c(n^2)},$$

where h is the constant term of $f(\tau)\tilde{H}(q)$. Then we have the following:

- (a) For each $f \in \mathcal{A}$, Ψ_f is an element of \mathcal{B} whose weight is $c(0)$;
- (b) the map $\Psi : \mathcal{A} \rightarrow \mathcal{B}$ given by $\Psi(f) := \Psi_f$ for $f \in \mathcal{A}$ is a group isomorphism of \mathcal{A} onto \mathcal{B} ;
- (c) the multiplicity of the zero of Ψ at a quadratic irrational τ of discriminant $D < 0$ is $\sum_{d>0} c(Dd^2)$.

REMARK 5.5. The product formula for the classical modular polynomial (for discriminant $D < 0$ whose degree is $H(-D)$)

$$(5.12) \quad \prod_{[\sigma]} (j(\tau) - j(\sigma)) = q^{-H(-D)} \prod_{n>0} (1 - q^n)^{c(n^2)}$$

holds, where σ runs over a complete set of representatives modulo Γ_1 for the imaginary quadratic irrationals which are roots of an equation of the form $a\sigma^2 + b\sigma + c = 0$ ($a, b, c \in \mathbb{Z}$) of the discriminant $b^2 - 4ac = D < 0$ (except that σ is a conjugate of one of the elliptic fixed points i or $(1 + i\sqrt{3})/2$ we have to replace the corresponding factor $j(\tau) - 1728$ or $j(\tau)$ by $(j(\tau) - 1728)^{1/2}$ or $j(\tau)^{1/3}$) and the exponents $c(n^2)$ are the coefficients of the uniquely determined nearly holomorphic modular form in \mathcal{A} . It is easy to check that the classical modular polynomial on the left hand side of (5.12) is contained in \mathcal{B} and that its corresponding element in \mathcal{A} is of the form $q^D + O(q)$.

Examples 5.6. (1) Let $f(\tau) := 12\theta(\tau) = 12 \sum_{n \in \mathbb{Z}} q^{n^2}$. It is easy to check that $f(\tau)$ is an element of \mathcal{A} and that $\Psi_f(\tau) = q \prod_{n>0} (1 - q^n)^{24}$ is a cusp form for Γ_1 of weight 12 known as the discriminant function.

(2) We put

$$F(\tau) := \sum_{n>0, n:\text{odd}} \sigma_1(n)q^n = q + 4q^3 + 6q^5 + 8q^7 + 13q^9 + \dots$$

We let

$$f(\tau) = 3F(\tau)\theta(\tau)(\theta(\tau)^4 - 2F(\tau))(\theta(\tau)^4 - 16F(\tau))E_6(4\tau)/\Delta(4\tau) + 168\theta(\tau),$$

where $\theta(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2}$, $\Delta(\tau)$ and $E_4(\tau)$ denote the discriminant function and the Eisenstein series of weight 4 respectively. (see Appendix A). It is easy to check that $f(\tau)$ is an element of \mathcal{A} and that $\Psi_f(\tau) = j(\tau)$ is the modular invariant. We also check that $\Psi_f(\tau) = j(\tau)$ has order 3 at the zero $\frac{1+i\sqrt{3}}{2}$ whose discriminant is -3 . Hence we obtain the modular product

$$j(\tau) = q^{-1}(1 - q)^{-744}(1 - q^2)^{80256}(1 - q^3)^{-12288744} \dots$$

(3) The Eisenstein series E_4 and E_6 are elements of \mathcal{B} . The elements of \mathcal{A} corresponding to E_4 and E_6 are given by

$$f_4(\tau) = q^{-3} + 4 - 240q + 26760q^4 - 85995q^5 + 1707264q^8 - 4096240q^9 + \dots$$

and

$$\begin{aligned} f_6(\tau) = & q^{-4} + 6 + 504q + 143388q^4 \\ & + 565760q^5 + 184373000q^8 + 51180024q^9 + O(q^{12}) \end{aligned}$$

respectively. Use the fact $E_4^3 = j \cdot \Delta$ for f_4 . The function $f_6(\tau)$ can be obtained from the theory of a generalized Kac-Moody algebra of rank 1 whose simple roots are all multiples of some root α of norm -2 and the simple roots are $n\alpha$ with $n \in \mathbb{Z}^+$ and multiplicity $504\sigma_3(n)$. Precisely,

$$\begin{aligned} f_6(\tau) = & (j(4\tau) - 876)\theta(\tau) \\ & - 2F(\tau)\theta(\tau)(\theta(\tau)^4 - 2F(\tau))(\theta(\tau)^4 - 16F(\tau))E_6(4\tau)/\Delta(4\tau), \end{aligned}$$

where $\theta(\tau)$ and $F(\tau)$ are defined in (2). Since $E_8 = E_4^2$, $E_{10} = E_4 \cdot E_6$ and $E_{14} = E_4^2 E_6$, their corresponding elements in \mathcal{A} are given by $2f_4$, $f_4 + f_6$ and $2f_4 + f_6$ respectively. The remaining Eisenstein series ($k \neq 4, 6, 8, 10, 14$, k : even, $k \geq 4$) are not elements of \mathcal{B} and hence they cannot be written as modular products. For instance, the modular products for E_4 , E_6 , E_8 , E_{10} and E_{14} are given by

$$E_4(\tau) = (1 - q)^{-240}(1 - q^2)^{26760}(1 - q^3)^{-4096240} \dots,$$

$$E_6(\tau) = (1 - q)^{504}(1 - q^2)^{143388}(1 - q^3)^{51180024} \dots,$$

$$E_8(\tau) = (1 - q)^{-480}(1 - q^2)^{53520}(1 - q^3)^{-8192480} \dots,$$

$$E_{10}(\tau) = (1 - q)^{264}(1 - q^2)^{170148}(1 - q^3)^{47083784} \dots$$

and

$$E_{14}(\tau) = (1 - q)^{24}(1 - q^2)^{196908}(1 - q^3)^{42987544} \dots$$

(4) Using the above theorem, we can show that there exist precisely 14 modular forms of weight 12 on Γ_1 which are contained in \mathcal{B} . Indeed, if

$$\begin{aligned} \Xi & := \{ n \in \mathbb{Z} \mid j(\tau) - n \in \mathcal{B} \} \\ & = \{ j(\tau) \in \mathbb{Z} \mid \tau \in H_1, \tau \text{ is imaginary quadratic} \}, \end{aligned}$$

only the modular forms $\Delta(\tau)(j(\tau) - n)$ (where $n \in \Xi$) and $\Delta(\tau)$ are modular forms of weight 12 in \mathcal{B} . It is well known that the elements of Ξ are

$$\begin{aligned} j\left(\frac{1+i\sqrt{3}}{2}\right) &= 0, & j(i) &= 2^6 \cdot 3^3, & j\left(\frac{1+i\sqrt{7}}{2}\right) &= -3^3 \cdot 5^3, & j(i\sqrt{2}) &= 2^6 \cdot 5^3, \\ j\left(\frac{1+i\sqrt{11}}{2}\right) &= -2^{15}, & j(i\sqrt{3}) &= 2^4 \cdot 3^3 \cdot 5^3, & j(2i) &= 2^3 \cdot 3^3 \cdot 11^3, \\ j\left(\frac{1+i\sqrt{19}}{2}\right) &= -2^{15} \cdot 3^3, & j\left(\frac{1+i\sqrt{27}}{2}\right) &= -2^{15} \cdot 3 \cdot 5^3, & j(i\sqrt{7}) &= 3^3 \cdot 5^3 \cdot 17^3, \\ j\left(\frac{1+i\sqrt{43}}{2}\right) &= -2^{18} \cdot 3^3 \cdot 5^3, & j\left(\frac{1+i\sqrt{67}}{2}\right) &= -2^{15} \cdot 3^3 \cdot 5^3 \cdot 11^3, \\ j\left(\frac{1+i\sqrt{163}}{2}\right) &= -2^{18} \cdot 3^3 \cdot 5^3 \cdot 23^3 \cdot 29^3. \end{aligned}$$

6. Final Remarks

In this final section we make some brief remarks on the fake monster Lie algebras, generalized Kac-Moody algebras of the arithmetic type, hyperbolic reflection groups and Jacobi forms. Finally we give some open problems.

6.1. The Fake Monster Lie Algebras

First of all we collect the properties of the *fake monster Lie algebra* M_Λ . (In [Bo5], M_Λ was called just the monster Lie algebra because the monster Lie algebra M defined in section 3 had not been discovered at that time yet.)

Let Λ be the Leech lattice of dimension 24. M_Λ is the generalized Kac-Moody Lie algebra with the following properties ($M_\Lambda 1$) – ($M_\Lambda 10$):

($M_\Lambda 1$) The root lattice L of M_Λ is $\Pi_{25,1} := \Lambda \oplus \Pi_{1,1}$.

($M_\Lambda 2$) $\rho = (0, 0, 1)$ is the Weyl vector of L with norm $\rho^2 = 0$. The real simple roots of M_Λ are the norm 2 vectors of the form $(\lambda, 1, \lambda^2/2 - 1)$, $\lambda \in \Lambda$, and the imaginary simple roots are the positive multiples of ρ each with multiplicity 24. (We observe that if r is a real simple root, then $(\rho, r) = -1$)

($M_\Lambda 3$) A nonzero vector $r \in L = \Pi_{25,1}$ is a root if and only if $r^2 \leq 2$, in which case it has multiplicity $p_{24}(1 - r^2/2)$, where $p_{24}(1 - r^2/2)$ is the number of partitions of $1 - r^2/2$ into 24 colours.

($M_\Lambda 4$) M_Λ has a $\Pi_{25,1}$ -grading. The piece $M_\Lambda(r)$ of degree $r \in \Pi_{25,1}$, $r \neq 0$ has dimension $p_{24}(1 - r^2/2)$.

($M_\Lambda 5$) M_Λ has an involution ω which acts as -1 on $\Pi_{25,1}$ and also on the piece $M_\Lambda(0)$ of degree $0 \in \Pi_{25,1}$.

($M_\Lambda 6$) M_Λ has a contravariant bilinear form (\cdot, \cdot) such that $M_\Lambda(k)$ is orthogonal to $M_\Lambda(l)$ with respect to (\cdot, \cdot) if $k \neq -l$, $k, l \in \Pi_{25,1}$ and such that (\cdot, \cdot) is positive definite on $M_\Lambda(k)$ for all $k \in \Pi_{25,1}$ with $k \neq 0$.

($M_\Lambda 7$) The denominator formula for M_Λ is given by

$$(6.1) \quad e^{-\rho} \sum_{w \in W} \sum_{n \in \mathbb{Z}} \det(w) \tau(n) e^{w(n\rho)} = \prod_{r \in L^+} (1 - e^r)^{p_{24}(1-r^2/2)},$$

where W is the Weyl group, L^+ is the set of all positive roots of M_Λ , and $\tau(n)$ is the Ramanujan tau function. (The discriminant function $\Delta(\tau)$ is the generating function of $\tau(n)$.) Indeed, L^+ is given by

$$L^+ = \{ v \in \Pi_{25,1} \mid v^2 \leq 2, (v, \rho) < 0 \} \cup \{ n\rho \mid n \in \mathbb{Z}^+ \}.$$

($M_\Lambda 8$) The universal central extension \hat{M}_Λ of M_Λ is a $\Pi_{25,1}$ -graded Lie algebra. If $0 \neq r \in \Pi_{25,1}$, then the piece $\hat{M}_\Lambda(r)$ of \hat{M}_Λ of degree r is mapped isomorphically to $M_\Lambda(r)$. The piece $\hat{M}_\Lambda(0)$ of degree 0, called the Cartan subalgebra of \hat{M}_Λ , can be represented naturally as the sum of a *one-dimensional* space for each vector of Λ and a space of dimension $24^2 = 576$ for each positive integer.

($M_\Lambda 9$) For each $r \in L^+$, we put

$$m(r) := \sum_{\substack{n > 0 \\ r/n \in \Pi_{25,1}}} \text{mult}(r/n) \cdot n.$$

Then for each $r \in L^+$, we have the following formula

$$(6.2) \quad (r + \rho)^2 m(r) = \sum_{\substack{\alpha, \beta \in L^+ \\ \alpha + \beta = r}} (\alpha, \beta) m(\alpha) m(\beta).$$

($M_\Lambda 10$) M_Λ is a $\text{Aut}(\hat{\Lambda})$ -module. In fact, $\text{Aut}(\hat{\Lambda})$ acts naturally on the vertex algebra of $\hat{\Lambda}$ and hence on M_Λ .

The detail for all the properties ($M_\Lambda 1$)-($M_\Lambda 10$) can be found in [Bo5].

REMARK 6.1. (a) M_Λ is essentially the space of physical vectors of the vertex algebra of $\hat{\Pi}_{25,1}$, where $\hat{\Pi}_{25,1}$ is the unique central extension of $\Pi_{25,1}$ by \mathbb{Z}_2 .

(b) M_Λ can be constructed from the vertex algebra of V_Λ of the central extension $\hat{\Lambda}$ of Λ by \mathbb{Z}_2 in the same way that the monster Lie algebra M was constructed from the monster vertex algebra V in section 3.

(c) The multiplicities $p_{24}(1+n)$ of the roots of M_Λ is given by the Rademacher's formula

$$p_{24}(1+n) = 2\pi n^{-13/2} \sum_{k>0} \frac{I_{13}(4\pi\sqrt{n}/k)}{k} \cdot \sum_{\substack{0 \leq h, h' \leq k \\ hh' \equiv -1 \pmod{k}}} e^{2\pi i(nh+h')/k},$$

where $I_{13}(z) := -iJ_{13}(iz)$ is the modified Bessel function of order 13. In particular, $p_{24}(1+n)$ is asymptotic to $2^{-1/2}n^{-27/4}e^{4\pi\sqrt{n}}$ for large n .

In [Bo6], Borchers constructed a family of Lie algebras and superalgebras, the so-called *monstrous Lie superalgebra* whose denominator formulas are twisted denominator formulas of the monster Lie algebra M . For each element g in the MONSTER G , we define the *monstrous Lie algebra* of g to be the generalized Kac-Moody superalgebra which has its root lattice $\Pi_{1,1}$ and simple roots $(1, n)$ with multiplicity $tr(g|_{V_n^\#})$. The denominator formula for the monstrous Lie superalgebra M_g of g is given by

$$\begin{aligned} (6.3) \quad T_g(p) - T_g(q) &= \sum_m tr(g|_{V_m^\#})p^m - \sum_n tr(g|_{V_n^\#})q^n \\ &= p^{-1} \prod_{m>0, n \in \mathbb{Z}} (1 - p^m q^n)^{\text{mult}_g(m,n)}. \end{aligned}$$

The multiplicity $\text{mult}_g(m, n)$ of the root $(m, n) \in \Pi_{1,1}$ is

$$(6.4) \quad \text{mult}_g(m, n) = \sum_{ds|(m,n,N)} \frac{\mu(s)}{ds} tr(g^d|_{V_{mn}^\#}).$$

where N is the order of g . We recall that the Thompson series $T_g(q)$ of g is the normalized generator for a genus zero function field of a discrete subgroup of $SL(2, \mathbb{R})$ containing the Hecke subgroup $\Gamma_0(hN)$, where h is a positive integer with $h|(24, N)$.

Furthermore Borchers (cf. [Bo6],) constructed a family of superalgebras whose denominator formulas are twisted ones of the fake monster Lie algebra M_Λ in the same way that he constructed a family of monstrous Lie superalgebras from the monster Lie algebra M .

Let g be an element of $\text{Aut}(\hat{\Lambda}) \cong 2^{24} \cdot \text{Aut}(\Lambda)$ of order N . We let

$$(6.5) \quad L := \{ \lambda \in \Lambda \mid g\lambda = \lambda \}$$

be the sublattice of Λ fixed by g . Then the dual L' of L is equal to the projection of Λ into the vector space $L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$ because Λ is unimodular. For simplicity we assume that any power g^n of g fixes all elements of $\hat{\Lambda}$ which are in the inverse image of Λ^{g^n} , where Λ^{g^n} is the set of elements in Λ fixed by g^n . According to [Bo3], there exists a reflection group W^g acting on L with following properties (W1)-(W2):

(W1) The positive roots of W^g are the sums of the conjugates of some positive real roots of $\Pi_{25,1}$.

(W2) Let ρ be the Weyl vector of W^g . The simple roots of W^g are the sums of orbits of simple roots of W that have positive norms and they are also the roots of W^g such that $(r, \rho) = -r^2/2$ with $\rho^2 = 0$.

Let \mathfrak{g}_g be a generalized Kac-Moody superalgebra with the following simple roots:

1. L is the root lattice of \mathfrak{g}_g .
2. The real simple roots are the simple roots of the reflection group W^g , which are the roots r with $(r, \rho) = -r^2/2$.
3. The imaginary simple roots are $n\rho$ ($n \in \mathbb{Z}^+$) with multiplicity $\text{mult}_g(n\rho)$ given by

$$\text{mult}_g(n\rho) = \sum_{ja_k=n} b_k$$

if g has a generalized cycle shape $a_1^{b_1} a_2^{b_2} \cdots$.

Then the denominator formula for the *fake monstrous superalgebra* \mathfrak{g}_g is given by

$$(6.6) \quad e^{-\rho} \sum_{w \in W^g} \det(w) w(\eta_g(e^\rho)) = \prod_{r \in L^+} (1 - e^r)^{\text{mult}_g(r)},$$

where $\eta_g(q)$ is the function defined by

$$(6.7) \quad \eta_g(q) := \eta(\varepsilon_1 q) \eta(\varepsilon_2 q) \cdots \eta(\varepsilon_{24} q)$$

if g has eigenvalues $\varepsilon_1, \dots, \varepsilon_{24}$ on $\Lambda_{\mathbb{R}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. It is easy to check that if g has a generalized cycle shape $a_1^{b_1} a_2^{b_2} \cdots$, then

$$\eta_g(q) = \eta(q^{a_1})^{b_1} \eta(q^{a_2})^{b_2} \cdots$$

Example 6.2. Let $p = 2, 3, 5, 7, 11, 23$ be six prime numbers such that $p+1$ divides 24. We let g be an element of $\text{Aut}(\hat{\Lambda})$ of order p corresponding to an element of $M_{24} \subset \text{Aut}(\Lambda)$ of cycle shape $1^{24/(p+1)} p^{24/(p+1)}$, where M_{24} is the Mathieu group. Then the denominator formula for the fake monstrous superalgebra (in fact, a Lie algebra) $\mathfrak{g}_p := \mathfrak{g}_g$ is given by

$$(6.8) \quad e^{-\rho} \sum_{w \in W^g} \det(w) w \left(e^\rho \prod_{n>0} (1 - e^{n\rho})^{24/(p+1)} (1 - e^{pn\rho})^{24/(p+1)} \right) \\ = \prod_{r \in L^+} (1 - e^r)^{p_g(1-r^2/2)} \prod_{r \in pL^+} (1 - e^r)^{p_g(1-r^2/2p)},$$

where L^+ denotes the set of all positive roots of \mathfrak{g}_p and $p_g(1+n)$ is defined by

$$(6.9) \quad \sum_{n>0} p_g(1+n)q^n = 1/\eta_g(q).$$

$\mathfrak{g}_2, \mathfrak{g}_3, \mathfrak{g}_5, \mathfrak{g}_7$ and \mathfrak{g}_{11} are called the *fake baby monster* Lie algebra, the *fake Fischer monster* Lie algebra, the *fake Harada-Norton monster* Lie algebra, the *fake Held monster* Lie algebra and the *fake Mathieu monster* Lie algebra respectively. We observe that the dimension of \mathfrak{g}_p ($p = 2, 3, 5, 7, 11, 23$) are 18, 14, 10, 8, 6, 1 respectively.

Example 6.3. Let \mathfrak{g}_{fC} be the *fake Conway Lie superalgebra* of rank 10. \mathfrak{g}_{fC} is the fake monstrous Lie superalgebra associated with an element $g \in \text{Aut}(\hat{\Lambda})$ of order 2 such that the descent g_0 of g to $\text{Aut}(\Lambda)$ is of order 2 and the lattice Λ^{g_0} of Λ fixed by g_0 is isomorphic to the lattice E_8 with all norms doubled. The lattice L of \mathfrak{g}_{fC} is the nonintegral lattice of determinant 1/4 all whose vectors have integral norm which is the dual lattice of the sublattice of even vectors of $I_{9,1}$. Here $I_{9,1} := \{(v, m, n) \mid v \in E_8, m, n \in \mathbb{Z}, m+n \text{ is even}\}$ is the lattice of dimension 10. Let W be the Weyl group of \mathfrak{g}_{fC} . In other words, W is the subgroup of $\text{Aut}(L)$ generated by the reflection of norm 1 vectors. The simple roots of W are the norm 1 vectors with $(r, \rho) = -1/2$. The simple roots of \mathfrak{g}_{fC} are the simple roots of W together with the positive multiple $n\rho$ ($n \in \mathbb{Z}^+$) of the Weyl vector $\rho = (0, 0, 1)$ each with multiplicity $8(-1)^n$. Here the multiplicity $-k < 0$ means a superroot of multiplicity k , so that the odd multiples of ρ are superroots. The multiplicity $\text{mult}(r)$ of the root $r = (v, m, n) \in L$ is given by

$$(6.10) \quad \text{mult}(r) = (-1)^{(m-1)(n-1)} p_g((1-r^2)/2) = (-1)^{m+n} |p_g((1-r^2)/2)|,$$

where $p_g(n)$ is defined by

$$\sum p_g(n) = q^{-1/2} \prod_{n>0} (1 - q^{n/2})^{-(-1)^n 8}.$$

Finally the denominator formula for the fake Conway sueralgebra \mathfrak{g}_{fC} is given by

$$(6.11) \quad e^{-\rho} \sum_{w \in W} \det(w) w \left(e^{\rho} \prod_{n>0} (1 - e^{n\rho})^{(-1)^n 8} \right) = \prod_{r \in L^+} (1 - e^r)^{\text{mult}(r)},$$

where L^+ denotes the set of positive roots.

6.2. Kac-Moody Algebras of the Arithmetic Type

Let $A = (a_{ij})$ be a symmetrizable generalized Cartan matrix of degree n and let $\mathfrak{g}(A)$ its associated Kac-Moody Lie algebra (see section 2). Then there exist a diagonal matrix $D = \text{diag}(\epsilon_1, \dots, \epsilon_n)$ with $\epsilon_i > 0$, $\epsilon_i \in \mathbb{Q}$ ($1 \leq i \leq n$) and a symmetric integral matrix $B = (b_{ij})$ such that

$$(6.12) \quad A = DB, \quad \text{g.c.d.}(\{b_{ij} \mid 1 \leq i, j \leq n\}) = 1.$$

We note that such matrices D and B are uniquely determined. Let

$$Q := \sum_{i=1}^n \mathbb{Z} \alpha_i, \quad Q_+ := \sum_{i=1}^n \mathbb{Z}_+ \alpha_i, \quad Q_- := -Q_+,$$

where $\alpha_1, \dots, \alpha_n$ are simple roots of A or $\mathfrak{g}(A)$. Then $Q = Q_+ \cup Q_-$ is a root lattice of A .

Now we have the *canonical symmetric bilinear form*

$$(6.13) \quad (\cdot, \cdot) : Q \times Q \longrightarrow \mathbb{Z}, \quad (\alpha_i, \alpha_j) = b_{ij} = a_{ij}/\epsilon_i.$$

Let Δ , Δ^+ and Δ^- be the set of all roots, positive roots, and negative roots of $\mathfrak{g}(A)$ respectively. We let W be the Weyl group of $\mathfrak{g}(A)$ generated by the fundamental reflections

$$(6.14) \quad r_{\alpha_i}(\beta) := \beta - 2 \frac{(\beta, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i, \quad \beta \in Q, \quad 1 \leq i \leq n.$$

It is clear that Δ is invariant under W . We let

$$(6.15) \quad K := \{ \alpha \in Q_+ \mid \alpha \neq 0, (\alpha, \alpha_i) \leq 0 \text{ for all } i, \text{ and } \text{supp}(\alpha) \text{ is connected} \},$$

where for $\alpha = \sum_{i=1}^n k_i \alpha_i \in Q_+$, $\text{supp}(\alpha)$ is defined to be the subset $\{ \alpha_i \mid k_i > 0 \}$ of the set $\{ \alpha_1, \dots, \alpha_n \}$, and $\text{supp}(\alpha)$ is said to be *connected* if there do not exist nonempty two sets A_1 and A_2 such that $\text{supp}(\alpha) = A_1 \cup A_2$ and $(\alpha, \beta) = 0$ for all $\alpha \in A_1$ and $\beta \in A_2$. Let Δ^{re} (resp. Δ^{im}) be the set of all real roots (resp. imaginary) roots of $\mathfrak{g}(A)$. Then it is easy to check that

$$(6.16) \quad \Delta^{re} = W(\alpha_1) \cup \dots \cup W(\alpha_n)$$

and

$$(6.17) \quad \Delta^{im} \cap Q_+ = W(K).$$

Definition 6.4. A generalized Cartan matrix A of degree n or its associated Kac-Moody Lie algebra $\mathfrak{g}(A)$ is said to be *of the arithmetic type* or *have the arithmetic type* if it is symmetrizable and indecompsable and also if for each $\beta \in Q$ with the property $(\beta, \beta) < 0$ there exist a positive integer $n(\beta) \in \mathbb{Z}^+$ and an imaginary root $\alpha \in \Delta^{im}$ such that

$$(6.18) \quad n(\beta)\beta \equiv \alpha \pmod{Q_0} \text{ on } Q,$$

where $Q_0 := \{ \gamma \in Q \mid (\gamma, \delta) = 0 \text{ for all } \delta \in Q \}$ denotes the kernel of (\cdot, \cdot) .

If we set $M := Q/Q_0$, then $(,)$ induces the *canonical nondegenerate, symmetric integral* bilinear form on the free \mathbb{Z} -module M defined by

$$(6.19) \quad S : M \times M \longrightarrow \mathbb{Z}.$$

We let $\pi : Q \longrightarrow M$ be the projection of Q onto M , and we denote by $\tilde{x} = \pi(x)$ the image of $x \in Q$ under π . We denote by (t_+, t_-, t_0) the signature of a symmetric matrix B .

The following theorem is due to V. V. Nikulin.

Theorem 6.5 ([N5], **Theorem 2.1**). A symmetrizable indecomposable generalized Cartan matrix A or its associated Kac-Moody Lie algebra $\mathfrak{g}(A)$ has the *arithmetic type* if and only if A has one of the following types (a), (b), (c) or (d):

- (a) The finite type case: $B > 0$.
- (b) The affine type case: $B \geq 0$ and B has the signature $(\ell, 0, 1)$.
- (c) The rank 2 hyperbolic case: B has the signature $(1, 1, 0)$.
- (d) The arithmetic hyperbolic type: B is hyperbolic of rank > 2 , equivalently, B has the signature $(\ell - 1, 1, k)$ with $\ell \geq 3$, and the index $[O(S) : \tilde{W}]$ is finite.

Here $O(S)$ and \tilde{W} denote the orthogonal group of S and the image of the Weyl group W under π respectively.

Now we assume that A is of the arithmetic hyperbolic type and that B has the signature $(t_+, 1, k)$ with $t_+ \geq 2$. We choose a subgroup \tilde{W} of $W(S)$ of finite index generated by reflections. We choose a fundamental polyhedron \mathcal{M} of \tilde{W} , and then let $P(\mathcal{M})_{pr}$ be the set of primitive elements of \mathcal{M} which are orthogonal to the faces of \mathcal{M} and directed outside.

Theorem 6.6 ([N5], **Theorem 4.5**). We assume that $S : M \times M \longrightarrow \mathbb{Z}$ is a reflexive primitive hyperbolic, symmetric integral bilinear form and that $\tilde{W} \subset W(S)$ satisfies the following conditions (6.20) and (6.20):

$$(6.20) \quad P(\mathcal{M})_{pr} \text{ generates } M;$$

$$(6.21) \quad P(\mathcal{M}_0)_{pr} \text{ generates } M,$$

where \mathcal{M}_0 is the fundamental polyhedron of $W(S)$.

In addition, we assume that we have a function

$$\lambda : P(\mathcal{M})_{pr} \longrightarrow \mathbb{Z}^+$$

satisfying the conditions (6.22) and (6.23):

$$(6.22) \quad S(\lambda(\alpha)\alpha, \lambda(\alpha)\alpha) \text{ divides } 2S(\lambda(\beta)\beta, \lambda(\alpha)\alpha) \text{ for all } \alpha, \beta \in P(\mathcal{M})_{pr};$$

$$(6.23) \quad \{ \lambda(\alpha)\alpha \mid \alpha \in P(\mathcal{M})_{pr} \} \text{ generates } M.$$

Then the data (S, \tilde{W}, λ) defines canonically a generalized Cartan matrix of the arithmetic hyperbolic type

$$A(S, \tilde{W}, \lambda) = (2S(\lambda(\beta)\beta, \lambda(\alpha)\alpha)/S(\lambda(\alpha)\alpha, \lambda(\alpha)\alpha), \quad \alpha, \beta \in P(\mathcal{M})_{pr}.$$

REMARK 6.7. (a) According to Nikulin (cf. [N3], [N4]) and Vinberg (cf. [V]), there exist only a finite number of isomorphism classes of reflexive primitive hyperbolic symmetric integral bilinear forms S of rank ≥ 3 , and the rank of S is less than 31. Therefore by Theorem 6.6, there are only *finite* Kac-Moody Lie algebras of the arithmetic hyperbolic type.

(2) In [K], a very special case of a generalized Cartan matrix A is considered. This matrix is called just *hyperbolic* there. This has the property that the fundamental polyhedron \mathcal{M} of \tilde{W} is a simplex. There exist only a finite list of these hyperbolic ones. These are characterized by the property: $0 \neq \alpha \in Q$ is an imaginary root if and only if $(\alpha, \alpha) < 0$.

(c) The complete list of the bilinear forms mentioned in Theorem 6.6 is not known yet.

Example 6.8. We consider an example of a *symmetric* generalized Cartan matrix A of the arithmetic hyperbolic type given by

$$(6.22) \quad A = (a_{ij}) = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Let $\mathcal{F} := \mathfrak{g}(A)$ be its associated Kac-Moody Lie algebra of the arithmetic hyperbolic type. Let \mathcal{F}_0 be the *affine* Kac-Moody Lie algebra of type $A_1^{(1)}$ with its Cartan matrix $A_0 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. Then it is known that

$$\mathcal{F}_0 \cong \mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} \cdot c,$$

which is a one-dimensional central extension of the loop algebra $\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}]$. We let \mathcal{F}_0^e be the semi-direct product of \mathcal{F}_0 and $\mathbb{C} \cdot d$, $d := -t \frac{d}{dt}$, whose bracket is defined as follows:

$$\begin{aligned} [x \otimes t^n, y \otimes t^m] &= [x, y] \otimes t^{m+n} + n \langle x, y \rangle \delta_{n, -m} c, \quad m, n \in \mathbb{Z}, \\ [d, x \otimes t^n] &= -n(x \otimes t^n), \quad n \in \mathbb{Z}, \\ [c, a] &= 0 \text{ for all } a \in \mathcal{F}_0^e, \text{ i.e., } c \text{ acts centrally,} \end{aligned}$$

where $x, y \in \mathfrak{sl}_2(\mathbb{C})$ and $\langle x, y \rangle := \text{tr}(xy) = 1/4 \text{tr}(\text{ad } x \text{ ad } y)$ denotes the Cartan-Killing form on the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. The fact that the Weyl group of \mathcal{F} is isomorphic to $PGL(2, \mathbb{Z})$ implies that \mathcal{F} is closely related to the theory of classical modular forms. In [F-F], Feingold and Frenkel constructed \mathcal{F} concretely and computed the Weyl-Kac denominator formula for \mathcal{F} explicitly. The denominator formula for \mathcal{F} is given by

$$(6.23) \quad \sum_{g \in PGL(2, \mathbb{Z})} \det(g) e^{2\pi i \sigma(gP^t gZ)} \\ = e^{2\pi i \sigma(PZ)} \prod_{0 \leq N \in S_2(\mathbb{Z})} \left(1 - e^{2\pi i \sigma(NZ)}\right)^{\text{mult}(N)} \prod_{N \in R} \left(1 - e^{2\pi i \sigma(NZ)}\right),$$

where $P = \begin{pmatrix} 3 & 1/2 \\ 1/2 & 2 \end{pmatrix}$, $Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in H_2$, $S_2(\mathbb{Z})$ denotes the set of all symmetric integral matrices of degree 2 and

$$R := \left\{ N = \begin{pmatrix} n_1 & n_3 \\ n_3 & n_2 \end{pmatrix} \in S_2(\mathbb{Z}) \mid n_1 n_2 - n_3^2 = -1, n_2 \geq 0, n_3 \leq n_1 + n_2, 0 \leq n_1 + n_2 \right\}.$$

We note that the root lattice of \mathcal{F} is isomorphic to $S_2(\mathbb{Z})$.

Let $\mathfrak{h} := \mathbb{C}h_1 \oplus \mathbb{C}h_2 \oplus \mathbb{C}h_3$ be a Cartan subalgebra of \mathcal{F} . We denote by $\alpha_1, \alpha_2, \alpha_3$ the elements of \mathfrak{h}^* defined by

$$(6.24) \quad \alpha_i(h_j) = a_{ij}, \quad 1 \leq i, j \leq 3.$$

We put

$$\gamma_1^* := \alpha_1/2, \quad \gamma_2^* := -\alpha_1 - \alpha_2 - \alpha_3, \quad \gamma_3^* := -\alpha_1 - \alpha_2$$

and

$$P^{++} := \{ n_1 \gamma_1^* + n_2 \gamma_2^* + n_3 \gamma_3^* \mid n_1, n_2, n_3 \in \mathbb{Z}_+, n_3 \geq n_2 \geq n_1 \geq 0 \}.$$

DEFINITION. (1) The number $m_1 + m_2$ in the weight $\lambda = m_1 \gamma_1^* + (m_1 + m_2) \gamma_2^* + m_3 \gamma_3^*$ is called the *level* of the weight λ .

(2) An irreducible standard \mathcal{F}_0^e -module or its character is called \mathcal{F} -dominant if the highest weight of this module lies in P^{++} . A \mathcal{F}_0^e -module or its character is called \mathcal{F} -dominant if each irreducible standard component is \mathcal{F} -dominant.

We let M_k be the complex vector space spanned by those \mathcal{F}_0^e -characters of the form

$$\chi(\tau, z, \omega) = \sum_{m \geq 0} \chi_m(\tau, z, \omega),$$

where for each $m \geq 0$, χ_m is the function satisfying the condition

$$(6.26) \quad \chi_m(\tau, z, \omega) = (-\tau)^{-k} \chi_m(-1/\tau, -z/\tau, \omega - z^2/\tau), \quad \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in H_2.$$

Let $M_k(m)$ be the subspace of M_k spanned by the \mathcal{F}_0^e -characters of level m satisfying the condition (6.26). We recall the results of J. Igusa (cf. [Ig1]) on Siegel modular forms of degree 2. We denote by $[\Gamma_2, k]$ (resp. $[\Gamma_2, k]_0$) the complex vector space of all Siegel modular forms (resp. cusp forms) of weight k on Γ_2 . Let E_k ($k \geq 4$, k : even) be the Eisenstein series of weight k on Γ_2 defined by

$$(6.27) \quad E_k(Z) := \sum_{C,D} \det(CZ + D)^{-k}, \quad Z \in H_2,$$

where (C, D) runs over the set of non-associated pairs of coprime symmetric matrices in $\mathbb{Z}^{(2,2)}$. Igusa proved that E_4, E_6, E_{10} and E_{12} are algebraically independent over \mathbb{C} and that

$$(6.28) \quad \bigoplus_{k=0}^{\infty} [\Gamma_2, k] = \mathbb{C}[E_4, E_6, E_{10}, E_{12}].$$

We define two cusp forms χ_{10} and χ_{12} of weight 10 and 12 by

$$(6.29) \quad \chi_{10} := -43876 \cdot 2^{-12} \cdot 3^{-5} \cdot 5^{-2} \cdot 7^{-1} \cdot 53^{-1} (E_4 E_6 - E_{10})$$

and

$$(6.30) \quad \chi_{12} := 131 \cdot 593 \cdot 2^{-13} \cdot 3^{-7} \cdot 5^{-3} \cdot 7^{-2} \cdot 337^{-1} (3^2 \cdot 7^2 E_4^3 - 2 \cdot 5^3 E_6^2 - 691 E_{12}).$$

Then according to (6.28), we have

$$(6.31) \quad \bigoplus_{k=0}^{\infty} [\Gamma_2, k] = \mathbb{C}[E_4, E_6, \chi_{10}, \chi_{12}].$$

For two nonnegative integers $k, m \geq 0$, we define the set

$$(6.32) \quad S(k, m) := \{ (a, b, c, d) \in (\mathbb{Z}_+)^4 \mid k = 4a + 6b + 10c + 12d, c + d = m \}.$$

We define the subspace $[\Gamma_2, k](m)$ of $[\Gamma_2, k]$ by

$$(6.33) \quad [\Gamma_2, k](m) := \sum_{(a,b,c,d) \in S(k,m)} \mathbb{C} E_4^a E_6^b \chi_{10}^c \chi_{12}^d.$$

Obviously $[\Gamma_2, k] = \sum_{m \geq 0} [\Gamma_2, k](m)$.

For $f(\tau, z, \omega) \in [\Gamma_2, k]$, we let

$$(6.34) \quad f(Z) := f(\tau, z, \omega) = \sum_{m \geq 0} \phi_m(\tau, z) e^{2\pi i m \omega}, \quad Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in H_2$$

be the Fourier-Jacobi expansion of f . As noted in section 4, $\phi_m(\tau, z)$ is a Jacobi form of weight k and m . Now for each non-negative integer $m \geq 0$ we define the linear map $L_m : [\Gamma_2, k] \longrightarrow M_k$ by

$$(6.35) \quad (L_m f)(Z) = (L_m f)(\tau, z, \omega) := \phi_m(\tau, z) e^{2\pi i m \omega}, \quad f \in [\Gamma_2, k], \quad Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in H_2,$$

where $\phi_m(\tau, z)$ ($m \geq 0$) is the Fourier-Jacobi coefficient of the expansion (6.34) of f .

DEFINITION. Let M'_k (resp. $M'_k(m)$) be the subspace of M_k (resp. $M_k(m)$) consisting of $PSL(2, \mathbb{Z})$ -invariant \mathcal{F}_0^e -characters which are \mathcal{F} -dominant.

In [F-F], Theorem 7.9, Feingold and Frenkel showed that L_m maps $[\Gamma_2, k](m)$ isomorphically onto $M'_k(m)$. Thus according to (6.33), we obtain , for each $m \geq 0$,

$$(6.36) \quad \dim_{\mathbb{C}} [\Gamma_2, k](m) = \dim_{\mathbb{C}} M'_k(m) = \#(S(k, m)),$$

where $\#(S)$ denotes the cardinality of the set S . Moreover we have the following ring-isomorphism

$$(6.37) \quad M' = \sum_{k \geq 0} \sum_{m \geq 0} M'_k(m) \cong \mathbb{C} [E_4, E_6, \chi_{10}, \chi_{12}].$$

Let $[\Gamma_2, k]^M$ be the Maass space. (See Appendix B.) Maass showed that

$$[\Gamma_2, k]^M = \mathbb{C} E_4 \oplus [\Gamma_2, k]_0 \quad \text{and} \quad \dim_{\mathbb{C}} [\Gamma_2, k]_0 = \#(S(k, 1)).$$

Also Maass showed that

$$(6.38) \quad [\Gamma_2, k]^M \cong M_k(1) \quad \text{and} \quad [\Gamma_2, k]_0 \cong M'_k(1).$$

The detail for (6.38) can be found in Appendix B, [E-Z] and [Ma2-4]. For $k \geq 4$, even, we have the simple dimensional formulas

$$(6.39) \quad \dim_{\mathbb{C}} M_k(1) = \left\lfloor \frac{k+2}{6} \right\rfloor \quad \text{and} \quad \dim_{\mathbb{C}} M'_k(1) = \left\lfloor \frac{k-4}{6} \right\rfloor = \#(S(k, 1)).$$

6.3. Open Problems

In this subsection, we give some open problems which should be investigated and give some comments. Those of Problem 1-6 are due to R. Borcherds (cf. [Bo6-7]).

Problem 1. Can the methods for constructing automorphic forms as infinite products in section 5 be used for semisimple Lie groups other than $O_{s+2,2}(\mathbb{R})$?

Problem 2. Are there a finite or infinite number of singular automorphic forms that can be written as modular products? Are there such singular modular forms on $O_{s+2,2}(\mathbb{R})$ for $s > 24$?

Problem 3. Interpret the automorphic forms that are modular products in terms of representation theory or the Langlands philosophy.

Problem 4. Extend Theorem 5.4 to higher levels.

Problem 5. Investigate the Lie algebras and the superalgebras coming from other elements of the MONSTER G or $\text{Aut}(\Lambda)$ and write down their denominator formulas explicitly in some nice form.

Problem 6. Are there any generalized Kac-Moody algebras other than the finite dimensional, affine, monstrous or fake monstrous ones, whose simple roots and root multiplicities can both be described explicitly?

Problem 7. Given a generalized Cartan matrix A of the arithmetic hyperbolic type, construct its associated Kac-Moody Lie algebra $\mathfrak{g}(A)$ of the same type explicitly. Give a relationship between the Kac-Moody algebras of the arithmetic hyperbolic type and classical mathematics. For instance, M. Yoshida showed that the Weyl group $W(A)$ of $\mathfrak{g}(A)$ of rank 3 are all hyperbolic triangle groups and that the semidirect product of the Weyl group $W(A)$ and the root lattice of $\mathfrak{g}(A)$ is isomorphic to a discrete subgroup of a parabolic subgroup of $Sp(2, \mathbb{R})$.

Problem 8. Develop the theory of Kac-Moody Lie algebras of the arithmetic hyperbolic type geometrically.

Problem 9. Give an analytic proof of the denominator formula (6.23) for \mathcal{F} analogous to that of the Jacobi's triple product identity.

Problem 10. Find the transformation behaviour of the denominator formula (6.23) for \mathcal{F} under the symplectic involution $Z \rightarrow -Z^{-1}$.

Problem 11. Apply the theory of the Kac-Moody Lie algebra \mathcal{F} to the study of the moduli space of principally polarized abelian surfaces.

Problem 12. Generalize the Maass correspondence to the Kac-Moody algebras of the arithmetic hyperbolic type other than \mathcal{F} ?

Appendix A. Classical Modular Forms

Here we present some well-known results on modular forms whose proofs can be found in many references, e.g., [Kob], [Ma1], [S], and [T].

Let H_1 be the upper half plane and let $\Gamma := SL(2, \mathbb{Z})$ be the elliptic modular group. For a positive integer $N \in \mathbb{Z}^+$, we define

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

and

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv 0 \pmod{N} \right\}$$

$\Gamma(N)$ (resp. $\Gamma_0(N)$) is called the *principal congruence subgroup* of level N (resp. the *Hecke subgroup* of level N). The subgroup Γ_θ of Γ generated by $\pm \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is called the *theta group*. A subgroup Γ_1 of Γ is called a *congruence subgroup* if Γ_1 contains $\Gamma(N)$ for some positive integer N . For instance, the Hecke subgroup $\Gamma_0(N)$ is a congruence subgroup because $\Gamma(N) \subset \Gamma_0(N) \subset \Gamma$. And $\Gamma(N)$ is a normal subgroup because it is the kernel of the reduction-modulo- N homomorphism $SL(2, \mathbb{Z}) \rightarrow SL(2, \mathbb{Z}/N\mathbb{Z})$. It is well known that the index of $\Gamma(N)$ in Γ is given by

$$(A.1) \quad [\Gamma : \Gamma(N)] = N^3 \prod_{p|N} (1 - p^{-2}).$$

The proof of (A.1) can be found in [Sh] pp.21-22. It was discovered around the 1880s that there are an infinite number of examples of noncongruence subgroups (cf. [Ma1] pp. 76-78). But $SL(n, \mathbb{Z})$ behaves quite differently for $n \geq 3$. In fact, it has been proved that if $n \geq 3$, every subgroup of $SL(n, \mathbb{Z})$ of finite index is a congruence subgroup (cf. [Bas]). A similar result for the Siegel modular group $Sp(n, \mathbb{Z})$ for $n \geq 2$ can be found in [Me].

For an integer $k \in \mathbb{Z}$, we denote by $[\Gamma, k]$ (resp. $[\Gamma, k]_0$) the vector space of all modular forms (resp. cusp forms) of weight k for the elliptic modular group Γ . Only for $k \geq 0$, k even, $[\Gamma, k]$ does not vanish.

For any positive integer k with $k \geq 2$, we put

$$(A.2) \quad G_{2k}(\tau) := \sum'_{m,n} \frac{1}{(m\tau + n)^{2k}}, \quad \tau \in H_1.$$

Here the symbol \sum' means that the summation runs over all pair of integers (m, n) distinct from $(0,0)$. Then $G_{2k} \in [\Gamma, 2k]$ and $G_{2k}(\infty) = 2\zeta(2k)$, where $\zeta(s)$ denotes

the Riemann zeta function. G_{2k} ($k \in \mathbb{Z}^+$, $k \geq 2$) is called the *Eisenstein series* of index $2k$. The Fourier expansion of G_{2k} ($k \geq 2$) is given by

$$(A.3) \quad G_{2k}(\tau) = 2\zeta(2k) + 2 \frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \quad \tau \in H_1,$$

where $q = e^{2\pi i \tau}$ and $\sigma_\ell(n) := \sum_{0 < d|n} d^\ell$.

We consider the following parabolic subgroup P of Γ given by

$$P := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma \right\}.$$

Then we can see easily that

$$G_{2k}(\tau) = 2\zeta(2k) \sum_{\gamma \in P \setminus \Gamma} \left(\frac{d(\gamma < \tau >)}{d\tau} \right)^k.$$

Here for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\tau \in H_1$, we set

$$\gamma < \tau > := (a\tau + b)(c\tau + d)^{-1}.$$

For a positive integer $k \geq 2$, we can see easily that

$$(A.4) \quad E_{2k}(\tau) \stackrel{\text{def}}{=} \frac{G_{2k}(\tau)}{2\zeta(2k)} = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$

where B_k ($k = 0, 1, 2, \dots$) denotes the k -th Bernoulli number defined by the formal power series expansion:

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

Then clearly $B_{2k+1} = 0$ for $k \geq 1$. The first few B_k are

$$B_0 = 1, \quad B_1 = -1/2, \quad B_2 = 1/6, \quad B_4 = -1/30, \quad B_6 = 1/42, \quad B_8 = -1/30, \quad B_{10} = 5/66, \\ B_{12} = -691/2730, \quad B_{14} = 7/6, \quad B_{16} = -3617/510, \quad B_{18} = 43867/798, \dots$$

Indeed, (A.4) follows immediately from the relation

$$\zeta(2k) = (-1)^{k-1} \frac{2^{2k-1} \pi^{2k}}{(2k)!} B_{2k}, \quad k = 1, 2, \dots.$$

For example,

$$\begin{aligned}
 E_4(\tau) &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n, & (240 = 2^4 \cdot 3 \cdot 5) \\
 E_6(\tau) &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n, & (504 = 2^3 \cdot 3^2 \cdot 7) \\
 E_8(\tau) &= 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n)q^n, & (480 = 2^5 \cdot 3 \cdot 5) \\
 E_{10}(\tau) &= 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n)q^n, & (264 = 2^3 \cdot 3 \cdot 11) \\
 E_{12}(\tau) &= 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^n, & (65520 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13) \\
 E_{14}(\tau) &= 1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n)q^n, & (24 = 2^3 \cdot 3).
 \end{aligned}$$

According to the argument on the dimension of $[\Gamma, k]$, we obtain the relation

$$(A.5) \quad E_4^2 = E_8, \quad E_4 E_6 = E_{10}.$$

These are equivalent to the identities:

$$\begin{aligned}
 \sigma_7(n) &= \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m), \\
 11\sigma_9(n) &= 21\sigma_5(n) - 10\sigma_3(n) + 5040 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_5(n-m).
 \end{aligned}$$

More generally, every E_{2k} can be expressed as a polynomial in E_4 and E_6 . For instance, $E_{14} = E_4^2 E_6$.

We put

$$(A.6) \quad g_2 := 60G_4, \quad \text{and} \quad g_3 := 140G_6.$$

Then it is obvious that

$$g_2 = \frac{(2\pi)^4}{2^2 \cdot 3} E_4, \quad \text{and} \quad g_3 = \frac{(2\pi)^6}{2^3 \cdot 3^3} E_6.$$

Since $g_2(\infty) = \frac{4}{3}\pi^4$ and $g_3(\infty) = \frac{8}{27}\pi^6$, we see that the *discriminant*

$$(A.7) \quad \tilde{\Delta} := g_2^3 - 27g_3^2$$

is a cusp form of weight 12, that is, $\tilde{\Delta} \in [\Gamma, 12]_0$. And we have

$$(A.8) \quad \begin{aligned} \tilde{\Delta}(\tau) &= (2\pi)^{12} \cdot 2^{-6} \cdot 3^{-3} (E_4(\tau)^3 - E_6(\tau)^2) \\ &= (2\pi)^{12} (q - 24q^2 + 252q^3 - 1472q^4 + \dots) \\ &= (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24} \quad (\text{Jacobi's identity}). \end{aligned}$$

In this article, we put $\Delta(\tau) := (2\pi)^{-12} \tilde{\Delta}(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$.

Fix $\tau \in H_1$. The Weierstrass \wp -function $\wp(z; \tau)$ is defined by

$$(A.9) \quad \wp(z; \tau) := \frac{1}{\tau^2} + \sum'_{m,n} \left\{ \frac{1}{(z - n - m\tau)^2} - \frac{1}{(n + m\tau)^2} \right\}, \quad z \in \mathbb{C}.$$

Then $\wp(z; \tau)$ is a meromorphic function with respect to $1, \tau$ with double poles at the points $n + m\tau$, $n, m \in \mathbb{Z}$. The map $\varphi_\tau : \mathbb{C} \rightarrow \mathbb{P}^2$ defined by

$$(A.10) \quad \varphi_\tau(z) := [1 : \wp(z; \tau) : \frac{d}{dz} \wp(z; \tau)], \quad z \in \mathbb{C}$$

induces an isomorphism of $X = \mathbb{C}/L_\tau$ with a nonsingular plane curve of the form

$$(A.11) \quad X_0 X_2^2 = 4X_1^3 + aX_0^2 X_1 + bX_0^3,$$

where a and b are suitable constants depending on τ and $L_\tau := \{m\tau + n \mid m, n \in \mathbb{Z}\}$ is the lattice in \mathbb{C} generated by 1 and τ . If we put $x = \wp(z; \tau)$ and $y = \frac{d}{dz} \wp(z; \tau)$, we have the differential equation

$$(A.12) \quad y^2 = 4x^3 - g_2(\tau)x - g_3(\tau).$$

Up to a numerical factor, $\tilde{\Delta}(\tau) := (g_2^3 - g_3^2)(\tau)$ is the discriminant of the polynomial $4x^3 - g_2(\tau)x - g_3(\tau)$. Since $\tilde{\Delta}(\tau) \neq 0$, $X_\tau = \mathbb{C}^2/L_\tau$ is a *nonsingular* elliptic curve. This story tells us as the reason why the function Δ is called the discriminant. We observe that the differential equation (A.12) shows that it is the inverse function for the elliptic integral in Weierstrass normal form, that is,

$$(A.13) \quad z - z_0 = \int_{\wp(z_0; \tau)}^{\wp(z; \tau)} (4w^3 - g_2(\tau)w - g_3(\tau))^{-1/2} dw.$$

The Ramanujan tau function $\tau(n)$ ($n \in \mathbb{Z}^+$) is defined by

$$(A.14) \quad \Delta(\tau) = (2\pi)^{-12} \tilde{\Delta}(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

The Dedekind eta function $\eta(\tau)$ is defined by

$$(A.15) \quad \eta(\tau) := q^{1/24} \prod_{n \geq 1} (1 - q^n), \quad \tau \in H_1.$$

Then $\eta(\tau)$ satisfies

$$\eta(\tau + 1) = \eta(\tau) \quad \text{and} \quad \eta(-1/\tau) = (\tau/i)^{1/2} \eta(\tau).$$

The Dedekind eta function $\eta(\tau)$ is related to the partition function $p(n)$ as follows:

$$(A.16) \quad q^{1/24} \eta(\tau)^{-1} = \prod_{n \geq 1} (1 - q^n)^{-1} = \sum_{n \geq 0} p(n) q^n,$$

where $p(n)$ is the number of partitions of n , i.e., the number of ways of writing

$$n = n_1 + \cdots + n_r, \quad n_j \in \mathbb{Z}^+ \quad (1 \leq j \leq r).$$

The *modular invariant* $J(\tau)$ is defined by

$$(A.17) \quad J := (60G_4)^3 / \tilde{\Delta} = g_2^3 / \tilde{\Delta} = (2\pi)^{12} \cdot 2^{-6} \cdot 3^{-3} E_4^3 / \tilde{\Delta},$$

The function $J(\tau)$ was first constructed by Julius Wilhelm Richard Dedekind (1831-1916) in 1877 and Felix Klein (1849-1925) in 1878. The modular invariant $J(\tau)$ has the following properties:

- (J1) $J(\tau)$ is a modular function. J is holomorphic in H_1 with a simple pole at ∞ , $J(i) = 1$ and $J\left(\frac{-1+\sqrt{3}i}{2}\right) = 0$.
- (J2) J defines a conformal mapping which is one-to-one from H_1/Γ onto \mathbb{C} , and hence J provides an identification of $H_1/\Gamma \cup \{\infty\}$ with the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$.
- (J3) The following are equivalent for a function f which is meromorphic on H_1 ;
 - (a) f is a modular function;
 - (b) f is a quotient of two modular forms of the same weight;
 - (c) f is a rational function of J , i.e., a quotient of polynomials in J . Thus J is called the *Hauptmodul* or the *fundamental function*.

The q -expansion of $j(\tau) := 1728J(\tau) = 2^6 \cdot 3^3 J(\tau)$, also called the modular invariant, is given by

$$(A.19) \quad j(\tau) = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n)q^n = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots .$$

We observe that $j(i) = 1728 = 2^6 \cdot 3^3$ and $j\left(\frac{1+\sqrt{3}i}{2}\right) = 0$. It was already mentioned that there is a surprising connection of the coefficients in (A.19) with the representations of the Fischer-Griess monster group. All of the early Fourier coefficients in (A.19) are simple linear combinations of degrees of characters of the MONSTER. This was first observed by John McKay and John Thompson. The modular invariant $J(\tau)$ is used to prove the small Picard theorem and to study an explicit reciprocity law for an imaginary quadratic number field.

For a positive definite symmetric real matrix S of degree n , we define the theta series

$$(A.20) \quad \theta_S(\tau) := \sum_{x \in \mathbb{Z}^n} e^{\pi i S[x]\tau}, \quad \tau \in H_1,$$

where $S[x] := {}^t x S x$ denotes the quadratic form associated to S . We can prove the transformation formula

$$(A.21) \quad \theta_{S^{-1}}(-1/\tau) = (\det S)^{1/2} (\tau/i)^{n/2} \theta_S(\tau).$$

It is known that if S is a positive definite symmetric even integral, unimodular matrix of degree n , then n is divided by 8 and $\theta_S(\tau) \in [\Gamma, n/2]$. In fact, for $n = 8$, there is only one positive definite symmetric even integral unimodular matrix up to equivalence modulo $GL(8, \mathbb{Z})$. For $n = 16$, there are two nonequivalent examples modulo $GL(16, \mathbb{Z})$. For $n = 24$, there are 24 nonequivalent examples modulo $GL(24, \mathbb{Z})$.

We consider a Jacobi function

$$(A.22) \quad \theta(\tau, z) := \sum_{n \in \mathbb{Z}} e^{\pi i (n^2 \tau + 2nz)}, \quad (\tau, z) \in H_1 \times \mathbb{C}$$

Then $\theta(\tau, z)$ satisfies the following properties:

- ($\theta.1$) $\theta(\tau, z)$ is an entire function on $H_1 \times \mathbb{C}$.
- ($\theta.2$) θ is quasi-periodic as a function of z in the following sense:
 - $\theta(\tau, z + n) = \theta(\tau, z)$ for all $n \in \mathbb{Z}$;
 - $\theta(\tau, z + n\tau) = e^{-\pi i (n^2 \tau + 2nz)} \theta(\tau, z)$ for all $n \in \mathbb{Z}$.

($\theta.3$) $\theta(\tau, z)$ satisfies the transformation formula

$$\theta(\tau, z) = (\tau/i)^{1/2} \sum_{n \in \mathbb{Z}} e^{-\pi i(n-z)^2/\tau}.$$

($\theta.4$) $\theta(\tau, (1 + \tau)/2) = 0$.

($\theta.5$) Fixing τ , the only zero of $\theta(z) := \theta(\tau, z)$ as a function of z in the period parallelogram on 1 and τ is $z = (1 + \tau)/2$. Moreover, this zero is simple.

For a proof of ($\theta.3$), use Poisson formula.

Appendix B. Kohnen Plus Space and Maass Space

Here we review the Kohnen plus space and the Maass space. And then we give isomorphisms of them with the vector spaces of Jacobi forms. For more detail we refer to [Koh], [Ma2-4].

We fix two positive integers n and m . Let

$$H_n := \{Z \in \mathbb{C}^{(n,n)} \mid Z = {}^tZ, \text{Im}Z > 0\}$$

be the Siegel upper half plane of degree n and let $\Gamma_n := Sp(n, \mathbb{Z})$ the Siegel modular group of degree n . That is,

$$\Gamma_n := \{g \in \mathbb{Z}^{(2n,2n)} \mid {}^t g J g = J\}, \quad J := \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$

Here E_n denotes the identity matrix of degree n . Then the real symplectic group $Sp(n, \mathbb{R})$ acts on H_n transitively. If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $Z \in H_n$,

$$(B.1) \quad M \langle Z \rangle := (AZ + B)(CZ + D)^{-1}.$$

Let \mathcal{M} be a positive definite, symmetric half integral matrix of degree m . For a fixed element $Z \in H_n$, we denote by $\Theta_{\mathcal{M}, Z}^{(n)}$ the vector space of all the functions $\theta : \mathbb{C}^{(m,n)} \rightarrow \mathbb{C}$ satisfying the condition

$$(B.2) \quad \theta(W + \lambda Z + \mu) = e^{-2\pi i \sigma(\mathcal{M}[\lambda]Z + 2{}^t W \mathcal{M} \lambda)}, \quad W \in \mathbb{C}^{(m,n)}$$

for all $\lambda, \mu \in \mathbb{Z}^{(m,n)}$. For brevity, we put $L := \mathbb{Z}^{(m,n)}$ and $\mathcal{L}_{\mathcal{M}} := L/(2\mathcal{M})L$. For each $\gamma \in \mathcal{L}_{\mathcal{M}} := L/(2\mathcal{M})L$, we define the theta series

$$(B.3) \quad \theta_{\gamma}(Z, W) := \sum_{\lambda \in L} e^{2\pi i \sigma(\mathcal{M}[\lambda + (2\mathcal{M})^{-1}\gamma]Z + 2{}^t W \mathcal{M}(\lambda + (2\mathcal{M})^{-1}\gamma))}, \quad (Z, W) \in H_n \times \mathbb{C}^{(m,n)}.$$

Then $\{\theta_\gamma(Z, W) \mid \gamma \in \mathcal{L}_M\}$ forms a basis for $\Theta_{\mathcal{M}, Z}^{(n)}$. For any Jacobi form $\phi(Z, W) \in J_{k, \mathcal{M}}(\Gamma_n)$, the function $\phi(Z, \cdot)$ with fixed Z is an element of $\Theta_{\mathcal{M}, Z}^{(n)}$ and $\phi(Z, W)$ can be written as a linear combination of theta series $\theta_\gamma(Z, W)$ ($\gamma \in \mathcal{L}_M$):

$$(B.4) \quad \phi(Z, W) = \sum_{\gamma \in \mathcal{L}_M} \phi_\gamma(Z) \theta_\gamma(Z, W).$$

Here $\phi = (\phi_\gamma(Z))_{\gamma \in \mathcal{L}_M}$ is a vector valued automorphic form with respect to theta multiplier system.

(I) Kohnen Plus Space (cf. [Ib], [Koh])

We consider the case: $m = 1$, $\mathcal{M} = E_m$, $L = \mathbb{Z}^{(1, n)} \cong \mathbb{Z}^n$. We consider the theta series

$$(B.5) \quad \theta^{(n)}(Z) := \sum_{\lambda \in L} e^{2\pi i \sigma(\lambda Z^t \lambda)} = \theta_0(Z, 0), \quad Z \in H_n.$$

We put

$$(B.6) \quad \Gamma_0^{(n)}(4) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \mid C \equiv 0 \pmod{4} \right\}.$$

Then $\Gamma_0^{(n)}(4)$ is a congruence subgroup of Γ_n . We define the automorphic factor $j : \Gamma_0^{(n)}(4) \times H_n \rightarrow \mathbb{C}^\times$ by

$$(B.7) \quad j(\gamma, Z) := \frac{\theta^{(n)}(\gamma \langle Z \rangle)}{\theta^{(n)}(Z)}, \quad \gamma \in \Gamma_0^{(n)}(4), \quad Z \in H_n.$$

Then we obtain the relation

$$(B.8) \quad j(\gamma, Z)^2 = \epsilon(\gamma) \cdot \det(CZ + D), \quad \epsilon(\gamma)^2 = 1$$

for any $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(4)$.

Now we define the Kohnen plus space $M_{k-\frac{1}{2}}^+(\Gamma_0^{(n)}(4))$ introduced by W. Kohnen (cf. [Koh]). $M_{k-\frac{1}{2}}^+(\Gamma_0^{(n)}(4))$ is the vector space consisting of holomorphic functions $f : H_n \rightarrow \mathbb{C}$ satisfying the following conditions:

- (a) $f(\gamma \langle Z \rangle) = j(\gamma, Z)^{2k-1} f(Z)$ for all $\gamma \in \Gamma_0^{(n)}(4)$;
- (b) f has the Fourier expansion

$$f(Z) = \sum_{T \geq 0} a(T) e^{2\pi i \sigma(TZ)},$$

where T runs over the set of semi-positive, half-integral symmetric matrices of degree n and $a(T) = 0$ unless $T \equiv -\mu^t \mu \pmod{4S_n^*(\mathbb{Z})}$ for some $\mu \in \mathbb{Z}^{(n,1)}$. Here we put

$$S_n^*(\mathbb{Z}) := \{T \in \mathbb{R}^{(n,n)} \mid T = {}^t T, \sigma(TS) \in \mathbb{Z} \text{ for all } S = {}^t S \in \mathbb{Z}^{(n,n)}\}.$$

For $\phi \in J_{k,1}(\Gamma_n)$, according to (B.4), we have

$$(B.9) \quad \phi(Z, W) = \sum_{\gamma \in L/2L} f_\gamma(Z) \theta_\gamma(Z, W), \quad Z \in H_n, W \in \mathbb{C}^n.$$

Now we put

$$(B.10) \quad f_\phi(Z) := \sum_{\gamma \in L/2L} f_\gamma(4Z), \quad Z \in H_n.$$

Then $f_\phi \in M_{k-\frac{1}{2}}^+(\Gamma_0^{(n)}(4))$.

Theorem 1 (Kohnen-Zagier ($n = 1$), Ibukiyama ($n > 1$)). *Suppose k is an even positive integer. We have the isomorphism*

$$J_{k,1}(\Gamma_n) \cong M_{k-\frac{1}{2}}^+(\Gamma_0^{(n)}(4))$$

$$\phi \mapsto f_\phi.$$

Furthermore the isomorphism is compatible with the action of Hecke operators.

(II) Maass Space

The Maass space or the Maaß's Spezialschar was introduced by H. Maass (1911-1993) to solve the Saito-Kurokawa conjecture. Let $k \in \mathbb{Z}^+$. We denote by $[\Gamma_2, k]$ the vector space of all Siegel modular forms of weight k and degree 2. We denote by $[\Gamma_2, k]^M$ the vector space of all Siegel modular forms $F : H_2 \rightarrow \mathbb{C}$, $F(Z) = \sum_{T \geq 0} a_F(T) e^{2\pi i \sigma(TZ)}$ in $[\Gamma_2, k]$ satisfying the following condition:

$$(B.11) \quad a_F \begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix} = \sum_{d \mid (n,r,m), d > 0} d^{k-1} a_F \begin{pmatrix} \frac{mn}{d^2} & \frac{r}{2d} \\ \frac{r}{2d} & 1 \end{pmatrix}$$

$$\text{for all } T = \begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix} \geq 0 \text{ with } n, r, m \in \mathbb{Z}.$$

The vector space $[\Gamma_2, k]^M$ is called the *Maass space* or the *Maaß's Spezialschar*.

For any F in $[\Gamma_2, k]$, we let

$$(B.12) \quad F(Z) = \sum_{m \geq 0} \phi_m(\tau, z) e^{2\pi i m \tau'}, \quad Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in H_2$$

be the Fourier-Jacobi expansion of F . Then for any $m \in \mathbb{Z}_+$ we obtain the linear map

$$(B.13) \quad \rho_m : [\Gamma_2, k] \longrightarrow J_{k,m}(\Gamma_1), \quad F \longmapsto \phi_m.$$

We denote that ρ_0 is nothing but the Siegel Φ -operator.

Maass (cf. [Ma 2-3]) showed that for k even, there exists a natural map $V : J_{k,1}(\Gamma_1) \longrightarrow [\Gamma_2, k]$ such that $\rho_1 \circ V$ is the identity. More precisely, we let $\phi \in J_{k,1}(\Gamma_1)$ with Fourier coefficients $c(n, r)$ ($n, r \in \mathbb{Z}$, $r^2 \leq 4n$) and we define for any $m \in \mathbb{Z}_{\geq 0}^+$

$$(B.13) \quad (V_m \phi)(\tau, z) := \sum_{n, r \in \mathbb{Z}, r^2 \leq 4mn} \left(\sum_{d|(n, r, m)} d^{k-1} c\left(\frac{mn}{d^2}, \frac{r}{d}\right) \right) e^{2\pi i(n\tau + rz)}.$$

It is easy to see that $V_1 \phi = \phi$ and $V_m \phi \in J_{k,m}(\Gamma_1)$. We define

$$(B.15) \quad (V\phi) \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} := \sum_{m \geq 0} (V_m \phi)(\tau, z) e^{2\pi i m \tau'}, \quad \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in H_2.$$

We denote by T_n ($n \in \mathbb{Z}^+$) the usual Hecke operators on $[\Gamma_2, k]$ resp. $[\Gamma_2, k]_0$. Here $[\Gamma_2, k]_0$ denote the vector subspace consisting of all cusp forms in $[\Gamma_2, k]$. For instance, if p is a prime, T_p and T_{p^2} are the Hecke operators corresponding to the two generators $\Gamma_2 \text{diag}(1, 1, p, p) \Gamma_2$ and $\Gamma_2 \text{diag}(1, p, p^2, p) \Gamma_2$ of the local Hecke algebra of Γ_2 at p respectively. We denote by $T_{J,n}$ ($n \in \mathbb{Z}^+$) the Hecke operators on $J_{k,m}(\Gamma_1)$ resp. $J_{k,m}^{\text{cusp}}(\Gamma_1)$ (cf. [E-Z]).

Theorem 2 (Maass [Ma 2-4], Eichler-Zagier [E-Z], Theorem 6.3). *Suppose k is an even positive integer. Then the map $\phi \mapsto V\phi$ gives an injection of $J_{k,1}(\Gamma_1)$ into $[\Gamma_2, k]$ which sends cusp forms to cusp forms and is compatible with the action of Hecke operators. The image of the map V is equal to the Maass space $[\Gamma_2, k]^M$. If p is a prime, one has*

$$T_p \circ V = V \circ (T_{J,p} + p^{k-2}(p+1))$$

and

$$T_{p^2} \circ V = V \circ (T_{J,p}^2 + p^{k-2}(p+1)T_{J,p} + p^{2k-2}).$$

In summary, we have the following isomorphisms

$$\begin{array}{ccccccc} [\Gamma_2, k]^M & \cong & J_{k,1}(\Gamma_1) & \cong & M_{k-\frac{1}{2}}^+(\Gamma_0^{(1)}(4)) & \cong & [\Gamma_1, 2k-2], \\ V\phi & \leftarrow & \phi & \rightarrow & f_\phi & & \end{array}$$

where the last isomorphism is the Shimura correspondence. And all the above isomorphisms are compatible with the action of Hecke operators.

REMARK. (1) $[\Gamma_2, k]^M = \mathbb{C}E_k^{(2)} \oplus [\Gamma_2, k]_0^M$, where $E_k^{(2)}$ is the Siegel-Eisenstein series of weight k on Γ_2 given by

$$E_k^{(2)}(Z) := \sum_{\{C,D\}} \det(CZ + D)^{-k}, \quad Z \in H_2$$

(sum over non-associated pairs of coprime symmetric matrices $C, D \in \mathbb{Z}^{(2,2)}$) and $[\Gamma_2, k]_0^M := [\Gamma_2, k]^M \cap [\Gamma_2, k]_0$.

(2) Maass proved that $\dim[\Gamma_2, k]^M = \lfloor \frac{k-4}{6} \rfloor$ for $k \geq 4$ even. It is known that $\dim[\Gamma_2, k] \sim 2^{-6} \cdot 3^{-3} \cdot 5^{-1} \cdot k^3$ as $k \rightarrow \infty$.

We observe that Theorem 2 implies that $[\Gamma_2, k]^M$ is invariant under all the Hecke operators and that it is annihilated by the operator

$$(B.16) \quad \mathcal{C}_p := T_p^2 - p^{k-2}(p+1)T_p - T_{p^2} + p^{2k-2}$$

for every prime p . We let $F \in [\Gamma_2, k]$ be a nonzero Hecke eigenform with $T_n F = \lambda_n F$ for $n \in \mathbb{Z}^+$. For a prime p , we put

$$Z_{F,p}(X) := 1 - \lambda_p X + (\lambda_p^2 - \lambda_{p^2} - p^{2k-4})X^2 - \lambda_p p^{2k-3} X^3 + p^{4k-6} X^4$$

so that $Z_{F,p}(p^{-s})$ ($s \in \mathbb{C}$) is the local spinor zeta function of F at p . We put

$$(B.17) \quad Z_F(s) := \prod_p Z_{F,p}(p^{-s}), \quad \operatorname{Re} s \gg 0.$$

Then we have

$$(B.18) \quad Z_F(s) = \zeta(2s - 2k + 4) \sum_{n \geq 1} \frac{\lambda_n}{n^s}, \quad \operatorname{Re} s \gg 0.$$

Theorem 3 (Saito-Kurokawa conjecture ; Andrianov [An], Maass [Ma 2-3], Zagier [Za]). *Let $k \in \mathbb{Z}^+$ be even and let F be a nonzero Hecke eigenform in $[\Gamma_2, k]^M$. Then there exists a unique normalized Hecke eigenform f in $[\Gamma_1, 2k - 2]$ such that*

$$Z_F(s) = \zeta(s - k + 1)\zeta(s - k + 2)L_f(s),$$

where $L_f(s)$ is the Hecke L -function attached to f .

Theorem 2 implies that $Z_F(s)$ has a pole at $s = k$ if F is a Hecke eigenform in $[\Gamma_2, k]_0^M$. If $F \in [\Gamma_2, k]_0$ is an eigenform, it was proved by Andrianov that $Z_F(s)$ has an analytic continuation to \mathbb{C} which is holomorphic everywhere if k is odd and is holomorphic except for a possible simple pole at $s = k$ if k is even. Moreover, the global function

$$(B.19) \quad Z_F^*(s) := (2\pi)^{-s}\Gamma(s)\Gamma(s - k + 2)Z_F(s)$$

is $(-1)^k$ -invariant under $s \mapsto 2k - 2 - s$. It was proved by Evdokimov and Oda that $Z_F(s)$ is holomorphic everywhere if and only if F is contained in the orthogonal complement of $[\Gamma_2, k]_0^M$ in $[\Gamma_2, M]$.

So far a generalization of the Maass space to higher genus $n > 2$ has not been given. There is a partial negative result by Ziegler (cf [Zi], Theorem 4.2). We will describe his result roughly. Let $F \in [\Gamma_{g+1}, k]$ ($g \in \mathbb{Z}^+$, k : even) be a Siegel modular form on H_{g+1} of weight k and let

$$F \begin{pmatrix} Z_1 & {}^tW \\ W & z_2 \end{pmatrix} = \sum_{m \geq 0} \Phi_{F,m}(Z_1, W) e^{2\pi i m z_2}, \quad \begin{pmatrix} Z_1 & {}^tW \\ W & z_2 \end{pmatrix} \in H_{g+1}, \text{ with } Z_1 \in H_g, z_2 \in H_1$$

be the Fourier-Jacobi expansion of F . For any nonnegative integer m , we consider the linear mapping

$$\rho_{g,m,k} : [\Gamma_{g+1}, k] \longrightarrow J_{k,m}(\Gamma_g)$$

defined by

$$\rho_{g,m,k}(F) := \Phi_{F,m}, \quad F \in [\Gamma_{g+1}, k].$$

Ziegler showed that for $g \geq 32$, the mapping

$$\rho_{g,1,16} : [\Gamma_{g+1}, 16] \longrightarrow J_{16,1}(\Gamma_g)$$

is not surjective.

Question: Is $\rho_{g,1,k}$ surjective for an integer $k \neq 16$?

Appendix C. The Orthogonal Group $O_{s+2,2}(\mathbb{R})$

A lattice is a free \mathbb{Z} -module of finite rank with a nondegenerate symmetric bilinear form with values in \mathbb{Q} . Let K be a positive definite unimodular even integral lattice of rank s with its associated symmetric matrix S_0 . Let $\Pi_{1,1}$ be the unique unimodular even integral Lorentzian lattice of rank 2 with its associated symmetric matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. If there is no confusion, we write $\Pi_{1,1} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

We define the lattices L and M by

$$(C.1) \quad L := K \oplus \Pi_{1,1} \quad \text{and} \quad M := \Pi_{1,1} \oplus L = \Pi_{1,1} \oplus K \oplus \Pi_{1,1}.$$

We put

$$K_{\mathbb{R}} := K \otimes_{\mathbb{Z}} \mathbb{R}, \quad L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}, \quad M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{C}.$$

We let

$$Q_K := S_0, \quad Q_L := \begin{pmatrix} 0 & 0 & -1 \\ 0 & S_0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad Q_M := \begin{pmatrix} 0 & 0 & \Pi_{1,1} \\ 0 & S_0 & 0 \\ \Pi_{1,1} & 0 & 0 \end{pmatrix}$$

be the unimodular even integral symmetric matrices associated with the lattices K , L and M respectively. The isometry group $O_M(\mathbb{R})$ of the quadratic space $(M_{\mathbb{R}}, Q_M)$ is defined by

$$(C.2) \quad O_M(\mathbb{R}) := \{ g \in GL(M_{\mathbb{R}}) \cong GL(s+4, \mathbb{R}) \mid {}^t g Q_M g = Q_M \}.$$

Then it is easy to see that $O_M(\mathbb{R})$ is isomorphic to the orthogonal group $O_{s+2,2}(\mathbb{R})$. Here for two nonnegative integers p and q with $p+q=n$, $O_{p,q}(\mathbb{R})$ is defined by

$$(C.3) \quad O_{p,q}(\mathbb{R}) := \{ g \in GL(p+q, \mathbb{R}) \mid {}^t g E_{p,q} g = E_{p,q} \},$$

where

$$E_{p,q} := \begin{pmatrix} E_p & 0 \\ 0 & -E_q \end{pmatrix}.$$

Indeed, Q_M is congruent to $E_{s+2,2}$ over \mathbb{R} , that is, $Q_M = {}^t a E_{s+2,2} a$ for some $a \in GL(s+4, \mathbb{R})$ and hence $O_M(\mathbb{R}) = a^{-1} O_{s+2,2}(\mathbb{R}) a$. For brevity, we write $O(p, \mathbb{R}) := O_{p,0}(\mathbb{R})$ and $SO(p, \mathbb{R}) := SO_{p,0}(\mathbb{R})$. Similarly, we have $O_L(\mathbb{R}) \cong O_{s+1,1}(\mathbb{R})$ and $O_K(\mathbb{R}) \cong O(s, \mathbb{R})$. We denote by $(,)_K$, $(,)_L$ and $(,)_M$ the nondegenerate symmetric bilinear forms on $K_{\mathbb{R}}$, $L_{\mathbb{R}}$ and $M_{\mathbb{R}}$ corresponding to Q_K , Q_L and Q_M respectively.

We let

$$(C.4) \quad D = D(M_{\mathbb{R}}) := \{ z \subset M_{\mathbb{R}} \mid \dim_{\mathbb{R}} z = 2, z \text{ is oriented and } (\cdot, \cdot)_M|_z < 0 \}$$

be the space of oriented negative two dimensional planes in $M_{\mathbb{R}}$. We observe that a negative two dimensional plane in $M_{\mathbb{R}}$ occurs twice in D with opposite orientation. Thus D may be regarded as a space consisting of two copies of the space of negative two dimensional planes in $M_{\mathbb{R}}$. For $z \in D$, the majorant associated to z is defined by

$$(C.5) \quad (\cdot, \cdot)_z := \begin{cases} (\cdot, \cdot)_M & \text{on } z^{\perp}; \\ -(\cdot, \cdot)_M & \text{on } z. \end{cases}$$

Then $(M_{\mathbb{R}}, (\cdot, \cdot)_z)$ is a positive definite quadratic space. It is easy to see that we have the orthogonal decomposition $M_{\mathbb{R}} = z^{\perp} \oplus z$ with respect to $(\cdot, \cdot)_z$ and that $(\cdot, \cdot)_M$ has the signature $(s+2, 0)$ on z^{\perp} and $(0, 2)$ on z .

According to Witt's theorem, $O_{s+2,2}(\mathbb{R})$ acts on D transitively. For a fixed element $z_0 \in D$, we denote by K_{∞} the stabilizer of $O_{s+2,2}(\mathbb{R})$ at z_0 . Then

$$(C.6) \quad D \cong O_{s+2,2}(\mathbb{R})/K_{\infty}$$

is realized as a homogeneous space. It is easily seen that K_{∞} is isomorphic to $O(s+2, \mathbb{R}) \times SO(2, \mathbb{R})$, which is a subgroup of the maximal compact subgroup $O(s+2, \mathbb{R}) \times O(2, \mathbb{R})$ of $O_M(\mathbb{R}) \cong O_{s+2,2}(\mathbb{R})$. It is also easy to check that $O_{s+2,2}(\mathbb{R})$ has four connected components. We denote by $SO_{s+2,2}(\mathbb{R})^0$ the identity component of $O_{s+2,2}(\mathbb{R})$. In fact, $SO_{s+2,2}(\mathbb{R})^0$ is the kernel of the spinor norm mapping

$$(C.7) \quad \rho : SO_{s+2,2}(\mathbb{R}) \longrightarrow \mathbb{R}^{\times}/(\mathbb{R}^{\times})^2.$$

Now we know that

$$(C.8) \quad D \cong O_{s+2,2}(\mathbb{R})/O(s+2, \mathbb{R}) \times SO(2, \mathbb{R})$$

has two connected components and the connected component D^0 containing the origin $o := z_0$ is realized as the homogeneous space as follows:

$$(C.9) \quad D^0 \cong O_{s+2,2}(\mathbb{R})/O(s, \mathbb{R}) \times O(2, \mathbb{R}) \cong SO_{s+2,2}(\mathbb{R})^0/SO(4, \mathbb{R}) \times SO(2, \mathbb{R}).$$

It is known that D^0 is a Hermitian symmetric space of noncompact type with complex dimension $s+2$. Let us describe a Hermitian structure on D^0 explicitly. For brevity, we write $G_{\mathbb{R}}^0 := SO_{s+2,2}(\mathbb{R})^0$ and $K_{\mathbb{R}}^0 := SO(s+2, \mathbb{R}) \times SO(2, \mathbb{R})$. Obviously $G_{\mathbb{R}}^0$ is the identity component of $O_{s+2,2}(\mathbb{R}) \cong O_M(\mathbb{R})$ and $K_{\mathbb{R}}^0$ is the

identity component of $O(s+2, \mathbb{R}) \times O(2, \mathbb{R}) \cong K_\infty$. For a positive integer n , the Lie algebra $\mathfrak{so}(n, \mathbb{R})$ of $SO(n, \mathbb{R})$ has dimension $(n-1)n/2$ and

$$(C.10) \quad \mathfrak{so}(n, \mathbb{R}) = \{ X \in \mathbb{R}^{(n,n)} \mid \sigma(X) = 0, {}^t X + X = 0 \}.$$

Then the Lie algebra \mathfrak{g} of $G_{\mathbb{R}}^0$ is given by

$$(C.11) \quad \mathfrak{g} = \left\{ \begin{pmatrix} A & C \\ {}^t C & B \end{pmatrix} \in \mathbb{R}^{(s+4, s+4)} \mid A \in \mathfrak{so}(s+2, \mathbb{R}), B \in \mathfrak{so}(2, \mathbb{R}), C \in \mathbb{R}^{(s+2, 2)} \right\}.$$

Let θ be the Cartan involution of $G_{\mathbb{R}}^0$ defined by

$$(C.12) \quad \theta(g) := E_{s+2, 2} g E_{s+2, 2}, \quad g \in G_{\mathbb{R}}^0.$$

Then $K_{\mathbb{R}}^0$ is the subgroup of $G_{\mathbb{R}}^0$ consisting of elements in $G_{\mathbb{R}}^0$ fixed by θ . We also denote by θ the differential of θ which is given by

$$(C.13) \quad \theta(X) = E_{s+2, 2} X E_{s+2, 2}, \quad X \in \mathfrak{g}.$$

Then \mathfrak{g} has the Cartan decomposition

$$(C.14) \quad \mathfrak{g} = \mathfrak{k} + \mathfrak{p},$$

where \mathfrak{k} and \mathfrak{p} denote the $(+1)$ -eigenspace and (-1) -eigenspace of θ respectively. More explicitly,

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathfrak{g} \mid A \in \mathfrak{so}(s+2, \mathbb{R}), B \in \mathfrak{so}(2, \mathbb{R}) \right\}$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & C \\ {}^t C & 0 \end{pmatrix} \mid C \in \mathbb{R}^{(s+2, 2)} \right\}.$$

The real dimension of \mathfrak{g} , \mathfrak{k} and \mathfrak{p} are $(s+3)(s+4)/2$, $(s^2+3s+4)/2$ and $2(s+2)$ respectively. Thus the real dimension of D^0 is $2(s+2)$. Since \mathfrak{p} is stable under the adjoint action of $K_{\mathbb{R}}^0$, i.e., $\text{Ad}(k)\mathfrak{p} = \mathfrak{p}$ for all $k \in K_{\mathbb{R}}^0$, $(\mathfrak{g}, \mathfrak{k}, \theta)$ is reductive. Thus the tangent space $T_o(D^0)$ of $D^0 \subset D$ at $o := z_0$ can be canonically identified with $\mathbb{R}^{2(s+2)}$ via

$$\begin{pmatrix} 0 & C \\ {}^t C & 0 \end{pmatrix} \longmapsto (x, y), \quad C = (x, y) \in \mathbb{R}^{(s+2, 2)} \cong \mathbb{R}^{2(s+2)}.$$

Then the adjoint action of $K_{\mathbb{R}}^0$ on $\mathfrak{p} \cong \mathbb{R}^{2(s+2)}$ is expressed as

$$(C.15) \quad \text{Ad} \left(\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \right) (x, y) = \begin{pmatrix} 0 & k_1 C {}^t k_2 \\ k_2 {}^t C k_1 & 0 \end{pmatrix},$$

where $k_1 \in SO(s+2, \mathbb{R})$, $k_2 \in SO(2, \mathbb{R})$ and $C = (x, y) \in \mathbb{R}^{(s+2, 2)}$. The Cartan-Killing form B of \mathfrak{g} is given by

$$(C.16) \quad B(X, Y) = (s+2)\sigma(XY), \quad X, Y \in \mathfrak{g}.$$

The restriction B_0 of B to \mathfrak{p} is given by

$$B_0((x, y), (x', y')) = 2(s+2)(\langle x, x' \rangle + \langle y, y' \rangle),$$

where \langle, \rangle is the standard inner product on \mathbb{R}^{s+2} . The restriction B_0 induces a $G_{\mathbb{R}}^0$ -invariant Riemannian metric g_0 on D^0 defined by

$$g_0(X, Y) := B_0(X, Y), \quad X, Y \in \mathfrak{p}.$$

It is easy to check that g_0 is invariant under the adjoint action of $K_{\mathbb{R}}$.

Now let J_0 be the complex structure on the real vector space \mathfrak{p} defined by

$$(C.17) \quad J_0((x, y)) := (-y, x), \quad (x, y) \in \mathfrak{p}.$$

We note that

$$J_0 = \text{Ad} \begin{pmatrix} E_{s+2} & 0 \\ 0 & I \end{pmatrix}, \quad I := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J_0^2 = \text{Id}_{\mathfrak{p}}.$$

It is easy to check that J_0 is $\text{Ad}(K_{\mathbb{R}})$ -invariant, i.e.,

$$J_0(\text{Ad}(k)X) = J_0(X) \quad \text{for all } k \in K_{\mathbb{R}} \text{ and } X \in \mathfrak{p}.$$

Hence J_0 induces an almost complex structure J on D^0 and also on D . J becomes a complex structure on D^0 via the natural identification

$$(C.18) \quad T_0(D^0) \cong \mathfrak{p} \cong \mathbb{R}^{2(s+2)} \cong \mathbb{C}^{s+2}, \quad (x, y) \longmapsto x + iy, \quad x, y \in \mathbb{R}^{s+2}.$$

Indeed, J is the pull-back of the standard complex structure on \mathbb{C}^{s+2} . The complexification $\mathfrak{p}_{\mathbb{C}} := \mathfrak{p} \otimes_{\mathbb{R}} \mathbb{C}$ has a canonical decomposition

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-,$$

where \mathfrak{p}_+ (resp. \mathfrak{p}_-) denotes the $(+i)$ -eigenspace (resp. $(-i)$ -eigenspace) of J_0 . Precisely, \mathfrak{p}_+ and \mathfrak{p}_- are given by

$$\mathfrak{p}_+ = \{ (x, -ix) \mid x \in \mathbb{C}^{s+2} \} \quad \text{and} \quad \mathfrak{p}_- = \{ (x, ix) \mid x \in \mathbb{C}^{s+2} \}.$$

Usually \mathfrak{p}_+ and \mathfrak{p}_- are called the *holomorphic* tangent space and the *anti-holomorphic* tangent space respectively. Moreover, it is easy to check that the Riemannian metric g_0 on D^0 is Hermitian with respect to the complex structure J , i.e., $g_0(JX, JY) = g_0(X, Y)$ for all smooth vector fields X and Y on D^0 . And D^0 has the canonical orientation induced by its complex structure.

In summary, we have

Theorem 1. D^0 is a Hermitian symmetric space of noncompact type with dimension $s + 2$. D^0 is realized as a bounded symmetric domain in \mathbb{C}^{s+2} and hence D is a union of two bounded symmetric domains in \mathbb{C}^{s+2} .

REMARK 2. We choose an orthogonal basis of z_0^\perp . We also choose a basis of z_0 which is properly oriented. Let

$$\tau_{z_0^\perp} := \text{diag}(E_{s+1}, -1) \quad \text{and} \quad \tau_{z_0} := \text{diag}(1, -1)$$

be the symmetries in the isometry groups $O(z_0^\perp)$ and $O(z_0)$ with respect to the last coordinates of z_0^\perp and z_0 respectively. We observe that τ_{z_0} reverses the orientation of z_0 and lies in $O(z_0) - SO(z_0)$. It is easy to check that

$$\rho(1_{z_0^\perp} \times \tau_{z_0}) = -1, \quad \rho(\tau_{z_0^\perp} \times 1_{z_0}) = 1,$$

where ρ is the spinor norm mapping defined by (C.7). It is easy to see that the set

$$(C.19) \quad \left\{ 1_{M_{\mathbb{R}}}, 1_{z_0^\perp} \times \tau_{z_0}, \tau_{z_0^\perp} \times 1_{z_0}, \tau_{z_0^\perp} \times \tau_{z_0} \right\}$$

is a complete set of coset representatives of $O_{s+2,2}(\mathbb{R})/SO_{s+2,2}(\mathbb{R})^0$. We note that the set (C.19) is contained in $O(s+2, \mathbb{R}) \times O(2, \mathbb{R})$ and so that (C.19) is a complete set of coset representatives of $O(s+2, \mathbb{R}) \times O(2, \mathbb{R})/(SO(s+2, \mathbb{R}) \times SO(2, \mathbb{R}))$. It is easy to see that the set $\{1_{M_{\mathbb{R}}}, 1_{z_0^\perp} \times \tau_{z_0}\}$ is a complete set of coset representatives of $(O(s+2, \mathbb{R}) \times O(2, \mathbb{R}))/ (O(s+2, \mathbb{R}) \times SO(2, \mathbb{R}))$. Thus we have

$$(C.20) \quad D = D^0 \cup (1_{z_0^\perp} \times \tau_{z_0})D^0.$$

The complex structure $-J_0$ on \mathfrak{p} determines the opposite almost complex structure on D^0 and the almost complex structure on the connected component $D - D^0$ is the one on D^0 carried by the element $1_{z_0^\perp} \times \tau_{z_0}$. The ground manifolds D^0 and $D - D^0$ may be regarded as the same one, but each carries the opposite almost complex structure.

REMARK 3. D^0 may be regarded as an open orbit of $G_{\mathbb{R}}^0$ in the complex projective quadratic space ${}^t z Q_M z = 0$ via the Borel embedding. (See [Bai] for detail.) D^0 is realized as a tube domain in \mathbb{C}^{s+2} given by (4.10) in section 4. For the explicit realization of D^0 as a bounded symmetric domain in \mathbb{C}^{s+2} , we refer to [Bai], [H] and [O].

Finally we present the useful equations for g to belong to $O_{s+2,2}(\mathbb{R})$. For $g \in O_{s+2,2}(\mathbb{R})$, we write

$$g = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

where $A_{11}, A_{13}, A_{31}, A_{33} \in \mathbb{R}^{(2,2)}$, $A_{22} \in \mathbb{R}^{(s,s)}$, $A_{12}, A_{32} \in \mathbb{R}^{(2,s)}$, and $A_{21}, A_{23} \in \mathbb{R}^{(s,2)}$. Then the condition ${}^t g Q_M g = Q_M$ is equivalent to the following equations given by

$$(C.21) \quad {}^t A_{11} \Pi_{1,1} A_{31} + {}^t A_{21} S_0 A_{32} + {}^t A_{31} \Pi_{1,1} A_{12} = 0,$$

$$(C.22) \quad {}^t A_{11} \Pi_{1,1} A_{32} + {}^t A_{21} S_0 A_{22} + {}^t A_{31} \Pi_{1,1} A_{12} = 0,$$

$$(C.23) \quad {}^t A_{11} \Pi_{1,1} A_{33} + {}^t A_{21} S_0 A_{23} + {}^t A_{31} \Pi_{1,1} A_{13} = \Pi_{1,1},$$

$$(C.24) \quad {}^t A_{12} \Pi_{1,1} A_{32} + {}^t A_{22} S_0 A_{22} + {}^t A_{32} \Pi_{1,1} A_{12} = S_0,$$

$$(C.25) \quad {}^t A_{12} \Pi_{1,1} A_{33} + {}^t A_{22} S_0 A_{23} + {}^t A_{32} \Pi_{1,1} A_{13} = 0,$$

and

$$(C.26) \quad {}^t A_{13} \Pi_{1,1} A_{33} + {}^t A_{23} S_0 A_{23} + {}^t A_{33} \Pi_{1,1} A_{13} = 0.$$

Appendix D. The Leech Lattice Λ

Here we collect some properties of the Leech lattice Λ . Most of the materials in this appendix can be found in [C-S].

The Leech lattice L is the unique positive definite unimodular even integral lattice of rank 24 with minimal norm 4. Λ was discovered by J. Leech in 1965. (cf. Notes on sphere packing, Can. J. Math. 19 (1967), 251-267.) It was realized by Conway, Parker and Sloane that the Leech lattice Λ has many strange geometric properties. Past three decades more than 20 constructions of Λ were found.

The following properties of Λ are well known:

(A1) The determinant of Λ is $\det \Lambda = 1$. The kissing number is $\tau = 196560$ and the packing radius is $\rho = 1$. The density is $\Delta = \pi^{12}/(12!) = 0.001930 \dots$. The covering radius is $R = \sqrt{2}$ and the thickness is $\Theta = (2\pi)^{12}/(12!) = 7.9035 \dots$.

(Λ2) There are 23 different types of deep hole one of which is the octahedral hole $8^{-1/2}(4, 0^{23})$ surrounded by 48 lattice points.

(Λ3) The Veronoi cell has 16969680 faces, 196560 corresponding to the minimal vectors and 16773120 to those of the next layer.

(Λ4) The automorphism group $\text{Aut}(\Lambda)$ of Λ has order

$$2^{22} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23 = 8315553613086720000.$$

$\text{Aut}(\Lambda)$ has the Mathieu group M_{24} as a subgroup. The automorphism group $\text{Aut}(\Lambda)$ is often denoted by Co_0 or θ because J.H. Conway first discovered this group.

For a given lattice L , we denote $N_m(L)$ by the number of vectors of norm m . Conway characterized the Leech lattice as follows (cf. A characterization of Leech's lattice, Invent. Math. 7 (1969), 137-142 or Chapter 12 in [C-S]) :

Theorem 1 (Conway). *Λ is the unique positive definite unimodular even integral lattice L with rank < 32 that satisfies any one of the following*

- (a) *L is not directly congruent to its mirror-image.*
- (b) *No reflection leaves L invariant.*
- (c) $N_2(L) = 0$.
- (d) $N_{2m}(L) = 0$ for some $m \geq 0$.

Theorem 2 (Conway). *If L is a unimodular even integral lattice with rank < 32 and $N_2(L) = 0$, then $L = \Lambda$ and $N_4(L) = 196560$, $N_6(L) = 16773120$, $N_8(L) = 398034000$.*

Now we review the Jacobi theta functions. For the present time being, we put $q = e^{i\pi\tau}$ and $\zeta = e^{i\pi z}$. (We note that we set $q = e^{2\pi i\tau}$ and $\zeta = e^{2\pi iz}$ at other places.) We define the Jacobi theta functions

$$(D.1) \quad \theta_1(\tau, z) := i^{-1} \sum_{n \in \mathbb{Z}} (-1)^n q^{(n+1/2)^2} \zeta^{2n+1},$$

$$(D.2) \quad \theta_2(\tau, z) := \sum_{n \in \mathbb{Z}} q^{(n+1/2)^2} \zeta^{2n+1},$$

$$(D.3) \quad \theta_3(\tau, z) := \sum_{n \in \mathbb{Z}} q^{n^2} \zeta^{2n},$$

$$(D.4) \quad \theta_4(\tau, z) := \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} \zeta^{2n},$$

where $\tau \in H_1$ and $z \in \mathbb{C}$. We also define the theta functions $\theta_k(\tau) := \theta_k(\tau, 0)$ for $k = 2, 3, 4$. Then it is easy to see that

$$(D.5) \quad \theta_2(\tau) = \sum_{n \in \mathbb{Z}} q^{(n+1/2)^2} = e^{\pi i \tau / 4} \theta_3(\tau, \tau/2),$$

$$(D.6) \quad \theta_4(\tau) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} = \theta_3(\tau, \pi/2) = \theta_3(\tau + 1).$$

According to the Poisson summation formula, we obtain

$$(D.7) \quad \theta_3(-1/\tau, z/\tau) = (-i\tau)^{1/2} e^{\pi i z^2 / \tau} \theta_3(\tau, z), \quad (\tau, z) \in H_1 \times \mathbb{C}.$$

And these theta functions can be written as infinite products as follows :

$$(D.8) \quad \theta_1(\tau, z) = 2 \sin \pi z \cdot q^{1/4} \prod_{n>0} (1 - q^{2n})(1 - q^{2n} \zeta^2)(1 - q^{2n} \zeta^{-2}),$$

$$(D.9) \quad \theta_2(\tau, z) = q^{1/4} \zeta \prod_{n>0} (1 - q^{2n})(1 + q^{2n} \zeta^2)(1 + q^{2n} \zeta^{-2}),$$

$$(D.10) \quad \theta_3(\tau, z) = \prod_{n>0} (1 - q^{2n})(1 + q^{2n} \zeta^2)(1 + q^{2n-1} \zeta^{-2}),$$

$$(D.11) \quad \theta_4(\tau, z) = \prod_{n>0} (1 - q^{2n})(1 - q^{2n-1} \zeta^2)(1 - q^{2n-1} \zeta^{-2}).$$

Thus the theta functions $\theta_k(\tau)$ ($2 \leq k \leq 4$) are written as infinite products :

$$(D.9') \quad \theta_2(\tau) = q^{1/4} \prod_{n>0} (1 - q^{2n})(1 + q^{2n})(1 + q^{2n-2}) = 2q^{1/4} \prod_{n>0} (1 - q^{2n})(1 + q^{2n})^2,$$

$$(D.10') \quad \theta_3(\tau) = \prod_{n>0} (1 - q^{2n})(1 + q^{2n-1})^2,$$

$$(D.11') \quad \theta_4(\tau) = \prod_{n>0} (1 - q^{2n})(1 - q^{2n-1})^2.$$

We note that the discriminant function $\Delta(\tau)$ is written as

$$(D.12) \quad \Delta(\tau) = q^2 \prod_{n>0} (1 - q^{2n})^{24} = \{ 1/2\theta_2(\tau)\theta_3(\tau)\theta_4(\tau) \}^8.$$

We observe that the theta function $\theta_3(\tau, z)$ is annihilated by the heat operator $H := \frac{\partial^2}{\partial z^2} - 4\pi i \frac{\partial}{\partial \tau}$. It is easy to check that $\theta_1(\tau, z)$ has zeros only at $m_1 + m_2\tau$ ($m_1, m_2 \in \mathbb{Z}$) and satisfies the equations

$$(D.13) \quad \theta_1(\tau, z + 1) = -\theta_1(\tau, z), \quad \theta_1(\tau, \tau + z) = -q^{-1}e^{-2\pi iz}\theta_1(\tau, z).$$

Now for a given positive definite lattice L , we define the *theta series* $\Theta_L(\tau)$ of a lattice L by

$$(D.14) \quad \Theta_L(\tau) := \sum_{\alpha \in L} q^{N(\alpha)} = \sum_{m \geq 0} N_m(L)q^m, \quad \tau \in H_1,$$

where $N(\alpha) := (\alpha, \alpha)$ denotes the norm of a vector $\alpha \in L$. We can also use (D.14) to define the theta series of a nonlattice packing L . The commonest examples of this appear when L is a translate of a lattice or a union of translates. Clearly $\theta_2(\tau) = \Theta_{\mathbb{Z}+1/2}(\tau)$, $\theta_3(\tau) = \Theta_{\mathbb{Z}}(\tau)$ and $\Theta_{\mathbb{Z}^n}(\tau) = \Theta_{\mathbb{Z}}(\tau)^n = \theta_3(\tau)^n$.

Returning to the Leech lattice L ,

$$\begin{aligned} \Theta_\Lambda(\tau) &= \Theta_{E_8}(\tau)^3 - 720 \Delta(\tau) \\ &= 1/8 \{ \theta_2(\tau)^8 + \theta_3(\tau)^8 + \theta_4(\tau)^8 \}^3 - 45/16 \{ \theta_2(\tau)\theta_3(\tau)\theta_4(\tau) \}^8 \\ &= 1/2 \{ \theta_2(\tau)^{24} + \theta_3(\tau)^{24} + \theta_4(\tau)^{24} \} - 69/16 \{ \theta_2(\tau)\theta_3(\tau)\theta_4(\tau) \}^8 \\ &= \sum_{m \geq 0} N_m(\Lambda)q^m = 1 + 196560q^4 + 16773120q^6 + \dots, \end{aligned}$$

where $\Theta_{E_8}(\tau)$ is the theta series of the exceptional lattice E_8 of rank 8. It is known that

$$(D.15) \quad N_m(\Lambda) = 65520/691 (\sigma_{11}(m/2) - \tau(m/2)).$$

The values of $N_m(\Lambda)$ for $0 \leq m \leq 100$, m : even can be found in [C-S], p. 135.

In the middle of 1980s, M. Koike, T. Kondo and T. Tasaka solved a special part of the Moonshine Conjectures for the Mathieu group M_{24} . For $g \in M_{24}$, we write

$$(D.16) \quad g = (n_1)(n_2) \cdots (n_s), \quad n_1 \geq \cdots \geq n_s \geq 1,$$

where (n_i) is a cycle of length n_i ($1 \leq i \leq s$). To each $g \in M_{24}$ of the form (D.16), we associate modular forms $\eta_g(\tau)$ and $\theta_g(\tau)$ defined by

$$(D.17) \quad \eta_g(\tau) := \eta(n_1\tau) \eta(n_2\tau) \cdots \eta(n_s\tau), \quad \tau \in H_1$$

and

$$(D.18) \quad \theta_g(\tau) := \sum_{\alpha \in \Lambda_g} e^{\pi i N(\alpha)\tau}, \quad \tau \in H_1,$$

where $\eta(\tau)$ is the Dedekind eta function and

$$(D.19) \quad \Lambda_g := \{ \alpha \in \Lambda \mid g \cdot \alpha = \alpha \}$$

is the positive definite even integral lattice of rank s . We observe that $\theta_g(\tau) = \Theta_{\Lambda_g}(\tau)$.

Theorem 3 ([Koi2]). *For any element $g \in M_{24}$ with $g \neq 12^2, 4^6, 2^{12}, 10^2 \cdot 2^2, 12 \cdot 6 \cdot 4 \cdot 2, 4^4 \cdot 2^4$, there exists a unique modular form $f_g(\tau) = 1 + \sum_{n \geq 0} a_g(n)q^{2n}$, $a_g(n) \in \mathbb{Z}$ satisfying the following conditions:*

(K1) *There exists an element $g_1 \in G$ such that $f_g(\tau)\eta_g(\tau)^{-1} = T_{g_1}(\tau) + c$ for some constant c , where G is the MONSTER and $T_{g_1}(\tau)$ denotes the Thompson series of $g_1 \in G$.*

(K2) *$a_g(1) = 0$, and $a_g(n)$ are nonnegative even integers for all $n \geq 1$.*

(K3) *If $g' = g^r$ for some $r \in \mathbb{Z}$, then $a_g(n) \leq a_{g'}(n)$ for all n .*

(K4) *$a_g(2)$ is equal to the cardinality of the set $\{ \alpha \in \Lambda_g \mid N(\alpha) = (\alpha, \alpha) = 4 \}$.*

Theorem 4 (Kondo and Tasaka [K-T]). *Let $g \in M_{24}$ be any element of the Mathieu group M_{24} . Then the function $\theta_g(\tau)\eta_g(\tau)^{-1}$ is a Hauptmodul for a genus 0 discrete subgroup of $SL(2, \mathbb{R})$. The function $\theta_g(\tau)$ is the unique modular form satisfying the conditions (K1)-(K4).*

For more detail on the Leech lattice Λ we refer to [Bo3], [C-S] and [Kon].

REFERENCES

- [A] G.E. Andrews, *A simple proof of Jacobi's triple product identity*, Proc. Amer. Math. Soc. **16** (1965), 333-334.
- [An] A.N. Andrianov, *Modular descent and the Saito-Kurokawa conjecture*, Invent. Math. **53** (1979), 267-280.
- [Bai] W.L. Baily, *Introductory Lectures on Automorphic Forms*, Iwanami Shoton, Publishers and Princeton Univ. Press, 1973.
- [Bas] H. Bass, *The congruence subgroup problem*, Proc. of Conf. in Local Fields (1967), Springer-Verlag, edited by T.A. Springer, 16-22.
- [Be] R. Bellman, *A Brief Introduction to Theta Functions*, Holts, Rinehart and Winston, Inc., New York, 1961.
- [Bo1] R.E. Borcherds, *Vertex algebras, Kac-Moody algebras and the monster*, Proc. Natl. Acad. Sci., USA. **83** (1986), 3068-3071.
- [Bo2] ———, *Generalized Kac-Moody algebras*, J. Algebra **115** (1988), 501-512.
- [Bo3] ———, *Lattices like the Leech Lattice*, J. Algebra **130** (1990), 219-234.
- [Bo4] ———, *Central Extensions of Generalized Kac-Moody Algebras*, J. Algebra **140** (1991), 330-335.
- [Bo5] ———, *The Monster Lie Algebra*, Adv. Math. **83** (1990), 30-47.
- [Bo6] ———, *Monstrous moonshine and monstrous Lie superalgebras*, Invent. Math. **109** (1992), 405-444.
- [Bo7] ———, *Automorphic forms on $O_{s+2,2}(\mathbb{R})$ and infinite products*, Invent. Math. (1995), 161-213.
- [BCQ] R.E. Borcherds, J.H. Conway and L. Queen, *A Monster Lie Algebra ?*, Adv. Math. **53** (1984), 75-79.
- [Ca-S] L. Carlitz and M.V. Subbarao, *A simple proof of Watson's quintuple product identity*, Proc. Amer. Math. Soc. **32**, No. 1 (1972), 42-44.
- [C-N] J.H. Conway and S. Norton, *Monstrous Moonshine*, Bull. London Math. Soc. **11** (1979), 308-339.
- [CCNPW] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.
- [C-S] J.H. Conway and N.J.A. Sloane, *Sphere Packings, Lattices and Groups*, Springer-Verlag, New York, 1988.
- [Ev] S.A. Evdokimov, *A characterization of the Maass space of Siegel cusp forms of genus 2 (in Russian)*, Mat. Sbornik (154) **112** (1980), 133-142.
- [E-Z] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progress in Math., vol. 55, Birkhäuser, Boston-Basel- Stuttgart, 1985.
- [F] E. Freitag, *Siegelschen Modulfunktionen*, Springer-Verlag, Berlin-Heidelberg-New York, 1983.
- [F-F] A.J. Feingold and I.B. Frenkel, *A Hyperbolic Kac-Moody Algebra and the Theory of Siegel Modular Forms of Genus 2*, Math. Ann. **263** (1983), 87-144.
- [FI] I.B. Frenkel, *Representations of Kac-Moody algebras and dual resonance models.*, In Flato et al. (eds.) Applications of group theory in theoretical physics, Lect. Appl. Math., Providence, RI. AMS. **21** (1985), 325-353.
- [FLM] I.B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Academic Press, Boston, 1988.
- [FLT] B. Fischer, D. Livingston and M.P. Thorne, *The characters of the "Monster" simple group*, Birmingham, 1978.
- [G1] V.G. Gritsenko, *Jacobi functions of n variables*, J. Soviet Math. **53** (1991), 243-252.

- [G2] ———, *Modular forms and moduli spaces of abelian and K3 surfaces*, St. Petersburg Math. J. **4** (1994).
- [G-T] P. Goddard and C.B. Thorne,, *Compatibility of the dual Pomeron with unitarity and the absence of ghosts in the dual resonance model*, Phys. Lett. B **40**, No. **2** (1972), 235-238.
- [G-Z] B.H. Gross and D. Zagier, *On Singular Moduli*, J. reine angew. Math. **355** (1985), 191-220.
- [H] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.
- [Ib] T. Ibukiyama, *On Jacobi forms and Siegel modular forms of half integral weights*, preprint (1991).
- [Ig1] J. Igusa, *On Siegel modular forms of genus two*, Amer. J. Math. **84** (1962), 175-200.
- [Ig2] ———, *Theta Functions*, Springer-Verlag, 1972.
- [K] V.G. Kac, *Infinite dimensional Lie algebras, 3rd ed.*, Cambridge Univ. Press, 1990.
- [KMW] V. Kac, R.V. Moody and M. Wakimoto, *On E_{10}* , preprint.
- [Kob] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Springer-Verlag, 1984.
- [Koh] W. Kohlen, *Modular forms of half integral weight on $\Gamma_0(4)$* , Math. Ann. **248** (1980), 249-266.
- [Koi1] M. Koike, *On Replication Formula and Hecke Operators*, Nagoya Univ., preprint.
- [Koi2] ———, *Mathieu Group M_{24} and Modular Forms*, Nagoya Math. J. **99** (1985), 147-157.
- [Kon] T. Kondo, *The automorphism group of Leech lattice and elliptic modular functions*, J. Math. Soc. Japan **37**, No. **2** (1985), 337-?.
- [K-T] T. Kondo and T. Tasaka, *The Theta Functions of Sublattices of the Leech Lattice*, Nagoya Math. J. **101** (1986), 151-179.
- [Ma1] H. Maass, *Lectures on Modular Functions of One Variable*, TIFR, Bombay, 1964.
- [Ma2] ———, *Über eine Spezialschar von Modulformen zweiten Grades I*, Invent. Math. **52** (1979), 95-104.
- [Ma3] ———, *Über eine Spezialschar von Modulformen zweiten Grades II*, Invent. Math. **53** (1979), 249-253.
- [Ma4] ———, *Über eine Spezialschar von Modulformen zweiten Grades III*, Invent. Math. **53** (1979), 255-265.
- [Mac] I.G. Macdonald, *Affine Root Systems and Dedekind's η -functions*, Invent. Math. **15** (1972), 91-143.
- [Me] J. Mennicke, *Zur Theorie der Siegelsche Modulgruppe*, Math. Ann. **159** (1965), 115-129.
- [Mu] D. Mumford, *Tata Lectures on Theta I*, vol. 28, Birkhäuser, Boston, 1983.
- [MNN] D. Mumford, M. Nori and P. Norman, *Tata Lectures on Theta III, Progress in Math.*, vol. 97, Birkhäuser, 1991.
- [N1] V.V. Nikulin, *Discrete Reflection Groups in Lobachevsky Spaces*, Proc. ICM, Warszawa (1983), 593-601.
- [N2] ———, *Discrete reflection Groups in Lobachevsky Spaces and Algebraic Surfaces*, Proc. ICM, Berkeley (1986), 654-669.
- [N3] ———, *On arithmetic groups generated by reflections in Lobachevsky spaces*, Math. USSR. Izv. English transl **16** (1981).
- [N4] ———, *On the classification of arithmetic groups generated by reflections in Lobachevsky spaces*, Math. USSR. Izv. English transl **18** (1982).

- [N5] ———, *A Lecture on Kac-Moody Lie Algebras of the Arithmetic Type*, preprint (1994).
- [No1] S.P. Norton, *More on moonshine, Computational group theory* (1984), Academic Press, London, 185-195.
- [No2] ———, *Generalized Moonshine*, Proc. Symp. Pure Math., AMS. **47** (1987), 208-209.
- [O] T. Oda, *On modular forms associated with indefinite quadratic forms of signature $(2, n-2)$* , Math. Ann. **231** (1977), 97-144.
- [R] H. Rademacher, *Topics in Analytic Number Theory*, Springer-Verlag, New York-Berlin-Heidelberg, 1973.
- [S] J.-P. Serre, *A Course in Arithmetic*, Springer-Verlag, 1973.
- [Sh] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton Univ. Press, Princeton, N.J., 1971.
- [S-V] M.V. Subbarao and M. Vidyasagar, *On Watson's quintuple product identity*, Proc. Amer. Math. Soc. **26** (1970), 23-27.
- [T] A. Terras, *Harmonic Analysis on Symmetric Spaces and Applications I*, Springer-Verlag, 1985.
- [Th1] J.G. Thompson, *A finiteness theorem for subgroups of $PSL(2, \mathbb{R})$ which is commensurable with $PSL(2, \mathbb{Z})$* , Proc. Symp. Pure Math. AMS. **37** (1979), 533-555.
- [Th2] ———, *Some numerology between the Fischer-Griess Monster and the elliptic modular function*, Bull. London Math. Soc. **11** (1979), 352-353.
- [W1] G.N. Watson, *Theorems stated by Ramanujan. VII: Theorems on continued fractions*, J. London Math. Soc. **4** (1929), 37-48.
- [W2] ———, *Ramanujan's Vermutung über Zerfallungszahlen*, J. Reine Angew. Math. **179** (1938), 97-128.
- [Y1] J.-H. Yang, *Remarks on Jacobi Forms of Higher Degree*, Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, The Pyungsan Institute for Mathematical Sciences, Seoul (1993), 33-58.
- [Y2] ———, *On Jacobi Forms of Higher Degree (Korean)*, Notices of the Pyungsan Institute for Math. Sciences, Seoul (1993), 3-29.
- [Y3] ———, *Singular Jacobi Forms*, Trans. of American Math. Soc. **347**, No. **6** (1995), 2041-2049.
- [Y4] ———, *Construction of Vector Valued Modular Forms from Jacobi Forms*, Can. J. Math. **47** (**6**) (1995), 1329-1339.
- [Za] D. Zagier, *Sur la conjecture de Saito-Kurokawa (d'après H. Maass). Sémin. Delange-Pisot-Poitou 1979-1980*, Progress in Math. Birkhäuser, Boston **12** (1980), 371-394.
- [Zi] C. Ziegler, *Jacobi Forms of Higher Degree*, Abh. Math. Sem. Univ. Hamburg **59** (1989), 191-224.

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