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A Note on Maass-Jacobi Forms II

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Abstract. This article is a continuation of the paper [21]. In this paper we deal with Maass-Jacobi forms on the Siegel-Jacobi space $\mathbb{H} \times \mathbb{C}^m$, where $\mathbb{H}$ denotes the Poincaré upper half plane and m is any positive integer.

1. Introduction

This article is a continuation of the paper [21]. Recently A. Pitale [14], K. Bringmann and O. Richter [4], and C. Conley and M. Raum [5] defined another notion of Maass-Jacobi forms and studied some properties of Maass-Jacobi forms. In [4], [14] and [21], the authors considered the case $n = m = 1$ and in [5], the authors dealt with the case $n = 1$ and m is arbitrary. In this paper, we consider mainly the case $n = 1$ and m is an arbitrary positive integer.

This paper is organized as follows. In Section 2, we give some useful geometric properties of the Siegel-Jacobi space $\mathbb{H} \times \mathbb{C}^m$. We study the invariant metrics, their Laplacians, a fundamental domain, geodesics, the scalar curvature and invariant differential forms on $\mathbb{H} \times \mathbb{C}^m$. In Section 3 we describe the center of the universal enveloping algebra of the complexfied Jacobi Lie algebra. This work is due to Conley and Raum [5]. In Section 4, we present some interesting and important results on invariant differential operators on the Siegel-Jacobi space $\mathbb{H} \times \mathbb{C}^m$. In Section 5, we discuss the notion of Maass-Jacobi forms introduced by J.-H. Yang [21]. Maass-Jacobi forms play an important role in the spectral theory of the Laplace operator on a fundamental domain for the Siegel-Jacobi space $\mathbb{H} \times \mathbb{C}^m$. In Section 6, we discuss the notion of Maass-Jacobi forms introduced by A. Pitale [14], Bringman-Richter [4] and Conley-Raum [5]. We describe the results obtained in [4] and [5]. More precisely the authors of [4] and [5] obtained an explicit Fourier expansion of

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the Poincaré series that is an example of harmonic Maass-Jacobi form. In Section 7, we discuss skew-holomorphic Jacobi forms introduced by N.-P. Skoruppa [18]. We describe the relation between cuspidal harmonic Maass-Jacobi forms and cuspidal skew-holomorphic Jacobi forms via the lowering operator $D_-^{(\mathcal{M})}$ (cf. (7.3)) In Section 8, we briefly review some results on covariant differential operators on the Siegel-Jacobi space $\mathbb{H}\times\mathbb{C}^m$ obtained by Conley and Raum [5]. In the final section we briefly mention two notions of Maass-Jacobi forms on the Siegel-Jacobi space $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ for the general case $n > 1$ and $m > 1$. Here $\mathbb{H}_n$ denotes the Siegel upper half plane of degree n. We present some natural problems related to the study of Maass-Jacobi forms.

Notations: We denote by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers respectively. $\mathbb{R}^\times$ denotes the set of all nonzero real numbers. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\mathrm{tr}(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. For a complex matrix A, $\overline{A}$ denotes the complex *conjugate* of A. For $A \in \mathbb{C}^{(k,l)}$ and $B \in \mathbb{C}^{(k,k)}$, we use the abbreviation $B\{A\} = {}^t\overline{A}BA$. For a positive integer n, I_n denotes the identity matrix of degree n. For a positive integer m and a commutative ring F, we denote by $S(m,F)$ the space of all $m \times m$ symmetric matrices with entries in F. For a complex number z, $|z|$ denotes the absolute value of z. For a complex number z, $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ denote the real part of z and the imaginary part of z respectively.

2. Geometric properties of the Siegel-Jacobi space $\mathbb{H} \times \mathbb{C}^m$

We fix a positive integer m throughout this paper and let

$$\mathbb{H} = \{\, \tau \in \mathbb{C} \mid \mathrm{Im}\, \tau > 0 \,\}$$

be the Poincaré upper half plane. Let $G = SL_2(\mathbb{R})$ be the special linear group of degree 2 and let

$$H_{\mathbb{R}}^{(m)} = \left\{ (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^m,\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\, {}^t\lambda \text{ symmetric } \right\}$$

be the Heisenberg group endowed with the following multiplication law

$$\big(\lambda, \mu; \kappa\big) \circ \big(\lambda', \mu'; \kappa'\big) = \big(\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda'\big)$$

with $\big(\lambda, \mu; \kappa\big), \big(\lambda', \mu'; \kappa'\big) \in H_{\mathbb{R}}^{(m)}$. We define the semidirect product of $SL_2(\mathbb{R})$ and $H_{\mathbb{R}}^{(m)}$

$$G^J = SL_2(\mathbb{R}) \ltimes H_{\mathbb{R}}^{(m)}$$

endowed with the following multiplication law

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot\big(M',(\lambda',\mu';\kappa')\big) = \big(MM',(\tilde\lambda+\lambda',\tilde\mu+\mu';\kappa+\kappa'+\tilde\lambda\,{}^t\mu'-\tilde\mu\,{}^t\lambda')\big)$$

with $M, M' \in SL_2(\mathbb{R}), (\lambda,\mu;\kappa),\ (\lambda',\mu';\kappa') \in H_{\mathbb{R}}^{(m)}$ and $(\tilde\lambda,\tilde\mu)=(\lambda,\mu)M'$. Then G^J acts on the Siegel-Jacobi space $\mathbb{H}\times\mathbb{C}^m$ of degree 1 and index m transitively by

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot(\tau,z) = \Big((a\tau+b)(c\tau+d)^{-1},(z+\lambda\tau+\mu)(c\tau+d)^{-1}\Big), \tag{2.1}$$

where $M=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{R}),\ (\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(m)}$, $\tau\in\mathbb{H}$ and $z={}^t(z_1,z_2,\cdots,z_m)\in\mathbb{C}^m$ with $z_i\in\mathbb{C}\,(1\le i\le m)$. We note that the Jacobi group G^J is *not* a reductive Lie group and that the homogeneous space $\mathbb{H}\times\mathbb{C}^m$ is not a symmetric space.

For a coordinate $(\tau,z)\in\mathbb{H}\times\mathbb{C}^n$, we write $\tau=x+i\,y$ with x real and $y>0$, and

$$z={}^t(z_1,z_2,\cdots,z_m),\quad z_j=u_j+i\,v_j,\quad u_j,\,v_j\ \text{real},\quad i=1,2,\cdots,m.$$

According to [23], for any two positive real numbers A and B, the following metric given by

$$\begin{aligned} ds^2_{m;A,B} &= \frac{1}{y^3}\left(A\,y\,+\,B\sum_{j=1}^m v_j^2\right)d\tau d\overline{\tau}\\ &\quad+\frac{B}{y^2}\left\{y\sum_{j=1}^m dz_jd\overline{z}_j-\sum_{j=1}^m v_j(d\tau d\overline{z}_j+d\overline{\tau}d\overline{z}_j)\right\}\\ &= \frac{1}{y^3}\left(A\,y\,+\,B\sum_{j=1}^m v_j^2\right)(dx^2+dy^2)\\ &\quad+\frac{B}{y^2}\left\{y\sum_{j=1}^m(du_j^2+dv_j^2)-2\sum_{j=1}^m v_j(dxdu_j+dydv_j)\right\} \end{aligned} \tag{2.2}$$

is a Kähler metric on $\mathbb{H}\times\mathbb{C}^m$ invariant under the action (2.1) of G^J.

We put

$$M_1 := \operatorname{tr}\left(y\frac{\partial}{\partial z}\,{}^t\!\left(\frac{\partial}{\partial\overline{z}}\right)\right)=y\sum_{j=1}^m\frac{\partial^2}{\partial z_j\partial\overline{z}_j}=\frac{y}{4}\left(\frac{\partial}{\partial u_j^2}+\frac{\partial}{\partial v_j^2}\right) \tag{2.3}$$

and

$$(2.4)\quad M_2 := y^2 \frac{\partial^2}{\partial\tau\partial\overline{\tau}} + \sum_{a,b=1}^{m} v_a v_b \frac{\partial^2}{\partial z_a \partial \overline{z}_b} + y \sum_{j=1}^{m} v_j \left(\frac{\partial^2}{\partial\tau\partial\overline{z}_j} + \frac{\partial^2}{\partial\overline{\tau}\partial z_j} \right)$$
$$= \frac{1}{4} \left\{ y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \sum_{a=1}^{m} v_a^2 \left(\frac{\partial^2}{\partial u_a^2} + \frac{\partial^2}{\partial v_a^2} \right) \right\}$$
$$+ \frac{1}{2} \sum_{1\leq a<b\leq m} v_a v_b \left(\frac{\partial^2}{\partial u_a \partial u_b} + \frac{\partial^2}{\partial v_a \partial v_b} \right).$$
$$+ \frac{y}{2} \sum_{j=1}^{m} v_j \left(\frac{\partial^2}{\partial x \partial u_j} + \frac{\partial^2}{\partial y \partial v_j} \right).$$

Then M_1 and M_2 are differential operators on $\mathbb{H} \times \mathbb{C}^m$ invariant under the action (2.1). The author [23] proved that

$$(2.5)\qquad \Delta_{m;A,B} := \frac{4}{B}\, M_1 + \frac{4}{A}\, M_2$$

is the Laplacian of $(\mathbb{H}\times\mathbb{C}^m, ds^2_{m;A,B})$. Furthermore the following $2(m+1)$-differential form

$$(2.6)\qquad dv = dx \wedge dy \wedge du_1 \wedge \cdots \wedge du_m \wedge dv_1 \wedge \cdots \wedge dv_m$$

is a G^J-invariant volume element on the Siegel-Jacobi space $\mathbb{H} \times \mathbb{C}^m$.

Let K^J be the stabilizer of G^J at $(i, 0)$. Then

$$K^J = \left\{ \left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, (0,0,R) \right) \,\Big|\, a^2 + b^2 = 1,\ a, b \in \mathbb{R},\ R = {}^tR \in \mathbb{R}^{(m,m)} \right\}.$$

Thus G^J/K^J is diffeomorphic to $\mathbb{H} \times \mathbb{C}^m$ via

$$gK^J \longmapsto g \cdot (i, 0) = \left(\frac{a\,i+b}{c\,i+d}, \frac{\lambda\,i+\mu}{c\,i+d} \right),$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(m)}$. The Siegel-Jacobi space $\mathbb{H} \times \mathbb{C}^m$ is a homogeneous space which is not symmetric. Let $\mathfrak{k}^J$ be the Lie algebra of K^J. Then the Lie algebra $\mathfrak{g}^J$ of G^J has the Cartan decomposition

$$(2.7)\qquad \mathfrak{g}^J = \mathfrak{k}^J + \mathfrak{p}^J,$$

where

$$\mathfrak{g}^J = \left\{ \left(\begin{pmatrix} x & y \\ z & -x \end{pmatrix}, (P,Q,R) \right) \,\Big|\, x, y, z \in \mathbb{R},\ P, Q \in \mathbb{R}^m,\ R = {}^tR \in \mathbb{R}^{(m,m)} \right\},$$
$$\mathfrak{k}^J = \left\{ \left(\begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}, (0,0,R) \right) \,\Big|\, x \in \mathbb{R},\ R = {}^tR \in \mathbb{R}^{(m,m)} \right\},$$
$$\mathfrak{p}^J = \left\{ \left(\begin{pmatrix} x & y \\ y & -x \end{pmatrix}, (P,Q,0) \right) \,\Big|\, x, y \in \mathbb{R},\ P, Q \in \mathbb{R}^m \right\}.$$

Lemma 2.1. *We have the relations*

$$[\mathfrak{k}^J, \mathfrak{k}^J] \subset \mathfrak{k}^J \quad \text{and} \quad [\mathfrak{k}^J, \mathfrak{p}^J] \subset \mathfrak{p}^J. \tag{2.8}$$

Proof. The Lie bracket operation on $\mathfrak{g}^J$ is given by

$$[(X_1,(P_1,Q_1,R_1)),(X_2,(P_2,Q_2,R_2))] = (X^*,(P^*,Q^*,R^*)), \tag{2.9}$$

where $X_1, X_2 \in \mathfrak{sl}_2(\mathbb{R}),\ P_1,Q_1,P_2,Q_2 \in \mathbb{R}^m,\ R_1 = {}^tR_1,\ R_2 = {}^tR_2 \in \mathbb{R}^{(m,m)}$,

$$\begin{aligned} X^* &= [X_1,X_2] = X_1X_2 - X_2X_1, \\ (P^*,Q^*) &= (P_1,Q_1)X_2 - (P_2,Q_2)X_1, \\ R^* &= P_1\,{}^tQ_2 - P_2\,{}^tQ_1 + Q_2\,{}^tP_1 - Q_1\,{}^tP_2. \end{aligned}$$

The relations (2.8) follow immediately from Formula (2.9). □

Remark 2.1. The relation

$$[\mathfrak{p}^J, \mathfrak{p}^J] \subset \mathfrak{k}^J$$

does not hold.

The vector space $\mathfrak{p}^J$ can be regarded as the tangent space of the Siegel-Jacobi space $\mathbb{H}\times\mathbb{C}^m \cong G^J/K^J$ at $(i,0)$. We define a complex structure I^J on the tangent space $\mathfrak{p}^J$ of $\mathbb{H}\times\mathbb{C}^m \cong G^J/K^J$ at $(i,0)$ by

$$I^J\left(\begin{pmatrix} x & y \\ y & -x \end{pmatrix},(P,Q,0)\right) = \left(\begin{pmatrix} y & -x \\ -x & -y \end{pmatrix},(Q,-P,0)\right). \tag{2.10}$$

Let

$$\mathfrak{p} = \left\{ \begin{pmatrix} x & y \\ y & -x \end{pmatrix} \in \mathbb{R}^{(2,2)} \,\middle|\, x,y\in\mathbb{R} \right\}$$

be the real vector space of dimension 2. Identifying $\mathfrak{p}$ with $\mathbb{C}$ via

$$\begin{pmatrix} x & y \\ y & -x \end{pmatrix} \longmapsto x + i\,y \in \mathbb{C}$$

and identifying $\mathbb{R}^m\times\mathbb{R}^m$ with $\mathbb{C}^m$ via

$$(P,Q) \longmapsto Q + i\,P, \quad P,Q\in\mathbb{R}^m,$$

we may regard the complex structure I^J as a real linear map on $\mathbb{C}\times\mathbb{C}^m$ defined by

$$I^J(x+i\,y, Q+i\,P) = (-y+i\,x, -P+i\,Q), \quad x+i\,y\in\mathbb{C},\ Q+i\,P\in\mathbb{C}^m. \tag{2.11}$$

Clearly I^J extends complex linearly on the complexification $\mathfrak{p}^J_{\mathbb{C}} = \mathfrak{p}^J\otimes_{\mathbb{R}}\mathbb{C}$ of $\mathfrak{p}^J$. Then $\mathfrak{p}^J_{\mathbb{C}}$ has a decomposition

$$\mathfrak{p}^J_{\mathbb{C}} = \mathfrak{p}^J_+ \oplus \mathfrak{p}^J_-, \tag{2.12}$$

where $\mathfrak{p}_+^J$ (resp. $\mathfrak{p}_-^J$) denotes the $(+i)$-eigenspace (resp. $(-i)$-eigenspace) of I^J. Precisely, both $\mathfrak{p}_+^J$ and $\mathfrak{p}_-^J$ are given by

$$\mathfrak{p}_+^J = \left\{ \left(\begin{pmatrix} x & i\,x \\ i\,x & -x \end{pmatrix}, (P, iP, 0) \right) \;\Big|\; x \in \mathbb{C},\; P \in \mathbb{C}^m \right\}$$

and

$$\mathfrak{p}_-^J = \left\{ \left(\begin{pmatrix} x & -i\,x \\ -i\,x & -x \end{pmatrix}, (P, -iP, 0) \right) \;\Big|\; x \in \mathbb{C},\; P \in \mathbb{C}^m \right\}.$$

Proposition 2.1. *Fix an element* $g = \big(M, (\lambda, \mu; \kappa)\big) \in G^J$ *with* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ *and* $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(m)}$. *We let* $(\tau_*, z_*) = g \cdot (\tau, z)$. *Let*

$$\mathbb{F}_g : \mathbb{H} \times \mathbb{C}^m \longrightarrow \mathbb{H} \times \mathbb{C}^m$$

be the biholomorphic mapping defined by the action (2.1) of g. *Then the differential mapping*

$$d\mathbb{F}_g : T_{(\tau,z)}\big(\mathbb{H} \times \mathbb{C}^m\big) \longrightarrow T_{(\tau_*,z_*)}\big(\mathbb{H} \times \mathbb{C}^m\big)$$

is given by

$$(w, \xi) \longmapsto \big(w(g), \xi(g)\big), \quad w \in \mathbb{C},\; \xi \in \mathbb{C}^m \tag{2.13}$$

with

$$w(g) = \frac{w}{(c\,\tau + d)^2} \qquad \text{and} \qquad \xi(g) = \frac{\xi}{c\,\tau + d} + \frac{w(d\,\lambda - c\,\mu - c\,z)}{(c\,\tau + d)^2}.$$

Here we identified $\mathfrak{p}^J$ *with* $\mathbb{C} \times \mathbb{C}^m$.

Proof. Let $\alpha(t) = \big(\tau(t), z(t)\big)$ $(-\epsilon < t < \epsilon,\ \epsilon > 0)$ be a smooth curve in $\mathbb{H} \times \mathbb{C}^m$ passing through $\alpha(0) = (\tau, z)$ with $\alpha'(0) = (w, \xi) \in T_{(\tau,z)}\big(\mathbb{H} \times \mathbb{C}^m\big)$. Then

$$\begin{aligned} \chi(t) :=&\ g \cdot \alpha(t) = \big(\tau(g;t), z(g;t)\big) \\ =&\ \left(\frac{a\,\tau(t) + b}{c\,\tau(t) + d}, \frac{z(t) + \lambda\,\tau(t) + \mu}{c\,\tau(t) + d} \right) \end{aligned}$$

is a smooth curve in $\mathbb{H} \times \mathbb{C}^m$ passing through $\chi(0) = (\tau_*, z_*)$. Then by an easy computation, we see that

$$\tau'(g;0) = \frac{\partial}{\partial t}\Big|_{t=0} \tau(g;t) = \frac{\tau'(0)}{(c\,\tau + d)^2} = \frac{w}{(c\,\tau + d)^2}$$

and

$$z'(g;0) = \frac{\partial}{\partial t}\Big|_{t=0} z(g;t) = \frac{\xi}{c\,\tau + d} + \frac{w(d\,\lambda - c\,\mu - c\,z)}{(c\,\tau + d)^2}.$$

□

Let $\Gamma_1 := SL_2(\mathbb{Z})$ be the elliptic modular group. We let

$$\Gamma_{1,m} := \Gamma_1 \ltimes H_{\mathbb{Z}}^{(m)}$$

be the arithmetic subgroup of G^J, where

$$H_{\mathbb{Z}}^{(m)} := \left\{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(m)} \mid \lambda, \mu, \kappa \textit{ are integral} \right\}$$

is a discrete subgroup of $H_{\mathbb{R}}^{(m)}$. Let $E_k := {}^t(0, \cdots, 1, 0, \cdots, 0)$ $(1 \le k \le m)$ be the $m \times 1$ matrix with the $(k,1)$-th entry 1 and other entries 0. For an element $\tau \in \mathbb{H}$, we set for brevity

$$F_k(\tau) := \tau\, E_k, \quad 1 \le k \le m.$$

Let

$$\mathcal{F} := \left\{ \tau \in \mathbb{H} \,\middle|\, |\tau| \ge 1, \ \ |\mathrm{Re}\, \tau| \le 1/2 \right\}$$

be a fundamental domain for $\Gamma_1 \backslash \mathbb{H}$. We refer to [16], pp. 78-79 for more detail. For each $\tau \in \mathcal{F}$, we define the subset P_τ of $\mathbb{C}^m$ by

$$P_\tau := \left\{ \sum_{k=1}^m \lambda_k E_k + \sum_{k=1}^m \mu_k F_k(\tau) \,\middle|\, 0 \le \lambda_k, \mu_k \le 1 \right\}.$$

For each $\tau \in \mathcal{F}$, we define the subset $\mathcal{D}_\tau$ of $\mathbb{H} \times \mathbb{C}^m$ by

$$\mathcal{D}_\tau := \{ (\tau, z) \in \mathbb{H} \times \mathbb{C}^m \mid z \in P_\tau \}.$$

Theorem 2.1. *The following subset*

$$\mathcal{F}_{[m]} := \bigcup_{\tau \in \mathcal{F}} \mathcal{D}_\tau \tag{2.14}$$

is a fundamental domain for $\Gamma_{1,m} \backslash (\mathbb{H} \times \mathbb{C}^m)$ with respect to the action (2.1).

Proof. Let (τ_*, z_*) be an arbitrary element of $\mathbb{H} \times \mathbb{C}^m$. We must find an element (τ, z) of $\mathcal{F}_{[m]}$ and $\gamma_* = (\gamma, (\lambda, \mu; \kappa)) \in \Gamma_{1,m}$ with $\gamma \in \Gamma_1 = SL_2(\mathbb{Z})$ such that $\gamma_* \cdot (\tau, z) = (\tau_*, z_*)$. Since $\mathcal{F}$ is a fundamental domain for $\Gamma_1 \backslash \mathbb{H}$, there is an element γ of Γ_1 and an element $\tau \in \mathcal{F}$ such that $\tau_* = \gamma \cdot \tau$. Here τ is unique up to the boundary of $\mathcal{F}$. We write

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 = SL_2(\mathbb{Z}).$$

We can find $\lambda, \mu \in \mathbb{Z}^m$ and $z \in P_\tau$ satisfying the equation

$$z + \lambda\, \tau + \mu = z_* (x\, \tau + d).$$

If we take $\gamma_* = (\gamma, (\lambda, \mu; 0)) \in \Gamma_{1,m}$, we see that $\gamma_* \cdot (\tau, z) = (\tau_*, z_*)$. Therefore

$$\mathbb{H} \times \mathbb{C}^m = \bigcup_{\gamma_* \in \Gamma_{1,m}} \gamma_* \cdot \mathcal{F}_{[m]}.$$

Let (τ, z) and $\gamma_* \cdot (\tau, z)$ be two elements of $\mathcal{F}_{[m]}$ with $\gamma_* = (\gamma, (\lambda, \mu; \kappa)) \in \Gamma_{1,m}$ with $\gamma \in \Gamma_1$. Then both τ and $\gamma \cdot \tau$ lie in $\mathcal{F}$. Therefore both of them either lie in the boundary of $\mathcal{F}$ or $\gamma = \pm I_2$. In the case that both τ and $\gamma \cdot \tau$ lie in the boundary of $\mathcal{F}$, both (τ, z) and $\gamma_* \cdot (\tau, z)$ lie in the boundary of $\mathcal{F}_{[m]}$. If $\gamma = \pm I_2$, we get

$$z \in P_\tau \qquad \text{and} \qquad \pm\,(z + \lambda\,\tau + \mu) \in P_\tau. \tag{2.15}$$

From the definition of P_τ and (2.16), we see that either $\lambda = \mu = 0,\ \gamma \neq -I_2$ or both z and $\pm(z + \lambda\,\tau + \mu)$ lie on the boundary of the parallelepiped P_τ. Hence either both (τ, z) and $\gamma_* \cdot (\tau, z)$ lie in the boundary of $\mathcal{F}_{[m]}$ or $\gamma_* = (I_2, (0, 0; \kappa)) \in \Gamma_{1,m}$. Consequently $\mathcal{F}_{[m]}$ is a fundamental domain for $\Gamma_{1,m}\backslash(\mathbb{H} \times \mathbb{C}^m)$ with respect to the action (2.1). □

Now we consider the Siegel-Jacobi space $\mathbb{H}_{1,1} := \mathbb{H} \times \mathbb{C}$ endowed with the Riemannian metric (cf. (2.2))

$$ds^2_{1;1,1} = \frac{y + v^2}{y^3}\,(dx^2 + dy^2) + \frac{1}{y}\,(du^2 + dv^2) - \frac{2v}{y^2}\,(dx\,du + dy\,dv),$$

where $\tau = x + i\,y$ with $x,\ y > 0$ real and $z = u + i\,v$ with u, v real are coordinates in $\mathbb{H}_{1,1}$. Then

$$E_1 := \frac{\partial}{\partial x}, \quad E_2 := \frac{\partial}{\partial y}, \quad E_3 := \frac{\partial}{\partial u}, \quad E_4 := \frac{\partial}{\partial v}$$

form a local frame field on $\mathbb{H}_{1,1}$. Let Γ^k_{ij} $(i, j, k = 1, 2, 3, 4)$ be the Christoffel symbols for the Riemannian connection ∇ determined uniquely by the Riemannian metric $ds^2_{1;1,1}$. That is,

$$\nabla_{E_i} E_j = \sum_{k=1}^{4} \Gamma^k_{ij}\,E_k, \qquad i, j = 1, 2, 3, 4.$$

Lemma 2.2. *For all $i, j, k = 1, 2, 3, 4$, $\Gamma^k_{ij} = \Gamma^k_{ji}$. The Christoffel symbols Γ^k_{ij}'s $(1 \leq i, j, k \leq 4)$ are given by*

$$\begin{aligned}
\Gamma^2_{11} &= \frac{2\,y + y^2}{2\,y^2}, \qquad \Gamma^1_{12} = \Gamma^2_{22} = -\,\frac{2\,y + v^2}{2\,y^2},\\
\Gamma^4_{11} &= \frac{v^3}{2\,y^3}, \qquad \Gamma^3_{12} = \Gamma^4_{22} = -\,\frac{v^3}{2\,y^3},\\
\Gamma^1_{14} &= \Gamma^1_{23} = \Gamma^2_{24} = \Gamma^4_{33} = \frac{v}{2\,y},\\
\Gamma^2_{13} &= \Gamma^3_{34} = \Gamma^4_{44} = -\,\frac{v}{2\,y}, \qquad \Gamma^4_{13} = \frac{y - v^2}{2\,y^2},\\
\Gamma^3_{14} &= \Gamma^3_{23} = \Gamma^4_{24} = -\,\frac{y - v^2}{2\,y^2}, \quad \Gamma^2_{33} = \frac{1}{2}, \quad \Gamma^1_{34} = \Gamma^2_{44} = -\,\frac{1}{2}
\end{aligned}$$

and all other $\Gamma_{ij}^k = 0$.

Proof. It is easy to prove the above lemma. We leave the proof to the reader. $\square$

Proposition 2.2. *Let* $\gamma(t) = \big(x(t) + i\,y(t), u(t) + i\,v(t)\big)$ *be a smooth curve in* $\mathbb{H}_{1,1}$. *For brevity we write*

$$\begin{aligned} \ddot{x} &= \frac{d^2x}{dt^2}, \quad \ddot{y} = \frac{d^2y}{dt^2}, \quad \ddot{u} = \frac{d^2u}{dt^2}, \quad \ddot{v} = \frac{d^2v}{dt^2}, \\ \dot{x} &= \frac{dx}{dt}, \quad \dot{y} = \frac{dy}{dt}, \quad \dot{u} = \frac{du}{dt}, \quad \dot{v} = \frac{dv}{dt}. \end{aligned}$$

Then the curve $\gamma(t)$ *is a geodesic in* $\mathbb{H}_{1,1}$ *with respect to the metric* $ds^2_{1;1,1}$ *if and only if it satisfies the following four differential equations*

$$\ddot{x} - \frac{2\,y + y^2}{2\,y^2}\,\dot{x}\,\dot{y} + \frac{v}{y}\,\dot{x}\,\dot{v} + \frac{v}{y}\,\dot{y}\,\dot{u} - \dot{u}\,\dot{v} = 0 \tag{2.16}$$

$$\ddot{y} + \frac{2\,y + y^2}{2\,y^2}\,\dot{x}^2 - \frac{2\,y + y^2}{2\,y^2}\,\dot{y}^2 + \frac{1}{2}\,\dot{u}^2 - \frac{1}{2}\,\dot{v}^2 - \frac{v}{y}\,\dot{x}\,\dot{u} + \frac{v}{y}\,\dot{y}\,\dot{v} = 0 \tag{2.17}$$

$$\ddot{u} - \frac{v^3}{y^3}\,\dot{x}\,\dot{y} - \frac{y - v^2}{y^2}\,\dot{x}\,\dot{v} - \frac{y - v^2}{y^2}\,\dot{y}\,\dot{u} - \frac{v}{y}\,\dot{u}\,\dot{v} = 0 \tag{2.18}$$

$$\ddot{v} + \frac{v^3}{2y^3}\,\dot{x}^2 - \frac{v^3}{2y^3}\,\dot{y}^2 + \frac{v}{2y}\,\dot{u}^2 - \frac{v}{2y}\,\dot{v}^2 + \frac{y - v^2}{y^2}\,\dot{x}\,\dot{u} - \frac{y - v^2}{y^2}\,\dot{y}\,\dot{v} = 0 \tag{2.19}$$

Proof. Using Lemma 2.2 and the geodesic equations, we obtain the above equations. $\square$

Remark 2.2. If $u = v = 0$, the equations (2.16)-(2.19) reduce to the following two equations

$$\ddot{x} - \frac{2}{y}\,\dot{x}\,\dot{y} = 0 \tag{2.20}$$

and

$$\ddot{y} + \frac{1}{y}\,\dot{x}^2 - \frac{1}{y}\,\dot{y}^2 = 0. \tag{2.21}$$

Thus these two equations (2.20) and (2.21) give geodesics in the Poincaré upper half plane $\mathbb{H}$ which are circles perpendicular to the x-axis or straight lines perpendicular to the x-axis. Therefore the curve $\gamma(t) = \big(x(t) + i\,y(t), 0\big)$ $(-\infty < t < \infty)$ such that $\alpha(t) = x(t) + i\,y(t)$ is a geodesic in $\mathbb{H}$ is a geodesic in $\mathbb{H}_{1,1}$ with respect to the metric $ds^2_{1;1,1}$.

Proposition 2.3. *Let $\gamma(t)$ be a geodesic in $\mathbb{H}_{1,1}$ joining two points $\gamma(0)=(\tau_1,0)$ and $\gamma(1)=(\tau_2,0)$ such that $\gamma(t)$ is contained in the subset $\{(\tau,0)\in\mathbb{H}_{1,1}\mid \tau\in\mathbb{H}\}$. Then the length ρ of the geodesic segment between $\gamma(0)=(\tau_1,0)$ and $\gamma(1)=(\tau_2,0)$ is given by*

$$\rho=\log\frac{1+R^{1/2}}{1-R^{1/2}}, \tag{2.22}$$

where $R:=R(\tau_1,\tau_2)$ is the cross-ratio of τ_1 and τ_2 defined by

$$R(\tau_1,\tau_2):=\frac{\tau_1-\tau_2}{\tau_1-\overline{\tau}_2}\cdot\frac{\overline{\tau}_1-\overline{\tau}_2}{\overline{\tau}_1-\tau_2}.$$

Proof. By remark 2.2, the length ρ is equal to the length ρ_0 of the geodesic in $\mathbb{H}$ joining τ_1 and τ_2 with respect to the Poincaré metric

$$ds^2=\frac{dx^2+dy^2}{y^2}.$$

It is well known that ρ_0 is given by the formula (2.22). We refer to [17] for the general case. □

Proposition 2.4. *Let (τ_1,z_1) and (τ_2,z_2) be two points in the Siegel-Jacobi space $\mathbb{H}\times\mathbb{C}^m$. Then there exists an element $g=\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix},(\lambda,\mu;\kappa)\right)\in G^J$ such that*

$$g\cdot(\tau_1,z_1)=(i,0)\qquad\text{and}\qquad g\cdot(\tau_2,z_2)=\left(i\,\delta,\frac{z_2+\lambda\,\tau_2+\mu}{c\,\tau_2+d}\right)$$

with $\delta>0$. Therefore the length of the geodesic joining (τ_1,z_1) to (τ_2,z_2) with respect to the Riemannian metric $ds^2_{m;A,B}$ is equal to that of the geodesic joining $(i,0)$ to $\left(i\,\delta,\frac{z_2+\lambda\,\tau_2+\mu}{c\,\tau_2+d}\right)$ with respect to the metric $ds^2_{m;A,B}$.

Proof. We see that there is an element $h=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{R})$ such that

$$h\cdot\tau_1=\frac{a\,\tau_1+b}{c\,\tau_1+d}=i\qquad\text{and}\qquad h\cdot\tau_2=\frac{a\,\tau_2+b}{c\,\tau_2+d}=i\,\delta$$

with $\delta>0$. We take

$$\lambda=-\frac{\operatorname{Im}z_1}{\operatorname{Im}\tau_1}\qquad\text{and}\qquad \mu=-\operatorname{Re}z_1+\frac{\operatorname{Re}\tau_1\cdot\operatorname{Im}z_1}{\operatorname{Im}\tau_1}$$

We easily see that the element

$$g=\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix},(\lambda,\mu;\kappa)\right)\in G^J$$

satisfies the condition

$$g\cdot(\tau_1,z_1)=(i,0)\qquad\text{and}\qquad g\cdot(\tau_2,z_2)=\left(i\,\delta,\,\frac{z_2+\lambda\,\tau_2+\mu}{c\,\tau_2+d}\right)$$

with $\delta>0$.

For each fixed element $g\in G^J$, according to the G^J-invariance of the metric $ds^2_{m;A,B}$, the map $\mathbb{F}_g$ of $\mathbb{H}\times\mathbb{C}^m$ defined by the action (2.1) of g is an isometry of $\mathbb{H}\times\mathbb{C}^m$ with respect to the metric $ds^2_{m;A,B}$. Consequently we obtain the second statement. □

Proposition 2.5. *The scalar curvature* $r(p)$ *of the Siegel-Jacobi space* $\big(\mathbb{H}_{1,1},\,ds^2_{1;1,1}\big)$ *is* -3 *for each point* p *of* $\mathbb{H}_{1,1}$.

Proof. Using Lemma 2.2, we obtain the scalar curvature $r(p)=-3$ for each point p of $\mathbb{H}_{1,1}$ by a tedious computation. □

Now we study differential forms on $\mathbb{H}\times\mathbb{C}^m$ invariant under the action (2.1) of $\Gamma_{1,m}$.

Proposition 2.6. *(a) Assume that*

$$\alpha=f(\tau,z)\,d\tau\,+\,\sum_{k=1}^{m}\phi_k(\tau,z)\,dz_k$$

is a differential 1*-form on* $\mathbb{H}\times\mathbb{C}^m$ *invariant under the action (2.1) of* $\Gamma_{1,m}$. *Then the functions* f *and* ϕ_k $(k=1,2,\cdots,m)$ *satisfy the following conditions*

$$f(\gamma\cdot(\tau,z))=(c\,\tau+d)^2f(\tau,z)\,+\,(c\,\tau+d)\sum_{k=1}^{m}(c\,z_k+c\,\mu_k-d\,\lambda_k)\,\phi_k(\tau,z)\tag{2.23}$$

and

$$\phi_k(\gamma\cdot(\tau,z))=(c\,\tau+d)\,\phi_k(\tau,z),\qquad k=1,2,\cdots,m\tag{2.24}$$

for all $\gamma=\left(\begin{pmatrix}a&b\\c&d\end{pmatrix},(\lambda,\mu;\kappa)\right)\in\Gamma_{1,m}$ *with* $\lambda={}^t(\lambda_1,\cdots,\lambda_m)\in\mathbb{Z}^m$ *and* $\mu={}^t(\mu_1,\cdots,\mu_m)\in\mathbb{Z}^m$.

(b) Let

$$\eta=\,d\tau\wedge dz_1\wedge dz_2\wedge\cdots\wedge dz_m$$

be a differential $(m+1)$*-form on* $\mathbb{H}\times\mathbb{C}^m$. *Assume that*

$$\theta=g(\tau,z)\,\eta^{\otimes\ell},\qquad \ell=1,2,3,\cdots,$$

is a differential $\ell(m+1)$*-form on* $\mathbb{H}\times\mathbb{C}^m$ *invariant under the action (2.1) of* $\Gamma_{1,m}$. *Then the function* g *satisfies the following condition*

$$g(\gamma\cdot(\tau,z))=(c\,\tau+d)^{\ell(m+2)}g(\tau,z)\tag{2.25}$$

for all $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu; \kappa) \right) \in \Gamma_{1,m}$.

(c) For $k = 1, 2, \cdots, m$, *we let*

$$\widetilde{\omega}_k = (-1)^{m-k}\, d\tau \wedge dz_1 \wedge \cdots \wedge dz_{k-1} \wedge \widehat{dz_k} \wedge dz_{k+1} \wedge \cdots \wedge dz_m$$

be a differential m*-form on* $\mathbb{H} \times \mathbb{C}^m$. *Assume that*

$$\beta = \sum_{k=1}^{m} a_k(\tau, z)\, \widetilde{\omega}_k \; + \; (-1)^m\, b(\tau, z)\, dz_1 \wedge \cdots \wedge dz_m$$

is a differential m*-form on* $\mathbb{H} \times \mathbb{C}^m$ *invariant under the action (2.1) of* $\Gamma_{1,m}$. *Then the functions* $a(\tau, z)$ *and* b_k $(k = 1, 2, \cdots, m)$ *satisfy the following conditions*

$$a_k(\gamma \cdot (\tau, z)) = (c\,\tau + d)^{m+1}\, a_k(\tau, z) - (c\,\tau + d)^m (c\, z_k + c\, \mu_k - d\, \lambda_k)\, b(\tau, z) \tag{2.26}$$

for $k - 1, 2, \cdots, m$ *and*

$$b(\gamma \cdot (\tau, z)) = (c\,\tau + d)^m\, b(\tau, z) \tag{2.27}$$

for all $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu; \kappa) \right) \in \Gamma_{1,m}$ *with* $\lambda = {}^t(\lambda_1, \cdots, \lambda_m) \in \mathbb{Z}^m$ *and* $\mu = {}^t(\mu_1, \cdots, \mu_m) \in \mathbb{Z}^m$.

Proof. For $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu; \kappa) \right) \in \Gamma_{1,m}$ with $\lambda = {}^t(\lambda_1, \cdots, \lambda_m) \in \mathbb{Z}^m$ and $\mu = {}^t(\mu_1, \cdots, \mu_m) \in \mathbb{Z}^m$ and $(\tau, z) \in \mathbb{H} \times \mathbb{C}^m$ with $z = {}^t(z_1, \cdots, z_m) \in \mathbb{C}^m$, we set $(\tau^*, z^*) = \gamma \cdot (\tau, z)$. In other words,

$$\tau^* = \frac{a\,\tau + b}{c\,\tau + d}, \qquad z_k^* = \frac{z_k + \lambda_k\,\tau + \mu_k}{c\,\tau + d}, \qquad k = 1, 2, \cdots, m.$$

Then we have

$$d\tau^* = \frac{d\tau}{(c\,\tau + d)^2} \tag{2.28}$$

and

$$dz_k^* = \left\{ \frac{\lambda_k}{c\,\tau + d} - \frac{c\,(z_k + \lambda_k\,\tau + \mu_k)}{(c\,\tau + d)^2} \right\} d\tau \, + \, \frac{dz_k}{c\,\tau + d}, \qquad k = 1, 2, \cdots, m. \tag{2.29}$$

Using the formulas (2.28) and (2.29), we obtain the desired results (a), (b) and (c). □

3. The center of the universal enveloping algebra of $\mathfrak{g}^J$

In this section we describe the center of the universal enveloping algebra of the complexication of the Jacobi Lie algebra $\mathfrak{g}^J$ explicitly.

Let $\mathfrak{g}^J_{\mathbb{C}}$ be the complexification of the Jacobi Lie algebra $\mathfrak{g}^J$. We put the 2×2 matrices

$$H=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\qquad E=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\qquad\text{and}\qquad F=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

Then $\{H,E,F\}$ is a basis of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Let $\epsilon_{ij}\,(1\le i\le m,\ j=1,2)$ be the $m\times 2$ matrices whose (i,j)-th entry is 1 and whose other entries are zero, and let E_{kl} be the $m\times m$ elementary matrix whose (k,l)-th entry is 1 and whose other entries are zero. We set $e_i:=\epsilon_{i1},\ f_i:=\epsilon_{i2}\ (1\le i\le m)$ and

$$R_{kl}:=\frac{1}{2}\,(E_{kl}\,+\,E_{ji}),\quad R_{kl}=R_{lk},\quad 1\le k,l\le m.$$

Then $\{H,E,F,e_i,f_i,R_{kl}\,|\,1\le i\le m,\ 1\le k\le l\le m\}$ is a basis for $\mathfrak{g}^J_{\mathbb{C}}$. It is easily seen that

$$\mathcal{Z}_m:=\Big\{(0,(0,0,R))\in\mathfrak{g}^J_{\mathbb{C}}\,|\,R=\,{}^tR\in\mathbb{C}^{(m,m)}\Big\}$$

is the center of $\mathfrak{g}^J_{\mathbb{C}}$.

Lemma 3.1. *We have the following.*

(1) $[H,E]=2\,E,\quad [H,F]=-2\,F,\quad [E,F]=H.$

(2) $[H,e_i]=-e_i,\quad [H,f_i]=f_i,\quad 1\le i\le m.$

(3) $[E,e_i]=f_i,\quad [E,f_i]=0,\quad 1\le i\le m.$

(4) $[F,e_i]=0,\quad [F,f_i]=-e_i,\quad 1\le i\le m.$

(5) $[e_i,f_j]=2\,R_{ij},\quad 1\le i,j\le m.$

Proof. The proof follows immediately from the fact that

$$\begin{aligned}(3.1)\quad &[(X_1,(P_1,Q_1,R_1)),(X_2,(P_2,Q_2,R_2))]\\ =\ &\Big([X_1,X_2],\big((P_1,Q_1)X_2-(P_2,Q_2)X_1,P_1\,{}^tQ_2-P_2\,{}^tQ_1+Q_2\,{}^tP_1-Q_1\,{}^tP_2\big)\Big),\end{aligned}$$

where $X_1,X_2\in\mathfrak{sl}_2(\mathbb{C}),\ [X_1,X_2]=X_1X_2-X_2X_1,\ P_i,Q_i\in\mathbb{C}^{(m,1)}\,(i=1,2),\ R_1,R_2\in\mathbb{C}^{(m,m)}$ with $R_1={}^tR_1$ and $R_2={}^tR_2$. □

Formally we put

$$e:=\begin{pmatrix}e_1\\ e_2\\ \vdots\\ e_m\end{pmatrix},\qquad f:=\begin{pmatrix}f_1\\ f_2\\ \vdots\\ f_m\end{pmatrix},$$

and

$$R := \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1m} \\ R_{21} & R_{22} & \cdots & R_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ R_{m1} & R_{m2} & \cdots & R_{mm} \end{pmatrix}, \quad R_{kl} = R_{lk}, \quad 1 \le k, l \le m.$$

Theorem 3.1. *The center $\mathcal{Z}_m\big(\mathfrak{g}^J_{\mathbb{C}}\big)$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}^J_{\mathbb{C}})$ of $\mathfrak{g}^J_{\mathbb{C}}$ is given by*

$$\mathcal{Z}_m\big(\mathfrak{g}^J_{\mathbb{C}}\big) = \mathbb{C}[\Omega_m, R_{kl} \mid 1 \le k \le l \le m].$$

That is, $\mathcal{Z}_m\big(\mathfrak{g}^J_{\mathbb{C}}\big)$ is a polynomial algebra on $1 + \frac{m(m+1)}{2}$ generators Ω_m, R_{kl} $(1 \le k \le l \le m)$. Here

$$\begin{aligned} \Omega_m : \;\; = \;\; & \det R \left\{ H^2 - (m+2)H \, + \, 4\, EF \right\} \\ & + \det R \left\{ E\, {}^t\!eR^{-1}e - \, {}^t\!fR^{-1}fF - \left(H - \frac{m+3}{2} \right) {}^t\!fR^{-1}e \right\} \\ & + \det R \left\{ \frac{1}{4}\, {}^t\!f \big({}^t\!fR^{-1}e\big) R^{-1}e - \frac{1}{4} \big({}^t\!eR^{-1}f\big)\big({}^t\!eR^{-1}e\big) \right\} \end{aligned}$$

is a Casimir operator of $\mathcal{U}(\mathfrak{g}^J\mathbb{C})$ of degree $m+2$.

Proof. Using the method computing the center of the universal enveloping algebra of a certain class of semidirect sum Lie algebras invented by Campoamer-Stursburg and Low [6] (cf. [2], [15]), Conley and Raum [5] proved the above theorem. We refer to [5] for the detail. □

Let $\gamma : G^J \times \big(\mathbb{H} \times \mathbb{C}^m\big) \longrightarrow \mathbb{C}^\times$ be a scalar cocycle with respect to the action (2.1). This means that γ is a smooth function satisfying the cocycle condition

$$\gamma(g_1 g_2, (\tau, z)) = \gamma(g_1, g_2 \cdot (\tau, z))\, \gamma(g_2, (\tau, z)) \tag{3.2}$$

for all $g_1, g_2 \in G^J$ and $(\tau, z) \in \mathbb{H} \times \mathbb{C}^m$. Then we get the map

$$\widehat{\gamma}(g) : G^J \longrightarrow C^\infty(\mathbb{H} \times \mathbb{C}^m)$$

defined by

$$\widehat{\gamma}(g)(\tau, z) := \gamma(g, (\tau, z)), \quad g \in G^J, \; (\tau, z) \in \mathbb{H} \times \mathbb{C}^m.$$

Then we obtain the right action $|_\gamma$ of G^J on $C^\infty(\mathbb{H} \times \mathbb{C}^m)$ defined by

$$(g \cdot f)(\tau, z) := \big(f|_\gamma[g^{-1}]\big)(\tau, z) := \gamma(g^{-1}, (\tau, z)) f(g^{-1} \cdot (\tau, z)), \tag{3.3}$$

where $g \in G^J$, $f \in C^\infty(\mathbb{H} \times \mathbb{C}^m)$ and $(\tau, z) \in \mathbb{H} \times \mathbb{C}^m$.

We note that the differential $d\widehat{\gamma}$ of $\widehat{\gamma}$ at the identity is given by

$$d\widehat{\gamma}(Y)(\tau, z) = \frac{d}{dt}\Big|_{t=0} \gamma(\exp(tY), (\tau, z)).$$

Therefore we have the differential right action $|_\gamma$ of $\mathfrak{g}^J_{\mathbb{C}}$ on $C^\infty(\mathbb{H}\times\mathbb{C}^m)$ defined by

$$(3.4)\qquad (\phi|_\gamma[Y])(\tau,z) := \frac{d}{dt}\Big|_{t=0}(\gamma(\exp(tY),(\tau,z))\phi(\exp(tY)\cdot(\tau,z)))$$

$$(3.5)\qquad = \gamma(Y,(\tau,z))\phi(\tau,z) + \frac{d}{dt}\Big|_{t=0}\phi(\exp(tY),(\tau,z)),$$

where $Y\in\mathfrak{g}^J_{\mathbb{C}}$ and $\phi\in C^\infty(\mathbb{H}\times\mathbb{C}^m)$. The action (3.4) extends to $\mathcal{U}(\mathfrak{g}^J_{\mathbb{C}})$ as usual, and elements of $\mathcal{U}(\mathfrak{g}^J_{\mathbb{C}})$ of order r act by differential operators of order $\le r$.

Let $\mathbb{D}_\gamma$ be the algebra of all differential operators D on $\mathbb{H}\times\mathbb{C}^m$ satisfying the following condition

$$(3.6)\qquad (D\phi)|_\gamma[g] = D\big(\phi|_\gamma[g]\big)$$

for all $g\in G^J$ and for all $\phi\in C^\infty(\mathbb{H}\times\mathbb{C}^m)$. Since G^J is connected, $\mathbb{D}_\gamma$ is the algebra of all differential operators $\mathbb{D}_\gamma$on $\mathbb{H}\times\mathbb{C}^m$ commuting with the $|_\gamma$-action of $\mathfrak{g}^J_{\mathbb{C}}$. In particular, the action $|_\gamma$ maps the center $\mathcal{Z}_m(\mathfrak{g}^J_{\mathbb{C}})$ of $\mathcal{U}(\mathfrak{g}^J_{\mathbb{C}})$ into the center $\mathcal{Z}_m(\mathbb{D}_\gamma)$ of $\mathbb{D}_\gamma$.

Throughout this section we let $\mathcal{M}$ be a positive definite half-integral symmetric matrix of degree m and let $k\in\mathbb{Z}^+$. We let $\gamma_{k,\mathcal{M}} : G^J\times(\mathbb{H}\times\mathbb{C}^m)\longrightarrow\mathbb{C}^\times$ be the canonical automorphic factor for G^J on $\mathbb{H}\times\mathbb{C}^m$ defined by

$$\gamma_{k,\,\mathcal{M}}\big((M,(\lambda,\mu;\kappa)),(\tau,z)\big) :$$
$$(3.7)\qquad = (c\tau+d)^k\, e^{2\pi i\,\mathcal{M}[z+\lambda\tau+\mu]\,c\,(c\tau+d)^{-1}}\, e^{-2\pi i\,\mathrm{tr}\left(\mathcal{M}(\tau\lambda\,{}^t\lambda+2\,\lambda\,{}^t z+\kappa+\mu\,{}^t\lambda)\right)},$$

where $(M,(\lambda,\mu;\kappa))\in G^J$ with $M=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{R})$, $(\lambda,\mu;\kappa)\in H^{(m)}_{\mathbb{R}}$ and $(\tau,z)\in\mathbb{H}\times\mathbb{C}^m$.

For brevity we write

$$\partial_\tau := \frac{\partial}{\partial\tau} = \frac12\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right),\qquad \partial_{\overline{\tau}} := \frac{\partial}{\partial\overline{\tau}} = \frac12\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right),$$
$$\partial_{z_j} := \frac{\partial}{\partial z_j} = \frac12\left(\frac{\partial}{\partial u_j} - i\,\frac{\partial}{\partial v_j}\right),\quad 1\le j\le m,$$
$$\partial_{\overline{z}_j} := \frac{\partial}{\partial \overline{z}_j} = \frac12\left(\frac{\partial}{\partial u_j} + i\,\frac{\partial}{\partial v_j}\right),\quad 1\le j\le m,$$
$$\partial_z := {}^t\big(\partial_{z_1},\partial_{z_2},\cdots,\partial_{z_m}\big),\quad \partial_{\overline{z}} := {}^t\big(\partial_{\overline{z}_1},\partial_{\overline{z}_2},\cdots,\partial_{\overline{z}_m}\big).$$

Lemma 3.2. *Let $\mathcal{M}$ and k be as above. We set $\widetilde{\mathcal{M}} := 2\pi i\,\mathcal{M}$. Then we have the following:*

$$(3.8)\qquad |_{\gamma_{k,\,\mathcal{M}}}[E] = 2\,\mathrm{Re}\,(\partial_\tau),$$

$$(3.9)\qquad |_{\gamma_{k,\,\mathcal{M}}}[F] = -2\,\mathrm{Re}\big(\tau\,(\tau\,\partial_\tau + {}^t z\,\partial_z)\big) - k\,\tau - \widetilde{\mathcal{M}}[z],$$

$$(3.10)\qquad |_{\gamma_{k,\,\mathcal{M}}}[H] = 2\,\mathrm{Re}\big(2\,\tau\,\partial_\tau + {}^t z\,\partial_z\big) + k,$$

$$(3.11)\qquad |_{\gamma_{k,\,\mathcal{M}}}[(0,(P,Q,R))] = 2\,\mathrm{Re}\big({}^t(P\,\tau+Q)\,\partial_z\big) + 2\,{}^tP\widetilde{\mathcal{M}}\,z + \mathrm{tr}\big(R\widetilde{\mathcal{M}}\big).$$

Proof. We observe that if $(X,(P,Q,R))\in \mathfrak{g}^J_{\mathbb C}$ with $X\in \mathfrak{sl}_2(\mathbb C),\ P,Q\in \mathbb C^{(m,1)}$ and $R={}^tR\in \mathbb C^{(m,m)}$, then

$$\exp\big((X,(P,Q,R))\big)=\Big(\exp(X),\big((P,Q)\,g(X),R-(P,Q)\,h(X)\,{}^t(-Q,P)\big)\Big), \tag{3.12}$$

where

$$\exp(t):=\sum_{n=0}^{\infty}\frac{t^n}{n!},\quad g(t):=\frac{e^t-1}{t}\quad \text{and}\quad h(t):=\frac{e^t-1-t}{t}.$$

Using the formula (3.12) we easily obtain the formulas (3.8)-(3.11). □.

Theorem 3.2.

$$|_{\gamma_{k,\,\mathcal M}}[\Omega_m]=\det\big(\widetilde{\mathcal M}\big)\ \big\{k(k-m-2)\,-\,2\,\mathcal C^{k,\,\mathcal M}\big\}, \tag{3.13}$$

where

$$\begin{aligned}\mathcal C^{k,\,\mathcal M}:\ =\ & -8\,y^2\partial_\tau\partial_{\overline\tau}\,+\,4\,i\left(k-\frac{m}{2}\right)y\,\partial_{\overline\tau}\\ &+2\,y^2\Big(\partial_{\overline\tau}\,\widetilde{\mathcal M}^{-1}[\partial_z]\,+\,\partial_\tau\,\widetilde{\mathcal M}^{-1}[\partial_{\overline z}]\Big)\,-\,8\,y\,\partial_\tau\,{}^tv\,\partial_{\overline z}\\ &-\frac{1}{2}\,y^2\left\{\widetilde{\mathcal M}^{-1}[\partial_{\overline z}]\,\widetilde{\mathcal M}^{-1}[\partial_z]-{}^t\big(\partial_{\overline z}\widetilde{\mathcal M}^{-1}\partial_z\big)^2\right\}\,+\,2\,y\,\big({}^tv\,\partial_{\overline z}\big)\,{}^t\partial_z\widetilde{\mathcal M}^{-1}\partial_u\\ &-\frac{i}{2}\,(2k-m+1)\,y\,{}^t\partial_{\overline z}\,\widetilde{\mathcal M}^{-1}\partial_u\,+\,2\,{}^tv\,\big({}^tv\,\partial_{\overline z}\big)\partial_{\overline z}\,+\,i\,(2k-m-1)\,{}^tv\,\partial_{\overline z}.\end{aligned}$$

The operator $\mathcal C^{k,\,\mathcal M}$ generates the image of the $|_{\gamma_{k,\mathcal M}}$-action of the center $\mathcal Z_m(\mathfrak g^J_{\mathbb C})$. In particular, $\mathcal C^{k,\,\mathcal M}$ is an element of the center of $\mathbb D_{\gamma_{k,\mathcal M}}$.

Proof. We write $\widetilde{\mathcal M}=\big(\widetilde{\mathcal M}_{pq}\big)$. According to (3.11), we have the relation $|_{\gamma_{k,\mathcal M}}[R_{pq}]=\widetilde{\mathcal M}_{pq}$ for all $1\le p\le q\le m$. The proof follows from Theorem 3.1. and Lemma 3.2. □

4. Invariant differential operators on $\mathbb H\times\mathbb C^m$

For brevity we put

$$T_{1,m}:=\mathbb C\times\mathbb C^m.$$

We define the real linear map $\Phi_m:\mathfrak p^J\longrightarrow T_{1,m}$ by

$$\Phi_m\left(\begin{pmatrix}x & y\\ y & -x\end{pmatrix},(P,Q,0)\right)=(x+i\,y,P+i\,Q), \tag{4.1}$$

where $\left(\begin{pmatrix}x & y\\ y & -x\end{pmatrix},(P,Q,0)\right)\in\mathfrak p^J$. Obviously Φ_m is a real linear isomorphism of $\mathfrak p^J$ onto $T_{1,m}$.

Let $S(m,\mathbb{R})$ denote the additive group consisting of all $m\times m$ real symmetric matrices. We define the group isomorphism $\theta_m : K^J \longrightarrow U(1)\times S(m,\mathbb{R})$ by

$$\theta_m\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix},(0,0;\kappa)\right)=(a+i\,b,\kappa), \tag{4.2}$$

where $\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix},(0,0;\kappa)\right)\in K^J$.

Theorem 4.1. *The adjoint representation Ad of K^J on $\mathfrak{p}^J$ is compatible with the natural action of $U(1)\times S(m,\mathbb{R})$ on $T_{1,m}=\mathbb{C}\times\mathbb{C}^m$ defined by*

$$(h,\kappa)\cdot(w,\xi):=(h^2w,h\,\xi),\quad h\in U(1),\ \kappa\in S(m,\mathbb{R}),\ w\in\mathbb{C},\ \xi\in\mathbb{C}^m \tag{4.3}$$

through the map Φ_m and θ_m. Precisely if $k^J\in K^J$ and $\alpha\in\mathfrak{p}^J$, then we have the following equality

$$\Phi_m\big(Ad(k^J)\alpha\big)=\theta_m(k^J)\cdot\Phi_m(\alpha). \tag{4.4}$$

We recall that we identified $\mathfrak{p}^J$ with $\mathbb{C}\times\mathbb{C}^m$.

Proof. We refer to [26] for the proof. □

The action (4.3) induces the action of $U(1)$ on the polynomial algebra $\mathrm{Pol}_{[m]}:=\mathrm{Pol}(T_{1,m})$. We denote by $\mathrm{Pol}_{[m]}^{U(1)}$ the subalgebra of $\mathrm{Pol}_{[m]}$ consisting of $U(1)$-invariants. We let $\mathbb{D}\big(\mathbb{H}\times\mathbb{C}^m\big)$ be the algebra of all differential operators invariant under the action (2.1) of G^J. According to [7], one gets a canonical linear bijection

$$\Theta_{[m]}:\mathrm{Pol}_{[m]}^{U(1)}\longrightarrow\mathbb{D}\big(\mathbb{H}\times\mathbb{C}^m\big) \tag{4.5}$$

of $\mathrm{Pol}_{[m]}^{U(1)}$ onto $\mathbb{D}\big(\mathbb{H}\times\mathbb{C}^m\big)$. But $\Theta_{[m]}$ is not multiplicative. The map $\Theta_{[m]}$ is described explicitly as follows. Let $\big\{\eta_\alpha\,|\,1\le\alpha\le 2(m+1)\big\}$ be a basis of $\mathfrak{p}^J$. If $P\in\mathrm{Pol}_{[m]}^{U(1)}$, then

$$\Big(\Theta_{[m]}(P)f\Big)(gK^J)=\left[P\left(\frac{\partial}{\partial t_\alpha}\right)f\left(g\exp\left(\sum_{\alpha=1}^{2(m+1)}t_\alpha\eta_\alpha\right)K^J\right)\right]_{(t_\alpha)=0}, \tag{4.6}$$

where $g\in G^J$ and $f\in C^\infty(\mathbb{H}\times\mathbb{C}^m)$.

Theorem 4.2. $\mathrm{Pol}_{[m]}^{U(1)}$ *is generated by*

$$q(w,\xi)=\,\mathrm{tr}\big(w\,\overline{w}\big), \tag{4.7}$$

$$\alpha_{kp}(w,\xi)=\,\mathrm{Re}\,\big(\xi\,{}^t\overline{\xi}\big)_{kp},\quad 1\le k\le p\le m, \tag{4.8}$$

$$\beta_{lq}(w,\xi)=\,\mathrm{Im}\,\big(\xi\,{}^t\overline{\xi}\big)_{lq},\quad 1\le l<q\le m, \tag{4.9}$$

$$f_{kp}(w,\xi)=\,\mathrm{Re}\,(\overline{w}\,\xi\,{}^t\xi)_{kp},\quad 1\le k\le p\le m, \tag{4.10}$$

$$g_{kp}(w,\xi)=\,\mathrm{Im}\,(\overline{w}\,\xi\,{}^t\xi)_{kp},\quad 1\le k\le p\le m, \tag{4.11}$$

where $w \in \mathbb{C}$ *and* $\xi \in \mathbb{C}^m$*.*

Proof. We refer to [9] or [26] for the general case. □

We let

$$w = r + i\,s \in \mathbb{C} \quad \text{and} \quad \xi = {}^t(\xi_1, \cdots, \xi_m) \in \mathbb{C}^m \text{ with } \xi_k = \zeta_k + i\,\eta_k,\ 1 \le k \le m,$$

where $r, s, \zeta_1, \eta_1, \cdots, \zeta_m, \eta_m$ are real. The invariants q, α_{kp}, β_{lq}, f_{kp} and g_{kp} are expressed in terms of $r, s, \zeta_k, \eta_l\ (1 \le k, l \le m)$ as follows:

$$\begin{aligned} q(w,\xi) &= r^2 + s^2, \\ \alpha_{kp}(w,\xi) &= \zeta_k\zeta_p + \eta_k\eta_p, \quad 1 \le k \le p \le m, \\ \beta_{lq}(w,\xi) &= \zeta_q\eta_l - \zeta_l\eta_q, \quad 1 \le l < q \le m, \\ f_{kp}(w,\xi) &= r(\zeta_k\zeta_p - \eta_k\eta_p) + s(\zeta_k\eta_p + \eta_k\zeta_p), \quad 1 \le k \le p \le m, \\ g_{kp}(w,\xi) &= r(\zeta_k\eta_p + \eta_k\zeta_p) - s(\zeta_k\zeta_p - \eta_k\eta_p), \quad 1 \le k \le p \le m. \end{aligned}$$

Theorem 4.3. *The* $\frac{m(m+1)}{2}$ *relations*

$$f_{kp}^2 + g_{kp}^2 = q\,\alpha_{kk}\,\alpha_{pp}, \quad 1 \le k \le p \le m \tag{4.12}$$

exhaust all the relations among a complete set of generators q, α_{kp}, β_{lq}, f_{kp} *and* g_{kp} *of* $\mathrm{Pol}_{[m]}^{U(1)}$ *with* $1 \le k \le p \le m$ *and* $1 \le l < q \le m$.

Theorem 4.4. *The action of* $U(1)$ *on* $Pol_{1,m}$ *is not multiplicity-free. In fact, if*

$$\mathrm{Pol}_{[m]} = \sum_{\sigma \in \widehat{U(1)}} m_\sigma\, \sigma,$$

then $m_\sigma = \infty$.

For the proofs of the above theorems we refer to [26].

We consider the case $m = 1$. For a coordinate (w, ξ) in $T_{1,1}$, we write $w = r + i\,s,\ \xi = \zeta + i\,\eta,\ r, s, \zeta, \eta$ real. The author [21] proved that the algebra $\mathrm{Pol}_{[1]}^{U(1)}$ is generated by

$$\begin{aligned} q(w,\xi) &= \frac{1}{4}\, w\,\overline{w} = \frac{1}{4}\big(r^2 + s^2\big), \\ \alpha(w,\xi) &= \xi\,\overline{\xi} = \zeta^2 + \eta^2, \\ \phi(w,\xi) &= \frac{1}{2}\,\mathrm{Re}\,\big(\xi^2\overline{w}\big) = \frac{1}{2}\,r\big(\zeta^2 - \eta^2\big) + s\,\zeta\eta, \\ \psi(w,\xi) &= \frac{1}{2}\,\mathrm{Im}\,\big(\xi^2\overline{w}\big) = \frac{1}{2}\,s\big(\eta^2 - \zeta^2\big) + r\,\zeta\eta. \end{aligned}$$

In [21], using Formula (3.6) the author calculated explicitly the images

$$D_1 = \Theta_{[1]}(q), \quad D_2 = \Theta_{[1]}(\alpha), \quad D_3 = \Theta_{[1]}(\phi) \quad \text{and} \quad D_4 = \Theta_{[1]}(\psi)$$

of q, α, ϕ and ψ under the Halgason map $\Theta_{[1]}$. We can show that the algebra $\mathbb{D}(\mathbb{H}\times\mathbb{C})$ is generated by the following differential operators

$$\begin{aligned} D_1 =& y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+v^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right) \\ &+2\,y\,v\left(\frac{\partial^2}{\partial x\partial u}+\frac{\partial^2}{\partial y\partial v}\right), \\ D_2 =& \, y\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right), \\ D_3 =& y^2\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial u^2}-\frac{\partial^2}{\partial v^2}\right)-2y^2\frac{\partial^3}{\partial x\partial u\partial v} \\ &-\left(v\frac{\partial}{\partial v}+1\right)D_2 \end{aligned}$$

and

$$\begin{aligned} D_4 =& y^2\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial v^2}-\frac{\partial^2}{\partial u^2}\right)-2\,y^2\frac{\partial^3}{\partial y\partial u\partial v} \\ &-v\frac{\partial}{\partial u}D_2, \end{aligned}$$

where $\tau = x+iy$ and $z = u+iv$ with real variables x, y, u, v. Moreover, we have

$$\begin{aligned} D_1D_2 - D_2D_1 =& \; 2\,y^2\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial u^2}-\frac{\partial^2}{\partial v^2}\right) \\ &-4\,y^2\frac{\partial^3}{\partial x\partial u\partial v}-2\left(v\frac{\partial}{\partial v}D_2+D_2\right). \end{aligned}$$

In particular, the algebra $\mathbb{D}(\mathbb{H}\times\mathbb{C})$ is not commutative. We refer to [1, 21] for more detail.

Recently Hiroyuki Ochiai [13] (see also [1]) proved the following result.

Theorem 4.5. *We have the following relations*

(*a*) $[D_1, D_2] = 2D_3$

(*b*) $[D_1, D_3] = 2D_1D_2 - 2D_3$

(*c*) $[D_2, D_3] = -D_2^2$

(*d*) $[D_4, D_1] = 0$

(*e*) $[D_4, D_2] = 0$

(*f*) $[D_4, D_3] = 0$

(g) $D_3^2 + D_4^2 = D_2 D_1 D_2$

These seven relations exhaust all the relations among the generators D_1, D_2, D_3 *and* D_4 *of* $\mathbb{D}(\mathbb{H} \times \mathbb{C})$.

Remark 4.1. According to Theorem 4.5, we see that D_4 is a generator of the center of $\mathbb{D}(\mathbb{H} \times \mathbb{C})$. We observe that the Lapalcian

$$\Delta_{1;A,B} = \frac{4}{A} D_1 + \frac{4}{B} D_2 \qquad \text{(see (2.5))}$$

of $(\mathbb{H} \times \mathbb{C}, ds^2_{1;A,B})$ does not belong to the center of $\mathbb{D}(\mathbb{H} \times \mathbb{C})$.

5. Maass-Jacobi Forms due to Yang

Using G^J-invariant differential operators on the Siegel-Jacobi space, we introduce a notion of Maass-Jacobi forms.

Definition 5.1. Let

$$\Gamma_{1,m} := SL_2(\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(m)}$$

be the discrete subgroup of G^J, where

$$H_{\mathbb{Z}}^{(m)} = \left\{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(m)} \mid \lambda, \mu, \kappa \text{ are integral} \right\}.$$

A smooth function $f : \mathbb{H} \times \mathbb{C}^m \longrightarrow \mathbb{C}$ is called a Maass-Jacobi form on $\mathbb{H} \times \mathbb{C}^m$ if f satisfies the following conditions (MJ1)-(MJ3) :

(MJ1) f is invariant under $\Gamma_{1,m}$.
(MJ2) f is an eigenfunction of the Laplacian $\Delta_{m;A,B}$ (cf. Formula (2.5)).
(MJ3) f has a polynomial growth, that is, there exist a constant $C > 0$ and a positive integer N such that

$$|f(x + iy, z)| \leq C\, |p(y)|^N \quad \text{as } y \longrightarrow \infty,$$

where $p(y)$ is a polynomial in y.

Remark 5.1. Let $\mathbb{D}_*$ be a commutative subalgebra of $\mathbb{D}(\mathbb{H} \times \mathbb{C}^m)$ containing the Laplacian $\Delta_{m;A,B}$. We say that a smooth function $f : \mathbb{H} \times \mathbb{C}^m \longrightarrow \mathbb{C}$ is a Maass-Jacobi form with respect to $\mathbb{D}_*$ if f satisfies the conditions $(MJ1)$, $(MJ2)_*$ and $(MJ3)$: the condition $(MJ2)_*$ is given by

$(MJ2)_*$ f is an eigenfunction of any invariant differential operator in $\mathbb{D}_*$.

It is natural to propose the following problems.

Problem A : Find all the eigenfunctions of $\Delta_{m;A,B}$.

Problem B : Construct Maass-Jacobi forms.

Problem C : Develop the spectral theory of the Laplacian $\Delta_{m;A,B}$ on a fundamental domain for the Siegel-Jacobi space $\mathbb{H} \times \mathbb{C}^m$ with respect to $\Gamma_{1,m}$.

If we find a *nice* eigenfunction ϕ of the Laplacian $\Delta_{m;A,B}$, we can construct a Maass-Jacobi form f_ϕ on $\mathbb{H} \times \mathbb{C}^m$ in the usual way defined by

$$f_\phi(\tau, z) := \sum_{\gamma \in \Gamma^\infty_{1,m} \backslash \Gamma_{1,m}} \phi\big(\gamma \cdot (\tau, z)\big), \tag{5.1}$$

where

$$\Gamma^\infty_{1,m} = \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu; \kappa) \right) \in \Gamma_{1,m} \,\Big|\, c = 0 \right\}$$

is a subgroup of $\Gamma_{1,m}$.

We consider the simple case $m = 1$ and $A = B = 1$. We take a coordinate $(\tau, z) \in \mathbb{H} \times \mathbb{C}$ with $\tau = x + iy,\ x \in \mathbb{R},\ y > 0$ and $z = u + iv,\ u, v$ real. A metric $ds^2_{1;1,1}$ on $\mathbb{H} \times \mathbb{C}$ given by

$$\begin{aligned} ds^2_{1;1,1} = & \frac{y + v^2}{y^3} \,(dx^2 + dy^2) + \frac{1}{y} \,(du^2 + dv^2) \\ & - \frac{2v}{y^2} \,(dx\, du + dy\, dv) \end{aligned}$$

is a G^J-invariant Kähler metric on $\mathbb{H} \times \mathbb{C}$. Its Laplacian $\Delta_{1;1,1}$ is given by

$$\begin{aligned} \Delta_{1;1,1} = & \; y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \\ & + (y + v^2) \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \\ & + 2\, y\, v \left(\frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v} \right). \end{aligned}$$

We provide some examples of eigenfunctions of $\Delta_{1;1,1}$.

(1) $h(x, y) = y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|a|y)\, e^{2\pi i a x}$ $(s \in \mathbb{C},\ a \neq 0)$ with eigenvalue $s(s-1)$. Here

$$K_s(z) := \frac{1}{2} \int_0^\infty \exp\left\{ -\frac{z}{2}(t + t^{-1}) \right\} t^{s-1}\, dt, \tag{5.2}$$

where $\operatorname{Re} z > 0$.

(2) $y^s,\ y^s x,\ y^s u$ $(s \in \mathbb{C})$ with eigenvalue $s(s-1)$.

(3) $y^s v,\ y^s uv,\ y^s xv$ with eigenvalue $s(s+1)$.

(4) $x,\ y,\ u,\ v,\ xv,\ uv$ with eigenvalue 0.

(5) All Maass wave forms.

We let $f;\mathbb{H}\times\mathbb{C}\longrightarrow\mathbb{C}$ be a Maass-Jacobi form with $\Delta_{1;1,1}f=\Lambda f$. Then f satisfies the following invariance relations

$$f(\tau+n,z)=f(\tau,z)\qquad\text{for all } n\in\mathbb{Z}$$

and

$$f(\tau,z+n_1\tau+n_2)=f(\tau,z)\qquad\text{for all } n_1,n_2\in\mathbb{Z}.$$

Therefore f is a smooth function on $\mathbb{H}\times\mathbb{C}$ which is periodic in x and u with period 1. So f has the following Fourier series

$$f(\tau,z)=\sum_{n\in\mathbb{Z}}\sum_{r\in\mathbb{Z}}c_{n,r}(y,v)\,e^{2\pi i\,(nx+ru)}. \tag{5.3}$$

For two fixed integers n and r, for brevity, we set $\varphi(y,v)=c_{n,r}(y,v)$. Then φ satisfies the following differential equation

$$\left[y^2\frac{\partial^2}{\partial y^2}+(y+v^2)\frac{\partial^2}{\partial v^2}+2\,yv\,\frac{\partial^2}{\partial y\partial v}-\left\{(A\,y+B\,v)^2+B^2\,y+\Lambda\right\}\right]\varphi=0, \tag{5.4}$$

where $A=2\pi n$ and $B=2\pi r$ are constants. We note that the function $\phi(y)=y^{\frac12}K_{s-\frac12}(2\pi|n|y)$ satisfies the the differential equation (5.4) with $\Lambda=s(s-1)$. Here $K_s(z)$ is the K-Bessel function defined by (5.2) (cf. [10], [19]).

6. Maass-Jacobi forms due to Pitale, Bringmann et al

We fix a positive integer m. Let $\mathcal{M}$ be a symmetric half-integral semi-positive definite matrix of degree m. Let $C^\infty(\mathbb{H}\times\mathbb{C}^m)$ be the algebra of all C^∞-functions on $\mathbb{H}\times\mathbb{C}^m$. For any nonnegative integer $k\in\mathbb{Z}$, we define the $|_{k,\mathcal{M}}$-slash action of G^J on $C^\infty(\mathbb{H}\times\mathbb{C}^m)$ as follows: If $f\in C^\infty(\mathbb{H}\times\mathbb{C}^m)$, and $(M,(\lambda,\mu;\kappa))\in G^J$ with $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{R})$ and $(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(m)}$,

$$\begin{aligned}&\big(f|_{k,\mathcal{M}}[(M,(\lambda,\mu;\kappa))]\big)(\tau,z):\\ =\;&(c\tau+d)^{-k}\,e^{-2\pi i\,\mathcal{M}[z+\lambda\tau+\mu]\,c\,(c\tau+d)^{-1}}\\ &\times e^{2\pi i\,\mathrm{tr}\left(\mathcal{M}(\tau\lambda\,{}^t\lambda+2\,\lambda\,{}^tz+\kappa+\mu\,{}^t\lambda)\right)}\,f\left(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\mu}{c\tau+d}\right),\end{aligned} \tag{6.1}$$

where $\tau\in\mathbb{H}$ and $z\in\mathbb{C}^m$. We recall the Siegel's notation $\alpha[\beta]={}^t\beta\alpha\beta$ for suitable matrices α and β. Let $\mathbb{D}_{k,\mathcal{M}}$ be the algebra of all differential operators D on $\mathbb{H}\times\mathbb{C}^m$ satisfying the following condition

$$(Df)|_{k,\mathcal{M}}[g]=D\big(f|_{k,\mathcal{M}}[g]\big) \tag{6.2}$$

for all $f\in C^\infty(\mathbb{H}\times\mathbb{C}^m)$ and for all $g\in G^J$. We recall the arithmetic subgroup $\Gamma_{1,m}$ of G^J defined by

$$\Gamma_{1,m}:=SL_2(\mathbb{Z})\ltimes H_{\mathbb{Z}}^{(m)}.$$

Definition 6.1. Let $\mathcal{C}^{k,\mathcal{M}}$ be the Casimir operator defined in Theorem 3.2. A smooth function $\phi : \mathbb{H} \times \mathbb{C}^m \longrightarrow \mathbb{C}$ is called a Maass-Jacobi form of weight k and index $\mathcal{M}$ if it satisfies the following conditions:

(MJ1*) $\phi|_{k,\mathcal{M}}[\gamma] = \phi$ for all $\gamma \in \Gamma_{1,m}$.

(MJ2*) ϕ is an eigenfunction of the Casimir operator $\mathcal{C}^{k,\mathcal{M}}$.

(MJ3*) For some $a > 0$,

$$\phi(\tau, z) = O\big(e^{ay}\, e^{2\,\pi\, i\, \mathcal{M}[v]/y}\big) \quad \text{as } y \longrightarrow \infty.$$

Furthermore if $\mathcal{C}^{k,\mathcal{M}}\phi = 0$, it is said to be a *harmonic* Maass-Jacobi form of weight k and index $\mathcal{M}$. We denote by $\mathbb{J}_{k,\mathcal{M}}$ the space of all harmonic Maass-Jacobi forms of weight k and index $\mathcal{M}$.

For the present being we let $\mathcal{M}$ be a positive definite integral even lattice of rank m and k an integer. We identify $\mathcal{M}$ with its Gram matrix with respect to a fixed basis, that is, a positive definite half-integral symmetric matrix of degree m. We write $|\mathcal{M}|$ for the determinant of the Gram matrix of $\mathcal{M}$. Throughout this section n will be an integer and r will be in $\mathbb{Z}^m$. For $r = {}^t(r_1, \cdots, r_m) \in \mathbb{Z}^m$ and $z = {}^t(z_1, \cdots, z_m) \in \mathbb{C}^m$, we put

$$\zeta^r := \prod_{j=1}^{m} e^{2\,\pi\, i\, r_j z_j},$$

where $\zeta = (\zeta_1, \cdots, \zeta_m)$ with $\zeta_j = e^{2\pi i z_j}$ $(1 \le j \le m)$. For $a \in \mathbb{C}$, we write $e(a) := e^{2\,\pi\, i\, a}$. For two vectors $\xi = {}^t(\xi_1, \cdots, \xi_m)$ and $\eta = {}^t(\eta_1, \cdots, \eta_m)$ in $\mathbb{C}^m$, we let

$$\langle \xi, \eta \rangle := \sum_{j=1}^{m} \xi_j\, \eta_j$$

be the standard scalar product.

We set

$$D = D_{\mathcal{M}}(n, r) := |\mathcal{M}|\big(4n - \mathcal{M}^{-1}[r]\big) \quad \text{and} \quad h = h_{\mathcal{M}}(r) := |\mathcal{M}|\, \mathcal{M}^{-1}[r]. \tag{6.3}$$

Let $M_{\nu,\mu}(w)$ be the usual M-Whittaker function, which is a solution to the following differential equation

$$\frac{\partial^2}{\partial w^2} f(w) + \left(-\frac{1}{4} + \frac{\nu}{w} + \frac{\frac{1}{4} - \mu^2}{w^2} \right) f(w) = 0. \tag{6.4}$$

For $s \in \mathbb{C}$, $\kappa \in \frac{1}{2}\mathbb{Z}$ and $t \in \mathbb{R}^\times$, we define the function

$$\mathcal{M}_{s,\kappa}(t) := |t|^{-\frac{\kappa}{2}}\, M_{\operatorname{sgn}(t)\frac{\kappa}{2},\, s-\frac{1}{2}}(|t|) \tag{6.5}$$

and

$$\phi^{(n,r)}_{k,\mathcal{M},s}(\tau,z) := \mathcal{M}_{s,k-\frac{m}{2}}\left(\frac{\pi\, Dy}{|\mathcal{M}|}\right) e^{2\,\pi\, i\,\left(\langle r,z\rangle + \frac{i}{4}\,\mathcal{M}^{-1}[r]y + nx\right)}. \tag{6.6}$$

We define the Poincaré series

$$P^{(n,r)}_{k,\mathcal{M},s}(\tau,z) := \sum_{\gamma\in\Gamma^{\infty}_{1,m}\backslash\Gamma_{1,m}} \left(\phi^{(n,r)}_{s,\mathcal{M},s}\Big|_{k,\mathcal{M}}[r]\right)(\tau,z). \tag{6.7}$$

Obviously $P^{(n,r)}_{k,\mathcal{M},s}$ is holomorphic in $\mathbb{C}^m$. It is easily seen that $P^{(n,r)}_{k,\mathcal{M},s}$ is an eigenfunction of the Casimir operator $\mathcal{C}^{k,\mathcal{M}}$ with eigenvalue

$$-2\,s(1-s) - \frac{1}{2}\left\{k^2 - k(m+2) + \frac{1}{4}\,m(m+4)\right\}.$$

For $s\in\mathbb{C},\ \kappa\in\frac{1}{2}\mathbb{Z}$ and $t\in\mathbb{R}^{\times}$, we set

$$\mathcal{W}_{s,\kappa}(t) := |t|^{-\frac{\kappa}{2}}\, W_{\mathrm{sgn}(t)\frac{\kappa}{2},\,s-\frac{1}{2}}(|t|), \tag{6.8}$$

where $W_{\nu,\mu}$ denotes the usual W-Whittaker function which is also a solution to the differential equation (6.4).

For $r\in\mathbb{Z}^m$, we define the theta series

$$\theta^{(r)}_{k,\mathcal{M}}(\tau,z) := \sum_{\lambda\in\mathbb{Z}^m} e^{2\,\pi\, i\,\mathcal{M}[\lambda]}\,\zeta^{2\mathcal{M}\lambda}\left\{e^{2\,\pi\, i\,\langle r,\lambda\rangle}\zeta^{r} \,+\, (-1)^k\, e^{-2\,\pi\, i\,\langle r,\lambda\rangle}\zeta^{r}\right\}. \tag{6.9}$$

Theorem 6.1(Bringmann-Richter [4] and Conley-Raum [5])**.** *The Poincaré series $P^{(n,r)}_{s,\mathcal{M},s}(\tau,z)$ has the Fourier expansion*

$$\begin{aligned} P^{(n,r)}_{k,\mathcal{M},s}(\tau,z) \;&=\; \mathcal{M}_{s,k-\frac{m}{2}}\left(\frac{\pi Dy}{|\mathcal{M}|}\right) e\left(\frac{-iDy}{4|\mathcal{M}|}\right)\theta^{(r)}_{k,\mathcal{M}}(\tau,z)\, e^{2\pi\, i\, n\,\tau} \\ &\quad + \sum_{n'\in\mathbb{Z},\, r'\in\mathbb{Z}^m} c_{y,s}(n',r')\, e^{2\,\pi\, i\, n'\tau}\,\zeta^{r'}. \end{aligned} \tag{6.10}$$

Here the coefficients $c_{y,s}(n',r')$ are

$$c_{y,s}(n',r') := b_{y,s}(n',r') \,+\, (-1)^k\, b_{y,s}(n',-r')$$

with $b_{y,s}$ depending on D and $D' = |\mathcal{M}|\,(4\,n' \,-\, \mathcal{M}^{-1}[r'])$ and $b_{y,s}(n',r')$ is given as follows:

(1) If $D'=0$, there is a constant $a_s(n',r')$ such that

$$b_{y,s}(n',r') = a_s(n',r')\,\frac{y^{1+\frac{m}{4}-\frac{k}{2}-s}}{\Gamma\left(s+\frac{k}{2}-\frac{m}{4}\right)\Gamma\left(s-\frac{k}{2}+\frac{m}{4}\right)}.$$

(2) If $DD' > 0$,

$$\begin{aligned} b_{y,s}(n',r') &= 2^{1-\frac{m}{2}}\,\pi\, i^{-k}\,|\mathcal{M}|^{-\frac{1}{2}}\,\frac{\Gamma(2s)}{\Gamma\left(s-\mathrm{sgn}(D')\left(\frac{k}{2}-\frac{m}{4}\right)\right)} \\ &\quad\times \left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{m+2}{4}}\, e\left(-\frac{i\,D'y}{4|\mathcal{M}|}\right)\,\mathcal{W}_{s,k-\frac{m}{2}}\left(\frac{\pi D' y}{|\mathcal{M}|}\right) \\ &\quad\times \sum_{c\in\mathbb{Z}^+} c^{-\frac{m+2}{2}}\,K_{c,\mathcal{M}}(n,r,n',r')\,J_{2s-1}\left(\frac{\pi\sqrt{DD'}}{c|\mathcal{M}|}\right), \end{aligned}$$

where Γ is the usual Gamma function, J_s is the usual J-Bessel function and $K_{c,\mathcal{M}}(n,r,n',r')$ is the Kloosterman sum defined by

$$(6.11)\quad \begin{aligned} K_{c,\mathcal{M}}(n,r,n',r') &:= e^{-\pi\, i\, c^{-1}\langle r,\,\mathcal{M}^{-1}r'\rangle} \\ &\quad\times \sum_{\substack{d\in(\mathbb{Z}/c\mathbb{Z})^\times,\\ \lambda\in\mathbb{Z}^m/c\mathbb{Z}^m}} e^{2\,\pi\, i\left(c^{-1}\bar{d}\mathcal{M}[\lambda]+n'd-\langle r',\lambda\rangle+\bar{d}\,n+\bar{d}\,\langle r,\lambda\rangle\right)}, \end{aligned}$$

where $\bar{d}$ is an integer inverse of d modulo c.

(3) If $DD' < 0$,

$$\begin{aligned} b_{y,s}(n',r') &= 2^{1-\frac{m}{2}}\,\pi\, i^{-k}\,|\mathcal{M}|^{-\frac{1}{2}}\,\frac{\Gamma(2s)}{\Gamma\left(s-\mathrm{sgn}(D')\left(\frac{k}{2}-\frac{m}{4}\right)\right)} \\ &\quad\times \left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{m+2}{4}}\, e\left(-\frac{i\,D'y}{4|\mathcal{M}|}\right)\,\mathcal{W}_{s,k-\frac{m}{2}}\left(\frac{\pi D' y}{|\mathcal{M}|}\right) \\ &\quad\times \sum_{c\in\mathbb{Z}^+} c^{-\frac{m+2}{2}}\,K_{c,\mathcal{M}}(n,r,n',r')\,I_{2s-1}\left(\frac{\pi\sqrt{DD'}}{c|\mathcal{M}|}\right), \end{aligned}$$

where I_s is the usual I-Bessel function.

Proof. We refer to [4] for the proof in the case $n = m = 1$ and to [5] in the case $n = 1$, m is arbitrary. □

Remark 6.1. If $s = \frac{k}{2} - \frac{m}{4}$ (resp. $s = 1 + \frac{m}{4} - \frac{k}{2}$) , then the Poincaré series $P^{(n,r)}_{k,\mathcal{M},s}(\tau,z)$ converges for $k > m+2$ (resp. $k < 0$). In both cases Poincaré series $P^{(n,r)}_{k,\mathcal{M},s}(\tau,z)$ is a *harmonic* Maass-Jacobi form of weight k and index $\mathcal{M}$ which is holomorphic in $\mathbb{C}^m$.

Remark 6.2. The Fourier coefficients $c^{(n,r)}_{y,s} = c^{(n,r)}_{k,\mathcal{M},s}$ of the Poincaré series $P^{(n,r)}_{k,\mathcal{M},s}(\tau,z)$ satisfy the the so-called *Zagier-type duality* with dual weights k and $m+2-k$. More precisely, if $D < 0$ and $D' < 0$, there is a constant $h_{k,s}$ depending only on k and s such that

$$(6.12)\qquad c^{(n,r)}_{k,\mathcal{M},s}(n',r') = h_{k,s}\, c^{(n',r')}_{m+2-k,\mathcal{M},s}(n,r)$$

while if $D < 0$ and $D' > 0$, there is a constant $\hat{h}_{k,s}$ depending only on k and s such that

$$c^{(n,r)}_{k,\mathcal{M},s}(n',r') = \hat{h}_{k,s}\, c^{(n',r')}_{m+2-k,\mathcal{M},s}(n,r). \tag{6.13}$$

7. Skew-Holomorphic Jacobi Forms

We define the *skew-slash action* of G^J on $C^\infty(\mathbb{H}\times\mathbb{C}^m)$ as follows: If $f \in C^\infty(\mathbb{H}\times\mathbb{C}^m)$, and $(M,(\lambda,\mu;\kappa)) \in G^J$ with $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{R})$ and $(\lambda,\mu;\kappa)\in H^{(m)}_{\mathbb{R}}$,

$$\begin{aligned} &\big(f|^{sk}_{k,\mathcal{M}}[(M,(\lambda,\mu;\kappa))]\big)(\tau,z): \\ = \;& (c\overline{\tau}+d)^{1-k}\,|c\tau+d|^{-1}\, e^{-2\,\pi\, i\,\mathcal{M}[z+\lambda\tau+\mu]\,c\,(c\tau+d)^{-1}} \\ & \times e^{2\,\pi\, i\,\mathrm{tr}\left(\mathcal{M}(\tau\lambda\,{}^t\lambda+2\,\lambda\,{}^tz+\kappa+\mu\,{}^t\lambda)\right)}\, f\left(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\mu}{c\tau+d}\right), \end{aligned} \tag{7.1}$$

where $\tau\in\mathbb{H}$ and $z\in\mathbb{C}^m$.

Definition 7.1. A smooth $f:\mathbb{H}\times\mathbb{C}^m \longrightarrow \mathbb{C}$ is said to be a *skew-holomorphic* Jacobi form of weight k and index $\mathcal{M}$ if it is real analytic in τ and is holomorphic in $z\in\mathbb{C}^m$ and satisfies the following conditions:

(SK1) $f|^{sk}_{k,\mathcal{M}}[\gamma] = f$ for all $\gamma\in\Gamma^J$.

(SK2) The Fourier expansion of f is of the form

$$f(\tau,z) = \sum_{\substack{n\in\mathbb{Z},\ r\in\mathbb{Z}^m \\ D\gg-\infty}} c(n,r)\, e^{\pi\, Dy/|\mathcal{M}|}\, e^{2\,\pi\, i\, n\,\tau}\,\zeta^r.$$

We denote by $\mathbb{J}^{sk}_{k,\mathcal{M}}$ the space of all skew-holomorphic Jacobi forms of weight k and index $\mathcal{M}$.

Remark 7.1. The notion of skew-holomorphic Jacobi forms was introduced by N.-P. Skoruppa [18].

Let

$$e_{n,r,\mathcal{M}}(\tau,z) := e^{2\,\pi\, i\,(n\tau+\langle r,z\rangle)}\, e^{\pi Dy/|\mathcal{M}|}.$$

We define the Poincaré series

$$P^{(n,r),sk}_{k,\mathcal{M}}(\tau,z) := \sum_{\gamma\in\Gamma^\infty_{1,m}\backslash\Gamma_{1,m}} \big(e_{n,r,\mathcal{M}}|^{sk}_{k,\mathcal{M}}[\gamma]\big)(\tau,z). \tag{7.2}$$

Theorem 7.1. *The Poincaré series $P^{(n,r),sk}_{k,\mathcal{M}}(\tau,z)$ defined in (7.2) is a cuspidal skew-holomorphic Jacobi form of weight k and index $\mathcal{M}$. And it has the Fourier* *expansion*

$$\begin{aligned} P_{k,\mathcal{M}}^{(n,r),sk}(\tau,z) &= e^{\pi D y/|\mathcal{M}|}\,\theta_{k-1,\mathcal{M}}^{(r)}(\tau,z)\,e^{2\pi i n\tau} \\ &\quad + \sum_{\substack{n'\in\mathbb{Z},\, r'\in\mathbb{Z}^m \\ D'>0}} c(n',r')\,e^{\pi D' y/|\mathcal{M}|}\,e^{2\pi i n'\tau}\,\zeta^{r'}, \end{aligned}$$

where $\theta_{k,\mathcal{M}}^{(r)}(\tau,z)$ *is defined in Formula (6.9) and the coefficients* $c(n',r')$ *are*

$$c(n',r') = b(n',r') + (-1)^k b(n',-r').$$

Here

$$\begin{aligned} b(n',r') : &= 2^{1-\frac{m}{2}}\,\pi\,i^{1-k}\left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{m+2}{4}} \\ &\quad \times \sum_{c\in\mathbb{Z}^+} c^{-\frac{m+2}{2}}\,K_{c,\mathcal{M}}(n,r,n',-r')\,J_{k-\frac{m+2}{2}}\left(\frac{\pi\sqrt{DD'}}{c|\mathcal{M}|}\right). \end{aligned}$$

Proof. The proof is analogous to that of Theorem 6.1. □

We define the following lowering operator

$$\begin{aligned} D_-^{(\mathcal{M})} &= \left(\frac{\tau-\overline{\tau}}{2\,i}\right)\left\{-(\tau-\overline{\tau})\,\partial_{\overline{\tau}} - {}^t(z-\overline{z})\partial_{\overline{z}} + \frac{\tau-\overline{\tau}}{8\,\pi\,i}\,\mathcal{M}^{-1}[\partial_{\overline{z}}]\right\} \\ &= -2\,i\,y\left(y\,\partial_{\overline{\tau}} + {}^t v\,\partial_{\overline{z}} - \frac{y}{8\,\pi\,i}\,\mathcal{M}^{-1}[\partial_{\overline{z}}]\right). \end{aligned} \tag{7.3}$$

We note that $D_-^{(\mathcal{M})}$ satisfies the following relation

$$\left(D_-^{(\mathcal{M})}\phi\right)\Big|_{k-2,\mathcal{M}}[\gamma] = D_-^{(\mathcal{M})}\left(\phi|_{k,\mathcal{M}}[\gamma]\right) \tag{7.4}$$

for all $\phi \in C^\infty(\mathbb{H}\times\mathbb{C}^m)$ and for all $\gamma\in\Gamma_{1,m}$.

Now we define the differential operator

$$\xi_{k,\mathcal{M}} := \left(\frac{\tau-\overline{\tau}}{2\,i}\right)^{k-\frac{5}{2}} D_-^{(\mathcal{M})} = y^{k-\frac{5}{2}}\,D_-^{(\mathcal{M})}. \tag{7.5}$$

It is easily seen that if f is a harmonic Maass-Jacobi form of weight k and index $\mathcal{M}$ which is holomorphic in $\mathbb{C}^m$, then the image $\xi_{k,\mathcal{M}}f$ of f under $\xi_{k,\mathcal{M}}$ is a skew-holomorphic Jacobi form of weight $3-k$ and index $\mathcal{M}$.

Theorem 7.2. *The Poincaré series* $P_{k,\mathcal{M}}^{(n,r),sk}(\tau,z)$ *span the space* $\mathbb{J}_{k,\mathcal{M}}^{sk,cusp}$ *of all cuspidal skew-holomorphic Jacobi forms of weight* k *and index* $\mathcal{M}$.

Proof. The proof can be found in [18]. □

Now we consider the special case $s = \frac{k}{2} - \frac{m}{4}$ and $s = 1 + \frac{m}{4} - \frac{k}{2}$.

Proposition 7.1. *The Poincaré series* $P^{(n,r)}_{k,\,\mathcal{M},\frac{k}{2}-\frac{m}{4}}$ *with* $k > 2+m$ *is meromorphic. If* $k < 0$,

$$\xi_{k,\mathcal{M}}\left(P^{(n,r)}_{k,\,\mathcal{M},1+\frac{m}{4}-\frac{k}{2}}\right) = c_{k,\mathcal{M}}\,P^{(n,r),sk}_{3-k,\,\mathcal{M}},$$

where $c_{k,\mathcal{M}}$ *is a constant depending on* k *and* $\mathcal{M}$.

Proof. We refer to [5], p. 18 for the proof. □

Proposition 7.2. *Let* $\mathbb{J}^{cusp,*}_{k,\,\mathcal{M}}$ *be the space of all cuspidal harmonic Maass-Jacobi forms of weight* k *and index* $\mathcal{M}$ *which are holomorphic in* $\mathbb{C}^m$. *Then we have the relation*

$$\xi_{k,\,\mathcal{M}}\left(\mathbb{J}^{cusp,*}_{k,\,\mathcal{M}}\right) = \mathbb{J}^{sk,cusp}_{k,\,\mathcal{M}}.$$

Proof. We refer to [5], p. 18 for the proof. □

8. Covariant differential operators on $\mathbb{H}\times\mathbb{C}^m$

Let G be a real Lie group, H a closed subgroup and V a finite dimensional complex vector space. For an element $x \in G$ we denote the coset xH by $\overline{x}$. A 1-cocycle of G on G/H with values in V is a smooth function $\alpha : G\times G/H \longrightarrow GL(V)$ satisfying the following condition

$$\alpha(g_1g_2,\overline{x}) = \alpha(g_2,\overline{x})\,\alpha(g_1,g_2\overline{x})$$

for all $g_1, g_2, x \in G$. The associated right action of G on $C^\infty\big(G/H\big)\otimes V$ is

$$f|_\alpha[g](\overline{x}) := \alpha(g,\overline{x})f(g\overline{x}), \qquad g, x \in G$$

and the associated representation of H on V is

$$\pi_\alpha(h) := \alpha(h,\overline{x}),$$

where $h \in H$ and e is the identity element of G.

Definition 8.1. Let V and V' be two finite dimensional complex vector spaces. Let α and α' be two 1-cocycles of G on G/H with values in V and V' respectively. A differential operator $D : C^\infty(G/H)\otimes V \longrightarrow C^\infty(G/H)\otimes V'$ is *covariant* from $|_\alpha$ to $|_{\alpha'}$ if for all $g \in G$ and $f \in C^\infty(G/H)\otimes V$, we have

$$D\big(f|_\alpha[g]\big) = (Df)|_{\alpha'}[g].$$

Let $\mathbb{D}_{\alpha,\alpha'}(G/H)$ be the space of all covariant differential operators from $|_\alpha$ to $|_{\alpha'}$ and $\mathbb{D}^q_{\alpha,\alpha'}(G/H)$ be the space of those of order $\le q$. When $\alpha = \alpha'$, we refer to such operators as $|_\alpha$-invariant, and we write simply $\mathbb{D}_\alpha(G/H)$ and $\mathbb{D}^q_\alpha(G/H)$

We consider our case

$$G^J = SL_2(\mathbb{R}) \ltimes H_{\mathbb{R}}^{(m)} \quad \text{and} \quad K^J = SO(2) \ltimes S(m,\mathbb{R}).$$

We observe that K^J is an abelian closed subgroup of G^J. We define the linear map $\xi : \mathfrak{g}_{\mathbb{C}}^J \longrightarrow \mathfrak{g}_{\mathbb{C}}^J$ by $\xi(X) = \widehat{X}$ with $X \in \mathfrak{g}_{\mathbb{C}}^J$, where

$$\begin{aligned} \widehat{H} : &= i\,(F-E), \qquad \widehat{E} := \frac{1}{2}\,\{H + i\,(E+F)\}, \qquad \widehat{F} := \frac{1}{2}\,\{H - i\,(E+F)\}, \\ \widehat{R}_{kl} : &= \frac{1}{2}\,R_{kl}, \qquad \widehat{e}_j := \frac{1}{2}\,(e_j - i\,f_j), \qquad \widehat{f}_j := \frac{1}{2}\,(e_j + i\,f_j). \end{aligned}$$

It is easy to see that there is a unique K^J-splitting

$$\mathfrak{g}_{\mathbb{C}}^J = \mathfrak{k}_*^J \oplus \mathfrak{p}_*^J, \tag{8.1}$$

where

$$\mathfrak{k}_*^J = \text{span}\{\widehat{H},\, \widehat{R}_{kl} \mid 1 \le k \le l \le m\,\}$$

and

$$\mathfrak{p}_*^J = \text{span}\{\widehat{E},\, \widehat{F},\, \widehat{e}_j,\, \widehat{f}_j \mid 1 \le j \le m\,\}.$$

We note that ξ is an automorphism of Lie algebras and so the given basis of $\mathfrak{p}_*^J$ is a K^J-eigenbasis : the $\widehat{H}$=weights of $\widehat{E}$, $\widehat{F}$, $\widehat{e}_j$ and $\widehat{f}_j$ are 1, -2, -1 and 1 respectively. We take the scalar valued 1-cocycle $\alpha := \gamma_{k,\mathcal{M}}$ defined by (3.7). We set $\mathcal{M} = (\mathcal{M}_{kl})$. We let $\pi_{k,\mathcal{M}} : K^J \longrightarrow GL_1(\mathbb{C})$ be the one-dimensional representation of K^J defined by

$$\pi_{k,\mathcal{M}}(h) := \gamma_{k,\mathcal{M}}(h,\overline{e})^{-1},$$

where $h \in K^J$ and $\overline{e} = (i,0) = eK^J$ with the identity element e in G^J. We remark that ξ maps the Casimir operator Ω_m to $\left(\frac{i}{2}\right)^m \Omega_m$.

Definition 8.2. Let $k \in \mathbb{Z}$ and $\mathcal{M} \in S(m,\mathbb{C})$. We define the raising operators X_+, Y_+ and the lowering operators X_- and Y_-:

$$\begin{aligned} X_+^{k,\mathcal{M}} : &= 2\,i\,\big(\partial_\tau + y^{-1}\,{}^t v \partial_z + y^{-2}\widetilde{\mathcal{M}}[v]\big), \qquad X_-^{k,\mathcal{M}} := -2\,i\,y\big(y\,\partial_{\overline{\tau}} + {}^t v\,\partial_{\overline{z}}\big), \\ Y_+^{k,\mathcal{M}} : &= i\,\partial_z + 2\,i\,y^{-1}\widetilde{\mathcal{M}}v, \qquad Y_-^{k,\mathcal{M}} := -\,i\,y\,\partial_{\overline{z}}, \qquad \widetilde{\mathcal{M}} := 2\,\pi\,i\,\mathcal{M}. \end{aligned}$$

We write $Y_{\pm,j}^{k,\mathcal{M}}$ for the j-th entry of $Y_{\pm}^{k,\mathcal{M}}$ $(1 \le j \le m)$.

For brevity, we write

$$\mathbb{D}(k,\mathcal{M};k',\mathcal{M}') := \mathbb{D}_{\gamma_{k,\mathcal{M}},\gamma_{k',\mathcal{M}'}}\big(G^J/K^J\big)$$

and

$$\mathbb{D}^q(k,\mathcal{M};k',\mathcal{M}') := \mathbb{D}^q_{\gamma_{k,\mathcal{M}},\gamma_{k',\mathcal{M}'}}\big(G^J/K^J\big),$$

where $k, k' \in \mathbb{Z},\ \mathcal{M}, \mathcal{M}' \in S(m,\mathbb{C})$, $q \in \mathbb{Z} \cup \{0\}$ and $G^J/K^J = \mathbb{H} \times \mathbb{C}^m$. We also write

$$\mathbb{D}_{k,\mathcal{M}} := \mathbb{D}(k,\mathcal{M};k,\mathcal{M}) \qquad \text{and} \qquad \text{and} \qquad \mathbb{D}^q_{k,\mathcal{M}} := \mathbb{D}^q(k,\mathcal{M};k,\mathcal{M}).$$

Conley and Raum [5] obtained the following three results.

Proposition 8.1. *(1) The spaces $\mathbb{D}^1(k,\mathcal{M};k\pm 2,\mathcal{M})$ are one-dimensional. In fact $\mathbb{D}^1(k,\mathcal{M};k\pm 2,\mathcal{M}) = \mathbb{C}X_{\pm}^{k,\mathcal{M}}$.*

(2) $\mathbb{D}^1(k,\mathcal{M};k\pm 1,\mathcal{M}) = \mathrm{Span}\{Y^{k,\mathcal{M}}_{\pm,j} \mid 1 \le j \le m\}$ are m-dimensional.

(3) $\mathbb{D}^0_{k,\mathcal{M}} = \mathbb{D}^1_{k,\mathcal{M}} = \mathbb{C}$.

(4) All other $\mathbb{D}^1(k,\mathcal{M};k',\mathcal{M}')$ are zero.

(5) We have the following commutation relations

$$[X_-, X_+] = -k, \quad [Y_{-,j}, Y_{+,j'}] = i\,\widetilde{\mathcal{M}}_{jj'}, \quad [X_-, Y_+] = -Y_-,$$
$$[Y_-, X_+] = Y_+, \quad [X_+, Y_+] = [X_-, Y_-] = 0.$$

Proposition 8.2. *Any covariant differential operator of order q may be expressed as a linear combination of products up to q raising and lowering operators. There is a unique such expression in which the raising operators are all to the left of the lowering operators. The expression of this form for the Casimir operator $\mathcal{C}^{k,\mathcal{M}}$ is*

$$\begin{aligned} \mathcal{C}^{k,\mathcal{M}} \;=\; & -2\,X_+X_- + i\left(X_+\,\widetilde{\mathcal{M}}^{-1}[Y_-] - \widetilde{\mathcal{M}}^{-1}[Y_+]X_-\right) \\ & -\frac{1}{2}\left\{\widetilde{\mathcal{M}}^{-1}[Y_+]\,\widetilde{\mathcal{M}}^{-1}[Y_-] - {}^tY_+\big({}^tY_+\,\widetilde{\mathcal{M}}^{-1}Y_-\big)\widetilde{\mathcal{M}}^{-1}Y_-\right\} \\ & -\frac{i}{2}\,(2\,k-m-3)\,{}^tY_+\,\widetilde{\mathcal{M}}^{-1}Y_-. \end{aligned} \tag{8.2}$$

Proposition 8.3. *The algebra $\mathbb{D}_{k,\mathcal{M}}$ is generated by $\mathbb{D}^3_{k,\mathcal{M}}$. Bases for $\mathbb{D}^2_{k,\mathcal{M}}$ and $\mathbb{D}^3_{k,\mathcal{M}}$ are given by*

$$\begin{aligned} \mathbb{D}^2_{k,\mathcal{M}} &= \mathrm{Span}\{1,\ X_+X_-,\ Y_{+,i}\,Y_{-,j} \mid 1 \le i,j \le m\}, \\ \mathbb{D}^3_{k,\mathcal{M}} &= \mathrm{Span}\{X_+\,Y_{-,i}\,Y_{-,j},\ Y_{+,i}\,Y_{+,j}\,X_- \mid 1 \le i \le j \le m\} \oplus \mathbb{D}^2_{k,\mathcal{M}}. \end{aligned}$$

Therefore we have

$$\dim_{\mathbb{C}} \mathbb{D}^2_{k,\mathcal{M}} = m^2 + 2 \qquad \text{and} \qquad \dim_{\mathbb{C}} \mathbb{D}^3_{k,\mathcal{M}} = 2\,m^2 + m + 2.$$

9. Final remarks

In this final section we briefly remark the general case $n > 1$ and $m > 1$.

We let

$$\mathbb{H}_n = \{\,\Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega, \quad \operatorname{Im}\Omega > 0\,\}$$

be the Siegel upper half plane of degree n and let

$$Sp(n,\mathbb{R}) = \{M \in \mathbb{R}^{(2n,2n)} \mid {}^tMJ_nM = J_n\ \}$$

be the symplectic group of degree n, where

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

$Sp(n,\mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{9.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $\Omega \in \mathbb{H}_n$.

For brevity, we write $G_n = Sp(n,\mathbb{R})$. The isotropy subgroup K_n at iI_n for the action (9.1) is a maximal compact subgroup given by

$$K_n = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \,\Big|\, A\,{}^tA + B\,{}^tB = I_n,\ A\,{}^tB = B\,{}^tA,\ A, B \in \mathbb{R}^{(n,n)} \right\}.$$

Let $\mathfrak{k}_n$ be the Lie algebra of K_n. Then the Lie algebra $\mathfrak{g}_n$ of G_n has a Cartan decomposition $\mathfrak{g}_n = \mathfrak{k}_n \oplus \mathfrak{p}_n$, where

$$\mathfrak{g}_n = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & -\,{}^tX_1 \end{pmatrix} \,\Big|\, X_1, X_2, X_3 \in \mathbb{R}^{(n,n)},\ X_2 = {}^tX_2,\ X_3 = {}^tX_3 \right\},$$

$$\mathfrak{k}_n = \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \,\Big|\, {}^tX + X = 0,\ Y = {}^tY \right\},$$

$$\mathfrak{p}_n = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \,\Big|\, X = {}^tX,\ Y = {}^tY,\ X, Y \in \mathbb{R}^{(n,n)} \right\}.$$

The subspace $\mathfrak{p}_n$ of $\mathfrak{g}_n$ may be regarded as the tangent space of $\mathbb{H}_n$ at iI_n.

We consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \{\,(\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\,{}^t\lambda \text{ symmetric }\}$$

endowed with the following multiplication law

$$(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') = (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda\,{}^t\mu' - \mu\,{}^t\lambda')$$

with $(\lambda,\mu;\kappa),(\lambda',\mu';\kappa')\in H_{\mathbb{R}}^{(n,m)}$. We define the semidirect product of $Sp(n,\mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$

$$G^J_{n,m}=Sp(n,\mathbb{R})\ltimes H_{\mathbb{R}}^{(n,m)}$$

endowed with the following multiplication law

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot\big(M',(\lambda',\mu';\kappa')\big)=\big(MM',(\tilde\lambda+\lambda',\tilde\mu+\mu';\kappa+\kappa'+\tilde\lambda\,{}^t\mu'-\tilde\mu\,{}^t\lambda')\big)$$

with $M,M'\in Sp(n,\mathbb{R}),(\lambda,\mu;\kappa),\,(\lambda',\mu';\kappa')\in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde\lambda,\tilde\mu)=(\lambda,\mu)M'$. Then $G^J_{n,m}$ acts on $\mathbb{H}_n\times\mathbb{C}^{(m,n)}$ transitively by

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot(\Omega,Z)=\Big(M\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}\Big), \tag{9.2}$$

where $M=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega,Z)\in\mathbb{H}_n\times\mathbb{C}^{(m,n)}$.

The stabilizer $K^J_{n,m}$ of $G^J_{n,m}$ at $(iI_n,0)$ for the action (9.2) is given by

$$K^J_{n,m}=\Big\{\big(k,(0,0;\kappa)\big)\ \big|\ k\in K_n,\ \kappa={}^t\kappa\in\mathbb{R}^{(m,m)}\Big\}.$$

Therefore $\mathbb{H}_n\times\mathbb{C}^{(m,n)}\cong G^J_{n,m}/K^J_{n,m}$ is a homogeneous space of *non-reductive type*. The Lie algebra $\mathfrak{g}^J_{n,m}$ of $G^J_{n,m}$ has a decomposition

$$\mathfrak{g}^J_{n,m}=\mathfrak{k}^J_{n,m}+\mathfrak{p}^J_{n,m},$$

where

$$\mathfrak{g}^J_{n,m}=\Big\{\big(Z,(P,Q,R)\big)\ \big|\ Z\in\mathfrak{g}_n,\ P,Q\in\mathbb{R}^{(m,n)},\ R={}^tR\in\mathbb{R}^{(m,m)}\Big\},$$

$$\mathfrak{k}^J_{n,m}=\Big\{\big(X,(0,0,R)\big)\ \big|\ X\in\mathfrak{k}_n,\ R={}^tR\in\mathbb{R}^{(m,m)}\Big\},$$

$$\mathfrak{p}^J_{n,m}=\Big\{\big(Y,(P,Q,0)\big)\ \big|\ Y\in\mathfrak{p}_n,\ P,Q\in\mathbb{R}^{(m,n)}\Big\}.$$

Thus the tangent space of the homogeneous space $\mathbb{H}_{n,m}$ at $(iI_n,0)$ is identified with $\mathfrak{p}^J_{n,m}$. We note that the Jacobi group $G^J_{n,m}$ is *not* a reductive Lie group and that the homogeneous space $\mathbb{H}_n\times\mathbb{C}^{(m,n)}$ is not a symmetric space. From now on, for brevity we write $\mathbb{H}_{n,m}=\mathbb{H}_n\times\mathbb{C}^{(m,n)}$, called the Siegel-Jacobi space of degree n and index m.

For a coordinate $(\Omega,Z)\in\mathbb{H}_{n,m}$ with $\Omega=(\omega_{\mu\nu})\in\mathbb{H}_n$ and $Z=(z_{kl})\in\mathbb{C}^{(m,n)}$, we put

$$\begin{aligned}
\Omega &= X+iY, \quad X=(x_{\mu\nu}), \quad Y=(y_{\mu\nu})\ \text{ real},\\
Z &= U+iV, \quad U=(u_{kl}), \quad V=(v_{kl})\ \text{ real},\\
d\Omega &= (d\omega_{\mu\nu}), \quad d\overline{\Omega}=(d\overline{\omega}_{\mu\nu}),\\
dZ &= (dz_{kl}), \quad d\overline{Z}=(d\overline{z}_{kl}),
\end{aligned}$$

$$\frac{\partial}{\partial \Omega} = \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial \omega_{\mu\nu}} \right), \quad \frac{\partial}{\partial \overline{\Omega}} = \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial \overline{\omega}_{\mu\nu}} \right),$$

$$\frac{\partial}{\partial Z} = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \cdots & \frac{\partial}{\partial z_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_{1n}} & \cdots & \frac{\partial}{\partial z_{mn}} \end{pmatrix}, \quad \frac{\partial}{\partial \overline{Z}} = \begin{pmatrix} \frac{\partial}{\partial \overline{z}_{11}} & \cdots & \frac{\partial}{\partial \overline{z}_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \overline{z}_{1n}} & \cdots & \frac{\partial}{\partial \overline{z}_{mn}} \end{pmatrix},$$

where δ_{ij} denotes the Kronecker delta symbol.

C. L. Siegel [17] introduced the symplectic metric ds_n^2 on $\mathbb{H}_n$ invariant under the action (9.1) of $Sp(n,\mathbb{R})$ given by

$$ds_n^2 = \sigma\Big(Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega}\Big) \tag{9.3}$$

and H. Maass [11] proved that the differential operator

$$\Delta_n = 4\,\sigma\left(Y\, {}^t\!\left(Y \frac{\partial}{\partial \overline{\Omega}} \right) \frac{\partial}{\partial \Omega} \right) \tag{9.4}$$

is the Laplacian of $\mathbb{H}_n$ for the symplectic metric ds_n^2. Here $\sigma(A)$ denotes the trace of a square matrix A. In [23], the author proved that for any two positive real numbers A and B, the following metric

$$\begin{aligned} ds^2_{n,m;A,B} \;=\;& A\,\sigma\Big(Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega}\Big) \\ & + B\,\Big\{ \sigma\Big(Y^{-1}\, {}^tV\, V\, Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega}\Big) + \sigma\Big(Y^{-1}\, {}^t(dZ)\, d\overline{Z}\Big) \\ & - \sigma\Big(V\, Y^{-1} d\Omega\, Y^{-1}\, {}^t(d\overline{Z})\Big) - \sigma\Big(V\, Y^{-1} d\overline{\Omega}\, Y^{-1}\, {}^t(dZ)\Big)\Big\} \end{aligned} \tag{9.5}$$

is a Riemannian metric on $\mathbb{H}_{n,m}$ which is invariant under the action (9.2) of the Jacobi group $G^J_{n,m}$.

The author [23] proved that for any two positive real numbers A and B, the Laplacian $\Delta_{n,m;A,B}$ of $(\mathbb{H}_{n,m}, ds^2_{n,m;A,B})$ is given by

$$\begin{aligned} \Delta_{n,m;A,B} \;=\;& \frac{4}{A}\left\{ \sigma\left(Y\, {}^t\!\left(Y \frac{\partial}{\partial \overline{\Omega}} \right) \frac{\partial}{\partial \Omega} \right) + \sigma\left(V Y^{-1}\, {}^tV\, {}^t\!\left(Y \frac{\partial}{\partial \overline{Z}} \right) \frac{\partial}{\partial Z} \right) \right. \\ & \left. + \sigma\left(V\, {}^t\!\left(Y \frac{\partial}{\partial \overline{\Omega}} \right) \frac{\partial}{\partial Z} \right) + \sigma\left({}^tV\, {}^t\!\left(Y \frac{\partial}{\partial \overline{Z}} \right) \frac{\partial}{\partial \Omega} \right) \right\} \\ & + \frac{4}{B}\,\sigma\left(Y \frac{\partial}{\partial Z}\, {}^t\!\left(\frac{\partial}{\partial \overline{Z}} \right) \right). \end{aligned} \tag{9.6}$$

Using $G^J_{n,m}$-invariant differential operators on the Siegel-Jacobi space $\mathbb{H}_{n,m}$, we introduce a notion of Maass-Jacobi forms.

Definition 9.1. Let

$$\Gamma_{n,m} := Sp(n,\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(n,m)}$$

be the discrete subgroup of G^J, where

$$H_{\mathbb{Z}}^{(n,m)} = \left\{ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)} \mid \lambda,\mu,\kappa \text{ are integral} \right\}.$$

A smooth function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is called a Maass-Jacobi form on $\mathbb{H}_{n,m}$ if f satisfies the following conditions (MJ1)-(MJ3) :

(MJ1) f is invariant under $\Gamma_{n,m}$.

(MJ2) f is an eigenfunction of the Laplacian $\Delta_{n,m;A,B}$ (cf. (9.6)).

(MJ3) f has a polynomial growth, that is, there exist a constant $C > 0$ and a positive integer N such that

$$|f(X+iY,Z)| \le C\,|p(Y)|^N \quad \text{as } \det Y \longrightarrow \infty,$$

where $p(Y)$ is a polynomial in $Y = (y_{ij})$.

Remark 9.1. Let $\mathbb{D}_*$ be a commutative subalgebra of $\mathbb{D}(\mathbb{H}_{n,m})$ containing the Laplacian $\Delta_{n,m;A,B}$. We say that a smooth function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is a Maass-Jacobi form with respect to $\mathbb{D}_*$ if f satisfies the conditions $(MJ1)$, $(MJ2)_*$ and $(MJ3)$: the condition $(MJ2)_*$ is given by

$(MJ2)_*$ f is an eigenfunction of any invariant differential operator in $\mathbb{D}_*$.

Let ρ be a rational representation of $GL(n,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbb{R}^{(m,m)}$ be a symmetric half-integral semi-positive definite matrix of degree m. Let $C^\infty(\mathbb{H}_{n,m}, V_\rho)$ be the algebra of all C^∞ functions on $\mathbb{H}_{n,m}$ with values in V_ρ. Let $J_{\rho,\mathcal{M}} : G^J_{n,m} \times \mathbb{H}_{n,m} \longrightarrow GL(V_\rho)$ be the canonical automorphic factor for $G^J_{n,m}$ on $\mathbb{H}_{n,m}$ given by

$$\begin{aligned}(9.7)\qquad J_{\rho,\mathcal{M}}(g,(\Omega,Z)) \;=\;& e^{2\pi i\,\mathrm{tr}(\mathcal{M}[Z+\lambda\Omega+\mu](C\Omega+D)^{-1}C)} \\ &\times e^{-2\pi i\,\mathrm{tr}(\mathcal{M}(\lambda\Omega\,{}^t\lambda\,+\,2\lambda\,{}^tZ\,+\,\kappa\,+\,\mu\,{}^t\lambda))}\rho(C\Omega+D),\end{aligned}$$

where $g = (M,(\lambda,\mu;\kappa)) \in G^J_{n.m}$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $(\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$. We recall the Siegel's notation $\alpha[\beta] = {}^t\beta\alpha\beta$ for suitable matrices α and β.

We define the $|_{\rho,\mathcal{M}}$-slash action of $G^J_{n,m}$ on $C^\infty(\mathbb{H}_{n,m}, V_\rho)$ as follows: If $f \in C^\infty(\mathbb{H}_{n,m}, V_\rho)$ and $g \in G^J_{n,m}$,

$$(9.8)\qquad \Big(f|_{\rho,\mathcal{M}}[g]\Big)(\Omega,Z) := J_{\rho,\mathcal{M}}(g,(\Omega,Z))^{-1} f(g\cdot(\Omega,Z)).$$

We define $\mathbb{D}_{\rho,\mathcal{M}}$ to be the algebra of all differential operators D on $\mathbb{H}_{n,m}$ satisfying the following condition

$$(Df)|_{\rho,\mathcal{M}}[g] = D(f|_{\rho,\mathcal{M}}[g]) \tag{9.9}$$

for all $f \in C^\infty(\mathbb{H}_{n,m}, V_\rho)$ and for all $g \in G^J_{n,m}$. We denote by $\mathcal{Z}_{\rho,\mathcal{M}}$ the center of $\mathbb{D}_{\rho,\mathcal{M}}$.

We define an another notion of Maass-Jacobi forms as follows.

Definition 9.2. A vector-valued smooth function $\phi : \mathbb{H}_{n,m} \longrightarrow V_\rho$ is called a Maass-Jacobi form on $\mathbb{H}_{n,m}$ of type ρ and index $\mathcal{M}$ if it satisfies the following conditions $(MJ1)_{\rho,\mathcal{M}},\ (MJ2)_{\rho,\mathcal{M}}$ and $(MJ3)_{\rho,\mathcal{M}}$:

$(MJ1)_{\rho,\mathcal{M}}$ $\phi|_{\rho,\mathcal{M}}[\gamma] = \phi$ for all $\gamma \in \Gamma_{n,m}$.

$(MJ2)_{\rho,\mathcal{M}}$ f is an eigenfunction of all differential operators in the center $\mathcal{Z}_{\rho,\mathcal{M}}$ of $\mathbb{D}_{\rho,\mathcal{M}}$.

$(MJ3)_{\rho,\mathcal{M}}$ f has a growth condition

$$\phi(\Omega, Z) = O\Big(e^{a \det Y} \cdot e^{2\pi\, \mathrm{tr}(\mathcal{M}[V]Y^{-1})}\Big)$$

as $\det Y \longrightarrow \infty$ for some $a > 0$.

The case $n = 1,\ m = 1$ and $\rho = \det^k (k = 0, 1, 2, \cdots)$ was studied by R. Bendt and R. Schmidt [1], A. Pitale [14] and K. Bringmann and O. Richter [4]. The case $n = 1,\ m =$arbitrary and $\rho = \det^k (k = 1, 2, \cdots)$ was dealt with by C. Conley and M. Raum [5]. In [5] the authors proved that the center $\mathcal{Z}_{\det^k,\mathcal{M}}$ of $\mathbb{D}_{\det^k,\mathcal{M}}$ is the polynomial algebra with one generator $\mathcal{C}^{k,\mathcal{M}}$ (cf. Theorem 3.2), the so-called *Casimir* operator which is a $|_{\det^k,\mathcal{M}}$-slash invariant differential operator of degree three for the case $n = m = 1$ or of degree four for the case $n = 1,\ m \geq 2$. As described in Section 6, Bringmann and Richter [4] considered the Poincaré series $\mathcal{P}^{(n,r)}_{k,\mathcal{M},s}$ (the case $n = m = 1$) (cf. (6.7)) that is a *harmonic* Maass-Jacobi form in the sense of Definition 9.2 and investigated its Fourier expansion and its Fourier coefficients. Here the *harmonicity* of $\mathcal{P}^{(n,r)}_{k,\mathcal{M},s}$ means that $\mathcal{C}^{k,\mathcal{M}}\mathcal{P}^{(n,r)}_{k,\mathcal{M},s} = 0$, i.e., $\mathcal{P}^{(n,r)}_{k,\mathcal{M},s}$ is an eigenfunction of $\mathcal{C}^{k,\mathcal{M}}$ with zero eigenvalue. Conley and Raum [5] generalized the results in [14] and [4] to the case $n = 1$ and m is an arbitrary positive integer.

Remark 9.2. In [3], Bringmann, Conley and Richter proved that the center of the algebra of differential operators invariant under the action of the Jacobi group over a complex quadratic field is generated by two Casimir operators of degree three. They also introduce an analogue of Kohnen's plus space for modular forms of half-integral weight over $K = \mathbb{Q}(i)$, and provide a lift from it to the space of Jacobi forms over K.

Definition 9.3. Let ρ and ρ' be two rational representations of $GL(n, \mathbb{C})$ on finite dimensional complex vector spaces V_ρ and V'_ρ respectively. Let $\mathcal{M}$ and $\mathcal{M}'$ be two symmetric half-integral semi-positive matrices of degree m. A differential operator

$T : C^\infty(\mathbb{H}_{n,m}) \otimes V_\rho \longrightarrow C^\infty(\mathbb{H}_{n,m}) \otimes V_{\rho'}$ is covariant from $|_{\rho,\,\mathcal{M}}$ to $|_{\rho',\,\mathcal{M}'}$ if T satisfies the following condition

$$T\Big(f|_{\rho,\mathcal{M}}[g]\Big) = (Tf)|_{\rho',\mathcal{M}'}[g]) \tag{9.10}$$

for all $f \in C^\infty(\mathbb{H}_{n,m}) \otimes V_\rho$ and for all $g \in G^J_{n,m}$.

Let $\mathbb{D}(\rho, \mathcal{M}; \rho', \mathcal{M}')$ be the space of all covariant differential operators on $\mathbb{H}_{n,m}$ from $|_{\rho,\,\mathcal{M}}$ to $|_{\rho',\,\mathcal{M}'}$, and let $\mathbb{D}^q(\rho, \mathcal{M}; \rho', \mathcal{M}')$ be the space of all covariant differential operators of order $\leq q$ on $\mathbb{H}_{n,m}$ from $|_{\rho,\,\mathcal{M}}$ to $|_{\rho',\,\mathcal{M}'}$. When $\rho = \rho'$ and $\mathcal{M} = \mathcal{M}'$, we refer to such differential operators as $|_{\rho,\,\mathcal{M}}$-invariant, and we write simply $\mathbb{D}_{\rho,\,\mathcal{M}}$ and $\mathbb{D}^q_{\rho,\,\mathcal{M}}$ instead of $\mathbb{D}(\rho, \mathcal{M}; \rho, \mathcal{M})$ and $\mathbb{D}^q(\rho, \mathcal{M}; \rho, \mathcal{M})$ respectively.

We present the natural problems.

Problem 1. Find the generators of the algebra $\mathbb{D}_{\rho,\,\mathcal{M}}$.

Problem 2. Find all the relations among a complete list of generators of $\mathbb{D}_{\rho,\,\mathcal{M}}$.

Finally we consider the special case that $\rho = \mathbf{1}$ is a trivial representation of $GL(n, \mathbb{C})$ and $\mathcal{M} = 0$. Let

$$T_{n,m} := S(m, \mathbb{C}) \times \mathbb{C}^{(m,n)}$$

be the complex vector space of dimension $\frac{n(n+1)}{2} + mn$. We obtain the natural action of $U(n)$ on $T_{n,m}$ given by

$$h \cdot (\omega, \zeta) := (h\,\omega\,{}^t h, \zeta\,{}^t h), \quad h \in U(n),\ \omega \in S(m, \mathbb{C}),\ \zeta \in \mathbb{C}^{(m,n)}. \tag{9.11}$$

We refer to [26] for a precise detail. Then the action (9.11) induces the action $\tau_{n,m}$ of $U(n)$ on the polynomial algebra $\mathrm{Pol}(T_{n,m})$ consisting of all polynomial functions on $T_{n,m}$. We denote by $\mathrm{Pol}(T_{n,m})^{U(n)}$ the subalgebra of $\mathrm{Pol}(T_{n,m})$ invariant under the action $\tau_{n,m}$ of $U(n)$. The we have the so-called Helgason map

$$\Theta_{n,m} : \mathrm{Pol}(T_{n,m})^{U(n)} \longrightarrow \mathbb{D}_{\mathbf{1},0} = \mathbb{D}(\mathbf{1}, 0; \mathbf{1}, 0)$$

defined by

$$\Big(\Theta_{n,m}(P) f\Big)(gK^J) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g \exp\left(\sum_{\alpha=1}^{N_\star} t_\alpha \eta_\alpha \right) K^J \right) \right]_{(t_\alpha)=0}, \tag{9.12}$$

where $N_\star = n(n+1) + 2mn$, $\big\{ \eta_\alpha \,|\, 1 \leq \alpha \leq N_\star \big\}$ is a basis of $\mathfrak{p}^J_{n,m}$ and $P \in \mathrm{Pol}(T_{n,m})^{U(n)}$. The map $\Theta_{n,m}$ is a linear bijection but is not multiplicative.

The following natural problems arise.

Problem 3. Find a complete list of explicit generators of $\mathrm{Pol}(T_{n,m})^{U(n)}$.

Problem 4. Find all the relations among a complete list of generators of $\mathrm{Pol}(T_{n,m})^{U(n)}$.

Problem 5. Find an easy or effective way to express the images of the above invariant polynomials or generators of $\mathrm{Pol}(T_{n,m})^{U(n)}$ under the Helgason map $\Theta_{n,m}$ explicitly.

Recently Problem 3 was solved completely in [9].

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