JAE-HYUN YANG

SELECTED PAPERS

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INHA UNIVERSITY
INCHEON, KOREA

KYUNG MOON SA

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Preface

This volume contains the papers selected among all my articles which have been published from 1986 to the present. The bibliography at the end of this volume gives the list of all my publications, including 8 books, 10 proceedings and other articles which are not reproduced here.

I hope that this volume will be very useful and helpful in the deep research of future mathematicians of the next generation. Finally I would like to give my deep and hearty thanks to Kyung Moon Sa for bringing out this publication in commemoration of my sixtieth birthday.

November, 2013
Jae-Hyun Yang

머리말

올해 회갑을 맞이하여 1985년부터 2013년까지 제가 발표했던 논문들 중에서 엄선하여 한 권의 책을 출판하게 되었습니다. 지난 29년 동안 발표한 논문과 기고문을 정리하여 보니 2,000 페이지가 넘었다는 사실을 알게 되었습니다. 이것을 모두 논문집으로 발간하려면 3권의 책으로 출판해야 합니다. 그러나 여러 고민 끝에 이들 논문과 기고문을 엄선해 한 권의 책으로 발간하는 것이 좋겠다는 의견이 있어 한 권의 책으로 발간하기로 하였습니다.

올해 그동안의 논문을 분류하여 보니, 저는 여러 분야를 다양하게 관심을 가지고 연구하여 왔다는 사실을 새삼 느끼게 되었습니다. 초창기에는 본인 자신의 수학을 세우며 창조했다기보다는 저명한 수학자들의 연구들을 두루 섭렵하였습니다. 지난 20여 년 전부터 저 나름대로의 수학을 만들려고 나름대로 노력해왔던 흔적이 보입니다. 앞으로 더욱더 아름답고, 깊고, 수준 높은 새로운 연구결과를 창출하여 대한민국 고유의 수학을 세계수학계에 길이길이 남기겠습니다.

이 논문집을 제작하는데 재정적으로 도움을 주신 경남대학의 손진우 교수님과 원광대학의 사범대학 학장 겸 교육대학원장이신 김용섭 교수님에게 깊은 감사의 마음을 보냅니다. 이 책을 출판하여 주신 경문사의 박문규 사장님과 김종원 이사님께도 감사의 뜻을 보냅니다. 저의 논문들을 정리하는데 헌신적으로 도와준 고기두 군, 이금규 양과 이 책을 훌륭하게 편집하여 주신 경문사의 편집부 및 여러 편집위원들에게도 깊은 감사의 마음을 보냅니다.

끝으로 지난 30여 년 동안 주위에서 저에게 관심을 가져주시고 물심양면으로 도와주신 모든 분들에게 진심으로 심심한 감사의 마음을 드립니다.

교수 양재현

2013년 11월 15일

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Curriculum Vitae

Jae-Hyun Yang

PERSONAL DATA

Birthday	December 6, 1953
Birthplace	Busan (or Pusan), Republic of Korea
Citizenship	Republic of Korea
Military Service	1977-78 (obligation)

EDUCATIONS

B.A.	Seoul National University, Korea	1976
M.A.	Seoul National University, Korea	1979
Ph.D.	University of California, Berkeley,	1984

POSITIONS

Assistant Professor	Inha University	1984-88
Associate Professor	Inha University	1988-93
Vice Dean	the College of Science, Inha University	1991-93
Professor	Inha University	1993-
Director	The Pyungsan Institute for Mathematical Sciences	1999-

VISITING POSITIONS

Visiting Professor	Heidlberg University	July-August, 1988
Visiting Scholar	Harvard University	September, 1988-August, 1989
Visiting Professor	Heidelberg University	Jan.-Feb. and July, 1991
Visiting Professor	Heidelberg University	July, 1992
Visiting Professor	Max-Planck Institute in Bonn	January-February, 1994
Visiting Professor	Harvard University	January-February, 1995
Visiting Scholar	Harvard University	September-November, 1996
Visiting Professor	RIMS in Kyoto University	December, 1996
Visiting Professor	Max-Planck Institute in Bonn	January-April, 1997
Visiting Professor	University of Tokyo	July, 2000

Visiting Professor	Harvard Univeristy	January, 2001
Visiting Professor	Institute for Advanced Study	February 2001
Visiting Professor	The Institute for Mathematical Sciences, Singapore	August 2002
Visiting Professor	the Hong Kong University of Science and Technology	February 2003
Visiting Professor	Seoul National University	January-August, 2004
Visiting Professor	Korea Institute for Advanced Study	July-August, 2005
Visiting Professor	RIMS in Kyoto University	Jan-Feb, 2007
Visiting Professor	RIMS in Kyoto University	Jan-Feb, 2007
Visiting Professor	Kyoto University	June-July, 2009
Visiting Professor	Kyushu University	July, 2010
Visiting Professor	Max-Planck Institute in Bonn	April-May, 2011

AWARDS and HONOR

- Summa cum laude, Busan High School, 1972
- Summa cum laude (Department of Mathematics), Seoul National University, 1976
- Fellow of the Korean Government, 1979
- Mathematical Society of Japan, Fellow, 1990
- Candidate (runner-up) for the National Science Prize, 1995
- Research Professor at Inha University, 1997-1998

ACTIVITIES

1. The principal organizer of International Symposium on Algebraic Geometry and Related Topics (Invited Speakers : E. Freitag, S. Mori, S. Mukai,K. Saito, W. Schmid, Y.-T. Siu, S.-T. Yau), Incheon, Republic of Korea, February 11-13, 1992.

2. The principal organizer of Symposium on Automorphic Forms and Related Topics (Invited Speakers : Takayuki Oda, Ichiro Satake, Ikuo Satake, Jae-Hyun Yang, Junesang Choi), Seoul, Republic of Korea, September 2-3, 1993.

3. The principal organizer of the 2nd Workshop on Number Theory and Relate Topics, Kyungju, Republic of Korea, April 1994.

4. The principal organizer of the 3rd Workshop on Number Theory, Algebraic Geometry and Related Topics, Pusan (or Busan), Republic of Korea, December 20-21, 1994.

5. The principal organizer of the Symposium on Number Theory, Geometry and Related Topics, Iksan, Republic of Korea, September 23-24, 1995.

6. The organizer of International Conference on Number Theory and Related Topics (Invited Speakers :Y.Ihara, C. Deninger, S. Kudla, B. Ramakrishnan, D. Prasad, S. Mochizuki), Yonsei University, Seoul, Republic of Korea, October 20-22, 1998.

7. The principal organizer of Summer School on Representation Theory of Lie Groups

(Invited Speakers : T. Kobayashi, W. Soergel, W. Schmid) Yonsei University, Seoul, Republic of Korea, July 20-22, 1999.

8. The principal organizer of the 2002 Seminar on Representation Theory of Lie Groups and Automorphic Forms, Incheon, Republic of Korea, February 5-6, 2002.

9. The organizer of the 2002 International Conference relating to the Clay Problems (Invited Speakers : S. M. Gonek, Cem Y. Yildirim, D. Prasad, K. Vilonen, J.-H.Yang) Chonju, Republic of Korea, July 9-11, 2002.

10. The principal organizer of the Workshop on Number Theory, Representation Theory and Geometry, Incheon, Republic of Korea, November 29-30, 2002.

11. The principal organizer of International Symposium on Representation Theory and Automorphic Forms (Invited Speakers : J.-S. Huang, T. Ikeda, T. Kobayashi, S. Miller, D. Ramakrishnan, W. Schmid, F. Shahidi, J.-H. Yang, K.-I. Yoshikawa), Seoul National University, Seoul, Korea, February 14-17, 2005.

12. The principal organizer of International Symposium on Automorphic Forms, L-Functions and Shimura Varieties (Invited Speakers : Jan H. Bruinier, Massaki Furusawa, Haruzo Hida, Takuya Konno, Shun-ichi Kimura, Dong-Uk Lee, V. Kumar Murty, Sung Myung, Byeong-Kweon Oh, Jae-Hyun Yang, Hiroyuki Yoshida), Inha University, Incheon, Korea, November 25-27, 2008.

13. The principal organizer of Workshop on Invariant Theory and Related Topics (Invited Speakers : Toru Umeda, Hiroyuki Ochiai, Akihito Wachi, Minoru Itoh, Jae-Hyun Yang, Young-Hoon Kiem, Donghoon Hyeon, Lin Han), Inha University, Incheon, Korea, February 17-19, 2010.

14. The principal organizer of Workshop on Number Theory and Related Topics, Inha University,Incheon, Republic of Korea, December 27-28, 2011.

15. The principal organizer of International Conference on Geometry, Number Theory and Representation Theory, Inha University, Incheon, Republic of Korea, October 10-12, 2012.

[A] Theory of Jacobi Forms

浜松(빈송: 하마마쓰: Hamamatsu)市의 어느 절에서 (2008년 2월)

数理解析研究所講究録805

保型形式と関連する
ゼータ関数の研究

京都大学数理解析研究所

1992年8月

プログラム

1月 6日 (月)

10:00 ～ 12:00 Tsuneo Arakawa (Rikkyo Univ.)
On Jacobi forms

13:30 ～ 14:30 Atsushi Murase (Kyoto Sangyo Univ.)
Takashi Sugano (Hiroshima Univ.)
Prehomogeneous affine spaces and dimension of Jacobi forms

14:40 ～ 15:30 Winfried Kohnen (Max-Planck-Inst.)
Non-holomorphic Eisenstein series on Siegel modular group and on Jacobi group

15:40 ～ 16:30 Akira Hori (RIMS)
L-function of Siegel wave form

16:40 ～ 17:30 Jae-Hyun Yang (Inha Univ.)
Some results on Jacobi forms of higher degree

1月 7日 (火)

10:00 ～ 12:00 Stephen Kudla (Univ. of Maryland)
Recent progress on the Siegel-Weil formula and applications

13:30 ～ 14:30 Tomoyoshi Ibukiyama (Osaka Univ.)
Differential operators on automorphic forms and invariant pluriharmonic polynomials

14:45 ～ 15:35 Takao Watanabe (Tohoku Univ.)
Theta lifting of cusp forms on the unitary group $U(d, d)$

15:50 ～ 16:40 Toyokazu Hiramatsu (Kobe Univ.), Tatsuo Okumoto (Kobe Univ.)
On zero-manifold of theta function of two variables and its application to arithmetic

1月 8日 (水)

10:00 ～ 12:00 Fumihiro Sato (Rikkyo Univ.)
On zeta functions of prehomogeneous vector spaces

13:30 ～ 14:30 Masakazu Muro (Gifu Univ.)
On residues of local or global zeta functions of prehomogeneous vector spaces

14:45 ～ 15:35 Yasuhiro Kajima (Nagoya Univ.)
On functional equations of local zeta functions of prehomogeneous vector spaces

15:50 ～ 16:40 Koichi Takase (Miyagi Univ. of Education)
On trinity of parabolic subgroup

1月 9日 (木)

10:00 ~ 10:50 Shoyu Nagaoka (Kinki Univ.)
Eisenstein series of low weight

11:00 ~ 12:00 Takayuki Oda (RIMS)
Specialization of Burau representation of Artin braid group

13:30 ~ 14:30 Yoshiyuki Kitaoka (Nagoya Univ.)
On Fourier coefficients of Klingen's Eisenstein series

14:45 ~ 15:35 Harutaka Koseki (Mie Univ.)
On contributions of elliptic elements to trace formula

15:50 ~ 16:40 Pia Bauer (Kyushu Univ.)
Some number theoretic results on ths Selberg trace formula for $PSL_2(\mathcal{O})$ of imaginary quadratic fields

1月 10日 (金)

10:00 ~ 10:50 Tetsuya Takahashi (Osaka Furitsu Univ.)
Characters of cuspidal unramified series for central simple algebras of prime degree

11:00 ~ 12:00 Kazuya Kato (Tokyo Univ.)
K_2 of modular curves ; "$L(E,1) \neq 0 \Rightarrow \sharp E(\mathbb{Q}) < \infty$"

保型形式と関連するゼータ関数の研究
研究集会報告集

1992年 1月 6日～ 1月10日

研究代表者　　高瀬 幸一(Koichi Takase)

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Some Results on Jacobi Forms of Higher Degree

JAE-HYUN YANG[1]

Abstract

In this article, the author gives some of his results on Jacobi forms of higher degree without proof. The proof can be found in the references [Y1] and [Y2].

1 Jacobi Forms

First of all, we introduce the notations. We denote by Z, R and C the ring of integers, the field of real numbers and the field of complex numbers respectively. We denote by Z^+ the set of all positive integers. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commuatative ring F. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. For $A \in F^{(k,l)}$, $\sigma(A)$ denotes the trace of A. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. E_n denotes the identity matrix of degree n. For any positive integer $g \in Z^+$, we let

$$H_g := \{\, Z \in C^{(g,g)} \mid Z = {}^tZ, \quad Im\, Z > 0 \,\}$$

the Siegel upper half plane of degree g. Let $Sp(g,R)$ and $Sp(g,Z)$ be the real symplectic group of degree g and the Siegel modular group of degree g respectively.

[1]This work was supported by KOSEF 901-0107-012-2 and TGRC-KOSEF 1991.

Let

$$(1.1) \qquad O_g(R^+) := \{M \in R^{(2g,2g)} \mid {}^tMJ_gM = \nu J_g \ \ for\ some\ \nu > 0\ \}$$

be the group of *similitudes* of degree g, where

$$J_g := \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}.$$

Let $M \in O_g(R^+)$. If ${}^tMJ_gM = \nu J_g$, we write $\nu = \nu(M)$. It is easy to see that $O_g(R^+)$ acts on H_g transitively by

$$M < Z >:= (AZ+B)(CZ+D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O_g(R^+)$ and $Z \in H_g$.

For $l \in Z^+$, we define

$$(1.2) \qquad O_g(l) := \{\, M \in Z^{(2g,2g)} \mid {}^tMJ_gM = lJ_g\ \}.$$

We observe that $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O_g(l)$ is equivalent to the conditions

$$(1.3) \qquad {}^tAC = {}^tCA,\ \ {}^tBD = {}^tDB,\ \ {}^tAD - {}^tCB = lE_g$$

or

$$(1.4) \qquad A\,{}^tB = B\,{}^tA,\ \ C\,{}^tD = D\,{}^tC,\ \ A\,{}^tD - B\,{}^tC = lE_g.$$

For two positive integers g and h, we consider the *Heisenberg group*

$$H_R^{(g,h)} := \{\, [(\lambda,\mu),\kappa] \mid \lambda,\, \mu \in R^{(h,g)},\ \ \kappa \in R^{(h,h)},\ \ \kappa + \mu\,{}^t\lambda\ symmetric\,\}$$

endowed with the following multiplication law

$$[(\lambda,\mu),\kappa] \circ [(\lambda',\mu'),\kappa'] \ := \ [(\lambda+\lambda',\mu+\mu'),\kappa+\kappa'+\lambda\,{}^t\mu' - \mu\,{}^t\lambda'].$$

We define the semidirect product of $O_g(R^+)$ and $H_R^{(g,h)}$

$$O_R^{(g,h)} =: O_g(R^+) \ltimes H_R^{(g,h)} \tag{1.5}$$

endowed with the following multiplication law

$$\begin{aligned}(1.6)\qquad & (M,[(\lambda,\mu),\kappa])\cdot(M',[(\lambda',\mu'),\kappa']) \\ := \; & (MM',[(\nu(M')^{-1}\tilde{\lambda}+\lambda',\nu(M')^{-1}\tilde{\mu}+\mu'),\nu(M')^{-1}\kappa+\kappa'+\nu(M')^{-1}(\tilde{\lambda}\,{}^t\mu'-\tilde{\mu}\,{}^t\lambda')]),\end{aligned}$$

with $M, M' \in O_g(R^+)$ and $(\tilde{\lambda},\tilde{\mu}) := (\lambda,\mu)M'$. Clearly the *Jacobi group* $G_R^{(g,h)} := Sp(g,R) \ltimes H_R^{(g,h)}$ is a normal subgroup of $O_R^{(g,h)}$. It is easy to see that $O_g(R^+)$ acts on $H_g \times C^{(h,g)}$ transitively by

$$(1.7)\; (M,[(\lambda,\mu),\kappa])\cdot(Z,W) := (M < Z >, \nu(W+\lambda Z+\mu)(CZ+D)^{-1}),$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O_g(R^+),\; \nu = \nu(M),\; (Z,W) \in H_g \times C^{(h,g)}$.

Let ρ be a rational representation of $GL(g,C)$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in R^{(h,h)}$ be a symmetric half integral matrix of degree h. We define

$$\begin{aligned}(1.8)\qquad & (f|_{\rho,\mathcal{M}}[(M,[(\lambda,\mu),\kappa])])(Z,W) \\ := \; & \exp\{-2\pi\nu i\sigma(\mathcal{M}[W+\lambda Z+\mu](CZ+D)^{-1}C)\} \\ & \times\exp\{2\pi\nu i\rho(\mathcal{M}(\lambda Z^t\lambda+2\lambda^tW+(\kappa+\mu^t\lambda)))\} \\ & \times\sigma(CZ+D)^{-1}f(M < Z >,\nu(W+\lambda Z+\mu)(CZ+D)^{-1}),\end{aligned}$$

where $\nu = \nu(M)$.

Lemma 1.1. Let $g_i = (M_i,[(\lambda_i,\mu_i),\kappa_i]) \in O_R^{(g,h)}\,(i=1,2)$. For any $f \in C^\infty(H_g \times C^{(h,g)}, V_\rho)$, we have

$$(f|_{\rho,\mathcal{M}}[g_1])|_{\rho,\nu(M_1)\mathcal{M}}[g_2] = f|_{\rho,\mathcal{M}}[g_1g_2]. \tag{1.9}$$

Definition 1.2. Let ρ and $\mathcal{M}$ be as above. Let

$$H_Z^{(g,h)} := \{\, [(\lambda,\mu),\kappa] \in H_R^{(g,h)} \,|\, \lambda,\mu \in Z^{(h,g)},\ \kappa \in Z^{(h,h)} \,\}.$$

A *Jacobi form* of index $\mathcal{M}$ with respect to ρ is a holomorphic function $f \in C^\infty(H_g \times C^{(h,g)}, V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho,\mathcal{M}}[\tilde\gamma] = f$ for all $\tilde\gamma \in \Gamma_g^J := Sp(g,Z) \ltimes H_Z^{(g,h)}$.

(B) f has a Fourier expansion of the following form :

$$f(Z,W) = \sum_{\substack{T\geq 0\\ \text{half-integral}}} \sum_{R\in Z^{(g,h)}} C(T,R)\exp(2\pi i\sigma(TZ+RW))$$

with $c(T,R) \neq 0$ only if $\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}\,{}^tR & \mathcal{M} \end{pmatrix} \geq 0$.

If $g \leq 2$, the condition (B) is superfluous by Koecher principle(see [Z] Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma_g)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ. In the special case $V_\rho = C$, $\rho(A) = (det\ A)^k$ ($k \in Z$, $A \in GL(g,C)$), we write $J_{k,\mathcal{M}}(\Gamma_g)$ instead of $J_{\rho,\mathcal{M}}(\Gamma_g)$ and call k the *weight* of a Jacobi form $f \in J_{k,\mathcal{M}}(\Gamma_g)$.

Ziegler([Zi] Theorem 1.8 or [E-Z] Theorem 1.1) proves that the vector space $J_{\rho,\mathcal{M}}(\Gamma_g)$ is finite dimensional.

2 Singular Jacobi Forms

In this section, we define the concept of singular Jacobi forms and characterize singular Jacobi forms.

Let $\mathcal{M}$ be a symmetric positive definite, half integral matrix of degree h. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ admits a Fourier expansion (see Definition

1.2 (B))

$$(2.1)\quad f(Z,W)=\sum_{T,R} c(T,R)e^{2\pi i\sigma(TZ)}\cdot e^{2\pi i\sigma(RW)},\quad Z\in H_g,\quad W\in C^{(h,g)}.$$

A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ is said to be *singular* if it admits a Fourier expansion such that the Fourier coefficient $c(T,R)$ is zero unless $det\,(4T - R\mathcal{M}^{-1}{}^tR) = 0$.

Example 2.1. Let $\mathcal{M} = {}^t\mathcal{M}$ be as above. Let $S \in Z^{(2k,2k)}$ be a symmetric positive definite integral matrix of degree $2k$ and $c \in Z^{(2k,h)}$. We consider the theta series

$$(2.2)\qquad \vartheta_{S,c}^{(g)}(Z,W) := \sum_{\lambda\in Z^{(2k,g)}} e^{\pi i\sigma(S[\lambda]Z+2S\lambda{}^t(cW))},\quad Z\in H_g,\quad W^{(h,g)}.$$

We assume that $2k < g + rank(\mathcal{M})$. Then $\vartheta_{S,c}(Z,W)$ is a singular Jacobi form in $J_{k,\mathcal{M}}(\Gamma_g)$, where $\mathcal{M} = \frac{1}{2}{}^tc\mathcal{M}c$. We note that if the Fourier coefficient $c(T,R)$ of $\vartheta_{S,c}^{(g)}$ is nonzero, there exists $\lambda \in Z^{(2k,g)}$ such that

$$\frac{1}{2}\,{}^t(\lambda,c)S(\lambda,c) = \begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M}\end{pmatrix}.$$

Thus

$$rank\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M}\end{pmatrix} \le 2k < g + rank(\mathcal{M}).$$

Therefore $det\,(4T - R\mathcal{M}^{-1}{}^tR) = 0$.

The following natural question arises:

Problem: *Characterize the singular Jacobi forms.*

The author([Y1]) gives some answers for this problem. He characterizes singular Jacobi forms by the *differential equation* and the *weight* of the representation ρ.

Now we define a very important differential operator characterizing *singular Jacobi forms.* We let

$$(2.3) \qquad \mathcal{P}_g := \{\, Y \in R^{(g,g)} \mid Y = {}^tY > 0 \,\}$$

be the open convex cone in the Euclidean space $R^{\frac{g(g+1)}{2}}$. We define the differential operator operator $M_{g,h,\mathcal{M}}$ on $\mathcal{P}_g \times R^{(h,g)}$ defined by

$$(2.4) \qquad M_{g,h,\mathcal{M}} := det\,(Y) \cdot det\left(\frac{\partial}{\partial Y} + \frac{1}{8\pi}{}^t\left(\frac{\partial}{\partial V}\right)\mathcal{M}^{-1}\left(\frac{\partial}{\partial V}\right)\right),$$

where $\frac{\partial}{\partial Y} = \left(\frac{(1+\delta_{\mu\nu})}{2}\frac{\partial}{\partial y_{\mu\nu}}\right)$ and $\frac{\partial}{\partial V} = \left(\frac{\partial}{\partial v_{kl}}\right)$.

Definition 2.2. An irreducible finite dimensional representation ρ of $GL(g, C)$ is determined uniquely by its highest weight $(\lambda_1, \cdots, \lambda_g) \in Z^g$ with $\lambda_1 \leq \cdots \leq \lambda_g$. We denote this representation by $\rho = (\lambda_1, \cdots, \lambda_g)$. The number $k(\rho) := \lambda_g$ is called the *weight* of ρ.

Theorem A. Let $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ be a Jacobi form of index $\mathcal{M}$ with respect to ρ. Then the following are equivalent:

(1) f is a *singular* Jacobi forms.

(2) f satisfies the *differential equation* $M_{g,h,\mathcal{M}} f = 0$.

Theorem B. Let $2\mathcal{M}$ be a symmetric positive definite, *unimodular* even matrix of degree h. Assume that ρ satisfies the following condition

$$(2.5) \qquad \rho(A) = \rho(-A) \quad \textit{for all } A \in GL(g, C).$$

Then any nonvanishing Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma_g)$ is *singular* if and only if $2k(\rho) < g + rank\,(\mathcal{M})$. Here $k(\rho)$ denotes the *weight* of ρ.

Conjecture. For general ρ and $\mathcal{M}$ without the above assumptions on them, *a nonvanishing Jacobi form* $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ *is singular if and only if*

$2k(\rho) < g + rank\,(\mathcal{M})$.

REMARKS. If $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ is a Jacobi form, we may write

$$(*) \qquad f(Z,W) = \sum_{a\in\mathcal{N}} f_a(Z)\cdot \vartheta_{2\mathcal{M},a,0}(Z,W), \quad Z\in H_g, \quad W \in C^{(h,g)},$$

where $\{\, f_a : H_g \longrightarrow V_\rho \,|\, a \in \mathcal{N}\,\}$ are uniquely determined holomorphic functions on H_g. A singular modular form of type ρ may be written as a finite sum of theta series $\vartheta_{S,P}(Z)$'s with pluriharmonic coefficients (cf. [F]). The following problem is quite interesting.

Problem. Describe the functions $\{f_a \,|\, a \in \mathcal{N}\,\}$ explicitely given by (*) when $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ is a *singular* Jacobi form.

3 The Siegel-Jacobi Operators

In this section, we investigate the Siegel-Jacobi operator and the action of Hecke operator on Jacobi forms. The Siegel-Jacobi operator

$$\Psi_{g,r} : J_{\rho,\mathcal{M}}(\Gamma_g) \longmapsto J_{\rho^{(r)},\mathcal{M}}(\Gamma_r)$$

is defined by

$$(\Psi_{g,r}f)(Z,W) := \lim_{t\to\infty} f\left(\begin{pmatrix} Z & 0 \\ 0 & itE_{g-r}\end{pmatrix}, (W,0)\right), \quad f \in J_{\rho,\mathcal{M}}(\Gamma_g),$$

$Z \in H_r$, $W \in C^{(h,r)}$ and $J_{\rho,\mathcal{M}}(\Gamma_g)$ denotes the space of all Jacobi forms of index $\mathcal{M}$ with respect to an irreducible rational finite dimesional representation ρ *of* $GL(g,C)$. We note that the above limit always exists because a Jacobi form f admits a Fourier expansion converging uniformly on any set of the form

$$\{(Z,W) \in H_g \times C^{(h,g)} \,|\, Im\, Z \geq Y_0 > 0, \; W \in K \subset C^{(h,g)} \; compact\}.$$

Here the representation $\rho^{(r)}$ of $GL(r,C)$ is defined as follows. Let $V_\rho^{(r)}$ be the subspace of V_ρ generated by $\{f(Z,W)\,|\,f\in J_{\rho,\mathcal{M}}(\Gamma_g),\,(Z,W)\in H_g\times C^{(h,g)}\}$. Then $V_\rho^{(r)}$ is invariant under

$$\left\{\begin{pmatrix} g & 0 \\ 0 & E_{g-r}\end{pmatrix} : g\in GL(r,C)\right\}.$$

Then we have a rational representation $\rho^{(r)}$ of $GL(r,C)$ on $V_\rho^{(r)}$ defined by

$$\rho^{(r)}(g)v := \rho\left(\begin{pmatrix} g & 0 \\ 0 & E_{g-r}\end{pmatrix}\right)v,\quad g\in GL(r,C),\quad v\in V_\rho^{(r)}.$$

In the Siegel case, we have the so-called Siegel $\Phi-$operator

$$\Phi=\Phi_{g,g-1}\,:\,[\Gamma_g,k]\longrightarrow[\Gamma_{g-1},k]$$

defined by

$$(\Phi f)(Z):=\lim_{t\to\infty} f\begin{pmatrix} Z & 0 \\ 0 & it\end{pmatrix},\quad f\in[\Gamma_g,k],\quad Z\in H_{g-1},$$

where $[\Gamma_g,k]$ denotes the vector space of all Siegel modular forms on H_g of weight k.

Here $[\Gamma_g,k]$ denotes the vector space of all Siegel modular forms on H_g of weight k.

The following properties of Φ are known :

(S1) If $k>2g$ and k is even, Φ is surjective.

(S2) If $2k<g$, then Φ is injective.

(S3) If $2k+1<g$, then Φ is bijective.

H. Maass([M1]) proved the statement (1) using Poincaré series. E. Freitag ([F2]) proved the statements (2) and (3) using the theory of singular modular forms.

The author([Y2]) proves the following theorems:

Theorem C. Let $2\mathcal{M} \in Z^{(h,h)}$ be a positive definite, unimodular symmetric even matrix of degree h. *We assume that* ρ *satisfies the condition (3.1)*:

$$(3.1) \qquad \rho(A) = \rho(-A) \quad for\ all\ A \in GL(g, C).$$

We also assume that ρ satisfies the condition $2k(\rho) < g + rank\,(\mathcal{M})$. Then the Siegel-Jacobi operator

$$\Psi_{g,g-1} : J_{\rho,\mathcal{M}}(\Gamma_g) \longrightarrow J_{\rho^{(g-1)},\mathcal{M}}(\Gamma_{g-1})$$

is injective. Here $k(\rho)$ denotes the *weight* of ρ.

Theorem D. Let $2\mathcal{M} \in Z^{(h,h)}$ be as above in Theorem A. *Assume that* ρ *satisfies the condition (3.1) and* $2k(\rho) + 1 < g + rank\,(\mathcal{M})$. Then The Siegel-Jacobi operator

$$\Psi_{g,g-1} : J_{\rho,\mathcal{M}}(\Gamma_g) \longrightarrow J_{\rho^{(g-1)},\mathcal{M}}(\Gamma_{g-1})$$

is an isomorphism.

Theorem E. Let $2\mathcal{M} \in Z^{(h,h)}$ be as above in Theorem A. *Assume that* $2k > 4g + rank\,(\mathcal{M})$ *and* $k \equiv 0 (mod\ 2)$. Then the Siegel-Jacobi operator

$$\Psi_{g,g-1} : J_{k,\mathcal{M}}(\Gamma_g) \longrightarrow J_{k,\mathcal{M}}(\Gamma_{g-1})$$

is surjecitve.

The proof of the above theorems is based on the important Shimura correspondence, the theory of singular modular forms and the result of H. Maass.

We recall

$$O_g(l) := \{ M \in Z^{(2g,2g)} \mid {}^tMJ_gM = lJ_g \}.$$

$O_g(l)$ is decomposed into finitely many double cosets *mod* Γ_g, i.e.,

(3.2) $$O_g(l) = \cup_{j=1}^m \Gamma_g g_j \Gamma_g \quad (disjoint\ union).$$

We define

(3.3) $$T(l) := \sum_{j=1}^m \Gamma_g g_j \Gamma_g \in \mathcal{H}^{(g)}, \quad the\ Hecke\ algebra.$$

Let $M \in O_g(l)$. For a Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$, we define

(3.4) $$f|_{\rho,\mathcal{M}}(\Gamma_g M \Gamma_g) := l^{gk(\rho)-\frac{g(g+1)}{2}} \sum_i f|_{\rho,\mathcal{M}}[(M_i,[(0,0),0])],$$

where $\Gamma_g M \Gamma_g = \cup_i^m \Gamma_g M_i$ (finite disjoint union) and $k(\rho)$ denotes the weight of ρ.

Theorem F. Let $M \in O_g(l)$ and $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$. Then

$$f|_{\rho,\mathcal{M}}(\Gamma_g M \Gamma_g) \in J_{\rho,l\mathcal{M}}(\Gamma_g).$$

For a prime p, we define

(3.5) $$O_{g,p} := \cup_{l=0}^{\infty} O_g(p^l).$$

Let $\check{\mathcal{L}}_{g,p}$ be the C-module generated by all left cosets $\Gamma_g M$, $M \in O_{g,p}$ and $\check{\mathcal{H}}_{g,p}$ the C-module generated by all double cosets $\Gamma_g M \Gamma_g$, $M \in O_{g,p}$. Then $\check{\mathcal{H}}_{g,p}$ is a commutative associative algebra. Since $j(\check{\mathcal{H}}_{g,p}) \subset \check{\mathcal{L}}_{g,p}$, we have a monomorphism $j : \check{\mathcal{H}}_{g,p} \longrightarrow \check{\mathcal{L}}_{g,p}$.

In a left coset $\Gamma_g M$, $M \in O_{g,p}$, we can choose a represntative M of the form

(3.6) $$M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \quad {}^tAD = p^{k_0}E_g, \quad {}^tBD = {}^tDB,$$

$$(3.7) \qquad A = \begin{pmatrix} a & \alpha \\ 0 & A^* \end{pmatrix}, \quad B = \begin{pmatrix} b & {}^t\beta_1 \\ \beta_2 & B^* \end{pmatrix}, \quad D = \begin{pmatrix} d & 0 \\ \delta & D^* \end{pmatrix},$$

where $\alpha,\ \beta_1,\ \beta_2,\ \delta \in Z^{g-1}$. Then we have

$$(3.8) \qquad M^* := \begin{pmatrix} A^* & B^* \\ 0 & D^* \end{pmatrix} \in O_{g-1,p}.$$

For any integer $r \in Z$, we define

$$(3.9) \qquad (\Gamma_g M)^* := \frac{1}{d^r}\Gamma_{g-1}M^*.$$

If $\Gamma_g M \Gamma_g = \cup_{j=1}^m \Gamma_g M_j$ (*disjoint union*), $M,\ M_j \in O_{g,p}$, then we define in a natural way

$$(3.10) \qquad (\Gamma_g M \Gamma_g)^* = \frac{1}{d^r}\sum_{j=1}^m \Gamma_{g-1}M_j^*.$$

We extend the above map (3.9) linearly on $\check{\mathcal{H}}_{g,p}$ and then we obtain an algebra homomorphism

$$(3.11) \qquad \begin{aligned} \check{\mathcal{H}}_{g,p} &\longrightarrow \check{\mathcal{H}}_{g-1,p} \\ T &\longmapsto T^* . \end{aligned}$$

It is known that the above map is a surjective map([ZH] Theorem 2).

Theorem G. Suppose we have

(a) a rational finite dimensional representation

$$\rho : GL(g, C) \longrightarrow GL(V_\rho),$$

(b) a rational finite dimensional representation

$$\rho_0 : GL(g-1, C) \longrightarrow GL(V_{\rho_0})$$

(c) a linear map $R : V_\rho \longrightarrow V_{\rho_0}$ satisfying the following properties (1) and (2):

(1) $R \circ \rho\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} = \rho_0(A) \circ R$ for all $A \in GL(g-1, C)$.

(2) $R \circ \rho\begin{pmatrix} a & 0 \\ 0 & E_{g-1} \end{pmatrix} = a^r R$ for some $a \in Z$.

Then for any $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ and $T \in \check{\mathcal{H}}_{g,p}$, we have

$$(R \circ \Psi_{g,g-1})(f|T) = R(\Psi_{g,g-1}f)|T^*,$$

where T^* is an element in $\check{\mathcal{H}}_{g-1,p}$ defined by (3.11).

Corollary. The Siegel-Jacobi operator is compatible with the action of $T \mapsto T^*$. Precisely, we have the following commutative diagram:

$$\begin{array}{ccc} J_{\rho,\mathcal{M}}(\Gamma_g) & \overset{\psi_{g,g-1}}{\longrightarrow} & J_{\rho(g-1),\mathcal{N}}(\Gamma_{g-1}) \\ \downarrow T & & \downarrow T^* \\ J_{\rho,\mathcal{N}}(\Gamma_g) & \overset{\psi_{g,g-1}}{\longrightarrow} & J_{\rho(g-1),\mathcal{N}}(\Gamma_{g-1}) \end{array} .$$

Here $\mathcal{N}$ is a certain symmetric half integral semipositive matrix of degree h.

Definition 3.2. Let $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ be a Jacobi form. Then we have a Fourier expansion given by (B) in Definition 1.2. A Jacobi form f is called a *cusp form* if $c(T,R) \neq 0$ implies $\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} > 0$. We denote by $J^{cusp}_{\rho,\mathcal{M}}(\Gamma_g)$ the vector space of all cusp forms in $J_{\rho,\mathcal{M}}(\Gamma_g)$.

Theorem H. Let $1 \leq r \leq g$. Assume $k(\rho) > g + r + rank\,(\mathcal{M}) + 1$ and $k(\rho)$ even. Then

$$J^{cusp}_{\rho,\mathcal{M}}(\Gamma_r) \subset \Psi_{g,r}(J_{\rho,\mathcal{M}}(\Gamma_g)).$$

4 Final Remarks

In this section we give some open problems which should be investigated and give some remarks.

Let

$$G_R^{(g,h)} := Sp(g,R) \ltimes H_R^{(g,h)}$$

be the *Jacobi group* of degree g. Let $\Gamma_g^J := Sp(g,Z) \ltimes H_Z^{(g,h)}$ be the discrete subgroup of $G_R^{(g,h)}$. For the case $g = h = 1$, the spectral theory for $L^2(\Gamma_1^J \backslash G_R^{(1,1)})$ had been investigated almost completely in [B1] and [B-B]. For general g and h, the spectral theory for $L^2(\Gamma_g^J \backslash G_R^{(g,h)})$ is not known yet.

Problem 1. Decompose the Hilbert space $L^2(\Gamma_g^J \backslash G_R^{(g,h)})$ into irreducible components of the Jacobi group $G_R^{(g,h)}$ for general g and h. In particular, classify all the irreducible unitary or admissible representations of the Jacobi group $G_R^{(g,h)}$ and establish the *Duality Theorem* for the Jacobi group $G_R^{(g,h)}$.

Problem 2. Give the *dimension formulae* for the vector space $J_{\rho,\mathcal{M}}(\Gamma_g)$ of Jacobi forms.

Problem 3. Construct Jacobi forms. Concerning this problem, discuss the *vanishing theorem* on the vector space $J_{\rho,\mathcal{M}}(\Gamma_g)$ of Jacobi forms.

Problem 4. Develope the theory of L-functions for the Jacobi group $G_R^{(g,h)}$. There are several attempts to establish L-functions in the context of the Jacobi group by Japanese mathematicians A. Mürase and T. Sugano using so-called the Whittaker-Shintani functions.

Problem 5. Give applications of Jacobi forms, for example in algebraic geometry and physics. In fact, Jacobi forms have found some applications in proving non-vanishing theorems for L-functions of modular forms [BFH], in the theory of Heeger points [GKS], in the theory of elliptic genera [Za] and in the string theory [C].

By a certain lifting, we may regard Jacobi forms as smooth functions on the Jacobi group $G_R^{(g,h)}$ which are invariant under the action of the discrete subgroup Γ_g^J and satisfy the differential equations and a certain growth condition.

Problem 6. Develope the theory of *automorphic forms* on the Jacobi group $G_R^{(g,h)}$. We observe that the Jacobi group is *not reductive.*

Finally for historical remarks on Jacobi forms, we refer to [B2].

References

[B1] R. Berndt, The Continuous Part of $L^2(\Gamma^J\backslash G^J)$ for the Jacobi Group G^J, Abh. Math. Sem. Univ. Hamburg **60**, 225-248 (1990).

[B2] R. Berndt, On Automorphic Forms for the Jacobi Group, Hamburg Univ., Heft 17 (1991).

[B-B] R. Berndt and S. Boecherer, Jacobi Forms and Discrete Series Representations of the Jacobi Group, Math. Z. **204**, 13-44 (1990).

[BFH] D. Bump, S. Friedberg and J. Hoffstein, Nonvanishing Theorems for L-functions of Modular Forms and their Derivatives.

[C] J. L. Cardy, Operator Content of Two-dimensional Conformally Invariant Theories, Nuclear Physics B **270**, 186-204 (1986).

[E-Z] Eichler, M. and Zagier, D., The Theory of Jacobi forms, Progress in math., **55**, Birkhäuser, Boston-Basel-Stuttgart (1985)

[F] E. Freitag, Thetareihen mit harmonischen Koeffizienten zur Siegelschen Modulgruppe, Math. Ann. **254**, 27-51 (1980)

[GKZ] B. Gross, W. Kohnen and D. Zagier, Heeger Points and the Derivative of L-series II. Math. Ann. **278**, 497-562 (1987).

[Y1] J.-H. Yang, The Differential Operators and Singular Jacobi Forms, preprint (1991).

[Y2] J.-H. Yang, The Siegel-Jacobi Operator, preprint (1992).

[Y-S] J.-H. Yang and Jin Woo Son, A note on Jacobi forms of higher degree, Jour. Korean Math. Soc., 341-358 (1991)

[Za] D. Zagier, Note on the Landweber-Stong Elliptic Genus, Springer Lecture Notes **1326**, 216-224.

[Zi] C. Ziegler, Jacobi Forms of Higher Degree, Abh. Math. Sem. Univ. Hamburg **59**, 191-224 (1989)

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Jacobi 형식에 관하여

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차 례

이 글에서는 Jacobi 형식의 이론의 최근 연구결과와 연구동향에 관해 간략하게 서술하고자 한다.

먼저, 여기서 자주 사용되는 기호들을 열거하겠다. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ 를 각각 정수환, 유리수체, 실수체, 복소수체라 하고 F 가 이 들 중의 하나일 때, $F^{(k,l)}$ 는 F 의 원을 원소로 갖는 $k\times l$ 행렬들의 집합을 나타낸다. $A\in F^{(k,l)}$ 일 때 ${}^tA$ 는 A 의 전치행렬을 나타낸다. B 가 정방행렬일때 $\sigma(B)$ 는 B 의 trace를 나타낸다. $A\in F^{(k,l)}$, $B\in F^{(k,k)}$ 일 때, $B[A] := {}^tABA$ 이다. $Sym_l(F)$ 는 $l\times l$ 대칭행렬 $S\in F^{(l,l)}$ 들의 집합을 나타내며 $Sym_l(F)^+$ 는 $l\times l$ positive 대칭행렬 $S\in F^{(l,l)}$들의 집합을 나타낸다. n 이 자연수일 때,

$$H_n := \{ Z \in \mathbb{C}^{(n,n)} \mid Z = {}^tZ, \ Im\, Z > 0 \}$$

을 차수가 n 인 Siegel 상반평면이라 하고

$$Sp(n,\mathbb{R}) := \{ g \in \mathbb{R}^{(2n,2n)} \mid {}^tg\, J_n\, g = J_n \}$$

를 차수가 n 인 symplectic 군이라 하자. 여기서,

$$J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad J_n := n \times n \text{ 단위행렬이다.}$$

그리고, $\Gamma_n := Sp(n,\mathbb{Z})$ 를 Siegel 모듈러 군 이라 놓자. $\rho : GL(n,\mathbb{C}) \to GL(V_\rho)$ 가 일반 선형군 $GL(n,\mathbb{C})$ 의 유한차원의 유리적 표현 (rational representation) 이라 할 때 $[\Gamma_n, \rho]$ 는 ρ 에 관한 V_ρ 의 벡터값을 갖는 Siegel 모듈러 형식들의 벡터공간을 나타낸다.

1. Jacobi 형식의 정의

Jacobi 형식의 원형은 Jacobi 세타급수

$$\theta(\tau, z) := \sum_{n \in \mathbb{Z}} e^{2\pi i (n^2\tau + 2nz)}, \quad \tau \in H_1, \ z \in \mathbb{C}$$

이다. 이 급수는 변수 τ 에 관해 보형함수이고 변수 z 에 관해 abel 함수이다. 그래서,

$$\text{Jacobi 형식} = \text{보형형식} + \text{세타급수(아벨함수)}$$

와 같이 간주할 수 있으며 Jacobi 형식은 두 변수에 관해 좋은 변환식을 갖는 함수라고 기대할 수 있다. 이제,

$$GSp(n,\mathbb{R})^+ := \{ g \in \mathbb{R}^{(2n,2n)} \mid {}^tg\, J_n\, g = \nu J_n \text{ for some } \nu > 0 \}$$

이라 놓자. $g \in GSp(n,\mathbb{R})^+$ 에 대해 ${}^tgJ_ng = \nu J_n$ 일때, $\nu := \nu(g)$ 이라 표기한다. $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp(n,\mathbb{R})^+$ 는 H_n 의 원 Z 에

$$g\langle Z \rangle := (AZ+B)(CZ+D)^{-1}$$

와 같이 작용한다. 두 자연수 m 과 n 에 대해

$$H_{\mathbb{R}}^{(n,m)} := \{ [(\lambda,\mu),\kappa] \mid \lambda,\mu \in \mathbb{R}^{(m,n)}, \kappa \in \mathbb{R}^{(m,m)}, \ \kappa + \mu\,{}^t\lambda \text{ 는 대칭행렬} \}$$

이라 두고 $H_{\mathbb{R}}^{(n,m)}$ 상에

$$[(\lambda,\mu),\kappa]\cdot[(\lambda',\mu'),\kappa']=[(\lambda+\lambda',\mu+\mu'),\kappa+\kappa'+\lambda{}^t\mu'-\mu{}^t\lambda']$$

와 같이 주어지는 곱을 정의하면 $H_{\mathbb{R}}^{(n,m)}$ 은 군이 되며 **Heisenberg** 군이라 일컬어진다. $GSp(n,\mathbb{R})^+$ 와 $H_{\mathbb{R}}^{(n,m)}$ 의 반직적 (semidirect product)

$$\hat{G}^J := GSp(n,\mathbb{R})^+ \propto H_{\mathbb{R}}^{(n,m)}$$

는 공간 $H_{n,m} := H_n\times\mathbb{C}^{(m,n)}$ 상에

$$g^J\cdot(Z,W) := (\ g<Z>,\ \nu(g)(W+\lambda Z+\mu)(CZ+D)^{-1}\ )$$

와 같이 작용한다. 여기서, $g^J = (g,[(\lambda,\mu),\kappa])\in\hat{G}^J$, $g=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in GSp(n,\mathbb{R})^+$ 이고 $Z\in H_n$, $W\in\mathbb{C}^{(m,n)}$ 이다. 이 작용이 추이적(transitive)임을 쉽게 알 수 있다. **Jacobi** 군 $G^J := Sp(n,\mathbb{R})\propto H_{\mathbb{R}}^{(n,m)}$ 은 $\hat{G}^J$ 의 정규부분군 이며 $H_{n,m}$ 상에 위와 같이 추이적으로 작용한다는 사실을 쉽게 보일 수 있다.

$\rho : GL(n,\mathbb{C})\to GL(V_\rho)$ 를 일반 선형군 $GL(n,\mathbb{C})$ 의 유한 차원의 유리적 표현(rational representation) 이라 하고 $M\in\mathbb{R}^{(m,m)}$ 을 $m\times m$ semi-positive 반정수(half integral) 행렬이라 한자. 그러면, Jacobi 군 G^J 의 **자연적 보형요소**(the canonical automorphic factor) $J_{M,\rho} : G^J\times H_{n,m}\to GL(V_\rho)$ 는

$$\begin{aligned} J_{M,\rho}(g^J,(Z,W)) &= e^{-2\pi i\,\sigma(M[W+\lambda Z+\mu](CZ+D)^{-1}C)}\\ &\times e^{-2\pi i\,\sigma(M(\lambda Z{}^t\lambda+2\lambda{}^tW+\mu{}^t\lambda+\kappa))}\cdot\rho(CZ+D)^{-1}\end{aligned}$$

으로 주어 진다. 여기서, $g^J = (g,[(\lambda,\mu),\kappa])\in G^J$, $g=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in Sp(n,\mathbb{R})$ 이고 $(Z,W)\in H_{n,m}$ 이다. 보형요소 $J_{M,\rho}$ 는 중요한 기하학적인 개념이며 이의 구성에 관해 [Ya8] 을 참고하길 바란다.

한편, $g^J\in G^J$ 는 $H_{n,m}$ 상의 C^∞ 벡터함수 $f\in C^\infty(H_{n,m},V_\rho)$ 에

$$(f|_{M,\rho}\,g^J)(Z,W) := J_{M,\rho}(g^J,(Z,W))f(g^J\cdot(Z,W))$$

와 같이 작용한다.

$$H_{\mathbb{Z}}^{(n,m)} := \{\,[(\lambda,\mu),\kappa]\in H_{\mathbb{R}}^{(n,m)}\mid \lambda,\mu,\kappa\ \textit{integral}\}$$

이라 두자. V_ρ 의 벡터값을 갖는 $H_{n,m}$ 상의 함수 f 가 표현 ρ 와 지표(index) M 의 $\Gamma_n^J := \Gamma_n \propto H_Z^{(n,m)}$ 에 관한 **Jacobi 형식**이란 아래의 조건 (i), (ii), (iii) 을 만족하는 경우이다.

(i) $f : H_{n,m} \to V_\rho$ 는 해석적(holomorphic) 함수이다.

(ii) 임의의 $\gamma^J \in \Gamma_n^J$ 에 관해 $f|_{M,\rho}\gamma^J = f$ 이다.

(iii) f 는 아래의 Fourier 전개

$$f(Z,W) = \sum_{T\geq 0}\sum_{R\in Z^{(n,m)}} c(T,R)\cdot e^{2\pi i\sigma(TZ+RW)}$$

를 갖는다. 단, $T\geq 0$ 는 반정수이고 $\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & M \end{pmatrix} \geq 0$ 이다.

$n\geq 2$ 인 경우는 Köcher 원리 (cf. [Zi], lemma 1.6)에 의해 조건 (iii) 은 불필요하다. 표현 ρ 와 지표 M 의 Γ_n^J 에 관한 Jacobi 형식들의 벡터공간을 $J_{\rho,M}(\Gamma_n)$ 이라 표기한다. $V_\rho = \mathbb{C}$, $\rho = \det^k$ $(k\in\mathbb{Z})$ 인 특수한 경우는 $J_{k,M}(\Gamma_n)$ 이라 표기하며 k 를 **무게** (weight) 라고 부른다. Ziegler (cf. [Zi] Theorem 1.8 또는 [EZ], $n=m=1$인 경우)에 의해 $J_{\rho,M}(\Gamma_n)$ 은 유한차원의 벡터공간임이 증명되었다.

Jacobi 형식의 역사에 관해 간략하게 설명하겠다. 이미, 앞에서 언급하였듯이 Jacobi 형식은 C.G.J. Jacobi 의 "Fundamenta nove theoriae functionum ellipticum" (Königsberg, 1829) 에서 소개된 Jacobi 세타급수에서 원형을 찾을 수 있다. 그 후, 많은 수학자들에 의해 다른 이름으로 연구되어 왔다. 1969년에 Piatetski-Sharpiro (cf. [PS]) 는 (Siegel) 모듈러 형식의 Fourier-Jacobi 전개와 모듈러 abel 함수의 체 (field) 에 관해 논의하며 이 체의 차원을 계산하고 있다. I. Satake (cf. [Sa1], [Sa2]) 는 세타 함수의 연구과정에서 야코비 군의 이산표현 (discrete series) 에 관해 논의 하고 있다. 1976년과 1978년에 G. Shimura (cf. [Sh1], [Sh2]) 는 Jacobi 세타급수를 이용하여 아벨함수의 복소곱 (complex multiplication) 의 이론을 전개하고 있다. 1982년에 Y.-S. Tai (cf. [Ta]) 는 Jacobi 형식의 어떤 벡터공간의 차원공식의 근사치를 계산하여 ppav (principally polarized abelian varieties) 의 모듈라이 공간 A_n 이 $n\geq 9$ 인 경우에는

general type 임을 증명하였다. 1983년에 Feingold 와 Frenkel (cf. [FF]) 의 논문에서 Jacobi 형식과 Kac-Moody 대수와의 밀접한 관계를 보이고 있다. 이에 관해, V. Kac 의 저서 [Kac] 를 참조하길 바란다. 기하학적인 측면에서는 Mumford 학파 (cf. [Mu1], [Mu2]) 와 J.-H. Yang (cf. [Ya9]-[Ya11]) 등 여러 대수기하학자들에 의해 연구되어 왔다. 1985년에 Eichler 와 Zagier 의 저서 [EZ] 의 출판 후 Jacobi 형식의 중요성이 널리 인식되어 광범위하게 연구되어 오고 있다. 예를 들면, R. Berndt, W. Kohnen, J. Kramer, N.-P. Skoruppa, D. Zagier 등에 의해 $n=m=1$ 인 경우에 집중적으로 연구되어 오고 있다. n, m 이 임의의 경우에는 T. Yamazaki (cf. [Y], n : 임의, $m=1$), H. Klingen (cf. [Kl1], [Kl2], n : 임의, $m=1$), A. Murase (cf. [Mur1]-[Mur3], [MS]), C. Ziegler (cf. [Zi]) 와 J.-H. Yang (cf. [Ya1]-[Ya8], [Ya11]-[Ya15]) 외에 지금은 많은 젊은 수학자들 (cf. [Ik])에 의해 연구되고 있다. A. Murase 와 T. Sugano 는 Whittaker-Shintani 함수를 사용하여 Jacobi 군에 부수되는 L-함수를 연구하고 있다. Jacobi 군 G^J 의 군 표현적인 입장에서의 연구는 R. Berndt, K. Takase 와 J.-H. Yang 등의 여러 수학자들에 의해 연구되고 있다. [BFH] 에서 저자들은 Jacobi 형식을 이용하여 모듈러 함수의 L-함수에 관한 비소멸정리 (nonvanishing theorem) 를 증명하였으며, Heegner point 이론 (cf. [GKS]), elliptic genera 의 이론 (cf. [Za1]) 과 현 이론 (cf. [Ca]) 등에서 Jacobi 형식의 이론이 유용하게 사용되고 있다.

2. Jacobi 형식의 구성 (construction)

이 절에서는 Jacobi 형식을 구성하는 방법을 몇 가지 소개하겠다.

(i) Siegel 보형형식의 Fourier-Jacobi 전개의 계수

차수가 $n+m$ 의 무게 k 인 Siegel 보형형식 $f\in[\Gamma_{n+m},k]$ 는 아래와 같이 주어지는 Fourier-Jacobi 전개

$$f\begin{pmatrix} Z_1 & {}^tZ_3 \\ Z_3 & Z_2 \end{pmatrix} = \sum_{T\geq 0,\ \text{반정수}} \phi_T(Z_1,Z_3)\, e^{2\pi i\sigma(TZ_2)}$$

를 갖는다. 여기서, $Z_1\in H_n$, $Z_2\in H_m$ 이다. 그러면, $\phi_T(Z_1,Z_3)\in J_{k,T}(\Gamma_n)$ 임을 쉽게 보일 수 있다.

(ii) Eisenstein 급수

$\Gamma_n^J := \Gamma_n \propto H_Z^{(n,m)}$ 의 부분군 $\Gamma_{n,\infty}^J$ 를 아래와 같이 정의한다.

$$\Gamma_{n,\infty}^J := \{ \gamma \in \Gamma_n^J \mid 1|_{k,S}\, \gamma = 1 \}.$$

임의의 $n \times m$ 반정수 대칭행렬 $S \geq 0$ 에 대해 Eisenstein 급수 $E_{k,S}^{(n)}(Z,W)$ 를 아래와 같이 형식적으로 정의하자.

$$\begin{aligned} E_{k,S}^{(n)}(Z,W) &:= \sum_{\gamma \in \Gamma_{n,\infty}^J \backslash \Gamma_n^J} (1|_{k,S}\, \gamma)(Z,W) \\ &= \sum_{\gamma \in \Gamma_{n,\infty}^J \backslash \Gamma_n^J} J_{S,k}(\gamma, (Z,W)) . \end{aligned}$$

k 가 짝수이고 $k > n + rank(S) + 1$ 이면 위의 Eisenstein 급수는 절대 수렴하며 $E_{k,S}^{(n)}(Z,W) \in J_{k,S}(\Gamma_n)$ 이다. 보다 일반적인 Klingen 식의 Eisenstein 급수를 구성할 수 있는데 이에 관해서는 Ziegler (cf. [Zi], pp 201-207) 또는 Yang (cf. [Ya18]) 을 참조하길 바란다.

(iii) 세타급수 (theta series)

먼저, $m \times m$ 반정수 대칭행렬 $S > 0$ 를 취하자.

$$Q := \begin{pmatrix} M & \frac{{}^t q}{2} \\ \frac{q}{2} & S \end{pmatrix}, \text{ half-integral, } Q > 0$$

($M \in \mathbb{Q}^{(m,m)}$, $M > 0$ 이고 반정수, $q \in \mathbb{Z}^{(l,m)}$) 에 대해 세타급수 $\theta_Q^{(n)}(Z,W)$ 를

$$\theta_Q^{(n)}(Z,W) := \sum_{G \in \mathbb{Z}^{(m+l,n)}} e^{2\pi i \sigma(Q[G]Z + 2\,{}^tWqSG)}$$

와 같이 정의한다. 여기서, $(Z,W) \in H_{n,l}$ 이다.

이 때, $2Q$ 가 even unimodular 이면 $\theta_Q^{(n)}(Z,W) \in J_{\frac{m+l}{2},S}(\Gamma_n)$ 이다.

Problem : 방금 언급한 Jacobi 형식을 구성하는 방법외에 다른 방법이 있는가 ?

3. Shimura 대응

$m \times m$ integral 대칭행렬 $S>0$ 와 $a, b \in \mathbb{Q}^{(m,n)}$ 에 대해 세타급수

(3.1) $$\theta_{S,a,b}(Z,W) := \sum_{\lambda \in \mathbb{Z}^{(m,n)}} e^{\pi i \sigma(S((\lambda+a)Z{}^t(\lambda+a)+2(\lambda+a){}^t(W+b)))}$$

를 정의한다.

$\Omega \in H_n$ 를 고정하고 $m \times m$ 반정수 대칭행렬 $M>0$ 를 취하자. 그러면, 격자 $L_\Omega := \mathbb{Z}^{(m,n)} \cdot \Omega + \mathbb{Z}^{(m,n)}$ 는 $\mathbb{C}^{(m,n)}$ 상에서

$$(\lambda\Omega+\mu) \cdot W = W+\lambda\Omega+\mu, \quad \lambda, \mu \in \mathbb{Z}^{(m,n)}, \ W \in \mathbb{C}^{(m,n)}$$

와 같이 properly discontinuously 작용한다.

$(W,\xi) \in \mathbb{C}^{(m,n)}$ 를 $(W+\lambda\Omega+\mu, e^{-2\pi i \sigma(M(\lambda\Omega{}^t\lambda+2\lambda{}^tW))} \cdot \xi)$ 와 동일하게 봄으로서 아벨다양체 상 $X_\Omega := \mathbb{C}^{(m,n)}/L_\Omega$ 의 해석적 선속 $\pounds_\Omega$ 를 얻는다. $\pounds_\Omega$ 는 ample 이고 $\dim_{\mathbb{C}} H^0(X_\Omega, \pounds_\Omega) = \{ \det(2M) \}^n$ 임을 쉽게 보일 수 있다. 실제로,

$$\left\{ \theta_{2M,a,0}(\Omega, W) \mid a \in (2M)^{-1}\mathbb{Z}^{(m,n)}/\mathbb{Z}^{(m,n)} \right\}$$

는 $H^0(X_\Omega, \pounds)$ 의 기저를 이룬다. 그러므로, $\Omega \in H_n$ 를 움직임으로써 임의의 Jacobi 형식 $f \in J_{\rho,M}(\Gamma_n)$ 에 대해

(3.2) $$f(Z,W) = \sum f_a(Z) \cdot \theta_{2M,a,0}(Z,W), \quad (Z,W) \in H_{n,m}$$

으로 표기 할 수 있음을 쉽게 보일 수 있다. 여기서, $a \in (2M)^{-1}\mathbb{Z}^{(m,n)}/\mathbb{Z}^{(m,n)}$ 이고 $f_a(Z)$ 는 유일하게 결정되는 H_n 상의 해석적 벡터함수이다. Poisson summation formula 를 이용하여 Jacobi 형식과 반정수 보형형식 사이의 Shimura correspondence 를 증명할 수 있다.

정리 3.1 (Shimura 대응) 벡터공간 $J_{\rho,M}(\Gamma_n)$ 와 어떤 반정수 Siegel 보형형식들의 벡터공간과 동형이다.

이 정리의 상세한 증명은 Yang (cf. [Ya18]) 또는 Ziegler (cf. [Zi]) 를 참조하길 바란다.

따름정리 $2M$ 이 unimodular 하고 ρ 가 $\rho(A)=\rho(-A)$ ($^{\forall}A\in GL(n,\mathbb{C})$) 인 조건을 만족한다고 하자. 그러면,

$$J_{\rho,M}(\Gamma_n) \cong [\Gamma_n,\ \rho \otimes \det^{-\frac{m}{2}}].$$

Shimura 대응은 다음절에서 응용될 것이다.

4. 반정수 보형형식과의 대응

이 절에서는 Kohnen space 와 Maaß space (또는 Maaß's Spezialschar) 를 소개하고 Jacobi 벡터공간과의 대응관계를 간략하게 서술하고자 한다.

3절에서 기술한 내용을 다소 되풀이 하지만 본 내용의 틀에 맞도록 기술하겠다. $m\times m$ 반정수 대칭행렬 $M>0$ 와 $Z\in H_n$ 에 대해

$$\theta(W+\lambda Z+\mu) = e^{-2\pi i\sigma(M[\lambda]Z+2{}^tWS\lambda)}, \quad \lambda,\mu\in\mathbb{Z}^{(m,n)}$$

의 변환식을 만족하는 모든 함수 $\theta:\mathbb{C}^{(m,n)}\to\mathbb{C}$ 들의 벡터공간을 $\Theta^{(n)}_{M,Z}$ 이라표기하자. 편의상, $L:=\mathbb{Z}^{(m,n)}$ 이라 놓자. $\gamma\in L/(2M)L$, $(Z,W)\in H_{n,m}$ 에 대해 세타급수 $\theta_\gamma(Z,W)$ 를

$$\theta_\gamma(Z,W) := \sum_{\lambda\in L} e^{2\pi i\sigma(M[\lambda+(2M)^{-1}\gamma]Z\ +\ 2{}^tWM(\lambda+(2M)^{-1}\gamma))}$$

이라 정의한다. 그러면,

(i) $\{\theta_\gamma(Z,W)\mid \gamma\in L/(2M)L\}$ 는 복소 벡터공간 $\Theta^{(n)}_{M,Z}$ 의 기저를 이룬다.

(ii) $\phi(Z,W)\in J_{k,M}(\Gamma_n)$ 이면 W 의 함수에 대해 $\phi(Z,\cdot)\in\Theta^{(n)}_{M,Z}$ 이다.

(iii) $\phi(Z,W)\in J_{k,M}(\Gamma_n)$ 는 세타급수 $\theta_\gamma(Z,W)$ 의 일차결합으로 나타낼 수 있다. 즉,

$$\phi(Z,W) = \sum_{\gamma\in L/(2M)L}\phi_\gamma(Z)\,\theta_\gamma(Z,W),\quad \phi_\gamma(Z)\in\mathbb{C}$$

여기서, $\phi = (\phi_\gamma(Z))_{\gamma\in L/(2M)L}$ 는 theta multiplier system 에 관련된 보형형식이다.

(I) Kohnen 공간 (cf. [Ib], [Ko1])

$l=1,\ S=1,\ L=\mathbb{Z}^{(1,n)}$ 인 경우에 관하여 생각하여 보자. 세타급수

$$\theta^{(n)}(Z) := \sum_{\lambda\in L} e^{2\pi i\sigma(\lambda Z{}^t\lambda)} = \theta_0(Z,0), \quad Z\in H_n$$

를 정의한다. 또

$$\Gamma_0^{(n)}(4) := \left\{\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in\Gamma_n \mid C \equiv 0 \mod 4\right\}$$

이라 정의하면 $\Gamma_0^{(n)}(4)$ 는 Γ_n 의 부분군이다. 보형요소 $j(M,Z)$ ($M\in\Gamma_0^{(n)}(4)$)를

$$j(M,Z) := \frac{\theta^{(n)}(M<Z>)}{\theta^{(n)}(Z)}, \quad M\in\Gamma_0^{(n)}(4)$$

와 같이 정의한다. 이 때, 임의의 $M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in\Gamma_0^{(n)}(4)$ 에 대해

(4.1) $$j(M,Z)^2 = \varepsilon(M)\cdot\det(CZ+D), \quad \varepsilon(M)^2 = 1$$

인 관계식을 얻는다. 1980년에 Kohnen 는 k 가 짝수 일 때 소위 일컬어지는 **Kohnen 공간** $M^+_{k-\frac{1}{2}}(\Gamma_0^{(n)}(4))$ 를 정의하였다.

함수 $f: H_n \to \mathbb{C}$ 가 $M^+_{k-\frac{1}{2}}(\Gamma_0^{(n)}(4))$ 의 원 (element) 이라 하는 것은 f 는 해석적 함수이며 아래의 두 조건 (4.2) 와 (4.3) 을 만족하는 경우이다.

(4.2) $f(M<Z>) = j(M,Z)^{2k-1} f(Z), \quad {}^\forall M\in\Gamma_0^{(n)}(4).$

(4.3) $f(Z)$ 는 아래의 Fourier 전개

$$f(Z) = \sum_{T\geq 0,\ \text{반정수}} a(T)\, e^{2\pi i\sigma(TZ)}$$

을 갖는다. 단, Fourier 계수 $a(T)$ 는

" $a(T)=0$ unless $T \equiv -\mu{}^t\mu \mod 4S_n^*(\mathbb{Z})$ for ${}^\exists\mu\in\mathbb{Z}^{(n,1)}$ "

의 조건을 만족한다. 여기서,

$$S_n^*(\mathbb{Z}) := \left\{T\in Sym_n(\mathbb{R}) \mid \sigma(TS)\in\mathbb{Z}\ \ {}^\forall S\in Sym_n(\mathbb{Z})\right\}.$$

임의의 $\phi \in J_{k,1}(\Gamma_n)$ $(l=1,\ S=1)$ 에 대해

$$\phi(Z,W) = \sum_{\gamma \in L/2L} f_\gamma(Z)\, \theta_\gamma(Z,W)$$

와 같이 표기할 수 있다. 이 때,

$$(4.4) \qquad f_\phi(Z) := \sum_{\gamma \in L/2L} f_\gamma(4Z)$$

이라 놓으면 $f_\phi \in M^+_{k-\frac{1}{2}}(\Gamma_0^{(n)}(4))$ 임을 증명할 수 있다.

정리 4.1 (Kohnen–Zagier ($n=1$), Ibukiyama ($n>1$))

$$J_{k,1}(\Gamma_n) \simeq M^+_{k-\frac{1}{2}}(\Gamma_0^{(n)}(4)).$$

여기서, 위의 동형사상에서의 대응관계는 $\phi \mapsto f_\phi$ 으로 주어진다. 게다가, 위의 동형사상은 Hecke 작용소의 작용과 양립 (compatible) 한다.

(II) *Maaß* 공간과 Saito–Kurokawa 예상

Maaß 공간의 개념은 Saito–Kurokawa 예상 (cf. [Ku]) 을 해결하기 위하여 소개되었다.

$F \in [\Gamma_2, k]$ 의 Siegel 보형형식이 *Maaß* 공간 $[\Gamma_2, k]^M$ 의 원이라 하는 것은 F 의 Fourier 계수 $a_F(T)$ ($T \in S_2^*(\mathbb{Z}), T \geq 0$) 에 대해 아래의 조건 (4.5)를 만족하는 경우이다.

$$(4.5) \qquad a_F\begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix} = \sum_{d|(n,r,m),\, d>0} d^{k-1} \cdot a_F\begin{pmatrix} \frac{mn}{d^2} & \frac{r}{2d} \\ \frac{r}{2d} & 1 \end{pmatrix},$$

$$\left({}^{\forall}T = \begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix} \in S_2^*(\mathbb{Z}),\ T \geq 0 \right).$$

Jacobi 형식 $\phi \in J_{k,1}(\Gamma_1)$ 에 대해 $F_\phi : H_2 \to \mathbb{C}$ 를

$$F_\phi\begin{pmatrix} z & v \\ v & w \end{pmatrix} := \sum_{n,m\in Z_{>0}, r\in Z, 4mn-r^2\geq 0} A\begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix} \cdot e^{2\pi i(nz+rw+mv)}$$

이라 정의한다. 여기서,

$$\phi(z,w) = \sum_{n,r\in Z, 4n-r^2\geq 0} c(n,r)\cdot e^{2\pi i\sigma(nz+rw)},$$

$$A\begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix} = \sum_{d\mid (n,r,m), d>0} d^{k-1}\cdot c\left(\frac{mn}{d^2}, \frac{r}{d}\right)$$

이다.

정리 4.2 (*Maaβ* [Ma1], Eichler–Zagier [EZ, Th. 6.3])

$$J_{k,1}(\Gamma_1) \simeq [\Gamma_2, k]^M.$$

여기서, 위의 동형사상의 대응관계는 $\phi \mapsto F_\phi$ 으로 주어진다. 게다가, 위의 동형사상은 Hecke 작용소의 작용과 양립 (compatible) 한다.

Saito–Kurokawa 예상은 *Maaβ* (cf. [Ma1]–[Ma3]), Andrianov (cf. [An])와 Eichler–Zagier (cf. [EZ]) 에 의해 해결되었다. 지금까지의 내용을 요약하면

$$\begin{array}{ccccccc} [\Gamma_2,k]^M & \simeq & J_{k,1}(\Gamma_1) & \simeq & M^+_{k-\frac{1}{2}}(\Gamma_0^{(1)}(4)) & \simeq_\iota & [\Gamma_1, 2k-2] \\ F_\phi & \leftarrow & \phi & \rightarrow & f_\phi & & \end{array}$$

으로 쉽게 표기할 수 있다. 여기서, ι 는 Shimura 대응이다. 위의 동형사상들은 Hecke 작용소의 작용과 양립한다.

5. 특이 Jacobi 형식

$M>0$ 를 $m\times m$ 반정수 행렬이라 하자. Jacobi 형식 $f\in J_{\rho,M}(\Gamma_n)$ 는 정의에 의해 Fourier 전개

$$f(Z,W) = \sum_{T,R} c(T,R)\cdot e^{2\pi i\sigma(TZ+RW)}.$$

를 갖는다. Fourier 계수 $c(T,R)$ 이 아래의 조건

$$\text{“ } c(T,R)\neq 0 \Rightarrow \det(4T-RM^{-1}\,{}^tR) = 0 \text{ ”}$$

을 만족할 때 f 를 **특이 (singular) Jacobi 형식**이라 한다.

Example : $c\in\mathbb{Z}^{(2k,m)}$ 이라 하자. 또, $S>0$ 를 $2k\times 2k$ integral 대칭행렬이라 하자. $2k<n+m$ 이면 아래와 같이 정의되는 세타급수

$$\theta^{(n)}_{S,c}(Z,W) := \sum_{\lambda\in\mathbb{Z}^{(2k,n)}} e^{\pi i\sigma(S[\lambda]Z+2S\lambda\,{}^t(cW))}$$

는 특이 Jacobi 형식이다. 정확히, $\theta^{(n)}_{S,c}(Z,W)\in J_{k,M}(\Gamma_n)$. 단, $M=\frac{1}{2}\,{}^tcSc$.

이제, 우리는 아래와 같은 자연스런 문제를 제기할 수 있다.

Problem : Characterize the singular Jacobi forms.

필자는 [Ya5] 에서 이 문제에 대한 해답을 주고 있다. 필자는 어떤 선형미분방정식과 표현 ρ 의 무게 (weight) 로서 특이 Jacobi 형식을 특징짓고 있다.

$$P_n := \{\, Y\in\mathbb{R}^{(n,n)} \mid Y={}^tY>0 \,\}$$

을 유크리드 공간 $\mathbb{R}^{\frac{n(n+1)}{2}}$ 안의 열린 convex cone 이라 하자.

공간 $P_n\times\mathbb{R}^{(m,n)}$ 상에 미분작용소 $M_{n,m,M}$ 를

$$M_{n,m,M} := \det(Y)\cdot\det\left(\frac{\partial}{\partial Y}+\frac{1}{8\pi}\,{}^t\!\left(\frac{\partial}{\partial V}\right)M^{-1}\left(\frac{\partial}{\partial V}\right)\right)$$

으로 정의한다. 여기서,

$$\frac{\partial}{\partial Y} := \left(\frac{1+\delta_{ab}}{2}\,\frac{\partial}{\partial y_{ab}}\right),\quad \frac{\partial}{\partial V} = \left(\frac{\partial}{\partial v_{kl}}\right),\quad Y=(y_{ab}),\quad V=(v_{kl})$$

정리 A (cf. Yang [Ya5]). Jacobi 형식 $f \in J_{\rho,M}(\Gamma_n)$ 에 대해 아래의 두 조건 (1)과 (2)는 동치이다.

(1) f 는 특이 Jacobi 형식이다.

(2) $M_{n,m,M} f = 0$.

정리 B (cf. Yang [Ya5]). $2M > 0$ 을 $m \times m$ even unimodular 대칭행렬이라 가정하자. 또, 표현 ρ 는

$$\rho(A) = \rho(-A) \quad {}^{\forall} A \in GL(n, \mathbb{C})$$

의 조건을 만족한다고 하자. 그러면, Jacobi 형식 $0 \neq f \in J_{\rho,M}(\Gamma_n)$ 에 대해 아래의 두 조건 (a)와 (b)는 동치이다.

(a) f 는 특이 Jacobi 형식이다.

(b) $2k(\rho) < n+m$.

여기서, $k(\rho)$ 는 표현 ρ 의 무게 (weight) 를 나타낸다.

Conjecture : For general ρ and M without the above assumptions on them, a *nonvanishing Jacobi* form $f \in J_{\rho,M}(\Gamma_n)$ *is singular* if *and only* if $2k(\rho) < n+m$.

Problem : Describe the functions $\{f_a \mid a \in (2M)^{-1}\mathbb{Z}^{(m,n)}/\mathbb{Z}^{(m,n)}\}$ explicitly given by (3.2) when $f \in J_{\rho,M}(\Gamma_n)$ is a *singular* Jacobi form.

필자는 이 문제에 대해 부분적인 해답을 주고 있다 (cf. [Ya6]).

6. Siegel–Jacobi 작용소

M 는 5절과 같다. Jacobi 형식 $f \in J_{\rho,M}(\Gamma_n)$ 에 대해

$$(\Psi_{n,r} f)(Z, W) := \lim_{t \to \infty} f\left(\begin{pmatrix} Z & 0 \\ 0 & I_{n-r} \end{pmatrix}, (W, 0) \right)$$

이라 정의한다. 단, $Z \in H_r$ 이고 $W \in \mathbb{C}^{(m,r)}$ 이다.

그러면, 상기의 극한값은 수렴한다. $GL(n, \mathbb{C})$ 의 rational 표현 (ρ, V_ρ) 에 대해

$$\{ (\Psi_{n,r}f)(Z,W) \mid f \in J_{\rho,M}(\Gamma_n), (Z,W) \in H_{r,m} \}$$

에 의해 생성되는 V_ρ 의 부분공간을 $V_\rho^{(r)}$ 으로 표기한다. 이 때, $GL(r,\mathbb{C})$ 의 rational 표현 $\rho^{(r)} : GL(r,\mathbb{C}) \to GL(V_\rho^{(r)})$ 을

$$\rho^{(r)}(g)v := \rho\left(\begin{pmatrix} g & 0 \\ 0 & I_{n-r} \end{pmatrix}\right)v, \quad g \in GL(r,\mathbb{C}), \quad v \in V_\rho^{(r)}$$

와 같이 정의한다. 그러면, 우리는 다음의 선형사상

$$\Psi_{n,r} : J_{\rho,M}(\Gamma_n) \to J_{\rho^{(r)},M}(\Gamma_n)$$

을 얻는다. 이 선형사상을 Siegel-Jacobi 작용소 라고 일컫는다. [Ya3] 와 [Ya7] 에서 필자는 이 작용소의 단사성 (injectivity) 또는 전사성 (surjectivity) 에 관해 논의하고 Siegel-Jacobi 작용소의 작용이 Jacobi 형식의 벡터공간 상에 정의되는 Hecke 작용소의 작용과 양립함을 증명하였다. 그리고, [Ya18] 에서 필자는 Siegel-Jacobi 작용소의 여러 성질을 조사하여 Jacobi 형식의 안정성 (stability) 에 관해 논의하고 있다. 상세한 기술은 참고문헌 [Ya3], [Ya7] 과 [Ya18] 을 참조하기 바란다.

7. Siegel 공식

이 절에서는 이차형식론의 Siegel 정리를 Jacobi 형식의 경우로 확장하고자 한다. $S \in S_l^*(\mathbb{Z})^+ := \{ T \in S_l^*(\mathbb{Z}) \mid T > 0 \}$ 를 고정한 후

$$Sym^*_{m+l}(S;\mathbb{Z}) := \left\{ Q = \begin{pmatrix} M & \frac{{}^tq}{2} \\ \frac{q}{2} & S \end{pmatrix} \mid M \in S_m^*(\mathbb{Z}),\ q \in \mathbb{Z}^{(l,m)} \right\}$$

이라 두자. 또한,

$$Sym^*_{m+l}(S;\mathbb{Z})^+ := \{ Q \in Sym^*_{m+l}(S;\mathbb{Z}) \mid Q > 0 \}$$

이라 놓자. 두 행렬 $Q, Q' \in Sym^*_{m+l}(S,\mathbb{Z})$ 가 같은 S-**class 에 속한다**고 일컫는 것은 어떤

$$\gamma = \begin{pmatrix} u & 0 \\ y & I_l \end{pmatrix} \quad (\text{단, } u \in SL(m,\mathbb{Z}),\ y \in \mathbb{Z}^{(l,m)})$$

가 존재하여 $Q' = {}^t\gamma Q \gamma$ 인 관계가 성립할 때이다. 그리고, 두 행렬

$Q,\ Q' \in Sym^*_{m+l}(S;\mathbb{Z})$ 가 같은 S-**genus** **에 속한다고** 일컫는 것은 임의의 소수 p 에 대해 어떤

$$\gamma_p = \begin{pmatrix} u_p & 0 \\ y_p & I_l \end{pmatrix} \quad (\text{단, } u_p \in GL(m, \mathbb{Z}_p),\ y_p \in \mathbb{Z}_p^{(l,m)})$$

가 존재하여 $Q' = {}^t\gamma_p Q \gamma_p$ 이고 $\operatorname{sign} Q = \operatorname{sign} Q'$ 인 관계가 성립할 때이다.
그러면,

(*) 주어진 Q 에 의해 결정되는 S-genus 는 유한갯수의 S-classes 로 이루어져 있다.

이 S-classes 의 갯수를 $H(Q)$ 로 표기하고 Q 의 S-**class number** 라고 부른다. 직교군 G 를

$$G := \left\{ \begin{pmatrix} a & 0 \\ x & I_l \end{pmatrix} \mid a \in SL(m,\mathbb{C}),\ x \in \mathbb{C}^{(l,m)},\ Q\left[\begin{pmatrix} a & 0 \\ x & I_l \end{pmatrix}\right] = Q \right\}$$

이라 정의한다. G 는 $\mathbb{Q}$ 위에서 정의되는 대수군 (an algebraic group) 이다.

$$G(\mathbb{Z}) := \left\{ \begin{pmatrix} a & 0 \\ x & I_l \end{pmatrix} \in G(\mathbb{Q}) \mid a \in SL(m,\mathbb{Z}),\ x \in \mathbb{Z}^{(l,m)} \right\}$$

이라 놓자. $G(\mathbb{Z}_p)$ 도 위와 유사하게 정의한다. adele 군 $G(\mathbb{A})$ 의 극대 부분군 $\amalg$ 는

$$\amalg = \prod_{p<\infty} G(\mathbb{Z}_p) \times G(\mathbb{R})$$

으로 표기할 수 있으며, $H(\mathbb{Q})$ 는 다음의 $G(\mathbb{A})$ 의 double coset decomposition (7.1) 에 의해 결정된다.

$$G(\mathbb{A}) = \coprod_{j=1}^{h} G(\mathbb{Q}) g_j \amalg, \quad h := H(\mathbb{Q}). \tag{7.1}$$

여기서, $g_j\ (1 \le j \le h)$ 는 $G(\mathbb{A})$ 의 원 (element) 이다.

표현의 갯수와 local density

자연수 m, n 이 주어져 있고 $m \ge n$ 이라 하자. $Q \in Sym^*_{m+l}(S;\mathbb{Z})^+$, $T \in Sym^*_{n+l}(S;\mathbb{Z})^+$ 에 대하여

$$A(Q;T) := \#\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^{(m+l,n)} \mid Q\left[\begin{pmatrix} x & 0 \\ y & I_l \end{pmatrix}\right] = T \right\},$$

$$A_{p^\nu}(Q;T) := \#\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in (\mathbb{Z}/p^\nu\mathbb{Z})^{(m+l,n)} \mid Q\left[\begin{pmatrix} x & 0 \\ y & I_l \end{pmatrix}\right] = T \mod p^\nu S^*_{n+l}(\mathbb{Z}) \right\}$$

이라 놓자. 그리고, local density $\alpha_p(Q;T)$ 를

$$\alpha_p(Q;T) := \lim_{\nu\to\infty} p^{-\nu(mn - \frac{n(n+1)}{2})} A_{p^\nu}(Q;T)$$

이라 정의한다. ∞-place 에 관한 local density $\alpha_\infty(Q;T)$ 는 적당히 정의할 수 있다.

$$E(Q) := \#\left\{ \begin{pmatrix} a \\ x \end{pmatrix} \mid a \in SL(m,\mathbb{Z}),\ x \in \mathbb{Z}^{(l,m)},\ Q\left[\begin{pmatrix} a & 0 \\ x & I_l \end{pmatrix}\right]\right\}$$

이라 놓자. 고전적인 경우 (cf. [Si]) 와 비슷하게 아래의 Siegel 공식을 얻는다.

정리 7.1 (Arakawa [Ar4]). $m \geq n$, Q, T 는 위와 같다고 하자. Q 에 의해 결정되는 S-genus 에 속하는 S-classes 의 완전 대표계를 Q_1, ⋯, Q_h ($h := H(Q)$) 이라 하자. 그러면,

$$\prod_{p\leq\infty} \alpha_p(Q;T) = \frac{\varepsilon \cdot \left(\sum_{j=1}^{h} \frac{A(Q_j;T)}{E(Q_j)} \right)}{\left(\sum_{j=1}^{h} \frac{1}{E(Q_j)} \right)}$$

인 관계식을 얻는다. 여기서,

$$\varepsilon = \begin{cases} 1 & \text{if } m > n+1 \text{ or } m = n = 1, \\ 2 & \text{if } m = n+1 \text{ or } m = n > 1. \end{cases}$$

위의 공식의 좌변의 무한곱은 수렴한다.

Arakawa 는 T. Ono (cf. [Ono]) 와 F. Sato (cf. [Sat]) 의 homogeneous 공간 상의 Siegel 공식을 사용하여 상기의 정리를 증명하고 있다.

고전적인 경우에 Siegel 이 얻었던 결과와 비슷하게 세타급수와 Eisenstein 급수와의 Siegel 공식을 얻을 수 있다.

정리 7.2 (Analytic Siegel's formula [Ar4]). 기호는 정리 7.1 과 동일하다. 그리고, $m>2n+l+2$, $\det(2Q)=1$ 이라 가정하자. 그러면,

$$\frac{\left(\sum_{j=1}^{h}\frac{\theta_{Q_j}^{(n)}(Z,W)}{E(Q_j)}\right)}{\left(\sum_{j=1}^{h}\frac{1}{E(Q_j)}\right)} = E^{(n)}_{\frac{m+l}{2},S}(Z,W)$$

인 관계식을 얻는다.

Example : $l=1$, $S=1$ 인 경우에 관해 생각하여 보자.

$$Q=\begin{pmatrix} M & \frac{{}^tq}{2} \\ \frac{q}{2} & 1 \end{pmatrix}, \quad \det(2Q)=1$$

이라 두면 $m\equiv 7 \pmod 8$ 이 된다. Q 에 대해 $\tilde{Q} = M - \frac{1}{4}\,{}^tqq$ 이라 놓고 세타급수 $\theta(\tilde{Q};z)$ 를

$$\theta(\tilde{Q};z) := \sum_{G_1\in Z^{(m,1)}} e^{8\pi i\,Q[G_1]z}, \quad z\in H_1$$

이라 정의하자. 그러면, $\theta(\tilde{Q};z)\in M^{+}_{\frac{m}{2}}(\Gamma_0(4))$ 이 된다. Q 에 의해 결정되는 S-genus 에 속하는 S-classes 의 완전대표계를 $Q_1,\cdots,Q_h$ ($h:=H(Q)$) 이라 하자. $m>5$ 일 때

$$\frac{\left(\sum_{j=1}^{h}\frac{\theta(\tilde{Q}_j;z)}{E(Q_j)}\right)}{\left(\sum_{j=1}^{h}\frac{1}{E(Q_j)}\right)} = \frac{1}{\zeta(2-m)}\,G^{+}_{\frac{m}{2}}(z) \tag{7.2}$$

인 관계식을 얻는다. $\zeta(w)$ 는 Riemann 제타함수이며 $G^{+}_{k-\frac{1}{2}}(z)$ 는 Cohen 함수 (cf. [Co], [EZ] p. 65) 라고 불리우는 함수로 Kohnen 공간 $M^{+}_{k-\frac{1}{2}}(\Gamma_0(4))$ 의 원(element) 이다. 그래서, 상기의 (7.2)는 반정수 보형형식에 관한 Siegel 공식으로 간주할 수 있다.

8. G^J 상의 보형형식과 G^J 의 군 표현론

먼저, cuspidal Jacobi 형식의 개념에 관해 설명하겠다. Jacobi 형식 $f \in J_{\rho,M}(\Gamma_n)$ 는 Fourier–Jacobi 전개 section 1. (iii) 을 갖는다. 이 때, 아래의 조건

$$c(T,R) \neq 0 \Rightarrow \begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & M \end{pmatrix} > 0$$

을 만족할 때 f 를 cuspidal Jacobi 형식이라 일컫는다. 이제, 복잡성을 피하기 위해 $m=n=1$ 인 경우만 다루겠다. 그러면, $H_1 \times \mathbb{C} \simeq G^J / K^J$ 는 homogeneous 공간이므로 $H_1 \times \mathbb{C}$ 상의 함수 f 는 Jacobi 군 G^J 상으로 lifting 할 수 있다. 즉, 이 대응 $f \mapsto \Phi_f$ 는

$$\Phi_f(g^J) := f(g^J \cdot (i,0))\, J_{M,\rho}(g^J,(i,0))$$

으로 주어진다. 단, $g^J \in G^J$ 이고 $K^J = SO(2) \propto \mathbb{R}$ 이다. 편의상, $G^J = SL(2,\mathbb{R}) \propto (\mathbb{R}^2 \cdot S^1)$, $K^J = SO(2) \propto S^1$ (단, $S^1 := \{ z \in \mathbb{C} \mid |z|=1 \}$ 으로 취할 수 있다. 아래의 성질 (1)–(4)를 만족하는 G^J 상의 C^∞ 함수 $\Phi : G^J \to \mathbb{C}$ 들의 벡터 공간을 $A_{k,l}(\Gamma^J)$ 이라 표시하자.

(1) $\Phi(rg) = \Phi(g) \quad \forall r \in \Gamma^J := SL(2,\mathbb{Z}) \propto \mathbb{Z}^2$

(2) $\Phi(g \cdot r(\theta,\zeta)) = \Phi(g)\, \zeta^l e^i k\theta \quad \forall r(\theta,\zeta) \in K^J$

(3) $X_-\Phi = Y_-\Phi = 0$

(4) 임의의 $g \in SL(2,\mathbb{Z})$ 에 대해 아래의 함수

$$g^J \mapsto \Phi(g \cdot g^J) y^{-\frac{k}{2}}$$

는 영역 $\{ z=x+iy \in H_1 \mid y>y_0 \}$ 에서 유계 (bounded) 이다. 여기서, X_- 와 Y_- 는 G^J 상의 적당한 미분작용소이다 (cf. [Be2]). 그러면,

(10.1) $\qquad J_{k,l}(\Gamma) \cong A_{k,l}(\Gamma^J), \quad \Gamma := SL(2,\mathbb{Z}).$

관계식 (10.1)는 Berndt (cf. [Be1]–[Be4], [B–B]) 에 의해 주어졌다.

$$L_0^2(\Gamma^J \backslash G^J) := \{ \Phi \in L^2(\Gamma^J \backslash G^J) : \Phi \text{ cuspidal} \}$$

이라 두자. lowest weight (k,m) 을 지닌 G^J 의 이산표현 (discrete series) $T_{k,l}$ 는

cuspidal Jacobi 형식과 밀접한 관계가 있다. 정의에 의해

$$T_{k,l}(r(\theta,\zeta))v = \zeta^l e^{ik\theta}v, \quad v \in V$$

그러면, 보형형식인 경우에 Gelfand 학파에 의해 얻어진 Duality Theorem 는 Jacobi 형식의 경우에도 성립한다는 사실이 Berndt (cf. [B-B]) 에 의해 증명되었다.

Duality Theorem (Berndt-Böcherer **[B-B]**). $m>0$ 에 대해 Jacobi 군 G^J 의 right regular representation (on $L_0^2(\Gamma \backslash G^J)$) 안에서의 $T_{k,l}$ 의 중복 (multiplicity) 을 $m_{k,l}$ 이라 놓자. 그러면,

$$m_{k,l} = \dim_{\mathbb{C}} J_{k,l}^{cusp}(\Gamma)$$

인 관계가 성립한다. $J_{k,l}^{cusp}(\Gamma)$ 는 벡터공간 $J_{k,l}(\Gamma)$ 안에 있는 cuspidal Jacobi 형식들로 이루어진 부분공간을 나타낸다.

임의의 m, n 인 경우의 Duality Theorem 는 필자 (cf. [Ya14]) 에 의해 증명되었다.

$L^2(\Gamma^J \backslash G^J)$ 의 연속부분 (continuous part) 는 [Be5] 를 참고하기 바란다 (단, $m=n=1$). 일반적인 경우, Takase (cf. [Ta1], [Ta2]) 는 spherical 함수를 사용하여 Jacobi 군 G^J 의 해석적 이산표현에 관해 연구하였다.

9. Whittaker 모델과 L-함수

보형형식에서의 L-함수 이론처럼 Jacobi 형식에서도 L-함수 이론을 전개할 수 있다. 가령, Jacobi 형식에 부수되는 L-함수라든가 Jacobi 군의 adelization $G^J(\mathbb{A})$ 에 부수되는 automorphic representation 에 관해 보형형식인 경우와 유사하게 이론을 전개할 수 있다. 이의 연구는 A. Murase, T. Sugano 등의 일본 수학자와 V. A. Gritsenko (cf. Introduction in the Theory of Zeta Functions, preprint) 등의 러시아 수학자들에 의해 연구되고 있다.

가령, Murase (cf. [Mur1], [Mur2]) 는 Hecke ring 을 도입하여 Satake 동형사상을 얻은 후 Satake 매개변수 (parameters) 를 이용하여 Jacobi 형식에 부수되는 소위, Shintani 제타함수를 정의한다. 그리고, 그는 이 제타함수가 해석적 접속성 (analytic continuation) 을 지님을 보이고 Rankin-Selberg 방법을 이용하여 함수적 등식

(functional equation) 을 발견하였다. $n=1$ 이고 임의의 m 인 경우에는 Sugano (cf. [Su]) 는 Jacobi 형식에 부수되는 L-함수를 정의한 후 이 L-함수의 연속적 접속성을 보이고 함수적 등식을 발견하였다. Murase 와 Sugano (cf. [MS]) 는 Shintani 의 출판되지 않은 강의록에 기술되어 있는 Shintani 의 아이디어를 사용하여 Shintani 의 예상들을 증명하였다. 상세한 것은 위에서 언급한 참고문헌을 참조하길 바란다.

10. 응 용

이 절에서는 Jacobi 형식의 응용으로서 두 가지를 소개하겠다.

(I) Non-surjectiveness 정리.

Siegel 모듈러 형식 $f\in[\Gamma_{n+1},k]$ 는 아래와 같은 Fourier-Jacobi 전개

(*) $$f\begin{pmatrix} Z_1 & {}^tW \\ W & z_2 \end{pmatrix} = \sum_{m=0}^{\infty} \Phi_{f,m}(Z_1, W)\cdot e^{2\pi i m z_2},$$

를 갖는다. 단 $Z_1\in H_n$ 이고 $z_2\in H_1$ 이다. 이 때,

(10.1) $$\alpha : [\Gamma_{n+1},k] \to J_{k,1}(\Gamma_n), \quad \alpha(f) := \Phi_{f,1}$$

으로 정의되는 선형사상을 생각하여 보자. 여기서, $f\in[\Gamma_{n+1},k]$는 Fourier-Jacobi 전개 (*) 를 갖는다. 이미, 2절에서 $\Phi_{f,1}\in J_{k,1}(\Gamma_n)$ 임을 언급하였으며 4절에서 $n=1$ 일 때 $[\Gamma_2,k]^M \cong J_{k,1}(\Gamma_1)$ 임을 설명하였다. $[\Gamma_2,k]^M$ 는 *Maaß* 의 Spezialschar 임을 상기하여라. 그러므로, $n=1$ 일 때는 선형사상 α 는 전사사상 (surjectiv map) 이다. 그러나, 임의의 n 에 대해서는 일반적으로 상기의 선형사상 α 는 전사사상이 아니라는 사실이 Ziegler 에 의해 증명되었다.

정리 10.1 (Ziegler [Zi], Theorem 4.2) $n\geq 32$ 일 때 선형사상

$$\alpha : [\Gamma_{n+1}, 16] \to J_{16,1}(\Gamma_n)$$

은 단사사상이 아니다. (즉, $k=16$ 인 경우).

Problem : For any n with $1<n<32$, is the above liner mapping α surjective ?

(II) Jacobi 형식으로 부터의 모듈러형식의 구성

두 자연수 m, n 을 고정하자. $\mathbb{C}^{(m,n)}$ 상에 정의된 동차다항식 (a homogeneous polynomial) P 에 대해

$$P(\partial_W) := P\left(\frac{\partial}{\partial W_{11}}, \cdots, \frac{\partial}{\partial W_{mn}}\right), \quad W=(W_{ij})\in\mathbb{C}^{(m,n)}$$

이라 두자. $M=(m_{pq})$ 를 $m\times m$ positive 반정수 대칭행렬이라 하고

$$\Delta_{i,j} := \sum_{p,q=1}^{m} 2m_{pq}\frac{\partial^2}{\partial W_{pi}\,\partial W_{qj}}, \quad 1\le i,j\le n$$

이라 놓자. 이 때, $\mathbb{C}^{(m,n)}$ 상의 다항식 P 가 있어

$$\Delta_{i,j}P=0, \quad 1\le i,j\le n$$

의 조건을 만족할 때 다항식 P 를 행렬 $(2M)^{-1}$ 에 관해 *pluriharmonic* 이라 한다.

Jacobi 형식 $f\in J_{\rho,M}(\Gamma_n)$ 와 pluriharmonic 동차다항식 P 에 대해

$$f_P(Z) := P(\partial_W)f(Z,W)|_{W=0}. \quad Z\in H_n,\ W\in\mathbb{C}^{(m,n)}$$

이라 두자. 그리고, 사상 $f_\tau : H_n \to V_\tau^*\otimes V_\rho$ 를

$$(f_\tau(Z))(P) := f_P(Z), \quad Z\in H_n, \quad P\in V_\tau$$

이라 정의한다. 여기서, $\tau : GL(n,\mathbb{C}) \to GL(V_\tau)$ 는 $GL(n,\mathbb{C})$ 의 기약 (irreducible) 표현이며 V_τ^* 는 이의 contragradient 표현이다. 상세한 것은 참고문헌 [Ya4, section 3] 을 참조하기 바란다. 필자는 아래의 결과를 증명하였다.

정리 10.2 (Yang [Ya4]). Jacobi 형식 $f\in J_{\rho,M}(\Gamma_n)$ 에 대해

$$f_\tau\in[\Gamma_n, \rho\otimes\check{\tau}]$$

이다. 단, $\check{\tau}$ 는 위에서 언급한 표현 τ 의 contragradient 표현을 나타낸다.

11. 끝 맺음말

끝으로, 이 절에서는 Jacobi 형식에 관해 앞으로 연구되어야 할 문제들을 열거하겠

다.

Problem 1. Decompose the Hilbert space $L^2(\Gamma^J \setminus G^J)$ into irreducible components for general m and n. Classify the irreducible unitary or admissible representations of the Jacobi group G^J.

Problem 2. Give the dimension formula for the vector space $J_{\rho,M}(\Gamma_n)$ of Jacobi forms and the vector space $J^{cusp}_{\rho,M}(\Gamma_n)$ of cuspidal Jacobi forms.

Problem 3. Construct Jacobi forms (cf. section 2).

Problem 4. Develope the theory of L-functions for the Jacobi group G^J (cf. section 9).

Problem 5. Give the applications of Jacobi forms in algebraic geometry, number theory and physics etc, (cf. [BFH], [Ca], [GKS], [Ya11], [Ya19], [Za1]).

Kramer (cf. [Kr1], [Kr2]) 는 Jacobi 형식의 연구를 기하학적 측면에서 연구하였으며 Ikuo Satake (cf. [Sa1]) 는 특이점 이론에 Jacobi 형식의 이론을 이용하고 있다. Heisenberg 군 $H_{\mathbb{R}}^{(n,m)}$ 의 군표현론 또는 기하학적인 측면에서는 필자에 의해 연구되었다 (cf. [Ya1], [Ya2], [Ya17]). E. Freitag 과 필자는 Jacobi 형식의 개념을 일반적인 관상영역 (tube domain) 상으로 확장하였다 (cf. [Ya16]). Jacobi 형식의 연구는 아벨다양체상의 벡터속의 이론과 밀접한 관계가 있음을 알 수 있다 (cf. [Ya9], [Ya10]). 또, Jacobi 형식은 Jones-Witten 이론에 응용되기도 한다 (cf. [At], G. Segal ; pp 17-35).

12. 참고문헌

[An] A. N. Andrianov, Modular descent and the Saito-Kurokawa conjecture, Invent. Math. 53 (1979), 267-280.

[Ar1] T. Arakawa, Real Analytic Eisenstein Series for the Jacobi Group, Abh. Math. Sem. Univ. Hamburg 60 (1990), 131-148.

[Ar2] T. Arakawa, Selberg Zeta Functions Associated with Theta Multiplier Systems of $SL_2(\mathbb{Z})$, preprint (1991).

[Ar3] T. Arakawa, On Jacobi Forms, Proc. RIMS Symp. on Automorphic Forms and associated Zeta Functions, Kyoto (1992), 1–18.

[Ar4] T. Arakawa, Siegel's formula for Jacobi forms, preprint.

[At] M. Atiyah, Oxford Seminar on Jones–Witten Theory (1988).

[Be1] R. Berndt, Some Differential Operators in the Theory of Jacobi Forms, IHES/M/84/10.

[Be2] R. Berndt, Die Jacobigruppe und die Wärmeleitungsgleichung, Math. Z. 191 (1986), 351–361.

[Be3] R. Berndt, Formes die Jacobi et quelques éléments de la théorie des représentations du group de Jacobi, Astérisque 147–148 (1987), 265–269.

[Be4] R. Berndt, Some Remarks on Automorphic Forms for the Jacobi Group, IHES/M/89/14.

[Be5] R. Berndt, The Continous Part of $L^2(\Gamma^J \setminus G^J)$ for the Jacobi Group G^J, Abh. Math. Sem. Univ. Hamburg 60 (1990), 225–248.

[Be6] R. Berndt, On Automorphic Forms for the Jacobi Group, preprint.

[BB] R. Berndt and S. Böcherer, Jacobi Forms and Discrete Series Representation of the Jacobi group, Math. Z. 204 (1990), 13–44.

[BO] A. Borel, Introduction to Automorphic Forms, Proc. Symp. Pure Math. Vol. IX. AMS (1966), 199–211.

[BFH] D. Bump, S. Friedberg and J. Hoffstein, Nonvanishing Theorems for L–functions of Modular Forms and Their Derivatives, Ann. Math. 131 (1990), 53–128.

[Ca] J. L. Cardy, Operator Content of Two–dimensional Conformally Invariant Theories, Nuclear Physics B 270 (1986), 186–204.

[Co] H. Cohen, Sums involving the values at negative integers of L–functions of quadratic characters, Math. Ann. 217 (1975), 271–285.

[E1] M. Eichler, Die Spur der Hecke Operatoren in gewissen Räumen von Jacobischen Modulformen, Abh. Math. Sem. Univ. Hamburg 54 (1984), 35–48.

[E2] M. Eichler, Eine neue Klasse von Modulformen und Modulfunktionen, Abh. Math. Sem. Univ. Hamburg 55 (1985), 53–68.

[E3] M. Eichler, Eine neue Klasse von Modulformen und Modulfunktionen, Abh. Math. Sem. Univ. Hamburg 57 (1987), 57-68.

[EZ] M. Eichler and D. Zagier, The Theory of Jacobi Forms, Birkhauser (1985).

[FF] A. J. Feingold and I. B. Frenkel, A Hyperbolic Kac-Moody Algebra and the Theory of Siegel Modular Forms of Genus 2, Math. Ann. 263 (1983), 87-144.

[Go1] R. Godement, A Theory of Spherical Functions I, Trans. Amer. Math. Soc. 73 (1952), 496-556.

[Go2] R. Godement, The Decomposition of $L^2(G/\Gamma)$ for $\Gamma = SL_2(\mathbb{Z})$, Proc. Symp. Pure Math. Vol. IX. AMS Providence (1966).

[GKS] B. Gross, W. Kohnen and D. Zagier, Heegner Points and the Derivative of L-series II, Math. Ann. 278 (1987), 497-562.

[Ib] T. Ibukiyama, On Jacobi forms and Siegel modular forms of half integral weight, preprint.

[Ik] T. Ikeda, On the Fourier-Jacobi coefficients of Eisenstein series, preprint.

[J] C. G. J. Jacobi, Fundamenta nova theoriae functionum ellipticum, Königsberg (1829).

[Kac] V. G. Kac, Infinite-dimensional Lie algebras, Birkhäuser, 1983.

[Kl1] H. Klingen, Metrisierungstheorie und Jacobiformen, Abh. Math. Sem. Univ. Hamburg 57 (1987), 165-178.

[Kl2] H. Klingen, Über Kernfunktionen für Jacobiformen und Siegelsche Modulformen, Math. Ann. 285 (1989), 405-416.

[Ko1] W. Kohnen, Modular forms of half integral weight on $\Gamma_0(4)$, Math. Ann. 248 (1980), 249-266.

[Ko2] W. Kohnen, Non Holomorphic Poincaré Type Series on Jacobi Groups, Math. Ann. (1991).

[Ko3] W. Kohnen, On growth estimates for Fourier coefficients of Jacobi forms and an application, Proc. RIMS Symp. on Automorphic Forms and associated Zeta Functions, Kyoto (1992), 29-35.

[KS] W. Kohnen and N.-P. Skoruppa, A Certain Dirichlet Series Attached to Siegel Modular Forms of Degree Two, Invent. Math. 95 (1989), 541-558.

(1986), 379-322.

[Kr2] J. Kramer, A Geometrical Approach to the Theory of Jacobi Forms, Compositio Math. 79 (1991), 1-19.

[Ku] N. Kurokawa, Examples of eigenvalues of Hecke operators on Siegel cusp forms of degree two, Invent. Math. 49 (1978), 149-165.

[Ma1] H. Maaß, Über eine Spezialschar von Modulformen zweiten Grades, Invent. Math. 52 (1979), 95-104.

[Ma2] H. Maaß, Über eine Spezialschar von Modulformen zweiten Grades (II), Invent. Math. 53 (1979), 249-253.

[Ma3] H. Maaß, Über eine Spezialschar von Modulformen zweiten Grades (III), Invent. Math. 53 (1979), 255-265.

[Mu1] D. Mumford, Abelian Varieties, Oxford Univ. Press (1970).

[Mu2] D. Mumford, Tata Lectures on Theta I (1983), II (1984), III (1991), Birkhäuser.

[Mur1] A. Murase, L-functions Attached to Jacobi Forms of Degree n. Part I, J. reine angew. Math. 401 (1989), 122-156.

[Mur2] A. Murase, L-functions Attached to Jacobi Forms of Degree n. Part II, Math. Ann. 290 (1991), 247-276.

[Mur3] A. Murase, On an Explicit Formula for the Whittaker-Shintani Functions on Sp, Abh. Math. Sem. Univ. Hamburg 61 (1991), 153-162.

[MS] A. Murase and T. Sugano, Whittaker-Shintani Functions on Symplectic Group of Fourier Jacobi Type, Compositio Math. 79 (1991), 321-349.

[Ono] T. Ono, Mean value theorem in adele geometry, J. Math. Soc. Japan 20 (1968), 275-288.

[PS] I. Piatetski-Sharpiro, Automorphic Functions and the Geometry of Classical Domains, Gordon Breach, New York (1969).

[Sa1] I. Satake, Fock Representations and Theta Functions, Ann. Math. Studies 66 (1971), 393-405.

[Sa2] I. Satake, Unitary representations of a semi-direct product of Lie groups on $\bar{\partial}$-cohomology spaces, Math. Ann. 190 (1971), 177-202.

[Sa3] I. Satake, Algebraic Structures of Symmetric Domains, Iwanami Shoten Publ. and Princeton Univ. Press (1980).

[SaI] Ikuo Satake, Automorphisms of the Extended Affine Root System and

Modular property for Flat Theta Invariants, RIMS preprint 759.
[Sat] F. Sato, Siegel's main theorem of homogeneous spaces, preprint.
[Sh1] G. Shimura, Theta Functions with Complex Multiplication, Duke Math. J. 43 (1976), 673–696.
[Sh2] G. Shimura, On Certain Reciprocity Laws for Theta Functions and Modular Forms, Acta Math. 141 (1978), 35–71.
[Si] C. L. Siegel, Über die analytische Theorie der quadratischen Formen, Ann. Math. 36 (1935), 527–606.
[SK1] N.–P. Skoruppa, Explicit Formulas for the Fourier Coefficients of Jacobi and Elliptic Modular Forms, Invent. Math. 102 (1990), 501–520.
[SK2] N.–P. Skoruppa, Heeger Cycles, Modular Forms and Jacobi Forms, Sém. de Théorie des Nombres, Bordeaux 3 (1991), 93–116.
[SZ1] N.–P. Skoruppa and D. Zagier, Jacobi Forms and a certain Space of Modular Forms, Invent. Math. 94 (1988), 113–146.
[SZ2] N.–P. Skoruppa and D. Zagier, A Trace Formula for Jacobi Forms, J. reine angew. Math. 393 (1989), 113–146.
[Su] T. Sugano, Jacobi forms and the theta lifting, preprint.
[T] Y.–S. Tai, On the Kodaira Dimension of the Moduli Space of Abelian Varieties, Invent. Math. 68 (1982), 425–439.
[Ta1] K. Takase, A Note on Automorphic Forms, J. reine angew. Math. 409 (1990), 138–171.
[Ta2] K. Takase, On Unitary Representations of the Jacobi Group, preprint (1991).
[Y] T. Yamazaki, Jacobi Forms and a Maass Relation for Eisenstein Series, J. Fac. Sci. Univ. Tokyo 33 (1986), 295–310.
[Ya1] J.–H. Yang, Harmonic Analysis on the Quotient Spaces of Heisenberg Groups, Nagoya Math. J. 123 (1991), 103–117.
[Ya2] J.–H. Yang, Harmonic Analysis on the Quotient Spaces of Heisenberg Groups (II), to appear in J. Number Theory.
[Ya3] J.–H. Yang, The Siegel–Jacobi Operator, to appear in Abh. Math. Sem. Univ. Hamburg 63 (1993).
[Ya4] J.–H. Yang, Construction of Vector Valued Modular Forms from Jacobi Forms, preprint.
[Ya5] J.–H. Yang, The Differential Operators and Singular Jacobi Forms, preprint.

preprint.

[Ya6] J.–H. Yang, Singular Jacobi Forms, preprint.

[Ya7] J.–H. Yang, Some Results on Jacobi Forms of Higher Degree, Proc. RIMS Symp. on Automorphic Forms and associated Zeta Functions, Kyoto (1992), 36–50.

[Ya8] J.–H. Yang, On Jacobi Forms of Higher Degree, to appear in Proc. RIMS Symp. on Infinite Root Systems and the Period Integrals, Kyoto.

[Ya9] J.–H. Yang, A Note on Holomorphic Vector Bundles over Complex Tori, Bull. KMS (1986), 149–154.

[Ya10] J.–H. Yang, Holomorphic Vector Bundles over Complex Tori, J. KMS (1989), 117–142.

[Ya11] J.–H. Yang (with J. W. Son), A Note on Jacobi Forms of Higher Degree, J. KMS (1991), 341–358.

[Ya12] J.–H. Yang, Vanishing Theorems on Jacobi Forms of Higher Degree, to appear in J. KMS (1993).

[Ya13] J.–H. Yang, Remarks on Automorphic Forms for the Jacobi Group, preprint.

[Ya14] J.–H. Yang, Harmonic Analysis for the Jacobi Group of Higher Degree (I), preprint.

[Ya15] J.–H. Yang, Harmonic Analysis for the Jacobi Group of Higher Degree (II), in preparation.

[Ya16] J.–H. Yang (with E. Freitag), Jacobi Forms on Tube Domains, preprint.

[Ya17] J.–H. Yang, Geometric Aspects on Heisenberg Groups, in preparation.

[Ya18] J.–H. Yang, Jacobi Forms of Higher Degree, 평산연구회 강의록, in preparation.

[Ya19] J.–H. Yang, Invariant Holomorphic Differential Forms on $H_2 \times \mathbb{C}^2$, unpublished.

[Za1] D. Zagier, Note on the Landweber–Stong Elliptic Genus, Springer Lecture Note 1326 (1988), 216–224.

[Za2] D. Zagier, Periods of Modular Forms and Jacobi Theta Functions, Invent. Math. 104 (1991), 449–465.

[Zi] C. Ziegler, Jacobi Forms of Higher Degree, Abh. Math. Sem. Univ. Hamburg 59 (1989), 191–224.

Abh. Math. Sem. Univ. Hamburg **63** (1993), 135–146

The Siegel-Jacobi Operator

By J.-H. YANG

1 Introduction

For any positive integer $g \in \mathbb{Z}^+$, we let H_g the Siegel upper half plane of degree g and let $\Gamma_g := \mathrm{Sp}(g, \mathbb{Z})$ the Siegel modular group of degree g. Let ρ be a rational finite dimensional representation of the general linear group $\mathrm{GL}(g, \mathbb{C})$ on V_ρ and let $\mathcal{M}$ be a symmetric half-integral semipositive matrix of degree h. Let $J_{\rho,\mathcal{M}}(\Gamma_g)$ be the vector space of all Jacobi forms on Γ_g of index $\mathcal{M}$ with respect to ρ (see Definition 2.1). For a positive integer r with $r < g$, we let $\rho^{(r)}: \mathrm{GL}(r, \mathbb{C}) \to \mathrm{GL}(V_\rho)$ be a rational representation of $\mathrm{GL}(r, \mathbb{C})$ defined by

$$\rho^{(r)}(a)v := \rho\left(\begin{pmatrix} a & 0 \\ 0 & E_{g-r} \end{pmatrix}\right)v, \quad a \in \mathrm{GL}(r, \mathbb{C}), v \in V_\rho .$$

The *Siegel-Jacobi operator* $\Psi_{g,r}: J_{\rho,\mathcal{M}}(\Gamma_g) \to J_{\rho^{(r)},\mathcal{M}}(\Gamma_r)$ is defined by

$$(\Psi_{g,r}f)(Z, W) := \lim_{t\to\infty} f\left(\begin{pmatrix} Z & 0 \\ 0 & itE_{g-r} \end{pmatrix}, (W, 0)\right),$$

where $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$, $Z \in H_r$ and $W \in \mathbb{C}^{(h,r)}$. We observe that the above limit always exists and the Siegel-Jacobi operator is a linear mapping (cf. [14]).

The aim of this paper is to investigate some properties of the Siegel-Jacobi operator. This article is organized as follows. In section **2**, we establish the notations and give a definition of Jacobi forms. In section **3**, we obtain the Shimura isomorphism based on ZIEGLER's work [14]. Using this isomorphism and the theory of singular modular forms, we obtain an injectivity or a surjectivity of the Siegel-Jacobi operator under certain conditions. In the final section, we define an action of the Hecke operator of Γ_g on $J_{\rho,\mathcal{M}}(\Gamma_g)$ and prove that the action of the Siegel-Jacobi operator on Jacobi forms is compatible with that of the Hecke algebra.

Notations. We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of rational numbers, the field of real numbers, and the field of complex numbers respectively. For $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(g, \mathbb{R})$ and $Z \in H_g$, we set $M\langle Z\rangle := (AZ + B)(CZ + D)^{-1}$. $[\Gamma_g, k]$ (resp. $[\Gamma_g, \rho]$) denotes the vector space of all Siegel modular forms of weight k (resp. of type ρ). We denote by

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$\mathbb{Z}^+$ the set of all positive integers. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. E_n denotes the identity matrix of degree n.

2 Jacobi Forms

In this section, we establish the notations and define the concept of Jacobi forms.

Let

$$\mathrm{GSp}(g,\mathbb{R})^+ = \{M \in \mathbb{R}^{(2g,2g)} \mid {}^tMJ_gM = \nu J_g \quad \text{for some} \quad \nu > 0\}$$

be the group of *similitudes* of degree g, where

$$J_g := \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}.$$

Let $M \in \mathrm{GSp}(g,\mathbb{R})^+$. If ${}^tMJ_gM = \nu J_g$, we write $\nu = \nu(M)$. It is easy to see that $\mathrm{GSp}(g,\mathbb{R})^+$ acts on H_g transitively by

$$M\langle Z\rangle := (AZ+B)(CZ+D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}(g,\mathbb{R})^+$ and $Z \in H_g$.

For two positive integers g and h, we consider the *Heisenberg group*

$$H_{\mathbb{R}}^{(g,h)} := \{[(\lambda,\mu),\kappa] \mid \lambda,\mu \in \mathbb{R}^{(h,g)}, \kappa \in \mathbb{R}^{(h,h)}, \kappa + \mu{}^t\lambda \quad \text{symmetric}\}$$

endowed with the following multiplication law

$$[(\lambda,\mu),\kappa] \circ [(\lambda',\mu'),\kappa'] := [(\lambda+\lambda',\mu+\mu'),\kappa+\kappa'+\lambda{}^t\mu'-\mu{}^t\lambda'].$$

We define the semidirect product of $\mathrm{GSp}(g,\mathbb{R})^+$ and $H_{\mathbb{R}}^{(g,h)}$

$$\hat{G}^J := \mathrm{GSp}(g,\mathbb{R})^+ \ltimes H_{\mathbb{R}}^{(g,h)}$$

endowed with the following multiplication law

$$\begin{aligned} &(M,[(\lambda,\mu),\kappa]) \cdot (M',[(\lambda',\mu'),\kappa']) \\ &:= (MM',[(\nu(M')^{-1}\tilde{\lambda}+\lambda',\nu(M')^{-1}\tilde{\mu}+\mu'),\nu(M')^{-1}\kappa+\kappa'+\nu(M')^{-1}(\tilde{\lambda}{}^t\mu'-\tilde{\mu}{}^t\lambda')]),\end{aligned}$$

with M, $M' \in \mathrm{GSp}\,(g,\mathbb{R})^+$ and $(\tilde{\lambda},\tilde{\mu}) := (\lambda,\mu)M'$. Clearly the *Jacobi group* $G^J := \mathrm{Sp}(g,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$ is a normal subgroup of $\hat{G}^J$. It is easy to see that $\hat{G}^J$ acts on $H_g \times \mathbb{C}^{(h,g)}$ transitively by

$$(2.1) \qquad (M,[(\lambda,\mu),\kappa]) \cdot (Z,W) := (M\langle Z\rangle, \nu(W+\lambda Z+\mu)(CZ+D)^{-1}),$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}(g,\mathbb{R})^+$, $\nu = \nu(M)$, $(Z,W) \in H_g \times \mathbb{C}^{(h,g)}$.

Let ρ be a rational representation of $GL(g,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbb{R}^{(h,h)}$ be a symmetric half integral semi-positive matrix of degree h. Let $C^\infty(H_g \times \mathbb{C}^{(h,g)}, V_\rho)$ be the algebra of all C^∞ functions on $H_g \times \mathbb{C}^{(h,g)}$ with values in V_ρ. For $f \in C^\infty(H_g \times \mathbb{C}^{(h,g)}, V_\rho)$, we define

$$\begin{aligned}(2.2) \qquad & (f|_{\rho,\mathcal{M}}[(M,[(\lambda,\mu),\kappa])])(Z,W) \\ := \;& e^{-2\pi v i\sigma(\mathcal{M}[W+\lambda Z+\mu](CZ+D)^{-1}C)} e^{2\pi v i\sigma(\mathcal{M}(\lambda Z{}^t\lambda+2\lambda{}^tW+(\kappa+\mu{}^t\lambda)))} \\ & \times \rho(CZ+D)^{-1} f(M\langle Z\rangle, v(W+\lambda Z+\mu)(CZ+D)^{-1}),\end{aligned}$$

where $v = v(M)$.

Definition 2.1. Let ρ and $\mathcal{M}$ be as above. Let

$$H_{\mathbb{Z}}^{(g,h)} := \{[(\lambda,\mu),\kappa] \in H_{\mathbb{R}}^{(g,h)} \mid \lambda,\mu \in \mathbb{Z}^{(h,g)}, \kappa \in \mathbb{Z}^{(h,h)}\}.$$

A *Jacobi form* of index $\mathcal{M}$ with respect to ρ on a subgroup $\Gamma \subset \Gamma_g$ of finite index is a holomorphic function $f \in C^\infty(H_g \times \mathbb{C}^{(h,g)}, V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho,\mathcal{M}}[\tilde{\gamma}] = f$ for all $\tilde{\gamma} \in \Gamma^J := \Gamma \ltimes H_{\mathbb{Z}}^{(g,h)}$.

(B) f has a Fourier expansion of the following form:

$$f(Z,W) = \sum_{\substack{T\geq 0 \\ \text{half-integral}}} \sum_{R\in\mathbb{Z}^{(g,h)}} c(T,R)\cdot e^{\frac{2\pi i}{\lambda_\Gamma}\sigma(TZ)} \cdot e^{2\pi i\sigma(RW)}$$

with some $\lambda_\Gamma \in \mathbb{Z}$ and $c(T,R) \neq 0$ only if

$$\begin{pmatrix} \frac{1}{\lambda_\Gamma}T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} \geq 0.$$

If $g \geq 2$, the condition (B) is superfluous by Koecher principle (see [14] Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ. In the special case $V_\rho = \mathbb{C}$, $\rho(A) = (\det A)^k$ ($k \in \mathbb{Z}$, $A \in GL(g,\mathbb{C})$), we write $J_{k,\mathcal{M}}(\Gamma)$ instead of $J_{\rho,\mathcal{M}}(\Gamma)$ and call k the *weight* of a Jacobi form $f \in J_{k,\mathcal{M}}(\Gamma)$.

Ziegler ([14] Theorem 1.8 or [2] Theorem 1.1) proves that the vector space $J_{\rho,\mathcal{M}}(\Gamma)$ is finite dimensional.

3 The Siegel-Jacobi Operator

Let (ρ, V_ρ) be a finite dimensional representation of $GL(g,\mathbb{C})$. For any positive integer r with $r < g$, we denote by $V_\rho^{(r)}$ the subspace of V_ρ generated by the values $\{\Psi_{g,r}f(Z,W) \mid f \in J_{\rho,\mathcal{M}}(\Gamma_g), (Z,W) \in H_g \times \mathbb{C}^{(h,g)}\}$. According to [10], $V_\rho^{(r)}$ is invariant under

$$\left\{\begin{pmatrix} a & 0 \\ 0 & E_{g-r} \end{pmatrix} : a \in GL(r,\mathbb{C})\right\}.$$

Then we have a rational representation $\rho^{(r)}$ of $\mathrm{GL}(r,\mathbb{C})$ on $V_\rho^{(r)}$ defined by

$$\rho^{(r)}(a)v := \rho\left(\begin{pmatrix} a & 0 \\ 0 & E_{g-r}\end{pmatrix}\right)v, \quad a \in \mathrm{GL}(r,\mathbb{C}), v \in V_\rho^{(r)}.$$

Following the argument of [10], we obtain

Lemma 3.1. *If (ρ, V_ρ) is irreducible, then $(\rho^{(r)}, V_\rho^{(r)})$ is also irreducible.*

Now we assume that $\mathcal{M}$ is a symmetric positive half-integral matrix of degree h. For any a, $b \in \mathbb{Q}^{(h,g)}$, we consider the theta series

$$\vartheta_{2\mathcal{M},a,b}(Z,W) := \sum_{\lambda\in\mathbb{Z}^{(h,g)}} e^{\pi i\sigma(2\mathcal{M}((\lambda+a)Z\,{}^t(\lambda+a)+2(\lambda+a)^t(W+b)))}$$

with characteristic (a,b) converging uniformly on any compact subset of $H_g \times \mathbb{C}^{(h,g)}$.

We fix an element $Z_0 \in H_g$. Let $\mathcal{N}$ be a complete system of representatives of the cosets $(2\mathcal{M})^{-1}\mathbb{Z}^{(h,g)}/\mathbb{Z}^{(h,g)}$. We denote by $T_{\mathcal{M}}(Z_0)$ the vector space of all holomorphic functions $\varphi:\mathbb{C}^{(h,g)} \to \mathbb{C}$ satisfying the condition

$$\varphi(W + \lambda Z_0 + \mu) = e^{-2\pi i\sigma(\mathcal{M}(\lambda Z_0{}^t\lambda+2\lambda^t W))}\varphi(W) \tag{3.1}$$

for every λ, $\mu \in \mathbb{Z}^{(h,g)}$. The functions $\{\vartheta_{2\mathcal{M},a,0}(Z_0,W) \mid a \in \mathcal{N}\}$ form a basis of $T_{\mathcal{M}}(Z_0)$ and its dimension is clearly $\{\det(2\mathcal{M})\}^g$. If f is a Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma_g)$, it is easy to see that each component of $\phi(W) := f(Z_0,W)$ satisfies the relation (3.1). So we may write

$$f(Z,W) = \sum_{a\in\mathcal{N}} f_a(Z)\cdot\vartheta_{2\mathcal{M},a,0}(Z,W), \quad Z \in H_g, W \in \mathbb{C}^{(h,g)}, \tag{3.2}$$

where $\{f_a: H_g \to V_\rho \mid a \in \mathcal{N}\}$ are uniquely determined holomorphic functions on H_g.

According to [14], we have

$$\begin{aligned} f_a(-Z^{-1}) &= \left\{\det\left(\frac{Z}{i}\right)\right\}^{-\frac{h}{2}}\cdot\{\rho(-Z)\}\cdot\{\det(2\mathcal{M})\}^{-\frac{g}{2}} \\ &\quad\times\sum_{b\in\mathcal{N}} e^{2\pi i\sigma(2\mathcal{M}a^t b)}\cdot f_b(Z) \end{aligned} \tag{3.3}$$

and

$$f_a(Z+S) = e^{-2\pi i\sigma(\mathcal{M}aS^t a)}\cdot f_a(Z), \quad S = {}^tS \in \mathbb{Z}^{(g,g)}. \tag{3.4}$$

By an easy argument, we see that the functions $\{f_a \mid a \in \mathcal{N}\}$ must have the Fourier expansion of the form

$$f_a(Z) = \sum_{\substack{T={}^tT\geq 0\\ \text{half-integral}}} c(T)\cdot e^{2\pi i\sigma(TZ)}. \tag{3.5}$$

Conversely, suppose there is given a family $\{f_a \mid a \in \mathcal{N}\}$ of holomorphic functions $f_a : H_g \to V_\rho$ satisfying the transformation laws (3.3), (3.4) and the cusp condition (3.5). Then we obtain a Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma_g)$ by defining $f(Z,W)$ via the equation (3.2).

So we obtain the Shimura isomorphism:

Theorem. (SHIMURA) *The equation* (3.2) *gives an isomorphism between* $J_{\rho,\mathcal{M}}(\Gamma_g)$ *and the vector space of* V_ρ*-valued Siegel modular forms of half integral weight satisfying the transformation laws* (3.3), (3.4) *and the cusp condition* (3.5).

Corollary 3.2. *Let* $2\mathcal{M}$ *be unimodular. We assume that* ρ *satisfies the following condition:*

$$(3.6) \qquad \rho(A) = \rho(-A) \quad \textit{for all} \quad A \in GL(g, \mathbb{C}).$$

Then we have

$$(3.7) \qquad J_{\rho,\mathcal{M}}(\Gamma_g) = [\Gamma_g, \tilde{\rho}] \cdot \vartheta_{2\mathcal{M},0,0}(Z,W) \cong [\Gamma_g, \tilde{\rho}],$$

where $\tilde{\rho} = \rho \otimes \det^{-\frac{h}{2}}$. *In particular, if* $k \cdot g$ *is even,*

$$(3.8) \qquad J_{k,\mathcal{M}}(\Gamma_g) = [\Gamma_g, k - \frac{h}{2}] \cdot \vartheta_{2\mathcal{M},0,0}(Z,W) \cong [\Gamma_g, k - \frac{h}{2}].$$

Proof. The proof of (3.7) follows from (3.3), (3.4) and (3.5). The representation $\det^k : \mathrm{GL}(g,\mathbb{C}) \to \mathbb{C}^\times$ defined by $\det^k(A) = (\det(A))^k$ satisfies the condition (3.6). Hence (3.8) follows from (3.7). □

Notations 3.3. In corollary 3.2, we denote the isomorphism of $J_{\rho,\mathcal{M}}(\Gamma_g)$ (resp. $J_{k,\mathcal{M}}(\Gamma_g)$) onto $[\Gamma_g, \tilde{\rho}]$ (resp. $[\Gamma_g, k - \frac{h}{2}]$) by

$$S_\rho : J_{\rho,\mathcal{M}}(\Gamma_g) \to [\Gamma_g, \tilde{\rho}] \quad (\text{resp. } S_{g,k} : J_{k,\mathcal{M}}(\Gamma_g) \to [\Gamma_g, k - \tfrac{h}{2}]).$$

We denote the Siegel operator by $\Phi_{g,r} : [\Gamma_g, \rho] \to [\Gamma_r, \rho^{(r)}]$, $0 < r < g$.

Definition 3.4. An irreducible finite dimensional representation ρ of $\mathrm{GL}(g,\mathbb{C})$ is determined by its highest weight $(\lambda_1, \lambda_2, \cdots, \lambda_g) \in \mathbb{Z}^g$ with $\lambda_1 \geq \cdots \geq \lambda_g$. We denote this representation by $\rho = (\lambda_1, \cdots, \lambda_g)$. The number $k(\rho) := \lambda_g$ is called the *weight* of ρ.

Theorem 3.5. *Let* $2\mathcal{M}$ *be a positive unimodular symmetric even matrix of degree* h. *We assume that* ρ *is irreducible and satisfies the condition* (3.6). *If* $2k(\rho) < g + \mathrm{rank}(\mathcal{M})$, *then the Siegel-Jacobi operator* $\Psi_{g,g-1}$ *is injective.*

Proof. By corollary 3.2, we have

$$(3.9) \qquad J_{\rho,\mathcal{M}}(\Gamma_g) = [\Gamma_g, \rho \otimes \det^{-\frac{h}{2}}] \cdot \vartheta_{2\mathcal{M},0,0}(Z,W).$$

By an easy computation, we have

$$S_{\rho^{(g-1)}} \circ \Psi_{g,g-1} = \Phi_{g,g-1} \circ S_{\rho}. \tag{3.10}$$

According to the assumption, the irreducible representation $\rho \otimes \det^{-\frac{h}{2}}$ of $GL(g, \mathbb{C})$ is *singular*, that is, $2k(\rho \otimes \det^{-\frac{h}{2}}) < g$. According to the well-known theory of singular modular forms ([10] Satz 4), every $f \in [\Gamma_g, \rho \otimes \det^{-\frac{h}{2}}]$ is a singular modular form. Thus the Siegel operator $\Phi_{g,g-1}$ is injective (see [11] for the proof of the injectivity of $\Phi_{g,g-1}$). Since S_ρ and $S_{\rho^{(g-1)}}$ are isomorphisms, the Siegel-Jacobi operator $\Psi_{g,g-1}$ is injective by (3.10). This completes the proof of Theorem 3.5. □

Theorem 3.6. *Let* $2\mathcal{M}$ *be as above in Theorem* 3.5. *Assume that* ρ *is irreducible and satisfies the condition* (3.6). *If* $2k(\rho) + 1 < g + \operatorname{rank}(\mathcal{M})$, *then the Siegel-Jacobi operator* $\Psi_{g,g-1}$ *is an isomorphism.*

Proof. By corollary 3.2, we have the relation (3.9). Similarly, we have the commutation relation (3.10). Since $2k(\rho \otimes \det^{-\frac{h}{2}}) + 1 < g$ by the assumption, according to the theory of singular modular forms (cf. [3] and [11]), the Siegel operator $\Phi_{g,g-1}$ is an isomorphism. Since S_ρ, $S_{\rho^{(g-1)}}$ and $\Phi_{g,g-1}$ are all isomorphisms, $\Psi_{g,g-1}$ is an isomorphism. □

Theorem 3.7. *Let* $2\mathcal{M}$ *be as above in Theorem* 3.5. *Assume that* $2k(\rho) > 4g + \operatorname{rank}(\mathcal{M})$ *and* $k \equiv 0 \pmod 2$. *Then the Siegel-Jacobi operator* $\Psi_{g,g-1} : J_{k,\mathcal{M}}(\Gamma_g) \to J_{k,\mathcal{M}}(\Gamma_{g-1})$ *is surjective.*

Proof. By corollary 3.2, we have

$$J_{k,\mathcal{M}}(\Gamma_g) = [\Gamma_g, k - \frac{h}{2}] \cdot \vartheta_{2\mathcal{M},0,0}(Z, W) \cong [\Gamma_g, k - \frac{h}{2}].$$

By the assumption, $2(k - \frac{h}{2}) > g$ and $k - \frac{h}{2} \equiv 0 \pmod 2$. According to Maass [6], the Siegel operator

$$\Phi_{g,g-1} : [\Gamma_g, k - \frac{h}{2}] \to [\Gamma_{g-1}, k - \frac{h}{2}]$$

is surjective. Consequently the surjectivity of the Siegel-Jacobi operator $\Psi_{g,g-1}$ follows immediately from the commutation relation

$$S_{g-1,k} \circ \Psi_{g,g-1} = \Phi_{g,g-1} \circ S_{g,k}.$$ □

4 Hecke Operator

In this section, we give the action of Hecke operators on Jacobi forms and prove that this action is compatible with that of the Siegel-Jacobi operator.

For a positive integer l, we define

$$O_g(l) := \{M \in \mathbb{Z}^{(2g,2g)} \mid {}^tMJ_gM = lJ_g\},$$

where

$$J_g := \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}.$$

$O_g(l)$ is decomposed into finitely many double cosets *mod* Γ_g, i.e.,

$$O_g(l) = \bigcup_{j=1}^{m} \Gamma_g g_j \Gamma_g \quad \text{(disjoint union)}.$$

We define

$$T(l) := \sum_{j=1}^{m} \Gamma_g g_j \Gamma_g \in \mathscr{H}^{(g)}, \quad \text{the Hecke algebra.}$$

Let $M \in O_g(l)$. For a Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$, we define

$$f|_{\rho,\mathcal{M}}(\Gamma_g M \Gamma_g) := l^{gk(\rho)-\frac{g(g+1)}{2}} \sum_{i=1}^{m} f|_{\rho,\mathcal{M}}[(M_i,[(0,0),0])],$$

where $\Gamma_g M \Gamma_g = \bigcup_i^m \Gamma_g M_i$ (finite disjoint union) and $k(\rho)$ denotes the weight of ρ. See (2.2) in section **2** for the definition of $f|_{\rho,\mathcal{M}}[(M_i,[(0,0),0])]$.

Proposition 4.1. *Let l be a positive integer. Let $M \in O_g(l)$ and $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$. Then*

$$f|_{\rho,\mathcal{M}}(\Gamma_g M \Gamma_g) \in J_{\rho,l\mathcal{M}}(\Gamma_g).$$

Proof. It is easy to compute it and so we omit the proof. □

For a prime p, we define

$$O_{g,p} := \bigcup_{l=0}^{\infty} O_g(p^l).$$

Let $\check{\mathscr{L}}_{g,p}$ be the $\mathbb{C}$-module generated by all left cosets $\Gamma_g M$, $M \in O_{g,p}$ and $\check{\mathscr{H}}_{g,p}$ the $\mathbb{C}$-module generated by all double cosets $\Gamma_g M \Gamma_g$, $M \in O_{g,p}$. Then $\check{\mathscr{H}}_{g,p}$ is a commutative associative algebra. We associate to a double coset

$$\Gamma_g M \Gamma_g = \bigcup_{i=1}^{m} \Gamma_g M_i, \quad M, M_i \in O_{g,p} \quad \text{(disjoint union)}$$

the element

$$j(\Gamma_g M \Gamma_g) = \sum_{i=1}^{m} \Gamma_g M_i \in \check{\mathscr{L}}_{g,p}.$$

We extend j linearly to the Hecke algebra $\check{\mathscr{H}}_{g,p}$ and then we have a monomorphism $j: \check{\mathscr{H}}_{g,p} \to \check{\mathscr{L}}_{g,p}$. We now define a bilinear mapping

$$\check{\mathscr{H}}_{g,p} \times \check{\mathscr{L}}_{g,p} \to \check{\mathscr{L}}_{g,p}$$

by

$$(\Gamma_g M \Gamma_g) \cdot (\Gamma_g M_0) = \sum_{i=1}^{m} \Gamma_g M_i M_0, \quad \text{where} \quad \Gamma_g M \Gamma_g = \bigcup_{i=1}^{m} \Gamma_g M_i.$$

This mapping is well defined because the definition does not depend on the choice of representatives.

Let $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ be a Jacobi form. For a left coset $L := \Gamma_g N$ with $N \in O_{g,p}$, we put

$$f|L := f|_{\rho,\mathcal{M}}[(N, [(0,0), 0])]. \tag{4.1}$$

We extend this operator (4.1) linearly to $\check{\mathscr{L}}_{g,p}$. If $T \in \check{\mathscr{H}}_{g,p}$, we write

$$f|T := f|j(T).$$

Obviously we have

$$(f|T)|L = f|(TL), \quad f \in J_{\rho,\mathcal{M}}(\Gamma_g).$$

In a left coset $\Gamma_g M$, $M \in O_{g,p}$, we can choose a representative M of the form

$$M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \quad {}^tAD = p^{k_0} E_g, {}^tBD = {}^tDB, \tag{4.2}$$

$$A = \begin{pmatrix} a & {}^t\alpha \\ 0 & A^* \end{pmatrix}, \quad B = \begin{pmatrix} b & {}^t\beta_1 \\ \beta_2 & B^* \end{pmatrix}, \quad D = \begin{pmatrix} d & 0 \\ \delta & D^* \end{pmatrix}, \tag{4.3}$$

where α, β_1, β_2, $\delta \in \mathbb{Z}^{g-1}$. Then we have

$$M^* := \begin{pmatrix} A^* & B^* \\ 0 & D^* \end{pmatrix} \in O_{g-1,p}. \tag{4.4}$$

For an integer $r \in \mathbb{Z}$, we define

$$(\Gamma_g M)^* := \frac{1}{d^r} \Gamma_{g-1} M^*. \tag{4.5}$$

If $\Gamma_g M \Gamma_g = \bigcup_{j=1}^{m} \Gamma_g M_j$ (disjoint union), M, $M_j \in O_{g,p}$, then we define in a natural way

$$(\Gamma_g M \Gamma_g)^* = \frac{1}{d^r} \sum_{j=1}^{m} \Gamma_{g-1} M_j^*. \tag{4.6}$$

We extend the above map (4.6) linearly on $\check{\mathscr{H}}_{g,p}$ and then we obtain an algebra homomorphism

$$\begin{array}{ccc} \check{\mathscr{H}}_{g,p} & \longrightarrow & \check{\mathscr{H}}_{g-1,p} \\ T & \longmapsto & T^{*}. \end{array} \tag{4.7}$$

It is known that the above map is a surjective map ([13] Theorem 2).

Let $\Psi^0_{g,r}: J_{\rho,\mathcal{M}}(\Gamma_g) \to J_{\rho_0^{(r)},\mathcal{M}}(\Gamma_r)$ be the *modified Siegel-Jacobi operator* defined by

$$(\Psi^0_{g,r}f)(Z,W) := \lim_{t\to\infty} f\left(\begin{pmatrix} itE_{g-r} & 0 \\ 0 & Z \end{pmatrix}, (0,W)\right), \quad (Z,W) \in H_r \times \mathbb{C}^{(h,r)},$$

where $\rho_0^{(r)}: \mathrm{GL}(r,\mathbb{C}) \to \mathrm{GL}(V_\rho)$ is a finite dimensional representation of $\mathrm{GL}(r,\mathbb{C})$ defined by

$$\rho_0^{(r)}(A) = \rho\begin{pmatrix} E_{g-r} & 0 \\ 0 & A \end{pmatrix}, \quad A \in \mathrm{GL}(r,\mathbb{C}).$$

The following theorem is a variant of the Siegel version [4].

Theorem 4.2. *Suppose we have*

(a) *a rational finite dimensional representation*

$$\rho: \mathrm{GL}(g,\mathbb{C}) \to \mathrm{GL}(V_\rho),$$

(b) *a rational finite dimensional representation*

$$\rho_0: \mathrm{GL}(g-1,\mathbb{C}) \to \mathrm{GL}(V_{\rho_0}),$$

(c) *a linear map* $R: V_\rho \to V_{\rho_0}$,

satisfying the following properties (1) *and* (2):

(1) $R \circ \rho\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} = \rho_0(A) \circ R$ *for all* $A \in \mathrm{GL}(g-1,\mathbb{C})$.

(2) $R \circ \rho\begin{pmatrix} a & 0 \\ 0 & E_{g-1} \end{pmatrix} = a^r R$ *for some* $r \in \mathbb{Z}$.

Then for any $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ *and* $T \in \check{\mathscr{H}}_{g,p}$, *we have*

$$(R \circ \Psi^0_{g,g-1})(f|T) = R(\Psi^0_{g,g-1}f)|T^{*}.$$

Proof. Let $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ be a Jacobi form. Then we have the Fourier expansion

$$f(Z,W) = \sum_{T,R} c(T,R) e^{2\pi i\sigma(TZ)} \cdot e^{2\pi i\sigma(RW)}.$$

By an easy computation, we have

$$(\Psi^0_{g,g-1}f)(Z,W)=\sum_{T,R} c\left(\begin{pmatrix}0&0\\0&T\end{pmatrix},\begin{pmatrix}0\\R\end{pmatrix}\right)\cdot e^{2\pi i\sigma(TZ+RW)},$$

where $(Z,W)\in H_{g-1}\times\mathbb{C}^{(h,g-1)}$, $T\in\mathbb{Q}^{(g-1,g-1)}$ runs over the set of all half integral matrices of degree $g-1$ and R runs over the set of all $(g-1)\times h$ integral matrices.

Lemma 4.3. *Let $f\in J_{\rho,\mathcal{M}}(\Gamma_g)$ be a Jacobi form. Then for any $\xi\in\mathbb{C}^{g-1}$,*

$$\Psi^0_{g,g-1}\left(\rho\begin{pmatrix}1&0\\\xi&E_{g-1}\end{pmatrix}f\right)=\Psi^0_{g,g-1}f\,.$$

Proof. Since ρ is rational, it suffices to show the above formula for *integral* $\xi\in\mathbb{Z}^{g-1}$. For convenience, we put

$$U=\begin{pmatrix}1&0\\\xi&E_{g-1}\end{pmatrix},\quad \xi\in\mathbb{Z}^{g-1}.$$

Then $M_U:=\begin{pmatrix}{}^tU^{-1}&0\\\xi&E_{g-1}\end{pmatrix}$ is an element in Γ_g. Since $f\in J_{\rho,\mathcal{M}}(\Gamma_g)$, we have $f|_{\rho,\mathcal{M}}[M_U]=f$ and hence

$$f(Z[U^{-1}],WU^{-1})=\rho(U)f(Z,W).$$

Thus we have

$$\begin{aligned}(\Psi^0_{g,g-1}(\rho(U)f))(Z,W)&=\lim_{t\to\infty}f\left(\begin{pmatrix}it+Z[\xi]&-{}^t\xi Z\\-Z\xi&Z\end{pmatrix},(-W\xi,W)\right)\\&=(\Psi^0_{g,g-1}f)(Z,W).\end{aligned}$$

Hence this completes the proof of the above lemma. □

Let $L:=\Gamma_gM\in\check{\mathscr{L}}_{g,p}$ $(M\in O_{g,p})$ be fixed, where M is of the form (4.2). We write $v:=v(M)=p^{k_0}$. Then we have

$$(f|L)(\tilde Z,\tilde W)=\rho(D)^{-1}f(\frac{1}{v}(\tilde Z[{}^tA]+A{}^tB),\tilde W{}^tA),$$

where $(\tilde Z,\tilde W)\in H_g\times\mathbb{C}^{(h,g)}$.

Therefore we have

$$\begin{aligned}&(\Psi^0_{g,g-1}(f|L))(Z,W)\\=\;&\rho(D)^{-1}\lim_{t\to\infty}f\left(\frac{1}{v}\begin{pmatrix}ita^2+Z[\alpha]&{}^t\alpha Z{}^tA^*\\A^*Z\alpha&Z[{}^tA^*]\end{pmatrix}+BD^{-1},(W\alpha,W{}^tA^*)\right)\\=\;&\rho(D)^{-1}(\Psi^0_{g,g-1}f)(\frac{1}{v}(Z[{}^tA^*]+B^{*t}A^*),W{}^tA^*).\end{aligned}$$

And we have

$$\begin{aligned} & d^r((\Psi^0_{g,g-1}f)|(\Gamma_g M)^*)(Z,W) \\ = \ & \rho\begin{pmatrix} 1 & 0 \\ 0 & D^* \end{pmatrix}(\Psi^0_{g,g-1}f)(\frac{1}{v}(Z[{}^tA^*]+B^{*t}A^*), W{}^tA^*). \end{aligned}$$

According to Lemma 4.3, we may take

$$D = \begin{pmatrix} d & 0 \\ 0 & D^* \end{pmatrix}.$$

Thus we have

$$(\Psi^0_{g,g-1}(f|L))(Z,W) = \rho\begin{pmatrix} d & 0 \\ 0 & D^* \end{pmatrix}\cdot\rho\begin{pmatrix} 1 & 0 \\ 0 & D^* \end{pmatrix}((\Psi^0_{g,g-1}f)|(\Gamma_{g-1}M^*))(Z,W).$$

Finally according to the assumption (c) in Theorem 4.2, we obtain

$$R(\Psi^0_{g,g-1}(f|(\Gamma_g M))) = R(\Psi^0_{g,g-1})|(\Gamma_g M)^*.$$

Hence for any $T \in \check{\mathscr{H}}_{g,p}$, we have

$$R(\Psi^0_{g,g-1}(f|T)) = R(\Psi^0_{g,g-1}f)|T^*.$$

This completes the proof of Theorem 4.2. □

References

[1] A.N. ANDRIANOV, Quadratic Forms and Hecke Operators, Springer-Verlag (1987).

[2] M. EICHLER and D. ZAGIER, The Theory of Jacobi Forms, Progress in Math. **55**, Birkhäuser, Boston-Basel-Stuttgart (1985).

[3] E. FREITAG, Thetareihen mit harmonischen Koeffizienten zur Siegelschen Modulgruppe, Math. Ann. **254** (1980), 27–51.

[4] E. FREITAG, Die Wirkung von Heckeoperatoren auf Thetareihen mit harmonischen Koeffizienten, Math. Ann. **258** (1982), 419–440.

[5] E. FREITAG, Siegelsche Modulfunktionen, Springer-Verlag, Heidelberg-Berlin-New York (1983).

[6] H. MAASS, Über die Darstellung der Modulformen n-ten Grades durch Poincarésche Reihen, Math. Ann. **135** (1961), 125–151.

[7] H. MAASS, Siegel Modular Forms and Dirichlet Series, Lecture Notes in Math. **216**, Springer-Verlag (1971).

[8] G. SHIMURA, On Modular Forms of Half Integral Weight, Ann. Math. **97** (1973), 440–481.

[9] G. SHIMURA, On Certain Reciprocity Laws for Theta Functions and Modular Forms, Acta Math. **141** (1979), 35–71.

[10] R. WEISSAUER, Vektorwertige Siegelsche Modulformen kleinen Gewichtes, J. reine angew. Math. **343** (1983), 184–202.

[11] R. Weissauer, Stabile Modulformen und Eisensteinreihen, Lecture Notes in Math. **1219**, Springer-Verlag (1986).

[12] T. Yamazaki, Jacobi Forms and a Maass Relation for Eisenstein Series, J. Fac. Sci. Uni. Tokyo, Sec. IA. Math. **33** (1986), 295–310.

[13] N.A. Zharkovskaya, The Siegel Operator and Hecke Operators, Functional Anal. Appl. **8** (1974), 113–120.

[14] C. Ziegler, Jacobi Forms of Higher Degree, Abh. Math. Sem. Univ. Hamburg **59** (1989), 191–224.

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VANISHING THEOREMS ON JACOBI FORMS OF HIGHER DEGREE

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1. Introduction

For any positive integer $g \in \mathbf{Z}^+$, we let

$$H_g := \{\, Z \in \mathbf{C}^{(g,g)} \,|\, {}^tZ = Z, \quad Im\, Z > 0 \,\}$$

be the Siegel upper half plane of degree g and $\Gamma_g := Sp(g, \mathbf{Z})$ be the Siegel modular group of degree g. Let ρ be an irreducible finite dimensional representation of $GL(g, \mathbf{C})$ and $\mathcal{M}$ be a symmetric half integral positive definite matrix of degree h. It is known ([Z] Theorem 1.8) that the vector space $J_{\rho,\mathcal{M}}(\Gamma_g)$ of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ_g is finite dimensional. For the precise definition of $J_{\rho,\mathcal{M}}(\Gamma_g)$, we refer to Definition 2.2. It is a natural question to ask under which conditions the vector space $J_{\rho,\mathcal{M}}(\Gamma_g)$ vanishes. In this paper, the author gives a vanishing theorem on $J_{\rho,\mathcal{M}}(\Gamma_g)$.

In section 2, we establish the notations and review some properties of Jacobi forms. In section 3, we define the Siegel-Jacobi operator and give the relation between the corank of a Jacobi form and the corank of ρ. In the final section, we establish the Shimura isomorphism and give a vanishing theorem on Jacobi forms using this isomorphism and the vanishing theorem on Siegel modular forms ([W] Satz 2).

Notations:. We denote by $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ the ring of integers, the field of real numbers, and the field of complex numbers respectively. For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbf{R})$ and $Z \in H_g$, we set $M < Z > := (AZ + B)(CZ + D)^{-1}$. $\Gamma_g := Sp(g, \mathbf{Z})$ denotes the Siegel modular group of

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degree g. $[\Gamma_g, k]$ (*resp.* $[\Gamma_g, \rho]$) denotes the vector space of all Siegel modular forms of weight k (*resp. of type* ρ). The symbol ":=" means that the expression on the right is the definition of that on the left. We denotes by $\mathbf{Z}^+$ the set of all positive integers. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. E_n denotes the identity matrix of degree n.

2. Jacobi Forms

In this section, we establish the notations and review some properties of Jacobi forms.

For two positive integers g and h, we consider the *Heisenberg group*

$$H_{\mathbf{R}}^{(g,h)} := \{\, [(\lambda,\mu),\kappa] \mid \lambda,\, \mu \in \mathbf{R}^{(h,g)},\ \kappa \in \mathbf{R}^{(h,h)},\ \kappa + \mu\,{}^t\lambda \ \textit{symmetric} \}$$

endowed with the following multiplication law

$$[(\lambda,\mu),\kappa] \circ [(\lambda',\mu'),\kappa'] \ := \ [(\lambda+\lambda',\mu+\mu'),\kappa+\kappa'+\lambda\,{}^t\mu' - \mu\,{}^t\lambda'].$$

We define the semidirect product of $Sp(g,\mathbf{R})$ and $H_{\mathbf{R}}^{(g,h)}$

$$G_{\mathbf{R}}^{(g,h)} := Sp(g,\mathbf{R}) \ltimes H_{\mathbf{R}}^{(g,h)} \tag{2.1}$$

endowed with the following multiplication law

$$\begin{aligned}(M,[(\lambda,\mu),\kappa])\cdot(M',[(\lambda',\mu'),\kappa'])\\ := (MM',[(\tilde{\lambda}+\lambda',\tilde{\mu}+\mu'),\kappa+\kappa'+(\tilde{\lambda}\,{}^t\mu' - \tilde{\mu}\,{}^t\lambda')]),\end{aligned}$$

with $M, M' \in Sp(g,\mathbf{R})$ and $(\tilde{\lambda},\tilde{\mu}) := (\lambda,\mu)M'$. The group $G_{\mathbf{R}}^{(g,h)} := Sp(g,\mathbf{R}) \ltimes H_{\mathbf{R}}^{(g,h)}$ is called the ***Jacobi group***. It is easy to see that $G_{\mathbf{R}}^{(g,h)}$ acts on $H_g \times \mathbf{C}^{(h,g)}$ transitively by

(2.3) $(M,[(\lambda,\mu),\kappa])\cdot(Z,W) := (M<Z>,(W+\lambda Z+\mu)(CZ+D)^{-1})$,

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbf{R})$, $(Z,W) \in H_g \times \mathbf{C}^{(h,g)}$.

Let ρ be a rational representation of $GL(g,\mathbf{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbf{R}^{(h,h)}$ be a symmetric half integral matrix of degree h. The *canonical automorphy factor* $I_{\rho,\mathcal{M}}$ for the action of $G_{\mathbf{R}}^{(g,h)}$ on $H_g \times \mathbf{C}^{(h,g)}$ is given by

$$I_{\rho,\mathcal{M}}(\hat{M},(Z,W)) := e^{-2\pi i\sigma(\mathcal{M}[W+\lambda Z+\mu](CZ+D)^{-1}C)} \times e^{2\pi i\sigma(\mathcal{M}(\lambda Z^t\lambda+2\lambda^t W+(\kappa+\mu^t\lambda)))}\rho(CZ+D)^{-1},$$

where $\hat{M} = (M,[(\lambda,\mu),\kappa])$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbf{R})$.
We denote by $C^\infty(H_g \times \mathbf{C}^{(h,g)}, V_\rho)$ the space of all smooth functions with values in V_ρ defined on $H_g \times \mathbf{C}^{(h,g)}$. Then we obtain an action of $G_{\mathbf{R}}^{(g,h)}$ on the space $C^\infty(H_g \times \mathbf{C}^{(h,g)}, V_\rho)$ by putting

(2.5) $$(f|_{\rho,\mathcal{M}}[\hat{M}])(Z,W) := I_{\rho,\mathcal{M}}(\hat{M},(Z,W))\, f(\hat{M}\cdot(Z,W)),$$

where $f \in C^\infty(H_g \times \mathbf{C}^{(h,g)})$.

A straightforward calculation yields the following.

LEMMA 2.1. *Let* $g_i = (M_i,[(\lambda_i,\mu_i),\kappa_i]) \in G_{\mathbf{R}}^{(g,h)}$ $(i=1,2)$. *For any* $f \in C^\infty(H_g \times \mathbf{C}^{(h,g)}, V_\rho)$, *we have*

(2.6) $$(f|_{\rho,\mathcal{M}}[g_1])|_{\rho,\mathcal{M}}[g_2] = f|_{\rho,\mathcal{M}}[g_1 g_2].$$

DEFINITION 2.2. Let ρ and $\mathcal{M}$ be as above. Let

$$H_{\mathbf{Z}}^{(g,h)} := \{\, [(\lambda,\mu),\kappa] \in H_{\mathbf{R}}^{(g,h)} \mid \lambda,\mu \in \mathbf{Z}^{(h,g)},\ \kappa \in \mathbf{Z}^{(h,h)} \,\}.$$

A *Jacobi form* of index $\mathcal{M}$ with respect to ρ on Γ is a holomorphic function $f \in C^\infty(H_g \times \mathbf{C}^{(h,g)}, V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho,\mathcal{M}}[\tilde{\gamma}] = f$ for all $\tilde{\gamma} \in \Gamma^J := \Gamma \propto H_{\mathbf{Z}}^{(g,h)}$.
(B) f has a Fourier expansion of the following form :

$$f(Z,W) = \sum_{\substack{T\geq 0 \\ \text{half-integral}}} \sum_{R\in \mathbf{Z}^{(g,h)}} c(T,R)\, e^{\frac{2\pi i}{\lambda_\Gamma}\sigma(TZ)} \cdot e^{2\pi i \sigma(RW)}$$

with $c(T,R) \neq 0$ only if $\begin{pmatrix} \frac{1}{\lambda_\Gamma}T & \frac{1}{2}R \\ \frac{1}{2}\,{}^tR & \mathcal{M} \end{pmatrix} \geq 0.$

If $g \geq 2$, the condition (B) is superfluous by Koecher principle(see [Z] Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ. In the special case $V_\rho = \mathbf{C},\ \rho(A) = (\det A)^k\,(k \in \mathbf{Z},\ A \in GL(g,\mathbf{C}))$, we write $J_{k,\mathcal{M}}(\Gamma)$ instead of $J_{\rho,\mathcal{M}}(\Gamma)$ and call k the *weight* of a Jacobi form $f \in J_{k,\mathcal{M}}(\Gamma)$.

Ziegler([Z] Theorem 1.8 or [E-Z] Theorem 1.1) proves that the vector space $J_{\rho,\mathcal{M}}(\Gamma_g)$ is finite dimensional.

From now on, we assume that Γ is a normal subgroup of Γ_g of finite index. If $M \in \Gamma_g$, then $\Gamma^M := M^{-1}\Gamma M$ is a subgroup of Γ_g of finite index. It is easy to show that if $f \in J_{\rho,\mathcal{M}}(\Gamma)$, then $f|_{\rho,\mathcal{M}}[M] \in J_{\rho,\mathcal{M}}(\Gamma^M)$. Thus $f|_{\rho,\mathcal{M}}[M]$ has the Fourier expansion of the form

$$(2.7) \qquad (f|_{\rho,\mathcal{M}}[M])(Z,W) = \sum_{T,R} c_M(T,R)\, e^{\frac{2\pi i}{\lambda}\sigma(TZ)} \cdot e^{2\pi i\sigma(RW)},$$

where T runs over the set of all semipositive half integral matrices of degree g, R runs over the set of $g \times h$ integral matrices and $\lambda = \lambda_{\Gamma^M} \in \mathbf{Z}$ is a suitable integer.

DEFINITION 2.3. Let ρ be an irreducible finite dimensional representation of $GL(g,\mathbf{C})$. Then ρ is determined uniquely by its highest weight $(\lambda_1,\cdots,\lambda_g) \in \mathbf{Z}^g$ with $\lambda_1 \geq \cdots \geq \lambda_g$. We denote this representation by $\rho = (\lambda_1,\cdots,\lambda_g)$. The number $k(\rho) := \lambda_g$ is called the *weight* of ρ. The number of λ_i's such that $\lambda_i = k(\rho) = \lambda_g\,(1 \leq i \leq g)$ is called the *corank* of ρ, denoted by *corank* (ρ).

DEFINITION 2.4. Let $f \in J_{\rho,\mathcal{M}}(\Gamma)$ be a nonvanishing Jacobi form of index $\mathcal{M}$ with respect to ρ on Γ. We define the *corank* of f as follows:

$$corank\,(f) := g - min_T\,(rank\,(T)),$$

where T runs over the set of all semipositive half integral matrices of degree g such that $c_M(T,R) \neq 0$ for at least one $M \in \Gamma_g$.

Let $T = (t_{ij})$ be a semipositive symmetric matrix of degree g. We write $r(T) = d$ if $t_{g-d,g-d}$ is the last diagonal element distinct from zero. Since $T \geq 0$, T must be of the form

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_1 \geq 0, \quad T_1 \in \mathbf{R}^{(g-d,g-d)}. \tag{2.8}$$

We note that T_1 is not invertible in general.

LEMMA 2.5. *Let $0 \neq f \in J_{\rho,\mathcal{M}}(\Gamma)$. Then for all T with $c_M(T,R) \neq 0$, we have $r(T) \leq corank\,(f)$. There exists $M \in \Gamma_g$ and T with $c_M(T,R) \neq 0$ such that*

$$r(T) = corank\,(\rho).$$

The proof of the above lemma is obvious.

3. The Siegel-Jacobi Operator

In this section, we define the Siegel-Jacobi operator and give the relation between the corank of a Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma)$ and that of ρ using the Siegel-Jacobi operator.

Let $\rho : GL(g,\mathbf{C}) \longrightarrow GL(V_\rho)$ be an irreducible rational representation of $GL(g,\mathbf{C})$ on a finite dimensional complex vector space V_ρ. Let $0 \leq r \leq g-1$. Now for a Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma)$ defined on $H_g \times \mathbf{C}^{(h,g)}$, we define $\Psi_{g,r} f \in \mathcal{O}(H_r \times \mathbf{C}^{(h,r)}, V_\rho)$ by

$$(\Psi_{g,r}f)(Z,W) := \lim_{t\to\infty} f\left(\begin{pmatrix} Z & 0 \\ 0 & itE_{g-r} \end{pmatrix}, (W,0)\right), \tag{3.1}$$

where $Z \in H_r$ and $W \in \mathbb{C}^{(h,r)}$. We note that the above limit (3.1) always exists because a Jacobi form f admits a Fourier expansion converging uniformly on any set of the form

$$\{(Z,W) \in H_g \times \mathbb{C}^{(h,g)} \,|\, Im\, Z \geq Y_0 > 0,\ W \in K \subset \mathbb{C}^{(h,g)}\ compact\}.$$

The operator $\Psi_{g,r}$ is called the *Siegel-Jacobi operator.*

As before, we assume that Γ is a normal subgroup of Γ_g of finite index. For $M \in \Gamma_g$, $\Gamma^M := M^{-1}\Gamma M$ is a subgroup of Γ_g of finite index. If $f \in J_{\rho,\mathcal{M}}(\Gamma)$, then $f|_{\rho,\mathcal{M}}[M]$ has a Fourier expansion of the form (2.7). Let $c_M(T,R)$ be a Fourier coefficient of $f|_{\rho,\mathcal{M}}[M]$ with $r(T) = g - r$. Let $\mathcal{U}$ be the subgroup of $GL(g,\mathbb{Z})$ consisting of $U \in GL(g,\mathbb{Z})$ such that

$$M_U = \begin{pmatrix} {}^tU^{-1} & 0 \\ 0 & U \end{pmatrix} \in \Gamma^M \subset \Gamma_g.$$

Since $(f|_{\rho,\mathcal{M}}[M])|_{\rho,\mathcal{M}}[M_U] = f|_{\rho,\mathcal{M}}[M]$, applying the Fourier expansion (2.7), we have

$$c_M(UT\,{}^tU, UR) = \rho(U)\, c_M(T,R). \tag{3.2}$$

For $k = 1, \cdots, g$, we let

$$G_{g,k} := \left\{ \begin{pmatrix} E_{g-k} & * \\ 0 & * \end{pmatrix} \in GL(g,\mathbb{C}) \right\}.$$

Then $G_{g,k} = GL(k,\mathbb{C}) \propto N$, where N is a unipotent radical of the group $G_{g,k}$. Then for any $U \in G_{g,g-r}$, we have

$$UT\,{}^tU = T, \quad UR = R. \tag{3.3}$$

Indeed, T is of the form (2.8). Since $\begin{pmatrix} \frac{1}{\lambda}T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} \geq 0$ with $\lambda = \lambda_{\Gamma^M}$, R is of the form

$$R = \begin{pmatrix} R_1 \\ 0 \end{pmatrix}, R_1 \in \mathbb{Z}^{(r,h)}.$$

According to (3.3), we obtain

$$c_M(T,R) = \rho(U)\, c_M(T,R),\ U \in \mathcal{U} \cap G_{g,g-r}. \tag{3.4}$$

We observe that the Zariski closure of $\mathcal{U} \cap G_{g,g-r}$ in $G_{g,g-r}$ contains the subgroup $G(g-r) := SL(g-r, \mathbf{C}) \propto N$. Thus $c_M(T,R)$ is invariant under all $U \in G(g-r)$ in the sense of (3.4). For $k = 1, \cdots, g$, we put

$$V_\rho^{G(k)} := \{, v \in V_\rho, |\rho(g)v = v \text{ for all } g \in G(k), \} \tag{3.5}$$

Here $G(k) := SL(k, \mathbf{C}) \propto N$. Then according to [W], we have

$$V_\rho^{G(k)} \cong \begin{cases} (\lambda_1, \cdots, \lambda_{g-k}) & if corank, (\rho) \geq k, \\ 0 & otherwise. \end{cases} \tag{3.6}$$

That is, if $V_\rho^{G(k)} \neq 0, V_\rho^{G(k)}$ is an irreducible finite dimensional representation of $GL(g-k, \mathbf{C})$.

Let $V_\rho^{(r)}$ be the subspace of V_ρ generated by the values $\{\, \Psi_{g,r} f(Z,W) \mid f \in J_{\rho,\mathcal{M}}(\Gamma),\ \ (Z,W) \in H_g \times \mathbf{C}^{(h,g)} \,\}$. If $V_\rho^{(r)} \neq 0$, according to (3.1), (3.4) and (3.6),

$$V_\rho^{G(g-r)} = V_\rho^{(r)}. \tag{3.7}$$

Thus $V_\rho^{(r)}$ is invariant under

$$\left\{ \left(\begin{pmatrix} g & 0 \\ 0 & E_{g-r} \end{pmatrix} \right) : g \in GL(r, \mathbf{C}) \right\}.$$

Then we have a rational representation $\rho^{(r)}$ of $GL(r, \mathbf{C})$ on $V_\rho^{(r)}$ defined by

$$\rho^{(r)}(g)v := \rho\left(\begin{pmatrix} g & 0 \\ 0 & E_{g-r} \end{pmatrix} \right) v,\ \ g \in GL(r, \mathbf{C}),\ \ v \in V_\rho^{(r)}. \tag{3.8}$$

So far we have proved

LEMMA 3.1. *Let ρ be irreducible. Then $(\rho^{(r)}, V_\rho^{(r)})$ is an irreducible finite dimensional representation of $GL(r, \mathbb{C})$.*

For all $0 \neq c_M(T,R) \in V_\rho^{G(g-r)}$, we have $r(T) \leq corank\,(\rho)$ by (3.6). By Lemma 2.5, we have $corank\,(f) \leq corank\,(\rho)$.

Thus we have

THEOREM 3.2. *Let $0 \neq f \in J_{\rho,\mathcal{M}}(\Gamma)$ be a nonvanishing Jacobi form of index $\mathcal{M}$ with respect to ρ on Γ. Then we have*

$$corank\,(f) \leq corank\,(\rho).$$

For more results on the Siegel-Jacobi operators, we refer to [Y1] or [Y2].

4. Vanishing Theorems

In this section, we establish the Shimura isomorphism and using this isomorphism we prove a vanishing theorem.

Let S be a symmetric, positive definite integral matrix of degree h and let $a, b \in \mathbb{Q}^{(h,g)}$. We consider

$$\vartheta_{S,a,b}(Z,W) := \sum_{\lambda \in Z^{(h,g)}} e^{\pi i \sigma(S((\lambda+a)Z\,{}^t(\lambda+a)+2(\lambda+a)\,{}^t(W+b)))} \tag{4.1}$$

with characteristic (a,b) converging uniformly on any compact subset of $H_g \times \mathbb{C}^{(h,g)}$.

Let $\mathcal{M}$ be a symmetric, positive definite and half-integral matrix of degree h and let $\mathcal{N}$ be a complete system of representatives of the cosets $(2\mathcal{M})^{-1}\mathbb{Z}^{(h,g)}/\mathbb{Z}^{(h,g)}$. We observe that $\#(\mathcal{N}) = \{det(2\mathcal{M})\}^g$. An easy application of the Poisson summation formula gives

LEMMA 4.1. *For $a \in \mathcal{N}$, we have*

$$\begin{aligned}(4.2)\quad & \vartheta_{2\mathcal{M},a,0}(-Z^{-1}, WZ^{-1}) \\ & = \{det(2\mathcal{M})\}^{-\frac{g}{2}} \left\{det\left(\frac{Z}{i}\right)\right\}^{\frac{h}{2}} e^{2\pi i\sigma(\mathcal{M}WZ^{-1}\,{}^tW)} \\ & \times \sum_{b\in\mathcal{N}} e^{-2\pi i\sigma(2\mathcal{M}b\,{}^ta)}\, \vartheta_{2\mathcal{M},b,0}(Z,W).\end{aligned}$$

Here we denote by

$$h(Z) := \sqrt{det\left(\frac{Z}{i}\right)}$$

the unique holomorphic function on H_g satisfying the following properties

$$\text{(a)}\qquad h(Z)^2 = det\left(\frac{Z}{i}\right)$$

$$\text{(b)}\qquad h(iE_g) = +1$$

and for any integer $r \in \mathbf{Z}$, we put

$$\left\{det\left(\frac{Z}{i}\right)\right\}^{\frac{r}{2}} := h(Z)^r = \left\{\sqrt{det\left(\frac{Z}{i}\right)}\right\}^r.$$

COROLLARY 4.2. *Let $2\mathcal{M}$ be unimodular. Then $\vartheta_{2\mathcal{M},0,0}(Z,W)$ is a Jacobi form of weight $\frac{h}{2}$ and index $\mathcal{M}$.*

We fix an element $Z_0 \in H_g$. We denote by $T_{\mathcal{M}}(Z_0)$ the vector space of all holomorphic functions $\varphi : \mathbf{C}^{(h,g)} \longrightarrow \mathbf{C}$ satisfying the condition

$$(4.3)\qquad \varphi(W + \lambda Z_0 + \mu) = e^{-2\pi i\sigma(\mathcal{M}(\lambda Z_0\,{}^t\lambda + 2\lambda\,{}^tW))}\varphi(W)$$

for every $\lambda, \mu \in \mathbf{Z}^{(h,g)}$. Then it is easy to show that the functions

$$(4.4)\qquad \{\,\vartheta_{2\mathcal{M},a,0}(Z_0,\, W)\,|\, a \in \mathcal{N}\,\}$$

form a basis of $T_{\mathcal{M}}(Z_0)$ and its dimension is clearly $\{det\,(2\mathcal{M})\}^g$ (cf. J. Igusa [I]). Let $\rho : GL(g,\mathbb{C}) \longrightarrow GL(V_\rho)$ be an irreducible rational representation of $GL(g,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. If f is a Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma)$, it is easy to see that each component of $\phi(W) := f(Z_0, W)$ satisfies the relation (4.3). So we may write

$$(4.5)\quad f(Z,W) = \sum_{a\in\mathcal{N}} f_a(Z)\cdot\vartheta_{2\mathcal{M},a,0}(Z,W),\quad Z\in H_g,\quad W\in\mathbb{C}^{(h,g)},$$

where $\mathcal{N}$ is a complete system of representatives of the cosets $(2\mathcal{M})^{-1}\mathbb{Z}^{(h,g)}/\mathbb{Z}^{(h,g)}$ and $\{f_a : H_g \longrightarrow V_\rho \,|\, a\in\mathcal{N}\}$ are uniquely determined holomorphic functions on H_g.

LEMMA 4.3. *Each $f_a(Z)\,(a\in\mathcal{N})$ is holomorphic.*

Proof. Since f and $\vartheta_{2\mathcal{M},a,0}(Z,W)\,(a\in\mathcal{N})$ are holomorphic,

$$\sum_{a\in\mathcal{N}} \frac{\partial f_a(Z)}{\partial \bar{Z}_{ij}}\cdot\vartheta_{2\mathcal{M},a,0}(Z,W) = 0,\quad Z=(Z_{ij}) = {}^tZ.$$

Since $\vartheta_{2\mathcal{M},a,0}(Z,W)(a\in\mathcal{N})$ form a basis of $T_{\mathcal{M}}(Z)$ as functions on $\mathbb{C}^{(h,g)}$, $\frac{\partial f_a(Z)}{\partial \bar{Z}_{ij}} = 0$ for all $Z\in H_g$ and $a\in\mathcal{N}$. Hence each $f_a(Z)\,(a\in\mathcal{N})$ is holomorphic.

According to Lemma 4.1, we have

(4.6)
$$f_a(-Z^{-1}) = \left\{det,\left(\frac{Z}{i}\right)\right\}^{-\frac{h}{2}}\cdot\{\rho(-Z)\}\cdot\{det,(2\mathcal{M})\}^{-\frac{g}{2}} \times \sum_{b\in\mathcal{N}} e^{2\pi i\sigma(2\mathcal{M}a^tb)}\cdot f_b(Z)$$

and

$$(4.7)\qquad f_a(Z+S) = e^{-2\pi i\sigma(\mathcal{M}aS^ta)}\cdot f_a(Z), S = {}^tS\in\mathbb{Z}^{(g,g)}.$$

We note that the Fourier coefficients $c(T,R)$ of $\vartheta_{2\mathcal{M},a,0}(Z,W)$ are given by

$$c(T,R) = \begin{cases} 1 & if\ ^{\exists}\lambda \in \mathbf{Z}^{(h,g)} s.t. \\ & {}^t(\lambda + a, E_h)\mathcal{M}(\lambda + a, E_h) = \begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}\,{}^tR & \mathcal{M} \end{pmatrix} \\ 0 & otherwise \end{cases}$$

We observe that $c(T,R) \neq 0$ implies $4T - R\mathcal{M}^{-1}\,{}^tR = 0$. By an easy argument, we see that the functions $\{f_a | a \in \mathcal{N}\}$ must have the Fourier expansions of the form

$$f_a(Z) = \sum_{\substack{T={}^tT\geq 0 \\ half\ integral}} c(T) \cdot e^{2\pi i\sigma(TZ)} \tag{4.8}$$

Conversely, suppose there is given a family $\{f_a | a \in \mathcal{N}\}$ of holomorphic functions $f_a : H_g \longrightarrow V_\rho$ satisfying the transformation laws (4.6), (4.7) and the cusp condition (4.8). Then we obtain a Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma_g)$ by defining $f(Z,W)$ via the equation (4.5). So far we have proved the Shimura isomorphism:

THEOREM 1 (SHIMURA). *The equation (4.5) gives an isomorphism between $J_{\rho,\mathcal{M}}(\Gamma_g)$ and the vector space of V_ρ-valued Siegel modular forms of half integral weight satisfying the transformation laws (4.6), (4.7) and the cusp condition (4.8).*

REMARK 4.4. Theorem 1 may be also formulated for Jacobi forms on a subgroup $\Gamma \subset \Gamma_g$ of finite index.

COROLLARY 4.5.. *If $2k < rank(\mathcal{M})$, then we have $J_{k,\mathcal{M}}(\Gamma) = 0$.*

Proof. The proof follows from the fact that the irreducible representation $(det)^{k-\frac{1}{2}rank(\mathcal{M})}$ of $GL(g,\mathbf{C})$ is not a polynomial representation. *q.e.d.*

COROLLARY 4.6. *Let* $2\mathcal{M}$ *be unimodular and* $k\cdot g$ *be odd. Then* $J_{k,\mathcal{M}}(\Gamma_g)=0$.

Proof. It follows immediately from (4.6) and the fact that $h\equiv 0\,(mod\,8)$. *q.e.d.*

COROLLARY 4.7. *Let* $2\mathcal{M}$ *be unimodular. We assume that* ρ *satisfies the following condition (4.9):*

$$\rho(A)=\rho(-A)\quad \text{for all } A\in GL(g,\mathbb{C}). \tag{4.9}$$

Then we have

$$J_{\rho,\mathcal{M}}(\Gamma)=[\Gamma,\tilde\rho]\cdot\vartheta_{2\mathcal{M},0,0}(Z,W)\cong[\Gamma,\tilde\rho], \tag{4.10}$$

where $\tilde\rho=\rho\otimes det^{-\frac{h}{2}}$. *In particular, if* $k\cdot g$ *is even,*

$$J_{k,\mathcal{M}}(\Gamma)=[\Gamma,k-\frac{h}{2}]\cdot\vartheta_{2\mathcal{M},0,0}(Z,W)\cong[\Gamma,k-\frac{h}{2}]. \tag{4.11}$$

Proof. The proof of (4.10) follows from (4.6), (4.7) and (4.8). The representation $det^k:GL(g,\mathbb{C})\longrightarrow\mathbb{C}^\times$ defined by $det^k(A)=(det\,(A))^k$ satisfies the condition (4.9). Hence (4.11) follows from (4.12). *q.e.d.*

EXAMPLE 4.8. We give several examples of the irreducible representations which satisfies the condition (4.9).

(a) If $k\cdot g$ is even, then the polynomial representation $\rho:GL(g,\mathbb{C})\longrightarrow\mathbb{C}^\times$ defined by $\rho(A):=(det\,A)^k\ (A\in GL(g,\mathbb{C}))$ satisfies the condition (4.9).

(b) The polynomial representation ρ of $GL(g,\mathbb{C})$ on the symmetric product $Symm^2(\mathbb{C}^g)$ of $\mathbb{C}^g$ defined by

$$\rho(A)Z:=AZ\,{}^tA,\quad A\in GL(g,\mathbb{C}),\quad Z\in Symm^2(\mathbb{C}^g)$$

satisfies the condition (4.9). It is obvious that ρ is irreducible. This representation is important geometrically because it is related with holomorphic 1-forms on H_g invariant under Γ_g.

(c) The polynomial representation ρ of $GL(g,\mathbf{C})$ on $Symm^2(\mathbf{C}^g)$ defined by

$$\rho(A)Z := (det\,A)^{g+1}A^{-1}Z\,{}^tA^{-1}, \quad A \in GL(g,\mathbf{C}), \quad Z \in Symm^2(\mathbf{C}^g)$$

satisfies the condition (4.9). It is easy to see that ρ is irreducible. This representation is also important geometrically because it is connected with holomorphic $(N-1)$-forms on H_g invariant under Γ_g, where $N = \frac{g(g+1)}{2} - 1$.

Now we prove a vanishing theorem on Jacobi forms.

THEOREM 2. *Let $2\mathcal{M}$ be an even unimodular positive definite matrix of degree h. Let $\rho = (\lambda_1, \cdots, \lambda_g)$ be an irreducible finite dimensional representation of $GL(g,\mathbf{C})$. Let $\lambda(\rho)$ be the number of λ_i's such that $\lambda_i = k(\rho)+1 = \lambda_g + 1,\ 1 \leq i \leq g$. Assume that ρ satisfies the following conditions:*

(a) ρ satisfies the condition (4.9);

(b) $\lambda(\rho) < 2\,(g - k(\rho) - corank\,(\rho)) + rank\,(\mathcal{M})$.

Then $J_{\rho,\mathcal{M}}(\Gamma_g) = 0$.

Proof. It is easily seen that $corank\,(\rho \otimes det^{-\frac{h}{2}}) = corank\,(\rho)$ and $\lambda(\rho \otimes det^{-\frac{h}{2}}) = \lambda(\rho)$. According to [W] Satz 2, we have $[\Gamma_g, \rho \otimes det^{-\frac{h}{2}}] = 0$. By corollary 4.7, we have $J_{\rho,\mathcal{M}}(\Gamma_g) = 0$. *q.e.d.*

COROLLARY 4.9. *Let $2\mathcal{M}$ be as above in Theorem 2. Assume that $2k(\rho) \leq g + rank\,(\mathcal{M}) - 2\,corank\,(\rho)$. Then $J_{\rho,\mathcal{M}}(\Gamma_g) = 0$.*

Proof. It follows immediately from Theorem 2 and the fact that $\lambda(\rho)$ is less than g. *q.e.d.*

References

[E-Z]. Eichler, M. and Zagier, D., *The Theory of Jacobi forms, Progress in Math. 55*, Birkhäuser, Boston-Basel-Stuttgart (1985).

[I]. J. Igusa, *Theta Functions*, Springer-Verlag (1972).

[S1]. G. Shimura, *On modular forms of half integral weight*, Ann. of Math. **97** (1973), 440–481.

[S2]. G. Shimura, *On certain reciprocity laws for theta functions and modular forms*, Acta Math. **141** (1979), 35–71.

[W]. R. Weissauer, *Vektorwertige Siegelsche Modulformen kleinen Gewichtes*, Jour. reine angew. Math. **343** (1983), 184–202.

[Y1]. J.-H. Yang, *The Siegel-Jacobi Operators*, to appear in Abh. Math. Sem. Univ. Hamburg **63** (1993).

[Y2]. J.-H. Yang, *Some Results on Jacobi Forms of Higher Degree*, to appear in the Proceedings of RIMS Symposium on Automorphic Forms and associated Zeta Functions, Kyoto (1992), 36–50.

[Z]. C. Ziegler, *Jacobi Forms of Higher Degree*, Abh. Math. Sem. Univ. Hamburg **59** (1989), 191–224.

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Automorphic Forms and Related Topics

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PREFACE

The workshop on "Automorphic Forms and Related Topics" held in Seoul, Korea during September 2-3, 1993 was organized to stimulate Korean young mathematicians to research more advanced areas in mathematics. This workshop was supported financially by the Pyungsan Institute for Mathematical Sciences which was recently established. The present volume contains 5 papers based on the talks given at Ehwa Women's University in Seoul.

We would like to give our sincere gratitude to all the invited speakers for their enthusiastic talks and their contributed articles. We also would like to thank all the participants who attended this workshop and in particular, the colleagues who helped us in preparing this workshop.

Jin-Woo Son
Jae-Hyun Yang
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  Lifting of supercuspidal representations of $GL(2, F)$
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- **Ichiro Satake :** Chuo University, Tokyo, Japan
  On zeta functions associated with cones and cusp singularities of the second kind
- **Ikuo Satake :** Osaka University, Osaka, Japan
  Flat structure for the simple elliptic singularity of type $\tilde{E}_6$ *and Jacobi forms*
- **Jae-Hyun Yang :** Inha University, Incheon, Korea
  Remarks on Jacobi forms of higher degree

List of Participants
* Invited Speaker

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(2) Y. S. Cho (Ehwa Women's University, Korea)
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REMARKS ON JACOBI FORMS OF HIGHER DEGREE

JAE-HYUN YANG

1. INTRODUCTION

A Jacobi form is an automorphic form on the Jacobi group, which is the semidirect product of the symplectic group $Sp(n,\mathbb{R})$ and the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ (cf. see section 2). Jacobi forms are useful because they are closely related to modular forms of half integral weight. The simplest case is when the symplectic group is $SL(2,\mathbb{R})$ and the Heisenberg group is three dimensional, that is, when $n = m = 1$. This case had been treated more or less systematically in [E-Z] and many papers of Zagier's school. But it seems to us that there is no systematic investigation of Jacobi forms of higher degree when $n > 1$ and $m > 1$. Some results could be found in [Y3]-[Y9] and [Zi].

The purpose of this paper, which is more or less of expository nature, is to provide some recent results of the author on Jacobi forms of higher degree when $n > 1$ and $m > 1$. Jacobi forms of higher degree have some nice properties which are not enjoyed by Jacobi forms of degree 1, e.g., the *singularity* of Jacobi forms. Here we talk about some results on *singular* Jacobi forms, the Siegel-Jacobi operator, construction of modular forms from Jacobi forms and the duality theorem for the Jacobi group. This paper is organized as follows. In section 2, we describe the geometric construction of the canonical automorphic factor for the Jacobi group which is needed in order to define the concept of Jacobi forms. In section 3, we define the concept of Jacobi forms and give some basic properties. In section 4, we investigate theta series and obtain the Shimura isomorphism based on Ziegler's work (cf. [Zi]). In section 5, we define the concept of singular Jacobi forms and introduce an important differential operator which characterizes singular Jacobi forms. Under some condition, we give a criterion that a Jacobi form is singular. In section 6, we discuss the Siegel-Jacobi operator and provide some properties of this operator. In section 7, as application, we construct new vector-valued modular forms from given Jacobi forms and provide some important identities. In section 8, we state the duality theorem for the Jacobi group of higher degree without proof and discuss the invariant theory on the Jacobi group G^J. Finally in section 9, we give some remarks and open problems concerning Jacobi forms.

Notations: We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers, and the field of complex numbers respectively. H_n denotes the Siegel upper half plane of degree

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n. For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $Z \in H_n$, we set $M{<}Z{>} := (AZ+B)(CZ+D)^{-1}$. $\Gamma_n := Sp(n,\mathbb{Z})$ denotes the Siegel modular group of degree n. $[\Gamma_n, k]$ (*resp.* $[\Gamma_n, \rho]$) denotes the vector space of all Siegel modular forms of weight k (respectively of type ρ). The symbol ":=" means that the expression on the right is the definition of that on the left. We denotes by $\mathbb{Z}^+$ the set of all positive integers. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. For $A \in F^{(k,k)}$, $\sigma(A)$ denotes the trace of A. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. E_n denotes the identity matrix of degree n.

Contents

2. The Canonical Automorphic Factor for the Jacobi Group

Let m and n be two fixed positive integers. It is well known that the automorphic group $Aut(H_{m+n})$ of the Siegel upper half plane of degree $m+n$ is given by

$$Aut\,(H_{m+n}) = Sp(m+n,\mathbb{R})/\{\pm E_{m+n}\}.$$

We observe that H_n is a rational boundary of H_{m+n} (cf. [N]). The normalizer $N(H_n) := \{\tilde{\sigma} \in Aut(H_{m+n}) \,:\, \tilde{\sigma}(H_n) \subset H_n\}$ of H_n is given by

$$N(H_n) = P(H_n)/\{\pm E_{m+n}\},$$

where

$$\begin{aligned} P(H_n) :&= \{g \in Sp(m+n,\mathbb{R}) \,:\, g(H_n) \subset H_n\} \\ &= \{[\sigma, u, (\lambda, \mu, \kappa)] \in Sp(m+n,\mathbb{R})\}. \end{aligned}$$

Here we put

$$[\sigma, u, (\lambda, \mu, \kappa)] := \begin{pmatrix} A & 0 & B & A\,{}^t\mu - B\,{}^t\lambda \\ u\lambda & u & u\mu & u\kappa \\ C & 0 & D & C\,{}^t\mu - D\,{}^t\lambda \\ 0 & 0 & 0 & {}^tu^{-1} \end{pmatrix},$$

where $\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$, $u \in GL(m, \mathbb{R})$, $\lambda, \mu \in \mathbb{R}^{(m,n)}$ and $\kappa \in \mathbb{R}^{(m,m)}$.

If $\begin{pmatrix} Z & {}^tW \\ W & T \end{pmatrix} \in H_{m+n}$ with $Z \in H_n$, $W \in \mathbb{R}^{(m,n)}$ and $T \in H_m$, we simply write

$$(Z, W, T) := \begin{pmatrix} Z & {}^tW \\ W & T \end{pmatrix}.$$

We denote the symplectic action of $N(H_n)$ on (Z, W, T) by

$$g \cdot (Z, W, T) := (\tilde{Z}, \tilde{W}, \tilde{T}), \qquad g \in N(H_n).$$

It is easy to see that $(\tilde{Z}, \tilde{W}, \tilde{T})$ is of the form

$$\begin{aligned} \tilde{Z} &= \sigma_g(Z), \\ \tilde{W} &= a(g; Z)(W) + b(g; Z), \\ \tilde{T} &= m_g(T) + c(g; Z, W), \end{aligned}$$

where $\sigma_g \in Aut(H_n)$, $m_g \in Aut(\mathcal{P}_m)$,

$$\begin{aligned} a(g; \cdot) &: H_n \longrightarrow GL(\mathbb{C}^{(m,n)}) \quad \text{holomorphic}, \\ b(g; \cdot) &: H_n \longrightarrow \mathbb{C}^{(m,n)} \quad \text{holomorphic}, \\ c(g; \cdot, \cdot) &: H_n \times \mathbb{C}^{(m,n)} \longrightarrow H_m \quad \text{holomorphic}. \end{aligned}$$

Here $\mathcal{P}_m := \{\, Y \in \mathbb{R}^{(m,m)} \mid Y = {}^tY > 0 \,\}$ is an open convex cone in $\mathbb{R}^{\frac{m(m+1)}{2}}$ and we set

$$Aut(\mathcal{P}_m) := \left\{ \xi \in GL(\mathbb{C}^{(m,m)}) \,\middle|\, \xi(\mathcal{P}_m) = \mathcal{P}_m \right\}.$$

Precisely, if $g = [\sigma, u, (\lambda, \mu, \kappa)] (mod\,\{\pm E_{m+n}\})$ with $\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ in $N(H_n) \subset Aut(H_{m+n})$, then we have

$$\begin{aligned} \sigma_g(Z) &= (AZ + B)(CZ + D)^{-1}, \quad m_g(T) = uT\,{}^tu, \\ a(g; Z)(W) &= uW(CZ + D)^{-1}, \\ b(g; Z) &= u(\lambda Z + \mu)(CZ + D)^{-1}, \\ c(g; Z, W) &= u\left\{\lambda\,{}^tW + \kappa - (W + \lambda Z + \mu)(CZ + D)^{-1}(C\,{}^tW + C\,{}^t\mu - D\,{}^t\lambda)\right\}{}^tu. \end{aligned}$$

Remark 2.1. In [PS], Piateski-Sharpiro called the mapping $(Z, W, T) \longmapsto (\tilde{Z}, \tilde{W}, \tilde{T})$ a *quasilinear* transformation.

From now on, we set

$$H_{n,m} := H_n \times \mathbb{C}^{(m,n)}.$$

We observe that $g = [\sigma, u, (\lambda, \mu, \kappa)](mod\{\pm E_{m+n}\}) \in N(H_n)$ acts on $H_{n,m}$ by

$$(Z, W) \longmapsto (\sigma_g(Z), a(g; Z)(W) + b(g; Z)).$$

The subgroup of $N(H_n)$ consisting of elements $g = [\sigma, u, (\lambda, \mu, \kappa)](mod\{\pm E_{m+n}\})$ with the property

$$m_g = \text{Identity} \quad on\ H_m$$

is called the *Jacobi group*, denoted by G^J. It follows immediately from the definition that

$$G^J = \{[\sigma, E_m, (\lambda, \mu, \kappa)] \in P(H_n)\}.$$

It is easy to see that G^J is the semidirect product of $Sp(n, \mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$, where

$$H_{\mathbb{R}}^{(n,m)} := \left\{ [E_n, E_m, (\lambda, \mu, \kappa)] := \begin{pmatrix} E_n & 0 & 0 & {}^t\mu \\ \lambda & E_m & \mu & \kappa \\ 0 & 0 & E_n & -{}^t\lambda \\ 0 & 0 & 0 & E_m \end{pmatrix} \in P(H_n) \right\}$$

is the nilpotent 2-step subgroup of $P(H_n)$, called the *Heisenberg group* equipped with the multiplication law

$$\begin{aligned}&[E_n, E_m, (\lambda, \mu, \kappa)] \circ [E_n, E_m, (\lambda', \mu', \kappa')] \\ &:= [E_n, E_m, (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda {}^t\mu' - \mu {}^t\lambda')].\end{aligned}$$

From now on, we simply write, if there is no confusion,

$$(\lambda, \mu, \kappa) := [E_n, E_m, (\lambda, \mu, \kappa)].$$

We observe that if $(\lambda, \mu, \kappa) \in H_{\mathbb{R}}^{(n,m)}$, $\kappa + \mu\, {}^t\lambda$ is symmetric. Now it is easy to see that the multiplcation law on G^J is given as follows:

$$\begin{aligned}&[\sigma, E_m, (\lambda, \mu, \kappa)] \circ [\sigma', E_m, (\lambda', \mu', \kappa')] \\ &:= [\sigma\sigma', E_m, (\tilde{\lambda} + \lambda, \tilde{\mu} + \mu, \kappa + \kappa' + \tilde{\lambda}\, {}^t\mu' - \tilde{\mu}\, {}^t\lambda)],\end{aligned}$$

where σ, $\sigma' \in Sp(n, \mathbb{R})$ and $(\tilde{\lambda}, \tilde{\mu}) := (\lambda, \mu) \cdot \sigma'$.

By a simple calculation, we see that the action of $[\sigma, E_m, (\lambda, \mu, \kappa)] \in G^J$ $\left(\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right)$ on (Z, W, T) is given by

$$\begin{aligned}\tilde{Z} &= (AZ + B)(CZ + D)^{-1}, \\ \tilde{W} &= W(CZ + D)^{-1} + (\lambda Z + \mu)(CZ + D)^{-1}, \\ \tilde{T} &= T + \lambda {}^tW + \kappa - (W + \lambda Z + \mu)(CZ + D)^{-1}(C\, {}^tW + C\, {}^t\mu - D\, {}^t\lambda).\end{aligned}$$

Now we consider another subgroup $\tilde{G}$ of G^J. By the definition, $\tilde{G}$ consists of elements of G^J whose action is of the following form:

$$(Z, W, T) \longmapsto (\sigma_g(Z), a(g; Z)(W), T + c(g; Z, W)), \quad c(g; Z, 0) = 0.$$

It is easily seen that

$$\tilde{G} = \left\{ [\sigma, E_m, (0,0,0)] := \begin{pmatrix} A & 0 & B & 0 \\ 0 & E_m & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & E_m \end{pmatrix} \in G^J \right\}.$$

Lemma 2.2. The map

$$J : \tilde{G} \times H_n \longrightarrow GL(\mathbb{C}^{(m,n)})$$

defined by

$$J(\tilde{\sigma}, Z) := a(\tilde{\sigma}; Z), \quad \tilde{\sigma} \in \tilde{G}, \quad Z \in H_n$$

is a factor of automorphy for $\tilde{G}$, that is, J satisfies the condition

$$J(\tilde{\sigma}_1\tilde{\sigma}_2, Z) = J(\tilde{\sigma}_1, \tilde{\sigma}_2 < Z >)\, J(\tilde{\sigma}_2, Z), \quad \tilde{\sigma}_1,\ \tilde{\sigma}_2 \in \tilde{G}, \quad Z \in H_n.$$

Proof. It is easy to prove it. We leave its proof to the reader. □

We note that the mapping

$$A(g, (Z, W)) := c(g; Z, W), \quad g \in G^J, \ (Z, W) \in H_{n,m} \tag{2.1}$$

is a summand of automorphy, i.e.,

$$A(g_1g_2, (Z, W)) = A(g_1, g_2 \cdot (Z, W)) + A(g_2, (Z, W)), \tag{2.2}$$

where g_1, $g_2 \in G^J$ and $(Z, W) \in H_{n,m}$.

Let

$$K_{\mathbb{C}} \subset GL(\mathbb{C}^{(m,n)})$$

be the complex Lie group generated by the linear mapping

$$\{ \ a(g; Z) : \ g \in G^J \ \}.$$

Then $K_{\mathbb{C}}$ is isomorphic to $GL(n, \mathbb{C})$.

Lemma 2.3. Let

$$\rho : GL(n,\mathbb{C}) \longrightarrow GL(V_\rho)$$

be a finite dimensional holomorphic representation of $GL(n,\mathbb{C})$ on a finite dimensional complex vector space V_ρ and let

$$\chi : \mathbb{C}^{(m,m)} \longrightarrow \mathbb{C}^\times$$

be a character on the additive group $\mathbb{C}^{(m,m)}$. Then the mapping

$$J_\rho : \tilde{G} \times H_n \longrightarrow GL(V_\rho)$$

defined by

$$J_\rho(\tilde{\sigma}, Z) := \rho(J(\tilde{\sigma}, Z)), \quad \tilde{\sigma} \in \tilde{G}, \quad Z \in H_n$$

is a factor of automorphy for $\tilde{G}$. Furthermore the mapping

$$J_{\chi,\rho}(g,(Z,W)) := \chi(c(g;Z,W))\,\rho(a(g;Z)), \quad g \in G^J$$

is a factor of automorphy for the Jacobi group G^J with respect to χ and ρ.

Proof. The proof of this first statement is obvious. The proof of the second statement follows immediately from the fact that $A(g,(Z,W)) := c(g;Z,W)$ is a summand of automorphy (cf. (2.1) and (2.2)) and that J_ρ is a factor of automorphy for $\tilde{G}$. □

Definition 2.4. J_ρ and $J_{\chi,\rho}$ are called the *canonical automorphic factor* for $\tilde{G}$ with respect to ρ and the *canonical automorphic factor* for G^J with respect to χ and ρ respectively.

Remark 2.5. Following the above argument, you can obtain the canonical automorphic factor for the Jacobi group in the case that the domain considered is a domain of tube type, i.e., a tube domain.

3. Jacobi Forms

In this section, we establish the notations and review some properties of Jacobi forms. Let

$$GSp(n,\mathbb{R})^+ := \{ M \in \mathbb{R}^{(2n,2n)} \mid {}^tMJ_nM = \nu J_n \ \ for\ some\ \nu > 0 \} \tag{3.1}$$

be the group of *similitudes* of degree n, where

$$J_n := \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$

Let $M \in GSp(n,\mathbb{R})^+$. If ${}^tMJ_nM = \nu J_n$, we write $\nu = \nu(M)$. It is easy to see that $GSp(n,\mathbb{R})^+$ acts on H_n transitively by

$$M < Z > := (AZ+B)(CZ+D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp(n,\mathbb{R})^+$ and $Z \in H_n$.

We define the semidirect product of $GSp(n,\mathbb{R})^+$ and $H_{\mathbb{R}}^{(n,m)}$

$$\hat{G}^J := GSp(n,\mathbb{R})^+ \propto H_{\mathbb{R}}^{(n,m)} \tag{3.2}$$

endowed with the following multiplication law

$$\begin{aligned}&[M,(\lambda,\mu,\kappa)]\circ[M',(\lambda',\mu',\kappa')]\\ &:=[MM',(\nu(M')^{-1}\tilde{\lambda}+\lambda',\,\nu(M')^{-1}\tilde{\mu}+\mu',\,\nu(M')^{-1}\kappa+\kappa'\\ &+\nu(M')^{-1}(\tilde{\lambda}\,{}^t\mu'-\tilde{\mu}\,{}^t\lambda'))],\end{aligned} \tag{3.3}$$

with $M,M'\in GSp(n,\mathbb{R})^+$ and $(\tilde{\lambda},\tilde{\mu}):=(\lambda,\mu)M'$. Clearly the *Jacobi group* $G^J := Sp(n,\mathbb{R})\propto H_{\mathbb{R}}^{(n,m)}$ is a normal subgroup of $\hat{G}^J$. The mapping

$$\hat{G}^J \ni [M,(\lambda,\mu),\kappa)] \longmapsto \begin{pmatrix} A & 0 & B & A\,{}^t\mu - B\,{}^t\lambda \\ \nu\lambda & \nu E_h & \nu\mu & \nu\kappa \\ C & 0 & D & C\,{}^t\mu - D\,{}^t\lambda \\ 0 & 0 & 0 & E_h \end{pmatrix} \in GSp(m+n,\mathbb{R})^+,$$

where $\nu := \nu(M)$ defines an embedding of $\hat{G}^J$ into the group $GSp(m+n,\mathbb{R})^+$ of similitudes of degree $m+n$, where $M=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in GSp(n,\mathbb{R})^+$. It is easy to see that $\hat{G}^J$ acts on $H_{n,m}:=H_n\times\mathbb{C}^{(m,n)}$ transitively by

$$[M,(\lambda,\mu,\kappa)]\cdot(Z,W) := (M<Z>,\nu(W+\lambda Z+\mu)(CZ+D)^{-1}), \tag{3.4}$$

where $M=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in GSp(n,\mathbb{R})^+$, $\nu=\nu(M)$, $(Z,W)\in H_{n,m}$.

Let ρ be a rational representation of $GL(n,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M}\in\mathbb{R}^{(m,m)}$ be a symmetric, semipositive half integral matrix of degree m. Let $C^\infty(H_{n,m},V_\rho)$ be the algebra of all C^∞ functions on $H_{n,m}$ with values in V_ρ. We define the action of $\hat{G}^J$ on $C^\infty(H_{n,m},V_\rho)$ by

$$\begin{aligned}&(f|_{\rho,\mathcal{M}}[M,(\lambda,\mu,\kappa)])(Z,W)\\ &:= e^{-2\pi\nu i\sigma(\mathcal{M}[W+\lambda Z+\mu](CZ+D)^{-1}C)}\\ &\times e^{2\pi\nu i\sigma(\mathcal{M}(\lambda Z^t\lambda+2\lambda^t W+(\kappa+\mu^t\lambda)))}\\ &\times \rho(CZ+D)^{-1}f(M<Z>,\nu(W+\lambda Z+\mu)(CZ+D)^{-1}),\end{aligned} \tag{3.5}$$

where $\nu=\nu(M)$.

Lemma 3.1. Let $f \in C^\infty(H_{n,m}, V_\rho)$. Let $M \in GSp(n,\mathbb{R})^+$, $\zeta,\ \zeta' \in H_{\mathbb{R}}^{(n,m)}$. We may regard $M,\ \zeta,\ \zeta'$ as the elements of $\hat{G}^J$ as follows:

$$M := [M,(0,0,0)] \in \hat{G}^J,$$
$$\zeta := [E_{2n},(\lambda,\mu,\kappa)] \in \hat{G}^J,$$
$$\zeta' := [E_{2n},(\lambda',\mu',\kappa')] \in \hat{G}^J.$$

We let

$$\tilde{g} := [M,\zeta] = [M,(\lambda,\mu,\kappa)] \in \hat{G}^J.$$

Then we have

(3.6) $\quad (f|_{\rho,\mathcal{M}}\zeta)|_{\rho,\mathcal{M}}M = f|_{\rho,\mathcal{M}}(\zeta \circ M)$

(3.7) $\quad (f|_{\rho,\mathcal{M}}M)|_{\rho,\nu(M)\mathcal{M}}\zeta M = f|_{\rho,\mathcal{M}}(M \circ \zeta M).$

(3.8) $\quad (f|_{\rho,\mathcal{M}}M)|_{\rho,\nu(M)\mathcal{M}}\zeta = f|_{\rho,\mathcal{M}}(M \circ \zeta) = f|_{\rho,\mathcal{M}}\tilde{g}.$

(3.9) $\quad (f|_{\rho,\mathcal{M}}\tilde{g})|_{\rho,\nu(M)\mathcal{M}}\zeta' = f|_{\rho,\mathcal{M}}(\tilde{g} \circ \zeta').$

Proof. First we observe that

$$\zeta \circ M = [M,(\nu(M)^{-1}(\lambda,\mu)M, \nu(M)^{-1}\kappa)],$$
$$\zeta M = [E_{2n},((\lambda,\mu)M,\kappa)],$$
$$M \circ \zeta M = [M,((\lambda,\mu)M,\kappa)].$$

A straightforward calculation yields (3.6), (3.7), (3.8) and (3.9). □

Corollary 3.2. Let $g_i = [M_i,(\lambda_i,\mu_i,\kappa_i)] \in \hat{G}^J\ (i=1,2)$. For any $f \in C^\infty(H_{n+m}, V_\rho)$, we have

$$(f|_{\rho,\mathcal{M}}g_1)|_{\rho,\nu(M_1)\mathcal{M}}g_2 = f|_{\rho,\mathcal{M}}(g_1 \circ g_2). \tag{3.10}$$

In particular, the Jacobi group $G^J := Sp(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$ acts on $C^\infty(H_{n+m}, V_\rho)$ as follows:

$$(f|_{\rho,\mathcal{M}}g_1)|_{\rho,\mathcal{M}}g_2 = f|_{\rho,\mathcal{M}}(g_1 \circ g_2), \quad g_1,\ g_2 \in G^J. \tag{3.11}$$

Proof. For $\tilde{g} = [M,(\lambda,\mu,\kappa)] \in \hat{G}^J$, we note that

$$[M,(\lambda,\mu,\kappa)] = [M,(0,0,0)] \circ [E_{2n},(\lambda,\mu,\kappa)].$$

Using (3.8), (3.6) and (3.9), we obtain the desired formula (3.10). □

Remarks 3.3. We note that

$$(f|_{\rho,\mathcal{M}}\zeta)|_{\rho,\mathcal{M}}M \neq (f|_{\rho,\mathcal{M}}M)|_{\rho,\nu(M)\mathcal{M}}\zeta M.$$

But if $M \in Sp(n,\mathbb{R})$, we have

$$(f|_{\rho,\mathcal{M}}\zeta)|_{\rho,\mathcal{M}}M = (f|_{\rho,\mathcal{M}}M)|_{\rho,\mathcal{M}}\zeta M.$$

Definition 3.4. Let ρ and $\mathcal{M}$ be as above. Let

$$H_{\mathbb{Z}}^{(n,m)} := \left\{ (\lambda,\mu,\kappa) \in H_{\mathbb{R}}^{(n,m)} \mid \lambda,\mu \in \mathbb{Z}^{(m,n)},\ \kappa \in \mathbb{Z}^{(m,m)} \right\}.$$

A *Jacobi form* of index $\mathcal{M}$ with respect to ρ on a subgroup $\Gamma \subset \Gamma_n$ of finite index is a holomorphic function $f \in C^\infty(H_{n,m}, V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho,\mathcal{M}}\tilde{\gamma} = f$ for all $\tilde{\gamma} \in \Gamma^J := \Gamma \propto H_{\mathbb{Z}}^{(n,m)}$.

(B) f has a Fourier expansion of the following form :

$$f(Z,W) = \sum_{\substack{T\geq 0 \\ \text{half-integral}}} \sum_{R\in\mathbb{Z}^{(n,m)}} c(T,R)\cdot e^{\frac{2\pi i}{\lambda_\Gamma}\sigma(TZ)}\cdot e^{2\pi i\sigma(RW)}$$

with some $\lambda_\Gamma \in \mathbb{Z}$ and $c(T,R) \neq 0$ only if $\begin{pmatrix} \frac{1}{\lambda_\Gamma}T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} \geq 0$. In addition, if a Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma)$ satisfies the strong condition that $c(T,R) \neq 0$ implies $\begin{pmatrix} \frac{1}{\lambda_\Gamma}T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} > 0$, it is called a *cuspidal Jacobi form.*

We denote by $J_{\rho,\mathcal{M}}(\Gamma)$ (*resp.* $J^{cusp}_{\rho,\mathcal{M}}(\Gamma)$) the vector space of all Jacobi forms (resp. cuspidal Jacobi forms) of index $\mathcal{M}$ with respect to ρ on Γ. In the special case $V_\rho = \mathbb{C}$, $\rho(A) = (det\,A)^k$ ($k \in \mathbb{Z}$, $A \in GL(n,\mathbb{C})$), we write $J_{k,\mathcal{M}}(\Gamma)$ instead of $J_{\rho,\mathcal{M}}(\Gamma)$ and call k the *weight* of a Jacobi form $f \in J_{k,\mathcal{M}}(\Gamma)$.

If $n \geq 2$, the condition (B) is superfluous by Köcher principle (cf. [Z] lemma 1.6). It is known that the vector space $J_{\rho,\mathcal{M}}(\Gamma)$ is finite dimensional (cf. [E-Z] Theorem 1.1 or [Z] Theorem 1.8).

Remark 3.5. Let ρ and $\mathcal{M}$ be as above. Let

$$\chi_{\mathcal{M}} : \mathbb{C}^{(m,m)} \longrightarrow \mathbb{C}^\times$$

be a character of the additive group $\mathbb{C}^{(m,m)}$ defined by $\chi_{\mathcal{M}}(t) := e^{2\pi i\sigma(\mathcal{M}t)}$ for $t \in \mathbb{C}^{(m,m)}$. The *canonical automorphic factor*

$$J_{\chi_{\mathcal{M}},\rho} := J_{\mathcal{M},\rho} : G^J \times H_{n,m} \longrightarrow GL(V_\rho) \tag{3.12}$$

for the Jacobi group G^J is given by

$$\begin{aligned} J_{\mathcal{M},\rho}(g,(Z,W)) &= e^{-2\pi i\sigma(\mathcal{M}[W+\lambda Z+\mu](CZ+D)^{-1}C)} \\ &\quad\times e^{2\pi i\sigma(\mathcal{M}(\lambda Z\,{}^t\lambda+2\lambda\,{}^tW+\kappa+\mu\,{}^t\lambda))}\rho(CZ+D)^{-1}, \end{aligned}$$

where $g = [\sigma, E_m, (\lambda,\mu,\kappa)] \in G^J$ with $\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $(Z,W) \in H_{n,m}$. Therefore the condition (A) in Definition 3.4 can be written as
$(A)'$ $\;(f(\tilde{\gamma}\cdot(Z,W)) = J_{\mathcal{M},\rho}(\tilde{\gamma},(Z,W))f(Z,W)$ for all $\tilde{\gamma} \in \Gamma^J$.

Remark 3.6. In [Y1]-[Y2], the author used theta series to give an explicit decomposition of the right regular representation of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ on $L^2(H_{\mathbb{Z}}^{(n,m)}\backslash H_{\mathbb{R}}^{(n,m)})$ into irreducibles.

Remark 3.7. For historical remarks on Jacobi forms, we refer to [B3] pp. 3-5.

4. Shimura Isomorphism

Let S be a symmetric, positive definite integral matrix of degree m and let $a, b \in \mathbb{Q}^{(m,n)}$. We consider

$$\vartheta_{S,a,b}(Z,W) := \sum_{\lambda \in \mathbb{Z}^{(m,n)}} e^{\pi i \sigma(S((\lambda+a)Z^t(\lambda+a)+2(\lambda+a)^t(W+b)))} \tag{4.1}$$

with characteristic (a, b) converging uniformly on any compact subset of $H_{n,m}$.

Let $\mathcal{M}$ be a symmetric, positive definite and half-integral matrix of degree m and let $\mathcal{N}$ be a complete system of representatives of the cosets $(2\mathcal{M})^{-1}\mathbb{Z}^{(m,n)}/\mathbb{Z}^{(m,n)}$. We observe that $\#(\mathcal{N}) = \{det(2\mathcal{M})\}^n$. An easy application of the Poisson summation formula gives

Lemma 4.1. For $a \in \mathcal{N}$, we have

$$\begin{aligned} \vartheta_{2\mathcal{M},a,0}(-Z^{-1}, WZ^{-1}) = \{det(2\mathcal{M})\}^{-\frac{n}{2}} \left\{ det\left(\frac{Z}{i}\right)\right\}^{\frac{m}{2}} e^{2\pi i \sigma(\mathcal{M}WZ^{-1\,t}W)} \\ \times \sum_{b \in \mathcal{N}} e^{-2\pi i \sigma(2\mathcal{M}b^ta)}\, \vartheta_{2\mathcal{M},b,0}(Z,W). \end{aligned} \tag{4.2}$$

Corollary 4.2. Let $2\mathcal{M}$ be unimodular. Then $\vartheta_{2\mathcal{M},0,0}(Z,W)$ is a Jacobi form of weight $\frac{m}{2}$ and index $\mathcal{M}$.

Lemma 4.3. Let $S = rI_m$ with $r > 0$. Then $\vartheta_{S,a,b}(Z,W)$ satisfies the heat equation

$$\sum_{k=1}^{m} \frac{\partial^2 \vartheta_{S,a,b}}{\partial W_{kp} \partial W_{kq}} = \frac{4\pi i r}{2-\delta_{pq}} \frac{\partial \vartheta_{S,a,b}}{\partial Z_{pq}}, \quad 1 \le p \le q \le n. \tag{4.3}$$

It is easy to prove it and so we omit its proof.

Lemma 4.4. Let $f \in J_{\rho,\mathcal{M}}(\Gamma)$ be a Jacobi form and let $c \in \mathbb{Z}^{(m,j)}$. Then the mapping $f^c : H_{n,j} \longrightarrow V_\rho$ defined by $f^c(Z,W) = f(Z,cW)\,(Z \in H_n,\ W \in \mathbb{C}^{(j,n)})$ defines a Jacobi form in $J_{\rho,\tilde{\mathcal{M}}}(\Gamma)$ with $\tilde{\mathcal{M}} = {}^tc\mathcal{M}c$.
The proof of lemma 4.4 is obvious.

Definition 4.5. Let $S \in \mathbb{Z}^{(2k,2k)}$ be a symmetric, positive definite unimodular and even matrix of degree $2k$ and let $c \in \mathbb{Z}^{(2k,m)}$. We define the theta series

$$\vartheta_{S,c}^{(n)}(Z,W) := \sum_{\lambda \in \mathbb{Z}^{(2k,n)}} e^{\pi i \sigma(S(\lambda Z^t\lambda + 2\lambda^t W^t c))}, \quad Z \in H_n, \quad W \in \mathbb{C}^{(m,n)}. \tag{4.4}$$

We observe that $\vartheta_{S,c}^{(n)}(Z,W) = \vartheta_{S,0,0}(Z,cW)$. Thus according to corollary 4.2 and lemma 4.4, $\vartheta_{S,c}^{(n)} \in J_{k,\mathcal{M}}(\Gamma_n)$ with $\mathcal{M} = \frac{1}{2}{}^tcSc$. We consider the ordinary theta series

$$\vartheta_S(Z) := \sum_{\lambda \in \mathbb{Z}^{(2k,n)}} e^{\pi i \sigma(S[\lambda]Z)}, \quad Z \in H_n. \tag{4.5}$$

We observe that $\vartheta_S(Z)=\vartheta^{(n)}_{S,0}(Z,W)$ and so $\vartheta_S(Z)\in J_{k,0}(\Gamma_n)$. In fact, $\vartheta_S(Z)$ is a Siegel modular form on H_n of weight k. It is easy to see that the Fourier coefficients $c(T,R)$ of $\vartheta^{(n)}_{S,c}$ are given by

$$c(T,R)=\#\{\lambda\in Z^{(2k,n)}\mid {}^t\lambda S\lambda=2T,\ {}^t\lambda Sc=R\}. \tag{4.6}$$

An easy calculation gives the following

Lemma 4.6. Let S_1 and S_2 be two symmetric positive definite integral matrices of degree $2k_1$ and $2k_2$ respectively. Let $c_1\in\mathbb{Z}^{(2k_1,m)}$ and $c_2\in\mathbb{Z}^{(2k_2,m)}$. Then

$$\vartheta^{(n)}_{S_1,c_1}(Z,W)\cdot\vartheta^{(n)}_{S_2,c_2}(Z,W)=\vartheta^{(n)}_{\begin{pmatrix}S_1&0\\0&S_2\end{pmatrix},\begin{pmatrix}c_1\\c_2\end{pmatrix}}(Z,W), \tag{4.7}$$

where $(Z,W)\in H_{n,m}$. Thus $\vartheta^{(n)}_{\begin{pmatrix}S_1&0\\0&S_2\end{pmatrix},\begin{pmatrix}c_1\\c_2\end{pmatrix}}\in J_{k_1+k_2,\mathcal{M}_1+\mathcal{M}_2}(\Gamma_n)$ with $\mathcal{M}_1=\frac{1}{2}{}^tc_1S_1c_1$ and $\mathcal{M}_2=\frac{1}{2}{}^tc_2S_2c_2$.

Now we fix an element $\Omega\in H_n$. We let $\mathcal{M}$ be a positive symmetric, half integral matrix of degree m. then the lattice $L_\Omega:=\mathbb{Z}^{(m,n)}\cdot\Omega+\mathbb{Z}^{(m,n)}$ in $\mathbb{C}^{(m,n)}$ acts on $\mathbb{C}^{(m,n)}$ properly discontinously by

$$(\lambda\Omega+\mu)\cdot W=W+\lambda\Omega+\mu,\quad \lambda,\mu\in\mathbb{Z}^{(m,n)},\quad W\in\mathbb{C}^{(m,n)}. \tag{4.8}$$

Identifying $(W,\xi)\in\mathbb{C}^{(m,n)}\times\mathbb{C}$ with

$$(W+\lambda\Omega+\mu,\ e^{-2\pi i\sigma(\mathcal{M}(\lambda\Omega\,{}^t\lambda+2\lambda{}^tW))}\xi),\quad \xi\in\mathbb{C},$$

we obtain a holomorphic line bundle $\mathcal{L}_\Omega$ over the abelian variety $X_\Omega:=\mathbb{C}^{(m,n)}/L_\Omega$. It is easy to see that $\mathcal{L}_\Omega$ is ample and $dim_{\mathbb{C}}H^0(X_\Omega,\mathcal{L}_\Omega)=\{det\,(2\mathcal{M})\}^n$. In fact,

$$\left\{\vartheta_{2\mathcal{M},a,0}(\Omega,W)\mid\ a\in(2\mathcal{M})^{-1}\mathbb{Z}^{(m,n)}/\mathbb{Z}^{(m,n)}\right\}$$

form a basis for $H^0(X_\Omega,\mathcal{L}_\Omega)$. Varying Ω in H_n, we obtain the theta series on $H_{n,m}$

$$\vartheta_{2\mathcal{M},a,0}(\Omega,W):=\sum_{\lambda\in\mathbb{Z}^{(m,n)}}e^{2\pi i\sigma(\mathcal{M}((\lambda+a)\Omega\,{}^t(\lambda+a)+2(\lambda+a){}^tW))}$$

converging uniformly on any compact subset of $H_{n,m}$.

If f is a Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma_n)$, it is easy to see that for a fixed element $\Omega\in H_n$ each component of $f(\Omega,W)$ represents a global section of $\mathcal{L}_\Omega$. Thus varying Ω in H_n, we may write

$$f(Z,W)=\sum_{a\in\mathcal{N}}f_a(Z)\cdot\vartheta_{2\mathcal{M},a,0}(Z,W),\quad (Z,W)\in H_{n,m}, \tag{4.9}$$

where $\mathcal{N}$ is a complete system of representatives of the cosets $(2\mathcal{M})^{-1}\mathbb{Z}^{(m,n)}/\mathbb{Z}^{(m,n)}$ and $\{f_a:H_n\longrightarrow V_\rho\mid a\in\mathcal{N}\}$ are uniquely determined holomorphic vector valued functions on H_n.

According to (4.2), we have

$$f_a(-Z^{-1}) = \left\{ det\left(\frac{Z}{i}\right)\right\}^{-\frac{m}{2}} \cdot \{det(2\mathcal{M})\}^{-\frac{n}{2}} \cdot \rho(-Z) \cdot \sum_{b\in\mathcal{N}} e^{2\pi i\sigma(2\mathcal{M}a\,{}^tb)} \cdot f_b(Z) \tag{4.10}$$

and

$$f_a(Z+S) = e^{-2\pi i\sigma(\mathcal{M}aS\,{}^ta)} \cdot f_a(Z), \quad S = {}^tS \in \mathbb{Z}^{(n,n)}. \tag{4.11}$$

We note that the Fourier coefficients $c(T,R)$ of $\vartheta_{2\mathcal{M},a,0}(Z,W)$ are given by

$$c(T,R) = \begin{cases} 1 & \text{if } \exists\lambda \in \mathbb{Z}^{(m,n)} \text{ s.t. } {}^t(\lambda+a,E_h)\mathcal{M}(\lambda+a,E_h) = \begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M}\end{pmatrix} \\ 0 & \text{otherwise}\end{cases}$$

We observe that $c(T,R) \neq 0$ implies $4T - R\mathcal{M}^{-1}{}^tR = 0$. By an easy argument, we see that the functions $\{f_a \,|\, a \in \mathcal{N}\}$ must have the Fourier expansions of the form

$$f_a(Z) = \sum_{\substack{T={}^tT\geq 0 \\ half\ integral}} c(T)\cdot e^{2\pi i\sigma(TZ)} \tag{4.12}$$

Conversely, suppose there is given a family $\{f_a \,|\, a \in \mathcal{N}\}$ of holomorphic functions $f_a : H_n \longrightarrow V_\rho$ satisfying the transformation laws (4.10), (4.11) and the cusp condition (4.12). Then we obtain a Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma_n)$ by defining $f(Z,W)$ via the equation (4.9). So far we have proved the Shimura correspondence:

Theorem 4.7 (Shimura). The equation (4.9) gives an isomorphism between $J_{\rho,\mathcal{M}}(\Gamma_n)$ and the vector space of V_ρ-valued Siegel modular forms of half integral weight satisfying the transformation laws (4.10), (4.11) and the cusp condition (4.12).

Remark 4.8. Theorem 4.7 may be also formulated for Jacobi forms on a subgroup $\Gamma \subset \Gamma_n$ of finite index.

Corollary 4.9. If $2k < rank\,(\mathcal{M})$, then we have $J_{k,\mathcal{M}}(\Gamma) = 0$.
Proof. The proof follows from the fact that the irreducible representation $(det)^{k-\frac{1}{2}rank\,(\mathcal{M})}$ of $GL(n,\mathbb{C})$ is not a polynomial representation. □

Corollary 4.10. Let $2\mathcal{M}$ be unimodular and $k\cdot n$ be odd. Then $J_{k,\mathcal{M}}(\Gamma_n) = 0$.

Proof. It follows immediately from (4.10) and the fact that $m \equiv 0\,(mod\,8)$. □

Corollary 4.11. Let $2\mathcal{M}$ be unimodular. *We assume that ρ satisfies the following condition (4.13):*

$$\rho(A) = \rho(-A) \quad \textit{for all } A \in GL(n,\mathbb{C}). \tag{4.13}$$

Then we have

$$J_{\rho,\mathcal{M}}(\Gamma_n) = [\Gamma_n, \tilde{\rho}] \cdot \vartheta_{2\mathcal{M},0,0}(Z,W) \cong [\Gamma_n, \tilde{\rho}], \tag{4.14}$$

where $\tilde{\rho} = \rho \otimes det^{-\frac{m}{2}}$. In particular, if $k \cdot n$ is even,

$$J_{k,\mathcal{M}}(\Gamma_n) = [\Gamma_n, k - \frac{m}{2}] \cdot \vartheta_{2\mathcal{M},0,0}(Z,W) \cong [\Gamma_n, k - \frac{m}{2}]. \tag{4.15}$$

Proof. The proof of (4.14) follows from (4.10), (4.11) and (4.12). The representation $det^k : GL(n,\mathbb{C}) \longrightarrow \mathbb{C}^\times$ defined by $det^k(A) = (det\,(A))^k$ satisfies the condition (4.13). Hence (4.15) follows from (4.14). □

Example 4.12. We give several examples of the irreducible representations which satisfies the condition (4.13).

(a) If $k \cdot n$ is even, then the polynomial representation $\rho : GL(n,\mathbb{C}) \longrightarrow \mathbb{C}^\times$ defined by $\rho(A) := (det\,(A))^k \ (A \in GL(n,\mathbb{C}))$ satisfies the condition (4.13).

(b) The polynomial representation ρ of $GL(n,\mathbb{C})$ on the symmetric product $Symm^2(\mathbb{C}^n)$ of $\mathbb{C}^n$ defined by

$$\rho(A)Z := AZ\,{}^tA, \quad A \in GL(n,\mathbb{C}), \quad Z \in Symm^2(\mathbb{C}^n)$$

satisfies the condition (4.13). It is obvious that ρ is irreducible. This representation is important geometrically because it is related to holomorphic 1-forms on H_n invariant under Γ_n.

(c) The polynomial representation ρ of $GL(n,\mathbb{C})$ on $Symm^2(\mathbb{C}^n)$ defined by

$$\rho(A)Z := (det\,A)^{n+1}A^{-1}Z\,{}^tA^{-1}, \quad A \in GL(n,\mathbb{C}), \quad Z \in Symm^2(\mathbb{C}^n)$$

satisfies the condition (4.13). It is easy to see that ρ is irreducible. This representation is also important geometrically because it is connected with holomorphic $(N-1)$-forms on H_n invariant under Γ_n, where $N = \frac{n(n+1)}{2} - 1$ (cf. [F2]).

The lattice $L := \mathbb{Z}^{(m,n)} \times \mathbb{Z}^{(m,n)}$ in $\mathbb{C}^{(m,n)}$ acts on $H_{n,m}$ by

$$(\lambda,\mu)\cdot(Z,W) = (Z, W + \lambda Z + \mu), \quad (\lambda,\mu) \in L, \quad (Z,W) \in H_{n,m}.$$

Then we have a universal family $p : \mathcal{X} \longrightarrow H_n$ of principally polarized abelian varieties over H_n. Then we observe that for each $\Omega \in H_n$, we have $p^{-1}(\Omega) \cong \mathbb{C}^{(m,n)}/L_\Omega = X_\Omega$.

We define the mapping

$$e_L : L \times H_{n,m} \longrightarrow \mathbb{C}^* \tag{4.16}$$

by

$$e_L((\lambda,\mu),(Z,W)) := e^{-2\pi i\sigma(\mathcal{M}(\lambda Z^t\lambda+2\lambda^t W))}.$$

Then e_L satisfies the 2-cocycle condition, i.e., $e_L \in H^1(L, \mathcal{O}^*_{H_{n,m}})$. Thus we obtain a holomorphic line bundle $M \longrightarrow \mathcal{X}$ over $\mathcal{X}$ identifying $((Z,W),\xi) \in H_{n,m} \times \mathbb{C}$ with

$$((\lambda,\mu)\cdot(Z,W),\ e_L((\lambda,\mu),(Z,W))\xi),\quad (\lambda,\mu)\in L,\quad \xi\in\mathbb{C}.$$

Then $\{\,\vartheta_{2\mathcal{M},a,0}(Z,W)\mid a\in\mathcal{N}\,\}$ form a basis for $H^0(\mathcal{X},M)$. We note that the restriction of M to $p^{-1}(\Omega)=X_\Omega$ coincides with a line bundle $\mathcal{L}_\Omega$ over X_Ω. The line bundle $\mathcal{L}_\Omega$ over X_Ω can be explicitly described according to Appell-Humbert Theorem([Mu] Chap. I).

5. Singular Jacobi Forms

In this section, we define the concept of singular Jacobi forms and introduce a differential operator $M_{n,m,\mathcal{M}}$ on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$, where $\mathcal{P}_n := \{\,Y \in \mathbb{R}^{(n,n)}\,|\,Y = {}^tY > 0\,\}$ denotes an open convex cone in $\mathbb{R}^{\frac{n(n+1)}{2}}$. We show that this differential operator characterizes singular Jacobi forms. Also we give a criterion that a Jacobi form is singular.

Let $\mathcal{M}$ be a symmetric positive definite, half integral matrix of degree m. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ admits a Fourier expansion (see Definition 3.4 (B))

$$f(Z,W)=\sum_{T,R} c(T,R)e^{2\pi i\sigma(TZ)}\cdot e^{2\pi i\sigma(RW)},\quad Z\in H_n,\quad W\in\mathbb{C}^{(m,n)}. \tag{5.1}$$

We note that if $\Gamma=\Gamma_n$, then $\lambda_\Gamma=1$. A Jacobi form $f\in J_{\rho,\mathcal{M}}(\Gamma_n)$ is said to be *singular* if it admits a Fourier expansion such that the Fourier coefficient $c(T,R)$ is zero unless $det\,(4T-R\mathcal{M}^{-1}{}^tR)=0$.

Lemma 5.1. Let T and $\mathcal{M}$ be two symmetric real matrices of degree n and m respectively. We assume that $\mathcal{M}$ is positive definite. Then

$$\begin{pmatrix} T & \frac12 R \\ \frac12 {}^tR & \mathcal{M}\end{pmatrix} \geq 0 \ \text{ if and only if }\ T\geq 0,\ 4T-R\mathcal{M}^{-1}\,{}^tR\geq 0.$$

Proof. The proof follows from the fact that

$$\begin{pmatrix} T & \frac12 R \\ \frac12 {}^tR & \mathcal{M}\end{pmatrix} = \begin{pmatrix} E_n & \frac12 R\mathcal{M}^{-1} \\ 0 & E_m\end{pmatrix}\begin{pmatrix} T-\frac14 R\mathcal{M}^{-1}{}^tR & 0 \\ 0 & \mathcal{M}\end{pmatrix}^t\begin{pmatrix} E_n & \frac12 R\mathcal{M}^{-1} \\ 0 & E_m\end{pmatrix}.$$

□

Example 5.2. Let $\mathcal{M} = {}^t\mathcal{M}$ be as above. Let $S \in \mathbb{Z}^{(2k,2k)}$ be a symmetric positive definite integral matrix of degree $2k$ and $c \in \mathbb{Z}^{(2k,m)}$. We consider the theta series

$$(5.2) \qquad \vartheta_{S,c}^{(n)}(Z,W) := \sum_{\lambda \in Z^{(2k,n)}} e^{\pi i \sigma(S(\lambda Z\, {}^t\lambda + 2\lambda\, {}^tW\, {}^tc))}, \quad Z \in H_n, \quad W \in \mathbb{C}^{(m,n)}.$$

We assume that $2k < n+\text{rank}(\mathcal{M})$. Then $\vartheta_{S,c}(Z,W)$ is a singular Jacobi form in $J_{k,\mathcal{M}}(\Gamma_n)$, where $\mathcal{M} = \frac{1}{2}{}^t c\mathcal{M}c$. We note that if the Fourier coefficient $c(T,R)$ of $\vartheta_{S,c}^{(n)}$ is nonzero, there exists $\lambda \in \mathbb{Z}^{(2k,n)}$ such that

$$(5.3) \qquad \frac{1}{2}{}^t(\lambda,c)S(\lambda,c) = \begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix}.$$

Thus

$$\text{rank}\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} \leq 2k < n + \text{rank}(\mathcal{M}).$$

Therefore $\det(4T - R\mathcal{M}^{-1}{}^tR) = 0$ by lemma 5.1. □

Now we define a differential operator $M_{n,m,\mathcal{M}}$ on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ defined by

$$(5.4) \qquad M_{n,m,\mathcal{M}} := \det(Y) \cdot \det\left(\frac{\partial}{\partial Y} + \frac{1}{8\pi}{}^t\left(\frac{\partial}{\partial V}\right)\mathcal{M}^{-1}\left(\frac{\partial}{\partial V}\right)\right),$$

where $\frac{\partial}{\partial Y} = \left(\frac{(1+\delta_{\mu\nu})}{2}\frac{\partial}{\partial y_{\mu\nu}}\right)$ and $\frac{\partial}{\partial V} = \left(\frac{\partial}{\partial v_{kl}}\right)$. For the detail of the construction of $M_{n,m,\mathcal{M}}$, we refer to [Y6].

Theorem 5.3. Let $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ be a Jacobi form of index $\mathcal{M}$ with respect to ρ. Then the following are equivalent:

(1) f is a *singular* Jacobi form.

(2) f satisfies the differential equation $M_{n,m,\mathcal{M}}f = 0$.

We refer to [Y6] for the proof.

Definition 5.4. An irreducible finite dimensional representation ρ of $GL(n,\mathbb{C})$ is determined uniquely by its highest weight $(\lambda_1,\cdots,\lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$. We denote this representation by $\rho = (\lambda_1,\cdots,\lambda_n)$. The number $k(\rho) := \lambda_n$ is called the *weight* of ρ.

Theorem 5.5. Let $2\mathcal{M}$ be a symmetric positive definite, *unimodular* even matrix of degree m. Assume that ρ is irreducible and satisfies the following condition

$$\rho(A) = \rho(-A) \quad \text{for all } A \in GL(n,\mathbb{C}).$$

Then any nonvanishing Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma_n)$ is *singular* if and only if $2k(\rho) < n + rank\,(\mathcal{M})$.

We refer to [Y6] for the proof.

Conjecture. For general ρ and $\mathcal{M}$ without the above assumptions on them, *a nonvanishing Jacobi form* $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ *is singular if and only if* $2k(\rho) < n + rank(\mathcal{M})$.

Remark. A singular modular form of type ρ may be written as a finite sum of theta series $\vartheta_{S,P}(Z)$'s with pluriharmonic coefficients (cf. [F1]). The following problem is quite interesting.

Problem. Describe the functions $\{f_a \,|\, a \in \mathcal{N}\}$ explicitely given by (4.9) when $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ is a *singular* Jacobi form.

6. The Siegel-Jacobi Operator

In this section, we investigate some properties of the Siegel-Jacobi operator.

Let ρ be a rational representation of $GL(n,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. For a positive integer r with $r < n$, we let $\rho^{(r)} : GL(r,\mathbb{C}) \longrightarrow GL(V_\rho)$ be a rational representation of $GL(r,\mathbb{C})$ defined by

$$\rho^{(r)}(a)v := \rho\left(\begin{pmatrix} a & 0 \\ 0 & E_{n-r}\end{pmatrix}\right)v, \quad a \in GL(r,\mathbb{C}), \quad v \in V_\rho.$$

We define the Siegel-Jacobi operator $\Psi_{n,r} : J_{\rho,\mathcal{M}}(\Gamma_n) \longrightarrow J_{\rho^{(r)},\mathcal{M}}(\Gamma_r)$ by

$$(\Psi_{n,r}f)(Z,W) := \lim_{t\to\infty} f\left(\begin{pmatrix} Z & 0 \\ 0 & itE_{n-r}\end{pmatrix}, (W,0)\right), \tag{6.1}$$

where $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$, $Z \in H_r$ and $W \in \mathbb{C}^{(m,r)}$. We observe that the above limit always exists and the Siegel-Jacobi operator is a linear mapping (cf. [Zi]).

Definition 6.1. An irreducible finite dimensional representation ρ of $GL(n,\mathbb{C})$ is determined by its highest weight $(\lambda_1,\cdots,\lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$. The number $k(\rho) := \lambda_n$ is called the *weight* of ρ.

Theorem 6.2. Let $2\mathcal{M}$ be a positive unimodular symmetric even matrix of degree m. We assume that ρ is irreducible and satisfies the condition

$$\rho(-A) = \rho(A) \qquad \text{for all } A \in GL(n,\mathbb{C}). \tag{6.2}$$

If $2k(\rho) < n + \text{rank}(\mathcal{M})$, then the Siegel-Jacobi operator $\Psi_{n,n-1}$ is injective.

Theorem 6.3. Let $2\mathcal{M}$ be as above in Theorem 6.2. Assume that ρ is irreducible and satisfies the condition (6.2). If $2k(\rho) + 1 < n + \text{rank}(\mathcal{M})$, then the Siegel-Jacobi operator $\Psi_{n,n-1}$ is an isomorphism.

Theorem 6.4. Let $2\mathcal{M}$ be as above in Theorem 6.2. Assume that $2k > 4n + \operatorname{rank}(\mathcal{M})$ and $k \equiv 0\,(mod\ 2)$. Then the Siegel-Jacobi operator $\Psi_{n,n-1} : J_{k,\mathcal{M}}(\Gamma_n) \longrightarrow J_{k,\mathcal{M}}(\Gamma_{n-1})$ is surjective.

The proofs of the above theorems are based on the Shimura isomorphism, the theory of singular modular forms and the result of H. Maass.

Theorem 6.5. Let $1 \leq r \leq n-1$. Assume that $k(\rho) > n + r + \operatorname{rank}(\mathcal{M}) + 1$ and $k(\rho)$ is even. Then we have

$$J^{cusp}_{\rho^{(r)},\mathcal{M}}(\Gamma_r) \subset \Psi_{n,r}(J_{\rho,\mathcal{M}}(\Gamma_n)).$$

Proof. The proof follows from the fact that for any cuspidal Jacobi form $f \in J^{cusp}_{\rho^{(r)},\mathcal{M}}(\Gamma_r)$, we have

$$\Psi_{n,r}(E^{(n)}_{\rho,\mathcal{M}}(Z,W,f)) = f(Z_1,W_1), \tag{6.3}$$

where $Z = \begin{pmatrix} Z_1 & * \\ * & * \end{pmatrix} \in H_n$ with $Z_1 \in H_r$ and $W = (W_1, *) \in \mathbb{C}^{(m,n)}$ with $W_1 \in \mathbb{C}^{(m,r)}$. Here $E^{(n)}_{\rho,\mathcal{M}}(Z,W,f) \in J_{\rho,\mathcal{M}}(\Gamma_n)$ denotes the Eisenstein series of Klingen's type corresponding to a cuspidal Jacobi form $f \in J^{cusp}_{\rho^{(r)},\mathcal{M}}(\Gamma_r)$. Following the idea of [Zi], pp. 208-209, we can prove (6.3). □

Corollary 6.6. Let $1 \leq r \leq n-1$. Assume that $k(\rho) > n + r + \operatorname{rank}(\mathcal{M}) + 1$ with $1 \leq r \leq n-1$ and $k(\rho)$ is even. For any cuspidal Jacobi form $f \in J^{cusp}_{\rho^{(r)},\mathcal{M}}(\Gamma_r)$, we have

$$\Psi_{n,r-1}(E^{(n)}_{\rho,\mathcal{M}}(*,*,f)) = 0. \tag{6.4}$$

In particular, $\Psi_{n,l}(E^{(n)}_{\rho,\mathcal{M}}(*,*,f)) = 0$ for any integral $l \in \mathbb{Z}$ with $1 \leq l \leq r-1$.

Proof. It follows immediately from (6.3) and Theorem 6.5. □

Remark 6.7. (1) We may define an action of the Hecke operator of Γ_n on $J_{\rho,\mathcal{M}}(\Gamma_n)$. We proved that the action of the Siegel-Jacobi operator on Jacobi forms is compatible with that of the Hecke algebra. We refer to [Y3] for the detail.
(2) Using the properties of the Siegel-Jacobi operator and the theory of singular Jacobi forms, we may introduce the concept of the so-called *stable Jacobi forms*. This concept is useful for the study of geometric properties of the universal family of principally polarized abelian varieties of dimension n. We refer to [Y8] for the detail.

7. Construction of Modular Forms from Jacobi Forms

In this section, we construct new vector valued modular forms by differentiating Jacobi forms with respect to toroidal variables and then evaluating at zero in the toroidal variables.

Let n and m be two positive integers and let $\mathcal{P}_{m,n} := \mathbb{C}[W_{11}, W_{12}, \cdots, W_{mn}]$ be the ring of complex valued polynomials on $\mathbb{C}^{(m,n)}$. For any homogeneous polynomial $P \in \mathcal{P}_{m,n}$, we put

$$P(\partial_W) := P\left(\frac{\partial}{\partial W_{11}}, \cdots, \frac{\partial}{\partial W_{mn}}\right). \tag{7.1}$$

Let S be a positive definite symmetric rational matrix of degree m. Let $T := (t_{pq})$ be the inverse of S. For each i, j with $1 \leq i, j \leq n$, we denote by $\Delta_{i,j}$ the following differential operator

$$\Delta_{i,j} := \sum_{p,q=1}^{m} t_{pq} \frac{\partial^2}{\partial W_{pi} \partial W_{qj}}, \quad 1 \leq i, j \leq n. \tag{7.2}$$

A polynomial P on $\mathbb{C}^{(m,n)}$ is said to be *harmonic* with respect to S if

$$\sum_{i=1}^{n} \Delta_{i,i} P = 0. \tag{7.3}$$

A polynomial P on $\mathbb{C}^{(m,n)}$ is called *pluriharmonic* with respect to S if

$$\Delta_{i,j} P = 0, \quad 1 \leq i, j \leq n. \tag{7.4}$$

If there is no confusion, we just write harmonic or pluriharmonic instead of harmonic or pluriharmonic with respect to S. Obviously a pluriharmonic polynomial is harmonic. We denote by $\mathcal{H}_{m,n}$ the space of all pluriharmonic polynomials on $\mathbb{C}^{(m,n)}$. The ring $\mathcal{P}_{m,n}$ of polynomials on $\mathbb{C}^{(m,n)}$ has a symmetric nondegenerate bilinear form $< P, Q >:= (P(\partial_W)Q)(0)$ for $P, Q \in \mathcal{P}_{m,n}$. It is easy to check that $<,>$ satisfies

$$< P, QR >=< Q(\partial_W)P, R >, \quad P, Q, R \in \mathcal{P}_{m,n}. \tag{7.5}$$

Lemma 7.1. $\mathcal{H}_{m,n}$ is invariant under the action of $GL(n, \mathbb{C}) \times O(S)$ given by

$$((A, B), P(W)) \longmapsto P({}^tBWA), \quad A \in GL(n, \mathbb{C}), \ B \in O(S). \tag{7.6}$$

Here $O(S) := \{ B \in GL(m, \mathbb{C}) \mid {}^tBSB = S \}$ denotes the orthogonal group of the quadratic form S.

Proof. See corollary 9.11 in [M-N-N]. □

Remark 7.2. In [K-V], Kashiwara and Vergne investigated an irreducible decomposition of the space of complex pluriharmonic polynomials defined on $\mathbb{C}^{(m,n)}$ under the action of (7.6). They showed that each irreducible component $\tau \otimes \lambda$ occurring in the decomposition of $\mathcal{H}_{m,n}$ under the action (7.6) has multiplicity one and the irreducible representation τ of $GL(n, \mathbb{C})$ is determined uniquely by the irreducible representation of $O(S)$.

We take $S := (2\mathcal{M})^{-1}$. According to lemma 7.1, there exists an irreducible subspace $V_\tau(\neq 0)$ invariant under the action of $GL(n,\mathbb{C})$ given by (7.6). We denote this represetation by τ. Then we have

$$(7.7) \qquad (\tau(A)P)(W) = P(WA), \quad A \in GL(n,\mathbb{C}), \quad P \in V_\tau, \quad W \in \mathbb{C}^{(m,n)}.$$

The action $\hat{\tau}$ of $GL(n,\mathbb{C})$ on V_τ^* is defined by

$$(7.8) \qquad (\hat{\tau}(A)^{-1}\zeta)(P) := \zeta(\tau({}^tA^{-1})P),$$

where $A \in GL(n,\mathbb{C})$, $\zeta \in V_\tau^*$ and $P \in V_\tau$.

Definition 7.3. Let $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ be a Jacobi form of index $\mathcal{M}$ with respect to ρ on Γ_n. Let $P \in V_\tau$ be a homogeneous pluriharmonic polynomial. We put

$$(7.9) \qquad f_P(Z) := P(\partial_W)f(Z,W)\Big|_{W=0}, \quad Z \in H_n, \quad W \in \mathbb{C}^{(m,n)}.$$

Now we define the mapping

$$f_\tau : H_n \longrightarrow V_\tau^* \otimes V_\rho$$

by

$$(7.10) \qquad (f_\tau(Z))(P) := f_P(Z), \quad Z \in H_n, \quad P \in V_\tau.$$

Theorem 7.4. Let τ and $\hat{\tau}$ be as before. Let $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ be a Jacobi form. Then $f_\tau(Z)$ is a modular form of type $\hat{\tau} \otimes \rho$, i.e., $f_\tau \in [\Gamma_n, \hat{\tau} \otimes \rho]$.

We refer to [Y9] for the proof.

Applications

We obtain important identities by applying theorem 7.4 to special Jacobi forms.

(I) Let $2\mathcal{M}$ be a positive definite symmetirc unimodular, even integral matrix of degree m. We consider the theta series

$$\theta_{2\mathcal{M}}(Z,W) := \sum_{\lambda \in \mathbb{Z}^{(m,n)}} e^{2\pi i \sigma(\mathcal{M}(\lambda Z\, {}^t\lambda + 2\lambda\, {}^tW))}.$$

Then $\theta_{2\mathcal{M}}(Z,W)$ is a Jacobi form of weight $\frac{m}{2}$ and index $\mathcal{M}$ (cf. corollary 4.2). We write $f(Z,W) := \theta_{2\mathcal{M}}(Z,W)$. By theorem 7.6, f_τ is Hom$(V_\tau,\mathbb{C})$-valued modular form of type $\hat{\tau} \otimes det^{\frac{m}{2}}$. In addition, for any homogeneous pluriharmonic P with respect to $(2\mathcal{M})^{-1}$, we obtain the following identity

$$\sum_{\lambda \in \mathbb{Z}^{(m,n)}} P(4\pi i \mathcal{M}\lambda\, {}^t(CZ+D)^{-1}) \cdot e^{2\pi i \sigma(\mathcal{M}\lambda(AZ+B)(CZ+D)^{-1}\, {}^t\lambda)}$$
$$= \{det(CZ+D)\}^{\frac{m}{2}} \sum_{\lambda \in \mathbb{Z}^{(m,n)}} P(4\pi i \mathcal{M}\lambda)\, e^{2\pi i \sigma(\mathcal{M}\lambda Z\, {}^t\lambda)}$$

for all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$ and $Z \in H_n$.

(II) Let $S \in \mathbb{Z}^{(2k,2k)}$ be a positive definite symmetirc, unimodular even matrix of degree $2k$. We choose an integral matrix $c \in \mathbb{Z}^{(2k,m)}$ such that ${}^t cSc$ is positive definite. We consider the following theta series

$$\theta_{S,c}(Z,W) := \sum_{\lambda \in \mathbb{Z}^{(2k,n)}} e^{\pi i \sigma(S(\lambda Z\, {}^t\lambda + 2\lambda\, {}^t(cW)))}.$$

Then $\theta_{S,c} \in J_{k,\mathcal{M}}(\Gamma_n)$ with $\mathcal{M} := \frac{1}{2}\, {}^t cSc$ because $\theta_{S,c}(Z,W) = \theta_{S,0,0}(Z,cW)$. We write $f(Z,W) := \theta_{S,c}(Z,W)$. Then by theorem 7.4, f_τ is a $\mathrm{Hom}(V_\tau,\mathbb{C})$-valued modular form of type $\hat{\tau} \otimes det^k$. Furthermore for any homogeneous pluriharmonic P with respect to $(2\mathcal{M})^{-1} = ({}^t cSc)^{-1}$, we obtain the following identity

$$\begin{aligned} &\sum_{\lambda \in \mathbb{Z}^{(2k,n)}} P(2\pi i\, {}^t cS\lambda\, {}^t(CZ+D)^{-1}) \cdot e^{\pi i \sigma(S\lambda(AZ+B)(CZ+D)^{-1}\, {}^t\lambda)} \\ &= \{det\,(CZ+D)\}^k \sum_{\lambda \in \mathbb{Z}^{(2k,n)}} P(2\pi i\, {}^t cS\lambda) \cdot e^{2\pi i \sigma(S\lambda Z\, {}^t\lambda)} \end{aligned}$$

for all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$ and $Z \in H_n$.

(III) In [Z], Ziegler defined the Eisenstein series $E^{(n)}_{k,\mathcal{M}}(Z,W)$ of Siegel type. Let $\mathcal{M}$ be a half integral positive definite symmetric matrix of degree m and let $k \in \mathbb{Z}^+$. We set

$$\Gamma_{n,0} := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \,\middle|\, C = 0 \right\}.$$

Let $\mathcal{R}$ be a complete system of representatives of the cosets $\Gamma_{n,0}\backslash\Gamma_n$ and Λ be a complete system of representatives of the cosets $\mathbb{Z}^{(m,n)}/(ker\,(\mathcal{M}) \cap \mathbb{Z}^{(m,n)})$, where $ker\,(\mathcal{M}) := \{\lambda \in \mathbb{R}^{(m,n)} \mid \mathcal{M} \cdot \lambda = 0\ \}$. The Eisenstein series $E^{(n)}_{k,\mathcal{M}}$ is defined by

$$\begin{aligned} E^{(n)}_{k,\mathcal{M}}(Z,W) :=& \sum_{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{R}} \det\,(CZ+D)^{-k} \cdot e^{2\pi i \sigma(\mathcal{M}W(CZ+D)^{-1}C\, {}^tW)} \\ &\cdot \sum_{\lambda \in \Lambda} e^{2\pi i \sigma(\mathcal{M}((AZ+B)(CZ+D)^{-1}\, {}^t\lambda + 2\lambda\, {}^t(CZ+D)^{-1}\, {}^tW))}, \end{aligned}$$

where $(Z,W) \in H_{n,m}$. Now we assume that $k > n+m+1$ and k is *even*. Then according to [Z], theorem 2.1, $E^{(n)}_{k,\mathcal{M}}(Z,W)$ is a nonvanishing Jacobi form in $J_{k,\mathcal{M}}(\Gamma_n)$. By theorem 7.4, $(E^{(n)}_{k,\mathcal{M}})_\tau$ is a $\mathrm{Hom}(V_\tau,\mathbb{C})$-valued modular form of type $\hat{\tau} \otimes \det^k$. We define the automorphic factor $j : Sp(n,\mathbb{R}) \times H_n \longrightarrow GL(n,\mathbb{C})$ by

$$j(g,Z) := cZ + d, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n,\mathbb{R}), \quad Z \in H_n.$$

Then for any homogeneous pluriharmonic polynomial P with respect to $(2\mathcal{M})^{-1}$, we obtain the following identity

$$\begin{aligned} & \det j(M,Z)^k \sum_{\gamma\in\mathcal{R}}\sum_{\lambda\in\Lambda} \det j(\gamma,Z)^{-k}\cdot P(4\pi i\mathcal{M}\lambda\, {}^tj(\gamma,Z)^{-1})\cdot e^{2\pi i\sigma(\mathcal{M}\cdot\gamma<Z>\cdot{}^t\lambda)} \\ &= \sum_{\gamma\in\mathcal{R}}\sum_{\lambda\in\Lambda} \det j(\gamma,M<Z>)^{-k}\cdot P(4\pi i\mathcal{M}\lambda\, {}^tj(\gamma M,Z)^{-1})\cdot e^{2\pi i\sigma(\mathcal{M}\cdot\gamma M<Z>\cdot{}^t\lambda)} \end{aligned}$$

for all $M\in\Gamma_n$ and $Z\in H_n$.

8. Harmonic Analysis for the Jacobi Group

In this section, we state the duality theorem for the Jacobi group of higher degree without proof. We refer to [Y7] for the proof and discuss the algebra of invariant differential operators on the Jacobi group G^J.

(I) Duality Theorem

Let (ρ, V_ρ) be an irreducible representation of $K = U(n)$ with highest weight $l := (l_1, l_2, \cdots, l_n) \in \mathbb{Z}^n$, $l_1 \geq l_2 \geq \cdots \geq l_n$. Then ρ is extended to a rational representation of $GL(n,\mathbb{C})$ which is also denoted by ρ. The representation space V_ρ has an hermitian inner product $(\ ,\ )$ such that $(\rho(g)u,v) = (u,\rho(g^*)v)$ for all $g\in GL(n,\mathbb{C})$, $u,v\in V_\rho$ and $g^* = {}^t\bar{g}$. Let $\mathcal{M}$ be a half integral positive definite symmetric matrix of degree m. For $(Z,W)\in H_{n,m}$, we write $Z = X+iY$, $W = U+iV$, $X,Y\in\mathbb{R}^{(n,n)}$, $U,V\in\mathbb{R}^{(m,n)}$. We put

$$\kappa_{\mathcal{M}}(Z,W) := e^{-4\pi\sigma({}^tV\mathcal{M}VY^{-1})}$$

and

$$d(Z,W) := (\det Y)^{-(m+n+1)}\, dX\,dY\,dU\,dV.$$

Then it is easy to see that $d(Z,W)$ is a G^J-invariant volume element on $H_{n,m}$. We denote by $E(\rho,\mathcal{M})$ the Hilbert space consisting of V_ρ-valued measurable functions φ on $H_{n,m}$ such that

$$|\varphi|^2 := \int_{H_{n,m}} (\rho(Im\, Z)\varphi(Z,W),\ \varphi(Z,W))\,\kappa_{\mathcal{M}}(Z,W)\,d(Z,W) < +\infty.$$

Let $\chi_{\mathcal{M}} : A \longrightarrow \mathbb{C}_1 = \{\, z\in\mathbb{C}\mid |z|=1\,\}$ be the unitary character of $A = \{\,[E_n,(0,0,\kappa)]\in G^J \mid \kappa = {}^t\kappa\in\mathbb{R}^{(m,m)}\,\}$ defined by $\chi_{\mathcal{M}}(\kappa) := e^{2\pi i\sigma(\mathcal{M}\kappa)}$, $\kappa\in A$. We define the automorphic factor $J_{\rho,\mathcal{M}} : G^J\times H_{n,m}\longrightarrow GL(V_\rho)$ by

$$\begin{aligned} J_{\rho,\mathcal{M}}(g,(Z,W)) :&= e^{2\pi i\sigma(\mathcal{M}[W+\lambda Z+\mu](CZ+D)^{-1}C)} \\ &\quad\times e^{-2\pi i\sigma(\mathcal{M}(\lambda Z\,{}^t\lambda+2\lambda\,{}^tW+\kappa+\mu\,{}^t\lambda))}\cdot\rho(CZ+D), \end{aligned}$$

where $g = [\sigma,(\lambda,\mu,\kappa)]\in G^J$ with $\sigma = \begin{pmatrix} A & B\\ C & D\end{pmatrix}\in G = Sp(n,\mathbb{R})$. The induced representation $\mathrm{Ind}_{K^J}^{G^J}(\rho\otimes\bar{\chi}_{\mathcal{M}})$ is realized on $E(\rho,\mathcal{M})$ by

$$\left(\mathrm{Ind}_{K^J}^{G^J}(\rho\otimes\bar{\chi}_{\mathcal{M}})(g)\varphi\right)(Z,W) := J_{\rho,\mathcal{M}}(g^{-1},(Z,W))^{-1}\,\varphi(g^{-1}\cdot(Z,W)),$$

where $g \in G^J$, $\varphi \in E(\rho, \mathcal{M})$ and

$$K^J := \left\{ [k, (0,0,\kappa)] \in G^J \mid k \in K,\ \kappa = {}^t\kappa \in \mathbb{R}^{(m,m)} \right\} \cong K \times A.$$

Let $H(\rho, \mathcal{M})$ be the subspace of $E(\rho, \mathcal{M})$ consisting of $\varphi \in E(\rho, \mathcal{M})$ which is holomorphic on $H_{n,m}$. Then $H(\rho, \mathcal{M})$ is a closed G^J-invariant subspace of $E(\rho, \mathcal{M})$. Let $\pi^{\rho,\mathcal{M}}$ be the restriction of $\mathrm{Ind}_{K^J}^{G^J}(\rho \otimes \bar{\chi}_{\mathcal{M}})$ to $H(\rho, \mathcal{M})$. We let the mapping $j : G \times H_n \longrightarrow GL(n, \mathbb{C})$ be the automorphic factor defined by

$$j(\sigma, Z) := CZ + D, \quad \sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G.$$

We define a unitary representation ρ_l of K by

$$\rho_l(k) := \rho(j(k, iE_n)), \quad k \in K.$$

Takase (cf. [T], Theorem 1.1) proved the following

Theorem 8.1. Suppose $l_n > n + \frac{1}{2}$. Then $H(\rho, \mathcal{M}) \neq 0$ and $\pi^{\rho,\mathcal{M}}$ is an irreducible unitary representation of G^J which is square integrable modulo A. The multiplicity of ρ_l is equal to one.

Let E_{ij} denote a square matrix of degree $2n$ with entry 1 where the i-th row and the j-th column meet, all other entries being 0. We put

$$H_i := E_{ii} - E_{n+i,n+i}\,(1 \leq i \leq n), \quad \mathfrak{h} := \sum_{i=1}^{n} \mathbb{C} H_i\,.$$

Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $e_j : \mathfrak{h} \longrightarrow \mathbb{C}\,(1 \leq j \leq n)$ be the linear form on $\mathfrak{h}$ defined by

$$e_j(H_i) := \delta_{ij}\,,$$

where δ_{ij} denotes the Kronecker delta symbol. The roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ are given by

$$\pm 2e_i\ (1 \leq i \leq n), \quad \pm e_k \pm e_l\ (1 \leq k < l \leq n).$$

The set Φ^+ of positive roots is given by

$$\Phi^+ = \{\, 2e_i\,(1 \leq i \leq n), \quad e_k + e_l\,(1 \leq k < l \leq n)\,\}.$$

Let

$$\mathfrak{g}_\alpha := \left\{ X \in \mathfrak{g} \;\middle|\; [H, X] = \alpha(H)X \quad \text{for all } H \in \mathfrak{h} \right\}$$

be the root space corresponding to a root α of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We put $\mathfrak{n} := \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha\,.$ We define

$$N^J := \left\{ [\exp X, (0, \mu; 0)] \in G^J \;\middle|\; X \in \mathfrak{n} \right\},$$

where exp : $\mathfrak{g} \longrightarrow G$ denotes the exponential mapping from $\mathfrak{g}$ to G. A subgroup N^g of G^J is said to be *horospherical* if it is conjugate to N^J, that is, $N^g = gN^Jg^{-1}$ for some $g \in G$. A horospherical subgroup N^g is said to be *cuspidal* for $\Gamma^J := \Gamma_n \propto H_{\mathbb{Z}}^{(n,m)}$ in G^J if $(N^g \cap \Gamma^J)\backslash N^g$ is compact. Let $L^2\left(\Gamma^J\backslash G^J;\rho\right)$ be the complex Hilbert space consisting of all Γ^J-invariant V_ρ-valued measurable functions Φ on G^J such that $\| \Phi \| < +\infty$, where $\| \ \|$ is the norm induced from the norm $| \ |$ on $E(\rho, \mathcal{M})$ by the lifting from $H_{n,m}$ to G^J. We denote by $L_0^2\left(\Gamma^J\backslash G^J;\rho\right)$ the subspace of $L^2\left(\Gamma^J\backslash G^J;\rho\right)$ consisting of functions φ on G^J such that $\varphi \in L^2\left(\Gamma^J\backslash G^J;\rho\right)$ and

$$\int_{N^g \cap \Gamma^J \backslash N g} \varphi(ng_0)\, dn = 0$$

for any cuspidal subgroup N^g of G^J and almost all $g_0 \in G^J$. Let R be the right regular representation of G^J on $L_0^2\left(\Gamma^J\backslash G^J;\rho\right)$ defined by

$$R(g_0)\varphi(g) := \varphi(gg_0), \quad g, g_0 \in G^J, \ \varphi \in L_0^2\left(\Gamma^J\backslash G^J;\rho\right).$$

Now we state the duality theorem for the Jacobi group G^J.

Duality Theorem. Let ρ be an irreducible representation of K with highest weight $l = (l_1, l_2, \cdots, l_n) \in \mathbb{Z}^n$, $l_1 \geq l_2 \geq \cdots \geq l_n$. Suppose $l_n > n + \frac{1}{2}$. and let $\mathcal{M}$ be a half integrable positive definite symmetric matrix of degree m. Then the multiplicity $m_{\rho,\mathcal{M}}$ of $\pi^{\rho,\mathcal{M}}$ in the right regular representation R of G^J in $L_0^2\left(\Gamma^J\backslash G^J;\rho\right)$ is equal to the dimension of $J_{\rho,\mathcal{M}}^{cusp}(\Gamma_n)$, that is,

$$m_{\rho,\mathcal{M}} = dim_{\mathbb{C}}\, J_{\rho,\mathcal{M}}^{cusp}(\Gamma_n).$$

Remark 8.2. In [B-B], Berndt and Böcherer proved the duality theorem for the Jacobi group in the case $m = n = 1$.

(II) Invariant differential operators on $H_{n,m}$

First of all, we review the results of G. Shimura (cf. [S3]). We let G/K be a hermitian symmetric space of classical and noncompact type. Here G is classical, usually a connected noncompact semisimple Lie group with finite center and K is a maximal compact subgroup of G. Let ρ be a continuous representation of K on a finite dimensional complex vector space V_ρ. We denote by $C^\infty(\rho)$ the set of all V_ρ-valued smooth functions $f : G \longrightarrow V_\rho$ by G such that

$$f(gk^{-1}) = \rho(k)f(g) \quad \text{for all } k \in K \text{ and } g \in G.$$

We let $\mathcal{D}(\rho)$ the ring of all left-invariant differential operators on G which map $C^\infty(\rho)$ into itself.

G. Shimura proved that if ρ is one-dimensional, then $\mathcal{D}(\rho)$ is commutative and there exists a canonically defined set of generators $L_1, L_2, \cdots, L_r$ for $\mathcal{D}(\rho)$ which are algebraically independent. Thus we have $\mathcal{D}(\rho) = \mathbb{C}[L_1, \cdots, L_r]$. Here r denotes the rank of the hermitian symmetric space G/K.

Returning to our case, since G^J acts on $H_{n,m} := H_n \times \mathbb{C}^{(m,n)}$ transitively, $H_{n,m}$ may be regarded as a homogeneous space G^J/K^J via the identification

$$G^J/K^J \longrightarrow H_{n,m}, \quad g \cdot K^J \longmapsto g \cdot (iE_n, 0), \quad g \in G^J,$$

where $K^J := U(n) \times Sym(m, \mathbb{R})$ is the stabilizer of G^J at $(iE_n, 0)$. Let τ be a continuous representation of K^J on a finite dimensional complex vector space V_τ. We denote by $C^\infty(\tau)$ the set of all V_τ-valued smooth functions on G^J such that

$$f(gk^{-1}) = \rho(k)f(g), \quad g \in G^J, \quad k \in K^J.$$

We let $\mathcal{D}(\tau)$ the ring of all left-invariant differential operators on G^J which map $C^\infty(\tau)$ into itself.

Problem: Discuss the 1st and 2nd main theorems for $\mathcal{D}(\tau)$ in the sense of invariant theory.

9. Final Remarks

In this section we give some open problems which should be investigated and give some remarks.

Let

$$G^J := Sp(n, \mathbb{R}) \propto H_{\mathbb{R}}^{(n,m)}$$

be the *Jacobi group* of degree n. Let $\Gamma^J := Sp(n, \mathbb{Z}) \propto H_{\mathbb{Z}}^{(n,m)}$ be the discrete subgroup of G^J. For the case $m = n = 1$, the spectral theory for $L^2\left(\Gamma^J \backslash G^J\right)$ of degree 1 had been investigated almost completely in [B1-2] and [B-B]. For general n and m, the spectral theory for $L^2\left(\Gamma^J \backslash G^J\right)$ is not known yet.

Problem 1. Decompose the Hilbert space $L^2\left(\Gamma^J \backslash G^J\right)$ into irreducible components of the Jacobi group G^J for general m and n. In particular, classify all the irreducible unitary or admissible representations of the Jacobi group G^J.

Problem 2. Give the *dimension formulae* for the vector space $J_{\rho,\mathcal{M}}(\Gamma_n)$ of Jacobi forms and the vector space $J_{\rho,\mathcal{M}}^{cusp}(\Gamma_n)$ of cuspidal Jacobi forms. Concerning this problem, discuss a vanishing theorem on the vector space of (cuspidal) Jacobi forms.

Problem 3. Construct Jacobi forms. Concerning this problem, we have several methods of construction:

(1) Fourier-Jacobi coefficients of Siegel modular forms.
(2) Eisenstein series of Klingen's type.
(3) Theta series.

It seems that so far we do not have any examples of Jacobi forms obtained without combining the above methods.

Problem 4. Develope the theory of L-functions for the Jacobi group G^J. There are several attempts to establish L-functions in the context of the Jacobi group by Japanese mathematicians A. Murase and T. Sugano using the so-called Whittaker-Shintani functions (cf. [Mur1-2]).

Problem 5. Give applications of Jacobi forms, for example in number theory, algebraic geometry and physics. In fact, Jacobi forms have found some applications in proving non-vanishing theorems for L-functions of modular forms [BFH], in the theory of Heeger points [GKS], in the theory of elliptic genera [Z] and in the string theory [C].

REFERENCES

[B1] R. Berndt, *Die Jacobigruppe und die Wärmeleitungsleichung*, Math. Z. **191** (1986), 351-361.

[B2] ———, *The continous part of* $L^2\left(\Gamma^J\backslash G^J\right)$, Abh. Math. Sem. Univ. Hamburg **60** (1990), 225-248.

[B3] ———, *On automorphic forms for the Jacobi group*, Hamburg Univ. Heft vol 17 (1991).

[B-B] R. Berndt and S. Boecherer, *Jacobi forms and discrete series representations of the Jacobi group*, Math. Z. **204** (1990), 13-44.

[BFH] D. Bump, S. Friedberg and J. Hoffstein, *Nonvanishing theorems for L-functions of modular forms and their derivatives*, Ann. Math. **131** (1990), 53-128.

[C] J. L. Cardy, *Operator content of two-dimensional conformally invariant theories*, Nuclear Physics B **270** (1986), 186-204.

[E-Z] Eichler, M. and Zagier, D., Progress in math. 55 (1985), *The theory of Jacobi forms*, Birkhäuser, Boston-Basel-Stuttgart.

[F1] E. Freitag, *Thetareihen mit harmonischen Koeffizienten zur Siegelschen Modulgruppe*, Math. Ann. **254** (1980), 27-51.

[F2] ———, *Siegelsche Modulfunktionen*, Springer-Verlag, Heidelberg-Berlin-New York, 1983.

[GKZ] B. Gross, W. Kohnen and D. Zagier, *Heeger points and the derivative of L-series II*, Math. Ann. **278** (1987), 497-562.

[K-V] M. Kashiwara and M. Vergne, *On the Segal-Shale-Weil representations and harmonic polynomials*, Invent. Math. **44** (1978), 1-47.

[Ma1] H. Maass, *Über die Darstellung der Modulformen n-ten durch Poincarésche Reihnen*, Math. Ann. **135** (1961), 125-151.

[Ma2] ———, *Siegel modular forms and Dirichlet series*, vol. 216, Springer-Verlag, Lecture Notes in Math, 1971.

[Mu] D. Mumford, *Abelian varieties*, Oxford University Press, 1985.

[M-N-N] D. Mumford, M. Nori and P. Norman, *Tata Lectures on Theta III* , *Progress in Math.*, vol. 97, Birkhäuser, 1991.

[Mur1] A. Murase, *L-functions attached to Jacobi forms of degree n (I)*, J. reine angew. Math. **401** (1989), 122-156.

[Mur2] ———, *L-functions attached to Jacobi forms of degree n (II)*, Math. Ann. **290** (1991), 247-276.

[S1] G. Shimura, *On modular forms of half integral weight*, Ann. Math. **97** (1973), 440-481.

[S2] ———, *On certain reciprocity laws for theta functions and modular forms*, Acta Math. **141** (1979), 35-71.

[S3] ———, *Invariant differential operators on hermitian symmetric spaces*, Ann. Math. **132** (1990), 237-272.

[T] K. Takase, *On unitary representations of Jacobi groups*, J. reine angew. Math. **430** (1992), 139-149.

[W1] R. Weissauer, *Vektorwertige Siegelsche Modularformen kleinen Gewichtes*, J. reine angew. Math. **343** (1983), 184-202.

[W2] ———, *Stabile Modulformen und Eisensteinreihnen, Lecture Notes in Math.*, Springer-Verlag.

[Y1] J.-H. Yang, *Harmonic analysis on the quotient spaces of Heisenberg groups*, Nagoya Math. J. **123** (1991), 103-117.

[Y2] ———, *Harmonic analysis on the quotient spaces of Heisenberg groups (II)*, to appear in J. Number Theory.
[Y3] ———, *The Siegel-Jacobi operator*, Abh. Math. Sem. Univ. Hamburg **63** (1993), 135-146.
[Y4] ———, *Some results on Jacobi forms of higher degree*, Proc. RIMS Symp. on Automorphic forms and associated Zeta functions, Kyoto (1992), 36-50.
[Y5] ———, *On Jacobi forms of higher degree*, to appear in Proc. RIMS Symp. on Infinite root systems and the period integrals, Kyoto.
[Y6] ———, *Singular Jacobi forms, preprint.*
[Y7] ———, *Harmonic analysis for the Jacobi group of higher degree, preprint.*
[Y8] ———, *Stable Jacobi forms, preprint.*
[Y9] ———, *Construction of vector valued modular forms from Jacobi forms, preprint.*
[Za1] D. Zagier, *Note on the Landweber-Stong Elliptic Genus, Springer Lecture Notes.*
[Za2] ———, *Periods of modular forms and Jacobi theta functions*, Invent. Math. **104** (1991), 449-465.
[Zh] N. A. Zharkovskaya, *The Siegel operator and Hecke operators*, Functional Anal. Appl. **8** (1974), 113-120.
[Zi] C. Ziegler, *Jacobi forms of higher degree*, Abh. Math. Sem. Univ. Hamburg **59** (1989), 191-224.

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SINGULAR JACOBI FORMS

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ABSTRACT. We introduce the differential operator $M_{g,h,\mathcal{M}}$ characterizing singular Jacobi forms. We also characterize singular Jacobi forms by the weight of the associated rational representation of the general linear group. And we provide eigenfunctions of the differential operator $M_{g,h,\mathcal{M}}$.

1. INTRODUCTION

Let g and h be two positive integers. Let $\mathcal{M}$ be a symmetric positive definite, half-integral matrix of degree h. For two positive integers k and l, we denote by $\mathbf{R}^{(k,l)}$ the space of all $k \times l$ matrices with entries in the field $\mathbf{R}$ of real numbers. We let

$$\mathscr{P}_g := \{Y \in \mathbf{R}^{(g,g)} \mid Y = {}^tY > 0\}$$

be the open convex cone of positive definite matrices of degree g in the Euclidean space $\mathbf{R}^{g(g+1)/2}$. We define the differential operator $M_{g,h,\mathcal{M}}$ on $\mathscr{P}_g \times \mathbf{R}^{(h,g)}$ defined by

$$M_{g,h,\mathcal{M}} := \det(Y) \cdot \det\left(\frac{\partial}{\partial Y} + \frac{1}{8\pi}\, {}^t\!\left(\frac{\partial}{\partial V}\right) \mathcal{M}^{-1} \left(\frac{\partial}{\partial V}\right)\right),$$

where

$$Y = (y_{\mu\nu}) \in \mathscr{P}_g, \quad V = (v_{kl}) \in \mathbf{R}^{(h,g)}, \quad \frac{\partial}{\partial Y} = \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial y_{\mu\nu}}\right)$$

and

$$\frac{\partial}{\partial V} = \left(\frac{\partial}{\partial v_{kl}}\right).$$

We note that this differential operator generalizes the differential operator $M_g := \det(Y) \cdot \det(\partial/\partial Y)$ on $\mathscr{P}_g$ which was introduced by H. Maass (cf. [M]). Using the differential operator M_g, Maass (cf. [M], pp. 202–204) proved that if a nonzero *singular* modular form of degree n and weight k exists, then $nk \equiv 0 \pmod 2$ and $0 < 2k \leq n-1$. The converse was proved by R. Weissauer (cf. [W], Satz 4).

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The aim of this paper is to characterize *singular Jacobi forms*. Singular Jacobi forms are defined to be the Jacobi forms which admit a Fourier expansion such that a Fourier coefficient $c(T, R)$ vanishes unless

$$\det\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} = 0.$$

For more detail, we refer to Definition 2.2. This paper is organized as follows. In Section 2, we review the notion of singular Jacobi forms which was introduced by Ziegler (cf. [Z], Definition 3.7) and establish the notations. In Section 3, we investigate some properties of the differential operator $M_{g,h,\mathcal{M}}$ to be used in the next section. In Section 4, we prove the main theorems. That is, we prove that singular Jacobi forms are characterized by $M_{g,h,\mathcal{M}}$ and the weight of the associated rational representation of the general linear group $GL(g, \mathbf{C})$. In the final section, we provide eigenfunctions of the above-mentioned differential operator $M_{g,h,\mathcal{M}}$.

Notations. We denote by $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ the ring of integers, the field of real numbers, and the field of complex numbers respectively. $\mathrm{Sp}(g, \mathbf{R})$ denotes the symplectic group of degree g. H_g denotes the Siegel upper half plane of degree g. For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(g, \mathbf{R})$ and $Z \in H_g$, we set $M\langle Z\rangle := (AZ + B)(CZ + D)^{-1}$. $\Gamma_g := \mathrm{Sp}(g, \mathbf{Z})$ denotes the Siegel modular group of degree g. $[\Gamma_g, k]$ (*resp.* $[\Gamma_g, \rho]$) denotes the vector space of all Siegel modular forms of weight k (*resp. of type* ρ). The symbol "$:=$" means that the expression on the right is the definition of that on the left. We denote by $\mathbf{Z}^+$ the set of all positive integers. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. For $A \in F^{(k,k)}$, $\sigma(A)$ denotes the trace of A. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. E_g denotes the identity matrix of degree g.

2. Singular Jacobi forms

In this section, we establish the notations and define the concept of singular Jacobi forms. Let

$$\mathrm{Sp}(g, \mathbf{R}) = \{M \in \mathbf{R}^{(2g,2g)} \mid {}^tMJ_gM = J_g\}$$

be the symplectic group of degree g, where

$$J_g := \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}.$$

It is easy to see that $\mathrm{Sp}(g, \mathbf{R})$ acts on H_g transitively by

$$M\langle Z\rangle := (AZ + B)(CZ + D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(g, \mathbf{R})$ and $Z \in H_g$. For two positive integers g and h, we consider the *Heisenberg group*

$$H_{\mathbf{R}}^{(g,h)} := \{[(\lambda, \mu), \kappa] \mid \lambda, \mu \in \mathbf{R}^{(h,g)}, \kappa \in \mathbf{R}^{(h,h)}, \ \kappa + \mu\,{}^t\lambda \text{ symmetric}\}$$

endowed with the following multiplication law

$$[(\lambda, \mu), \kappa] \circ [(\lambda', \mu'), \kappa'] := [(\lambda + \lambda', \mu + \mu'), \kappa + \kappa' + \lambda\,{}^t\mu' - \mu\,{}^t\lambda'].$$

We define the semidirect product of $\mathrm{Sp}(g, \mathbf{R})$ and $H_{\mathbf{R}}^{(g,h)}$

$$G^J := \mathrm{Sp}(g, \mathbf{R}) \propto H_{\mathbf{R}}^{(g,h)}$$

endowed with the following multiplication law

$$\begin{aligned}(M, [(\lambda, \mu), \kappa]) \cdot (M', [(\lambda', \mu'), \kappa'])\\ := (MM', [(\tilde{\lambda} + \lambda', \tilde{\mu} + \mu'), \kappa + \kappa' + \tilde{\lambda}\,{}^t\mu' - \tilde{\mu}\,{}^t\lambda']),\end{aligned}$$

with $M, M' \in \mathrm{Sp}(g, \mathbf{R})$ and $(\tilde{\lambda}, \tilde{\mu}) := (\lambda, \mu)M'$. It is easy to see that G^J acts on $H_g \times \mathbf{C}^{(h,g)}$ transitively by

$$(2.1)\qquad (M, [(\lambda, \mu), \kappa]) \cdot (Z, W) := (M\langle Z\rangle, (W + \lambda Z + \mu)(CZ + D)^{-1}),$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(g, \mathbf{R})$, $[(\lambda, \mu), \kappa] \in H_{\mathbf{R}}^{(g,h)}$ and $(Z, W) \in H_g \times \mathbf{C}^{(h,g)}$.

Let ρ be a rational representation of $GL(g, \mathbf{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbf{R}^{(h,h)}$ be a symmetric half-integral semipositive definite matrix of degree h. Let $C^\infty(H_g \times \mathbf{C}^{(h,g)}, V_\rho)$ be the algebra of all C^∞ functions on $H_g \times \mathbf{C}^{(h,g)}$ with values in V_ρ. For $f \in C^\infty(H_g \times \mathbf{C}^{(h,g)}, V_\rho)$, we define

$$\begin{aligned}&(f|_{\rho,\mathcal{M}}[(M, [(\lambda, \mu), \kappa])])(Z, W)\\ (2.2)\qquad &:= e^{-2\pi i \sigma(\mathcal{M}[W+\lambda Z+\mu](CZ+D)^{-1}C)} \cdot e^{2\pi i \sigma(\mathcal{M}(\lambda Z\,{}^t\lambda + 2\lambda\,{}^tW + (\kappa + \mu\,{}^t\lambda)))}\\ &\times \rho(CZ + D)^{-1} f(M\langle Z\rangle, (W + \lambda Z + \mu)(CZ + D)^{-1}),\end{aligned}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(g, \mathbf{R})$ and $[(\lambda, \mu), \kappa] \in H_{\mathbf{R}}^{(g,h)}$.

Definition 2.1. Let ρ and $\mathcal{M}$ be as above. Let

$$H_{\mathbf{Z}}^{(g,h)} := \{[(\lambda, \mu), \kappa] \in H_{\mathbf{R}}^{(g,h)} \mid \lambda, \mu \in \mathbf{Z}^{(h,g)}, \kappa \in \mathbf{Z}^{(h,h)}\}.$$

A *Jacobi form* of index $\mathcal{M}$ with respect to ρ on Γ_g is a holomorphic function $f \in C^\infty(H_g \times \mathbf{C}^{(h,g)}, V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho,\mathcal{M}}[\tilde{\gamma}] = f$ for all $\tilde{\gamma} \in \Gamma_g^J := \Gamma_g \propto H_{\mathbf{Z}}^{(g,h)}$.

(B) f has a Fourier expansion of the following form:

$$f(Z, W) = \sum_{\substack{T \geq 0\\ \text{half-integral}}} \sum_{R \in \mathbf{Z}^{(g,h)}} c(T, R) \cdot e^{2\pi i \sigma(TZ)} \cdot e^{2\pi i \sigma(RW)}$$

with $c(T, R) \neq 0$ only if $\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} \geq 0$.

If $g \geq 2$, the condition (B) is superfluous by the Koecher principle (cf. [Z], Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma_g)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ_g. In the special case $V_\rho = \mathbf{C}$, $\rho(A) = (\det A)^k$ ($k \in \mathbf{Z}$, $A \in GL(g, \mathbf{C})$), we write $J_{k,\mathcal{M}}(\Gamma_g)$ instead of $J_{\rho,\mathcal{M}}(\Gamma_g)$ and call k the *weight* of a Jacobi form $f \in J_{k,\mathcal{M}}(\Gamma_g)$.

Ziegler (cf. [Z], Theorem 1.8 or [E-Z], Theorem 1.1) proves that the vector space $J_{\rho,\mathcal{M}}(\Gamma_g)$ is finite dimensional.

Definition 2.2. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ is said to be *singular* if it admits a Fourier expansion such that a Fourier coefficient $c(T, R)$ vanishes unless $\det\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} = 0$.

Example 2.3. Let $S \in \mathbf{Z}^{(2k,2k)}$ be a symmetric, positive definite, unimodular even integral matrix and $c \in \mathbf{Z}^{(2k,h)}$. We define the theta series

$$(2.3)\quad \vartheta_{S,c}^{(g)}(Z, W) := \sum_{\lambda \in \mathbf{Z}^{(2k,g)}} e^{\pi\{\sigma(S\lambda Z\,{}^t\lambda)+2\sigma({}^tcS\lambda\,{}^tW)\}}, \quad Z \in H_g, \quad W \in \mathbf{C}^{(h,g)}.$$

We put $\mathcal{M} := \frac{1}{2}{}^tcSc$. We assume that $2k < g + \operatorname{rank}(\mathcal{M})$. Then it is easy to see that $\vartheta_{S,c}^{(g)}$ is a singular Jacobi form in $J_{k,\mathcal{M}}(\Gamma_g)$ (cf. [Z], p. 212).

Remark 2.4. Without loss of generality, we may assume that $\mathcal{M}$ is a *positive definite* symmetric, half-integral matrix of degree h (cf. [Z], Theorem 2.4). From now on, throughout this paper $\mathcal{M}$ is assumed to be positive definite.

3. THE DIFFERENTIAL OPERATOR $M_{g,h,\mathcal{M}}$

Let $\mathcal{P}_g$ be the open convex cone of positive definite matrices of degree g in $\mathbf{R}^{g(g+1)/2}$ defined in the introduction.

From now on, for $Y = (y_{\mu\nu}) \in \mathcal{P}_g$ and $V = (v_{kl}) \in \mathbf{R}^{(h,g)}$, we write

$$dY = (dy_{\mu\nu}), \quad dV = (dv_{kl}), \quad 1 \le \mu, \nu \le g, \ 1 \le k \le h, \ 1 \le l \le g,$$
$$\frac{\partial}{\partial Y} = \left(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial y_{\mu\nu}}\right), \quad \frac{\partial}{\partial V} = \left(\frac{\partial}{\partial v_{kl}}\right).$$

For a real matrix X of degree g and an integer k with $1 \le k \le g$, we denote by $C_k(X)$ the matrix of minors of degree k. We define the differential operator $M_{k,h}$ on $\mathcal{P}_g \times \mathbf{R}^{(h,g)}$ by

$$(3.1)\quad M_{k,h} := \sigma\left(C_k(Y)C_k\left(\frac{\partial}{\partial Y} + \frac{1}{8\pi}\,{}^t\!\left(\frac{\partial}{\partial V}\right)\left(\frac{\partial}{\partial V}\right)\right)\right), \qquad 1 \le k \le g.$$

Following the notations of H. Maass (cf. [M], p. 67), the differential operator $M_{k,h}$ may be expressed as

$$M_{k,h} = \sum_{\substack{1\le\alpha_1<\cdots<\alpha_k\le g\\ 1\le\beta_1<\cdots<\beta_k\le g}} \begin{pmatrix}\alpha_1\cdots\alpha_k\\ \beta_1\cdots\beta_k\end{pmatrix}_Y \cdot \begin{pmatrix}\beta_1\cdots\beta_k\\ \alpha_1\cdots\alpha_k\end{pmatrix}_{\frac{\partial}{\partial Y}+\frac{1}{8\pi}{}^t(\frac{\partial}{\partial V})(\frac{\partial}{\partial V})}, \qquad 1 \le k \le g.$$

In particular, we are interested in the following differential operator

$$(3.2)\qquad M_{g,h} := \det(Y)\cdot\det\left(\frac{\partial}{\partial Y} + \frac{1}{8\pi}\,{}^t\!\left(\frac{\partial}{\partial V}\right)\left(\frac{\partial}{\partial V}\right)\right).$$

Lemma 3.1. *Let $T = {}^tT \in \mathbf{R}^{(h,g)}$ and $R \in \mathbf{R}^{(g,h)}$. Then we have*

$$(3.3)\qquad \frac{\partial}{\partial Y}e^{-2\pi\sigma(TY)} = -2\pi e^{-2\pi\sigma(TY)}\cdot T$$

and

$$(3.4)\quad M_{g,h}(e^{-2\pi\sigma(TY+RV)}) = \left(-\frac{\pi}{2}\right)^g \det(Y\cdot(4T - R\,{}^tR))\cdot e^{-2\pi\sigma(TY+RV)}.$$

Proof. (3.3) follows from an easy computation. We set

$$(3.5)\qquad P := (P_{\mu\nu}) := \frac{\partial}{\partial Y} + \frac{1}{8\pi}\,{}^t\!\left(\frac{\partial}{\partial V}\right)\left(\frac{\partial}{\partial V}\right).$$

Then we have

$$P_{\mu\mu} := \frac{\partial}{\partial y_{\mu\mu}} + \frac{1}{8\pi}\sum_{k=1}^{h}\frac{\partial^2}{\partial v_{k\mu}^2}, \qquad 1 \le \mu \le g, \tag{3.6}$$

and

$$P_{\mu\nu} := \frac{1}{2}\frac{\partial}{\partial y_{\mu\nu}} + \frac{1}{8\pi}\sum_{k=1}^{h}\frac{\partial^2}{\partial v_{k\mu}\partial v_{k\nu}}, \qquad 1 \le \mu < \nu \le g. \tag{3.7}$$

We note that if $T = (t_{\mu\nu})$, $R = (r_{\mu k})$ and $V = (v_{k\nu})$, then

$$\sigma(TY) = \sum_{\mu=1}^{g} t_{\mu\mu}y_{\mu\mu} + 2\sum_{\mu<\nu} t_{\mu\nu}y_{\mu\nu}, \quad \sigma(RV) = \sum_{\mu=1}^{g}\sum_{k=1}^{h} r_{\mu k}v_{k\mu}.$$

By an easy calculation, we get

$$P_{\mu\nu}(e^{-2\pi\sigma(TY+RV)}) = -\frac{\pi}{2}\left(4t_{\mu\nu} - \sum_{k=1}^{h} r_{\mu k}r_{\nu k}\right)\cdot e^{-2\pi\sigma(TY+RV)}. \tag{3.8}$$

Thus we get

$$\det(P)(e^{-2\pi\sigma(TY+RV)}) = \left(-\frac{\pi}{2}\right)^g \det(4T - R\,{}^tR)\cdot e^{-2\pi\sigma(TY+RV)}. \tag{3.9}$$

Consequently we obtain the desired result (3.4). □

Now we let $\mathcal{M}$ be a symmetric positive definite, half-integral matrix of degree h. We define the differential operator $M_{g,h,\mathcal{M}}$ on $\mathcal{P}_g \times R^{(h,g)}$ by

$$M_{g,h,\mathcal{M}} := \det(Y)\cdot\det\left(\frac{\partial}{\partial Y} + \frac{1}{8\pi}\,{}^t\!\left(\mathcal{M}^{-1/2}\frac{\partial}{\partial V}\right)\left(\mathcal{M}^{-1/2}\frac{\partial}{\partial V}\right)\right). \tag{3.10}$$

By changing the coordinate V by $\widehat{V} = \mathcal{M}^{1/2}V$, we obtain $\partial/\partial\widehat{V} = \mathcal{M}^{-1/2}\partial/\partial V$. Thus (3.10) may be written as

$$M_{g,h,\mathcal{M}} = \det(Y)\cdot\det\left(\frac{\partial}{\partial Y} + \frac{1}{8\pi}\,{}^t\!\left(\frac{\partial}{\partial\widehat{V}}\right)\left(\frac{\partial}{\partial\widehat{V}}\right)\right). \tag{3.11}$$

Theorem 3.2. *Let $T = {}^tT \in \mathbf{R}^{(h,g)}$ and $R \in \mathbf{R}^{(g,h)}$. Then we have*

$$\begin{aligned} &M_{g,h,\mathcal{M}}(e^{-2\pi\sigma(TY+RV)}) \\ &\quad = \left(-\frac{\pi}{2}\right)^g \det(Y\cdot(4T - R\mathcal{M}^{-1}\,{}^tR))\cdot e^{-2\pi\sigma(TY+RV)}. \end{aligned} \tag{3.12}$$

Proof. If we set $\widehat{R} = R\mathcal{M}^{-1/2}$, then $\widehat{R}\widehat{V} = RV$. Applying (3.11) to $e^{-2\pi\sigma(TY+RV)} = e^{-2\pi\sigma(TY+\widehat{R}\widehat{V})}$ and using Lemma 3.1, we obtain the desired result (3.12). □

4. Proof of main theorems

First we prove that a singular Jacobi form is characterized by the differential operator $M_{g,h,\mathcal{M}}$.

Theorem 4.1. *Let $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ be a Jacobi form of index $\mathcal{M}$ with respect to a rational representation ρ of $GL(g, \mathbf{C})$. Then the following conditions are equivalent:*

(1) *f is a singular Jacobi form.*

(2) *f satisfies the differential equation $M_{g,h,\mathcal{M}} f = 0$.*

Proof. First we observe that for a Fourier coefficient $c(T, R)$ of $f(Z, W)$, we have

$$\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}\,{}^tR & \mathcal{M} \end{pmatrix} = \begin{pmatrix} E_g & \frac{1}{2}R\mathcal{M}^{-1} \\ 0 & E_h \end{pmatrix} \begin{pmatrix} T - \frac{1}{4}R\mathcal{M}^{-1}\,{}^tR & 0 \\ 0 & \mathcal{M} \end{pmatrix} {}^t\begin{pmatrix} E_g & \frac{1}{2}R\mathcal{M}^{-1} \\ 0 & E_h \end{pmatrix}.$$

Thus it follows immediately that $\det\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} = 0$ if and only if

$$\det(4T - R\mathcal{M}^{-1}\,{}^tR) = 0.$$

Suppose $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ is singular. Then according to Theorem 3.2, we have

$$M_{g,h,\mathcal{M}} f(Z, W) = \left(-\frac{\pi}{2}\right)^g \det(Y) \sum_{T,R} c(T, R) \det(4T - R\mathcal{M}^{-1}\,{}^tR) \times e^{2\pi i\sigma(TZ)} \cdot e^{2\pi i\sigma(RW)}.$$

Since f singular, $c(T, R) \neq 0$ implies $\det(4T - R\mathcal{M}^{-1}\,{}^tR) = 0$. Hence we obtain the equation $M_{g,h,\mathcal{M}} f = 0$.

Conversely, we assume $M_{g,h,\mathcal{M}} f = 0$. Then

$$\left(-\frac{\pi}{2}\right)^g \det(Y)\, c(T, R) \det(4T - R\mathcal{M}^{-1}\,{}^tR) = \int_0^1 \cdots \int_0^1 M_{g,h,\mathcal{M}} f(Z, W) \cdot e^{-2\pi i\sigma(TZ+RW)} d[X]d[U],$$

where $Z = X + iY$, $W = U + iV$ with real $X = (x_{\mu\nu})$, $Y = (y_{\mu\nu})$, $U = (u_{kl})$, $V = (v_{kl})$ and

$$d[X]d[U] = dx_{11}dx_{12}\cdots dx_{g-1,g}dx_{gg}du_{11}\cdots du_{h,g-1}du_{hg}.$$

According to the assumption, we have for any T and R

$$c(T, R) \cdot \det(4T - R\mathcal{M}^{-1}\,{}^tR) = 0.$$

This means that $c(T, R) \neq 0$ implies $\det(4T - R\mathcal{M}^{-1}\,{}^tR) = 0$. Hence f is singular. □

Let S be a symmetric positive definite integral matrix of degree h and let $a, b \in \mathbf{Q}^{(h,g)}$. We consider

$$\vartheta_{S,a,b}(Z, W) := \sum_{\lambda \in \mathbf{Z}^{(h,g)}} e^{\pi i\sigma\{S((\lambda+a)Z\,{}^t(\lambda+a)+2(\lambda+a)\,{}^t(W+b))\}} \tag{4.1}$$

with characteristic (a, b) converging uniformly on any compact subset of $H_g \times \mathbf{C}^{(h,g)}$.

If f is a Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma_g)$, then according to [Z], we may write

$$f(Z, W) = \sum_{a\in\mathcal{N}} f_a(Z) \cdot \vartheta_{2\mathcal{M},a,0}(Z, W), \quad Z \in H_g, \quad W \in \mathbf{C}^{(h,g)}, \tag{4.2}$$

where $\mathcal{N}$ is a complete system of representatives of $(2\mathcal{M})^{-1}\mathbf{Z}^{(h,g)}/\mathbf{Z}^{(h,g)}$ and $\{f_a\colon H_g \to V_\rho \mid a \in \mathcal{N}\}$ are uniquely determined holomorphic vector valued functions on H_g.

According to Yang (cf. [Y], Corollary 3.2), we have

Proposition 4.2. *Let $2\mathcal{M}$ be unimodular. We assume that ρ satisfies the following condition*:

$$\rho(A) = \rho(-A) \quad \textit{for all } A \in GL(g, \mathbf{C}). \tag{4.3}$$

Then we have

$$J_{\rho,\mathcal{M}}(\Gamma_g) = [\Gamma_g, \tilde{\rho}] \cdot \vartheta_{2\mathcal{M},0,0}(Z, W) \cong [\Gamma_g, \tilde{\rho}], \tag{4.4}$$

where $\tilde{\rho} = \rho \otimes \det^{-h/2}$.

Notation 4.3. In Propositoin 4.2, we denote the isomorphism of $J_{\rho,\mathcal{M}}(\Gamma_g)$ onto $[\Gamma_g, \rho \otimes \det^{-h/2}]$ by

$$S_{\rho,\mathcal{M}}: J_{\rho,\mathcal{M}}(\Gamma_g) \to [\Gamma_g, \rho \otimes \det^{-h/2}].$$

Definition 4.4. An irreducible finite dimensional representation ρ of $GL(g, \mathbf{c})$ is determined uniquely by its highest weight $(\lambda_1, \lambda_2, \dots, \lambda_g) \in \mathbf{Z}^g$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_g$. We denote this representation by $\rho = (\lambda_1, \lambda_2, \dots, \lambda_g)$. The number $k(\rho) := \lambda_g$ is called the *weight* of ρ.

Theorem 4.5. *Let $2\mathcal{M}$ be a symmetric, positive definite, unimodular even matrix of degree h. Assume that ρ is irreducible and satisfies the condition* (4.3). *Then a nonvanishing Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma_g)$ is singular if and only if $2k(\rho) < g+h$.*

Proof. According to Proposition 4.2, we have

$$J_{\rho,\mathcal{M}}(\Gamma_g) = [\Gamma_g, \rho \otimes \det^{-h/2}] \cdot \vartheta_{2\mathcal{M},0,0}(Z, W).$$

For any $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$, $f = S_{\rho,\mathcal{M}}(f) \cdot \vartheta_{2\mathcal{M},0,0}(Z, W)$. First of all, we observe that the Fourier coefficients $b(T, R)$ of $\vartheta_{2\mathcal{M},0,0}(Z, W)$ is given by

$$b(T, R) = \begin{cases} 1 & \text{if } \exists \lambda \in \mathbf{Z}^{(h,g)} \text{ s.t. } T = \mathcal{M}[\lambda],\ {}^tR = 2\mathcal{M}\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously we have $4T - R\mathcal{M}^{-1}\,{}^tR = 0$ for T, R with $b(T, R) \neq 0$. Let $a(T)$ and $c(T, R)$ be the Fourier coefficients of $S_{\rho,\mathcal{M}}(f)(Z)$ and $f(Z, W)$ respectively. If $c(T, R) \neq 0$, then $c(T, R) = a(T_1)b(T_2, R)$ with $T = T_1 + T_2$ because T_2 is uniquely determined by R.

Now we suppose that $f(Z, W) \neq 0$ is singular. If $a(T_1) \neq 0$ for some half integral $T_1 \geq 0$, then there exist $T_2 \geq 0$ and $R \in \mathbf{Z}^{(g,h)}$ such that $b(T_2, R) \neq 0$ and hence $c(T_1 + T_2, R) = a(T_1)b(T_2, R) \neq 0$ is the Fourier coefficient of $f(Z, W)$. By assumption and the fact that $4T_2 - R\mathcal{M}^{-1}\,{}^tR = 0$, we have

$$\det(4(T_1 + T_2) - R\mathcal{M}^{-1}\,{}^tR) = \det(4T_1) = 0.$$

Hence $S_{\rho,\mathcal{M}}(f) \neq 0$ is singular. According to [W], Satz 4, we obtain the condition $2k(\rho) < g+h$. Conversely, suppose $2k(\rho) < g+h$. Then, according to [W], Satz 4, $S_{\rho,\mathcal{M}}(f)$ is singular. If $c(T, R) \neq 0$, then we have $c(T, R) = a(T_1)b(T_2, R)$ for uniquely determined half-integral T_1 and T_2 with $T = T_1 + T_2$. Since $a(T_1) \neq 0$ and $S_{\rho,\mathcal{M}}(f)$ is singular, $\det(T_1) = 0$. Using the fact that $4T_2 - R\mathcal{M}^{-1}\,{}^tR = 0$, we obtain

$$\det(4T - R\mathcal{M}^{-1}\,{}^tR) = \det(4T_1) = 0.$$

Hence $f(Z, W)$ is singular. This completes the proof. □

Remark 4.6. For general ρ and $\mathcal{M}$ without the above assumptions on them, it is possible to prove that *a nonvanishing Jacobi form* $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ *is singular if and only if* $2k(\rho) < g + h$ *since* [W], Satz 4 also holds for normal subgroups of finite index in Γ_g.

Remark 4.7. Ziegler (cf. [Z], Theorem 3.12) proved that a strongly singular Jacobi form may be written as a linear combination of theta series $\vartheta_{S,c}^{(g)}$ (cf. (2.3)).

Finally we prove

Theorem 4.8. *Let* $\mathcal{M}$ *be a symmetric, positive definite half-integral matrix of degree* h. *Then for all* $a, b \in \mathbf{Q}^{(h,g)}$, *the theta series* $\vartheta_{2\mathcal{M},a,b}(Z, W)$ *satisfies the differential equation*

$$M_{g,h,\mathcal{M}}\vartheta_{2\mathcal{M},a,b}(Z, W) = 0. \tag{4.5}$$

Proof. For each $\lambda \in \mathbf{Z}^{(h,g)}$, we put

$$T_\lambda := {}^t(\lambda + a)\mathcal{M}(\lambda + a), \quad R_\lambda := 2\,{}^t(\lambda + a)\mathcal{M}.$$

According to Theorem 3.2, we have

$$\begin{aligned} &M_{g,h,\mathcal{M}}\vartheta_{2\mathcal{M},a,b}(Z, W) \\ &= \left(-\frac{\pi}{2}\right)^g \det(Y) \cdot \sum_{\lambda \in \mathbf{Z}^{(h,g)}} \det(4T_\lambda - R_\lambda \mathcal{M}^{-1}\,{}^tR_\lambda) \\ &\quad \cdot e^{2\pi i \sigma\{\mathcal{M}((\lambda+a)Z\,{}^t(\lambda+a)+2(\lambda+a)\,{}^t(W+b))\}}. \end{aligned}$$

It is easy to show that $\det(4T_\lambda - R_\lambda\mathcal{M}^{-1}\,{}^tR_\lambda) = 0$ for all $\lambda \in \mathbf{Z}^{(h,g)}$. Hence we obtain the equation (4.5). □

5. Eigenfunctions of $M_{g,h,\mathcal{M}}$

In this section, we give eigenfunctions of the differential operator $M_{g,h,\mathcal{M}}$.

For $Y \in \mathcal{P}_g$, we let $Y = T[Q]$ be the Jacobian decomposition of Y such that

$$T = \begin{pmatrix} t_1 & 0 & \dots & 0 \\ 0 & t_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t_g \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix},$$

where T is a diagonal matrix with all $t_\nu > 0$ $(\nu = 1, \dots, g)$ and Q is an upper triangular matrix with ones in the main diagonal (cf. [M]). We call the t_ν $(1 \le \nu \le g)$ and the elements q_{kl} $(1 \le k < l \le g)$ of Q *Jacobian coordinates* of Y.

For $s = (s_1, \dots, s_g) \in \mathbf{C}^g$, we define the function $f_s(Y)$ on $\mathcal{P}_g$ by

$$f_s(Y) := \prod_{k=1}^{g} t_k^{s_k + k/2 - (g+1)/4}, \quad Y = T[Q] \in \mathcal{P}_g.$$

We put

$$\varepsilon := (\varepsilon_{11}, \dots, \varepsilon_{1g}, \dots, \varepsilon_{h1}, \dots, \varepsilon_{hg}) \in \mathbf{Z}_2^{hg}, \qquad \mathbf{Z}_2 = \{0, 1\}.$$

That is, $\varepsilon_{ij} = 0$ or 1 for $1 \le i \le h$ and $1 \le j \le g$.

Theorem 5.1. *Let $\mathscr{M}$ be a half-integral positive definite symmetric matrix of degree h. Let $\mathscr{M}^{1/2}$ be the unique positive definite symmetric matrix such that $(\mathscr{M}^{1/2})^2 = \mathscr{M}$. We put $\mathscr{M}^{1/2} := (\alpha_{ij})$, $1 \le i, j \le h$. Then for each $s = (s_1, \ldots, s_g) \in \mathbf{C}^g$ and $\varepsilon = (\varepsilon_{11}, \ldots, \varepsilon_{h1}, \ldots, \varepsilon_{hg}) \in \mathbf{Z}_2^{hg}$, the function $f_{s,\varepsilon,\mathscr{M}}(Y, V)$ on $\mathscr{P}_g \times \mathbf{R}^{(h,g)}$ defined by*

$$f_{s,\varepsilon,\mathscr{M}}(Y, V) := f_s(Y) \cdot \left(\sum_{k=1}^{h} \alpha_{1k} v_{k1}\right)^{\varepsilon_{11}} \cdots \left(\sum_{k=1}^{h} \alpha_{ik} v_{kj}\right)^{\varepsilon_{ij}} \cdots \left(\sum_{k=1}^{h} \alpha_{hk} v_{kg}\right)^{\varepsilon_{hg}}$$

is an eigenfunction of the differential operator $M_{g,h,\mathscr{M}}$ with the eigenvalue $\lambda_{s,\varepsilon,\mathscr{M}} = \prod_{k=1}^{g}(s_k + (g-1)/4)$.

Proof. We leave the proof to the interested reader. □

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References

[E-Z] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progress in Math., 55, Birkhäuser, Boston, Basel, and Stuttgart, 1985.

[M] H. Maass, *Siegel modular forms and Dirichlet series*, Lecture Notes in Math., vol. 216, Springer-Verlag, Berlin and New York, 1971.

[W] R. Weissauer, *Vektorwertige Siegelsche Modulformen kleinen Gewichtes*, J. Reine Angew. Math. **343** (1983), 184–202.

[Y] J.-H. Yang, *The Siegel-Jacobi operator*, Abh. Math. Sem. Univ. Hamburg **63** (1993), 135–146.

[Z] C. Ziegler, *Jacobi forms of higher degree*, Abh. Math. Sem. Univ. Hamburg **59** (1989), 191–224.

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CONSTRUCTION OF VECTOR VALUED MODULAR FORMS FROM JACOBI FORMS

JAE-HYUN YANG

ABSTRACT. We give a geometrical construction of the canonical automorphic factor for the Jacobi group and construct new vector valued modular forms from Jacobi forms by differentiating them with respect to toroidal variables and then evaluating at zero.

1. **Introduction.** For given two fixed positive integers n and m, we let

$$H_n := \{Z \in \mathbb{C}^{(n,n)} \mid Z = {}^tZ,\ \operatorname{Im} Z > 0\}$$

be the Siegel upper half plane of degree n and let Γ_n be the Siegel modular group of degree n. Let

$$\mathcal{P}_{m,n} := \mathbb{C}[W_{11}, \ldots, W_{mn}], \quad W = (W_{kl}) \in \mathbb{C}^{(m,n)}$$

be the ring of polynomial functions on $\mathbb{C}^{(m,n)}$. Here $\mathbb{C}^{(n,n)}$ (resp. $\mathbb{C}^{(m,n)}$) denotes the space of all complex $n \times n$ (resp. $m \times n$)-matrices (see notation below). For any homogeneous polynomial $P \in \mathcal{P}_{m,n}$, we define the differential operator $P(\partial_W)$ on $\mathbb{C}^{(m,n)}$ as follows:

$$P(\partial_W) := P\Big(\frac{\partial}{\partial W_{11}}, \ldots, \frac{\partial}{\partial W_{mn}}\Big).$$

In this paper, the author proves that if P is a *homogeneous pluriharmonic* polynomial in $\mathcal{P}_{m,n}$ and $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ (see Definition 3.1) is a *Jacobi form* of index $\mathcal{M}$ with respect to a rational representation ρ of the general group $\mathrm{GL}(n, \mathbb{C})$, then the following function

$$P(\partial_W)f(Z, W)|_{W=0}$$

yields a vector valued modular form with respect to a new rational representation of $\mathrm{GL}(n, \mathbb{C})$. For precise details, we refer to Definition 5.1 and Main Theorem in Section 5. In [M-N-N] (*cf.* pp. 147–156), the authors proved the similar result for theta functions. Our result is a generalization of their result because theta functions are special examples of Jacobi forms.

This paper is organized as follows. In Section 2, we provide a geometrical construction of the canonical automorphic factor for the Jacobi group. In Section 3, we review Jacobi forms and establish the notation. In Section 4, we review pluriharmonic polynomials and obtain some properties to be used in the subsequent sections. In Section 5, we shall prove

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the main theorem. In Section 6, we obtain two identities by applying the main theorem to Jacobi forms.

NOTATION. We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers, and the field of complex numbers respectively. $\Gamma_n := \mathrm{Sp}(n,\mathbb{Z})$ denotes the Siegel modular group of degree n. The symbol ":=" means that the expression on the right is the definition of that on the left. We denotes by $\mathbb{Z}^+$ the set of all positive integers. $F^{(k,l)}$ denotes the set of all $k\times l$ matrices with entries in a commutative ring F. For a square matrix $A\in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. For $A\in F^{(k,l)}$ and $B\in F^{(k,k)}$, we set $B[A]={}^tABA$. For any $M\in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. E_n denotes the identity matrix of degree n.

2. **The canonical automorphic factor for the Jacobi group.** Let m and n be two fixed positive integers. It is well known that the automorphism group $\mathrm{Aut}(H_{m+n})$ of the Siegel upper half plane of degree $m+n$ is given by

$$\mathrm{Aut}(H_{m+n})=\mathrm{Sp}(m+n,\mathbb{R})/\{\pm E_{m+n}\}.$$

We observe that H_n is a rational boundary of H_{m+n} (*cf.* [N]). The normalizer $N(H_n):=\{\tilde\sigma\in\mathrm{Aut}(H_{m+n}):\tilde\sigma(H_n)\subset H_n\}$ of H_n is given by

$$N(H_n)=P(H_n)/\{\pm E_{m+n}\},$$

where

$$\begin{aligned}P(H_n):&=\{g\in\mathrm{Sp}(m+n,\mathbb{R}):g(H_n)\subset H_n\}\\&=\{[\sigma,u,(\lambda,\mu,\kappa)]\in\mathrm{Sp}(m+n,\mathbb{R})\}.\end{aligned}$$

Here we put

$$[\sigma,u,(\lambda,\mu,\kappa)]:=\begin{pmatrix}A&0&B&A{}^t\mu-B{}^t\lambda\\u\lambda&u&u\mu&u\kappa\\C&0&D&C{}^t\mu-D{}^t\lambda\\0&0&0&{}^tu^{-1}\end{pmatrix},$$

where $\sigma=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\mathrm{Sp}(n,\mathbb{R})$, $u\in GL(m,\mathbb{R})$, $\lambda,\mu\in\mathbb{R}^{(m,n)}$ and $\kappa\in\mathbb{R}^{(m,m)}$.

If $\begin{pmatrix}Z&{}^tW\\W&T\end{pmatrix}\in H_{m+n}$ with $Z\in H_n$, $W\in\mathbb{R}^{(m,n)}$ and $T\in H_m$, we simply write

$$(Z,W,T):=\begin{pmatrix}Z&{}^tW\\W&T\end{pmatrix}.$$

We denote the symplectic action of $N(H_n)$ on (Z,W,T) by

$$g\cdot(Z,W,T):=(\tilde Z,\tilde W,\tilde T),\quad g\in N(H_n).$$

It is easy to see that $(\tilde Z,\tilde W,\tilde T)$ is of the form

$$\begin{aligned}\tilde Z&=\sigma_g(Z),\\\tilde W&=a(g;Z)(W)+b(g;Z),\\\tilde T&=m_g(T)+c(g;Z,W),\end{aligned}$$

where $\sigma_g \in \mathrm{Aut}(H_n)$, $m_g \in \mathrm{Aut}(\mathcal{P}_m)$,

$$\begin{aligned} a(g;\cdot) &: H_n \longrightarrow \mathrm{GL}(\mathbb{C}^{(m,n)}) \quad \text{holomorphic}, \\ b(g;\cdot) &: H_n \longrightarrow \mathbb{C}^{(m,n)} \quad \text{holomorphic}, \\ c(g;\cdot,\cdot) &: H_n \times \mathbb{C}^{(m,n)} \longrightarrow H_m \quad \text{holomorphic}. \end{aligned}$$

Here $\mathcal{P}_m := \{Y \in \mathbb{R}^{(m,m)} \mid Y = {}^tY > 0\}$ is an open convex cone in $\mathbb{R}^{\frac{m(m+1)}{2}}$ and we set

$$\mathrm{Aut}(\mathcal{P}_m) := \{\xi \in \mathrm{GL}(\mathbb{C}^{(m,m)}) \mid \xi(\mathcal{P}_m) = \mathcal{P}_m\}.$$

REMARK 2.1. In [PS], Piateski-Sharpiro called the *mapping* $(Z,W,T) \longmapsto (\tilde{Z},\tilde{W},\tilde{T})$ a *quasilinear* transformation.

From now on, we set

$$H_{n,m} := H_n \times \mathbb{C}^{(m,n)}.$$

We observe that $g = [\sigma, u, (\lambda,\mu,\kappa)](\mathrm{mod}\{\pm E_{m+n}\}) \in N(H_n)$ acts on $H_{n,m}$ by

$$(Z,W) \longmapsto \big(\sigma_g(Z), a(g;Z)(W) + b(g;Z)\big).$$

The subgroup of $N(H_n)$ consisting of elements $g = [\sigma, u, (\lambda,\mu,\kappa)](\mathrm{mod}\{\pm E_{m+n}\})$ with the property

$$m_g = \text{Identity} \quad \text{on } H_m$$

is called the *Jacobi group*, denoted by G^J. It follows immediately from the definition that

$$G^J = \{[\sigma, E_m, (\lambda,\mu,\kappa)] \in P(H_n)\}.$$

It is easy to see that G^J is the semidirect product of $\mathrm{Sp}(n,\mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$, where

$$H_{\mathbb{R}}^{(n,m)} := \{[E_n, E_m, (\lambda,\mu,\kappa)] \in P(H_n)\}$$

is the nilpotent 2-step subgroup of $P(H_n)$, called the *Heisenberg group*. For some results on $H_{\mathbb{R}}^{(n,m)}$, we refer to [Y1]–[Y2].

Now we consider another subgroup $\tilde{G}$ of G^J. By the definition, $\tilde{G}$ consists of elements of G^J whose action is of the following form:

$$(Z,W,T) \longmapsto \big(\sigma_g(Z), a(g;Z)(W), T + c(g;Z,W)\big), \quad c(g;Z,0) = 0.$$

LEMMA 2.2. *The map*

$$J\colon \tilde{G} \times H_n \longrightarrow \mathrm{GL}(\mathbb{C}^{(m,n)})$$

defined by

$$J(\tilde{\sigma}, Z) := a(\tilde{\sigma}; Z), \quad \tilde{\sigma} \in \tilde{G},\ Z \in H_n$$

is a factor of automorphy for $\tilde{G}$.

PROOF. It is easy to prove it. We leave its proof to the reader. ∎

We note that the mapping

$$A\big(g,(Z,W)\big) := c(g;Z,W), \quad g \in G^J,\ (Z,W) \in H_{n,m} \tag{2.1}$$

is a summand of automorphy, *i.e.*,

$$A\big(g_1g_2,(Z,W)\big) = A\big(g_1,g_2\cdot(Z,W)\big) + A\big(g_2,(Z,W)\big), \tag{2.2}$$

where $g_1, g_2 \in G^J$ and $(Z,W) \in H_{n,m}$. We let

$$K_{\mathbb{C}} \subset \mathrm{GL}(\mathbb{C}^{(m,n)})$$

be the complex Lie group generated by the linear mapping

$$\{a(g;Z) : g \in G^J\}.$$

Then $K_{\mathbb{C}}$ is isomorphic to $\mathrm{GL}(n,\mathbb{C})$.

LEMMA 2.3. *Let*

$$\rho\colon \mathrm{GL}(n,\mathbb{C}) \longrightarrow \mathrm{GL}(V_\rho)$$

be a finite dimensional holomorphic representation of $\mathrm{GL}(n,\mathbb{C})$ *on a finite dimensional complex vector space* V_ρ *and let* $\chi\colon \mathbb{C}^{(m,n)} \longrightarrow \mathbb{C}^\times$ *be a character on the additive group* $\mathbb{C}^{(m,m)}$. *Then the mapping* $J_\rho\colon \tilde{G}\times H_n \longrightarrow \mathrm{GL}(V_\rho)$ *defined by*

$$J_\rho(\tilde{\sigma},Z) := \rho\big(J(\tilde{\sigma},Z)\big), \quad \tilde{\sigma}\in\tilde{G},\ Z\in H_n$$

is a factor of automorphy for $\tilde{G}$. *Furthermore the mapping*

$$J_{\chi,\rho}\big(g,(Z,W)\big) := \chi\big(c(g;Z,W)\big)\rho\big(a(g;Z)\big), \quad g\in G^J$$

is a factor of automorphy for the Jacobi group G^J *with respect to* χ *and* ρ.

PROOF. The proof of this first statement is obvious. The proof of the second statement follows immediately from the fact that $A\big(g,(Z,W)\big) := c(g;Z,W)$ is a summand of automorphy (*cf.* (2.1) and (2.2)) and that J_ρ is a factor of automorphy for $\tilde{G}$. ∎

DEFINITION 2.4. J_ρ and $J_{\chi,\rho}$ are called the *canonical automorphic factor* for $\tilde{G}$ with respect to ρ and the *canonical automorphic factor* for G^J with respect to χ and ρ respectively.

3. **Jacobi forms.** In this section, we establish the notation and define the concept of Jacobi forms.

Let

$$\mathrm{Sp}(n,\mathbb{R}) = \{M\in\mathbb{R}^{(2n,2n)} \mid {}^tMJ_nM = J_n\}$$

be the symplectic group of degree n, where

$$J_n := \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$

It is easy to see that $\mathrm{Sp}(n,\mathbb{R})$ acts on H_n transitively by

$$M\langle Z\rangle := (AZ+B)(CZ+D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{R})$ and $Z \in H_n$.

For two positive integers n and m, we recall that the Jacobi group $G^J := \mathrm{Sp}(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$ is the semidirect product of the symplectic group $\mathrm{Sp}(n,\mathbb{R})$ and the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ endowed with the following multiplication law

$$\big(M,(\lambda,\mu,\kappa)\big)\cdot\big(M',(\lambda',\mu',\kappa')\big) := \big(MM',(\tilde{\lambda}+\lambda',\tilde{\mu}+\mu',\kappa+\kappa'+\tilde{\lambda}\,{}^t\mu'-\tilde{\mu}\,{}^t\lambda')\big)$$

with $M,M' \in \mathrm{Sp}(n,\mathbb{R})$, $(\lambda,\mu,\kappa),(\lambda',\mu',\kappa') \in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde{\lambda},\tilde{\mu}) := (\lambda,\mu)M'$. It is easy to see that G^J acts on $H_{n,m} := H_n \times \mathbb{C}^{(m,n)}$ transitively by

$$\big(M,(\lambda,\mu,\kappa)\big)\cdot(Z,W) := \big(M\langle Z\rangle,(W+\lambda Z+\mu)(CZ+D)^{-1}\big), \tag{3.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{R})$, $(\lambda,\mu,\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(Z,W) \in H_{n,m}$.

Let ρ be a rational representation of $\mathrm{GL}(n,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbb{R}^{(m,m)}$ be a symmetric half-integral semi-positive definite matrix of degree m. Let $C^\infty(H_{n,m},V_\rho)$ be the algebra of all C^∞ functions on $H_{n,m}$ with values in V_ρ. For $f \in C^\infty(H_{n,m},V_\rho)$, we define

$$\begin{aligned}\Big(f|_{\rho,\mathcal{M}}\big[\big(M,(\lambda,\mu,\kappa)\big)\big]\Big)(Z,W) \\ := e^{-2\pi i\sigma(\mathcal{M}[W+\lambda Z+\mu](CZ+D)^{-1}C)} \times e^{2\pi i\sigma(\mathcal{M}(\lambda Z\,{}^t\lambda+2\lambda\,{}^tW+(\kappa+\mu\,{}^t\lambda)))} \\ \times \rho(CZ+D)^{-1}f\big(M\langle Z\rangle,(W+\lambda Z+\mu)(CZ+D)^{-1}\big),\end{aligned} \tag{3.2}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{R})$, $(\lambda,\mu,\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(Z,W) \in H_{n,m}$.

DEFINITION 3.1. Let ρ and $\mathcal{M}$ be as above. Let

$$H_{\mathbb{Z}}^{(n,m)} := \{(\lambda,\mu,\kappa) \in H_{\mathbb{R}}^{(n,m)} \mid \lambda,\mu \in \mathbb{Z}^{(m,n)},\kappa \in \mathbb{Z}^{(m,m)}\}.$$

A *Jacobi form* of index $\mathcal{M}$ with respect to ρ on Γ_n is a holomorphic function $f \in C^\infty(H_{n,m},V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho,\mathcal{M}}[\tilde{\gamma}] = f$ for all $\tilde{\gamma} \in \Gamma_n^J := \Gamma_n \ltimes H_{\mathbb{Z}}^{(n,m)}$.

(B) f has a Fourier expansion of the following form:

$$f(Z,W) = \sum_{\substack{T\geq 0 \\ \text{half-integral}}} \sum_{R\in\mathbb{Z}^{(n,m)}} c(T,R)\cdot e^{2\pi i\sigma(TZ)}\cdot e^{2\pi i\sigma(RW)}$$

with $c(T,R) \neq 0$ only if $\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} \geq 0$.

If $n \geq 2$, the condition (B) is superfluous by Köcher principle (*cf.* [Z] Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma_n)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ_n. Ziegler (*cf.* [Z] Theorem 1.8 or [E-Z] Theorem 1.1) proves that the vector space $J_{\rho,\mathcal{M}}(\Gamma_n)$ is finite dimensional. For more results on Jacobi forms with $n > 1$ and $m > 1$, we refer to [Y3]–[Y6] and [Z].

4. **Pluriharmonic polynomials.** We review pluriharmonic polynomials of matrix arguments and collect some properties to be used in the next section (*cf.* [K-V] and [M-N-N]).

Let n and m be two positive integers and let $\mathcal{P}_{m,n} := \mathbb{C}[W_{11}, W_{12}, \ldots, W_{mn}]$ be the ring of complex valued polynomials on $\mathbb{C}^{(m,n)}$. For any homogeneous polynomial $P \in \mathcal{P}_{m,n}$, we put

$$P(\partial_W) := P\Big(\frac{\partial}{\partial W_{11}}, \ldots, \frac{\partial}{\partial W_{mn}}\Big). \tag{4.1}$$

Let S be a positive definite symmetric rational matrix of degree m. Let $T := (t_{pq})$ be the inverse of S. For each i,j with $1 \leq i,j \leq n$, we denote by $\triangle_{i,j}$ the following differential operator

$$\triangle_{i,j} := \sum_{p,q=1}^{m} t_{pq} \frac{\partial^2}{\partial W_{pi} \partial W_{qj}}, \quad 1 \leq i, j \leq n. \tag{4.2}$$

A polynomial P on $\mathbb{C}^{(m,n)}$ is said to be *harmonic* with respect to S if

$$\sum_{i=1}^{n} \triangle_{i,i} P = 0. \tag{4.3}$$

A polynomial P on $\mathbb{C}^{(m,n)}$ is called *pluriharmonic* with respect to S if

$$\triangle_{i,j} P = 0, \quad 1 \leq i, j \leq n. \tag{4.4}$$

If there is no confusion, we just write harmonic or pluriharmonic instead of harmonic or pluriharmonic with respect to S. Obviously a pluriharmonic polynomial is harmonic. We denote by $\mathcal{H}_{m,n}$ the space of all pluriharmonic polynomials on $\mathbb{C}^{(m,n)}$. The ring $\mathcal{P}_{m,n}$ of polynomials on $\mathbb{C}^{(m,n)}$ has a symmetric nondegenerate bilinear form $\langle P, Q \rangle := \big(P(\partial_W)Q\big)(0)$ for $P, Q \in \mathcal{P}_{m,n}$. It is easy to check that $\langle\, ,\, \rangle$ satisfies

$$\langle P, QR \rangle = \langle Q(\partial_W)P, R \rangle, \quad P, Q, R \in \mathcal{P}_{m,n}. \tag{4.5}$$

LEMMA 4.1. *$\mathcal{H}_{m,n}$ is invariant under the action of* $\mathrm{GL}(n,\mathbb{C}) \times O(S)$ *given by*

$$\big((A,B), P(W)\big) \longmapsto P({}^tBWA), \quad A \in \mathrm{GL}(n,\mathbb{C}),\ B \in O(S). \tag{4.6}$$

Here $O(S) := \{B \in \mathrm{GL}(m,\mathbb{C}) \mid {}^tBSB = S\}$ *denotes the orthogonal group of the quadratic form* S.

PROOF. See Corollary 9.11 in [M-N-N]. ∎

REMARK 4.2. In [K-V], Kashiwara and Vergne investigated an irreducible decomposition of the space of complex pluriharmonic polynomials defined on $\mathbb{C}^{(m,n)}$ under the action of (4.6). They showed that each irreducible component $\tau \otimes \lambda$ occurring in the decomposition of $\mathcal{H}_{m,n}$ under the action (4.6) has multiplicity one and the irreducible representation τ of $\mathrm{GL}(n,\mathbb{C})$ is determined uniquely by the irreducible representation of $O(S)$.

LEMMA 4.3. *If P is pluriharmonic, then we have*

$$P(\partial_W)e^{\sigma(WC^tWS^{-1})} = P(2S^{-1}WC)e^{\sigma(WC^tWS^{-1})}$$

for all complex symmetric matrices $C \in \mathbb{C}^{(n,n)}$ of degree n. We recall that $\sigma(A)$ denotes the trace of a square matrix.

PROOF. We set $h(W) := \sigma(WC^tWS^{-1})$. We observe that $h(\partial_W)P = 0$. Indeed,

$$\begin{aligned} h(W) &= \sum_{i,k,l,m} W_{ik}c_{kl}W_{ml}t_{mi} \\ &= \sum_{k,l} c_{kl}\Big(\sum_{i,m} t_{mi}W_{ml}W_{ik}\Big) \\ &= \sum_{k,l} c_{kl}h_{lk}. \end{aligned}$$

Thus $h(\partial_W)P = \sum_{k,l} c_{kl}\big(h_{lk}(\partial_W)P\big) = \sum_{k,l} c_{kl}\triangle_{l,k}P = 0$. We put $\varphi(W) := e^{h(W)}$. Then $f(W) := \varphi(W+A) = \varphi(W)\varphi(A)\eta(W)$, where $A \in \mathbb{C}^{(m,n)}$ and $\eta(W) := e^{\sigma(2WC^tAS^{-1})}$.

$$\begin{aligned} P(\partial_W)\varphi(W)|_{W=A} &= P(\partial_W)f(W)|_{W=0} \\ &= \varphi(A)\big(P(\partial_W)\varphi(W)\eta(W)\big)\Big|_{W=0} \\ &= \varphi(A)P(\partial_W)\eta(W)|_{W=0}. \end{aligned}$$

Indeed, since $h(\partial_W)P = 0$, we have

$$\begin{aligned} P(\partial_W)\big(\varphi(W)\eta(W)\big)\Big|_{W=0} &= \langle P, \varphi\cdot\eta\rangle = \langle \varphi(\partial_W)P, \eta\rangle \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\langle h^n(\partial_W)P, \eta\rangle \\ &= \langle P, \eta\rangle = P(\partial_W)\eta(W)|_{W=0}. \end{aligned}$$

By an easy computation, we obtain

$$P(\partial_W)\eta(W) = P(2S^{-1}AC)\eta(W).$$

Finally, we have

$$P(\partial_W)\varphi(W)|_{W=A} = \varphi(A)\cdot P(2S^{-1}AC)\eta(0).$$

Hence we obtain the desired result. ■

5. **Proof of Main Theorem.** Throughout this section we fix a rational representation ρ of $\mathrm{GL}(n,\mathbb{C})$ on a finite dimensional complex vector space V_ρ and a positive definite symmetric, half-integral matrix $\mathcal{M}$ of degree m once and for all.

We set $S := (2\mathcal{M})^{-1}$. As in the previous section, we denote by $\mathcal{H}_{m,n}$ the vector space of all pluriharmonic polynomials with respect to S on $\mathbb{C}^{(m,n)}$. According to Lemma 4.1, there exists an irreducible subspace $V_\tau(\neq 0)$ invariant under the action of $\mathrm{GL}(n,\mathbb{C})$ given by (4.6). We denote this representation by τ. Then we have

(5.1) $$\big(\tau(A)P\big)(W) = P(WA), \quad A \in \mathrm{GL}(n,\mathbb{C}),\ P \in V_\tau,\ W \in \mathbb{C}^{(m,n)}.$$

The action $\hat{\tau}$ of $GL(n,\mathbb{C})$ on V_τ^* is defined by

$$(5.2)\qquad \big(\hat{\tau}(A)^{-1}\zeta\big)(P) := \zeta\big(\tau({}^tA^{-1})P\big),$$

where $A \in GL(n,\mathbb{C})$, $\zeta \in V_\tau^*$ and $P \in V_\tau$.

DEFINITION 5.1. Let $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ be a Jacobi form of index $\mathcal{M}$ with respect to ρ on Γ_n. Let $P \in V_\tau$ be a homogeneous pluriharmonic polynomial. We put

$$(5.3)\qquad f_P(Z) := P(\partial_W)f(Z,W)|_{W=0}, \quad Z \in H_n,\ W \in \mathbb{C}^{(m,n)}.$$

Now we define the mapping

$$f_\tau : H_n \longrightarrow V_\tau^* \otimes V_\rho$$

by

$$(5.4)\qquad \big(f_\tau(Z)\big)(P) := f_P(Z), \quad Z \in H_n,\ P \in V_\tau.$$

DEFINITION 5.2. A holomorphic function $f: H_n \longrightarrow V_\rho$ is called a *modular form of type ρ on Γ_n* if

$$f(M\langle Z\rangle) = \rho(CZ+D)f(Z), \quad Z \in H_n$$

for all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$. If $n = 1$, the additional cuspidal condition will be added. We denote by $[\Gamma_n, \rho]$ the vector space of all modular forms of type ρ on Γ_n.

MAIN THEOREM. *Let τ and $\hat{\tau}$ be as before. Let $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ be a Jacobi form. Then $f_\tau(Z)$ is a modular form of type $\hat{\tau} \otimes \rho$, i.e., $f_\tau \in [\Gamma_n, \hat{\tau} \otimes \rho]$.*

PROOF. Let

$$f(Z,W) = \sum_{T,R} c(T,R)e^{2\pi i\sigma(TZ)} \cdot e^{2\pi i\sigma(RW)}$$

be a Fourier expansion of $f(Z,W)$. Then we have

$$P(\partial_W)f(Z,W) = \sum_{T,R} P(2\pi i\,{}^tR)\cdot c(T,R)\cdot e^{2\pi i\sigma(TZ+RW)}$$

and

$$(5.5)\qquad f_P(Z) := P(\partial_W)f(Z,W)|_{W=0} = \sum_{T,R} P(2\pi i\,{}^tR)\cdot e^{2\pi i\sigma(TZ)}\cdot c(T,R)$$

Since $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$, we have the following transformation law

$$(5.6)\qquad f\big(M\langle Z\rangle, W(CZ+D)^{-1}\big) = e^{2\pi i\sigma(\mathcal{M}W(CZ+D)^{-1}C\,{}^tW)}\cdot \rho(CZ+D)f(Z,W)$$

for all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$. Applying $P(\partial_W)$ to (5.6), according to Lemma 4.3, we have

$$\begin{aligned} P(\partial_W)&f\big(M\langle Z\rangle, W(CZ+D)^{-1}\big) \\ &= P\big(4\pi i\mathcal{M}W(CZ+D)^{-1}C\big)e^{2\pi i\sigma(\mathcal{M}W(CZ+D)^{-1}C\,{}^tW)} \\ &\qquad \times \rho(CZ+D)f(Z,W) + h(Z,W) + e^{2\pi i\sigma(\mathcal{M}W(CZ+D)^{-1}C\,{}^tW)} \\ &\qquad \times \sum_{T,R} P(2\pi i\,{}^tR)\cdot\rho(CZ+D)c(T,R)\cdot e^{2\pi i\sigma(TZ+RW)}, \end{aligned}$$

where $h(Z, W)$ is a V_ρ-valued function on $H_{n,m}$ whose restriction to $W = 0$ vanishes. Here we used the fact that $(CZ + D)^{-1}C$ is a complex symmetric matrix of degree n and Lemma 4.3. If we evaluate this at $W = 0$, P being homogeneous, we have

$$(5.7)\quad P(\partial_W)f\big(M\langle Z\rangle, W(CZ+D)^{-1}\big)\big|_{W=0} = \sum_{T,R} P(2\pi i{}^tR)\cdot e^{2\pi i\sigma(TZ)}\cdot \rho(CZ+D)c(T,R).$$

On the other hand,

$$\begin{aligned} &P(\partial_W)f\big(M\langle Z\rangle, W(CZ+D)^{-1}\big)\big|_{W=0}\\ &\quad = P(\partial_W)\sum_{T,R} c(T,R)e^{2\pi i\sigma(T\cdot M\langle Z\rangle)}\cdot e^{2\pi i\sigma(RW(CZ+D)^{-1})}|_{W=0}\\ &\quad = \sum_{T,R} P\big(2\pi i{}^tR{}^t(CZ+D)^{-1}\big)\cdot e^{2\pi i\sigma(T\cdot M\langle Z\rangle)}\cdot c(T,R). \end{aligned}$$

Thus according to (5.7), we have

$$(5.8)\quad \sum_{T,R}\tilde P(2\pi i{}^tR)\cdot e^{2\pi i\sigma(T\cdot M\langle Z\rangle)}\cdot c(T,R)\sum_{T,R} P(2\pi i{}^tR)\cdot e^{2\pi i\sigma(TZ)}\cdot\rho(CZ+D)c(T,R),$$

where $\tilde P(W) := P\big(W{}^t(CZ+D)^{-1}\big)$. By (5.5), (5.8) implies

$$(5.9)\qquad f_{\tilde P}(M\langle Z\rangle) = \rho(CZ+D)f_P(Z),$$

that is,

$$(5.10)\qquad \big(f_\tau(M\langle Z\rangle)\big)(\tilde P) = \rho(CZ+D)f_\tau(Z)(P).$$

Since $\tilde P = \tau\big({}^t(CZ+D)^{-1}\big)P$, we have from (5.9)

$$\big((\hat\tau^{-1}\otimes 1_{V_\rho})(CZ+D)f_\tau(M\langle Z\rangle)\big)(P) = \big((1_{V_\tau^*}\otimes\rho)(CZ+D)f_\tau(Z)\big)(P),$$

where $1_{V_\tau^*}$ (resp. 1_{V_ρ}) denotes the trivial representation of $\mathrm{GL}(n,\mathbb{C})$ on V_τ^* (resp. V_ρ). Hence we obtain

$$(5.11)\qquad f_\tau(M\langle Z\rangle) = (\hat\tau\otimes\rho)(CZ+D)f_\tau(Z)$$

for all $M = \begin{pmatrix} A & B\\ C & D\end{pmatrix}\in\Gamma_n$. Therefore f_τ is a $\mathrm{Hom}(V_\tau, V_\rho)$-valued modular form of type $\hat\tau\otimes\rho$. ■

6. **Applications.** In this final section, we obtain important identities by applying the main theorem to two special Jacobi forms.

(I) Let $S\in\mathbb{Z}^{(2k,2k)}$ be a positive definite symmetric, unimodular even matrix of degree $2k$. We choose an integral matrix $c\in\mathbb{Z}^{(2k,m)}$ such that ${}^tcSc$ is positive definite. We consider the following theta series

$$\theta_{S,c}(Z,W) := \sum_{\lambda\in\mathbb{Z}^{(2k,n)}} e^{\pi i\sigma(S(\lambda Z{}^t\lambda + 2\lambda{}^t(cW)))}.$$

Then $\theta_{S,c} \in J_{k,\mathcal{M}}(\Gamma_n)$ with $\mathcal{M} := \frac{1}{2}{}^tcSc$ (*cf.* [Z], p. 212). We write $f(Z,W) := \theta_{S,c}(Z,W)$. Then by Main Theorem, f_T is a $\mathrm{Hom}(V_\tau, \mathbb{C})$-valued modular form of type $\hat{\tau} \otimes \det^k$. Furthermore, according to (5.9), for any homogeneous pluriharmonic P with respect to $(2\mathcal{M})^{-1} = ({}^tcSc)^{-1}$, we obtain the following identity

$$\sum_{\lambda \in \mathbb{Z}^{(2k,n)}} P\big(2\pi i\, {}^tcS\lambda\, {}^t(CZ+D)^{-1}\big) \cdot e^{\pi i \sigma(S\lambda(AZ+B)(CZ+D)^{-1}\,{}^t\lambda)}$$
$$= \{\det(CZ+D)\}^k \sum_{\lambda \in \mathbb{Z}^{(2k,n)}} P(2\pi i\, {}^tcS\lambda) \cdot e^{\pi i \sigma(S\lambda Z\,{}^t\lambda)}$$

for all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$ and $Z \in H_n$.

(II) In [Z], Ziegler defined the Eisenstein series $E^{(n)}_{k,\mathcal{M}}(Z,W)$ of Siegel type. Let $\mathcal{M}$ be a half integral positive definite symmetric matrix of degree m and let $k \in \mathbb{Z}^+$. We set

$$\Gamma_{n,0} := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \,\middle|\, C = 0 \right\}.$$

Let $\mathcal{R}$ be a complete system of representatives of the cosets $\Gamma_{n,0} \backslash \Gamma_n$ and Λ be a complete system of representatives of the cosets $\mathbb{Z}^{(m,n)} / \big(\ker(\mathcal{M}) \cap \mathbb{Z}^{(m,n)}\big)$, where $\ker(\mathcal{M}) := \{\lambda \in \mathbb{R}^{(m,n)} \mid \mathcal{M} \cdot \lambda = 0\}$. The Eisenstein series $E^{(n)}_{k,\mathcal{M}}$ is defined by

$$E^{(n)}_{k,\mathcal{M}}(Z,W) := \sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathcal{R}} \det(CZ+D)^{-k} \cdot e^{2\pi i \sigma(\mathcal{M} W (CZ+D)^{-1} C\, {}^tW)}$$
$$\cdot \sum_{\lambda \in \Lambda} e^{2\pi i \sigma(\mathcal{M}((AZ+B)(CZ+D)^{-1}\,{}^t\lambda + 2\lambda\,{}^t(CZ+D)^{-1}\,{}^tW))},$$

where $(Z,W) \in H_{n,m}$. Now we assume that $k > n+m+1$ and k is *even*. Then according to [Z], Theorem 2.1, $E^{(n)}_{k,\mathcal{M}}(Z,W)$ is a nonvanishing Jacobi form in $J_{k,\mathcal{M}}(\Gamma_n)$. By Main Theorem, $(E^{(n)}_{k,\mathcal{M}})_T$ is a $\mathrm{Hom}(V_\tau, \mathbb{C})$-valued modular form of type $\hat{\tau} \otimes \det^k$. We define the automorphic factor j: $\mathrm{Sp}(n,\mathbb{R}) \times H_n \longrightarrow GL(n,\mathbb{C})$ by

$$j(g,Z) := cZ + d, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{R}),\ Z \in H_n.$$

Then according to (5.9), for any homogeneous pluriharmonic polynomial P with respect to $(2\mathcal{M})^{-1}$, we obtain the following identity

$$\det j(M,Z)^k \sum_{\gamma \in \mathcal{R}} \sum_{\lambda \in \Lambda} \det j(\gamma,Z)^{-k} \cdot P\big(4\pi i \mathcal{M}\lambda\, {}^tj(\gamma,Z)^{-1}\big) \cdot e^{2\pi i \sigma(\mathcal{M} \cdot \gamma\langle Z\rangle \cdot {}^t\lambda)}$$
$$= \sum_{\gamma \in \mathcal{R}} \sum_{\lambda \in \Lambda} \det j(\gamma, M\langle Z\rangle)^{-k} \cdot P\big(4\pi i \mathcal{M}\lambda\, {}^tj(\gamma M, Z)^{-1}\big) \cdot e^{2\pi i \sigma(\mathcal{M} \cdot \gamma M\langle Z\rangle \cdot {}^t\lambda)}$$

for all $M \in \Gamma_n$ and $Z \in H_n$.

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References

[E-Z] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progress in Math. **55**, Birkhäuser, Boston, Basel, Stuttgart, 1985.

[K-V] M. Kashiwara and M. Vergne, *On the Segal-Shale-Weil Representations and Harmonic Polynomials,* Invent. Math. **44**(1978), 1–47.

[M-N-N] D. Mumford, M. Nori and P. Norman, *Tata Lectures on Theta III*, Progr. Math. **97**, Birkhäuser, 1991.

[N] Y. Namikawa, *Toroidal compactification of Siegel spaces*, Springer-Verlag **812**, 1980.

[PS] I. Piateski-Sharpiro, *Automorphic Functions and the Geometry of Classical Domains*, Gordan-Breach, New York, 1966.

[Y1] J.-H. Yang, *Harmonic Analysis on the Quotient Spaces of Heisenberg Groups,* Nagoya Math. J. **123**(1991), 103–117.

[Y2] ———, *Harmonic Analysis on the Quotient Spaces of Heisenberg Groups II,* J. Number Theory (1) **49**(1994), 63–72.

[Y3] ———, *Some Results on Jacobi Forms of Higher Degree,* Proc. RIMS Symp. on Automorphic Forms and Associated Zeta Functions, Kyoto **805**, 1992, 36–50.

[Y4] ———, *The Siegel-Jacobi Operator,* Abh. Math. Sem. Univ. Hamburg **63**(1993), 135–146.

[Y5] ———, *Remarks on Jacobi Forms of Higher Degree,* Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, The Pyungsan Institute for Mathematical Sciences, Seoul, 1993, 33–58.

[Y6] ———, *Singular Jacobi Forms,* Trans. Amer. Math. Soc, **347**(1995), 2041–2049.

[Z] C. Ziegler, *Jacobi Forms of Higher Degree,* Abh. Math. Sem. Univ. Hamburg **59**(1989), 191–224

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KAC-MOODY ALGEBRAS, THE MONSTROUS MOONSHINE, JACOBI FORMS AND INFINITE PRODUCTS

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Table of Contents

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1. Introduction

Recently R. E. Borcherds obtained some quite interesting results in [Bo6-7]. First he solved the Moonshine Conjectures made by Conway and Norton ([C-N]). Secondly he constructed automorphic forms on the orthogonal group $O_{s+2,2}(\mathbb{R})$ which are modular products and then wrote some of the well-known meromorphic modular forms as infinite products. Modular products roughly mean infinite products whose exponents are the coefficients of certain nearly holomorphic modular forms. The theory of Jacobi forms plays an important role in his second work in [Bo7]. More than 10 years ago Feingold and Frenkel ([F-F]) realized the connection between the theory of a special hyperbolic Kac-Moody Lie algebra of the type $HA_1^{(1)}$ and that of Jacobi forms of degree one and then generalized the results of H. Maass to higher levels. So far the relationship between the theory of Jacobi forms of higher degree and that of other hyperbolic Kac-Moody algebras has not been developed yet. The work of Borcherds in [Bo7] gives a light on the possibility for the relationship between them. This fact urged me to write a somewhat supplementary or expository note on Borcherds' recent works which is useful for my research on Jacobi forms although I am not an expert in the theory of Kac-Moody Lie algebras and lattices. I hope that this note will be useful for the readers who are interested in these interesting subjects. I learned a lot about these subjects while I had been preparing this article. I had given a lecture on some of these materials at the 4th Symposium of the Pyungsan Institute for Mathematical Sciences held at the Wonkwang University in September, 1995.

As mentioned above, the purpose of this paper is to give a survey of Borcherds' recent results in [Bo6-7] to the core. This article is organized as follows. In section 2, we collect some of the well-known properties of Kac-Moody Lie algebras, e.g., the Weyl-Kac character formula, the root multiplicity and so on. In the appendix, we discuss the generalized Kac-Moody Lie algebras introduced by Borcherds roughly. In section 3, we give a sketchy survey on the Moonshine Conjectures solved by Borcherds ([Bo6]). We discuss the monster Lie algebra and the no-ghost theorem. This section is completely based on the article [Bo6]. In section 4, we review some of the theory of Jacobi forms and discuss singular Jacobi forms briefly. We present Borcherds' construction of nearly holomorphic Jacobi forms by making use of the concept of affine vector systems. In section 5, we give a brief history of infinite products and present the work of Borcherds that expressed modular forms in the Kohnen "plus" space of weight 1/2 as infinite products. For instance, we write the well-known modular forms like the discriminant function $\Delta(\tau)$, the modular invariant $j(\tau)$ and the Eisenstein series $E_k(\tau)$ ($k \geq 4$, k : *even*) as infinite products explicitly. In the final section, we discuss the fake monster Lie algebras and Kac-Moody Lie algebras of the arithmetic hyperbolic type defined by V. V. Nikulin ([N5]). As an example, we explain the generalization of Maass correspondence to higher levels which was done by A. J. Feingold and I. B. Frenkel ([F-F]). Finally we also give some open problems which have to be investigated. In the appendix A, we collect some of the well-known properties of classical modular forms. In the appendix B, we briefly discuss the Kohnen "plus" space and the Maass "Spezialschar" which are essential for the understanding of the works in [Bo7] and [F-F]. In the appendix C, we discuss the geometrical aspect of the orthogonal group $O_{s+2,2}(\mathbb{R})$ briefly. In the final appendix, we collect some of the well-known properties of the Leech lattice Λ. Also we briefly discuss the Jacobi theta functions.

Finally I would like to give my deep thanks to TGRC-KOSEF for its financial support on this work. I also would like to give my hearty thanks to my Korean colleagues for their interest in this work.

Notations: We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of rational numbers, the field of real numbers, and the field of complex numbers respectively. $\mathbb{Z}^+$ and $\mathbb{Z}_+$ denote the set of all positive integers and the set of nonnegative integers respectively. For a positive integer n, $\Gamma_n := Sp(n,\mathbb{Z})$ denotes the Siegel modular group of degree n. The symbol ":=" means that the expression on the right is the definition of that on the left. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. E_n denotes the identity matrix of degree n. We denote by Λ the Leech lattice. $\Pi_{1,1}$ denotes the unique unimodular even integral Lorentzian lattice of rank 2. We denote by G the MONSTER group. For $g \in G$, $T_g(q)$ denotes the Thompson series of g. M and $V^\natural$ denote the monster Lie algebra and the moonshine module respectively. $\eta(\tau)$ denotes the Dedekind eta function. $\tau(n)$ denotes the Ramanujan function. Usually ρ denotes the Weyl vector. We denote by $[\Gamma_n, k]$ (resp. $[\Gamma, k]_0$) the complex vector space of all Siegel modular forms (resp. cusp forms) on H_n of weight k with respect to Γ_n. We denote by $[\Gamma_2, k]^M$ the Maass space or the Maass Spezialschar.

2. Kac-Moody Lie Algebras

In this section, we review the basic definitions and properties of Kac-Moody Lie algebras.

An $n \times n$ matrix $A = (a_{ij})$ is called a *generalized Cartan matrix* if it satisfies the following conditions: (i) $a_{ii} = 2$ for $i = 1, \cdots, n$; (ii) a_{ij} are nonpositive integers for $i \neq j$; (iii) $a_{ij} = 0$ implies $a_{ji} = 0$. An indecomposable generalized Cartan matrix is said to be *of finite type* if all its principal minors are positive, *of affine type* if all its proper principal minors are positive and $\det A = 0$, and is said to be *of indefinite type* if A is of neither finite type nor affine type. A is said to be *of* *hyperbolic type* if it is of indefinite type and all of its proper principal submatrices are of finite type or affine type, and to be *of almost hyperbolic type* if it is of indefinite type and at least one of its proper principal submatrices is of finite or affine type.

A matrix A is called *symmetrizable* if there exists an invertible diagonal matrix $D = \mathrm{diag}(q_1, \cdots, q_n)$ with $q_i > 0$, $q_i \in \mathbb{Q}$ such that DA is symmetric. If A is an $n \times n$ matrix of rank l, then *a realization* of A is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, where $\mathfrak{h}$ is a complex vector space of dimension $2n - l$, $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ and $\Pi^\vee = \{\alpha_1^\vee, \cdots, \alpha_n^\vee\}$ are linearly independent subsets of $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively such that $\alpha_j(\alpha_i^\vee) = a_{ij}$ for $1 \leq i, j \leq n$.

Definition 2.1. The *Kac-Moody Lie algebra* $\mathfrak{g}(A)$ associated with the generalized Cartan matrix A is the Lie algebra generated by the elements e_i, f_i $(i = 1, 2, \cdots, n)$ and $\mathfrak{h}$ with the defining relations

$$\begin{aligned}
&[h, h'] = 0 \qquad \text{for all } h, h' \in \mathfrak{h},\\
&[e_i, f_j] = \delta_{ij}\alpha_i^\vee \qquad \text{for } 1 \leq i, j \leq n,\\
&[h, e_i] = \alpha_i(h)e_i, \quad [h, f_i] = -\alpha_i(h)f_i \qquad \text{for } i = 1, 2, \cdots, n,\\
&(ad\, e_i)^{1-a_{ij}}(e_j) = \ (ad\, f_i)^{1-a_{ij}}(f_j) = 0 \qquad \text{for } i \neq j.
\end{aligned}$$

The elements of Π (resp. $\Pi^\vee$) are called *simple roots* (resp. *simple coroots*) of $\mathfrak{g}(A)$.

Let $Q := \sum_{i=1}^n \mathbb{Z}\alpha_i$, $Q_+ := \sum_{i=1}^n \mathbb{Z}_+\alpha_i$ and $Q_- := -Q_+$. Q is called the *root lattice*. For $\alpha := \sum_{i=1}^n k_i\alpha_i \in Q$ the number $\mathrm{ht}(\alpha) := \sum_{i=1}^n k_i$ is called the *height* of α. We define a *partial ordering* $\geq$ on $\mathfrak{h}^*$ by $\lambda \geq \mu$ if $\lambda - \mu \in Q_+$. For each $\alpha \in \mathfrak{h}^*$, we put

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g}(A) \mid [h, X] = \alpha(h)X \text{ for all } h \in \mathfrak{h}\}.$$

If $\mathfrak{g}_\alpha \neq 0$, α is called a *root* and $\mathfrak{g}_\alpha$ is called the *root space* attached to α. The number $\mathrm{mult}\,\alpha := \dim\, \mathfrak{g}_\alpha$ is called the *multiplicity* of α. The Kac-Moody Lie algebra $\mathfrak{g}(A)$ has the following root space decomposition with respect to $\mathfrak{h}$:

$$\mathfrak{g}(A) = \sum_{\alpha \in Q} \mathfrak{g}_\alpha \qquad \text{(direct sum)}. \tag{2.1}$$

A root α with $\alpha > 0$ (resp. $\alpha < 0$) is called *positive* (resp. *negative*). All roots are either positive or negative. We denote by Δ, Δ^+ and Δ^- the set of all roots, positive roots and negative roots respectively.

Definition 2.2. Let $\mathfrak{g}(A)$ be a symmetrizable Kac-Moody Lie algebra associated with a symmetrizable generalized Cartan matrix $A = (a_{ij})$. A $\mathfrak{g}(A)$-module V is

$\mathfrak{h}-$*diagonalizable* if $V = \oplus_{\mu\in\mathfrak{h}^*} V_\mu$, where V_μ is the weight space of weight μ given by

$$V_\mu := \{v \in V \mid hv = \mu(h)v \text{ for all } h \in \mathfrak{h}\} \neq 0.$$

The number $\text{mult}_V \mu := \dim V_\mu$ is called the *multiplicity* of weight μ. When all the weight spaces are finite-dimensional, we define the *character* of V by

$$\text{ch}\, V := \sum_{\mu\in\mathfrak{h}^*} (\dim V_\mu) e^\mu = \sum_{\mu\in\mathfrak{h}^*} (\text{mult}_V \mu) e^\mu. \tag{2.2}$$

An $\mathfrak{h}$-diagonalizable $\mathfrak{g}(A)$-module V is said to be *integrable* if all the Chevalley generators $e_i, f_i (i = 1, 2, \cdots, n)$ are locally nilpotent on V. A $\mathfrak{g}(A)$-module V is called a *highest weight module* with highest weight $\Lambda \in \mathfrak{h}^*$ if there exists a nonzero vector $v \in V$ such that (i) $e_i v = 0$ for all $i = 1, 2, \cdots, n$; (ii) $hv = \Lambda(h)v$ for all $h \in \mathfrak{h}$; and (iii) $U(\mathfrak{g}(A))\, v = V$. A vector v is called a *highest weight vector.* Here $U(\mathfrak{g}(A))$ denotes the universal enveloping algebra of $\mathfrak{g}(A)$.

Let $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) be the subalgebra of $\mathfrak{g}(A)$ generated by $e_1, \cdots, e_n$ (resp. $f_1, \cdots, f_n$). Then we have the *triangular decomposition*

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \qquad \text{(direct sum of vector spaces)}.$$

Let $\mathfrak{b}_+ := \mathfrak{h} + \mathfrak{n}_+$ be the *Borel subalgebra* of $\mathfrak{g}(A)$. For a fixed $\Lambda \in \mathfrak{h}^*$, we let $\mathbb{C}(\Lambda)$ be the one-dimensional $\mathfrak{b}_+$-module with the $\mathfrak{b}_+$-action defined by

$$\mathfrak{n}_+ \cdot 1 = 0 \text{ and } h \cdot 1 = \Lambda(h)1 \text{ for all } h \in \mathfrak{h}.$$

The induced module

$$M(\Lambda) := U(\mathfrak{g}(A)) \otimes_{U(\mathfrak{b}_+)} \mathbb{C}(\Lambda) \tag{2.3}$$

is called the *Verma module* with highest weight Λ. It is known that every $\mathfrak{g}(A)$-module with highest weight Λ is a qutoient of $M(\Lambda)$ and $M(\Lambda)$ contains a unique proper maximal submodule $M'(\Lambda)$.

We put

$$L(\Lambda) := M(\Lambda)/M'(\Lambda). \tag{2.4}$$

Then we can show that $L(\Lambda)$ is an irreducible $\mathfrak{g}(A)$-module.

We set

$$P := \{\lambda \in \mathfrak{h}^* \,|< \lambda, \alpha_i^\vee > \in \mathbb{Z} \text{ for } i = 1, \cdots, n\},$$
$$P_+ := \{\lambda \in P \,|< \lambda, \alpha_i^\vee > \geq 0 \text{ for } i = 1, \cdots, n\},$$
$$P_{++} := \{\lambda \in P \,|< \lambda, \alpha_i^\vee > > 0 \text{ for } i = 1, \cdots, n\}.$$

The set P is called the *weight lattice.* Elements from P (resp. P_+ or P_{++}) are called *integral weights* (resp. *dominant* or *regular dominant integral weights*). We observe that $Q \subset P$ and $P_{++} \subset P_+ \subset P$. If Λ is an element of P_+, i.e., Λ is a dominant integral weight, then $L(\Lambda)$ is *integrable* (cf. [K], p. 171) and the *Weyl-Kac character formula* for $L(\Lambda)$ is given by

$$\operatorname{ch} L(\Lambda) = \frac{\sum_{w\in W} \epsilon(w) e^{w(\Lambda+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{\operatorname{mult}\alpha}} \tag{2.5}$$

Here $\epsilon(w) := (-1)^{\ell(w)} = \det_{\mathfrak{h}^*} w$ for $w \in W$, W the *Weyl group* of $\mathfrak{g}(A)$ and ρ is an element of $\mathfrak{h}^*$ such that $< \rho, \alpha_i^\vee >= 1$ for $i = 1, \cdots, n$. We recall that $W \subset \operatorname{Aut}(\mathfrak{h}^*)$ is the subgroup generated by the reflections $\sigma_i(\lambda) := \lambda - \lambda(\alpha_i^\vee)\alpha_i$ $(1 \leq i \leq n)$.

We set $\Lambda = 0$ in (2.5). Since $L(0)$ is the trivial module over $\mathfrak{g}(A)$, we obtain the following denominator identity or denominator formula:

$$\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{\operatorname{mult}\alpha} = \sum_{w\in W} \epsilon(w) e^{w(\rho)-\rho}. \tag{2.6}$$

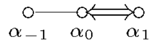
Substituting (2.6) into (2.5), we obtain another form of the Weyl-Kac charater formula:

$$\operatorname{ch} L(\Lambda) = \frac{\sum_{w\in W} \epsilon(w) e^{w(\Lambda+\rho)}}{\sum_{w\in W} \epsilon(w) e^{w(\rho)}}. \tag{2.7}$$

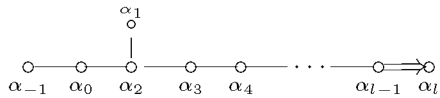
Of course, in the case when $\mathfrak{g}(A)$ is a finite dimensional semisimple Lie algebra, then (2.7) is the classical Weyl character formula and (2.6) is the Weyl denominator identity. We remark that an integrable highest weight module $L(\Lambda)$ over $\mathfrak{g}(A)$ is unitarizable and conversely if $L(\Lambda)$ is irreducible, then $\Lambda \in P_+$(cf. [K], p.196).

Let $A = (a_{ij})$ be a generalized Cartan matrix. We associate to A a graph $\mathcal{S}(A)$ called the Dynkin diagram of A as follows. If $a_{ij}a_{ji} \leq 4$ and $|a_{ij}| \geq |a_{ji}|$, the vertices i and j are connected by $|a_{ij}|$ lines, and these lines are equipped with an arrow pointing toward i if $|a_{ij}| > 1$. If $a_{ij}a_{ji} > 4$, the vertices i and j are connected by a bold-faced line equipped with an ordered pair of integers $|a_{ij}|$, $|a_{ji}|$. We list some of hyperbolic Kac-Moody Lie algebras.

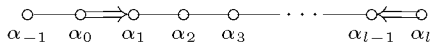
$HA_1^{(1)}$:



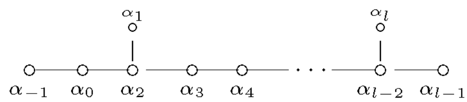
$HB_l^{(1)},\ l \geq 3$:



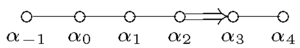
$HC_l^{(1)},\ l \geq 2$:



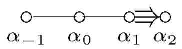
$HD_l^{(1)},\ l \geq 4$:



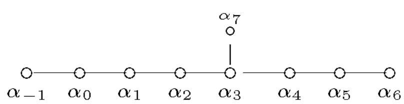
$HF_4^{(1)}$:



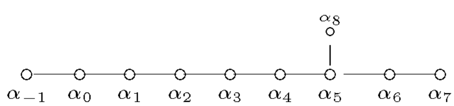
$HG_2^{(1)}$:



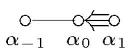
$HE_7^{(1)}$:



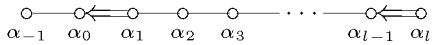
$HE_8^{(1)}$:



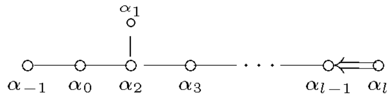
$HA_2^{(2)}$:



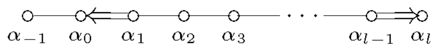
$HA_{2l}^{(2)},\ l \geq 2$:



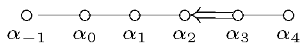
$HA_{2l-1}^{(2)},\ l \geq 3$:



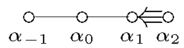
$HD_{l+1}^{(2)},\ l \geq 2$:



$HE_6^{(2)}$:



$HD_4^{(3)}$:



Let $A = (a_{ij})_{i,j=-1,0,\cdots,\ell}$ be a hyperbolic generalized Cartan matrix whose corresponding affine submatrix of A is given by $A_0 = (a_{kl})_{k,l=0,1,\cdots,\ell}$. We can realize the hyperbolic Kac-Moody Lie algebra $\mathfrak{g}(A)$ as the minimal graded Lie algebra $L = \oplus_{n\in\mathbb{Z}} L_n$ with local part $V + \mathfrak{g}(A_0) + V^*$, where $V = L(-\alpha_{-1})$ is the basic representation of the affine Kac-Moody Lie algebra $\mathfrak{g}(A_0)$ and V^* is the contragredient of V. Thus $L = G/I$, and $L_n = G_n/I_n$, where $G = \oplus_{n\in\mathbb{Z}} G_n$ is the maximal graded Lie algebra with local part $V + \mathfrak{g}(A_0) + V^*$ and $I = \oplus_{n\in\mathbb{Z}} I_n$ is the maximal graded ideal of G intersecting the local part trivially. Each L_n ($n \in \mathbb{Z}$) is a $\mathfrak{g}(A_0)$-module. (By definition, $G = \oplus_{n\in\mathbb{Z}} G_n$ is called a *graded Lie algebra* if G is a Lie algebra and $[G_i, G_j] \subset G_{i+j}$ for all $i, j \in \mathbb{Z}$.) A graded Lie algebra $G = \oplus_{n\in\mathbb{Z}} G_n$ is called *irreducible* if the represantation ϕ_{-1} of G_0 on G_{-1} defined by $\phi_{-1}(x_0)x_{-1} = [x_0, x_{-1}]$ for all $x_0 \in G_0$ and $x_{-1} \in G_{-1}$ is irreducible. A graded Lie algebra $G = \oplus_{n\in\mathbb{Z}} G_n$ is said to be *maximal* (resp. *minimal*) if for any other graded Lie algebra $G' = \oplus_{n\in\mathbb{Z}} G'_n$, every isomorphism of the local parts of G and G' can be extended to an epimorphism of G onto G' (resp. of G' onto G). Kac (cf. [K]) proved that for any local Lie algebra $G_{-1} \oplus G_0 \oplus G_1$, there exist unique up to isomorphism maximal and minimal graded Lie algebras whose local parts are isomorphic to a given Lie algebra $G_{-1} \oplus G_0 \oplus G_1$.

Example 2.3. Let

$$A = (a_{ij})_{i,j=-1,0,1} := \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

be the hyperbolic generalized Cartan matrix. We can realize the corresponding hyperbolic Kac-Moody Lie algebra $\mathfrak{g}(A) := HA_1^{(1)}$ as the minimal graded Lie algebra $L = \oplus_{n\in\mathbb{Z}} L_n$ with local part $V + \mathfrak{g}(A_0) + V^*$, where $A_0 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ and $V := L(-\alpha_{-1})$ is the basic representation of the affine Kac-Moody Lie algebra $\mathfrak{g}(A_0) := A_1^{(1)}$. The dimensions $\dim(L_{-n})_\alpha$ for $0 \le n \le 5$ were computed by A. J. Feingold, I. B. Frenkel, S.-J. Kang and etc. For instance, $\dim(L_0)_\alpha = 1$ and

$$\dim(L_{-1})_\alpha = p\left(1 - \frac{(\alpha,\alpha)}{2}\right),$$

where p is the partition function defined by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{\phi(q)}, \qquad \phi(q) := \prod_{n\ge 1}(1-q^n). \tag{2.8}$$

Example 2.4. Let

$$A = (a_{ij})_{i,j=-1,0,1} := \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -4 \\ 0 & -1 & 2 \end{pmatrix}$$

be the generalized Cartan matrix of hyperbolic type. We can realize $\mathfrak{g}(A) := HA_2^{(2)}$ as the minimal graded Lie algebra $L = \oplus_{n\in\mathbb{Z}} L_n$. The dimensions $\dim L_{-n}$ $(1 \le n \le 3)$ were computed by A. J. Feingold and S. J. Kang.

Example 2.5. Kac-Moody-Wakimoto (cf.[KMW]) considered the hyperbolic Kac-Moody Lie algebra $HE_8^{(1)} = E_{10}$. Using the modular invariant property of level 2 string functions, they computed root multiplicities of G_{-2} and I_{-2}. Thus they obtained the formula

$$\dim(L_{-2})_\alpha = \xi\left(3 - \frac{(\alpha,\alpha)}{2}\right),$$

where $\xi(n)$ is defined by the relation

$$\sum_{n=0}^{\infty} \xi(n)q^n = \frac{1}{\phi(q)^8}\left(1 - \frac{\phi(q^2)}{\phi(q^4)}\right). \tag{2.9}$$

REMARK 2.6. In [Fr], Frenkel conjectured that for a hyperbolic Kac-Moody Lie algebra $\mathfrak{g}$, we have

$$\dim \mathfrak{g}_\alpha \leq p^{(\ell-2)}\left(1-\frac{(\alpha,\alpha)}{2}\right),$$

where ℓ is the size of the generalized Cartan matrix of $\mathfrak{g}$ and the function $p^{(\ell-2)}(n)$ is defined by

$$\sum_{n=0}^{\infty} p^{(\ell-2)}(n)q^n = \frac{1}{\phi(q)^{\ell-2}} = \prod_{n\geq 1}(1-q^n)^{2-\ell}. \tag{2.10}$$

But this conjecture does not hold for E_{10} (cf. [KMW]). This conjecture is true for $HA_n^{(1)}$. We observe that $HA_n^{(1)}$ is of hyperbolic type for $n \leq 7$ and that $HA_n^{(1)}$ is of almost hyperbolic type for $n \geq 8$.

Appendix. Generalized Kac-Moody Algebras

Let I be a countable index set. A real matrix $A = (a_{ij})_{i,j\in I}$ is called a *Borcherds-Cartan matrix* if it satisfies the following conditions:

(BC1) $a_{ii} = 2$ or $a_{ii} \leq 0$ for all $i \in I$;
(BC2) $a_{ij} \leq 0$ if $i \neq j$ and $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$;
(BC3) $a_{ij} = 0$ implies $a_{ji} = 0$.

Let $I^{re} := \{i \in I \mid a_{ii} = 2\}$ and $I^{im} := \{i \in I \mid a_{ii} \leq 0\}$. Let $\underline{m} = (m_i \mid i \in I)$ be the *charge* of A, i.e., $m_i = 1$ for all $i \in I^{re}$ and $m_j \in \mathbb{Z}^+$ for all $j \in I^{im}$. A Borcherds-Cartan matrix A is said to be *symmetrizable* if there exists a diagonal matrix $D = \mathrm{diag}(\delta_i \mid i \in I)$ with $\delta_i > 0$ $(i \in I)$ such that DA is symmetric.

Definition 2.7. The *generalized Kac-Moody algebra* $\mathfrak{g} = \mathfrak{g}(A, \underline{m})$ with a symmetrizable Borcherds-Cartan matrix A of charge $\underline{m} = (m_i \mid i \in I)$ is the complex Lie algebra generated by the elements h_i, d_i $(i \in I)$, e_{ik}, f_{ik} $(i \in I,\ k = 1, \cdots, m_i)$ with the defining relations:

$$\begin{aligned}
&[h_i, h_j] = [h_i, d_j] = [d_i, d_j] = 0,\\
&[h_i, e_{j\ell}] = a_{ij}e_{j\ell},\ \ [h_i, f_{j\ell}] = -a_{ij}f_{j\ell},\\
&[d_i, e_{j\ell}] = \delta_{ij}e_{j\ell},\ \ [d_i, f_{j\ell}] = -\delta_{ij}f_{j\ell},\\
&[e_{ik}, f_{j\ell}] = \delta_{ij}\delta_{k\ell}h_i,\\
&(ad\, e_{ik})^{1-a_{ij}}(e_{j\ell}) = (ad\, f_{ik})^{1-a_{ij}}(f_{j\ell}) = 0 \quad \text{if } a_{ii} = 2 \text{ and } i \neq j,\\
&[e_{ik}, e_{j\ell}] = [f_{ik}, f_{j\ell}] = 0 \quad \text{if } a_{ii} = 0
\end{aligned}$$

for all $i, j \in I,\ k = 1, \cdots, m_i,\ \ell = 1, \cdots, m_j$.

The subalgebra $\mathfrak{h} := \left(\sum_{i\in I}\mathbb{C}h_i\right) \oplus \left(\sum_{i\in I}\mathbb{C}d_i\right)$ is called the *Cartan subalgebra* of $\mathfrak{g}$. For each $j \in I$, we define a *linear functional* $\alpha_j \in \mathfrak{h}^*$ by

$$\alpha_j(h_i) = a_{ij}, \quad \alpha_j(d_i) = \delta_{ij} \quad \text{for all } i, j \in I.$$

Let $\Pi := \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ and $\Pi^\vee := \{h_i \mid i \in I\} \subset \mathfrak{h}$. The elements of Π (resp. $\Pi^\vee$) are called the *simple roots* (resp. *simple coroots*) of $\mathfrak{g}$. We set

$$Q := \sum_{i\in I}\mathbb{Z}\alpha_i, \quad Q^+ := \sum_{i\in I}\mathbb{Z}_+\alpha_i, \quad Q^- := -Q^+.$$

Q is called the *root lattice* of $\mathfrak{g}$. We define a partial ordering $\leq$ on $\mathfrak{h}^*$ by $\lambda \leq \mu$ if $\mu - \lambda \in Q^+$. For $\alpha \in \mathfrak{h}^*$, we put

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid [h, X] = \alpha(h)X \quad \text{for all } h \in \mathfrak{h}\}.$$

If $\mathfrak{g}_\alpha \neq 0$ and $\alpha \neq 0$, α is called *root* of $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ is called the *root space attached to the root* α. The generalized Kac-Moody algebra $\mathfrak{g}$ has the root decomposition

$$\mathfrak{g} = \sum_{\alpha\in Q}\mathfrak{g}_\alpha \qquad \text{(direct sum)}. \tag{2.11}$$

We observe that $\mathfrak{g}_{\alpha_i} = \sum_{k=1}^{m_i}\mathbb{C}e_{i,k}$ and $\mathfrak{g}_{-\alpha_i} = \sum_{k=1}^{m_i}\mathbb{C}f_{i,k}$. The number $\operatorname{mult}\alpha := \dim \mathfrak{g}_\alpha$ is called the *multiplicity* of α. A root α with $\alpha > 0$ (with $\alpha < 0$) is said to be *positive* (resp. *negative*). We denote by $\Delta, \Delta^+, \Delta^-$ the set of all roots, positive roots, and negative roots respectively. We set

$$\mathfrak{n}^+ := \sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha, \qquad \mathfrak{n}^- := \sum_{\alpha\in\Delta^-}\mathfrak{g}_\alpha. \tag{2.12}$$

Then we have the triangular decomposition: $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$.

Since A is symmetrizable, there exists a symmetric linear form $(\ |\ )$ on $\mathfrak{h}^*$ satisfying the condition $(\alpha_i \mid \alpha_j) = \delta_i a_{ij}$ for all $i, j \in I$. We say that a root α is *real* if $(\alpha \mid \alpha) > 0$ and *imaginary* if $(\alpha \mid \alpha) \leq 0$. In particular, the simple root α_i is real if $a_{ii} = 2$ and imaginary if $a_{ii} \leq 0$. We note that the imaginary simple roots may have muliplicity > 1. For each $i \in I^{re}$, we let $\sigma_i \in \operatorname{Aut}(\mathfrak{h}^*)$ be the *simple reflection* on $\mathfrak{h}^*$ defined by $\sigma_i(\lambda) := \lambda - \lambda(h_i)\alpha_i$ for $\lambda \in \mathfrak{h}^*$. The subgroup W of $\operatorname{Aut}(\mathfrak{h}^*)$ generated by the σ_i's ($i \in I^{re}$) is called the *Weyl group* of $\mathfrak{g}$.

Let

$$P_G^+ := \{\lambda \in \mathfrak{h}^* \mid \lambda(h_i) \geq 0 \ \text{ for all } i \in I, \ \lambda(h_i) \in \mathbb{Z}_+ \ \text{ if } a_{ii} = 2\}.$$

For $\lambda \in P_G^+$, we let $V(\lambda)$ be the irreducible highest weight module over $\mathfrak{g}$ with highest weight λ. We denote by T the set of all imaginary simple roots counted with multiplicities. We choose $\rho \in \mathfrak{h}^*$ such that $\rho(h_i) = \frac{1}{2}a_{ii}$ for all $i \in I$. Then we have the *Weyl-Kac-Borcherds character formula* [Bo1] :

$$\text{ch}\, V(\lambda) = \frac{\sum_{w\in W}\sum_{F\subset T,\, F\perp\lambda}(-1)^{\ell(w)+|F|}e^{w(\lambda+\rho-s(F))-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{\text{mult}\,\alpha}}, \tag{2.13}$$

where F runs over all the finite subsets of T such that any two distinct elements of F are mutually orthogonal, $\ell(w)$ denotes the length of $w \in W$, $|F| := \text{Card}(F)$ and $s(F)$ denotes the sum of elements in F. For $\lambda = 0$, we obtain the *denominator identity*:

$$\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{\text{mult}\,\alpha} = \sum_{\substack{w\in W\\ F\subset T}}(-1)^{\ell(w)+|F|}e^{w(\rho-s(F))-\rho}. \tag{2.14}$$

Remark 2.8. The notion of a generalized Kac-Moody algebra was introduced by Borcherds in his study of the vertex algebras and the moonshine conjecture [Bo1-3]. As mentioned above, the structure and the representation theory of generalized Kac-Moody algebras are very similar to those of Kac-Moody algebras. The main difference is that a generalized Kac-Moody algebra may have imaginary *simple* roots.

3. The Moonshine Conjectures and The Monster Lie Alegebra

In this section, we give a construction of the *Monster Lie algebra* M and a sketchy proof of the *Moonshine Conjectures* due to R. E. Borcherds [Bo6].

The *Fischer-Griess monster sporadic simple group* G, briefly the MONSTER, is the largest among the 26 sporadic finite simple groups of order

$$2^{46}\cdot 3^{20}\cdot 5^9\cdot 7^6\cdot 11^2\cdot 13^3\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 41\cdot 47\cdot 59\cdot 71.$$

It is known that the dimension of the smallest nontrivial irreducible representation of the MONSTER is 196883 ([FLT]). It was observed by John McKay that $1 + 196883 = 196884$, which is the first nontrivial coefficient of the elliptic modular function $j_*(q) := j(q) - 744$, where $j(q)$ is the *modular invariant*:

$$j_*(q) = \sum_{n\geq -1} c(n)q^n = q^{-1} + 196884q + 21493760q^2 + \cdots. \tag{3.1}$$

Later J.G. Thompson [Th2] found that the early coefficients of the elliptic modular function $j_*(q)$ are simple linear combinations of the irreducible character degrees of

G. Motivated by these observations, J. H. Conway and S. Norton [C-N] conjectured that there is an infinite dimensional graded representation $V^\natural = \sum_{n\geq -1} V_n^\natural$ of the MONSTER G with $\dim V_n^\natural = c(n)$ such that for any element $g \in G$, the Thompson series

$$T_g(q) := \sum_{n\geq -1} tr(g|_{V_n^\natural})q^n, \qquad c_g(n) := tr(g|_{V_n^\natural}) \tag{3.2}$$

is a *Hauptmodul* for a genus 0 discrete subgroup of $SL(2,\mathbb{R})$. It is known that there are 194 conjugacy classes of the MONSTER G. Only 171 of the Thompson series $T_g(q)$, $g \in G$ are distinct. Conway reports on this strange and remarkable phenomenon as follows: "Because these new links are still completely unexplained, we refer to them collectively as the 'moonshine' properties of the MONSTER, intending the word to convey our feelings that they are seen in a dim light, and that the whole subject is rather vaguely illicit!". Therefore the above-mentioned conjectures had been called the *moonshine conjectures.* Recently these conjectures were proved to be true by Borcherds [Bo6] by constructing the so-called *monster Lie algebra.* In his proof, he made use of the *natural* graded representation $V^\natural := \sum_{n\geq -1} V_n^\natural$ of the MONSTER G, called the *moonshine module* or the *Monster vertex algebra*, which was constructed by I.B. Frenkel, J. Lepowsky and A. Meurman [FLM]. (The vector space $V^\natural$ and $V_n^\natural$ are denoted by $V^\natural$ and $V_{-n}^\natural$ respectively in [FLM].) The *graded dimension* $\dim_* V^\natural := \sum_{n\geq -1}(\dim V_n^\natural)q^n$ of the moonshine module $V^\natural$ is given by $\dim_* V^\natural = T_1(q) = j(q) - 744$.

Let $\Pi_{1,1} \cong \mathbb{Z}^2$ be the 2-dimensional even Lorentzian lattice with the matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. The *Monster Lie algebra* M constructed by Borcherds has the following properties:

(M1) M is a $\mathbb{Z}^2$-graded generalized Kac-Moody Lie algebra with Borcherds-Cartan matrix $A = (-(i+j))_{i,j\in I}$ of charge $\underline{m} = ((c(i)|\, i \in I)$, where $I = \{-1\} \cup \{i|\, i \geq 1\}$. The root lattice of M is $\Pi_{1,1} \cong \mathbb{Z}^2$.

(M2) M is a $\mathbb{Z}^2$-graded representation of the MONSTER G such that $M_{(0,0)} \cong \mathbb{R}^2$ and $M_{(m,n)} \cong V_{mn}^\natural$ for all $(m,n) \neq (0,0)$. In particular, $\dim M_{(m,n)} = \dim V_{mn}^\natural = c(mn)$ for all $(m,n) \neq (0,0)$.

(M3) The only *real simple* root of M is $(1,-1)$ and the *imaginary simple* roots of M are of the form $(1,i)$ for $i \geq 1$ with multiplicity $c(i)$.

(M4) $tr(g|_{M_{(m,n)}}) = tr(g|_{V_{mn}^\natural}) = c_g(mn)$ for all $g \in G$ and $(m,n) \neq (0,0)$.

(M5) M has a *contravariant* bilinear form $(\ ,\ )_0$ which is positive definite on the piece $M_{(m,n)}$ of degree $(m,n) \neq (0,0)$. (By a contravariant bilinear form we mean that there is an involution σ on M such that $(u,v) := -(u,\sigma(v))_0$ is invariant and $(u,v) = 0$ if $\deg(u) + \deg(v) \neq 0$.)

We denote by $e_{-1} := e_{1,-1},\, e_{i,k(i)}$ and $f_{-1} := f_{-1,1},\, f_{i,k(i)}\, (\,i \in I,\ k(i) = 1, 2, \cdots, c(i)\,)$ the positve and negative simple root vectors of M respectively. Then we have

$$\begin{aligned} M_{(1,-1)} &= \mathbb{C}e_{-1}, \qquad M_{(-1,1)} = \mathbb{C}f_{-1}, \\ M_{(1,i)} &= \mathbb{C}e_{i,1} \oplus \mathbb{C}e_{i,2} \oplus \cdots \oplus \mathbb{C}e_{i,c(i)}, \\ M_{(-1,-i)} &= \mathbb{C}f_{i,1} \oplus \mathbb{C}f_{i,2} \oplus \cdots \oplus \mathbb{C}f_{i,c(i)} \quad (i \geq 1). \end{aligned}$$

Consider a basis of $M_{(1,i)}$ consisting of the eigenvectors $v_{i,k(i)}(g)$ of an element $g \in G$ with eigenvalues $\lambda_{i,k(i)}(g)$, where $k(i) = 1, 2, \cdots, c(i)$. Since $M_{(1,i)} \cong V_i\ (i \geq 1)$ as G-modules, we have

$$\sum_{k(i)=1}^{c(i)} \lambda_{i,k(i)}(g) = tr(g|_{M_{(1,i)}}) = tr(g|_{V_i^\sharp}) = c_g(i) \tag{3.3}$$

for all $g \in G$ and $i \geq 1$. In addition, since $M_{(1,-1)} \cong M_{(-1,1)} \cong V_{-1}^\sharp \cong \mathbb{R}^2$ is the trivial G-module, we have $g \cdot e_{-1} = e_{-1},\ g \cdot f_{-1} = f_{-1}$ for all $g \in G$.

For small degrees M looks like Fig. 1.

$$\begin{array}{ccccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 0 & 0 & 0 & 0 & V_3^\sharp & V_6^\sharp & V_9^\sharp & \cdots \\
\cdots & 0 & 0 & 0 & 0 & V_2^\sharp & V_4^\sharp & V_6^\sharp & \cdots \\
\cdots & 0 & 0 & V_{-1}^\sharp & 0 & V_1^\sharp & V_2^\sharp & V_3^\sharp & \cdots \\
\cdots & 0 & 0 & 0 & \mathbb{R}^2 & 0 & 0 & 0 & \cdots \\
\cdots & V_3^\sharp & V_2^\sharp & V_1^\sharp & 0 & V_{-1}^\sharp & 0 & 0 & \cdots \\
\cdots & V_6^\sharp & V_4^\sharp & V_2^\sharp & 0 & 0 & 0 & 0 & \cdots \\
\cdots & V_9^\sharp & V_6^\sharp & V_3^\sharp & 0 & 0 & 0 & 0 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Fig. 1

Now we give a construction of the Monster Lie algebra. First of all we define the notion of vertex algebras.

Definition 3.1. A *vertex algebra* V over $\mathbb{R}$ is a real vector space with an infinite number of bilinear products, written as $u_n v$ for $u, v \in V$, $n \in \mathbb{Z}$, such that

(V1) $u_n v = 0$ for n sufficiently large (depending on u and v),

(V2) for all $u, v, w \in V$ and $m, n, q \in \mathbb{Z}$, we have

$$\sum_{i\in\mathbb{Z}} \binom{m}{i} (u_{q+i}v)_{m+n-i}w = \sum_{i\in\mathbb{Z}} (-1)^i \binom{q}{i} \left(u_{m+q-i}(v_{n+i}w) - (-1)^q v_{n+q-i}(u_{m+i}w)\right),$$

(V3) there is an element $1 \in V$ such that $v_n 1 = 0$ if $n \geq 0$ and $v_{-1}1 = v$.

Example 3.2. (1) For each even lattice L, there is a vertex algebra V_L associated with L constructed by Borcherds [Bo1]. Let $\hat{L}$ be the central extension of L by $\mathbb{Z}_2$, i.e., the double cover of L. The underliying vector space of the vertex algebra V_L is given by $V_L = \mathbb{Q}(\hat{L}) \otimes S(\otimes_{i>0} L_i)$, where $\mathbb{Q}(\hat{L})$ is the twisted group ring of $\hat{L}$ and $S(\oplus_{i>0} L_i)$ is the ring of polynomials over the sum of an infinite number of copies L_i of $L \otimes \mathbb{R}$.
(2) Let V be a commutative algebra over $\mathbb{R}$ with derivation D. Then V becomes a vertex algebra by defining

$$u_n v := \begin{cases} \frac{D^{-n-1}(u)v}{(-n-1)!} & \text{for } n < 0 \\ 0 & \text{for } n \geq 0. \end{cases}$$

Conversely any vertex algebra over $\mathbb{R}$ for which $u_n v = 0$ for $n \geq 0$ arises from a commutative algebra in this way.
(3) Let V and W be two vertex algebras. Then the tensor product $V \otimes W$ as vector spaces becomes a vertex algebra if we define the multiplication by

$$(a \otimes b)_n (c \otimes d) := \sum_{i\in\mathbb{Z}} (a_i c) \otimes (b_{n-1-i} d), \quad n \in \mathbb{Z}.$$

We note that the identity element of $V \otimes W$ is given by $1_V \otimes 1_W$.
(4) The moonshine module $V^\natural$ is a vertex algebra.

Definition 3.3. Let V be a vertex algebra over $\mathbb{R}$. A *conformal vector* of *dimension* or *central charge* $c \in \mathbb{R}$ of V is defined to be an element ω of V satisfying the following three conditions:

(1) $\omega_0 v = D(v)$ for all $v \in V$;

(2) $\omega_1 \omega = 2\omega,\ \omega_3 \omega = c/2,\ \omega_i \omega = 0$ for $i = 2$ or $i > 3$;

(3) any element of V is a sum of eigenvectors of the operator $L_0 := \omega_1$ with integral eigenvalues.

Here D is the operator of V defined by $D(v) := v_{-2}1$ for all $v \in V$. If v is an eigenvector of L_0, then its eigenvalue $\lambda(v)$ is called the *conformal weight* of v and v is called a *conformal vector* of conformal weight $\lambda(v)$. If vertex algebras V and W have conformal vectors ω_V and ω_W of dimension m and n respectively, then

$\omega_V \otimes \omega_W$ is a conformal vector of the vertex algebra $V \otimes W$ of dimension $m+n$. It is known that the vertex algebra V associated with any c-dimensional even lattice has a *canonical* conformal vector ω of dimension c. We define the operators $L_i\,(\,i \in \mathbb{Z}\,)$ of V by

$$L_i := \omega_{i+1}, \quad i \in \mathbb{Z}. \tag{3.4}$$

These operators satisfy the relations

$$[L_i,\, L_j] = (i-j)L_{i+j} + \binom{i+1}{3}\frac{c}{2}\delta_{i+j,0}, \quad i,j \in \mathbb{Z} \tag{3.5}$$

and so make V into a module over a *Virasoro algebra* spanned by a central element c and $L_i\,(i \in \mathbb{Z})$. We observe that the operator L_{-1} is equal to the operator D. We define

$$P^i = \{v \in V \mid L_0(v) = \omega_1 v = iv,\; L_k(v) = 0 \text{ for } k > 0\,\},\quad i \in \mathbb{Z}. \tag{3.6}$$

Then the space $P^1/(DV \cap P^1)$ is a subalgebra of the Lie algebra V/DV, which is equal to P^1/DP^0 for the vertex algebra V_L or for the Monster vertex algebra $V^\natural$. Here DV denotes the image of V under D. It is known that the algebra P^1/DP^0 is a generalized Kac-Moody algebra. The structure of a Lie algebra on V/DV is given by the bracket: $[u,v] := u_0 v(u,v \in V)$.

The vertex algebra V_L associated with an even lattice L has a real valued symmetric bilinear form $(\,,)$ such that if u has degree k, the adjoint u_n^* of the operator u_n with respect to $(\,,)$ is given by

$$u_n^* = (-1)^k \sum_{j \geq 0} \frac{L_1^j(\sigma(u))_{2k-j-n-2}}{j!}, \tag{3.7}$$

where σ is the automorphism of V_L defined by

$$\sigma(e^w) := (-1)^{(w,w)/2}(e^w)^{-1} \tag{3.8}$$

for e^w an element of the twisted group ring of L corresponding to the vector $w \in L$. If a vertex algebra has a bilinear form with the above properties, we say that *the bilinear form is compatible with the conformal vector.*

Definition 3.4. A *vertex operator algebra* is defined to be a vertex algebra with a conformal vector ω such that the eigenspaces of the operator $L_0 := \omega_1$ are all finite dimensional and their eigenvalues are all nonnegative integers.

For example, the Monster vertex algebra $V^\natural$ is a vertex operator algebra whose conformal vector spans the subspace $V_1^\natural$ fixed by the MONSTER G. The vertex algebra $V_{\Pi_{1,1}}$ associated with the 2-dimensional even unimodular Lorentzian lattice $\Pi_{1,1}$ is *not* a vertex operator algebra.

We recall the properties of the Monster vertex algebra $V^\natural$.

($V^\natural 1$) $V^\natural$ is a vertex algebra over $\mathbb{R}$ with conformal vector ω of dimension 24 and a positive definite symmetric bilinear form $(\,,)$ such that the adjoint of u_n is given by the expression (3.7), where σ is the trivial automorphism of $V^\natural$.

($V^\natural 2$) $V^\natural$ is a sum of eigenspaces $V_n^\natural$ of the operator $L_0 := \omega_1$, where $V_n^\natural$ is the eigenspace on which L_0 has eigenvalue $n+1$ and $\dim V_n^\natural = c(n)$. Thus $V^\natural$ is a vertex operator algebra in the sense of Definition 3.4.

($V^\natural 3$) The MONSTER G acts faithfully and homogeneously on $V^\natural$ preserving the vertex algebra structure, the conformal vector ω and the bilinear form $(\,,)$. The first few representations $V_n^\natural$ of the MONSTER G are decomposed as

$$\begin{aligned}
V_{-1}^\natural &= \chi_1, \quad V_0^\natural = 0,\\
V_1^\natural &= \chi_1 + \chi_2,\\
V_2^\natural &= \chi_1 + \chi_2 + \chi_3,\\
V_3^\natural &= 2\chi_1 + 2\chi_2 + \chi_3 + \chi_4,\\
V_5^\natural &= 4\chi_1 + 5\chi_2 + 3\chi_3 + 2\chi_4 + \chi_5 + \chi_6 + \chi_7,
\end{aligned}$$

where χ_n $(1 \le n \le 7)$ are the first seven irreducible representations of G, indexed in order of increasing dimension.

($V^\natural 4$) For $g \in G$, the Thompson series $T_g(q)$ is a *completely replicable function*, i.e., satisfies the identity

(3.9)
$$p^{-1}\exp\left(-\sum_{i>0}\sum_{m\in\mathbb{Z}^+,\,n\in\mathbb{Z}} tr(g^i|_{V_{mn}^\natural})p^{mi}q^{mi}/i\right) = \sum_{m\in\mathbb{Z}} tr(g|_{V_m^\natural})p^m - \sum_{n\in\mathbb{Z}} tr(g|_{V_n^\natural})q^n.$$

We remark that the properties ($V^\natural 1$), ($V^\natural 2$) and ($V^\natural 3$) characterize $V^\natural$ completely as a graded representation of G.

Construction of the Monster Lie algebra M: The tensor product $V := V^\natural \otimes V_{\Pi_{1,1}}$ of $V^\natural$ and $V_{\Pi_{1,1}}$ is also a vertex algebra. Then P^1/DP^0 is a Lie algebra with an *invariant* bilinear form $(\,,)$ and an involution τ. Here P^1 and $D := L_1$ are defined by (3.4) and (3.6), and τ is the involution on V induced by the trivial automorphism of $V^\natural$ and the involutuon ω of $V_{\Pi_{1,1}}$. Let $R := \{v \in V \mid (u,v) = 0 \ \text{ for } u \in V\}$ be the radical of $(\,,)$. It is easy to see that DP is a proper subset of R. We define M to be the quotient of the Lie algebra P^1/DP^0 by R. The $\Pi_{1,1}$-grading of $V_{\Pi_{1,1}}$ induces a $\Pi_{1,1}$-grading on M. According to the *no-ghost theorem*, $M_{(m,n)}$ is isomorphic to the piece $V^\natural_{mn}$ of degree $1-(m,n)^2/2 = 1+mn$ if $(m,n) \neq (0,0)$ and $M_{(0,0)} \cong V^\natural_0 \oplus \mathbb{R}^2 \cong \mathbb{R}^2$. And if $v \in M$ is nonzero and homogeneous of nonzero degree in $\Pi_{1,1}$, then $(v, \tau(v)) > 0$. M satisfies the properties (M1)-(M5).

REMARK 3.5. The construction of the Monster Lie algebra M from a vertex algebra can be carried out for any vertex algebra with a conformal vector, but it is only when this vector has *dimension* 24 that we can apply the no-ghost theorem to identify the *homogeneous* pieces of M with those of $V^\natural$. The important thing is that the bilinear form $(\,,)$ on M is positive definite on any piece of nonzero degree, and thus need not be true if the conformal vector has dimension greater than 24, even if the inner product is positive definite.

Problem. Are there some other ways to construct the Monster Lie algebra ?

SKETCHY PROOF OF THE MOONSHINE CONJECTURES : The proof is divided into two steps.

STEP I. The Thompson series are determined by the first 5 coefficients $c_g(i)$, $1 \leq i \leq 5$ for all $g \in G$ because of the identities (3.9).

STEP II. The Hauptmoduls listed in Conway and Norton [C-N, Table 2] satisfy the identities (3.9) and have the same first 5 coefficients of the Thompson series.

The proof of step I is done by comparing the coefficients of p^2 and p^4 of both sides of the identities (3.9) and so obtaining the recursion formulas among $c_g(i)$. The proof of step II follows from the result of Norton [No1] and Koike [Koi1] that the modular functions associated with elements of the MONSTER G also satisfy the identites (3.9) and hence satisfy the same recursion formulas. Roughly we explain how Conway and Norton [C-N] associate to an element of G a modular function of genus 0. Let g be an element of G corresponding to an element of odd order in $\mathrm{Aut}(\Lambda)$ with Leech lattice Λ such that g fixes no nonzero vectors. Let $\epsilon_1, \cdots, \epsilon_{24}$ be eigenvalues of g on the real vector space $\Lambda \otimes \mathbb{R}$. We define

$$\eta_g(q) := \eta_g(\epsilon_1 q) \cdots \eta(\epsilon_{24} q), \quad q := e^{2\pi i \tau},\ \tau \in H_1, \tag{3.10}$$

where $\eta(q) := q^{1/24} \prod_{n \geq 1}(1-q^n)$ is the Dedekind eta function and $H_1 := \{z \in \mathbb{C} |\ \Im z > 0\}$ is the Poincaré upper half plane. We put

$$j_g(q) := \frac{1}{\eta_g(q)} - \frac{1}{\eta_g(0)}.$$

Then $j_g(q)$ is the modular function of genus 0. $j_g(q)$ is the modular function associated with an element g of G by Conway and Norton.

Appendix: The No-Ghost Theorem

Here we discuss the NO-GHOST THEOREM. First of all, we describe the concept of a *Virasoro algebra*.

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathbb{F}[t, t^{-1}]$ be the commutative associative algebra of Laurent polynomials in an indeterminate t, i.e., the algebra of finite linear combinations of integral powers of t. Let $p(t) \in \mathbb{F}[t, t^{-1}]$ and we consider the derivation $D_{p(t)}$ of $\mathbb{F}[t, t^{-1}]$ defined by

$$D_{p(t)} := p(t)\frac{d}{dt}. \tag{1}$$

The vector space $\mathfrak{a}$ spanned by all the derivations of type (1) has the Lie algebra structure with respect to the natural Lie bracket

$$[D_p,\, D_q] = D_{pq'-p'q} \quad \text{for all } p, q \in \mathbb{F}[t, t^{-1}]. \tag{2}$$

We choose the following basis $\{d_n | \, n \in \mathbb{Z}\}$ of $\mathfrak{a}$ defined by

$$d_n := -t^{n+1}\frac{d}{dt}, \quad n \in \mathbb{Z}. \tag{3}$$

By (2), we have the commutation relation

$$[d_m, d_n] = (m-n)d_{m+n}, \quad m, n \in \mathbb{Z}. \tag{4}$$

It is easily seen that $\mathfrak{a}$ is precisely the Lie algebra consisting of all derivations of $\mathbb{F}[t, t^{-1}]$.

Now we consider the one-dimensional central extension $\mathfrak{b}$ of $\mathfrak{a}$ by $\mathbb{F}c$ with basis consisting of a central element c and elements L_n, $n \in \mathbb{Z}$, corresponding to the basis d_n, $n \in \mathbb{Z}$, of $\mathfrak{a}$. We define the bilinear map $[\,,\,]_* : \mathfrak{b} \times \mathfrak{b} \longrightarrow \mathfrak{b}$ by

$$[c, \mathfrak{b}]_* = [\mathfrak{b}, c]_* = [c, c]_* = 0$$

and

$$[L_m, L_n]_* = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}\,c \tag{5}$$

for all $m, m \in \mathbb{Z}$. Then $[\,,\,]_*$ is anti-symmetric and satisfy the Jacobi identity. Thus $(\mathfrak{b}, [\,,\,]_*)$ has the Lie algebra structure. The Lie algebra $\mathfrak{b}$ is called a *Virasoro algebra.* It is not difficult to see that the extension $\mathfrak{b}$ of the Lie algebra $\mathfrak{a}$ is the unique nontrivial one-dimensional central extension up to isomorphism.

Now we state the *no-ghost theorem* and give its sketchy proof.

The No-Ghost Theorem. Let V be a vertex algebra with a nondegenerate bilinear form $(\,,\,)_V$. Suppose that V is acted on by a Virasoro algebra $\mathfrak{b}$ in such a way that the adjoint of L_k with respect to $(\,,\,)_V$ is $L_{-k}\,(k \in \mathbb{Z})$, the central element of $\mathfrak{b}$ *acts as multiplication by* 24, any vector of V is a sum of eigenvectors of L_0 with nonnegative integral eigenvalues, and all the eigenspaces of L_0 are finite dimensional. We let $V^k := \{v \in V \,|\, L_0(v) = kv\}\,(k \in \mathbb{Z}_+)$ be the k-eigenspace of L_0. Assume that V is acted on by a group G which preserves all this strucutre. Let $V_{\Pi_{1,1}}$ be the vertex algebra associated with the two-dimensional even unimodular Lorentzian lattice $\Pi_{1,1}$ so that $V_{\Pi_{1,1}}$ is $\Pi_{1,1}$-graded, has a bilinear form $(\,,\,)_{1,1}$ and is acted on by the Virasoro algebra $\mathfrak{b}$ as mentioned in this section. We let

$$P^1 := \{\, v \in V \otimes V_{\Pi_{1,1}} \,|\, L_0(v) = v,\;\; L_k(v) = 0 \;\text{ for all } k > 0 \,\}$$

and let P^1_r be the subspace of P^1 of degree $r \in \Pi_{1,1}$. All these spaces inherit an action of G from the action of G on V and the trivial action of G on $V_{\Pi_{1,1}}$. Let $(\,,\,) := (\,,\,)_V \otimes (\,,\,)_{1,1}$ be the tensor product of $(\,,\,)_V$ and $(\,,\,)_{1,1}$ and let

$$R := \{\, v \in V \otimes V_{\Pi_{1,1}} \,|\, (u,v) = 0 \;\text{ for all } u \in V \otimes V_{\Pi_{1,1}} \,\}$$

be the null space of $(\,,)$. Then as G-modules with an invariant bilinear form,

$$P^1_r/R \cong \begin{cases} V^{1-(r,r)/2}, & \text{for } r \neq 0 \\ V^1 \oplus \mathbb{R}^2, & \text{for } r = 0. \end{cases}$$

Remark. (1) The name "no-ghost theorem" comes from the fact that in the original statement of the theorem in [G-T], V was a part of the underlying vector space of the vertex algebra associated with a positive definite lattice so that the inner product on V^i was positive definite, and thus P^1_r had no *ghosts*, i.e., *vectors of negative norm* for $r \neq 0$.
(2) If we take the the moonshine module $V^\natural$ as V, then $V^\natural_n$ corresponds to V^{n+1} for all $n \in \mathbb{Z}$.

A sketchy proof : Fix a certain nonzero vector $r \in \Pi_{1,1}$ and some norm 0 vector $w \in \Pi_{1,1}$ with $(r,w) \neq 0$. We have an action of the Virasoro algebra on $V \otimes V_{\Pi_{1,1}}$ generated by its conformal vector. The operators L_m of the Virasoro algebra $\mathfrak{b}$ satisfy the relations

$$[L_m, L_n]_* = (m-n)L_{m+n} + \frac{26}{12}(m^3 - m)\delta_{m+n,0}, \quad m,n \in \mathbb{Z}, \tag{6}$$

and the adjoint of L_m is L_{-m}. Here 26 comes from the fact that the central element c acts on V as multiplication by 24 and the dimension of $\Pi_{1,1}$ is two. We define the operators K_m, $m \in \mathbb{Z}$ by

$$K_m := v_{m-1}, \tag{7}$$

where $v := e_{-2}^{-w}e^w$ is an element of the vertex algebra $V_{\Pi_{1,1}}$ and e^w is an element of the group ring of the double cover of $\Pi_{1,1}$ corresponding to $w \in \Pi_{1,1}$ and e^{-w} is its inverse. Then these operators satisfy the relations

$$[L_m, K_n]_* = -nK_{m+n}, \quad [K_m, K_n]_* = 0 \tag{8}$$

for all $m, n \in \mathbb{Z}$. (8) follows from the fact that the adjoint of K_m is K_{-m} and $(w, w) = 0$.

Now we define the subspaces T^1 and Ve^r of $V \otimes V_{\Pi_{1,1}}$ by

$$T^1 := \{\, v \in V \otimes V_{\Pi_{1,1}} \,|\, \deg(v) = r,\ L_0(v) = v,\ L_m(v) = K_m(v) = 0 \text{ for all } m > 0 \,\}$$

and $Ve^r := V \otimes e^r$. Then we can prove that

$$T^1 \cong V^{1-(r,r)/2}e^r \quad \text{and} \quad T^1 \cong P^1/R. \tag{9}$$

We leave the proof of (9) to the reader. Consequently we have the desired result

$$P^1/R \cong V^{1-(r,r)/2}e^r \cong V^{1-(r,r)/2}.$$

For the case $r = 0$, we leave the detail to the reader.

Finally we remark that in [FI] Frenkel uses the no-ghost theorem to prove some results about Kac-Moody algebras.

4. Jacobi Forms

In this section, we discuss Jacobi forms associated to the symplectic group $Sp(g, \mathbb{R})$ and those associated to the orthogonal group $O_{s+2,2}(\mathbb{R})$ respectively. We also discuss the differences between them.

I. Jacobi forms associated to $Sp(g, \mathbb{R})$.

An exposition of the theory of Jacobi forms associated to the symplectic group $Sp(g, \mathbb{R})$ can be found in [E-Z], [Y1]-[Y4] and [Zi].

In this subsection, we establish the notations and define the concept of Jacobi forms associated to the symplectic group. For any positive integer $g \in \mathbb{Z}^+$, we let

$$Sp(g, \mathbb{R}) = \{M \in \mathbb{R}^{(2g,2g)} \mid {}^tMJ_gM = J_g\}$$

be the symplectic group of degree g, where

$$J_g := \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}.$$

It is easy to see that $Sp(g,\mathbb{R})$ acts on H_g transitively by

$$M < Z > := (AZ+B)(CZ+D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R})$ and $Z \in H_g$. For two positive integers g and h, we consider the *Heisenberg group*

$$H_{\mathbb{R}}^{(g,h)} := \{[(\lambda,\mu),\kappa] \mid \lambda,\mu \in \mathbb{R}^{(h,g)}, \kappa \in \mathbb{R}^{(h,h)}, \kappa + \mu\,{}^t\lambda \text{ symmetric}\}$$

endowed with the following multiplication law

$$[(\lambda,\mu),\kappa] \circ [(\lambda',\mu'),\kappa'] := [(\lambda+\lambda',\mu+\mu'),\kappa+\kappa'+\lambda\,{}^t\mu' - \mu\,{}^t\lambda'].$$

We define the semidirect product of $Sp(g,\mathbb{R})$ and $H_{\mathbb{R}}^{(g,h)}$

$$G^J := Sp(g,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$$

endowed with the following multiplication law

$$\begin{aligned}(M,&[(\lambda,\mu),\kappa]) \cdot (M',[(\lambda',\mu'),\kappa']) \\ &:= (MM',[(\widetilde{\lambda}+\lambda',\widetilde{\mu}+\mu'),\kappa+\kappa'+\widetilde{\lambda}\,{}^t\mu' - \widetilde{\mu}\,{}^t\lambda']),\end{aligned}$$

with $M,\ M' \in Sp(g,\mathbb{R})$ and $(\widetilde{\lambda},\widetilde{\mu}) := (\lambda,\mu)M'$. It is easy to see that G^J acts on $H_g \times \mathbb{C}^{(h,g)}$ transitively by

$$(M,[(\lambda,\mu),\kappa]) \cdot (Z,W) := (M < Z >, (W+\lambda Z+\mu)(CZ+D)^{-1}), \tag{4.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R}),\ [(\lambda,\mu),\kappa] \in H_{\mathbb{R}}^{(g,h)}$ and $(Z,W) \in H_g \times \mathbb{C}^{(h,g)}$.

Let ρ be a rational representation of $GL(g,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbb{R}^{(h,h)}$ be a symmetric half-integral semipositive definite matrix of degree h. Let $C^\infty(H_g \times \mathbb{C}^{(h,g)}, V_\rho)$ be the algebra of all C^∞ functions on $H_g \times \mathbb{C}^{(h,g)}$ with values in V_ρ. For $f \in C^\infty(H_g \times \mathbb{C}^{(h,g)}, V_\rho)$, we define

$$\begin{aligned}(f|_{\rho,\mathcal{M}}&[(M,[(\lambda,\mu),\kappa])])(Z,W) \\ &:= e^{-2\pi i\sigma(\mathcal{M}[W+\lambda Z+\mu](CZ+D)^{-1}C)} \cdot e^{2\pi i\sigma(\mathcal{M}(\lambda Z\,{}^t\lambda + 2\lambda\,{}^tW + (\kappa+\mu\,{}^t\lambda)))} \\ &\times \rho(CZ+D)^{-1} f(M < Z >, (W+\lambda Z+\mu)(CZ+D)^{-1}),\end{aligned} \tag{4.2}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R})$ and $[(\lambda,\mu),\kappa] \in H_{\mathbb{R}}^{(g,h)}$.

Definition 4.1. Let ρ and $\mathcal{M}$ be as above. Let

$$H_{\mathbb{Z}}^{(g,h)} := \{[(\lambda,\mu),\kappa] \in H_{\mathbb{R}}^{(g,h)} \mid \lambda,\mu \in \mathbb{Z}^{(h,g)}, \kappa \in \mathbb{Z}^{(h,h)}\}.$$

A *Jocobi form* of index $\mathcal{M}$ with respect to ρ on Γ_g is a holomorphic function $f \in C^\infty(H_g \times \mathbb{C}^{(h,g)}, V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho,\mathcal{M}}[\widetilde{\gamma}] = f$ for all $\widetilde{\gamma} \in \Gamma_g^J := \Gamma_g \ltimes H_{\mathbb{Z}}^{(g,h)}$.
(B) f has a Fourier expansion of the following form

$$f(Z,W) = \sum_{\substack{T \geq 0 \\ \text{half-integral}}} \sum_{R \in \mathbb{Z}^{(g,h)}} c(T,R) \cdot e^{2\pi i \sigma(TZ)} \cdot e^{2\pi i \sigma(RW)}$$

with $c(T,R) \neq 0$ only if $\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} \geq 0$.

Moreover if $c(T,R) \neq 0$ implies $\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} > 0$, f is called a *cusp Jacobi form.*

If $g \geq 2$, the condition (B) is superfluous by the Koecher principle (cf. [Zi], Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma_g)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ_g. In the special case $V_\rho = \mathbb{C}$, $\rho(A) = (\det A)^k (k \in \mathbb{Z}, A \in GL(g,\mathbb{C}))$, we write $J_{k,\mathcal{M}}(\Gamma_g)$ instead of $J_{\rho,\mathcal{M}}(\Gamma_g)$ and call k the *weight* of a Jacobi form $f \in J_{k,\mathcal{M}}(\Gamma_g)$.

Ziegler (cf. [Zi], Theorem 1.8 or [E-Z], Theorem 1.1) proves that the vector space $J_{\rho,\mathcal{M}}(\Gamma_g)$ is finite dimensional.

Definition 4.2. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ is said to be *singular* if it admits a Fourier expansion such that a Fourier coefficient $c(T,R)$ vanishes unless $\det \begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} = 0$.

Example 4.3. Let $S \in \mathbb{Z}^{(2k,2k)}$ be a symmetric, positive definite, unimodular even integral matrix and $c \in \mathbb{Z}^{(2k,h)}$. We define the theta series

$$\vartheta_{S,c}^{(g)}(Z,W) := \sum_{\lambda \in \mathbb{Z}^{(2k,g)}} e^{\pi\{\sigma(S\lambda Z^t\lambda) + 2\sigma({}^tcS\lambda{}^tW)\}}, \quad Z \in H_g,\ W \in \mathbb{C}^{(h,g)}. \tag{4.3}$$

We put $\mathcal{M} := \frac{1}{2}{}^tcSc$. We assume that $2k < g + \operatorname{rank}(\mathcal{M})$. Then it is easy to see that $\vartheta_{S,c}^{(g)}$ is a singular Jacobi form in $J_{k,\mathcal{M}}(\Gamma_g)$ (cf. [Zi], p. 212).

Remark 4.4. Without loss of generality, we may assume that $\mathcal{M}$ is a *positive definite* symmetric, half-integral matrix of degree h (cf. [Zi], Theorem 2.4).

¿From now on, throughout this paper $\mathcal{M}$ is assumed to be positive definite.

Definition 4.5. An irreducible finite dimensional representation ρ of $GL(g,\mathbb{C})$ is determined uniquely by its highest weight $(\lambda_1,\lambda_2,\cdots,\lambda_g)\in\mathbb{Z}^g$ with $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_g$. We denote this representation by $\rho=(\lambda_1,\lambda_2,\cdots,\lambda_g)$. The number $k(\rho):=\lambda_g$ is called the *weight* of ρ.

The author (cf. [Y3]) proved that singular Jacobi forms in $J_{\rho,\mathcal{M}}(\Gamma_g)$ are characterized by their singular weights.

Theorem 4.6 (Yang [Y3]). *Let $2\mathcal{M}$ be a symmetric, positive definite even integral matrix of degree h. Assume that ρ is an irreducible representation of $GL(g,\mathbb{C})$. Then a nonvanishing Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma_g)$ is singular if and only if $2k(\rho)<g+h$. Only the nonnegative integers k with $0\leq k\leq\frac{g+h}{2}$ can be the weights of singular Jacobi forms in $J_{k,\mathcal{M}}(\Gamma_g)$. These integers are called singular weights in $J_{k,\mathcal{M}}(\Gamma_g)$.*

Proof. The proof can be found in [Y3], Theorem 4.5. □

II. Jacobi forms associated to $O_{s+2,2}(\mathbb{R})$

An exposition of the theory of Jacobi forms associated to the orthogonal group can be found in [Bo7], [G1] and [G2].

First we fix a positive integer s. We let L_0 be a positive definite even integral lattice with a quadratic form Q_0 and let $\Pi_{1,1}$ be the nonsingular even integral lattice with its associated symmetric matrix $I_2:=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$. We define the following lattices L_1 and M by

$$L_1:=L_0\oplus\Pi_{1,1}\quad\text{and}\quad M:=\Pi_{1,1}\oplus L_1. \tag{4.4}$$

Then L_1 and M are nonsingular even integral lattices of $(s+1,1)$ and $(s+2,2)$ respectively. From now on we denote by Q_0, Q_1, Q_M (resp. $(\ ,\ )_0$, $(\ ,\ )_1$, $(\ ,\ )_M$) the quadradic forms (resp. the nondegenerate symmetric bilinar forms) associated with the lattices L_0, L_1, M respectively. We also denote by S_0, S_1, and S_M the nonsingular symmetric even integral matrices associated with the lattices L_0, L and M respectively. Thus S_1 and S_M are given by

$$S_1:=\begin{pmatrix}0&0&-1\\0&S_0&0\\-1&0&0\end{pmatrix}\quad\text{and}\quad S_M:=\begin{pmatrix}0&0&I_2\\0&S_0&0\\I_2&0&0\end{pmatrix}, \tag{4.5}$$

where $I_2:=\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$. We let $M_{\mathbb{R}}:=M\otimes_{\mathbb{Z}}\mathbb{R}$ and $M_{\mathbb{C}}:=M\otimes_{\mathbb{Z}}\mathbb{C}$ be the quadratic spaces over $\mathbb{R}$ and $\mathbb{C}$ respectively. We let

$$O(M_{\mathbb{R}},S_M):=\{g\in GL(M_{\mathbb{R}})\mid {}^tgS_Mg=S_M\} \tag{4.6}$$

be the real orthogonal group of the quadratic space $(M_{\mathbb{R}}, Q_M)$. We denote by O_M the isometry group of the lattice (M, Q_M). Then O_M is an algebraic group defined over $\mathbb{Z}$. We observe that S_M is congruent to $E_{s+2,2}$ over $\mathbb{R}$, i.e., $S_M = {}^t a E_{s+2,2} a$ for some $a \in GL(s+4, \mathbb{R})$, where

$$E_{s+2,2} := \begin{pmatrix} E_{s+2} & 0 \\ 0 & -E_2 \end{pmatrix}. \tag{4.7}$$

Then it is easy to see that $O(M_{\mathbb{R}}, S_M) = a^{-1} O(M_{\mathbb{R}}, E_{s+2,2}) a$. Now for brevity we write $O(M_{\mathbb{R}})$ simply instead of $O(M_{\mathbb{R}}, S_M)$. Obviously $O(M_{\mathbb{R}})$ is isomorphic to the real orthogonal group

$$O_{s+2,2}(\mathbb{R}) := \{ g \in GL(s+4, \mathbb{R}) \mid {}^t g E_{s+2,2} g = E_{s+2,2} \}. \tag{4.8}$$

$O(M_{\mathbb{R}})$ has four connected components. Let $G_{\mathbb{R}}^0$ be the identity component of $O(M_{\mathbb{R}})$ and let $K_{\mathbb{R}}^0$ be its maximal compact subgroup. Then the pair $(G_{\mathbb{R}}^0, K_{\mathbb{R}}^0)$ of the real semisimple groups isomorphic to the pair $(SO(s+2,2)^0, SO(s+2,\mathbb{R}) \times SO(2,\mathbb{R}))$ is a symmetric pair of type (BDI) (cf. [H] 445-446). The homogeneous space $X := G_{\mathbb{R}}^0 / K_{\mathbb{R}}^0$ is a Hermitian symmetric space of noncompact type of dimension $s+2$ (cf. see Appendix C). Indeed, X is a bounded symmetric domain of type IV in the Cartan classification. It is known that X is isomorphic to a $G_{\mathbb{R}}^0$-orbit in the projective space $\mathbb{P}(M_{\mathbb{C}})$. Precisely, if we let $D := \{ z \in \mathbb{P}(M_{\mathbb{C}}) \mid (z,z)_M = 0,\ (z, \overline{z})_M < 0 \}$, then

$$D \cong G_{\mathbb{R}}^0 x_0 \cup G_{\mathbb{R}}^0 \overline{x_0} = D^+ \cup \overline{D^+}, \quad D^+ := G_{\mathbb{R}}^0 x_0, \tag{4.9}$$

where $\overline{x_0}$ denotes the complex conjugation of x_0 in $\mathbb{P}(M_{\mathbb{C}})$. We shall denote by $G_{\mathbb{R}}$ the subgroup of $O(M_{\mathbb{R}})$ preserving the domain D^+. It is known that $D^+ \cong G_{\mathbb{R}}^0 / K_{\mathbb{R}}^0$ may be realized as a tube domain in $\mathbb{C}^{s+2}$ given by

$$\mathcal{D} := \left\{ {}^t Z = (\omega, z, \tau) \in \mathbb{C}^{s+2} \mid \omega \in H_1,\ \tau \in H_1,\ S_1[\operatorname{Im} Z] < 0 \right\}, \tag{4.10}$$

where $\operatorname{Im} Z$ denotes the imaginary part of the column vector Z. An embedding of the tube domain $\mathcal{D}$ into the projective space $\mathbb{P}(M_{\mathbb{C}})$, called the Borel embedding, is of the following form

$$p(Z) = p({}^t(\omega, z_1, \cdots, z_s, \tau)) = {}^t(\frac{1}{2} S_1[Z] : \omega : z_1 : \cdots : z_s : \tau : 1) \in \mathbb{P}(M_{\mathbb{C}}). \tag{4.11}$$

$G_{\mathbb{R}}$ acts on $\mathcal{D}$ transitively as follows: if $g = (g_{kl}) \in G_{\mathbb{R}}$ with $1 \le k, l \le s+4$ and $Z = {}^t(\omega, z_1, \cdots, z_s, \tau) \in \mathcal{D}$, then

$$g < Z > := (\widetilde{\omega}, \widetilde{z}_1, \cdots, \widetilde{z}_s, \widetilde{\tau}), \tag{4.12}$$

where

$$\widetilde{\omega} := (\frac{1}{2}g_{2,1}S_1[Z] + g_{2,2}\omega + \sum_{l=3}^{s+2} g_{2,l}z_{l-2} + g_{2,s+3}\tau + g_{2,s+4})\, J(g,Z)^{-1},$$

$$\widetilde{z}_k := (\frac{1}{2}g_{k+2,1}S_1[Z] + g_{k+2,2}\omega + \sum_{l=3}^{s+2} g_{k+2,l}z_{l-2} + g_{k+2,s+3}\tau + g_{k+2,s+4})\, J(g,Z)^{-1},\ 1 \le k \le s,$$

$$\widetilde{\tau} := (\frac{1}{2}g_{s+3,1}S_1[Z] + g_{s+3,2}\omega + \sum_{l=3}^{s+2} g_{s+3,l}z_{l-2} + g_{s+3,s+3}\tau + g_{s+3,s+4})\, J(g,Z)^{-1}.$$

Here we put

$$(4.13)\quad J(g,Z) := \frac{1}{2}g_{s+4,1}S_1[Z] + g_{s+4,2}\omega + \sum_{l=3}^{s+2} g_{s+4,l}z_{l-2} + g_{s+4,s+3}\tau + g_{s+4,s+4}.$$

It is easily seen that

$$(4.14)\qquad p(g < Z >)J(g,Z) = g \cdot p(Z) \quad (\cdot \text{ is the matrix multiplication})$$

and that $J : G_{\mathbb{R}} \times \mathcal{D} \to GL(1,\mathbb{C}) = \mathbb{C}^\times$ is the automorphic factor, i.e.,

$$J(g_1g_2, Z) = J(g_1, g_2 < Z >)J(g_2, Z)$$

for all $g_1,\ g_2 \in G_{\mathbb{R}}$ and $Z \in \mathcal{D}$.

Let $O_M(\mathbb{Z})$ be the isometry group of the lattice M. Then $\Gamma_M := G_{\mathbb{R}} \cap O_M(\mathbb{Z})$ is an arithmetic subgroup of $G_{\mathbb{R}}$.

Definition 4.7. Let k be an integer. A holomorphic function f on $\mathcal{D}$ is a modular form of weight k with respect to Γ_M if it satisfies the following transformation behaviour

$$(4.15)\qquad (f|_k\gamma)(Z) := J(\gamma, Z)^{-k} f(\gamma < Z >) = f(Z)$$

for all $\gamma \in \Gamma_M$ and $Z \in \mathcal{D}$. For a subgroup Γ of Γ_M of finite index, a modular form with respect to Γ can be defined in the same way.

We denote by $M_k(\Gamma)$ the vector space consisting of all modular forms of weight k with respect to Γ. We now introduce the concept of cusp forms for Γ_M. First of all we note that the realization D^+ of our tube domain $\mathcal{D}$ in the projective space $\mathbb{P}(M_{\mathbb{C}})$ is obtained as a subset of the quadric D in $\mathbb{P}(M_{\mathbb{C}})$ (cf. see (4.10)). A maximal connected complex analytic set X in $\overline{D^+} \setminus D^+$ is called a *boundary component* of D^+, where $\overline{D^+}$ denotes the closure of D^+ in $\mathbb{P}(M_{\mathbb{C}})$. The normalizer

$N(X) := \{g \in G_{\mathbb{R}} \mid g(X) = X\}$ of a boundary component X of D^+ is a maximal parabolic subgroup of $G_{\mathbb{R}}$. X is called a *rational boundary component* if the normalizer $N(X)$ of X is defined over $\mathbb{Q}$. A modular form with respect to Γ_M is called a *cusp form* if it vanishes on every rational boundary component of D^+. It is well known that any rational boundary component X of D^+ corresponds to a primitive isotropic sublattice S of M via $X = X_S := \mathbb{P}(S \otimes \mathbb{C}) \cap \overline{D^+}$. Since the lattice M contains only isotropic lines and planes, there exist two types of rational boundary components, which are points and curves.

The orthogonal group $G_{\mathbb{R}}$ has the rank two and so there are two types of maximal parabolic subgroups in Γ_M. Therefore there are two types of Fourier expansions of modular forms. A subgroup of Γ_M fixing a null sublattice of M of rank one is called a *Fourier group.* A subgroup of Γ_M fixing a null sublattice of M of rank two is called a *Jacobi parabolic group**. Both a Fourier group and a Jacobi parabolic group are maximal parabolic subgroups of Γ_M.

Let $f \in M_k(\Gamma_M)$ be a modular form of weight k with respect to Γ_M. Since the following $\gamma_\ell(\ell \in L_1 \cong \mathbb{Z}^{s+2})$ defined by

$$\gamma_\ell := \begin{pmatrix} 1 & {}^t\alpha & b \\ 0 & E_{s+2} & \ell \\ 0 & 0 & 1 \end{pmatrix}, \quad b := \frac{1}{2}S_1[\ell],\ \alpha = S_1\ell \tag{4.16}$$

are elements of Γ_M, $f(Z+\ell) = f(Z)$ for all $\ell \in L_1$. We note that $\gamma_\ell(Z) = Z + \ell$ for all $Z \in \mathcal{D}$ and $J(\gamma_\ell, Z) = 1$. Hence we have a Fourier expansion

$$f(Z) = \sum_{\ell} a(\ell) e^{2\pi i ({}^t\ell S_1 Z)}, \tag{4.17}$$

where ℓ runs over the set $\{\ell \in \widetilde{L_1} \mid i\ell \in \mathcal{D},\ S_1[\ell] \geq 0\}$. Here $\widehat{L_1}$ denotes the dual lattice of L_1, that is,

$$\widehat{L_1} := \{\ell \in L_1 \otimes_{\mathbb{Z}} \mathbb{Q} \mid {}^t\ell S_1 \alpha \in \mathbb{Z} \text{ for all } \alpha \in L_1\}.$$

We let

$$f(Z) = f(\omega, z, \tau) = \sum_{m \geq 0} \phi_m(\tau, z) e^{2\pi i m \omega} \tag{4.18}$$

be the Fourier-Jacobi expansion of f with respect to the variable w. Obviously the Fourier-Jacobi coefficient

$$\phi_0(\tau, z) = \lim_{v \to \infty} f(iv, z, \tau) \tag{4.19}$$

*In [Bo7], this group was named just a Jacobi group. The definition of a Jacobi group is different from ours.

depends only on τ. We can show that the Fourier-Jacobi coefficients $\phi_m(\tau,z)$ $(m \geq 0)$ satisfies the following functional equations

$$\phi_m\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=(c\tau+d)^k e^{\pi i m \frac{cS_0[z]}{c\tau+d}}\phi_m(\tau,z) \tag{4.20}$$

and

$$\phi_m(\tau,z+x\tau+y)=e^{-2\pi i m({}^t x S_0 z+\frac{1}{2}S_0[x]\tau)}\phi_m(\tau,z) \tag{4.21}$$

for all $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_1 = SL(2,\mathbb{Z})$ and all $x,\ y\in\mathbb{Z}^s$.

Now we define the Jacobi forms associated to the orthogonal group. First we choose the following basis of M such that

$$M=\mathbb{Z}e_1\oplus\mathbb{Z}e_2\oplus L_0\oplus\mathbb{Z}e_{-2}\oplus\mathbb{Z}e_{-1}, \tag{4.22}$$

where e_1,e_2,e_{-1},e_{-2} are four isotropic vectors with $(e_i,e_j)=\delta_{i,-j}$. Let $P_\mathbb{R}$ be the Jacobi parabolic subgroup of $G_\mathbb{R}$ preserving the isotropic plane $\mathbb{R}e_1\oplus\mathbb{R}e_2$. Then it is easily seen that an element g of $P_\mathbb{R}$ is given by the following form:

$$g=\begin{pmatrix} A^0 & X_1 & Y \\ 0 & U & X \\ 0 & 0 & A\end{pmatrix},\quad X_1\in\mathbb{R}^{(2,n)}, Y\in\mathbb{R}^{(2,2)}, X\in\mathbb{R}^{(n,2)}, \tag{4.23}$$

$$A\in GL_2(\mathbb{R})^+,\quad S_0[U]=S_0,\quad A^0=I\,{}^tA^{-1}I,$$
$$X_1=I\,{}^tA^{-1}\,{}^tXS_0U,\quad {}^tYIA+{}^tAIY=S_0[X],$$

where $I:=I_2=\begin{pmatrix} 0 & -1 \\ -1 & 0\end{pmatrix}$. We denote by GO_L for the general orthogonal group or conformal group of the lattice L consisting of linear transformations multiplying the quadratic form by an invertible element of a lattice L. We let $K:=\mathbb{Z}e_1\oplus\mathbb{Z}e_2$ be the 2-dimensional primitive null sublattice of M. Then we have a homomorphism $\pi_P: P_\mathbb{R}\longrightarrow GO_K(\mathbb{R})\times GO_{L_0}(\mathbb{R})$ defined by

$$\begin{pmatrix} A_0 & X_1 & Y \\ 0 & U & X \\ 0 & 0 & A\end{pmatrix}\longmapsto (A^0,U). \tag{4.24}$$

The connected component of the kernel of π_P is called a *Heisenberg group*, denoted by $\mathrm{Heis}(M_\mathbb{R})$. It is easy to see that $\mathrm{Heis}(M_\mathbb{R})$ consists of the following elements

$$\{X;r\}=\{x,y;r\}=\begin{pmatrix} 1 & 0 & {}^tyS_0 & {}^txS_0y-r & \frac{1}{2}S_0[y] \\ 0 & 1 & {}^txS_0 & \frac{1}{2}S_0[x] & r \\ 0 & 0 & E_s & x & y \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1\end{pmatrix}, \tag{4.25}$$

where $X = (x, y)$, with $x, y \in \mathbb{R}^{(s,1)}$ and $r \in \mathbb{R}$. The multiplication on Heis$(M_\mathbb{R})$ is given by
(4.26)
$$\{X_1; r_1\}\{X_2; r_2\} := \{X_1 + X_2; r_1 + r_2 + {}^t x_1 S_0 y_2\},\ X_1 = (x_1, y_1),\ X_2 = (x_2, y_2).$$

We let $G^J_\mathbb{R}$ be the subgroup of $P_\mathbb{R}$ generated by the following elements

$$\{A\} := \mathrm{diag}(A^0, E_s, A),\quad A \in SL(2, \mathbb{R}),\ A^0 = I\,{}^tA^{-1} I \tag{4.27}$$

and $\{X; r\}$ in Heis$(M_\mathbb{R})$. $G^J_\mathbb{R}$ is called the (real) *Jacobi group* of the lattice M. We observe that $G^J_\mathbb{R}$ is isomorphic to the semidirect product of $SL(2, \mathbb{R})$ and Heis$(M_\mathbb{R})$. It is easy to check that

$$\{X; r\}\{A\} = \{A\}\{XA; r + \frac{1}{2}({}^t x_A S_0 y_A - {}^t x S_0 y)\}, \tag{4.28}$$

where x_A and y_A are the columns of the matrix XA. We see easily that Heis$(M_\mathbb{R})$ is a normal subgroup of the Jacobi group $G^J_\mathbb{R}$ and the center C^J of $G^J_\mathbb{R}$ consists of all elements $\nabla(r) := \{0, 0; r\}$, $r \in \mathbb{R}$. According to (4.12), the actions of $\{A\}$ and $\{x, y; r\}$ on $\mathcal{D}$ are given by as follows:

(4.29)
$$\{A\} < Z >= {}^t\Big(\omega - \frac{cS_0[z]}{2(c\tau + d)}, \frac{{}^tz}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\Big);$$
(4.30)
$$\{x, y; r\} < Z >= {}^t(\omega + r + {}^tS_0 z + \frac{1}{2}S_0[x]\tau, {}^t(z + x\tau + y), \tau),$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$ and $Z = {}^t(\omega, z, \tau) \in \mathcal{D}$. ¿From (4.29) and (4.30), we can define the action of the Jacobi group $G^J_\mathbb{R}$ on the (τ, z)-domain $H_1 \times \mathbb{C}^s$, which we denote by $g < (\tau, z) >$, $g \in G^J_\mathbb{R}$.

Let k and m be two integers. For $g \in G^J_\mathbb{R}$ and $Z = {}^t(\omega, {}^tz, \tau) \in \mathcal{D}$, we denote by $\omega(g; Z)$ the ω-component of $g < Z >$. Now we define the mapping $J_{k,m}$: $G^J_\mathbb{R} \times (H_1 \times \mathbb{C}^s) \to GL(1, \mathbb{C}) = \mathbb{C}^\times$ by

$$J_{k,m}(g, (\tau, z)) := J(g, Z)^k e^{-2\pi i m\omega(g;Z)} \cdot e^{2\pi i m\omega}, \tag{4.31}$$

where $g \in G^J_\mathbb{R}$, $Z = {}^t(\omega, {}^tz, \tau) \in \mathcal{D}$ and $J(g, Z)$ is the automorphic factor defined by (4.13). $J_{k,m}$ is well-defined, i.e., it is independent of the choice of $Z = {}^t(w, {}^tz, \tau) \in \mathcal{D}$ with given $(\tau, z) \in H_1 \times \mathbb{C}^s$. It is easy to check that $J_{k,m}$ is an automorphic factor for the Jacobi group $G^J_\mathbb{R}$. In particular, we have

$$J_{k,m}(\{A\}, (\tau, z)) = e^{\pi i \frac{cmS_0[z]}{c\tau + d}} \cdot (c\tau + d)^k,\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \tag{4.32}$$

and

$$J_{k,m}(\{x,y;r\},(\tau,z)) = e^{-2\pi i m(r+{}^t x S_0 z+\frac{1}{2}S_0[x]\tau)}. \tag{4.33}$$

We have a natural action of $G^J_{\mathbb{R}}$ on the algebra $C^\infty(H_1\times\mathbb{C}^s)$ of all C^∞ functions on $H_1\times\mathbb{C}^s$ given by

$$(\phi|_{k,m}g)(\tau,z) := J_{k,m}(g,(\tau,z))^{-1}\phi(g<(\tau,z)>), \tag{4.34}$$

where $\phi\in C^\infty(H_1\times C^s)$, $g\in G^J_{\mathbb{R}}$ and $(\tau,z)\in H_1\times\mathbb{C}^s$. We let $\Gamma^J_M := \Gamma_M\cap G^J_{\mathbb{R}}$ (cf. Definition 4.7). Then Γ^J_M is a discrete subgroup of $G^J_{\mathbb{R}}$ which acts on $H_1\times\mathbb{C}^s$ properly discontinuously.

Definition 4.8. Let k and m be nonnegative integers. A holomorphic function $\phi: H_1\times\mathbb{C}^s\to\mathbb{C}$ is called a Jacobi form of weight k and index m on Γ^J_M if ϕ satisfies the following functional equation

$$\phi|_{k,m}\gamma=\phi \quad \text{for all} \quad \gamma\in\Gamma^J_M \tag{4.35}$$

and $f(\tau,z)$ has a Fourier expansion

$$\phi(\tau,z)=\sum_{n\in\mathbb{Z}}\sum_{\ell\in\widehat{L_0}} c(n,\ell)e^{2\pi i(n\tau+{}^t\ell S_0 z)} \tag{4.36}$$

with $c(n,\ell)\neq 0$ only if $2nm-S_0[\ell]\geq 0$. Here $\widehat{L_0}$ is the dual lattice of L_0, i.e.,

$$\widehat{L_0} := \{\ell\in L_0\otimes\mathbb{Q}\mid {}^t\ell S_0\alpha\in\mathbb{Z}\ \text{ for all } \alpha\in L_0\}.$$

A Jacobi form ϕ of weight k and index m is called a *cusp form* if $c(n,\ell)\neq 0$ implies $2nm-S_0[\ell]>0$. We denote by $J_{k,m}(\Gamma^J_M)$ (resp. $J^{\text{cusp}}_{k,m}(\Gamma^J_M)$) the vector space of all Jacobi forms (resp. cusp forms) of weight k and index m on Γ^J_M.

REMARK 4.9. (1) $J_{k,m}(\Gamma^J_M)$ is finite dimensional.
(2) The Fourier-Jacobi coefficients ϕ_m of a modular form f (cf. (4.17)) are Jacobi forms of weight k and index m on Γ^J_M. (cf. (4.20) and (4.21)).
(3) If $\phi\in J_{k,m}(\Gamma^J_M)$, the function $f_\phi(\omega,z,\tau) := \phi(\tau,z)e^{2\pi i m\omega}$ is a modular form with respect to the subgroup of finite index of the integral Jacobi parabolic subgroup $P_{\mathbb{Z}} := P_{\mathbb{R}}\cap\Gamma_M$.

Let m be a nonnegative integer and let $G_m(L_0) := \widehat{L_0}/mL_0$ be the *discriminant group* of the lattice L_0. For each $h\in G_m(L_0)$, we define the theta function $\vartheta_{S_0,m,h} := \vartheta_{L_0,m,h}$

$$\vartheta_{L_0,m,h}(\tau,z) := \sum_{\ell\in L_0} e^{\pi i m(S_0[\ell+\frac{h}{m}]\tau+2\,{}^t(\ell+\frac{h}{m})S_0 z)}, \tag{4.37}$$

where $(\tau, z) \in H_{1,s} := H_1 \times \mathbb{C}^s$. Any Jacobi form $\phi \in J_{k,m}(\Gamma_M^J)$ can be written as

$$\phi(\tau, z) = \sum_{h \in G_m(L_0)} \phi_h(\tau) \vartheta_{L_0,m,h}(\tau, z) \tag{4.38}$$

with

$$\phi_h(\tau) := \sum_{r \geq 0} c((2r + qS_0[h])(2qm)^{-1}, h) e^{2\pi i \frac{r\tau}{qm}},$$

where each $r \geq 0$ satisfies the condition $2r \equiv -qS_0[h] (\mathrm{mod}\, 2qm)$, $c(n, \ell)$ denotes the Fourier coefficients of $\phi(\tau, z)$ and q is the level of the quadratic form S_0. We can rewrite (4.38) as follows :

$$\phi(\tau, z) = {}^t\Phi(\tau) \cdot \Theta_{L_0,m}(\tau, z), \tag{4.39}$$

where

$$\Phi(\tau) := (\phi_h(\tau))_{h \in G_m(L_0)} \quad \text{and} \quad \Theta_{L_0,m} := (\vartheta_{L_0,m,h})_{h \in G_m(L_0)}. \tag{4.40}$$

Then we can show that for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, the theta function $\Theta_{L_0,m}$ satisfies the following transformation formula

$$\Theta_{L_0,m}\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = e^{\pi i m \frac{cS_0[z]}{c\tau + d}} \cdot (c\tau + d)^{\frac{s}{2}} \cdot \chi(M) \Theta_{L_0,m}(\tau, z), \tag{4.41}$$

where $\chi(M)$ is a certain unitary matrix of degree $|G_m(L_0)|$ (cf. [G2], p.9 and [O], p.105). And $\Phi(\tau)$ satisfies the following functional equations :

$$\Phi(\tau + 1) = e^{-\pi i \frac{S_0[h]}{m}} \Phi(\tau), \quad \Phi(-1/\tau) = \tau^{k - \frac{s}{2}} \overline{U(J)} \Phi(\tau), \tag{4.42}$$

where $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and

$$U(J) := (\det S_0)^{-\frac{1}{2}} \left(\frac{i}{m}\right)^{\frac{s}{2}} \left(e^{-2\pi i \frac{{}^t g S_0 h}{m}}\right)_{g,h \in G_m(L_0)} \tag{4.43}$$

We note that the finite group $G_m(L_0)$ may be regarded as the quadratic space equipped with the quadratic form q_{m,L_0} defined by

$$q_{m,L_0}(h + mL_0) := (h + mL_0, h + mL_0) \in (h, h)_0 + 2\mathbb{Z}. \tag{4.44}$$

¿From (4.41) and (4.42) it follows that $\Phi(\tau)$ is a vector-valued modular form of a half-integral weight and that the vector space of Jacobi forms of index m depends only on the quadratic space $(G_m(L_0), q_{m,L_0})$.

Lemma 4.10. *Let M_1 and M_2 be two even integral lattices of dimension s_1 ans s_2. We assume that the quadratic spaces $(G_{m_1}(M_1), q_{m_1,M_1})$ and $(G_{m_2}(M_2), q_{m_2,M_2})$ are isomorphic. Then we have the isomorphism*

$$J_{k,m_1}(\Gamma^J_{M_1}) \cong J_{k+\frac{s_2-s_1}{2},m_2}(\Gamma^J_{M_2}).$$

Proof. The proof is done if the map

$$ {}^t\Phi(\tau) \cdot \Theta_{M_1,m_1}(\tau, z) \longmapsto {}^t\Phi(\tau) \cdot \Theta_{M_2,m_2}(\tau, z) \tag{4.45}$$

is an isomorphism of $J_{k,m_1}(\Gamma^J_{M_1})$ onto $J_{k+\frac{s_2-s_1}{2},m_2}(\Gamma^J_{M_2})$. The isomorphism can be proved using (4.41) and $s_1 \equiv s_2 \pmod 8$. □

Now we discuss the concept of singular modular forms and singular Jacobi forms.

Definition 4.11. A modular form f with respect to Γ_M (or a Jacobi form ϕ of index with respect to Γ^J_M) is said to be *singular* if its Fourier coefficients satisfy the following condition that

$$a(n, \ell, m) \neq 0 \text{ (or } c(n, \ell) \neq 0) \text{ implies } 2nm - S_0[\ell] = 0,$$

where $a(n, \ell, m)$ and $c(n, \ell)$ denote the Fourier coefficients of f and ϕ in their Fourier expansions respectively.

We consider the *differential operators* D and $\widehat{D}$ defined by

$$D := \frac{\partial^2}{\partial\omega\partial\tau} - \frac{1}{2} S_0[\frac{\partial}{\partial z}] \tag{4.46}$$

and

$$\widehat{D} := \frac{\partial}{\partial z} - \frac{1}{4\pi i m} S_0[\frac{\partial}{\partial z}]. \tag{4.47}$$

Then it is easy to see that if f if a singular modular form and if ϕ is a Jacobi form of index m, then $Df = 0$ and $\widehat{D}\phi = 0$. We can also show easily that any Jacobi form with respect to Γ^J_M has its weight $s/2$ and that any Jacobi form of weight $s/2$ with respect to Γ^J_M is singular. From this fact, we see that a weight of a singular modular form with respect to Γ_M is either 0 or $s/2$.

For each positive integer m, we let $M(m) := \Pi_{1,1} \oplus L_0 \oplus \Pi_{1,1}$ be the lattice with its associated symmetric matrix given by

$$S(m) := \begin{pmatrix} 0 & 0 & I_2 \\ 0 & mS_0 & 0 \\ I_2 & 0 & 0 \end{pmatrix}, \quad I_2 := \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{4.48}$$

We let $\Gamma^J_m := \Gamma^J_{M(m)}$ be the integral Jacobi group of the lattice $M(m)$. It follows immediately from the definitions that if $\phi \in J_{k,m}(\Gamma^J_M)$, then $\phi \in J_{k,1}(\Gamma^J_m)$. The existence of a nonconstant singular Jacobi form of index 1 with respect to Γ^J_M guarantees the unimodularity of the lattice M. Precisely, we have

Proposition 4.12 ([G2], Lemma 4.5). *Let M be a maximal even integral lattice. This means that M is not a sublattice of any even integral lattice. Then a nonconstant singular Jacobi form of index 1 with respect to Γ_M^J exists if and only if the lattice M is unimodular.*

Proof. The proof can be found in [G2], p.21. But we write his proof here. Let $\phi(\tau, z)$ be a nonconstant singular Jacobi form of index 1 with respect to Γ_M^J. According to (4.39), $\phi(\tau, z) = \Phi(\tau) \cdot \Theta_{L_0,1}(\tau, z_0)$. Thus the components $\phi_h(\tau)$ ($h \in G_1(L_0)$) of Φ are constants because their weights are all zero. By (4.42), we have

$$\phi_h(\tau + 1) = e^{-\pi i S_0[h]} \phi_h(\tau), \quad h \in G_1(L_0).$$

Therefore the components ϕ_h are not zero only for the isotrophic vectors h in the group $G_1(L_0) = \widehat{L_0}/L_0$. Since $M = \Pi_{1,1} \oplus L_0 \oplus \Pi_{1,1}$ is maximal, there exists only the trivial isotropic element $h = 0$ in $G_1(L_0)$. Again by (4.42), we obtain that $|G_1(L_0)| = 1$ and so L_0 is unimodular. Hence the lattice M is unimodular. □

Example 4.13. We assume that M is a *unimodular* even integral lattice of signature $(s + 2, 2)$. Then the theta series

$$\vartheta(\tau, z) := \sum_{\lambda \in L} e^{\pi i (S_0[\lambda]\tau + 2\,{}^t\lambda S_0 z)}, \quad (\tau, z) \in H_{1,s} \tag{4.49}$$

is a singular Jacobi form of weight $s/2$ and index 1 with respect to Γ_M^J. The arithmetic lifting f_ϑ of $\vartheta(\tau, z)$ defined by

$$f_\vartheta(\omega, z, \tau) := \frac{(s/2 - 1)!\xi(s/2)}{(2\pi i)^{s/2}} + \sum_{\substack{n,m \geq 0, \lambda \in L \\ 2nm = S_0[\lambda] \\ (n,m) \neq (0,0)}} \sigma_{s/2-1}(n, m; \lambda)\, e^{2\pi i (n\tau + {}^t\lambda S_0 z + m\omega)} \tag{4.50}$$

is a singular modular form of weight $s/2$ with respect to Γ_M^J, where $\sigma_{s/2-1}(n, m; \lambda)$ denotes the sum of $(s/2 - 1)$-powers of all common divisors of the numbers n, m and the vector $\lambda \in L$. For more detail, we refer to Theorem 3.1 and Example 4.4 in [G2].

As we have seen so far, automorphic forms on the real symplectic group and those on the real orthogonal group have different geometric objects, different automorphic factors (cf. (4.12)), and somewhat different properties. For instance, in case of the orthogonal group $O_{s+2,2}(\mathbb{R})$, there is a gap between 0 and $s/2$ such that there exist no modular forms and no Jacobi forms with weights in this gap. By the way, this phenomenon does not happen for automorphic forms and Jacobi forms in the case of the symplectic group $Sp(g, \mathbb{R})$ because all integers less than half the largest singular weight are also singular weights. For more detail, we refer to [F] for singular modular forms and to [Y3] for singular Jacobi forms. In both cases the number of singular weights is equal to the real rank of the corresponding Lie group. Nonetheless the properties of Jacobi forms for the orthogonal group are similar to those of Jacobi forms for the symplectic group. For example, the Fourier coefficient $c(n,\ell)$ of a Jacobi form of weight k and index m for $O_{s+2,2}(\mathbb{R})$ depends only on the number $2mn - S_0[\ell]$ (which is the norm of the vector (n,ℓ,m) in the lattice $\widehat{L_1}$) and the equivalence class of ℓ in the discriminant group $G_m(L_0) = \widehat{L_0}/L_0$ (cf. compare Theorem 2.2 in [E-Z] with our case). We observe that the automorphic factors for the Jacobi groups for both cases are quite similar (cf. see (4.2) and (4.31)-(4.33)). The expression of Jacobi forms in terms of (4.38) or (4.39) are similar to that of Jacobi forms for the symplectic group (cf. [E-Z], [Y1], and [Zi]).

Remark 4.14. In [Bo7], R. Borcherds investigates automorphic forms and Jacobi forms for $O_{s+2,2}(\mathbb{R})$ which are either nearly holomorphic or meromorphic.

Meromorphic functions with all poles at cusps are called *nearly holomorphic ones.*

Borcherds' construction of Jacobi forms : Let K be a positive definite integral lattice of dimension s. A function $c : K \to \mathbb{Z}^+ \cup \{0\}$ is said to be a *vector system* if it satisfies the following three properties (1)–(3) :

(1) The set $\{v \in K | c(v) \neq 0\}$ if finite.
(2) $c(v) = c(-v)$ for all $v \in K$.
(3) The function taking λ to $\sum_{v\in K} c(v)(\lambda, v)^2$ is constant on the sphere of norm 1 vectors $\lambda \in K \otimes \mathbb{R}$.

We will write V for the *multiset* of vectors in a vector system and so we think of V as containing $c(v)$ copies of each vector $v \in K$. And we write $\sum_{v\in V} f(v)$ instead of $\sum_{v\in K} c(v)f(v)$. The vector system is said to be *trivial* if it only contains vectors of zero norm.

The hyperplanes orthogonal to the vectors of a vector system V divides $K_{\mathbb{R}} := K \otimes \mathbb{R}$ into cones which we call the *Weyl chambers* of V. We note that unlike the case of root systems, the Weyl chamber of V need not be all the same type. If we choose a fixed Weyl chamber W, then we can define the *positive* and *negative* vectors of V by saying that v is positive or negative, denoted by $v > 0$ or $v < 0$ if $(v,\lambda) > 0$ or $(v,\lambda) < 0$ for some vector λ in the interior W^0 of W. It is easy to check that the concept of positivity and negativity does not depend on the choice of a vector λ in W^0. Obviously every nonzero vector of the vector system V is either positive or negative.

We define the *Weyl vector* $\rho = \rho_W$ of W by

$$\rho := \frac{1}{2} \sum_{\substack{v\in V \\ v>0}} v.$$

We define d to be the number of vectors in V and define $k := \frac{c(0)}{2}$. The rational number k is called the *weight* of V. We define the *index* m of V by

$$m := (2 \dim K)^{-1} \sum_{v \in V} (v, v)$$

We can show that the index m of V is a nonnegative integer. If V is a vector system in K, we define the *(untwisted) affine vector system* of V to be the multiset of vectors $(v, n) \in K \oplus \mathbb{Z}$ with $v \in V$. We say that (v, n) is *positive* if either $n > 0$ or $n = 0,\ v > 0$. It can be seen that the Weyl vectors for different Weyl chambers differ by elements of K.

Borcherds (cf. [Bo7], p.183) define the function $\psi(\tau, z)$ on $H_1 \times K_{\mathbb{C}}$ with $K_{\mathbb{C}} = K \otimes \mathbb{C} \cong \mathbb{C}^s$ by

$$\psi(\tau, z) := q^{\frac{d}{24}} \zeta^{-\rho} \prod_{(v,n)>0} (1 - q^n \zeta^v), \quad (\tau, z) \in H_1 \times K_{\mathbb{C}}, \tag{4.51}$$

where (v, n) runs over the set of all positive vectors in the affine vector system of V, $q^a := e^{2\pi i a \tau}$ and $\zeta^v = e^{2\pi i (z,v)}$. Then $\psi(\tau, z)$ is a *nearly holomorphic Jacobi form* of weight k and index m. Thus ψ can be written as a finite sum of theta functions times nearly holomorphic modular forms. In fact, ψ satisfies the following transformation formulas:

$$\begin{aligned}
\psi(\tau + 1, z) &= e^{\frac{\pi i d}{12}} \psi(\tau, z), \\
\psi(-1/\tau, z/\tau) &= (-i)^{d/2-k} (\tau/i)^k e^{\pi i m (z,z)/\tau} \psi(\tau, z), \\
\psi(\tau, z + \mu) &= (-1)^{2(\rho,\mu)} \psi(\tau, z), \\
\psi(\tau, z + \lambda\tau) &= (-1)^{2(\rho,\lambda)} q^{-m(\lambda,\lambda)/2} \zeta^{-m\lambda} \psi(\tau, z)
\end{aligned}$$

for all $\lambda, \mu \in \widehat{K}$ (the dual of K).

5. Infinite Products and Modular Forms

In [Bo7], R. Borcherds constructed automorphic forms on $O_{s+2,2}(\mathbb{R})^0$ which are modular products and using the theory of these automorphic theory expressed some meromorphic modular forms for $SL(2, \mathbb{Z})$ with certain conditions as infinite products. Roughly speaking, a modular product means an infinite product whose exponents are the coefficients of a certain nearly holomorphic modular form. For instance, he wrote modular forms as the modular invariant j and the Eisenstein series E_4 and E_6 as infinite products. These results tell us implicitly that the denominator function of a generalized Kac-Moody algebra is sometimes an automorphic form on $O_{s+2,2}(\mathbb{R})^0$ which is a modular product. In this section we discuss Borcherds' results just mentioned in some detail.

We shall start by giving some well-known classical product identities. First we give some of the product identities of L. Euler (1707-83) which are

$$\sum_{n\geq 0} \frac{(-1)^n q^{n(n+1)/2} z^n}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n>0}(1-q^n z), \tag{5.1}$$

$$\sum_{n\geq 0} \frac{z^n}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n>0}(1-q^n z)^{-1}, \tag{5.2}$$

$$\sum_{n\in\mathbb{Z}} (-1)^n q^{3(n+1/6)^2/2} = q^{1/24} \prod_{n>0}(1-q^n). \tag{5.3}$$

A similiar product identity due to C. F. Gauss (1777-1855) is

$$\sum_{n\in\mathbb{Z}} q^{n^2} = (1+q^2)(1-q^2)(1+q^3)^2(1-q^4)\cdots . \tag{5.4}$$

Both of (5.3) and (5.4) are special cases of the so-called Jacobi's triple product identity [C. G. J. Jacobi (1804-51)]

$$\sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} z^n = \prod_{n>0}(1-q^{2n})(1-q^{2n-1}z)(1-q^{2n-1}z^{-1}) \tag{5.5}$$

if we choose z to be some fixed power of q. In fact, if you replace q and z in (5.5) by $q^{3/2}$ and $q^{1/2}$ respectively, you obtain the identity (5.3), and if you replace z in (5.5) by -1, you get the identity (5.4).

The quintuple product identity derived by G. N. Watson (1886-?) is

$$\begin{aligned}\sum_{n\in\mathbb{Z}} q^{(3n^2+n)/2}(z^{3n} - z^{-3n-1}) \\ = \prod_{n>0}(1-q^n)(1-q^n z)(1-q^{n-1}z^{-1})(1-q^{2n-1}z^2)(1-q^{2n-1}z^{-2}).\end{aligned} \tag{5.6}$$

Historically speaking, in 1929 Watson (cf. [W1]) derived the identity (5.6) in the course of proving some of Ramanujan's theorems on continued fractions. In 1938, Watson (cf. [W2]) proved the following identity :

$$\begin{aligned}\sum_{n\in\mathbb{Z}} q^{n(3n+2)}(z^{-3n} - z^{3n+2}) \\ = \prod_{n>0}(1-q^{2n})(1-q^{2n-2}z^2)(1-q^{2n}z^{-2})(1+q^{2n-1}z)^{-1}(1+q^{2n-1}z^{-1}).\end{aligned} \tag{5.7}$$

Subbarao and Vidyasagar (cf. [S-V]) showed that the identities (5.6) and (5.7) are equivalent. The two identities (5.1) and (5.2) of Euler are easily established (cf.[Be], p. 49). G. E. Andrews showed that the Jacobi's triple product identity (5.5) can be obtained easily from the identities (5.1) and (5.2) in his short paper [A]. Carlitz and Subbarao (cf. [C-S]) gave a simple proof of the quintuple product identity (5.7).

The following denominator formula for a finite dimensional simple Lie algebra $\mathfrak{g}$

$$e^{\rho} \sum_{w \in W} \det(w)\, e^{-w(\rho)} = \prod_{\alpha > 0} (1 - e^{\alpha}) \tag{5.8}$$

is due to Hermann Weyl (1885-1955), where W is the Weyl group of $\mathfrak{g}$, ρ is the Weyl vector and the product runs over the set of all positive roots. Macdonald (cf. [Mac]) observed that the Weyl denominator formula is just a statement about finite root systems, and then generalized this formula to *affine* root systems producing the so-called *Macdonald identities.* He noticed that the Jacobi's triple product identity is just the Macdonald identity for the simplest affine root system. Kac observed that the Macdonald identities are just the denominator formulas for the Kac-Moody Lie algebras in the early 1970s. Thereafter he obtained the so-called *Weyl-Kac character formulas* for representations of the affine Kac-Moody algebras generalizing the Weyl character formula (see (2.5)-(2.7) and [K], p. 173). The Weyl-Kac character formula for the affine Kac-Moody algebra is given as follows:

$$e^{\rho} \sum_{w \in W} \det(w)\, e^{-w(\rho)} = \prod_{\alpha > 0} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}, \tag{5.9}$$

where $\mathrm{mult}(\alpha)$ is the multiplicity of the root α. For more detail we refer to (2.6) and [K]. For instance, the Jacobi's triple product identity is just the Weyl-Kac character formula for the affine Kac-Moody algebra $SL_2(\mathbb{R}[z, z^{-1}])$ and the Weyl-Kac character formulas for the affine Kac-Moody algebras $SL_n(\mathbb{R}[z, z^{-1}])$ are just the Macdonald identities. It seems that the Weyl-Kac character formula is true for non-affine Kac-Moody algebras. Borcherds obtained the so-called *Weyl-Kac-Borcherds character formula* for a generalized Kac-Moody algebra (cf. (2.13)-(2.14)). The Weyl-Kac character formula is proved by the Euler-Poincaré principle applied to the cohomology of the Lie subalgebra E of $\mathfrak{g}$ associated to the positive roots of the Kac-Moody algebra $\mathfrak{g}$.

It seems to the author that Borcherds was the first one that discovered that the denominator functions of the generalized Kac-Moody algebras which could be written as infinite products are often automorphic forms on the orthogonal group $O_{s+2,2}(\mathbb{R})^0$. Moreover he gave a method of constructing automorphic forms on $O_{s+2,2}(\mathbb{R})^0$ through modular forms of weight $-s/2$ with integer coefficients and obtained the connection between the Kohnen's "plus" space of weight $1/2$ and the space of modular forms on Γ_1 satisfying some conditions.

Now we are in a position to describe his works on infinite products related to automorphic forms on the orthogonal group $O_{s+2,2}(\mathbb{R})$.

We let L be the unimodular even integral Lorentzian lattice $\mathrm{II}_{s+1,1}$ of dimension $s+2$ and let $M := L \oplus \mathrm{II}_{1,1}$, where $\mathrm{II}_{1,1}$ is the unique 2-dimensional unimodular even integral Lorentzian lattice with its inner product matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. We choose a negative norm vector α in $L_{\mathbb{R}} := L \otimes \mathbb{R}$. We say that a vector v in $L_{\mathbb{R}}$ is *positive*, denoted by $v > 0$ if $(v, \alpha) > 0$.

Theorem 5.1 (Borcherds [Bo7], Theorem 10.1). Let $f(\tau) = \sum_n c(n) q^n$ be a nearly holomorphic modular form of weight $-s/2$ for Γ_1 with integer coefficients, with $24|c(0)$ if $s = 0$. Then there is a unique vector $\delta \in L$ such that

$$\Phi(v) := e^{-2\pi i(\delta, v)} \prod_{r>0,\, r \in L} \left(1 - e^{-2\pi i (r,v)}\right)^{c(-r^2/2)}, \quad v \in \Omega \tag{5.10}$$

is a meromorphic automorphic form of weight $c(0)/2$ for $O_M(\mathbb{Z})^0 \cong O_{s+2,2}(\mathbb{Z})^0$, where $r^2 := (r, r)$ and

$$\Omega := \{\, z \in M \otimes \mathbb{C} \,|\, (z, z) = 0,\ (z, \bar{z}) > 0 \,\}.$$

Remark 5.2. Borcherds showed that all the zeros and poles of Φ lie on the rational quadratic divisors and computed the multiplicites of the zeros of Φ. Roughly speaking a rational quadratic divisor means the zero set of $a(y, y) + (b, y) + c = 0$ with $a, c \in \mathbb{Z}$ and $b \in L$.

Definition 5.3. We define the function $H : \mathbb{Z}_+ \longrightarrow \mathbb{Q}$ by

$$H(n) := \begin{cases} \text{the Hurwitz class number of the discriminant } -n \text{ if } n > 0\,; \\ -1/12 \text{ if } n = 0. \end{cases}$$

We note that

$$\tilde{H}(q) := \sum_{n \geq 0} H(n) q^n = -1/12 + q^3/3 + q^4/2 + q^7 + q^8 + q^{11} + (4/3) q^{12} + \cdots.$$

Now we state a very interesting result.

Theorem 5.4 (Borcherds [Bo7], Theorem 14.1). Let $\mathcal{A}$ be the additive group consisting of nearly holomorphic modular forms of weight $1/2$ for $\Gamma_0(4)$ whose coefficients are integers and satisfy the Kohnen's "plus space" condition. We also let $\mathcal{B}$ be the multiplicative group consisting of meromorphic modular forms for some characters of Γ_1 of integral weight with leading coefficient 1 whose coefficients are integers and all of whose zeros and poles are either cusps or imaginary quadratic irrationals. To each $f(\tau) = \sum_n c(n) q^n$ in $\mathcal{A}$ we associate the function $\Psi_f : H_1 \longrightarrow \mathbb{C}$ defined by

$$\Psi_f(\tau) := q^{-h} \prod_{n>0} (1 - q^n)^{c(n^2)}, \tag{5.11}$$

where h is the constant term of $f(\tau) \tilde{H}(q)$. Then we have the following:

(a) For each $f \in \mathcal{A}$, Ψ_f is an element of $\mathcal{B}$ whose weight is $c(0)$;

(b) the map $\Psi : \mathcal{A} \longrightarrow \mathcal{B}$ given by $\Psi(f) := \Psi_f$ for $f \in \mathcal{A}$ is a group isomorphism of $\mathcal{A}$ onto $\mathcal{B}$;

(c) the multiplicity of the zero of Ψ at a quadratic irrational τ of discriminant $D < 0$ is $\sum_{d>0} c(Dd^2)$.

REMARK 5.5. The product formula for the classical modular polynomial (for discriminant $D < 0$ whose degree is $H(-D)$)

$$\prod_{[\sigma]}(j(\tau) - j(\sigma)) = q^{-H(-D)} \prod_{n>0}(1 - q^n)^{c(n^2)} \tag{5.12}$$

holds, where σ runs over a complete set of representatives modulo Γ_1 for the imaginary quadratic irrationals which are roots of an equation of the form $a\sigma^2 + b\sigma + c = 0$ ($a, b, c \in \mathbb{Z}$) of the discriminant $b^2 - 4ac = D < 0$ (except that σ is a conjugate of one of the elliptic fixed points i or $(1+i\sqrt{3})/2$ we have to replace the corresponding factor $j(\tau) - 1728$ or $j(\tau)$ by $(j(\tau) - 1728)^{1/2}$ or $j(\tau)^{1/3}$) and the exponents $c(n^2)$ are the coefficients of the uniquely determined nearly holomorphic modular form in $\mathcal{A}$. It is easy to check that the classical modular polynomial on the left hand side of (5.12) is contained in $\mathcal{B}$ and that its corresponding element in $\mathcal{A}$ is of the form $q^D + O(q)$.

Examples 5.6. (1) Let $f(\tau) := 12\theta(\tau) = 12\sum_{n\in\mathbb{Z}} q^{n^2}$. It is easy to check that $f(\tau)$ is an element of $\mathcal{A}$ and that $\Psi_f(\tau) = q\prod_{n>0}(1 - q^n)^{24}$ is a cusp form for Γ_1 of weight 12 known as the discriminant function.

(2) We put

$$F(\tau) := \sum_{n>0,\, n:\text{odd}} \sigma_1(n)q^n = q + 4q^3 + 6q^5 + 8q^7 + 13q^9 + \cdots .$$

We let

$$f(\tau) = 3F(\tau)\theta(\tau)(\theta(\tau)^4 - 2F(\tau))\,(\theta(\tau)^4 - 16F(\tau))\,E_6(4\tau)/\Delta(4\tau) + 168\theta(\tau),$$

where $\theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}$, $\Delta(\tau)$ and $E_4(\tau)$ denote the discriminant function and the Eisenstein series of weight 4 respectively. (see Appendix A). It is easy to check that $f(\tau)$ is an element of $\mathcal{A}$ and that $\Psi_f(\tau) = j(\tau)$ is the modular invariant. We also check that $\Psi_f(\tau) = j(\tau)$ has order 3 at the zero $\frac{1+i\sqrt{3}}{2}$ whose discriminant is -3. Hence we obtain the modular product

$$j(\tau) = q^{-1}(1-q)^{-744}(1-q^2)^{80256}(1-q^3)^{-12288744} \cdots .$$

(3) The Eisenstein series E_4 and E_6 are elements of $\mathcal{B}$. The elements of $\mathcal{A}$ corresponding to E_4 and E_6 are given by

$$f_4(\tau) = q^{-3} + 4 - 240q + 26760q^4 - 85995q^5 + 1707264q^8 - 4096240q^9 + \cdots$$

and

$$f_6(\tau) = q^{-4} + 6 + 504q + 143388q^4 + 565760q^5 + 184373000q^8 + 51180024q^9 + O(q^{12})$$

respectively. Use the fact $E_4^3 = j \cdot \Delta$ for f_4. The function $f_6(\tau)$ can be obtained from the theory of a generalized Kac-Moody algebra of rank 1 whose simple roots are all multiples of some root α of norm -2 and the simple roots are $n\alpha$ with $n \in \mathbb{Z}^+$ and multiplicity $504\sigma_3(n)$. Precisely,

$$f_6(\tau) = (j(4\tau) - 876)\theta(\tau) - 2F(\tau)\theta(\tau)(\theta(\tau)^4 - 2F(\tau))(\theta(\tau)^4 - 16F(\tau))E_6(4\tau)/\Delta(4\tau),$$

where $\theta(\tau)$ and $F(\tau)$ are defined in (2). Since $E_8 = E_4^2,\ E_{10} = E_4 \cdot E_6$ and $E_{14} = E_4^2 E_6$, their corresponding elements in $\mathcal{A}$ are given by $2f_4,\ f_4 + f_6$ and $2f_4 + f_6$ respectively. The remaining Eisenstein series ($k \neq 4, 6, 8, 10, 14,\ k$: even, $k \geq 4$) are not elements of $\mathcal{B}$ and hence they cannot be written as modular products. For instance, the modular products for $E_4,\ E_6,\ E_8,\ E_{10}$ and E_{14} are given by

$$E_4(\tau) = (1-q)^{-240}(1-q^2)^{26760}(1-q^3)^{-4096240} \cdots,$$

$$E_6(\tau) = (1-q)^{504}(1-q^2)^{143388}(1-q^3)^{51180024} \cdots,$$

$$E_8(\tau) = (1-q)^{-480}(1-q^2)^{53520}(1-q^3)^{-8192480} \cdots,$$

$$E_{10}(\tau) = (1-q)^{264}(1-q^2)^{170148}(1-q^3)^{47083784} \cdots$$

and

$$E_{14}(\tau) = (1-q)^{24}(1-q^2)^{196908}(1-q^3)^{42987544} \cdots.$$

(4) Using the above theorem, we can show that there exist precisely 14 modular forms of weight 12 on Γ_1 which are contained in $\mathcal{B}$. Indeed, if

$$\begin{aligned} \Xi :&= \{\, n \in \mathbb{Z} \mid j(\tau) - n \in \mathcal{B} \,\} \\ &= \{\, j(\tau) \in \mathbb{Z} \mid \tau \in H_1,\ \tau \text{ is imaginary quadratic} \,\}, \end{aligned}$$

only the modular forms $\Delta(\tau)(j(\tau)-n)$ (where $n \in \Xi$) and $\Delta(\tau)$ are modular forms of weight 12 in $\mathcal{B}$. It is well known that the elements of Ξ are

$$j\left(\frac{1+i\sqrt{3}}{2}\right)=0, \quad j(i)=2^6\cdot 3^3, \quad j\left(\frac{1+i\sqrt{7}}{2}\right)=-3^3\cdot 5^3, \quad j(i\sqrt{2})=2^6\cdot 5^3,$$
$$j\left(\frac{1+i\sqrt{11}}{2}\right)=-2^{15}, \quad j(i\sqrt{3})=2^4\cdot 3^3\cdot 5^3, \quad j(2i)=2^3\cdot 3^3\cdot 11^3,$$
$$j\left(\frac{1+i\sqrt{19}}{2}\right)=-2^{15}\cdot 3^3, \quad j\left(\frac{1+i\sqrt{27}}{2}\right)=-2^{15}\cdot 3\cdot 5^3, \quad j(i\sqrt{7})=3^3\cdot 5^3\cdot 17^3,$$
$$j\left(\frac{1+i\sqrt{43}}{2}\right)=-2^{18}\cdot 3^3\cdot 5^3, \quad j\left(\frac{1+i\sqrt{67}}{2}\right)=-2^{15}\cdot 3^3\cdot 5^3\cdot 11^3,$$
$$j\left(\frac{1+i\sqrt{163}}{2}\right)=-2^{18}\cdot 3^3\cdot 5^3\cdot 23^3\cdot 29^3.$$

6. Final Remarks

In this final section we make some brief remarks on the fake monster Lie algebras, generalized Kac-Moody algebras of the arithmetic type, hyperbolic reflection groups and Jacobi forms. Finally we give some open problems.

6.1. The Fake Monster Lie Algebras

First of all we collect the properties of the *fake monster Lie algebra* M_Λ. (In [Bo5], M_Λ was called just the monster Lie algebra because the monster Lie algebra M defined in section 3 had not been discovered at that time yet.)

Let Λ be the Leech lattice of dimension 24. M_Λ is the generalized Kac-Moody Lie algebra with the following properties $(M_\Lambda 1)-(M_\Lambda 10)$:

$(M_\Lambda 1)$ The root lattice L of M_Λ is $\mathrm{II}_{25,1} := \Lambda \oplus \mathrm{II}_{1,1}$.

$(M_\Lambda 2)$ $\rho=(0,0,1)$ is the Weyl vector of L with norm $\rho^2=0$. The real simple roots of M_Λ are the norm 2 vectors of the form $(\lambda,1,\lambda^2/2-1)$, $\lambda\in\Lambda$, and the imaginary simple roots are the positive multiples of ρ each with multiplicity 24. (We observe that if r is a real simple root, then $(\rho,r)=-1$)

$(M_\Lambda 3)$ A nonzero vector $r\in L=\mathrm{II}_{25,1}$ is a root if and only if $r^2\le 2$, in which case it has multiplicity $p_{24}(1-r^2/2)$, where $p_{24}(1-r^2/2)$ is the number of partitions of $1-r^2/2$ into 24 colours.

$(M_\Lambda 4)$ M_Λ has a $\mathrm{II}_{25,1}$-grading. The piece $M_\Lambda(r)$ of degree $r\in\mathrm{II}_{25,1}, r\neq 0$ has dimension $p_{24}(1-r^2/2)$.

($M_\Lambda 5$) M_Λ has an involution ω which acts as -1 on $\mathrm{II}_{25,1}$ and also on the piece $M_\Lambda(0)$ of degree $0 \in \mathrm{II}_{25,1}$.

($M_\Lambda 6$) M_Λ has a contravariant bilinear form $(\,,\,)$ such that $M_\Lambda(k)$ is orthogonal to $M_\Lambda(l)$ with respect to $(\,,\,)$ if $k \neq -l,\ k, l \in \mathrm{II}_{25,1}$ and such that $(\,,\,)$ is positive definite on $M_\Lambda(k)$ for all $k \in \mathrm{II}_{25,1}$ with $k \neq 0$.

($M_\Lambda 7$) The denominator formula for M_Λ is given by

$$e^{-\rho} \sum_{w\in W} \sum_{n\in \mathbb{Z}} \det(w)\, \tau(n)\, e^{w(n\rho)} = \prod_{r\in L^+} (1-e^r)^{p_{24}(1-r^2/2)}, \tag{6.1}$$

where W is the Weyl group, L^+ is the set of all positive roots of M_Λ, and $\tau(n)$ is the Ramanujan tau function. (The discriminant function $\Delta(\tau)$ is the generating function of $\tau(n)$.) Indeed, L^+ is given by

$$L^+ = \left\{ v \in \mathrm{II}_{25,1} \,|\, v^2 \leq 2,\ (v,\rho) < 0 \right\} \cup \left\{ n\rho \,|\, n \in \mathbb{Z}^+ \right\}.$$

($M_\Lambda 8$) The universal central extension $\hat{M}_\Lambda$ of M_Λ is a $\mathrm{II}_{25,1}$-graded Lie algebra. If $0 \neq r \in \mathrm{II}_{25,1}$, then the piece $\hat{M}_\Lambda(r)$ of $\hat{M}_\Lambda$ of degree r is mapped isomorphically to $M_\Lambda(r)$. The piece $\hat{M}_\Lambda(0)$ of degree 0, called the Cartan subalgebra of $\hat{M}_\Lambda$, can be represented naturally as the sum of a *one-dimensional* space for each vector of Λ and a space of dimension $24^2 = 576$ for each positive integer.

($M_\Lambda 9$) For each $r \in L^+$, we put

$$m(r) := \sum_{\substack{n>0 \\ r/n\in \mathrm{II}_{25,1}}} \mathrm{mult}(r/n)\cdot n.$$

Then for each $r \in L^+$, we have the following formula

$$(r+\rho)^2 m(r) = \sum_{\substack{\alpha,\beta\in L^+ \\ \alpha+\beta=r}} (\alpha,\beta)\, m(\alpha)\, m(\beta). \tag{6.2}$$

($M_\Lambda 10$) M_Λ is a $\mathrm{Aut}(\hat{\Lambda})$-module. In fact, $\mathrm{Aut}(\hat{\Lambda})$ acts naturally on the vertex algebra of $\hat{\Lambda}$ and hence on M_Λ.

The detail for all the properties ($M_\Lambda 1$)-($M_\Lambda 10$) can be found in [Bo5].

REMARK 6.1. (a) M_Λ is essentially the space of physical vectors of the vertex algebra of $\hat{\mathrm{II}}_{25,1}$, where $\hat{\mathrm{II}}_{25,1}$ is the unique central extension of $\mathrm{II}_{25,1}$ by $\mathbb{Z}_2$.
(b) M_Λ can be constructed from the vertex algebra of V_Λ of the central extension $\hat{\Lambda}$ of Λ by $\mathbb{Z}_2$ in the same way that the monster Lie algebra M was constructed from the monster vertex algebra V in section 3.

(c) The multiplicities $p_{24}(1+n)$ of the roots of M_Λ is given by the Rademacher's formula

$$p_{24}(1+n) = 2\pi n^{-13/2} \sum_{k>0} \frac{I_{13}(4\pi\sqrt{n}/k)}{k} \cdot \sum_{\substack{0\le h,h'\le k \\ hh'\equiv -1 \,(\mathrm{mod}\, k)}} e^{2\pi i(nh+h')/k},$$

where $I_{13}(z) := -iJ_{13}(iz)$ is the modified Bessel function of order 13. In particular, $p_{24}(1+n)$ is asymptotic to $2^{-1/2}n^{-27/4}e^{4\pi\sqrt{n}}$ for large n.

In [Bo6], Borcherds constructed a family of Lie algebras and superalgebrs, the so-called *monstrous Lie superalgebra* whose denominator formulas are twisted denominator formulas of the monster Lie algebra M. For each element g in the MONSTER G, we define the *monstrous Lie algebra* of g to be the genrealized Kac-Moody superalgebra which has its root lattice $\text{II}_{1,1}$ and simple roots $(1,n)$ with multiplicity $tr(g|_{V_n^\sharp})$. The denominator formula for the monstrous Lie superalgebra M_g of g is given by

$$(6.3) \quad T_g(p) - T_g(q) = \sum_m tr(g|_{V_m^\sharp})p^m - \sum_n tr(g|_{V_n^\sharp})q^n = p^{-1} \prod_{m>0,\, n\in\mathbb{Z}} (1-p^m q^n)^{\mathrm{mult}_g(m,n)}.$$

The multiplicity $\mathrm{mult}_g(m,n)$ of the root $(m,n) \in \text{II}_{1,1}$ is

$$(6.4) \qquad \mathrm{mult}_g(m,n) = \sum_{ds|(m,n,N)} \frac{\mu(s)}{ds}\, tr(g^d|_{V_{mn}^\sharp}).$$

where N is the order of g. We recall that the Thompson series $T_g(q)$ of g is the normalized generator for a genus zero function field of a discrete subgroup of $SL(2,\mathbb{R})$ containing the Hecke subgroup $\Gamma_0(hN)$, where h is a positive integer with $h|(24,N)$.

Furthermore Borcherds (cf. [Bo6],) constructed a family of superalgebras whose denominator formulas are twisted ones of the fake monster Lie algebra M_Λ in the same way that he constructed a family of monstrous Lie superalgebras from the monster Lie algebra M.

Let g be an element of $\mathrm{Aut}(\hat{\Lambda}) \cong 2^{24} \cdot \mathrm{Aut}(\Lambda)$ of order N. We let

$$(6.5) \qquad L := \{\, \lambda \in \Lambda \mid g\lambda = \lambda \,\}$$

be the sublattice of Λ fixed by g. Then the dual L' of L is equal to the projection of Λ into the vector space $L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$ because Λ is unimodular. For simplicity we assume that any power g^n of g fixes all elements of $\hat{\Lambda}$ which are in the inverse image of Λ^{g^n}, where Λ^{g^n} is the set of elements in Λ fixed by g^n. According to [Bo3], there exists a reflection group W^g acting on L with following properties (W1)-(W2) :

(W1) The positive roots of W^g are the sums of the conjugates of some positive real roots of $\Pi_{25,1}$.

(W2) Let ρ be the Weyl vector of W^g. The simple roots of W^g are the sums of orbits of simple roots of W that have positive norms and they are also the roots of W^g such that $(r,\rho)=-r^2/2$ with $\rho^2=0$.

Let $\mathfrak{g}_g$ be a generalized Kac-Moody superalgebra with the following simple roots:

1. L is the root lattice of $\mathfrak{g}_g$.
2. The real simple roots are the simple roots of the reflection group W^g, which are the roots r with $(r,\rho)=-r^2/2$.
3. The imaginary simple roots are $n\rho$ ($n\in\mathbb{Z}^+$) with multiplicity $\mathrm{mult}_g(n\rho)$ given by

$$\mathrm{mult}_g(n\rho)=\sum_{ja_k=n} b_k$$

if g has a generalized cycle shape $a_1^{b_1}a_2^{b_2}\cdots$.

Then the denominator formula for the *fake monstrous superalgebra* $\mathfrak{g}_g$ is given by

$$e^{-\rho}\sum_{w\in W^g}\det(w)\,w\,(\eta_g(e^\rho))=\prod_{r\in L^+}(1-e^r)^{\mathrm{mult}_g(r)}, \tag{6.6}$$

where $\eta_g(q)$ is the function defined by

$$\eta_g(q):=\eta(\varepsilon_1 q)\eta(\varepsilon_2 q)\cdots\eta(\varepsilon_{24} q) \tag{6.7}$$

if g has eigenvalues $\varepsilon_1,\cdots,\varepsilon_{24}$ on $\Lambda_{\mathbb{R}}:=\Lambda\otimes_{\mathbb{Z}}\mathbb{R}$. It is easy to check that if g has a generalized cycle shape $a_1^{b_1}a_2^{b_2}\cdots$, then

$$\eta_g(q)=\eta(q^{a_1})^{b_1}\eta(q^{a_2})^{b_2}\cdots.$$

Example 6.2. Let $p=2,3,5,7,11,23$ be six prime numbers such that $p+1$ divides 24. We let g be an element of $\mathrm{Aut}(\hat{\Lambda})$ of order p corresponding to an element of $M_{24}\subset\mathrm{Aut}(\Lambda)$ of cycle shape $1^{24/(p+1)}p^{24/(p+1)}$, where M_{24} is the Mathieu group. Then the denominator formula for the fake monstrous superalgebra (in fact, a Lie algebra) $\mathfrak{g}_p:=\mathfrak{g}_g$ is given by

$$\begin{aligned} e^{-\rho}\sum_{w\in W^g}\det(w)\,w\left(e^\rho\prod_{n>0}(1-e^{n\rho})^{24/(p+1)}(1-e^{pn\rho})^{24/(p+1)}\right)\\ =\prod_{r\in L^+}(1-e^r)^{p_g(1-r^2/2)}\prod_{r\in pL^+}(1-e^r)^{p_g(1-r^2/2p)}, \end{aligned} \tag{6.8}$$

where L^+ denotes the set of all positive roots of $\mathfrak{g}_p$ and $p_g(1+n)$ is defined by

$$\sum_{n>0} p_g(1+n)q^n = 1/\eta_g(q). \tag{6.9}$$

$\mathfrak{g}_2$, $\mathfrak{g}_3$, $\mathfrak{g}_5$, $\mathfrak{g}_7$ and $\mathfrak{g}_{11}$ are called the *fake baby* monster Lie algebra, the *fake Fischer* monster Lie algebra, the *fake Harada - Norton* monster Lie algebra, the *fake Held* monster Lie algebra and the *fake Mathieu* monster Lie algebra respectively. We observe that the dimension of $\mathfrak{g}_p\,(\,p\,=\,2,3,5,7,11,23\,)$ are $18, 14, 10, 8, 6, 1$ respectively.

Example 6.3. Let $\mathfrak{g}_{fC}$ be the *fake Conway Lie superalgebra* of rank 10. $\mathfrak{g}_{fC}$ is the fake monstrous Lie superalgebra associated with an element $g \in \mathrm{Aut}(\hat{\Lambda})$ of order 2 such that the descent g_0 of g to $\mathrm{Aut}(\Lambda)$ is of order 2 and the lattice Λ^{g_0} of Λ fixed by g_0 is isomorphic to the lattice E_8 with all norms doubled. The lattice L of $\mathfrak{g}_{fC}$ is the nonintegral lattice of determinant $1/4$ all whose vectors have integral norm which is the dual lattice of the sublattice of even vectors of $I_{9,1}$. Here $I_{9,1} := \{\,(v,m,n)\,|\,v\in E_8,\ m,n\in\mathbb{Z},\ m+n \text{ is even}\,\}$ is the lattice of dimension 10. Let W be the Weyl group of $\mathfrak{g}_{fC}$. In other words, W is the subgroup of $\mathrm{Aut}(L)$ generated by the reflection of norm 1 vectors. The simple roots of W are the norm 1 vectors with $(r,\rho) = -1/2$. The simple roots of $\mathfrak{g}_{fC}$ are the simple roots of W together with the positive multiple $n\rho\,(\,n\in\mathbb{Z}^+\,)$ of the Weyl vector $\rho = (0,0,1)$ each with multiplicity $8(-1)^n$. Here the multiplicity $-k<0$ means a superroot of multiplicity k, so that the odd multiples of ρ are superroots. The multiplicity $\mathrm{mult}(r)$ of the root $r=(v,m,n)\in L$ is given by

$$\mathrm{mult}(r) = (-1)^{(m-1)(n-1)}p_g\left((1-r^2)/2\right) = (-1)^{m+n}|p_g((1-r^2)/2)|, \tag{6.10}$$

where $p_g(n)$ is defined by

$$\sum p_g(n) = q^{-1/2}\prod_{n>0}(1-q^{n/2})^{-(-1)^n 8}.$$

Finally the denominator formula for the fake Conway sueralgebra $\mathfrak{g}_{fC}$ is given by

$$e^{-\rho}\sum_{w\in W}\det(w)\,w\left(e^{\rho}\prod_{n>0}(1-e^{n\rho})^{(-1)^n 8}\right) = \prod_{r\in L^+}(1-e^r)^{\mathrm{mult}(r)}, \tag{6.11}$$

where L^+ denotes the set of positive roots.

6.2. Kac-Moody Algebras of the Arithmetic Type

Let $A = (a_{ij})$ be a symmetrizable generalized Cartan matrix of degree n and let $\mathfrak{g}(A)$ its associated Kac-Moody Lie algebra (see section 2). Then there exist a diagonal matrix $D = \operatorname{diag}(\epsilon_1, \cdots, \epsilon_n)$ with $\epsilon_i > 0$, $\epsilon_i \in \mathbb{Q}\,(\,1 \leq i \leq n\,)$ and a symmetric integral matrix $B = (b_{ij})$ such that

$$A = DB, \quad \text{g.c.d.}(\{\, b_{ij} \mid 1 \leq i, j \leq n\}) = 1. \tag{6.12}$$

We note that such matrices D and B are uniquely determined. Let

$$Q := \sum_{i=1}^{n} \mathbb{Z}\,\alpha_i, \quad Q_+ := \sum_{i=1}^{n} \mathbb{Z}_+\alpha_i, \quad Q_- := -Q_+,$$

where $\alpha_1, \cdots, \alpha_n$ are simple roots of A or $\mathfrak{g}(A)$. Then $Q = Q_+ \cup Q_-$ is a root lattice of A.

Now we have the *canonical symmetric* bilinear form

$$(\ ,\ ) \,:\, Q \times Q \longrightarrow \mathbb{Z}, \quad (\alpha_i, \alpha_j) = b_{ij} = a_{ij}/\epsilon_i. \tag{6.13}$$

Let Δ, Δ^+ and Δ^- be the set of all roots, positive roots, and negative roots of $\mathfrak{g}(A)$ respectively. We let W be the Weyl group of $\mathfrak{g}(A)$ generated by the fundamental reflections

$$r_{\alpha_i}(\beta) := \beta - 2\frac{(\beta, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i, \quad \beta \in Q, \quad 1 \leq i \leq n. \tag{6.14}$$

It is clear that Δ is invariant under W. We let

$$K := \{\, \alpha \in Q_+ \mid \alpha \neq 0,\ (\alpha, \alpha_i) \leq 0 \text{ for all } i, \text{ and } \operatorname{supp}(\alpha) \text{ is connected}\,\}, \tag{6.15}$$

where for $\alpha = \sum_{i=1}^{n} k_i\alpha_i \in Q_+$, $\operatorname{supp}(\alpha)$ is defined to be the subset $\{\, \alpha_i \,|\, k_i > 0\,\}$ of the set $\{\, \alpha_1, \cdots, \alpha_n\,\}$, and $\operatorname{supp}(\alpha)$ is said to be *connected* if there do not exist nonempty two sets A_1 and A_2 such that $\operatorname{supp}(\alpha) = A_1 \cup A_2$ and $(\alpha, \beta) = 0$ for all $\alpha \in A_1$ and $\beta \in A_2$. Let Δ^{re} (resp. Δ_{im}) be the set of all real roots (resp. imaginary) roots of $\mathfrak{g}(A)$. Then it is easy to check that

$$\Delta^{re} = W(\alpha_1) \cup \cdots \cup W(\alpha_n) \tag{6.16}$$

and

$$\Delta^{im} \cap Q_+ = W(K). \tag{6.17}$$

Definition 6.4. A generalized Cartan matrix A of degree n or its associated Kac-Moody Lie algebra $\mathfrak{g}(A)$ is said to be *of the arithmetic type* or *have the arithmetic type* if it is symmetrizable and indecompsable and also if for each $\beta \in Q$ with the property $(\beta, \beta) < 0$ there exist a positive integer $n(\beta) \in \mathbb{Z}^+$ and an imaginary root $\alpha \in \Delta^{im}$ such that

$$n(\beta)\beta \equiv \alpha \mod Q_0 \ \text{ on } Q, \tag{6.18}$$

where $Q_0 := \{\, \gamma \in Q \mid (\gamma, \delta) = 0 \text{ for all } \delta \in Q\,\}$ denotes the kernel of $(\ ,\ )$.

If we set $M := Q/Q_0$, then $(\ ,\ )$ induces the *canonical nondegenerate, symmetric integral* bilinear form on the free $\mathbb{Z}$-module M defined by

$$S : M \times M \longrightarrow \mathbb{Z}. \tag{6.19}$$

We let $\pi : Q \longrightarrow M$ be the projection of Q onto M, and we denote by $\tilde{x} = \pi(x)$ the image of $x \in Q$ under π. We denote by (t_+, t_-, t_0) the signature of a symmetric matrix B.

The following theorem is due to V. V. Nikulin.

Theorem 6.5 ([N5], Theorem 2.1). A symmetrizable indecomposable generalized Cartan matrix A or its associated Kac-Moody Lie algebra $\mathfrak{g}(A)$ has the *arithmetic type* if and only if A has one of the following types (a), (b), (c) or (d) :

(a) The finite type case: $B > 0$.

(b) The affine type case: $B \geq 0$ and B has the signature $(\ell, 0, 1)$.

(c) The rank 2 hyperbolic case: B has the signature $(1, 1, 0)$.

(d) The arithmetic hyperbolic type: B is hyperbolic of rank > 2, equivalently, B has the signature $(\ell - 1, 1, k)$ with $\ell \geq 3$, and the index $[O(S) : \tilde{W}]$ is finite.

Here $O(S)$ and $\tilde{W}$ denote the orthogonal group of S and the image of the Weyl group W under π respectively.

Now we assume that A is of the arithmetic hyperbolic type and that B has the signature $(t_+, 1, k)$ with $t_+ \geq 2$. We choose a subgroup $\tilde{W}$ of $W(S)$ of finite index generated by reflections. We choose a fundamental polyhedron $\mathcal{M}$ of $\tilde{W}$, and then let $P(\mathcal{M})_{pr}$ be the set of primitive elements of $\mathcal{M}$ which are orthogonal to the faces of $\mathcal{M}$ and directed outside.

Theorem 6.6 ([N5], Theorem 4.5). We assume that $S : M \times M \longrightarrow \mathbb{Z}$ is a reflexive primitive hyperbolic, symmetric integral bilinear form and that $\tilde{W} \subset W(S)$ satisfies the following conditions (6.20) and (6.20) :

$$P(\mathcal{M})_{pr} \text{ generates } M\,; \tag{6.20}$$

$$P(\mathcal{M}_0)_{pr} \text{ generates } M, \tag{6.21}$$

where $\mathcal{M}_0$ is the fundamental polyhedron of $W(S)$.

In additon, we assume that we have a function

$$\lambda : P(\mathcal{M})_{pr} \longrightarrow \mathbb{Z}^+$$

satisfying the conditions (6.22) and (6.23).:

$$S(\lambda(\alpha)\alpha, \lambda(\alpha)\alpha) \text{ divides } 2\,S(\lambda(\beta)\beta, \lambda(\alpha)\alpha) \text{ for all } \alpha, \beta \in P(\mathcal{M})_{pr}\,; \tag{6.22}$$

$$\{\, \lambda(\alpha)\alpha \mid \alpha \in P(\mathcal{M})_{pr} \,\} \text{ generates } M. \tag{6.23}$$

Then the data $(S, \tilde{W}, \lambda)$ defines canonically a generalized Cartan matrix of the arithmetic hyperbolic type

$$A(S, \tilde{W}, \lambda) = (\,2\,S(\lambda(\beta)\beta,\, \lambda(\alpha)\alpha)/S(\lambda(\alpha)\alpha, \lambda(\alpha)\alpha)\,, \quad \alpha,\, \beta \in P(\mathcal{M})_{pr}.$$

Remark 6.7. (a) According to Nikulin (cf. [N3], [N4]) and Vinberg (cf. [V]), there exist only a finite number of isomorphism classes of reflexive primitive hyperbolic symmetric integral bilinear forms S of rank ≥ 3, and the rank of S is less than 31. Therefore by Theorem 6.6, there are only *finite* Kac-Moody Lie algebras of the arithmetic hyperbolic type.

(2) In [K], a very special case of a generlized Cartan matrix A is considered. This matrix is called just *hyperbolic* there. This has the property that the fundamental polyhedron $\mathcal{M}$ of $\tilde{W}$ is a simplex. There exist only a finite list of these hyperbolic ones. These are characterized by the property : $0 \neq 0 \in Q$ is an imaginary root if and only if $(\alpha, \alpha) < 0$.

(c) The complete list of the bilinear forms mentioned in Theorem 6.6 is not known yet.

Example 6.8. We consider an example of s *symmetric* generalized Cartan matrix A of the arithmetic hyperbolic type given by

$$A = (a_{ij}) = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{6.22}$$

Let $\mathcal{F} := \mathfrak{g}(A)$ be its associated Kac-Moody Lie algebra of the arithmetic hyperbolic type. Let $\mathcal{F}_0$ be the *affine* Kac-Moody Lie algebra of type $A_1^{(1)}$ with its Cartan matrix $A_0 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. Then it is known that

$$\mathcal{F}_0 \cong s\ell_2(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\cdot c,$$

which is a one-dimensional central extension of the loop algebra $s\ell_2(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}]$. We let $\mathcal{F}_0^e$ be the semi-direct product of $\mathcal{F}_0$ and $\mathbb{C}\cdot d,\ d := -t\frac{d}{dt}$, whose bracket is defined as follows :

$$\begin{aligned} &[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{m+n} + n < x, y > \delta_{n,-m} c, \quad m, n \in \mathbb{Z}, \\ &[d,\, x \otimes t^n] = -n(x \otimes t^n), \quad n \in \mathbb{Z}, \\ &[c, a] = 0 \text{ for all } a \in \mathcal{F}_0^e, \text{ i.e., } c \text{ acts centrally,} \end{aligned}$$

where $x, y \in s\ell_2(\mathbb{C})$ and $< x, y >:= \mathrm{tr}\,(xy) = 1/4\,\mathrm{tr}\,(\,\mathrm{ad}\,x\,\mathrm{ad}\,y\,)$ denotes the Cartan-Killing form on the Lie algebra $s\ell_2(\mathbb{C})$. The fact that the Weyl group of $\mathcal{F}$ is isomorphic to $PGL(2,\mathbb{Z})$ implies that $\mathcal{F}$ is closely related to the theory of classical modular forms. In [F-F], Feingold and Frenkel constructed $\mathcal{F}$ concretely and computed the Weyl-Kac denominator formula for $\mathcal{F}$ explicitly. The denominator formula for $\mathcal{F}$ is given by

$$\begin{aligned}(6.23)\qquad &\sum_{g\in PGL(2,\mathbb{Z})} \det\,(g)\, e^{2\pi i\sigma(gP\,{}^tgZ)} \\ &= e^{2\pi i\sigma(PZ)} \prod_{0\le N\in S_2(\mathbb{Z})} \left(1 - e^{2\pi i\sigma(NZ)}\right)^{\mathrm{mult}(N)} \prod_{N\in R} \left(1 - e^{2\pi i\sigma(NZ)}\right),\end{aligned}$$

where $P = \begin{pmatrix} 3 & 1/2 \\ 1/2 & 2 \end{pmatrix}$, $Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in H_2$, $S_2(\mathbb{Z})$ denotes the set of all symmetric integral matrices of degree 2 and

$$R := \left\{ N = \begin{pmatrix} n_1 & n_3 \\ n_3 & n_2 \end{pmatrix} \in S_2(\mathbb{Z}) \mid n_1n_2 - n_3^2 = -1,\ n_2 \ge 0,\ n_3 \le n_1 + n_2,\ 0 \le n_1 + n_2 \right\}.$$

We note that the root lattice of $\mathcal{F}$ is isomorphic to $S_2(\mathbb{Z})$.

Let $\mathfrak{h} := \mathbb{C}h_1 \oplus \mathbb{C}h_2 \oplus \mathbb{C}h_3$ be a Cartan subalgebra of $\mathcal{F}$. We denote by α_1, α_2, α_3 the elements of $\mathfrak{h}^*$ defined by

$$(6.24)\qquad \alpha_i(h_j) = a_{ij}, \quad 1 \le i, j \le 3.$$

We put

$$\gamma_1^* := \alpha_1/2, \quad \gamma_2^* := -\alpha_1 - \alpha_2 - \alpha_3, \quad \gamma_3^* := -\alpha_1 - \alpha_2$$

and

$$P^{++} := \{\, n_1\gamma_1^* + n_2\gamma_2^* + n_3\gamma_3^* \mid n_1, n_2, n_3 \in \mathbb{Z}_+,\ n_3 \ge n_2 \ge n_1 \ge 0 \,\}.$$

Definition. (1) The number $m_1 + m_2$ in the weight $\lambda = m_1\gamma_1^* + (m_1 + m_2)\gamma_2^* + m_3\gamma_3^*$ is called the *level* of the weight λ.
(2) An irreducible standard $\mathcal{F}_0^e$-module or its character is called $\mathcal{F}$-*dominant* if the highest weight of this module lies in P^{++}. A $\mathcal{F}_0^e$-module or its character is called $\mathcal{F}$-*dominant* if each irreducible standard component is $\mathcal{F}$-dominant.

We let M_k be the complex vector space spanned by those $\mathcal{F}_0^e$-characters of the form

$$\chi(\tau, z, \omega) = \sum_{m\ge 0} \chi_m(\tau, z, \omega),$$

where for each $m \geq 0$, χ_m is the function satisfying the condition

$$(6.26)\qquad \chi_m(\tau, z, \omega) = (-\tau)^{-k}\chi_m(-1/\tau, -z/\tau, \omega - z^2/\tau), \quad \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in H_2.$$

Let $M_k(m)$ be the subspace of M_k spanned by the $\mathcal{F}_0^e$-characters of level m satisfying the condition (6.26). We recall the results of J. Igusa (cf. [Ig1]) on Siegel modular forms of degree 2. We denote by $[\Gamma_2, k]$ (resp. $[\Gamma_2, k]_0$) the complex vector space of all Siegel modular forms (resp. cusp forms) of weight k on Γ_2. Let E_k ($k \geq 4$, k : even) be the Eisenstein series of weight k on Γ_2 defined by

$$(6.27)\qquad E_k(Z) := \sum_{C,D} \det(CZ + D)^{-k}, \quad Z \in H_2,$$

where (C, D) runs over the set of non-associated pairs of coprime symmetric matrices in $\mathbb{Z}^{(2,2)}$. Igusa proved that E_4, E_6, E_{10} and E_{12} are algebraically independent over $\mathbb{C}$ and that

$$(6.28)\qquad \oplus_{k=0}^{\infty}[\Gamma_2, k] = \mathbb{C}[E_4, E_6, E_{10}, E_{12}].$$

We define two cusp forms χ_{10} and χ_{12} of weight 10 and 12 by

$$(6.29)\qquad \chi_{10} := -43876 \cdot 2^{-12} \cdot 3^{-5} \cdot 5^{-2} \cdot 7^{-1} \cdot 53^{-1} (E_4E_6 - E_{10})$$

and

$$(6.30)\quad \chi_{12} := 131 \cdot 593 \cdot 2^{-13} \cdot 3^{-7} \cdot 5^{-3} \cdot 7^{-2} \cdot 337^{-1}(3^2 \cdot 7^2 E_4^3 - 2 \cdot 5^3 E_6^2 - 691E_{12}).$$

Then according to (6.28), we have

$$(6.31)\qquad \oplus_{k=0}^{\infty}[\Gamma_2, k] = \mathbb{C}[E_4, E_6, \chi_{10}, \chi_{12}].$$

For two nonnegative integers $k, m \geq 0$, we define the set

$$(6.32)\quad S(k, m) := \left\{ (a, b, c, d) \in (\mathbb{Z}_+)^4 \mid k = 4a + 6b + 10c + 12d,\ c + d = m \right\}.$$

We define the subspace $[\Gamma_2, k](m)$ of $[\Gamma_2, k]$ by

$$(6.33)\qquad [\Gamma_2, k](m) := \sum_{(a,b,c,d)\in S(k,m)} \mathbb{C}\, E_4^a E_6^b \chi_{10}^c \chi_{12}^d.$$

Obviously $[\Gamma_2, k] = \sum_{m\geq 0}[\Gamma_2, k](m)$.

For $f(\tau, z, \omega) \in [\Gamma_2, k]$, we let

$$(6.34) \qquad f(Z) := f(\tau, z, \omega) = \sum_{m \geq 0} \phi_m(\tau, z)\, e^{2\pi i m\omega}, \quad Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in H_2$$

be the Fourier-Jacobi expansion of f. As noted in section 4, $\phi_m(\tau, z)$ is a Jacobi form of weight k and m. Now for each non-negative integer $m \geq 0$ we define the linear map $L_m : [\Gamma_2, k] \longrightarrow M_k$ by
(6.35)

$$(L_m f)(Z) = (L_m f)(\tau, z, \omega) := \phi_m(\tau, z)\, e^{2\pi i m\omega}, \quad f \in [\Gamma_2, k], \quad Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in H_2,$$

where $\phi_m(\tau, z)\,(\, m \geq 0\,)$ is the Fourier-Jacobi coefficient of the expansion (6.34) of f.

DEFINITION. Let M_k' (resp. $M_k'(m)$) be the subspace of M_k (resp. $M_k(m)$) consisting of $PSL(2, \mathbb{Z})$-invariant $\mathcal{F}_0^e$-characters which are $\mathcal{F}$-dominant.

In [F-F], Theorem 7.9, Feingold and Frenkel showed that L_m maps $[\Gamma_2, k](m)$ isomorphically onto $M_k'(m)$. Thus according to (6.33), we obtain , for each $m \geq 0$,

$$(6.36) \qquad \dim_{\mathbb{C}} [\Gamma_2, k](m) = \dim_{\mathbb{C}} M_k'(m) = \sharp(S(k, m)),$$

where $\sharp(S)$ denotes the cardinality of the set S. Moreover we have the following ring-isomorphism

$$(6.37) \qquad M' = \sum_{k \geq 0} \sum_{m \geq 0} M_k'(m) \cong \mathbb{C}\,[E_4, E_6, \chi_{10}, \chi_{12}].$$

Let $[\Gamma_2, k]^M$ be the Maass space. (See Appendix B.) Maass showed that

$$[\Gamma_2, k]^M = \mathbb{C}\, E_4 \oplus [\Gamma_2, k]_0 \quad \text{and} \quad \dim_{\mathbb{C}} [\Gamma_2, k]_0 = \sharp(S(k, 1)).$$

Also Maass showed that

$$(6.38) \qquad [\Gamma_2, k]^M \cong M_k(1) \quad \text{and} \quad [\Gamma_2, k]_0 \cong M_k'(1).$$

The detail for (6.38) can be found in Appendix B, [E-Z] and [Ma2-4]. For $k \geq 4$, even, we have the simple dimensional formulas

$$(6.39) \qquad \dim_{\mathbb{C}} M_k(1) = \left[\frac{k+2}{6} \right] \quad \text{and} \quad \dim_{\mathbb{C}} M_k'(1) = \left[\frac{k-4}{6} \right] = \sharp(S(k, 1)).$$

6.3. Open Problems

In this subsection, we give some open problems which should be investigated and give some comments. Those of Problem 1-6 are due to R. Borcherds (cf. [Bo6-7]).

Problem 1. Can the methods for constructing automorphic forms as infinite products in section 5 be used for semisimple Lie groups other than $O_{s+2,2}(\mathbb{R})$?

Problem 2. Are there a finite or infinite number of singular automorphic forms that can be written as modular products ? Are there such singular modular forms on $O_{s+2,2}(\mathbb{R})$ for $s > 24$?

Problem 3. Interprete the automorphic forms that are modular products in terms of representation theory or the Langlands philosophy.

Problem 4. Extend Theorem 5.4 to higher levels.

Problem 5. Investigate the Lie algebras and the superalgebras coming from other elements of the MONSTER G or Aut (Λ) and write down their denominator formulas explicitly in some nice form.

Problem 6. Are there any generalized Kac-Moody algebras other than the finite dimensional, affine, monstrous or fake monstrous ones, whose simple roots and root multiplicities can both be described explicitly ?

Problem 7. Given a generalized Cartan matrix A of the arithmetic hyperbolic type, construct its associated Kac-Moody Lie algebra $\mathfrak{g}(A)$ of the same type explicitly. Give a relationship between the Kac-Moody algebras of the arithmetic hyperbolic type and classical mathematics. For instance, M. Yoshida showed that the Weyl group $W(A)$ of $\mathfrak{g}(A)$ of rank 3 are all hyperbolic triangle groups and that the semidirect product of the Weyl group $W(A)$ and the root lattice of $\mathfrak{g}(A)$ is isomorphic to a discrete subgroup of a parabolic subgroup of $Sp(2,\mathbb{R})$.

Problem 8. Develope the theory of Kac-Moody Lie algebras of the arithmetic hyperbolic type geometrically.

Problem 9. Give an analytic proof of the denominator formula (6.23) for $\mathcal{F}$ analogous to that of the Jacobi's triple product identity.

Problem 10. Find the transformation behaviour of the denominator formula (6.23) for $\mathcal{F}$ under the symplectic involution $Z \to -Z^{-1}$.

Problem 11. Apply the theroy of the Kac-Moody Lie algebra $\mathcal{F}$ to the study of the moduli space of principally polarized abelian surfaces.

Problem 12. Generalize the Maass correspondence to the Kac-Moody algebras of the arithmetic hyperbolic type other than $\mathcal{F}$?

Appendix A. Classical Modular Forms

Here we present some well-known results on modular forms whose proofs can be found in many references, e.g., [Kob], [Ma1], [S], and [T].

Let H_1 be the upper half plane and let $\Gamma := SL(2,\mathbb{Z})$ be the elliptic modular group. For a positive integer $N \in \mathbb{Z}^+$, we define

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ (\mathrm{mod} N) \right\}$$

and

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;\middle|\; c \equiv 0 \ (\mathrm{mod} N) \right\}$$

$\Gamma(N)$ (resp. $\Gamma_0(N)$) is called the *principal congruence subgroup* of *level* N (resp. the *Hecke subgroup* of level N). The subgroup Γ_θ of Γ generated by $\pm\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is called the *theta group*. A subgroup Γ_1 of Γ is called a *congruence subgroup* if Γ_1 contains $\Gamma(N)$ for some positive integer N. For instance, the Hecke subgroup $\Gamma_0(N)$ is a congruence subgroup because $\Gamma(N) \subset \Gamma_0(N) \subset \Gamma$. And $\Gamma(N)$ is a normal subgroup because it is the kernel of the reduction-modulo-N homomorphism $SL(2,\mathbb{Z}) \to SL(2,\mathbb{Z}/N\mathbb{Z})$. It is well known that the index of $\Gamma(N)$ in Γ is given by

$$\text{(A.1)} \qquad [\Gamma : \Gamma(N)] = N^3 \prod_{p|N} (1 - p^{-2}).$$

The proof of (A.1) can be found in [Sh] pp.21-22. It was discovered around the 1880s that there are an infinite number of examples of noncongruence subgroups (cf. [Ma1] pp. 76-78). But $SL(n,\mathbb{Z})$ behaves quite differently for $n \geq 3$. In fact, it has been proved that if $n \geq 3$, every subgroup of $SL(n,\mathbb{Z})$ of finite index is a congruence subgroup (cf. [Bas]). A similar result for the Siegel modular group $Sp(n,\mathbb{Z})$ for $n \geq 2$ can be found in [Me].

For an integer $k \in \mathbb{Z}$, we denote by $[\Gamma, k]$ (resp. $[\Gamma, k]_0$) the vector space of all modular forms (resp. cusp forms) of weight k for the elliptic modular group Γ. Only for $k \geq 0$, k even, $[\Gamma, k]$ does not vanish.

For any positive integer k with $k \geq 2$, we put

$$\text{(A.2)} \qquad G_{2k}(\tau) := {\sum_{m,n}}' \frac{1}{(m\tau + n)^{2k}}, \qquad \tau \in H_1.$$

Here the symbol $\sum'$ means that the summation runs over all pair of integers (m, n) distinct from (0,0). Then $G_{2k} \in [\Gamma, 2k]$ and $G_{2k}(\infty) = 2\zeta(2k)$, where $\zeta(s)$ denotes the Riemann zeta function. G_{2k} ($k \in \mathbb{Z}^+,\ k \geq 2$) is called the *Eisenstein series* of *index* $2k$. The Fourier expansion of G_{2k} ($k \geq 2$) is given by

$$G_{2k}(\tau) = 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n, \qquad \tau \in H_1, \tag{A.3}$$

where $q = e^{2\pi i\tau}$ and $\sigma_\ell(n) := \sum_{0<d|n} d^\ell$.

We consider the following parabolic subgroup P of Γ given by

$$P := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma \right\}.$$

Then we can see easily that

$$G_{2k}(\tau) = 2\zeta(2k)\sum_{\gamma\in P\backslash\Gamma}\left(\frac{d(\gamma < \tau >)}{d\tau}\right)^k.$$

Here for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\tau \in H_1$, we set

$$\gamma < \tau > := (a\tau + b)(c\tau + d)^{-1}.$$

For a positive integer $k \geq 2$, we can see easily that

$$E_{2k}(\tau) :\overset{\text{def}}{=} \frac{G_{2k}(\tau)}{2\zeta(2k)} = 1 - \frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n, \tag{A.4}$$

where B_k ($k = 0, 1, 2, \cdots$) donotes the $k-$th Bernoulli number defined by the formal power series expansion:

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

Then clearly $B_{2k+1} = 0$ for $k \geq 1$. The first few B_k are

$B_0 = 1,\ B_1 = -1/2,\ B_2 = 1/6,\ B_4 = -1/30,\ B_6 = 1/42,\ B_8 = -1/30,\ B_{10} = 5/66,$
$B_{12} = -691/2730,\ B_{14} = 7/6,\ B_{16} = -3617/510,\ B_{18} = 43867/798, \cdots$

Indeed, (A.4) follows immediately from the relation

$$\zeta(2k) = (-1)^{k-1}\frac{2^{2k-1}\pi^{2k}}{(2k)!}B_{2k}, \quad k = 1, 2, \cdots.$$

For example,

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \quad (240 = 2^4 \cdot 3 \cdot 5)$$
$$E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \quad (504 = 2^3 \cdot 3^2 \cdot 7)$$
$$E_8(\tau) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n, \quad (480 = 2^5 \cdot 3 \cdot 5)$$
$$E_{10}(\tau) = 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n, \quad (264 = 2^3 \cdot 3 \cdot 11)$$
$$E_{12}(\tau) = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n, \quad (65520 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13)$$
$$E_{14}(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n) q^n, \quad (24 = 2^3 \cdot 3).$$

According to the argument on the dimension of $[\Gamma, k]$, we obtain the relation

(A.5) $$E_4^2 = E_8, \quad E_4 E_6 = E_{10}.$$

These are equivalent to the identities:

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{n=1}^{n-1} \sigma_3(m)\sigma_3(n-m),$$
$$11\sigma_9(n) = 21\sigma_5(n) - 10\sigma_3(n) + 5040 \sum_{m=1}^{n-1} \sigma_3(n)\sigma_5(n-m).$$

More generally, every E_{2k} can be expressed as a polynomial in E_4 and E_6. For instance, $E_{14} = E_4^2 E_6$.

We put

(A.6) $$g_2 := 60 G_4, \text{ and } g_3 := 140 G_6.$$

Then it is obvious that

$$g_2 = \frac{(2\pi)^4}{2^2 \cdot 3} E_4, \text{ and } g_3 = \frac{(2\pi)^6}{2^3 \cdot 3^3} E_6.$$

Since $g_2(\infty) = \frac{4}{3}\pi^4$ and $g_3(\infty) = \frac{8}{27}\pi^6$, we see that the *discriminant*

$$\tilde{\Delta} := g_2^3 - 27g_3^2 \tag{A.7}$$

is a cusp form of weight 12, that is, $\tilde{\Delta} \in [\Gamma, 12]_0$. And we have

$$\begin{aligned} \tilde{\Delta}(\tau) &= (2\pi)^{12} \cdot 2^{-6} \cdot 3^{-3}(E_4(\tau)^3 - E_6(\tau)^2) \\ &= (2\pi)^{12}(q - 24q^2 + 252q^3 - 1472q^4 + \cdots) \\ &= (2\pi)^{12} q \prod_{n=1}^{\infty} (1-q^n)^{24} \qquad \text{(Jacobi's identity).} \end{aligned} \tag{A.8}$$

In this article, we put $\Delta(\tau) := (2\pi)^{-12}\tilde{\Delta}(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$.

Fix $\tau \in H_1$. The Weierstrass $\wp$-function $\wp(z;\tau)$ is defined by

$$\wp(z;\tau) := \frac{1}{\tau^2} + \sum_{m,n}{}' \left\{ \frac{1}{(z-n-m\tau)^2} - \frac{1}{(n+m\tau)^2} \right\}, \quad z \in \mathbb{C}. \tag{A.9}$$

Then $\wp(z;\tau)$ is a meromorphic function with respect to $1,\tau$ with double poles at the points $n+m\tau,\ n,m \in \mathbb{Z}$. The map $\varphi_\tau : \mathbb{C} \to \mathbb{P}^2$ defined by

$$\varphi_\tau(z) := [1 : \wp(z;\tau) : \frac{d}{dz}\wp(z;\tau)], \quad z \in \mathbb{C} \tag{A.10}$$

induces an isomorphism of $X = \mathbb{C}/L_\tau$ with a nonsingular plane curve of the form

$$X_0X_2^2 = 4X_1^3 + aX_0^2X_1 + bX_0^3, \tag{A.11}$$

where a and b are suitable constants depending on τ and $L_\tau := \{m\tau + n \mid m,n \in \mathbb{Z}\}$ is the lattice in $\mathbb{C}$ generated by 1 and τ. If we put $x = \wp(z;\tau)$ and $y = \frac{d}{dz}\wp(z;\tau)$, we have the differential equation

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau). \tag{A.12}$$

Up to a numerical factor, $\tilde{\Delta}(\tau) := (g_2^3 - g_3^2)(\tau)$ is the discriminant of the polynomial $4x^3 - g_2(\tau)x - g_3(\tau)$. Since $\tilde{\Delta}(\tau) \neq 0,\ X_\tau = \mathbb{C}^2/L_\tau$ is a *nonsingular* elliptic curve. This story tells us as the reason why the function Δ is called the discrimant. We observe that the differential equation (A.12) shows that it is the inverse function for the elliptic integral in Weierstrass normal form, that is,

$$z - z_0 = \int_{\wp(z_0;\tau)}^{\wp(z;\tau)} (4w^3 - g_2(\tau)w - g_3(\tau))^{-1/2} dw. \tag{A.13}$$

The Ramanujan tau function $\tau(n)$ $(n \in \mathbb{Z}^+)$ is defined by

$$\Delta(\tau) = (2\pi)^{-12}\tilde{\Delta}(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n. \tag{A.14}$$

The Dedekind eta function $\eta(\tau)$ is defined by

$$\eta(\tau) := q^{1/24}\prod_{n\geq 1}(1-q^n), \qquad \tau \in H_1. \tag{A.15}$$

Then $\eta(\tau)$ satisfies

$$\eta(\tau+1) = \eta(\tau) \qquad \text{and} \qquad \eta(-1/\tau) = (\tau/i)^{1/2}\eta(\tau).$$

The Dedekind eta function $\eta(\tau)$ is related to the partition function $p(n)$ as follows:

$$q^{1/24}\eta(\tau)^{-1} = \prod_{n\geq 1}(1-q^n)^{-1} = \sum_{n\geq 0}p(n)q^n, \tag{A.16}$$

where $p(n)$ is the number of partitions of n, i.e., the number of ways of writing

$$n = n_1 + \cdots + n_r, \quad n_j \in \mathbb{Z}^+ \ \ (1 \leq j \leq r).$$

The *modular invariant* $J(\tau)$ is defined by

$$J := (60G_4)^3/\tilde{\Delta} = g_2^3/\tilde{\Delta} = (2\pi)^{12}\cdot 2^{-6}\cdot 3^{-3}E_4^3/\tilde{\Delta}, \tag{A.17}$$

The function $J(\tau)$ was first constructed by Julius Wilhelm Richard Dedekind (1831-1916) in 1877 and Felix Klein (1849-1925) in 1878. The modular invariant $J(\tau)$ has the following properties:

(J1) $J(\tau)$ is a modular function. J is holomorphic in H_1 with a simple pole at ∞, $J(i) = 1$ and $J\left(\frac{-1+\sqrt{3}i}{2}\right) = 0$.

(J2) J defines a conformal mapping which is one-to-one from H_1/Γ onto $\mathbb{C}$, and hence J provides an identification of $H_1/\Gamma \cup \{\infty\}$ with the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$.

(J3) The following are equivalent for a function f which is meromorphic on H_1;
(a) f is a modular function;
(b) f is a quotient of two modular forms of the same weight;
(c) f is a rational function of J, i.e., a quotient of polynomials in J. Thus J is called the *Hauptmodul* or the *fundamental function.*

The q-expansion of $j(\tau) := 1728J(\tau) = 2^6 \cdot 3^3 J(\tau)$, also called the modular invariant, is given by

$$j(\tau) = q^{-1} + 744 + \sum_{n=1}^{\infty} c(n)q^n = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots . \tag{A.19}$$

We observe that $j(i) = 1728 = 2^6 \cdot 3^3$ and $j\left(\frac{1+\sqrt{3}i}{2}\right) = 0$. It was already mentioned that there is a surprising connction of the coefficients in (A.19) with the representations of the Fischer-Griess monster group. All of the early Fourier coefficients in (A.19) are simple linear combinations of degrees of characters of the MONSTER. This was first observed by John Mckay and John Thompson. The modular invariant $J(\tau)$ is used to prove the small Picard theorem and to study an explicit reciprocity law for an imaginary quadratic number field.

For a positive definite symmetric real matrix S of degree n, we define the theta series

$$\theta_S(\tau) := \sum_{x \in \mathbb{Z}^n} e^{\pi i S[x]\tau}, \qquad \tau \in H_1, \tag{A.20}$$

where $S[x] := {}^t x S x$ denotes the quadratic form associated to S. We can prove the transformation formula

$$\theta_{S^{-1}}(-1/\tau) = (\det S)^{1/2} (\tau/i)^{n/2} \theta_S(\tau). \tag{A.21}$$

It is known that if S is a positive definite symmetric even integral, unimodular matrix of degree n, then n is divided by 8 and $\theta_S(\tau) \in [\Gamma, n/2]$. In fact, for $n = 8$, there is only one positive definite symmetric even integral unimodular matrix up to equivalence modulo $GL(8,\mathbb{Z})$. For $n = 16$, there are two nonequivalent examples modulo $GL(16,\mathbb{Z})$. For $n = 24$, there are 24 nonequivalent examples modulo $GL(24,\mathbb{Z})$.

We consider a Jacobi function

$$\theta(\tau, z) := \sum_{n \in \mathbb{Z}} e^{\pi i (n^2 \tau + 2nz)}, \qquad (\tau, z) \in H_1 \times \mathbb{C} \tag{A.22}$$

Then $\theta(\tau, z)$ satisfies the following properties:

(θ.1) $\theta(\tau, z)$ is an entire function on $H_1 \times \mathbb{C}$.

(θ.2) θ is quasi-periodic as a function of z in the following sense:
$\theta(\tau, z+n) = \theta(\tau, z)$ for all $n \in \mathbb{Z}$;
$\theta(\tau, z+n\tau) = e^{-\pi i (n^2\tau + 2nz)} \theta(\tau, z)$ for all $n \in \mathbb{Z}$.

(θ.3) $\theta(\tau, z)$ satisfies the transformation formula

$$\theta(\tau, z) = (\tau/i)^{1/2} \sum_{n \in \mathbb{Z}} e^{-\pi i (n-z)^2/\tau}.$$

(θ.4) $\theta(\tau, (1+\tau)/2) = 0$.

(θ.5) Fixing τ, the only zero of $\theta(z) := \theta(\tau, z)$ as a function of z in the period parallelogram on 1 and τ is $z = (1+\tau)/2$. Moreover, this zero is simple.

For a proof of (θ.3), use Poisson formula.

Appendix B. Kohnen Plus Space and Maass Space

Here we review the Kohnen plus space and the Maass space. And then we give isomorphisms of them with the vector spaces of Jacobi forms. For more detail we refer to [Koh], [Ma2-4].

We fix two positive integers n and m. Let

$$H_n := \{ Z \in \mathbb{C}^{(n,n)} \mid Z = {}^tZ, \ \mathrm{Im} Z > 0 \}$$

be the Siegel upper half plane of degree n and let $\Gamma_n := Sp(n, \mathbb{Z})$ the Siegel modular group of degree n. That is,

$$\Gamma_n := \{ g \in \mathbb{Z}^{(2n,2n)} \mid {}^tgJg = J \}, \quad J := \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$

Here E_n denotes the identity matrix of degree n. Then the real symplectic group $Sp(n, \mathbb{R})$ acts on H_n transitively. If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $Z \in H_n$,

$$M < Z > := (AZ + B)(CZ + D)^{-1}. \tag{B.1}$$

Let $\mathcal{M}$ be a positive definite, symmetric half integral matrix of degree m. For a fixed element $Z \in H_n$, we denote by $\Theta^{(n)}_{\mathcal{M},Z}$ the vector space of all the functions $\theta : \mathbb{C}^{(m,n)} \to \mathbb{C}$ satisfying the condition

$$\theta(W + \lambda Z + \mu) = e^{-2\pi i \sigma(\mathcal{M}[\lambda] Z + 2 {}^tW \mathcal{M} \lambda)}, \quad W \in \mathbb{C}^{(m,n)} \tag{B.2}$$

for all λ, $\mu \in \mathbb{Z}^{(m,n)}$. For brevity, we put $L := \mathbb{Z}^{(m,n)}$ and $\mathcal{L}_{\mathcal{M}} := L/(2\mathcal{M})L$. For each $\gamma \in \mathcal{L}_{\mathcal{M}} := L/(2\mathcal{M})L$, we define the theta series

(B.3)

$$\theta_\gamma(Z, W) := \sum_{\lambda \in L} e^{2\pi i \sigma(\mathcal{M}[\lambda + (2\mathcal{M})^{-1}\gamma] Z + 2 {}^tW \mathcal{M}(\lambda + (2\mathcal{M})^{-1}\gamma))}, \quad (Z, W) \in H_n \times \mathbb{C}^{(m,n)}.$$

Then $\{\theta_\gamma(Z,W)\mid \gamma\in\mathcal{L}_\mathcal{M}\}$ forms a basis for $\Theta^{(n)}_{\mathcal{M},Z}$. For any Jacobi form $\phi(Z,W)\in J_{k,\mathcal{M}}(\Gamma_n)$, the function $\phi(Z,\cdot)$ with fixed Z is an element of $\Theta^{(n)}_{\mathcal{M},Z}$ and $\phi(Z,W)$ can be written as a linear combination of theta series $\theta_\gamma(Z,W)$ $(\gamma\in\mathcal{L}_\mathcal{M})$:

$$\phi(Z,W)=\sum_{\gamma\in\mathcal{L}_\mathcal{M}}\phi_\gamma(Z)\theta_\gamma(Z,W). \tag{B.4}$$

Here $\phi=(\phi_\gamma(Z))_{\gamma\in\mathcal{L}_\mathcal{M}}$ is a vector valued automorphic form with respect to theta mutiplier system.

(I) Kohnen Plus Space (cf. [Ib], [Koh])

We consider the case: $m=1,\ \mathcal{M}=E_m,\ L=\mathbb{Z}^{(1,n)}\cong\mathbb{Z}^n$. We consider the theta series

$$\theta^{(n)}(Z):=\sum_{\lambda\in L}e^{2\pi i\sigma(\lambda Z\,{}^t\lambda)}=\theta_0(Z,0),\quad Z\in H_n. \tag{B.5}$$

We put

$$\Gamma_0^{(n)}(4):=\left\{\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\Gamma_n\ \middle|\ C\equiv 0\,(\mathrm{mod}\,4)\right\}. \tag{B.6}$$

Then $\Gamma_0^{(n)}(4)$ is a congruence subgroup of Γ_n. We define the automorphic factor $j:\Gamma_0^{(n)}(4)\times H_n\to\mathbb{C}^\times$ by

$$j(\gamma,Z):=\frac{\theta^{(n)}(\gamma<Z>)}{\theta^{(n)}(Z)},\quad \gamma\in\Gamma_0^{(n)}(4),\ Z\in H_n. \tag{B.7}$$

Then we obtain the relation

$$j(\gamma,Z)^2=\epsilon(\gamma)\cdot\det(CZ+D),\quad \epsilon(\gamma)^2=1 \tag{B.8}$$

for any $\gamma=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\Gamma_0^{(n)}(4)$.

Now we define the Kohnen plus space $M^+_{k-\frac12}(\Gamma_0^{(n)}(4))$ introduced by W. Kohnen (cf. [Koh]). $M^+_{k-\frac12}(\Gamma_0^{(n)}(4))$ is the vector space consisting of holomorphic functions $f:H_n\to\mathbb{C}$ satisfying the following conditions:

(a) $f(\gamma<Z>)=j(\gamma,Z)^{2k-1}f(Z)$ for all $\gamma\in\Gamma_0^{(n)}(4)$;
(b) f has the Fourier expansion

$$f(Z)=\sum_{T\geq 0}a(T)e^{2\pi i\sigma(TZ)},$$

where T runs over the set of semi-positive, half-integral symmetric matrices of degree n and $a(T) = 0$ unless $T \equiv -\mu{}^t\mu \bmod 4S_n^*(\mathbb{Z})$ for some $\mu \in \mathbb{Z}^{(n,1)}$. Here we put

$$S_n^*(\mathbb{Z}) := \{T \in \mathbb{R}^{(n,n)} \mid T = {}^tT,\ \sigma(TS) \in \mathbb{Z} \text{ for all } S = {}^tS \in \mathbb{Z}^{(n,n)}\}.$$

For $\phi \in J_{k,1}(\Gamma_n)$, according to (B.4), we have

$$\text{(B.9)} \qquad \phi(Z,W) = \sum_{\gamma \in L/2L} f_\gamma(Z)\theta_\gamma(Z,W), \quad Z \in H_n,\ W \in \mathbb{C}^n.$$

Now we put

$$\text{(B.10)} \qquad f_\phi(Z) := \sum_{\gamma \in L/2L} f_\gamma(4Z), \quad Z \in H_n.$$

Then $f_\phi \in M^+_{k-\frac{1}{2}}(\Gamma_0^{(n)}(4))$.

Theorem 1 (Kohnen-Zagier ($n = 1$), Ibukiyama ($n > 1$)). *Suppose k is an even positive integer. We have the isomorphism*

$$J_{k,1}(\Gamma_n) \cong M^+_{k-\frac{1}{2}}(\Gamma_0^{(n)}(4))$$
$$\phi \mapsto f_\phi.$$

Furthermore the isomorphism is compatible with the action of Hecke operators.

(II) Maass Space

The Maass space or the Maaß's Spezialschar was introduced by H. Maass (1911-1993) to solve the Saito-Kurokawa conjecture. Let $k \in \mathbb{Z}^+$. We denote by $[\Gamma_2, k]$ the vector space of all Siegel modular forms of weight k and degree 2. We denote by $[\Gamma_2, k]^M$ the vector space of all Siegel modular forms $F : H_2 \to \mathbb{C}$, $F(Z) = \sum_{T \geq 0} a_F(T) e^{2\pi i \sigma(TZ)}$ in $[\Gamma_2, k]$ satisfying the following condition:

$$\text{(B.11)} \qquad a_F\begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix} = \sum_{d|(n,r,m),\, d>0} d^{k-1} a_F\begin{pmatrix} \frac{mn}{d^2} & \frac{r}{2d} \\ \frac{r}{2d} & 1 \end{pmatrix}$$
$$\text{for all } T = \begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix} \geq 0 \text{ with } n, r, m \in \mathbb{Z}.$$

The vector space $[\Gamma_2, k]^M$ is called the *Maass space* or the *Maaß's Spezialschar.*

For any F in $[\Gamma_2, k]$, we let

(B.12) $$F(Z) = \sum_{m\geq 0} \phi_m(\tau,z) e^{2\pi i m\tau'},\ Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in H_2$$

be the Fourier-Jacobi expansion of F. Then for any $m \in \mathbb{Z}_+$ we obtain the linear map

(B.13) $$\rho_m : [\Gamma_2, k] \longrightarrow J_{k,m}(\Gamma_1), \quad F \longmapsto \phi_m.$$

We denote that ρ_0 is nothing but the Siegel $\Phi-$operator.

Maass (cf. [Ma 2-3]) showed that for k even, there exists a natural map $V : J_{k,1}(\Gamma_1) \longrightarrow [\Gamma_2, k]$ such that $\rho_1 \circ V$ is the identity. More precisely, we let $\phi \in J_{k,1}(\Gamma_1)$ with Fourier coefficients $c(n,r)$ $(n, r \in \mathbb{Z},\ r^2 \leq 4n)$ and we define for any $m \in \mathbb{Z}^+_{\geq 0}$

(B.13) $$(V_m\,\phi)(\tau,z) := \sum_{n,r\in\mathbb{Z},\ r^2\leq 4mn} \left(\sum_{d|(n,r,m)} d^{k-1} c\left(\frac{mn}{d^2}, \frac{r}{d}\right)\right) e^{2\pi i(n\tau+rz)}.$$

It is easy to see that $V_1\phi = \phi$ and $V_m\phi \in J_{k,m}(\Gamma_1)$. We define

(B.15) $$(V\phi)\begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} := \sum_{m\geq 0} (V_m\phi)(\tau,z) e^{2\pi i m\tau'}, \quad \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in H_2.$$

We denote by T_n $(n \in \mathbb{Z}^+)$ the usual Hecke operators on $[\Gamma_2, k]$ resp. $[\Gamma_2, k]_0$. Here $[\Gamma_2, k]_0$ denote the vector subspace consisting of all cusp forms in $[\Gamma_2, k]$. For instance, if p is a prime, T_p and T_{p^2} are the Hecke operators corresponding to the two generators $\Gamma_2\,\mathrm{diag}(1,1,p,p)\,\Gamma_2$ and $\Gamma_2\,\mathrm{diag}(1,p,p^2,p)\,\Gamma_2$ of the local Hecke algebra of Γ_2 at p respectively. We denote by $T_{J,n}$ $(n \in \mathbb{Z}^+)$ the Hecke operators on $J_{k,m}(\Gamma_1)$ resp. $J^{\mathrm{cusp}}_{k,m}(\Gamma_1)$ (cf. [E-Z]).

Theorem 2 (Maass [Ma 2-4], Eichler-Zagier [E-Z], Theorem 6.3). *Suppose k is an even positive integer. Then the map $\phi \mapsto V\phi$ gives an injection of $J_{k,1}(\Gamma_1)$ into $[\Gamma_2, k]$ which sends cusp forms to cusp forms and is compatible with the action of Hecke operators. The image of the map V is equal to the Maass space $[\Gamma_2, k]^M$. If p is a prime, one has*

$$T_p \circ V = V \circ (T_{J,p} + p^{k-2}(p+1))$$

and

$$T_{p^2} \circ V = V \circ (T^2_{J,p} + p^{k-2}(p+1)T_{J,p} + p^{2k-2}).$$

In summary, we have the following isomorphisms

$$\begin{array}{ccccccc} [\Gamma_2,k]^M & \cong & J_{k,1}(\Gamma_1) & \cong & M^+_{k-\frac{1}{2}}(\Gamma_0^{(1)}(4)) & \cong & [\Gamma_1,2k-2], \\ V\phi & \leftarrow & \phi & \rightarrow & f_\phi & & \end{array}$$

where the last isomorphism is the Shimura correspondence. And all the above isomorphisms are compatible with the action of Hecke operators.

REMARK. (1) $[\Gamma_2,k]^M = \mathbb{C}E_k^{(2)} \oplus [\Gamma_2,k]_0^M$, where $E_k^{(2)}$ is the Siegel-Eisenstein series of weight k on Γ_2 given by

$$E_k^{(2)}(Z) := \sum_{\{C,D\}} \det(CZ+D)^{-k}, \quad Z \in H_2$$

(sum over non-associated pairs of coprime symmetric matrices $C, D \in \mathbb{Z}^{(2,2)}$) and $[\Gamma_2,k]_0^M := [\Gamma_2,k]^M \cap [\Gamma_2,k]_0$.
(2) Maass proved that $\dim[\Gamma_2,k]^M = \left[\frac{k-4}{6}\right]$ for $k \geq 4$ even. It is known that $\dim[\Gamma_2,k] \sim 2^{-6}\cdot 3^{-3}\cdot 5^{-1}\cdot k^3$ as $k \to \infty$.

We observe that Theorem 2 implies that $[\Gamma_2,k]^M$ is invariant under all the Hecke operators and that it is annihilated by the operator

$$\text{(B.16)} \qquad \mathcal{C}_p := T_p^2 - p^{k-2}(p+1)T_p - T_{p^2} + p^{2k-2}$$

for every prime p. We let $F \in [\Gamma_2,k]$ be a nonzero Hecke eigenform with $T_nF = \lambda_n F$ for $n \in \mathbb{Z}^+$. For a prime p, we put

$$Z_{F,p}(X) := 1 - \lambda_p X + (\lambda_p^2 - \lambda_{p^2} - p^{2k-4})X^2 - \lambda_p p^{2k-3}X^3 + p^{4k-6}X^4$$

so that $Z_{F,p}(p^{-s})$ $(s \in \mathbb{C})$ is the local spinor zeta function of F at p. We put

$$\text{(B.17)} \qquad Z_F(s) := \prod_p Z_{F,p}(p^{-s}), \quad \operatorname{Re} s \gg 0.$$

Then we have

$$\text{(B.18)} \qquad Z_F(s) = \zeta(2s-2k+4)\sum_{n\geq 1}\frac{\lambda_n}{n^s}, \quad \operatorname{Re} s \gg 0.$$

Theorem 3 (Saito-Kurokawa conjecture ; Andrianov [An], Maass [Ma 2-3], Zagier [Za]). *Let $k \in \mathbb{Z}^+$ be even and let F be a nonzero Hecke eigenform in $[\Gamma_2, k]^M$. Then there exists a unique normalized Hecke eigenform f in $[\Gamma_1, 2k-2]$ such that*

$$Z_F(s) = \zeta(s-k+1)\zeta(s-k+2)L_f(s),$$

where $L_f(s)$ is the Hecke L-function attached to f.

Theorem 2 implies that $Z_F(s)$ has a pole at $s = k$ if F is a Hecke eigenform in $[\Gamma_2, k]_0^M$. If $F \in [\Gamma_2, k]_0$ is an eigenform, it was proved by Andrianov that $Z_F(s)$ has an analytic continuation to $\mathbb{C}$ which is holomorphic everywhere if k is odd and is holomorphic except for a possible simple pole at $s = k$ if k is even. Moreover, the global function

$$Z_F^*(s) := (2\pi)^{-s}\Gamma(s)\Gamma(s-k+2)Z_F(s) \tag{B.19}$$

is $(-1)^k$-invariant under $s \mapsto 2k-2-s$. It was proved by Evdokimov and Oda that $Z_F(s)$ is holomorphic everywhere if and only if F is contained in the orthogonal complement of $[\Gamma_2, k]_0^M$ in $[\Gamma_2, M]$.

So far a generalization of the Maass space to higher genus $n > 2$ has not been given. There is a partial negative result by Ziegler (cf [Zi], Theorem 4.2). We will describe his result roughly. Let $F \in [\Gamma_{g+1}, k]$ ($g \in \mathbb{Z}^+$, k : even) be a Siegel modular form on H_{g+1} of weight k and let

$$F\begin{pmatrix} Z_1 & {}^tW \\ W & z_2 \end{pmatrix} = \sum_{m\geq 0} \Phi_{F,m}(Z_1, W)\, e^{2\pi i m z_2}, \quad \begin{pmatrix} Z_1 & {}^tW \\ W & z_2 \end{pmatrix} \in H_{g+1}, \text{ with } Z_1 \in H_g,\ z_2 \in H_1$$

be the Fourier-Jacobi expansion of F. For any nonnegative integer m, we consider the linear mapping

$$\rho_{g,m,k} : [\Gamma_{g+1}, k] \longrightarrow J_{k,m}(\Gamma_g)$$

defined by

$$\rho_{g,m,k}(F) := \Phi_{F,m}, \quad F \in [\Gamma_{g+1}, k].$$

Ziegler showed that for $g \geq 32$, the mapping

$$\rho_{g,1,16} : [\Gamma_{g+1}, 16] \longrightarrow J_{16,1}(\Gamma_g)$$

is not surjective.

Question: Is $\rho_{g,1,k}$ surjective for an integer $k \neq 16$?

Appendix C. The Orthogonal Group $O_{s+2,2}(\mathbb{R})$

A lattice is a free $\mathbb{Z}$-module of finite rank with a nondegerate symmetric bilinear form with values in $\mathbb{Q}$. Let K be a positive definite unimodular even integral lattice of rank s with its associated symmetric matrix S_0. Let $\Pi_{1,1}$ be the unique unimodular even integral Lorentzian lattice of rank 2 with its associated symmetric matrix $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. If there is no confusion, we write $\Pi_{1,1} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

We define the lattices L and M by

$$L := K \oplus \Pi_{1,1} \quad \text{and} \quad M := \Pi_{1,1} \oplus L = \Pi_{1,1} \oplus K \oplus \Pi_{1,1}. \tag{C.1}$$

We put

$$K_{\mathbb{R}} := K \otimes_{\mathbb{Z}} \mathbb{R}, \qquad L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}, \qquad M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{C}.$$

We let

$$Q_K := S_0, \quad Q_L := \begin{pmatrix} 0 & 0 & -1 \\ 0 & S_0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad Q_M := \begin{pmatrix} 0 & 0 & \Pi_{1,1} \\ 0 & S_0 & 0 \\ \Pi_{1,1} & 0 & 0 \end{pmatrix}$$

be the unimodular even integral symmetric matrices associated with the lattices K, L and M respectively. The isometry group $O_M(\mathbb{R})$ of the quadratic space $(M_{\mathbb{R}}, Q_M)$ is defined by

$$O_M(\mathbb{R}) := \left\{ g \in GL(M_{\mathbb{R}}) \cong GL(s+4, \mathbb{R}) \mid {}^t g Q_M g = Q_M \right\}. \tag{C.2}$$

Then it is easy to see that $O_M(\mathbb{R})$ is isomorphic to the orthogonal group $O_{s+2,2}(\mathbb{R})$. Here for two nonnegative integers p and q with $p+q=n$, $O_{p,q}(\mathbb{R})$ is defined by

$$O_{p,q}(\mathbb{R}) := \left\{ g \in GL(p+q, \mathbb{R}) \mid {}^t g E_{p,q} g = E_{p,q} \right\}, \tag{C.3}$$

where

$$E_{p,q} := \begin{pmatrix} E_p & 0 \\ 0 & -E_q \end{pmatrix}.$$

Indeed, Q_M is congruent to $E_{s+2,2}$ over $\mathbb{R}$, that is, $Q_M = {}^t a E_{s+2,2} a$ for some $a \in GL(s+4, \mathbb{R})$ and hence $O_M(\mathbb{R}) = a^{-1} O_{s+2,2}(\mathbb{R}) a$. For brevity, we write $O(p, \mathbb{R}) := O_{p,0}(\mathbb{R})$ and $SO(p, \mathbb{R}) := SO_{p,0}(\mathbb{R})$. Similiarly, we have $O_L(\mathbb{R}) \cong O_{s+1,1}(\mathbb{R})$ and $O_K(\mathbb{R}) \cong O(s, \mathbb{R})$. We denote by $(\ ,\ )_K$, $(\ ,\ )_L$ and $(\ ,\ )_M$ the nondegenerate symmetric bilinear forms on $K_{\mathbb{R}}$, $L_{\mathbb{R}}$ and $M_{\mathbb{R}}$ corresponding to Q_K, Q_L and Q_M respectively.

We let

$$D = D(M_{\mathbb{R}}) := \{\, z \subset M_{\mathbb{R}} \mid \dim_{\mathbb{R}} z = 2,\ z \text{ is oriented and } (\ ,\ )_M|_z < 0\} \tag{C.4}$$

be the space of oriented negative two dimensional planes in $M_{\mathbb{R}}$. We observe that a negative two dimensional plane in $M_{\mathbb{R}}$ occurs twice in D with opposite orientation. Thus D may be regarded as a space consisting of two copies of the space of negative two dimensional planes in $M_{\mathbb{R}}$. For $z \in D$, the majorant associated to z is defined by

$$(\ ,\ )_z := \begin{cases} (\ ,\ )_M & \text{on } z^{\perp}\,; \\ -(\ ,\ )_M & \text{on } z. \end{cases} \tag{C.5}$$

Then $(M_{\mathbb{R}},\,(\ ,\ )_z)$ is a positive definite quadratic space. It is easy to see that we have the orthogonal decomposition $M_{\mathbb{R}} = z^{\perp} \oplus z$ with respect to $(\ ,\ )_z$ and that $(\ ,\ )_M$ has the signature $(s+2,0)$ on $z^{\perp}$ and $(0,2)$ on z.

According to Witt's theorem, $O_{s+2,2}(\mathbb{R})$ acts on D transitively. For a fixed element $z_0 \in D$, we denote by K_∞ the stabilizer of $O_{s+2,2}(\mathbb{R})$ at z_0. Then

$$D \cong O_{s+2,2}(\mathbb{R})/K_\infty \tag{C.6}$$

is realized as a homogeneous space. It is easily seen that K_∞ is isomorphic to $O(s+2,\mathbb{R}) \times SO(2,\mathbb{R})$, which is a subgroup of the maximal compact subgroup $O(s+2,\mathbb{R}) \times O(2,\mathbb{R})$ of $O_M(\mathbb{R}) \cong O_{s+2,2}(\mathbb{R})$. It is also easy to check that $O_{s+2,2}(\mathbb{R})$ has four connected components. We denote by $SO_{s+2,2}(\mathbb{R})^0$ the identity component of $O_{s+2,2}(\mathbb{R})$. In fact, $SO_{s+2,2}(\mathbb{R})^0$ is the kernel of the spinor norm mapping

$$\rho\,:\, SO_{s+2,2}(\mathbb{R}) \longrightarrow \mathbb{R}^{\times}/(\mathbb{R}^{\times})^2. \tag{C.7}$$

Now we know that

$$D \cong O_{s+2,2}(\mathbb{R})/O(s+2,\mathbb{R}) \times SO(2,\mathbb{R}) \tag{C.8}$$

has two connected components and the connected component D^0 containing the origin $o := z_0$ is realized as the homogeneous space as follows:

$$D^0 \cong O_{s+2,2}(\mathbb{R})/O(s,\mathbb{R}) \times O(2,\mathbb{R}) \cong SO_{s+2,2}(\mathbb{R})^0/SO(4,\mathbb{R}) \times SO(2,\mathbb{R}). \tag{C.9}$$

It is known that D^0 is a Hermitian symmetric space of noncompact type with complex dimension $s+2$. Let us describe a Hermitian structure on D^0 explicitly. For brevity, we write $G^0_{\mathbb{R}} := SO_{s+2,2}(\mathbb{R})^0$ and $K^0_{\mathbb{R}} := SO(s+2,\mathbb{R}) \times SO(2,\mathbb{R})$. Obviously $G^0_{\mathbb{R}}$ is the identity component of $O_{s+2,2}(\mathbb{R}) \cong O_M(\mathbb{R})$ and $K^0_{\mathbb{R}}$ is the identity component of $O(s+2,\mathbb{R})\times O(2,\mathbb{R})\cong K_\infty$. For a positive integer n, the Lie algebra $\mathfrak{so}(n,\mathbb{R})$ of $SO(n,\mathbb{R})$ has dimension $(n-1)n/2$ and

$$\mathfrak{so}(n,\mathbb{R}) = \{\, X \in \mathbb{R}^{(n,n)} \mid \sigma(X)=0,\ {}^tX + X = 0\,\}. \tag{C.10}$$

Then the Lie algebra $\mathfrak{g}$ of $G^0_{\mathbb{R}}$ is given by

(C.11)

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & C \\ {}^tC & B \end{pmatrix} \in \mathbb{R}^{(s+4,s+4)} \mid A \in \mathfrak{so}(s+2,\mathbb{R}),\ B \in \mathfrak{so}(2,\mathbb{R}),\ C \in \mathbb{R}^{(s+2,2)} \right\}.$$

Let θ be the Cartan involution of $G^0_{\mathbb{R}}$ defined by

$$\theta(g) := E_{s+2,2}\, g\, E_{s+2,2}, \quad g \in G^0_{\mathbb{R}}. \tag{C.12}$$

Then $K^0_{\mathbb{R}}$ is the subgroup of $G^0_{\mathbb{R}}$ consisting of elements in $G^0_{\mathbb{R}}$ fixed by θ. We also denote by θ the differential of θ which is given by

$$\theta(X) = E_{s+2,2}\, X\, E_{s+2,2}, \quad X \in \mathfrak{g}. \tag{C.13}$$

Then $\mathfrak{g}$ has the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \tag{C.14}$$

where $\mathfrak{k}$ and $\mathfrak{p}$ denote the $(+1)$-eigenspace and (-1)-eigenspace of θ respectively. More explicitly,

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathfrak{g} \mid A \in \mathfrak{so}(s+2,\mathbb{R}),\ B \in \mathfrak{so}(2,\mathbb{R}) \right\}$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & C \\ {}^tC & 0 \end{pmatrix} \mid C \in \mathbb{R}^{(s+2,2)} \right\}.$$

The real dimension of $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}$ are $(s+3)(s+4)/2$, $(s^2+3s+4)/2$ and $2(s+2)$ respectively. Thus the real dimension of D^0 is $2(s+2)$. Since $\mathfrak{p}$ is stable under the adjoint action of $K^0_{\mathbb{R}}$, i.e., $\mathrm{Ad}(k)\mathfrak{p}=\mathfrak{p}$ for all $k\in K^0_{\mathbb{R}}$, $(\mathfrak{g},\mathfrak{k},\theta)$ is reductive. Thus the tangent space $T_o(D^0)$ of $D^0\subset D$ at $o:=z_0$ can be canonically identified with $\mathbb{R}^{2(s+2)}$ via

$$\begin{pmatrix} 0 & C \\ {}^tC & 0 \end{pmatrix} \longmapsto (x,y), \quad C=(x,y)\in \mathbb{R}^{(s+2,2)} \cong \mathbb{R}^{2(s+2)}.$$

Then the adjoint action of $K^0_{\mathbb{R}}$ on $\mathfrak{p}\cong\mathbb{R}^{2(s+2)}$ is expressed as

$$\mathrm{Ad}\left(\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}\right)(x,y) = \begin{pmatrix} 0 & k_1 C\, {}^tk_2 \\ k_2\, {}^tC\, {}^tk_1 & 0 \end{pmatrix}, \tag{C.15}$$

where $k_1 \in SO(s+2,\mathbb{R}),\ k_2 \in SO(2,\mathbb{R})$ and $C = (x,y) \in \mathbb{R}^{(s+2,2)}$. The Cartan-Killing form B of $\mathfrak{g}$ is given by

$$B(X,Y) = (s+2)\,\sigma(XY), \quad X,Y \in \mathfrak{g}. \tag{C.16}$$

The restriction B_0 of B to $\mathfrak{p}$ is given by

$$B_0((x,y),(x',y')) = 2(s+2)(< x, x' > + < y, y' >),$$

where $<\,,\,>$ is the standard inner product on $\mathbb{R}^{s+2}$. The restriction B_0 induces a $G^0_{\mathbb{R}}$-invariant Riemannian metric g_0 on D^0 defined by

$$g_0(X,Y) := B_0(X,Y), \quad X,Y \in \mathfrak{p}.$$

It is easy to check that g_0 is invariant under the adjoint action of $K_{\mathbb{R}}$.

Now let J_0 be the complex structure on the real vector space $\mathfrak{p}$ defined by

$$J_0((x,y)) := (-y,x), \quad (x,y) \in \mathfrak{p}. \tag{C.17}$$

We note that

$$J_0 = \mathrm{Ad}\begin{pmatrix} E_{s+2} & 0 \\ 0 & I \end{pmatrix}, \quad I := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J_0^2 = \mathrm{Id}_{\mathfrak{p}}.$$

It is easy to check that J_0 is $\mathrm{Ad}(K_{\mathbb{R}})$-invariant, i.e.,

$$J_0(\mathrm{Ad}(k)X) = J_0(X) \ \text{ for all } k \in K_{\mathbb{R}} \text{ and } X \in \mathfrak{p}.$$

Hence J_0 induces an almost complex structure J on D^0 and also on D. J becomes a complex structure on D^0 via the natural identification

$$T_0(D^0) \cong \mathfrak{p} \cong \mathbb{R}^{2(s+2)} \cong \mathbb{C}^{s+2}, \quad (x,y) \longmapsto x+iy, \quad x,y \in \mathbb{R}^{s+2}. \tag{C.18}$$

Indeed, J is the pull-back of the standard complex structure on $\mathbb{C}^{s+2}$. The complexification $\mathfrak{p}_{\mathbb{C}} := \mathfrak{p} \otimes_{\mathbb{R}} \mathbb{C}$ has a canonical decomposition

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-,$$

where $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) denotes the $(+i)$-eigenspace (resp. $(-i)$-eigenspace) of J_0. Precisely, $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are given by

$$\mathfrak{p}_+ = \{\,(x,-ix) \mid x \in \mathbb{C}^{s+2}\,\} \quad \text{and} \quad \mathfrak{p}_- = \{\,(x,ix) \mid x \in \mathbb{C}^{s+2}\,\}.$$

Usually $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are called the *holomorphic* tangent space and the *anti-holomorphic* tangent space respectively. Moreover, it is easy to check that the Riemannian metric g_0 on D^0 is Hermitian with respect to the complex structure J, i.e., $g_0(JX,JY) = g_0(X,Y)$ for all smooth vector fields X and Y on D^0. And D^0 has the canonical orientation induced by its complex structure.

In summary, we have

Theorem 1. D^0 is a Hermitian symmetric space of noncompact type with dimension $s+2$. D^0 is realized as a bounded symmetric domain in $\mathbb{C}^{s+2}$ and hence D is a union of two bounded symmetric domains in $\mathbb{C}^{s+2}$.

REMARK 2. We choose an orthogonal basis of $z_0^\perp$. We also choose a basis of z_0 which is properly oriented. Let

$$\tau_{z_0^\perp} := \mathrm{diag}\,(E_{s+1}, -1) \quad \text{and} \quad \tau_{z_0} := \mathrm{diag}\,(1, -1)$$

be the symmetries in the isometry groups $O(z_0^\perp)$ and $O(z_0)$ with respect to the last coordinates of $z_0^\perp$ and z_0 respectively. We observe that τ_{z_0} reverses the orientation of z_0 and lies in $O(z_0) - SO(z_0)$. It is easy to check that

$$\rho(1_{z_0^\perp} \times \tau_{z_0}) = -1, \quad \rho(\tau_{z_0^\perp} \times 1_{z_0}) = 1,$$

where ρ is the spinor norm mapping defined by (C.7). It is easy to see that the set

$$\left\{ 1_{M_\mathbb{R}},\ 1_{z_0^\perp} \times \tau_{z_0},\ \tau_{z_0^\perp} \times 1_{z_0},\ \tau_{z_0^\perp} \times \tau_{z_0} \right\} \tag{C.19}$$

is a complete set of coset representatives of $O_{s+2,2}(\mathbb{R})/SO_{s+2,2}(\mathbb{R})^0$. We note that the set (C.19) is contained in $O(s+2,\mathbb{R})\times O(2,\mathbb{R})$ and so that (C.19) is a complete set of coset representatives of $O(s+2,\mathbb{R})\times O(2,\mathbb{R})/(SO(s+2,\mathbb{R})\times SO(2,\mathbb{R}))$. It is easy to see that the set $\{ 1_{M_\mathbb{R}},\ 1_{z_0^\perp}\times\tau_{z_0} \}$ is a complete set of coset representatives of $(O(s+2,\mathbb{R}) \times O(2,\mathbb{R}))/(O(s+2,\mathbb{R}) \times SO(2,\mathbb{R}))$. Thus we have

$$D = D^0 \cup (1_{z_0^\perp} \times \tau_{z_0}) D^0. \tag{C.20}$$

The complex structure $-J_0$ on $\mathfrak{p}$ determines the opposite almost complex structure on D^0 and the almost complex structure on the connected component $D - D^0$ is the one on D^0 carried by the element $1_{z_0^\perp} \times \tau_{z_0}$. The ground manifolds D^0 and $D - D^0$ may be regarded as the same one, but each carries the opposite almost complex structure.

REMARK 3. D^0 may be regarded as an open orbit of $G_\mathbb{R}^0$ in the complex projective quadratic space ${}^t z Q_M z = 0$ via the Borel embedding. (See [Bai] for detail.) D^0 is realized as a tube domain in $\mathbb{C}^{s+2}$ given by (4.10) in section 4. For the explicit realization of D^0 as a bounded symmetric domain in $\mathbb{C}^{s+2}$, we refer to [Bai], [H] and [O].

Finally we present the useful equations for g to belong to $O_{s+2,2}(\mathbb{R})$. For $g \in O_{s+2,2}(\mathbb{R})$, we write

$$g = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

where $A_{11},\, A_{13},\, A_{31},\, A_{33} \in \mathbb{R}^{(2,2)}$, $A_{22} \in \mathbb{R}^{(s,s)}$, $A_{12},\, A_{32} \in \mathbb{R}^{(2,s)}$, and $A_{21},\, A_{23} \in \mathbb{R}^{(s,2)}$. Then the condition ${}^t g Q_M g = Q_M$ is equivalent to the following equations given by

$$\text{(C.21)} \qquad {}^tA_{11}\Pi_{1,1}A_{31} + {}^tA_{21}S_0A_{32} + {}^tA_{31}\Pi_{1,1}A_{12} = 0,$$

$$\text{(C.22)} \qquad {}^tA_{11}\Pi_{1,1}A_{32} + {}^tA_{21}S_0A_{22} + {}^tA_{31}\Pi_{1,1}A_{12} = 0,$$

$$\text{(C.23)} \qquad {}^tA_{11}\Pi_{1,1}A_{33} + {}^tA_{21}S_0A_{23} + {}^tA_{31}\Pi_{1,1}A_{13} = \Pi_{1,1},$$

$$\text{(C.24)} \qquad {}^tA_{12}\Pi_{1,1}A_{32} + {}^tA_{22}S_0A_{22} + {}^tA_{32}\Pi_{1,1}A_{12} = S_0,$$

$$\text{(C.25)} \qquad {}^tA_{12}\Pi_{1,1}A_{33} + {}^tA_{22}S_0A_{23} + {}^tA_{32}\Pi_{1,1}A_{13} = 0,$$

and

$$\text{(C.26)} \qquad {}^tA_{13}\Pi_{1,1}A_{33} + {}^tA_{23}S_0A_{23} + {}^tA_{33}\Pi_{1,1}A_{13} = 0.$$

Appendix D. The Leech Lattice Λ

Here we collect some properties of the Leech lattice Λ. Most of the materials in this appendix can be found in [C-S].

The Leech lattice L is the unique positive definite unimodular even integral lattice of rank 24 with minimal norm 4. Λ was discovered by J. Leech in 1965. (cf. Notes on sphere packing, Can. J. Math. 19 (1967), 251-267.) It was realized by Conway, Parker and Sloane that the Leech lattice Λ has many strange geometric properties. Past three decades more than 20 constructions of Λ were found.

The following properties of Λ are well known :

(Λ1) The determinant of Λ is $\det \Lambda = 1$. The kissing number is $\tau = 196560$ and the packing radius is $\rho = 1$. The density is $\Delta = \pi^{12}/(12!) = 0.001930\cdots$. The covering radius is $R = \sqrt{2}$ and the thickness is $\Theta = (2\pi)^{12}/(12!) = 7.9035\cdots$.

(Λ2) There are 23 different types of deep hole one of which is the octahedral hole $8^{-1/2}(4, 0^{23})$ surrounded by 48 lattice points.

(Λ3) The Veronoi cell has 16969680 faces, 196560 corresponding to the minimal vectors and 16773120 to those of the next layer.

(Λ4) The automorphism group $\mathrm{Aut}(\Lambda)$ of Λ has order

$$2^{22} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23 = 8315553613086720000.$$

$\mathrm{Aut}(\Lambda)$ has the Mathieu group M_{24} as a subgroup. The automorphism group $\mathrm{Aut}(\Lambda)$ is often denoted by Co_0 or $\cdot 0$ because J.H. Conway first discovered this group.

For a given lattice L, we denote $N_m(L)$ by the number of vectors of norm m. Conway characterized the Leech lattice as follows (cf. A characterization of Leech's lattice, Invent. Math. 7 (1969), 137-142 or Chapter 12 in [C-S]) :

Theorem 1 (Conway). *Λ is the unique positive definite unimodular even integral lattice L with rank < 32 that satisfies any one of the following*

(a) L is not directly congruent to its mirror-image.

(b) No reflection leaves L invariant.

(c) $N_2(L) = 0$.

(d) $N_{2m}(L) = 0$ for some $m \geq 0$.

Theorem 2 (Conway). *If L is a unimodular even integral lattice with rank < 32 and $N_2(L) = 0$, then $L = \Lambda$ and $N_4(L) = 196560$, $N_6(L) = 16773120$, $N_8(L) = 398034000$.*

Now we review the Jacobi theta functions. For the present time being, we put $q = e^{i\pi\tau}$ and $\zeta = e^{i\pi z}$. (We note that we set $q = e^{2\pi i\tau}$ and $\zeta = e^{2\pi iz}$ at other places.) We define the Jacobi theta functions

$$\theta_1(\tau, z) := i^{-1} \sum_{n\in\mathbb{Z}} (-1)^n q^{(n+1/2)^2} \zeta^{2n+1}, \tag{D.1}$$

$$\theta_2(\tau, z) := \sum_{n\in\mathbb{Z}} q^{(n+1/2)^2} \zeta^{2n+1}, \tag{D.2}$$

$$\theta_3(\tau, z) := \sum_{n\in\mathbb{Z}} q^{n^2} \zeta^{2n}, \tag{D.3}$$

$$\theta_4(\tau,z) := \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} \zeta^{2n}, \tag{D.4}$$

where $\tau \in H_1$ and $z \in \mathbb{C}$. We also define the theta functions $\theta_k(\tau) := \theta_k(\tau, 0)$ for $k = 2, 3, 4$. Then it is easy to see that

$$\theta_2(\tau) = \sum_{n\in\mathbb{Z}} q^{(n+1/2)^2} = e^{\pi i\tau/4}\theta_3(\tau, \tau/2), \tag{D.5}$$

$$\theta_4(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} = \theta_3(\tau, \pi/2) = \theta_3(\tau+1). \tag{D.6}$$

According to the Poisson summation formula, we obtain

$$\theta_3(-1/\tau, z/\tau) = (-i\tau)^{1/2} e^{\pi i z^2/\tau}\theta_3(\tau,z), \quad (\tau,z)\in H_1\times\mathbb{C}. \tag{D.7}$$

And these theta functions can be written as infinite products as follows:

$$\theta_1(\tau,z) = 2\sin\pi z\cdot q^{1/4}\prod_{n>0}(1-q^{2n})(1-q^{2n}\zeta^2)(1-q^{2n}\zeta^{-2}), \tag{D.8}$$

$$\theta_2(\tau,z) = q^{1/4}\zeta\prod_{n>0}(1-q^{2n})(1+q^{2n}\zeta^2)(1+q^{2n}\zeta^{-2}), \tag{D.9}$$

$$\theta_3(\tau,z) = \prod_{n>0}(1-q^{2n})(1+q^{2n}\zeta^2)(1+q^{2n-1}\zeta^{-2}), \tag{D.10}$$

$$\theta_4(\tau,z) = \prod_{n>0}(1-q^{2n})(1-q^{2n-1}\zeta^2)(1-q^{2n-1}\zeta^{-2}). \tag{D.11}$$

Thus the theta functions $\theta_k(\tau)\,(\,2\le k\le 4\,)$ are written as infinite products:

$$\theta_2(\tau) = q^{1/4}\prod_{n>0}(1-q^{2n})(1+q^{2n})(1+q^{2n-2}) = 2q^{1/4}\prod_{n>0}(1-q^{2n})(1+q^{2n})^2, \tag{D.9'}$$

$$\theta_3(\tau) = \prod_{n>0}(1-q^{2n})(1+q^{2n-1})^2, \tag{D.10'}$$

(D.11')
$$\theta_4(\tau) = \prod_{n>0}(1-q^{2n})(1-q^{2n-1})^2.$$

We note that the discriminant function $\Delta(\tau)$ is written as

(D.12)
$$\Delta(\tau) = q^2\prod_{n>0}(1-q^{2n})^{24} = \{\,1/2\theta_2(\tau)\theta_3(\tau)\theta_4(\tau)\,\}^8\,.$$

We observe that the theta function $\theta_3(\tau,z)$ is annihilated by the heat operator $H := \frac{\partial^2}{\partial z^2} - 4\pi i\frac{\partial}{\partial\tau}$. It is easy to check that $\theta_1(\tau,z)$ has zeros only at $m_1 + m_2\tau\,(\,m_1, m_2 \in \mathbb{Z}\,)$ and satisfies the equations

(D.13)
$$\theta_1(\tau, z+1) = -\theta_1(\tau,z), \quad \theta_1(\tau,\tau+z) = -q^{-1}e^{-2\pi iz}\theta_1(\tau,z).$$

Now for a given positive definite lattice L, we define the *theta series* $\Theta_L(\tau)$ of a lattice L by

(D.14)
$$\Theta_L(\tau) := \sum_{\alpha\in L} q^{N(\alpha)} = \sum_{m\geq 0} N_m(L)q^m, \quad \tau\in H_1,$$

where $N(\alpha) := (\alpha,\alpha)$ denotes the norm of a vector $\alpha\in L$. We can also use (D.14) to define the theta series of a nonlattice packing L. The commonest examples of this appear when L is a translate of a lattice or a union of translates. Clearly $\theta_2(\tau) = \Theta_{\mathbb{Z}+1/2}(\tau)$, $\theta_3(\tau) = \Theta_{\mathbb{Z}}(\tau)$ and $\Theta_{\mathbb{Z}^n}(\tau) = \Theta_{\mathbb{Z}}(\tau)^n = \theta_3(\tau)^n$.

Returing to the Leech lattice L,

$$\begin{aligned}\Theta_\Lambda(\tau) &= \Theta_{E_8}(\tau)^3 - 720\,\Delta(\tau)\\ &= 1/8\left\{\,\theta_2(\tau)^8+\theta_3(\tau)^8+\theta_4(\tau)^8\,\right\}^3 - 45/16\,\{\,\theta_2(\tau)\theta_3(\tau)\theta_4(\tau)\,\}^8\\ &= 1/2\left\{\,\theta_2(\tau)^{24}+\theta_3(\tau)^{24}+\theta_4(\tau)^{24}\,\right\} - 69/16\,\{\,\theta_2(\tau)\theta_3(\tau)\theta_4(\tau)\,\}^8\\ &= \sum_{m\geq 0} N_m(\Lambda)q^m = 1 + 196560q^4 + 16773120q^6 + \cdots,\end{aligned}$$

where $\Theta_{E_8}(\tau)$ is the theta series of the exceptional lattice E_8 of rank 8. It is known that

(D.15)
$$N_m(\Lambda) = 65520/691\,\left(\,\sigma_{11}(m/2) - \tau(m/2)\right).$$

The values of $N_m(\Lambda)$ for $0\leq m\leq 100$, m : even can be found in [C-S], p. 135.

In the middle of 1980s, M. Koike, T. Kondo and T. Tasaka solved a special part of the Moonshine Conjectures for the Mathieu group M_{24}. For $g \in M_{24}$, we write

$$g = (n_1)(n_2)\cdots(n_s), \quad n_1 \geq \cdots \geq n_s \geq 1, \tag{D.16}$$

where (n_i) is a cycle of length n_i ($1 \leq i \leq s$). To each $g \in M_{24}$ of the form (D.16), we associate modular forms $\eta_g(\tau)$ and $\theta_g(\tau)$ defined by

$$\eta_g(\tau) := \eta(n_1\tau)\,\eta(n_2\tau)\cdots\eta(n_s\tau), \quad \tau \in H_1 \tag{D.17}$$

and

$$\theta_g(\tau) := \sum_{\alpha \in \Lambda_g} e^{\pi i N(\alpha)\tau}, \quad \tau \in H_1, \tag{D.18}$$

where $\eta(\tau)$ is the Dedekind eta function and

$$\Lambda_g := \{\, \alpha \in \Lambda \mid g \cdot \alpha = \alpha \,\} \tag{D.19}$$

is the positive definite even integral lattice of rank s. We observe that $\theta_g(\tau) = \Theta_{\Lambda_g}(\tau)$.

Theorem 3 ([Koi2]). *For any element $g \in M_{24}$ with $g \neq 12^2,\ 4^6,\ 2^{12},\ 10^2 \cdot 2^2,\ 12{\cdot}6{\cdot}4{\cdot}2,\ 4^4{\cdot}2^4$, there exists a unique modular form $f_g(\tau) = 1 + \sum_{n\geq 0} a_g(n)q^{2n},\ a_g(n) \in \mathbb{Z}$ satisfying the following conditions :*

(K1) There exists an element $g_1 \in G$ such that $f_g(\tau)\eta_g(\tau)^{-1} = T_{g_1}(\tau) + c$ for some constant c, where G is the MONSTER and $T_{g_1}(\tau)$ denotes the Thompson series of $g_1 \in G$.

(K2) $a_g(1) = 0$, and $a_g(n)$ are nonnegative even integers for all $n \geq 1$.

(K3) If $g' = g^r$ for some $r \in \mathbb{Z}$, then $a_g(n) \leq a_{g'}(n)$ for all n.

(K4) $a_g(2)$ is equal to the cardinality of the set $\{\, \alpha \in \Lambda_g \mid N(\alpha) = (\alpha, \alpha) = 4 \,\}$.

Theorem 4 (Kondo and Tasaka [K-T]). *Let $g \in M_{24}$ be any element of the Mathieu group M_{24}. Then the function $\theta_g(\tau)\,\eta_g(\tau)^{-1}$ is a Hauptmodul for a genus 0 discrete subgroup of $SL(2,\mathbb{R})$. The function $\theta_g(\tau)$ is the unique modular form satisfying the conditions (K1)-(K4).*

For more detail on the Leech lattice Λ we refer to [Bo3], [C-S] and [Kon].

References

[A] G.E. Andrews, *A simple proof of Jacobi's triple product identity*, Proc. Amer. Math. Soc. **16** (1965), 333-334.

[An] A.N. Andrianov, *Modular descent and the Saito-Kurokawa conjecture*, Invent. Math. **53** (1979), 267-280.

[Bai] W.L. Baily, *Introductory Lectures on Automorphic Forms*, Iwanami Shoton, Publishers and Princeton Univ. Press, 1973.

[Bas] H. Bass, *The congruence subgroup problem, Proc. of Conf. in Local Fields* (1967), Springer-Verlag, edited by T.A. Springer, 16-22.

[Be] R. Bellman, *A Brief Introduction to Theta Functions*, Holts, Rinehart and Winston, Inc., New York, 1961.

[Bo1] R.E. Borcherds, *Vertex algebras, Kac-Moody algebras and the monster*, Proc. Natl. Acad. Sci., USA. **83** (1986), 3068-3071.

[Bo2] ———, *Generalized Kac-Moody algebras*, J. Algebra **115** (1988), 501-512.

[Bo3] ———, *Lattices like the Leech Lattice*, J. Algebra **130** (1990), 219-234.

[Bo4] ———, *Central Extensions of Generalized Kac-Moody Algebras*, J. Algebra **140** (1991), 330-335.

[Bo5] ———, *The Monster Lie Algebra*, Adv. Math. **83** (1990), 30-47.

[Bo6] ———, *Monstrous moonshine and monstrous Lie superalgebras*, Invent. Math. **109** (1992), 405-444.

[Bo7] ———, *Automorphic forms on $O_{s+2,2}(\mathbb{R})$ and infinite products*, Invent. Math. (1995), 161-213.

[BCQ] R.E. Borcherds, J.H. Conway and L. Queen, *A Monster Lie Algebra ?*, Adv. Math. **53** (1984), 75-79.

[Ca-S] L. Carlitz and M.V. Subbarao, *A simple proof of Watson's quintuple product identity*, Proc. Amer. Math. Soc. **32, No. 1** (1972), 42-44.

[C-N] J.H. Conway and S. Norton, *Monstrous Moonshine*, Bull. London Math. Soc. **11** (1979), 308-339.

[CCNPW] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

[C-S] J.H. Conway and N.J.A. Sloane, *Sphere Packings, Lattices and Groups*, Springer-Verlag, New York, 1988.

[Ev] S.A. Evdokimov, *A characterization of the Maass space of Siegel cusp forms of genus* 2 *(in Russian)*, Mat. Sbornik (154) **112** (1980), 133-142.

[E-Z] M. Eichler and D. Zagier, *The Theory of Jacobi Forms, Progress in Math.*, vol. 55, Birkhäuser, Boston-Basel- Stuttgart, 1985.

[F] E. Freitag, *Siegelschen Modulfunktionen*, Springer-Verlag, Berlin-Heidelberg-New York, 1983.

[F-F] A.J. Feingold and I.B. Frenkel, *A Hyperbolic Kac-Moody Algebra and the Theory of Siegel Modular Forms of Genus 2*, Math. Ann. **263** (1983), 87-144.

[FI] I.B. Frenkel, *Representations of Kac-Moody algebras and dual resonance models.*, In Flato et al. (eds.) Applications of group theory in theoretical physics, Lect. Appl. Math., Providence, RI. AMS. **21** (1985), 325-353.

[FLM] I.B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Academic Press, Boston, 1988.

[FLT] B. Fischer, D. Livingston and M.P. Thorne, *The characters of the "Monster" simple group*, Birmingham, 1978.

[G1] V.G. Gritsenko, *Jacobi functions of n variables*, J. Soviet Math. **53** (1991), 243-252.

[**G2**] ______, *Modular forms and moduli spaces of abelian and K3 surfaces*, St. Petersburg Math. J. **4** (1994).

[**G-T**] P. Goddard and C.B. Thorne,, *Compatibility of the dual Pomeron with unitarity and the absence of ghosts in the dual resonance model*, Phys. Lett. B **40, No. 2** (1972), 235-238.

[**G-Z**] B.H. Gross and D. Zagier, *On Singular Moduli*, J. reine angew. Math. **355** (1985), 191-220.

[**H**] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.

[**Ib**] T. Ibukiyama, *On Jacobi forms and Siegel modular forms of half integral weughts*, preprint (1991).

[**Ig1**] J. Igusa, *On Siegel modular forms of genus two*, Amer. J. Math. **84** (1962), 175-200.

[**Ig2**] ______, *Theta Functions*, Springer-Verlag, 1972.

[**K**] V.G. Kac, *Infinite dimensional Lie algebras, 3rd ed.*, Cambridge Univ. Press, 1990.

[**KMW**] V. Kac, R.V. Moody and M. Wakimoto, *On* E_{10}, preprint.

[**Kob**] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Springer-Verlag, 1984.

[**Koh**] W. Kohnen, *Modular forms of half integral weight on* $\Gamma_0(4)$, Math. Ann. **248** (1980), 249-266.

[**Koi1**] M. Koike, *On Replication Formula and Hecke Operators*, Nagoya Univ., preprint.

[**Koi2**] ______, *Mathieu Group* M_{24} *and Modular Forms*, Nagoya Math. J. **99** (1985), 147-157.

[**Kon**] T. Kondo, *The automorphism group of Leech lattice and elliptic modular functions*, J. Math. Soc. Japan **37, No. 2** (1985), 337-?.

[**K-T**] T. Kondo and T. Tasaka, *The Theta Functions of Sublattices of the Leech Lattice*, Nagoya Math. J. **101** (1986), 151-179.

[**Ma1**] H. Maass, *Lectures on Modular Functions of One Variable*, TIFR, Bombay, 1964.

[**Ma2**] ______, *Über eine Spezialschar von Modulformen zweiten Grades I*, Invent. Math. **52** (1979), 95-104.

[**Ma3**] ______, *Über eine Spezialschar von Modulformen zweiten Grades II*, Invent. Math. **53** (1979), 249-253.

[**Ma4**] ______, *Über eine Spezialschar von Modulformen zweiten Grades III*, Invent. Math. **53** (1979), 255-265.

[**Mac**] I.G. Macdonald, *Affine Root Systems and Dedekind's* η*-functions*, Invent. Math. **15** (1972), 91-143.

[**Me**] J. Mennicke, *Zur Theorie der Siegelsche Modulgruppe*, Math. Ann. **159** (1965), 115-129.

[**Mu**] D. Mumford, *Tata Lectures on Theta I*, vol. 28, Birkhäuser, Boston, 1983.

[**MNN**] D. Mumford, M. Nori and P. Norman, *Tata Lectures on Theta III, Progress in Math.*, vol. 97, Birkhäuser, 1991.

[**N1**] V.V. Nikulin, *Discrete Reflection Groups in Lobachevsky Spaces*, Proc. ICM, Warszawa (1983), 593-601.

[**N2**] ______, *Discrete reflection Groups in Lobachevsky Spaces and Algebraic Surfaces*, Proc. ICM, Berkeley (1986), 654-669.

[**N3**] ______, *On arithmetic groups generated by reflections in Lobachevsky spaces*, Math. USSR. Izv. English transl **16** (1981).

[**N4**] ______, *On the classification of arithmetic groups generated by reflections in Lobachevsky spaces*, Math. USSR. Izv. English transl **18** (1982).

[**N5**] ______, *A Lecture on Kac-Moody Lie Algebras of the Arithmetic Type*, preprint (1994).

[**No1**] S.P. Norton, *More on moonshine, Computational group theory* (1984), Academic Press, London, 185-195.

[**No2**] ______, *Generalized Moonshine*, Proc. Symp. Pure Math., AMS. **47** (1987), 208-209.

[**O**] T. Oda, *On modular forms associated with indefinite quadratic forms of signature (2,n-2)*, Math. Ann. **231** (1977), 97-144.

[**R**] H. Rademacher, *Topics in Analytic Number Theory*, Springer-Verlag, New York-Berlin-Heidelberg, 1973.

[**S**] J.-P. Serre, *A Course in Arithmetic*, Springer-Verlag, 1973.

[**Sh**] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton Univ. Press, Princeton, N.J., 1971.

[**S-V**] M.V. Subbarao and M. Vidyasagar, *On Watson's quintuple product identity*, Proc. Amer. Math. Soc. **26** (1970), 23-27.

[**T**] A. Terras, *Harmonic Analysis on Symmetric Spaces and Applications I*, Springer-Verlag, 1985.

[**Th1**] J.G. Thompson, *A finiteness theorem for subgroups of* $PSL(2,\mathbb{R})$ *which is commensurable with* $PSL(2,\mathbb{Z})$, Proc. Symp. Pure Math. AMS. **37** (1979), 533-555.

[**Th2**] ______, *Some numerology between the Fischer-Griess Monster and the elliptic modular function*, Bull. London Math. Soc. **11** (1979), 352-353.

[**W1**] G.N. Watson, *Theorems stated by Ramanujan. VII: Theorems on continued fractions*, J. London Math. Soc. **4** (1929), 37-48.

[**W2**] ______Ramanujan's Vermutung über Zerfallungsanzahlen, J. Reine Angew. Math. **179** (1938), 97-128.

[**Y1**] J.-H. Yang, *Remarks on Jacobi Forms of Higher Degree*, Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, The Pyungsan Institute for Mathematical Sciences, Seoul (1993), 33-58.

[**Y2**] ______, *On Jacobi Forms of Higher Degree (Korean)*, Notices of the Pyungsan Institute for Math. Sciences, Seoul (1993), 3-29.

[**Y3**] ______, *Singular Jacobi Forms*, Trans. of American Math. Soc. **347, No. 6** (1995), 2041-2049.

[**Y4**] ______, *Construction of Vector Valued Modular Forms from Jacobi Forms*, Can. J. Math. **47 (6)** (1995), 1329-1339.

[**Za**] D. Zagier, *Sur la conjecture de Saito-Kurokawa (d'après H. Maass). Sém. Delange-Pisot-Poitou 1979-1980*, Progress in Math. Birkhäuser, Boston **12** (1980), 371-394.

[**Zi**] C. Ziegler, *Jacobi Forms of Higher Degree*, Abh. Math. Sem. Univ. Hamburg **59** (1989), 191-224.

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Number theory, geometry and related topics (Iksan City, 1995), 13–82, *Pyungsan Inst. Math. Sci., Seoul*, 1996.

This is an excellent survey article about the recent impressive works of R. E. Borcherds. The results of Borcherds presented in this article are (i) construction of the Monster Lie algebra, (ii) proof of the Moonshine conjectures, (iii) construction of automorphic forms on the orthogonal group $O_{s+2,2}(\mathbf{R})$, (iv) infinite product representations of meromorphic modular forms, and (v) construction of Jacobi forms.

The article is organized as follows: In §2, some properties of Kac-Moody Lie algebras relevant to the subsequent discussions are collected. In §3, Borcherds' proof of the Moonshine conjectures is discussed. In §4, the theory of Jacobi forms, especially Borcherds' construction of almost holomorphic Jacobi forms, is presented. Finally, in §5, Borcherds' infinite product formula and modular forms are discussed. In the final remarks, open problems are formulated. There are several appendices; in Appendix A, classical modular forms are discussed; in Appendix B, the Kohnen "plus" space and the Maass "Spezialschar" are briefly discussed; in Appendix C, geometric aspects of the orthogonal group $O_{s+2,2}(\mathbf{R})$ are presented, and in the final appendix, some well-known properties of the Leech lattice and Jacobi theta functions are discussed.

{For the entire collection see MR1404965 (97b:11002)}

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A Geometrical Theory of Jacobi Forms of Higher Degree

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In this paper, we give a survey of a geometrical theory of Jacobi forms of higher degree. And we present some geometric results and discuss some geometric problems to be investigated in the future.

1. Introduction

A Jacobi form is an automorphic form on the Jacobi group, which is the semi-direct product of the symplectic group $Sp(g,\mathbb{R})$ and the Heisenberg group $H_{\mathbb{R}}^{(g,h)}$ (see section 2). Jacobi forms are very useful because they are closely related to modular forms of half integral weight and the theory of the moduli space of abelian varieties. The simplest case is when the symplectic group is $SL(2,\mathbb{R})$ and the Heisenberg group is three dimensional, that is, $g = h = 1$. This case had been treated more or less systematically in [21] and many papers of Zagier's school. But it seems to us that there is no systematic investigation of Jacobi forms of higher degree when $g > 1$ and $h > 1$. Some results could be found in [17], [79]-[89] and [94].

The purpose of this paper is to give a survey of a geometrical theory of Jacobi forms of higher degree. And we present some geometric results and discuss some geometric problems which should be investigated in the future. In Section 2, we review the notion of Jacobi forms and establish the notations. In Section 3, we present a brief historical remark and some motivation on Jacobi forms. In Section 4, we review the toroidal compactifications of the Siegel modular variety and the universal abelian variety. In Section 5, we introduce the automorphic vector bundle $E_{\rho,\mathcal{M}}$ associated with the canonical automorphic factor $J_{\mathcal{M},\rho}$ for the Jacobi group $G_{g,h}^J$ and then discuss the properties of $E_{\rho,\mathcal{M}}$ related to Jacobi forms. In Section 6, we give some open problems related to Wang's result(cf. [63]). In Section 7, we describe the boundary of the Satake compactification in terms of the languages of Jacobi forms. These results are essentially due to Igusa [35]. In Section 8, we

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provide you with some characterizations of *singular* Jacobi forms due to Yang [85]. We roughly explain that the study of *singular* Jacobi forms is closely related to the invariant theory of the action of the group $GL(g,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$(cf. (9.1)) and to the geometry of the universal abelian variety. In Section 9, we introduce some results of the Siegel-Jacobi operator. We describe implicitly that the Siegel-Jacobi operator plays an important role in the study of the universal abelian variety. In Section 10, we present $G_{g,h}^J$-invariant Kähler metrics and $G_{g,h}^J$-invariant differential operators on the Siegel-Jacobi space $H_g \times \mathbb{C}^{(h,g)}$. We introduce the notion of Maass-Jacobi forms. In the final section, we give a brief remark on some recent geometric results. In appendix A, we talk about subvarieties of the Siegel modular variety and present several problems. In appendix B, we describe why the study of *singular modular forms* is closely related to that of the geometry of the Siegel modular variety. Finally I would like to give my hearty thanks to Professor Tadao Oda and Dr. Hiroyuki Ito for inviting me to Sendai and giving me a chance to give a lecture at the conference on Hodge Theory and Algebraic Geometry.

Notations: We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers, and the field of complex numbers respectively. H_g denotes the Siegel upper half plane of degree g. $\Gamma_g := Sp(g,\mathbb{Z})$ denotes the Siegel modular group of degree g. The symbol ":=" means that the expression on the right is the definition of that on the left. We denote by $\mathbb{Z}^+$ the set of all positive integers. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. E_n denotes the identity matrix of degree n. For a commutative ring K, we denote by $S_\ell(K)$ the vector space of symmetric matrices of degree ℓ with entries in K. For a positve integer g and an integer k, we denote by $[\Gamma_g, k]$ the vector space of all Siegel modular forms on H_g of weight k.

2. Jacobi Forms

In this section, we establish the notations and define the concept of Jacobi forms.

Let

$$Sp(g,\mathbb{R}) = \{ M \in \mathbb{R}^{(2g,2g)} \mid {}^tMJ_gM = J_g \}$$

be the symplectic group of degree g, where

$$J_g := \left(\begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix} \right).$$

It is easy to see that $Sp(g,\mathbb{R})$ acts on H_g transitively by

$$M < Z > := (AZ+B)(CZ+D)^{-1},$$

where $M = \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) \in Sp(g,\mathbb{R})$ and $Z \in H_g$.

For two positive integers g and h, we recall that the Jacobi group $G^J_{g,h} := Sp(g,\mathbb{R}) \ltimes H^{(g,h)}_{\mathbb{R}}$ is the semidirect product of the symplectic group $Sp(g,\mathbb{R})$ and the Heisenberg group $H^{(g,h)}_{\mathbb{R}}$ endowed with the following multiplication law

$$(M,(\lambda,\mu,\kappa))\cdot(M',(\lambda',\mu',\kappa')) := \,(MM',(\tilde{\lambda}+\lambda',\tilde{\mu}+\mu',\kappa+\kappa'+\tilde{\lambda}\,{}^t\!\mu'-\tilde{\mu}\,{}^t\!\lambda'))$$

with $M, M' \in Sp(g,\mathbb{R}), (\lambda,\mu,\kappa),\ (\lambda',\mu',\kappa') \in H^{(g,h)}_{\mathbb{R}}$ and $(\tilde{\lambda},\tilde{\mu}) := (\lambda,\mu)M'$. It is easy to see that $G^J_{g,h}$ acts on $H_{g,h} := H_g \times \mathbb{C}^{(h,g)}$ transitively by

$$(2.1) \qquad (M,(\lambda,\mu,\kappa))\cdot(Z,W) := (M<Z>,(W+\lambda Z+\mu)(CZ+D)^{-1}),$$

where $M = \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) \in Sp(g,\mathbb{R}),\ (\lambda,\mu,\kappa) \in H^{(g,h)}_{\mathbb{R}}$ and $(Z,W) \in H_{g,h}$.

Let ρ be a rational representation of $GL(g,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbb{R}^{(h,h)}$ be a symmetric half-integral semi-positive definite matrix of degree h. Let $C^\infty(H_{g,h},V_\rho)$ be the algebra of all C^∞ functions on $H_{g,h}$ with values in V_ρ. For $f \in C^\infty(H_{g,h},V_\rho)$, we define

$$(2.2) \qquad \begin{aligned} & (f|_{\rho,\mathcal{M}}[(M,(\lambda,\mu,\kappa))])(Z,W) \\ := \ & e^{-2\pi i\sigma(\mathcal{M}[W+\lambda Z+\mu](CZ+D)^{-1}C)} \times e^{2\pi i\sigma(\mathcal{M}(\lambda Z\,{}^t\!\lambda+2\lambda\,{}^t\!W+(\kappa+\mu\,{}^t\!\lambda)))} \\ & \times\rho(CZ+D)^{-1}f(M<Z>,(W+\lambda Z+\mu)(CZ+D)^{-1}), \end{aligned}$$

where $M = \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) \in Sp(g,\mathbb{R}),\ (\lambda,\mu,\kappa) \in H^{(g,h)}_{\mathbb{R}}$ and $(Z,W) \in H_{g,h}$.

Definition 2.1. Let ρ and $\mathcal{M}$ be as above. Let

$$H^{(g,h)}_{\mathbb{Z}} := \{\,(\lambda,\mu,\kappa) \in H^{(g,h)}_{\mathbb{R}} \,|\, \lambda,\mu \in \mathbb{Z}^{(h,g)},\ \kappa \in \mathbb{Z}^{(h,h)}\,\}.$$

Let Γ be a discrete subgroup of Γ_g of finite index. A *Jacobi form* of index $\mathcal{M}$ with respect to ρ on Γ is a holomorphic function $f \in C^\infty(H_{g,h},V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho,\mathcal{M}}[\tilde{\gamma}] = f$ for all $\tilde{\gamma} \in \Gamma^J := \Gamma \ltimes H^{(g,h)}_{\mathbb{Z}}$.

(B) f has a Fourier expansion of the following form :

$$f(Z,W) = \sum_{\substack{T\geq 0 \\ \text{half-integral}}} \ \sum_{R\in\mathbb{Z}^{(g,h)}} c(T,R)\cdot e^{\frac{2\pi i}{\lambda_\Gamma}\,\sigma(TZ)}\cdot e^{2\pi i\sigma(RW)}$$

with some nonzero integer $\lambda_\Gamma \in \mathbb{Z}$ and $c(T,R) \neq 0$ only if $\left(\begin{pmatrix} \frac{1}{\lambda_\Gamma}T & \frac{1}{2}R \\ \frac{1}{2}\,{}^t\!R & \mathcal{M} \end{pmatrix}\right) \geq 0$.

If $g \geq 2$, the condition (B) is superfluous by Köcher principle (cf. [94] Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ. Ziegler(cf. [94] Theorem 1.8 or [21] Theorem 1.1) proves that the vector space $J_{\rho,\mathcal{M}}(\Gamma)$ is finite dimensional. For more results on Jacobi forms with $g>1$ and $h>1$, we refer to [17], [79]-[89] and [94].

Definition 2.2. A Jacobi form $f\in J_{\rho,\mathcal{M}}(\Gamma)$ is said to be a *cusp* (or *cuspidal*) form if $\begin{pmatrix} \frac{1}{\lambda_\Gamma}T & \frac12 R \\ \frac12 {}^tR & \mathcal{M}\end{pmatrix} > 0$ for any $T,\ R$ with $c(T,R)\neq 0$. A Jacobi form $f\in J_{\rho,\mathcal{M}}(\Gamma)$ is said to be *singular* if it admits a Fourier expansion such that a Fourier coefficient $c(T,R)$ vanishes unless $\det \begin{pmatrix} \frac{1}{\lambda_\Gamma}T & \frac12 R \\ \frac12 {}^tR & \mathcal{M}\end{pmatrix} = 0.$

Example 2.3. Let $S\in\mathbb{Z}^{(2k,2k)}$ be a symmetric, positive definite, unimodular even integral matrix and $c\in\mathbb{Z}^{(2k,h)}$. We define the theta series

$$\vartheta^{(g)}_{S,c}(Z,W):=\sum_{\lambda\in\mathbb{Z}^{(2k,g)}} e^{\pi i\{\sigma(S\lambda Z\,{}^t\lambda)+2\sigma({}^tcS\lambda\,{}^tW)\}},\quad Z\in H_g,\ W\in\mathbb{C}^{(h,g)}. \tag{2.2}$$

We put $\mathcal{M}:=\frac12 {}^tcSc$. We assume that $2k<g+rank\,(\mathcal{M})$. Then it is easy to see that $\vartheta^{(g)}_{S,c}$ is a singular Jacobi form in $J_{k,\mathcal{M}}(\Gamma_g)$(cf. [94] p.212).

3. Historical Remarks

In this section, we will make brief historical remarks on Jacobi forms.

In 1985, the names Jacobi group and Jacobi forms got kind of standard by the classic book [21] by Eichler and Zagier to remind of Jacobi's "Fundamenta nova theoriae functionum ellipticorum", which appeared in 1829 ([36]). Before [21] these objects appeared more or less explicitly and under different names in the work of many authors.

In 1969 Pyatetski-Shapiro [52] discussed the Fourier-Jacobi expansion of Siegel modular forms and the field of modular abelian functions. He gave the dimension of this field in the higher degree.

About the same time Satake [55]-[56] introduced the notion of "groups of Harish-Chandra type" which are non reductive but still behave well enough so that he could determine their canonical automorphic factors and kernel functions.

Shimura [57]-[58] gave a new foundation of the theory of complex multiplication of abelian functions using Jacobi theta functions.

Kuznetsov [41] constructed functions which are almost Jacobi forms from ordinary elliptic modular functions.

Starting 1981, Berndt [4]-[6] published some papers which studied the field of arithmetic Jacobi functions, ending up with a proof of Shimura reciprocity law for the field of these functions with arbitrary level. Furthermore he investigated the discrete series for the Jacobi group $G^J_{g,h}$ and developed the spectral theory for $L^2(\Gamma^J\backslash G^J_{g,h})$ in the case $g=h=1$([9],[11]). Recently he [10] studied the L-

functions and the Whittaker models for the Jacobi forms.

The connection of Jacobi forms to modular forms was given by Maass, Andrianov, Kohnen, Shimura, Eichler and Zagier. This connection is pictured as follows. For k even, we have the following isomorphisms

$$[\Gamma_2, k]^M \cong J_{k,1}(\Gamma_1) \cong M^+_{k-\frac{1}{2}}(\Gamma_0^{(1)}(4)) \cong [\Gamma_1, 2k-2].$$

Here $[\Gamma_2, k]^M$ denotes the Maass's Spezialschar, $M^+_{k-\frac{1}{2}}(\Gamma_0^{(1)}(4))$ denotes the Kohnen space and $[\Gamma_1, 2k-2]$ denotes the vector space consisting of elliptic modular forms of weight $2k-2$. For a precise detail, we refer to [42]-[44], [1], [21], [37] and [81].

In 1982 Tai [60] gave asymptotic dimension formulae for certain spaces of Jacobi forms for arbitrary g and $h = 1$ and used these ones to show that the moduli A_g of principally polarized abelian varieties of dimension g is *of general type* for $g \geq 9$.

Feingold and Frenkel [23] essentially discussed Jacobi forms in the context of Kac-Moody Lie algebras generalizing the Maass correspondence to higher level. Gritsenko [30] studied Fourier-Jacobi expansions and a non-commutative Hecke ring in connection with the Jacobi group.

After 1985 the theory of Jacobi forms for $g = h = 1$ had been studied more or less systematically by the Zagier school. A large part of the theory of Jacobi forms of higher degree was investigated by Dulinski [17], Kramer [40], Yamazaki [69], Yang [79]-[89] and Ziegler [94].

There were several attempts to establish L-functions in the context of the Jacobi group by Murase [47]-[48] and Sugano [50] using the so-called "Whittaker-Shintani functions".

Recently Kramer [40] developed an arithmetic theory of Jacobi forms of higher degree. Runge [54] discussed some part of the geometry of Jacobi forms for arbitrary g and $h = 1$. Quite recently T. Arakawa and B. Heim [2] studied the iterated Petersson scalar product of a diagonal-restricted real analytic Jacobi Eisenstein series of degree (3,1) against elliptic Jacobi forms generalizing Garrett's result in the case of Siegel Eisenstein series of degree 3.

For a good survey on some motivation and background for the study of Jacobi forms, we refer to [10].

4. Review on Toroidal Compactifications of the Siegel Space and the Universal Abelian Variety

In this section, we will make a brief review on toroidal compactification of the Siegel space and the universal abelian variety. We refer to [3], [22] and [51] for more detail.

I. A toroidal compactification of the Siegel modular variety

First we realize H_g as a bounded symmetric domain $D_g := \{ W \in \mathbb{C}^{(g,g)} \mid W = {}^tW,\ \ E_g - Z\bar{Z} > 0 \}$ (called the generalized unit disc of degree g) in $S_g(\mathbb{C})$ via the transformation $\Phi : H_g \longrightarrow D_g$ given by

$$\Phi(Z) := (Z - iE_g)(Z + iE_g)^{-1}, \quad Z \in H_g.$$

Indeed, it is a Harish-Chandra realization of a homogeneous space. The inverse Φ^{-1} of Φ given by

$$\Phi^{-1}(Z) := i\,(E_g + W)(E_g - W)^{-1}, \quad W \in D_g$$

is called the *generalized Cayley transformation.*

Let $\bar{D}_g$ be the topological closure of D_g in $S_g(\mathbb{C})$. Then $\bar{D}_g$ is the disjoint union of all boundary components of D_g. Let

$$F_r := \left\{ \begin{pmatrix} Z_1 & 0 \\ 0 & E_{g-r} \end{pmatrix} \in \bar{D}_g \mid Z_1 \in D_r \right\}, \quad 0 \le r \le g$$

be the standard rational boundary components of D_g. Then any boundary component F of D_g is of the form $F = g \cdot F_r$ for some $g \in Sp(g,\mathbb{R})$ and some r with $0 \le r \le g$. In addition, if F is a rational boundary component of D_g, then it is of the form $F = \gamma \cdot F_r$ for some $\gamma \in Sp(g,\mathbb{Z})$ and some r with $0 \le r \le g$. We note that $F_0 = \{ E_g \}$ and $F_g = D_g$. We set

$$D_g^* := \cup_{0 \le r \le g}\, Sp(g,\mathbb{Z}) \cdot F_r\,. \tag{4.1}$$

Then D_g^* is clearly the union of all rational boundary components of D_g and is called the *rational closure* of D_g. We let $\Gamma_g := Sp(g,\mathbb{Z})$ for brevity. Then we obtain the so-called Satake-Baily-Borel compactification $A_g^* := \Gamma_g \backslash D_g^*$ of $A_g := \Gamma_g \backslash D_g$. Let F be a *rational* boundary component of D_g. We denote by $P(F)$, $W(F)$, $U(F)$ the parabolic subgroup associated with F, the unipotent radical of $P(F)$ and the center of $W(F)$ respectively. We set $V(F) := W(F)/U(F)$. Since $P(g \cdot F) = gP(F)g^{-1}$ for $g \in Sp(g,\mathbb{R})$, it is enough to investigate the structures of these groups for the standard rational boundary components F_r ($0 \le r \le g$).

Now we take $F = F_r$ for some r with $0 \le r \le g$. We define $D(F) := U(F)_{\mathbb{C}} \cdot D_g \subset \hat{D}_g$. Here $\hat{D}_g := B \backslash Sp(g,\mathbb{R})_{\mathbb{C}}$ is the compact dual of D_g with B a parabolic subgroup of $Sp(g,\mathbb{R})_{\mathbb{C}}$. It is obvious that $U(F)_{\mathbb{C}} \cong S_{g-r}(\mathbb{C})$ and $D(F) \cong F \times V(F) \times U(F)_{\mathbb{C}}$ analytically. We observe that $U(F)$ acts on $D(F)$ as the linear translation on the factor $U(F)_{\mathbb{C}}$. The isomorphism $\varphi : D(F) \longrightarrow F \times V(F) \times U(F)_{\mathbb{C}}$ is given by

$$\varphi\left(\begin{pmatrix} Z_1 & Z_2 \\ * & Z_3 \end{pmatrix}\right) := (Z_1, Z_2, Z_3), \quad Z_1 \in D_r,\ Z_2 \in \mathbb{C}^{(r,g-r)},\ Z_3 \in S_{g-r}(\mathbb{C}).$$

We define the mapping $\Phi_F : D(F) \longrightarrow U(F)$ by

(4.2)

$$\Phi_F((Z_1, Z_2, Z_3)) := \mathrm{Im}\, Z_3 \ - \ {}^t(\mathrm{Im}\, Z_2)\,(\mathrm{Im}\, Z)^{-1}\,(\mathrm{Im}\, Z_2), \quad (Z_1, Z_2, Z_3) \in D(F).$$

Then $D_g \cong H_g$ is characterized by $\Phi_F(Z) > 0$ for all $Z \in D_g$. This is the realization of a Siegel domain of the third kind. We let $C(F)$ be the cone of real positive symmetric matrices of degree $g-r$ in $U(F) \cong S_{g-r}(\mathbb{R})$. Clearly we have $D_g = \Phi^{-1}(C(F))$. We define

$$G_h(F) := \mathrm{Aut}\,(F) \quad (\,\text{modulo finite group}\,)$$

and

$$G_l(F) := \mathrm{Aut}\,(U(F),\, C(F)).$$

Then it is easy to see that

$$P(F) \,=\, (\,G_h(F) \times G_l(F)\,) \ltimes W(F) \quad (\,\text{the semidirect product}\,)$$

We obtain the natural projections $p_h : P(F) \longrightarrow G_h(F)$ and $p_l : P(F) \longrightarrow G_l(F)$.

Step I : Partial compactification for a rational bounadry component.

Now we let Γ be an arithmetic subgroup of $Sp(g, \mathbb{R})$. We let

$$\begin{aligned} \Gamma(F) :\quad & = \Gamma \cap P(F), \\ \bar{\Gamma}(F) :\quad & = p_l(\Gamma(F)) \subset G_l(F), \\ U_\Gamma(F) :\quad & = \Gamma \cap U(F), \quad \text{a lattice in } U(F), \\ W_\Gamma(F) :\quad & = \Gamma \cap W(F). \end{aligned}$$

We note that $\bar{\Gamma}(F)$ is an arithmetic subgroup of $G_l(F)$.

Let $\Sigma_F = \{\,\sigma_\alpha^F\,\}$ be a $\bar{\Gamma}(F)$-admissible polyhedral decomposition of $C(F)$. We set $D(F)' := D(F)/U(F)_{\mathbb{C}}$. Since $D(F)' \cong F \times V(F)$, the projection $\pi_F : D(F) \longrightarrow D(F)'$ is a principal $U(F)_{\mathbb{C}}$-bundle over $D(F)'$. The map

$$(4.3) \qquad \pi_{F,\Gamma} : U_\Gamma(F)\backslash D(F) \cong F \times V(F) \times (U_\Gamma(F)\backslash U(F)_{\mathbb{C}}) \longrightarrow D(F)'$$

is a principal $T(F)$-bundle with the structure group $T(F) := U_\Gamma(F)\backslash U(F)_{\mathbb{C}} \cong (\mathbb{C}^*)^q$, where $q = \dfrac{(g-r)(g-r+1)}{2}$. Let X_{Σ_F} be a normal torus embedding of $T(F)$. We note that X_{Σ_F} is determined by Σ_F. Then we obtain a fibre bundle

$$(4.4) \qquad \mathcal{X}(\Sigma_F) := (U_\Gamma(F)\backslash D(F)) \times_{T(F)} \mathrm{X}_{\Sigma_F}$$

over $D(F)'$ with fibre X_{Σ_F}. We denote by $\mathbf{X}\,(\Sigma_F)$ the interior of the closure of $U_\Gamma(F)\backslash D_g$ in $\mathcal{X}(\Sigma_F)$ (because $D_g \subset D(F)$). $\mathbf{X}(\Sigma_F)$ has a fibrewise $T(F)$-orbit decomposition $\coprod_\mu O(\mu)$ such that

(i) each $O(\mu)$ is an algebraic torus bundle over $D(F)'$,
(ii) $\sigma_\mu \prec \sigma_\nu$ iff $\overline{O(\mu)} \supseteq O(\nu)$,
(iii) $\dim \sigma_\mu + \dim O(\mu) = \dim D(F)$,
(iv) for $\sigma_\mu = 0$, $O(\mu) = U_\Gamma(F)\backslash D(F)$.

We define

$$O(F) := \bigcup_{\sigma_\alpha^F \cap C(F) \neq \emptyset} O(\alpha) \subset \mathbf{X}(\Sigma_F)$$

and

$$\bar{O}(F) := \Gamma(F)/U_\Gamma(F)\backslash O(F).$$

We note that $O(F_g) = D_g$ and $\bar{O}(F_g) = \Gamma\backslash D_g$. We set

$$\mathbf{Y}(\Sigma_F) := \Gamma(F)/U_\Gamma(F)\backslash \mathbf{X}(\Sigma_F)\,. \tag{4.5}$$

We note that $\Gamma(F)/U_\Gamma(F)$ acts on $\mathbf{Y}(\Sigma_F)$ properly discontinuously. Then we can show that $\mathbf{Y}(\Sigma_F)$ has a canonical quotient structure of a normal analytic space and $\bar{O}(F)$ is a closed analytic set in $\mathbf{Y}(\Sigma_F)$.

Step II : Gluing.

Let $\Sigma := \{\, \Sigma_F \mid F$ is a rational boundary component of $D_g \,\}$ be a Γ-admissible family of polyhedral decompositions. We put

$$\widetilde{(\Gamma\backslash D_g)} := \cup_{F:\mathrm{rational}} \mathbf{X}(\Sigma_F).$$

We define the equivalence relation $\backsim$ on $(\Gamma\tilde{\backslash} D_g)$ as follows:

$$X_1 \backsim X_2, \quad X_1 \in \mathbf{X}(\Sigma_{F_1}), \quad X_2 \in \mathbf{X}(\Sigma_{F_2})$$

iff there exist a rational boundary component F, an element $\gamma \in \Gamma$ such that $F_1 \prec F$, $\gamma F_2 \prec F$ and there exists an element $X \in \mathbf{X}(\Sigma_F)$ such that $\pi_{F,F_1}(X) = X_1$, $\pi_{F,F_2}(X) = \gamma X_2$, where

$$\pi_{F,F_1} : \mathbf{X}(\Sigma_F) \longrightarrow \mathbf{X}(\Sigma_{F_1}), \quad \pi_{F,F_2} : \mathbf{X}(\Sigma_F) \longrightarrow \mathbf{X}(\Sigma_{\gamma F_2}).$$

The space $\overline{(\Gamma\backslash D_g)} := \widetilde{(\Gamma\backslash D_g)}/\backsim$ is called the *toroidal compactification* of $\Gamma\backslash D_g$ associated with Σ. It is known that $\overline{(\Gamma\backslash D_g)}$ is a Hausdorff analytic variety containing $\Gamma\backslash D_g$ as an open dense subset. For a *neat* arithmetic subgroup Γ, we can obtain a smooth projective toroidal compactification of $\Gamma\backslash D_g$.

II. A toroidal compactification of the universal abelian variety

For a positive integer $g \in \mathbb{Z}^+$, we put $X := \mathbb{Z}^g$. Let $B(X)$ be the $\mathbb{Z}$-module of integral valued symmetric bilinear forms on X and let $B(X)_\mathbb{R} := B(X) \otimes_\mathbb{Z} \mathbb{R}$. Let $C(X) \subset B(X)_\mathbb{R}$ be the convex cone of all positive semi-positive symmetric bilinear forms on $X_\mathbb{R}$ whose radicals are defined over $\mathbb{Q}$. We let X^* be the dual of X. For a positive integer $s \in \mathbb{Z}^+$, we let

$$\tilde{B}_s(X) := B(X) \times (X^*)^s \quad \text{and} \quad \tilde{B}_s(X)_\mathbb{R} := \tilde{B}_s(X) \otimes_\mathbb{Z} \mathbb{R}.$$

Then the semidirect product $GL(X) \ltimes X^s$ acts on $\tilde{B}_s(X)_\mathbb{R}$ in the natural way and the projection $\tilde{B}_s(X)_\mathbb{R} \longrightarrow B(X)_\mathbb{R}$ is equivariant with respect to the canonical morphism $GL(X) \ltimes X^s \longrightarrow GL(X)$. Inside $\tilde{B}_s(X)_\mathbb{R}$ we obtain the cone $\tilde{C}_s(X)$ consisting of $q = (b; \ell_1, \cdots, \ell_s) \in \tilde{B}_s(X)_{\mathbb{R}}$ such that $b \in C(X)$ and each ℓ_j vanishes on the radical of b.

Let a $GL(X)$-admissible polyhedral cone decomposition $\mathcal{C} = \{\sigma_\alpha\}$ of $C(X)$ be given. A $GL(X) \ltimes X^s$-admissible polyhedral cone decomposition $\tilde{\mathcal{C}} = \{\tau_\beta\}$ of $\tilde{C}_s(X)$ relative to $\mathcal{C} = \{\sigma_\alpha\}$ is defined to be a collection $\tilde{\mathcal{C}} = \{\tau_\beta\}$ such that

(1) each τ_β is a non-degenerate rational polyhedral cone which is open in the smallest $\mathbb{R}$-subspace containing it;

(2) any face of a $\tau_\beta \in \tilde{\mathcal{C}}$ belongs to $\tilde{\mathcal{C}}$;

(3) $\tilde{C}_s(X) = \cup_{\tau_\beta \in \tilde{\mathcal{C}}} \tau_\beta$;

(4) $\tilde{\mathcal{C}}$ is invariant under the action of $GL(X) \ltimes X^s$ and there are only finitely many $GL(X) \ltimes X^s$-orbits;

(5) any $\tau_\beta \in \tilde{\mathcal{C}}$ maps into a $\sigma_\alpha \in \mathcal{C}$ under the natural projection $\tilde{C}_s(X) \longrightarrow C(X)$.

We call $\tilde{\mathcal{C}}$ *equidimensional* if in (5) of the above definition each $\tau_\beta \in \tilde{\mathcal{C}}$ maps onto a $\sigma_\alpha \in \mathcal{C}$. Again, $\tilde{\mathcal{C}}$ is called *smooth* or *regular* if each $\tau_\beta \in \tilde{\mathcal{C}}$ is generated by part of a $\mathbb{Z}$-basis of $\tilde{B}_s(X)$. According to the reduction theory [3], there exists a smooth equidimensional $GL(X) \ltimes X^s$-admissible polyhedral cone decomposition $\tilde{\mathcal{C}}$ of $\tilde{C}_s(X)$ relative to $\mathcal{C}$. Let F be the split torus $\tilde{B}_s(X)_{\mathbb{R}} \otimes_{\mathbb{Z}} G_{\mathbf{m}}$. The choice of a polyhedral cone decomposition $\tilde{\mathcal{C}} = \{\tau_\beta\}$ of $\tilde{C}_s(X)$ as above provides us with a torus embedding $F \hookrightarrow \bar{F}$. Then $\bar{F}$ is stratified by F-orbits and $GL(X) \ltimes X^s$ acts on $\bar{F}$ preserving this stratification. Therefore we obtain the toroidal compactification $\bar{A}_{g,s}$ of the universal abelian variety $A_{g,s} := \Gamma^J_{g,s} \backslash H_g \times \mathbb{C}^{(s,g)}$ with $\Gamma^J_{g,s} := \Gamma_g \ltimes H^{(g,h)}_{\mathbb{Z}}$. We collect some properties of the toroidal compactification $\bar{A}_{g,s}$.

(a) $\bar{A}_{g,s}$ is a Haudorff analytic variety containing $A_{g,s}$ as an open dense subset.

(b) $\bar{A}_{g,s}$ has a stratification parametrized by the $GL(X) \ltimes X^s$-orbits of cones $\tau_\beta \in \tilde{\mathcal{C}}$.

(c) The toroidal compactification $\bar{A}_{g,s}$ depends on the choice of a smooth equidimensional $GL(X) \ltimes X^s$-admissible polyhedral cone decomposition $\tilde{\mathcal{C}} = \{\tau_\beta\}$ of $\tilde{C}_s(X)$ relative to $\mathcal{C}$. In order to indicate this dependence we write $\bar{A}_{g,s}(\tilde{\mathcal{C}})$ instead of $\bar{A}_{g,s}$. The natural projection $\pi : A_{g,s} \longrightarrow A_g$ extends to a proper morphism $\bar{\psi} : \bar{A}_{g,s} \longrightarrow \bar{A}_g$.

Now we recall [22], p. 197 that an *admissible homogeneous principal polarization function* of $\{\tau_\beta\} \longrightarrow \{\sigma_\alpha\}$ is a piecewise linear function $\tilde{\phi} : \tilde{C}_s(X) \longrightarrow \mathbb{R}$ satisfying the following conditions

(P1) $\tilde{\phi}$ is continuous and $GL(X)$-invariant;
(P2) $\tilde{\phi}$ takes rational values on $\tilde{B}_s(X) \cap \tilde{C}_s(X)$ with bounded denominators;
(P3) $\tilde{\phi}$ is homogeneous, i.e., $\tilde{\phi}(t \cdot q) = t \cdot \tilde{\phi}(q)$ for all real $t \geq 0$ and all $q \in \tilde{C}_s(X)$;
(P4) $\tilde{\phi}$ is linear on each $\tau_\beta \in \tilde{\mathcal{C}}$;

(P5) $\tilde{\phi}$ is convex in the sense that

$$\tilde{\phi}(t \cdot q + (1-t) \cdot q') \geq t \cdot \tilde{\phi}(q) + (1-t) \cdot \tilde{\phi}(q')$$

for all $t \in \mathbb{R}$ with $0 \leq t \leq 1$ and any $q, q' \in \tilde{C}_s(X)$.

(P6) $\tilde{\phi}$ is strictly convex, that is, for each $\sigma_\alpha \in \mathcal{C} = \{\sigma_\alpha\}$ and each $\tau_\beta \in \tilde{\mathcal{C}} = \{\tau_\beta\}$ lying over σ_α, there exist a finite number of linear functionals $\ell_i : \tilde{B}_s(X) \longrightarrow \mathbb{R}$, $1 \leq i \leq m$ with $\ell_i \geq \tilde{\phi}$ on the preimage of σ_α for each i and

$$\tau_\beta = \{\, q \in \tilde{C}_s(X) \,|\, q \text{ lies over } \sigma_\alpha \text{ and } \tilde{\phi}(q) = \ell_i(q) \text{ for each } i \,\}.$$

(P7) There exists a rational positive number r such that for each $\mu = (\mu_1, \cdots, \mu_s) \in X^s$, the function

$$\tilde{\phi} - \tilde{\phi} \circ T_\mu \,:\, q \longmapsto f(q) - f(\mu \cdot q)$$

is equal to r times (restriction to $\tilde{C}_s(X)$ of) the linear functional $\tilde{\chi}_\mu$ on $\tilde{B}_s(X)$, where for $q = (b; \ell_1, \cdots, \ell_s) \in \tilde{C}_s(X)$,

$$\tilde{\chi}_\mu(q) := \sum_{1 \leq i \leq s} a_i(\mu_i) = \sum_{1 \leq i \leq s} \{\, b(\mu_i, \mu_i) + 2 \cdot \ell_i(\mu_i)\}.$$

The conditions (P1)-(P7) above constitute a kind of convexity conditions on $\{\tau_\beta\} \longrightarrow \{\sigma_\alpha\}$. They imply that the morphism $\bar{A}_{g,s} \longrightarrow \bar{A}_g$ attached to $\{\tau_\beta\} \longrightarrow \{\sigma_\alpha\}$ is *projective.* Indeed, the theory of torus embeddings shows that an admissible homogeneous principal polarization function $\tilde{\phi} : \tilde{C}_s(X) \longrightarrow \mathbb{R}$ gives rise to an invertible sheaf $\bar{\mathcal{L}}(\tilde{\phi})$, which is ample on $\bar{A}_{g,s}(\tilde{\mathcal{C}})$ relative to $\bar{A}_g(\mathcal{C})$.

5. The Automorphic Vector Bundle $E_{\rho,\mathcal{M}}$

Let ρ and $\mathcal{M}$ be as before in section 2. Assume that Γ is a subgroup of $\Gamma_g := Sp(g, Z)$ of finte index which acts freely on H_g and $-E_{2g} \notin \Gamma$. Then $\Gamma^J := \Gamma \ltimes H_{\mathbb{Z}}^{(g,h)}$ acts on $H_{g,h} := H_g \times \mathbb{C}^{(h,g)}$ properly discontinuously. We consider the automorphic factor $J_{\mathcal{M},\rho} : G^J_{g,h} \times H_{g,h} \longrightarrow GL(V_\rho)$ defined by

$$J_{\mathcal{M},\rho}(\tilde{g}, (Z, W)) := \begin{array}{l} e^{2\pi i \sigma(\mathcal{M}[W + \lambda Z + \mu](CZ+D)^{-1}C)} \\ \times e^{-2\pi i \sigma(\mathcal{M}(\lambda Z\, {}^t\lambda + 2\lambda^t W + \kappa + \mu\, {}^t\lambda))} \rho(CZ+D), \end{array}$$

where $\tilde{g} = (M, (\lambda, \mu, \kappa)) \in G^J$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbb{R})$. Then $J_{\mathcal{M},\rho}$ defines the automorphic vector bundle $E_{\rho,\mathcal{M}} := H_{g,h} \times_{\Gamma^J} V_\rho$ over $A_{g,h,\Gamma} := \Gamma^J \backslash H_{g,h}$. By the definition, Jacobi forms in $J_{\rho,\mathcal{M}}(\Gamma)$ may be considered as holomorphic sections of the vector bundle $E_{\rho,\mathcal{M}}$ with some additional cusp condition. For $g \geq 2$, this additional condition may be dropped according to Köcher principle. Let $\bar{A}_{g,h,\Gamma}$ be a toroidal compactification given by a regular Γ-admissible family Σ of polyhedral decompositions.

Without proof we provides our results.

Theorem 5.1. *$A_{g,h,\Gamma}$ is contained in $\bar{A}_{g,h,\Gamma}$ as a Zariski open subset. $E_{\rho,\mathcal{M}}$ can be extended uniquely to the holomorphic vector bundle $\bar{E}_{\rho,\mathcal{M}}$ over $\bar{A}_{g,h,\mathcal{M}}$. And $H^i(A_{g,h,\Gamma}, E_{\rho,\mathcal{M}})$
$\cong H^i(\bar{A}_{g,h,\mathcal{M}}, \bar{E}_{\rho,\mathcal{M}})$. In particular, the dimension of $J_{\rho,\mathcal{M}}$ is finite dimensional.*
vspace0.1in

Definition 5.2. Let ρ be an irreducible rational representation of $GL(g,\mathbb{C})$ with its highest weight $(\lambda_1, \lambda_2, \cdots, \lambda_g)$. We call the number of j ($1 \le j \le g$) such that $\lambda_j = \lambda_g$ the *corank* of ρ which is denoted by corank(ρ). The number $k(\rho) := \lambda_g$ is called the *weight* of ρ.

Theorem 5.3. *Let $2\mathcal{M}$ be an even unimodular positive definite matrix of degree h. Let ρ be an irreducible finite dimensional representation of $GL(g,\mathbb{C})$ with highest weight $\rho = (\lambda_1, \cdots, \lambda_g)$. Let $\lambda(\rho)$ be the number of $\lambda_i's$ such that $\lambda_i = k(\rho) + 1 = \lambda_g + 1,\ 1 \le i \le g$. Assume that ρ satisfies the following conditions :*

$$\begin{array}{ll} [a] & \rho(A) = \rho(-A) \quad \textit{for all } A \in GL(g,\mathbb{C}), \\ [b] & \lambda(\rho) < 2(g - k(\rho) - \textit{corank}(\rho)) + h. \end{array}$$

Then $H^0(A_{g,h,\Gamma}, E_{\rho,\mathcal{M}}) = 0$.

Proof. The proof can be found in [80].

Corollary 5.4. *Let $2\mathcal{M}$ be as above in Theorem 5.3. Assume that $2k(\rho) \le g + h - 2corank(\rho)$. Then $H^0(A_{g,h,\Gamma}, E_{\rho,\mathcal{M}}) = 0$.*

Remark 5.5. N.-P. Skoruppa [Sk] proved that $J_{1,m}(\Gamma_1) = 0$ for any nonnegative integer m. It is interesting to give the geometric proofs of this fact and Theorem 5.3.

We give the following open problems :

Problem 1. Give the explicit dimension formula or estimate for $H^0(A_{g,h,\Gamma}, E_{\rho,\mathcal{M}})$.

Problem 2. Compute the cohomology groups $H^k(A_{g,h,\Gamma},\, E_{\rho,\mathcal{M}})$ explicitly. Here $0 \le k \le \dfrac{g(g+2h+1)}{2}$.

Problem 3. Under which conditions is $E_{\rho,\mathcal{M}}$ *ample*?

Problem 4. Discuss the analogue of Hirzebruch's proportionality theorem for $E_{\rho,\mathcal{M}}$(cf. [45]).

6. Smooth Compactification of Siegel Moduli Spaces and Open Problems

Let $\Gamma_g(k)$ be the principal congruence subgroup of $Sp(g,Z)$ of level k and let H_g be the Siegel upper-half plane of degree g. We assume that $k \ge 3$. This implies that $\Gamma_g(k)$ is a *neat* arithmetic subgroup. Let $\bar{X}$ be the toroidal compactification of $X := \Gamma_g(k)\backslash H_g$ from $\Gamma_g(k)$-admissible family given by the central cone decomposition $\sum_{cent}$ or a refinement of $\sum_{cent}$. Then the boundary $D := \bar{X} - X = \sum_{i=1}^m D_i$ is a divisor of $\bar{X}$ with normal crossing, that is, each D_i is an irreducible smooth divisor of $\bar{X}$ and $D_1, \cdots, D_m$ intersect transversally. If $g \leq 4$, we have the following results obtained by Wang [63].

Theorem 6.1. *(1) Each divisor is algebraically isomorphic to*

$$\bar{Y}_{g-1} := \overline{\Gamma_{g-1}^J(k)\backslash(H_{g-1} \times C^{g-1})}.$$

Here $\Gamma_{g-1}^J(k) := \Gamma_{g-1}(k) \ltimes (kZ)^{g-1}$ *is the Jacobi modular group acting on the homogeneous space* $W_{g-1} := H_{g-1} \times C^{g-1}$ *in a usual way and* $\bar{Y}_{g-1}$ *is the compactification of the universal family* $Y_{g-1} := \Gamma_{g-1}^J(k)\backslash(H_{g-1} \times C^{g-1})$ *of abelian varieties induced from the same* $\Gamma_g(k)$*-admissible family.*

(2) All D_i *intersect along the boundary* $\bar{Y}_{g-1} - Y_{g-1}$*.*

We have several *natural* questions.

Problem 6.2. Describe $\bar{Y}_{g-1}$ and $\bar{Y}_{g-1} - Y_{g-1}$ explicitly in terms of Jacobi forms. More generally,*describe* $\bar{Y}_r$ and $\bar{Y}_r - Y_r$ when $Y_r := \Gamma_r(k) \ltimes H_Z^{(r,k)}\backslash H_r \times C^{(r,k)}$ $(1 \leq r \leq g)$.

Problem 6.3. Describe the field of meromorphic functions on $\bar{Y}_{g-1}$ or $\bar{Y}_r$.

Problem 6.4. Can any $\Gamma_{g-1}^J(k)$-invariant or $\Gamma_r^J(k)$-invariant meromorphic function on Y_{g-1} or Y_r be expressed by a quotient of two Jacobi forms of the same weight and index?

7. The Boundary of the Satake Compactification

Let Γ be a discrete subgroup of $Sp(g, \mathbb{Q})$ which is commensurable with Γ_g. We denote by $M_k(\Gamma)$ the complex vector space consisting of Siegel modular forms of weight k with respect to Γ ($k \in \mathbb{Z}$). These vector spaces generate a positively graded ring

$$M(\Gamma) := \oplus_{k \geq 0} M_k(\Gamma)$$

which are integrally closed and of finite type over $M_0(\Gamma) = \mathbb{C}$. The projective variety $A_{g,\Gamma}^*$ associated with $M(\Gamma)$ contains a Zariski open subset which is complex analytically isomorphic to $A_{g,\Gamma} := \Gamma\backslash H_g$. In addition, the boundary $\partial A_{g,\Gamma}^* := A_{g,\Gamma}^* - A_{g,\Gamma}$ is a disjoint union of a finite number of rational boundary components of H_g.

From now on, we let $\Gamma := \Gamma_g(k)$ be the principal congruence subgroup of Γ_g of level k. We write $g = p + q$ for $0 \leq p < g$. We write an element Z of H_g as

$$\begin{pmatrix} \tau & W \\ {}^tW & T \end{pmatrix}, \quad \tau \in H_p, \ W \in \mathbb{C}^{(p,q)}, \ \ T \in H_q,$$

or simply $Z = (\tau, W, T)$. The Siegel operator $\Phi : M(\Gamma_g(k)) \longrightarrow M(\Gamma_p(k))$ defined by

$$(7.1) \qquad (\Phi f)(\tau) := \lim_{\operatorname{Im} T \to 0} f\left(\begin{pmatrix} \tau & W \\ * & T \end{pmatrix}\right) = \lim_{c \to 0} f\left(\begin{pmatrix} \tau & 0 \\ 0 & icE_q \end{pmatrix}\right)$$

is a weight-preserving homomorphism which is almost surjective in the sense that it is surjective for all large weights. Thus we have a canonical holomorphic embedding $\Phi^* : A^*_{p,\Gamma_p(k)} \longrightarrow A^*_{g,\Gamma_g(k)}$. We can see that the image of $A_{p,\Gamma_p(k)} = \Gamma_p(k)\backslash H_p$ is a quasi-projective subvariety of $A^*_{g,\Gamma_g(k)}$ and that $Sp(g,\, \mathbb{Z}/k\mathbb{Z})$ acts on $A^*_{g,\Gamma_g(k)}$ as automorphisms. $Sp(g,\, \mathbb{Z}/k\mathbb{Z})$ transforms $\Phi^*(A_{p,\Gamma_p(k)})$ to its conjugates. Thus we have

$$\begin{aligned} \partial A^*_{g,\Gamma_g(k)} : \ &= A^*_{g,\Gamma_g(k)} - A_{g,\Gamma_g(k)} \\ &= \textstyle\bigcup_{\gamma \in Sp(g,\, \mathbb{Z}/k\mathbb{Z})} \coprod_{l=0}^{g-1} \gamma \cdot \Phi^*(A_{l,\Gamma_l(k)}) \end{aligned}$$

So in order to investigate the boundary $\partial A^*_{g,\Gamma_g(k)}$, it is enough to investigate the boundary points in the image $\Phi^*(A_{p,\Gamma_p(k)})$ of $A_{p,\Gamma_p(k)} = \Gamma_p(k)\backslash H_p$ under Φ^* for $0 \le p < g$.

Omitting the detail, we state the following results.

Theorem 7.1(Igusa). *Let τ_0 be an element of H_p. Then the analytic local ring $\mathcal{O}$ of $A^*_{g,\Gamma_g(k)}$ at the image point of τ_0 under Φ^* consists of convergent series of the following form*

$$f(\tau, W, T) = \sum_{\mathcal{M}} \left(\sum_u \phi_{\mathcal{M}}(\tau, W\, {}^t u)\, e^{\frac{2\pi i \sigma(\mathcal{M}[u]T)}{k}} \right), \qquad \phi_{\mathcal{M}} \in J_{0,\mathcal{M}}(\Gamma_g(k)),$$

where $\mathcal{M}$ runs over the equivalent classes of inequivalent half-integral semi-positive symmetric matrices of degree q, $\phi_{\mathcal{M}}$ is a holomorphic function defined on $V \times \mathbb{C}^{(q,p)}$ for some open neighborhood V of τ_0 in H_p and u runs over distinct $\mathcal{M}[u]$ for $u \in GL(q, \mathbb{Z})(k)$.

Theorem 7.2(Igusa). *The ideal I in $\mathcal{O}$ associated with the boundary $\partial A^*_{g,\Gamma_g(k)} = A^*_{g,\Gamma_g(k)} - A_{g,\Gamma_g(k)}$ consists of convergent series*

$$\sum_{\mathcal{M}} \left(\sum_u \phi_{\mathcal{M}}(\tau, W\, {}^t u)\, e^{\frac{2\pi i \sigma(\mathcal{M}[u]T)}{k}} \right), \qquad \phi_{\mathcal{M}} \in J_{0,\mathcal{M}}(\Gamma_g(k)),$$

where $\mathcal{M}$ runs over inequivalent symmetric positive definite half-integral matrices of degree q, $\phi_{\mathcal{M}}$ is a holomorphic function defined on $V \times \mathbb{C}^{(q,p)}$ for some open neighborhood V of τ_0 in H_p and u runs over distinct $\mathcal{M}[u]$ for $u \in GL(q, \mathbb{Z})(k)$.

8. Singular Jacobi Forms

In this section, we discuss the notion of singular Jacobi forms. Without loss of generality we may assume that $\mathcal{M}$ is positive definite. For simplicity, we consider the case that Γ is the Siegel modular group Γ_g of degree g.

Let g and h be two positive integers. We recall that $\mathcal{M}$ is a symmetric positive definite, half-integral matrix of degree h. We let

$$\mathcal{P}_g := \{Y \in \mathbb{R}^{(g,g)} \,|\, Y = {}^tY > 0\}$$

be the open convex cone of positive definite matrices of degree g in the Euclidean space $\mathbb{R}^{\frac{g(g+1)}{2}}$. We define the differential operator $M_{g,h,\mathcal{M}}$ on $\mathcal{P}_g \times \mathbb{R}^{(h,g)}$ defined by

$$M_{g,h,\mathcal{M}} := \det(Y) \cdot \det\left(\frac{\partial}{\partial Y} + \frac{1}{8\pi}\,{}^t\!\left(\frac{\partial}{\partial V}\right)\mathcal{M}^{-1}\left(\frac{\partial}{\partial V}\right)\right),$$

where

$$Y = (y_{\mu\nu}) \in \mathcal{P}_g, \quad V = (v_{kl}) \in \mathbb{R}^{(h,g)}, \quad \frac{\partial}{\partial Y} = \left(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial y_{\mu\nu}}\right)$$

and

$$\frac{\partial}{\partial V} = \left(\frac{\partial}{\partial v_{kl}}\right).$$

Yang [85] characterized singular Jacobi forms as follows:

Theorem 8.1. *Let $f \in J_{\rho,\mathcal{M}}(\Gamma_g)$ be a Jacobi form of index $\mathcal{M}$ with respect to a finite dimensional rational representation ρ of $GL(g,\mathbb{C})$. Then the following conditions are equivalent :*
(1) f is a singular Jacobi form.
(2) f satisfies the differential equation $M_{g,h,\mathcal{M}}f = 0$.

Theorem 8.2. *Let ρ be an irreducible finite dimensional representation of $GL(g,\mathbb{C})$. Then there exists a nonvanishing singular Jacobi form in $J_{\rho,\mathcal{M}}(\Gamma_g)$ if and only if $2k(\rho) < g+h$. Here $k(\rho)$ denotes the weight of ρ.*

For the proofs of the above theorems we refer to [85], Theorem 4.1 and Theorem 4.5.

Exercise 8.3. Compute the eigenfunctions and the eigenvalues of $M_{g,h,\mathcal{M}}$(cf. [85], pp. 2048-2049).

Now we consider the following group $GL(g,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$ equipped with the multiplication law

$$\begin{aligned}&(A,(\lambda,\mu,\kappa)) * (B,(\lambda',\mu',\kappa'))\\ &= (AB,\,(\lambda B+\lambda',\,\mu\,{}^tB^{-1}+\mu',\,\kappa+\kappa'+\lambda B\,{}^t\mu'-\mu\,{}^tB^{-1}\,{}^t\lambda')),\end{aligned}$$

where $A, B \in GL(g,\mathbb{R})$ and $(\lambda,\mu,\kappa), (\lambda',\mu',\kappa') \in H_{\mathbb{R}}^{(g,h)}$. We observe that $GL(g,\mathbb{R})$ acts on $H_{\mathbb{R}}^{(g,h)}$ on the right as automorphisms. And we have the canonical action of

$GL(g,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$ on $\mathcal{P}_g \times \mathbb{R}^{(h,g)}$ defined by

$$(A,(\lambda,\mu,\kappa)) \circ (Y,V) := (AY\,{}^tA,\ (V+\lambda Y+\mu)\,{}^tA), \tag{8.1}$$

where $A \in GL(g,\mathbb{R}), (\lambda,\mu,\kappa) \in H_{\mathbb{R}}^{(g,h)}$ and $(Y,V) \in \mathcal{P}_g \times \mathbb{R}^{(h,g)}$.

Lemma 8.4. *$M_{g,h,\mathcal{M}}$ is invariant under the action of $GL(g,\mathbb{R}) \ltimes \{\, (0,\mu,0)\, | mu \in \mathbb{R}^{(h,g)}\, \}$.*

Proof. It follows immediately from the direct calculation.

We have the following natural questions.

Problem 8.5. Develope the invariant theory for the action of $GL(g,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$ on $\mathcal{P}_g \times \mathbb{R}^{(h,g)}$.

Problem 8.6. Discuss the application of the theory of singular Jacobi forms to the geometry of the universal abelian variety as that of singular modular forms to the geometry of the Siegel modular variety (see Appendix B).

9. The Siegel-Jacobi Operator

Let ρ and $\mathcal{M}$ be the same as in the previous sections. For positive integers r and g with $r < g$, we let $\rho^{(r)} : GL(r,\mathbb{C}) \longrightarrow GL(V_\rho)$ be a rational representation of $GL(r,\mathbb{C})$ defined by

$$\rho^{(r)}(a)v := \rho\left(\begin{pmatrix} a & 0 \\ 0 & E_{g-r} \end{pmatrix} \right) v, \quad a \in GL(r,\mathbb{C}), \quad v \in V_\rho.$$

The Siegel-Jacobi operator $\Psi_{g,r} : J_{\rho,\mathcal{M}}(\Gamma_g) \longrightarrow J_{\rho^{(r)},\mathcal{M}}(\Gamma_r)$ is defined by

$$(\Psi_{g,r}f)(Z,W) := \lim_{t\to\infty} f\left(\begin{pmatrix} Z & 0 \\ 0 & itE_{g-r} \end{pmatrix}, (W,0) \right), \tag{9.1}$$

where $f \in J_{\rho,\mathcal{M}}(\Gamma_g),\ Z \in H_r$ and $W \in \mathbb{C}^{(h,r)}$. It is easy to check that the above limit always exists and the Siegel-Jacobi operator is a linear mapping. Let $V_\rho^{(r)}$ be the subspace of V_ρ spanned by the values $\{\, (\Psi_{g,r}f)(Z,W)\, |\, f \in J_{\rho,\mathcal{M}}(\Gamma_g),\ (Z,W) \in H_r \times \mathbb{C}^{(h,r)}\, \}$. Then $V_\rho^{(r)}$ is invariant under the action of the group

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & E_{g-r} \end{pmatrix} \ :\ a \in GL(r,\mathbb{C}) \right\} \cong GL(r,\mathbb{C}).$$

We can show that if $V_\rho^{(r)} \neq 0$ and (ρ, V_ρ) is irreducible, then $(\rho^{(r)}, V_\rho^{(r)})$ is also irreducible.

Theorem 9.1. *The action of the Siegel-Jacobi operator is compatible with that of that of the Hecke operator.*

We refer to [83] for a precise detail on the Hecke operators and the proof of the above theorem.

Problem 9.2. Discuss the injectivity, surjectivity and bijectivity of the Siegel-Jacobi operator.

This problem was partially discussed by Yang [83] and Kramer [40] in the special cases. For instance, Kramer [40] showed that if g is arbitrary, $h = 1$ and $\rho : GL(g, \mathbb{C}) \longrightarrow \mathbb{C}^\times$ is a one-dimensional representation of $GL(g, \mathbb{C})$ defined by $\rho(a) := (\det(a))^k$ for some $k \in \mathbb{Z}^+$, then the Siegel-Jacobi operator

$$\Psi_{g,g-1} : J_{k,m}(\Gamma_g) \longrightarrow J_{k,m}(\Gamma_{g-1})$$

is surjective for $k \gg m \gg 0$.

Theorem 9.3. *Let $1 \leq r \leq g-1$ and let ρ be an irreducible finite dimensional representation of $GL(g, \mathbb{C})$. Assume that $k(\rho) > g + r + rank(\mathcal{M}) + 1$ and that k is even. Then*

$$J^{cusp}_{\rho^{(r)},\mathcal{M}}(\Gamma_r) \subset \Psi_{g,r}(J_{\rho,\mathcal{M}}(\Gamma_g)).$$

Here $J^{cusp}_{\rho^{(r)},\mathcal{M}}(\Gamma_r)$ denotes the subspace consisting of all cuspidal Jacobi forms in $J_{\rho^{(r)},\mathcal{M}}(\Gamma_r)$.

Idea of Proof. For each $f \in J^{\rm cusp}_{\rho^{(r)},\mathcal{M}}(\Gamma_r)$, we can show by a direct computation that

$$\Psi_{g,r}(E^{(g)}_{\rho,\mathcal{M}}(Z, W; f)) = f,$$

where $E^{(g)}_{\rho,\mathcal{M}}(Z, W; f)$ is the Eisenstein series of Klingen's type associated with a cusp form f. For a precise detail, we refer to [94].

Remark 9.4. Dulinski [17] decomposed the vector space $J_{k,\mathcal{M}}(\Gamma_g)\,(k \in \mathbb{Z}^+)$ into a direct sum of certain subspaces by calculating the action of the Siegel-Jacobi operator on Eisenstein series of Klingen's type explicitly.

For two positive integers r and g with $r \leq g-1$, we consider the bigraded ring

$$J^{(r)}_{*,*}(\ell) := \oplus_{k=0}^{\infty} \oplus_{\mathcal{M}} J_{k,\mathcal{M}}(\Gamma_r(\ell))$$

and

$$M^{(r)}_{*}(\ell) := \oplus_{k=0}^{\infty} J_{k,0}(\Gamma_r(\ell)) = \oplus_{k=0}^{\infty}[\Gamma_r(\ell), k],$$

where $\Gamma_r(\ell)$ denotes the principal congruence subgroup of Γ_r of level ℓ and $\mathcal{M}$ runs over the set of all symmetric semi-positive half-integral matrices of degree h. Let

$$\Psi_{r,r-1,\ell} : J_{k,\mathcal{M}}(\Gamma_r(\ell)) \longrightarrow J_{k,\mathcal{M}}(\Gamma_{r-1}(\ell))$$

be the Siegel-Jacobi operator defined by (9.1).

Problem 9.5. Investigate Proj $J^{(r)}_{*,*}(\ell)$ over $M^{(r)}_{*}(\ell)$ and the quotient space

$$Y_r(\ell) := (\Gamma_r(\ell) \ltimes (\ell\mathbb{Z})^2)\backslash(H_r \ltimes \mathbb{C}^r)$$

for $1 \le r \le g-1$.

The difficulty to this problem comes from the following facts (A) and (B):

(A) $J^{(r)}_{*,*}(\ell)$ is not finitely generated over $M^{(r)}_*(\ell)$.

(B) $J^{\rm cusp}_{k,\mathcal{M}}(\Gamma_r(\ell)) \neq \ker \Psi_{r,r-1,\ell}$ in general.

These are the facts different from the theory of Siegel modular forms. We remark that Runge([54], pp. 190-194) discussed some parts about the above problem.

10. Invariant Metrics on the Siegel-Jacobi Space

For a brevity, we write $H_{g,h} := H_g \times \mathbb{C}^{(h,g)}$. For a coordinate $(Z,W) \in H_{g,h}$ with $Z = (z_{\mu\nu}) \in H_g$ and $W = (w_{kl}) \in \mathbb{C}^{(h,g)}$, we put

$$\begin{aligned} Z &= X + iY, & X &= (x_{\mu\nu}), & Y &= (y_{\mu\nu}) \ \text{ real},\\ W &= U + iV, & U &= (u_{kl}), & V &= (v_{kl}) \ \text{ real},\\ dZ &= (dz_{\mu\nu}), & dX &= (dx_{\mu\nu}), & dY &= (dy_{\mu\nu}),\\ dW &= (dw_{kl}), & dU &= (du_{kl}), & dV &= (dv_{kl}), \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial Z} &= \left(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial z_{\mu\nu}}\right), & \frac{\partial}{\partial \overline{Z}} &= \left(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial \overline{z}_{\mu\nu}}\right),\\ \frac{\partial}{\partial X} &= \left(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial x_{\mu\nu}}\right), & \frac{\partial}{\partial Y} &= \left(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial y_{\mu\nu}}\right), \end{aligned}$$

$$\frac{\partial}{\partial W} := \begin{pmatrix} \frac{\partial}{\partial w_{11}} & \cdots & \frac{\partial}{\partial w_{h1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial w_{1g}} & \cdots & \frac{\partial}{\partial w_{hg}} \end{pmatrix}, \quad \frac{\partial}{\partial \overline{W}} := \begin{pmatrix} \frac{\partial}{\partial \overline{w}_{11}} & \cdots & \frac{\partial}{\partial \overline{w}_{h1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \overline{w}_{1g}} & \cdots & \frac{\partial}{\partial \overline{w}_{hg}} \end{pmatrix},$$

$$\frac{\partial}{\partial U} := \begin{pmatrix} \frac{\partial}{\partial u_{11}} & \cdots & \frac{\partial}{\partial u_{h1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial u_{1g}} & \cdots & \frac{\partial}{\partial u_{hg}} \end{pmatrix}, \quad \frac{\partial}{\partial V} := \begin{pmatrix} \frac{\partial}{\partial v_{11}} & \cdots & \frac{\partial}{\partial v_{h1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial v_{1g}} & \cdots & \frac{\partial}{\partial v_{hg}} \end{pmatrix}.$$

We let

$$T_g := \left\{ z \in \mathbb{C}^{(g,g)} \mid z = {}^t z \right\}$$

be the vector space of all $g \times g$ complex *symmetric* matrices. The unitary group $K := U(g)$ of degree g acts on the complex vector space $T_g \times \mathbb{C}^{(h,g)}$ by

(10.1) $$k \cdot (z,w) := (\, k\, z\, {}^t k,\ w\, {}^t k\,), \qquad k \in U(g),\ z \in T_g,\ w \in \mathbb{C}^{(h,g)}.$$

Then this action induces naturally the action ρ of $U(g)$ on the polynomial algebra $\mathrm{Pol}_{h,g} := \mathrm{Pol}\,(T_g \times \mathbb{C}^{(h,g)})$. We denote by $\mathrm{Pol}^K_{h,g}$ the subalgebra of $\mathrm{Pol}_{h,g}$ consisting of all K-invariants of the action ρ of $K := U(g)$. We also denote by $\mathbb{D}(H_{g,h})$ the algebra of all differential operators on $H_{g,h}$ which is invariant under the action (2.1) of the Jacobi group $G^J_{g,h}$. Then we can show that there exists a natural linear bijection

(10.2) $$\Phi : \mathrm{Pol}^K_{h,g} \longrightarrow \mathbb{D}(H_{g,h})$$

of $\mathrm{Pol}_{h,g}^K$ onto $\mathbb{D}(H_{g,h})$.

Theorem 10.1. *The algebra $\mathbb{D}(H_{g,h})$ is generated by the images under the mapping Φ of the following invariants*

$$\text{(I1)} \qquad p_j(z,w) := \sigma((z\bar{z})^j), \qquad 1 \le j \le g,$$

$$\text{(I2)} \qquad \psi_k^{(1)}(z,w) := (w\,{}^t\bar{w})_{kk}, \qquad 1 \le k \le h,$$

$$\text{(I3)} \qquad \psi_{kp}^{(2)}(z,w) := \mathrm{Re}(w\,{}^t\bar{w})_{kp}, \qquad 1 \le k < p \le h,$$

$$\text{(I4)} \qquad \psi_{kp}^{(2)}(z,w) := \mathrm{Im}(w\,{}^t\bar{w})_{kp}, \qquad 1 \le k < p \le h,$$

$$\text{(I5)} \qquad f_{kp}^{(1)}(z,w) := \mathrm{Re}(w\bar{z}\,{}^t w)_{kp}, \qquad 1 \le k \le p \le h$$

and

$$\text{(I6)} \qquad f_{kp}^{(2)}(z,w) := \mathrm{Im}(w\bar{z}\,{}^t w)_{kp}, \qquad 1 \le k \le p \le h.$$

In particular, $\mathbb{D}(H_{g,h})$ is not commutative.

Theorem 10.1'. *The algebra $\mathbb{D}(H_{1,1})$ is generated by the following differential operators*

$$\begin{aligned} D := \;& y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) \\ & + 2\,y\,v\left(\frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v}\right), \end{aligned}$$

$$\Psi := y\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right),$$

$$D_1 := 2\,y^2\,\frac{\partial^3}{\partial x \partial u \partial v} - y^2\,\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}\right) + \left(v\,\frac{\partial}{\partial v} + 1\right)\Psi$$

and

$$D_2 := y^2\,\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial v^2} - \frac{\partial^2}{\partial u^2}\right) - 2\,y^2\,\frac{\partial^3}{\partial y \partial u \partial v} - v\,\frac{\partial}{\partial u}\Psi,$$

where $\tau = x + iy$ and $z = u + iv$ with real variables x, y, u, v. Moreover, we have

$$\begin{aligned} [D,\,\Psi] := D\Psi - \Psi D = \;& 2\,y^2\,\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}\right) - 4\,y^2\,\frac{\partial^3}{\partial x \partial u \partial v} \\ & - 2\left(v\,\frac{\partial}{\partial v}\Psi + \Psi\right). \end{aligned}$$

In particular, the algebra $\mathbb{D}(H_{1,1})$ is not commutative.

Theorem 10.2. *The following metric*

$$\begin{aligned} ds_{g,h}^2 := \;& \sigma\left(Y^{-1}dZ\,Y^{-1}d\overline{Z}\right) + \sigma\left(Y^{-1}\,{}^tV\,V\,Y^{-1}dZ\,Y^{-1}d\overline{Z}\right) \\ & + \sigma\left(Y^{-1}\,{}^t(dW)\,d\overline{W}\right) \\ & \sigma\left(Y^{-1}dZ\,Y^{-1}\,{}^t(d\overline{W})\,V + Y^{-1}d\overline{Z}\,Y^{-1}\,{}^t(dW)\,V\right) \end{aligned} \tag{10.3}$$

is a Riemannian metric on the Siegel-Jacobi space $H_{g,h}$ which is invariant under the action (1.2) of the Jacobi group $G^J_{g,h}$. Also the above metric is a Kähler metric. The Laplace-Beltrami operator $\Delta_{g,h}$ of the Siegel-Jacobi space $(H_{g,h},\, ds^2_{g,h})$ is given by

$$\begin{aligned}\Delta_{h,g} = \quad & 4\,\sigma\left(Y\,\tfrac{\partial}{\partial Z}\,Y\,\tfrac{\partial}{\partial \overline{Z}}\right) + 4\,\sigma\left(Y\,\tfrac{\partial}{\partial W}\,{}^t\left(\tfrac{\partial}{\partial \overline{W}}\right)\right)\\ & +4\,\sigma\left(\tfrac{\partial}{\partial W}\,V\,\tfrac{\partial}{\partial \overline{W}}\,V\right)\\ & +4\,\sigma\left(\tfrac{\partial}{\partial Z}\,Y\,\tfrac{\partial}{\partial \overline{W}}\,V + \tfrac{\partial}{\partial \overline{Z}}\,Y\,\tfrac{\partial}{\partial W}\,V\right).\end{aligned} \tag{10.4}$$

The following differential form

$$dv := (\,detY\,)^{-(g+h+1)}\,[dX]\wedge[dY]\wedge[dU]\wedge[dV]$$

is a $G^J_{g,h}$-invariant volume element on $H_{g,h}$, where

$$[dX] := \wedge_{\mu\le\nu}dx_{\mu\nu},\quad [dY] := \wedge_{\mu\le\nu}dy_{\mu\nu},\quad [dU] := \wedge_{k,l}du_{kl}\quad \text{and}\quad [dV] := \wedge_{k,l}dv_{k,l}.$$

Theorem 10.3. *The automorphism group of $H_{g,h}$ is isomorphic to the group $Sp(g,\mathbb{R})\ltimes\left(\mathbb{R}^{(h,g)}\times\mathbb{R}^{(h,g)}\right)$ equipped with the multiplication*

$$(M,(\lambda,\mu))\cdot(M',(\lambda',\mu)) := (MM',\,(\tilde\lambda+\lambda',\tilde\mu+\mu')),$$

where $M,M'\in Sp(g,\mathbb{R})$, $\lambda,\mu\in\mathbb{R}^{(h,g)}$ and $(\tilde\lambda,\tilde\mu) := (\lambda,\mu)M'$.

Theorem 10.4. *The scalar curvature of the Siegel-Jacobi space $(H_1\times\mathbb{C},\,ds^2)$ is -3.*

We note that according to Theorem 2, the metric ds^2 is given by

$$\begin{aligned} ds^2 := ds^2_{1,1} = \quad & \tfrac{y+v^2}{y^3}\,(dx^2+dy^2) + \tfrac{1}{y}\,(du^2+dv^2)\\ & -\tfrac{2v}{y^2}\,(dxdu+dydv)\end{aligned} \tag{10.5}$$

on $H_1\times\mathbb{C}$ which is invariant under the action (2.1) of the Jacobi group $G^J_{1,1} = SL(2,\mathbb{R})\ltimes H^{(1,1)}_{\mathbb{R}}$, where $z=x+iy\in H_1$ and $w=u+iv\in\mathbb{C}$ with x,y,u,v real coordinates.

Remark 10.5. The Poincaré upper half plane H_1 is a two dimensional Riemannian manifold with the Poincaré metric

$$ds^2_0 := \frac{dx^2\,+\,dy^2}{y^2},\quad z=x+iy\in H_1 \text{ with } x,y \text{ real}.$$

It is easy to see that the Gaussian curvature is -1 everywhere and H_1 is an *Einstein manifold.* In fact, if we denote by $S_0(X,Y)$ the Ricci curvature of $(H_1,\,ds^2_0)$, then we have

$$S_0(X,Y) = -g_0(X,Y)\quad \text{for all } X,Y\in\mathcal{X}(H_1),$$

where $\mathcal{X}(H_1)$ denotes the algebra of all smooth vector fields on H_1 and $g_0(X,Y)$ is the inner product on the tangent bundle $T(H_1)$ induced by the Poincaré metric ds_0^2. But the Siegel-Jacobi space $H_1 \times \mathbb{C}$ is *not* an Einstein manifold. Indeed, if we denote by $S(X,Y)$ the Ricci curvature of $(H_1 \times \mathbb{C},\, ds^2)$ and $E_1 := \frac{\partial}{\partial x}$, we can see without difficulty that there does not exist a constant c such that

$$S(E_1,E_1) \,=\, c\,g(E_1,E_1) \,=\, c\,g_{11} \,=\, c\,\frac{y+v^2}{y^3},.$$

where $g=(g_{ij})$ is the inner product on the tangent bundle $T(H_1 \times \mathbb{C})$ induced by the metric (10.5).

Now we will introduce the notion of *Maass-Jacobi forms.*

Definition 10.6. A smooth function $f : H_{g,h} \longrightarrow \mathbb{C}$ is called a *Maass-Jacobi form* on $H_{g,h}$ if f satisfies the following conditions (MJ1)-(MJ3) :

(MJ1) f is invariant under $\Gamma_{g,h}^J := \Gamma_g \ltimes H_Z^{(g,h)}$.

(MJ2) f is an eigenfunction of the Lapalce-Beltrami operator $\Delta_{n,m}$.

(MJ3) f has a polynomial growth.

Here $\Gamma_g := Sp(g,Z)$ denotes the Siegel modular group of degree g and and

$$H_Z^{(g,h)} := \Big\{ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(g,h)} \mid \lambda,\mu,\kappa \text{ integral} \Big\}.$$

For more details on Maass-Jacobi forms in the case $g=h=1$, we refer to [89].

11. Final Remarks

In [32] and [34], Gritsenko, Hulek and Sankaran gave applications of Jacobi forms of degree 1 in the study of the moduli space of abelian surfaces with a certain polarization. We refer to [7],[9],[11],[61],[62] for the representation theory of the Jacobi group.

Appendix A. Subvarieties of the Siegel Modular Variety

Here we assume that the ground field is the complex number field $\mathbb{C}$.

Definition A.1. A nonsingular variety X is said to be *rational* if X is birational to a projective space $P^n(\mathbb{C})$ for some integer n. A nonsingular variety X is said to be *stably rational* if $X \times P^k(\mathbb{C})$ is birational to $P^N(\mathbb{C})$ for certain nonnegative integers k and N. A nonsingular variety X is called *unirational* if there exists a dominant rational map $\varphi : P^n(\mathbb{C}) \longrightarrow X$ for a certain positive integer n, equivalently if the function field $\mathbb{C}(X)$ of X can be embedded in a purely transcendental extension $\mathbb{C}(z_1,\cdots,z_n)$ of $\mathbb{C}$.

Remarks A.2. (1) It is easy to see that the rationality implies the stably rationality and that the stably rationality implies the unirationality.

(2) If X is a Riemann surface or a complex surface, then the notions of rationality, stably rationality and unirationality are equivalent one another.

(3) Griffiths and Clemens(cf. Ann. of Math. 95(1972), 281-356) showed that most of cubic threefolds in $P^4(\mathbb{C})$ are unirational but *not* rational.

The following natural questions arise :

QUESTION 1. Is a stably rational variety *rational*? Indeed, the question was raised by Bogomolov.

QUESTION 2. Is a general hypersurface $X \subset P^{n+1}(\mathbb{C})$ of degree $d \leq n+1$ *unirational*?

Definition A.3. Let X be a nonsingular variety of dimension n and let K_X be the canonical divisor of X. For each positive integer $m \in \mathbb{Z}^+$, we define the *m-genus* $P_m(X)$ of X by

$$P_m(X) := \dim_{\mathbb{C}} H^0(X, \mathcal{O}(mK_X)).$$

The number $p_g(X) := P_1(X)$ is called the *geometric genus* of X. We let

$$N(X) := \left\{ m \in \mathbb{Z}^+ \mid P_m(X) \geq 1 \right\}.$$

For the present, we assume that $N(X)$ is nonempty. For each $m \in N(X)$, we let $\{\phi_0, \cdots, \phi_{N_m}\}$ be a basis of the vector space $H^0(X, \mathcal{O}(mK_X))$. Then we have the mapping $\Phi_{mK_X} : X \longrightarrow P^{N_m}(\mathbb{C})$ by

$$\Phi_{mK_X}(z) := (\phi_0(z) : \cdots : \phi_{N^m}(z)), \quad z \in X.$$

We define the *Kodaira dimension* $\kappa(X)$ of X by

$$\kappa(X) := \max \left\{ \dim_{\mathbb{C}} \Phi_{mK_X}(X) \mid m \in N(X) \right\}.$$

If $N(X)$ is empty, we put $\kappa(X) := -\infty$. Obviously $\kappa(X) \leq \dim_{\mathbb{C}} X$. A nonsingular variety X is said to be *of general type* if $\kappa(X) = \dim_{\mathbb{C}} X$. A singular variety Y in general is said to be rational, stably rational, unirational or of general type if any nonsingular model X of Y is rational, stably rational, unirational or of general type respectively. We define

$$P_m(Y) := P_m(X) \quad \text{and} \quad \kappa(Y) := \kappa(X).$$

A variety Y of dimension n is said to be *of logarithmic general type* if there exists a smooth compactification $\tilde{Y}$ of Y such that $D := \tilde{Y} - Y$ is a divisor with normal crossings only and the transcendence degree of the logarithmic canonical ring

$$\oplus_{m=0}^{\infty} H^0(\tilde{Y}, \, m(K_{\tilde{Y}} + [D]))$$

is $n+1$, i.e., the *logarithmic Kodaira dimension* of Y is n. We observe that the notion of being of logarithmic general type is weaker than that of being of general type.

Let $A_g := \Gamma_g \backslash H_g$ be the Siegel modular variety of degree g, that is, the moduli space of principally polarized abelian varieties of dimension g. So far it has been proved that A_g is of general type for $g \geq 7$. At first Freitag [24] proved this fact when g is a multiple of 24. Tai [60] proved this for $g \geq 9$ and Mumford [46] proved this fact for $g \geq 7$. On the other hand, A_g is known to be unirational for $g \leq 5$: Donagi [16] for $g = 5$, Clemens [15] for $g = 4$ and classical for $g \leq 3$. For $g = 3$, using the moduli theory of curves, Riemann [53], Weber [65] and Frobenius [28] showed that $A_3(2) := \Gamma_3(2)\backslash H_3$ is a rational variety and moreover gave 6 generators of the modular function field $K(\Gamma_3(2))$ written explicitly in terms of derivatives of odd theta functions at the origin. So A_3 is a unirational variety with a Galois covering of a rational variety of degree $[\Gamma_3 : \Gamma_3(2)] = 1,451,520$. Here $\Gamma_3(2)$ denotes the principal congruence subgroup of Γ_3 of level 2. Furthermore it was shown that A_3 is stably rational(cf. [38], [12]). For a positive integer k, we let $\Gamma_g(k)$ be the principal congruence subgroup of Γ_g of level k. Let $A_g(k)$ be the moduli space of abelian varieties of dimension g with k-level structure. It is classically known that $A_g(k)$ is of logarithmic general type for $k \geq 3$(cf. [45]). Wang [64] proved that $A_2(k)$ is of general type for $k \geq 4$. On the other hand, van der Geer [29] showed that $A_2(3)$ is rational. The remaining unsolved problems are summarized as follows:

Problem 1. Is A_3 rational?

Problem 2. Are $A_4,\ A_5$ stably rational or rational?

Problem 3. Discuss the (uni)rationality of A_6.

Problem 4. What type of varieties are $A_g(k)$ for $g \geq 3$ and $k \geq 2$?

We already mentioned that A_g is of general type if $g \geq 7$. It is natural to ask if the subvarieties of A_g ($g \geq 7$) are of general type, in particular the subvarieties of A_g of codimension one. Freitag [Fr3] showed that there exists a certain bound g_0 such that for $g \geq g_0$, each irreducible subvariety of A_g of codimension one is of general type. Weissauer [Wei2] proved that every irreducible divisor of A_g is of general type for $g \geq 10$. Moreover he proved that every subvariety of codimension $\leq g - 13$ in A_g is of general type for $g \geq 13$. We observe that the smallest known codimension for which there exist subvarieties of A_g for large g which are not of general type is $g - 1$. $A_1 \times A_{g-1}$ is a subvariety of A_g of codimension $g - 1$ which is not of general type.

Remark A.4. Let $\mathcal{M}_g$ be the coarse moduli space of curves of genus g over $\mathbb{C}$. Then $\mathcal{M}_g$ is an analytic subvariety of A_g of dimension $3g - 3$. It is known that $\mathcal{M}_g$ is unirational for $g \leq 10$. So the Kodaira dimension $\kappa(\mathcal{M}_g)$ of $\mathcal{M}_g$ is $-\infty$ for $g \leq 10$. Harris and Mumford [H-M] proved that $\mathcal{M}_g$ is of general type for odd g

with $g \geq 25$ and $\kappa(\mathcal{M}_{23}) \geq 0$.

Appendix B. Singular Modular Forms

Let ρ be a rational representation of $GL(g, \mathbb{C})$ on a finite dimensional complex vector space V_ρ. A holomorphic function $f : H_g \longrightarrow V_\rho$ with values in V_ρ is called a modular form of type ρ if it satisfies

$$f(M < Z >) = \rho(CZ + D) f(Z)$$

for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$ and $Z \in H_g$. We denote by $[\Gamma_g, \rho]$ the vector space of all modular forms of type ρ. A modular form $f \in [\Gamma_g, \rho]$ of type ρ has a Fourier series

$$f(Z) = \sum_{T \geq 0} a(T) e^{2\pi i (TZ)}, \quad Z \in H_g,$$

where T runs over the set of all semipositive half-integral symmetric matrices of degree g. A modular form f of type ρ is said to be *singular* if a Fourier coefficient $a(T)$ vanishes unless $\det(T) = 0$.

Freitag [25] proved that every singular modular form can be written as a finite linear combination of theta series with harmonic coefficients and proposed the problem to characterize singular modular forms. Weissauer [66] gave the following criterion.

Theorem B.1. *Let ρ be an irreducible rational representation of $GL(g, \mathbb{C})$ with its highest weight $(\lambda_1, \cdots, \lambda_g)$. Let f be a modular form of type ρ. Then the following are equivalent:*

(a) f is singular.

(b) $2\lambda_g < g$.

Now we describe how the concept of singular modular forms is closely related to the geometry of the Siegel modular variety. Let X be the Satake compactification of the Siegel modular variety $A_g = \Gamma_g \backslash H_g$. Then A_g is embedded in X as a quasiprojective algebraic subvariety of codimension g. Let X_s be the smooth part of A_g and $\tilde{X}$ the desingularization of X. Without loss of generality, we assume $X_s \subset \tilde{X}$. Let $\Omega^p(\tilde{X})$ (resp. $\Omega^p(X_s)$) be the space of holomorphic p-form on $\tilde{X}$ (resp. X_s). Freitag and Pommerening [27] showed that if $g > 1$, then the restriction map

$$\Omega^p(\tilde{X}) \longrightarrow \Omega^p(X_s)$$

is an isomorphism for $p < \dim_{\mathbb{C}} \tilde{X} = \frac{g(g+1)}{2}$. Since the singular part of A_g is at least codimension 2 for $g > 1$, we have an isomorphism

$$\Omega^p(\tilde{X}) \cong \Omega^p(H_g)^{\Gamma_g}.$$

Here $\Omega^p(H_g)^{\Gamma_g}$ denotes the space of Γ_g-invariant holomorphic p-forms on H_g. Let $\mathrm{Sym}^2(\mathbb{C}^g)$ be the symmetric power of the canonical representation of $GL(g,\mathbb{C})$ on $\mathbb{C}^n$. Then we have an isomorphism

$$\Omega^p(H_g)^{\Gamma_g} \longrightarrow [\Gamma_g, \wedge^p \mathrm{Sym}^2(\mathbb{C}^g)].$$

Theorem B.2([66]). *Let ρ_α be the irreducible representation of $GL(g,\mathbb{C})$ with highest weight*

$$(g+1,\cdots,g+1,g-\alpha,\cdots,g-\alpha)$$

such that corank$(\rho_\alpha)=\alpha$ for $1\leq\alpha\leq g$. If $\alpha=-1$, we let $\rho_\alpha=(g+1,\cdots,g+1)$. Then

$$\Omega^p(H_g)^{\Gamma_g}=\begin{cases}[\Gamma_g,\rho_\alpha], & \textit{if } p=\frac{g(g+1)}{2}-\frac{\alpha(\alpha+1)}{2}\\ 0, & \textit{otherwise.}\end{cases}$$

Remark B.3. *If $2\alpha > g$, then any $f\in[\Gamma_g,\rho_\alpha]$ is singular. Thus if $p<\frac{g(3g+2)}{8}$, then any Γ_g-invariant holomorphic p-form on H_g can be expressed in terms of vector valued theta series with harmonic coefficients. It can be shown with a suitable modification that the just mentioned statement holds for a sufficiently small congruence subgroup of Γ_g.*

Thus the natural question is to ask how to determine the Γ_g-invariant holomorphic p-forms on H_g for the nonsingular range $\dfrac{g(3g+2)}{8}\leq p\leq\dfrac{g(g+1)}{2}$. Weissauer [68] answered the above question for $g=2$. For $g>2$, the above question is still open. It is well know that the vector space of vector valued modular forms of type ρ is finite dimensional. The computation or the estimate of the dimension of $\Omega^p(H_g)^{\Gamma_g}$ is interesting because its dimension is finite even though the quotient space A_g is noncompact.

Finally we will mention the results due to Weisauer [67]. We let Γ be a congruence subgroup of Γ_2. According to Theorem B.2, Γ-invariant holomorphic forms in $\Omega^2(H_2)^\Gamma$ are corresponded to modular forms of type (3,1). We note that these invariant holomorphic 2-forms are contained in the *nonsingular range*. And if these modular forms are not cusp forms, they are mapped under the Siegel Φ-operator to cusp forms of weight 3 with respect to some congruence subgroup (dependent on Γ) of the elliptic modular group. Since there are finitely many cusps, it is easy to deal with these modular forms in the adelic version. Observing these facts, he showed that any 2-holomorphic form on $\Gamma\backslash H_2$ can be expressed in terms of theta series with harmonic coefficients associated to binary positive definite quadratic forms. Moreover he showed that $H^2(\Gamma\backslash H_2,\mathbb{C})$ has a pure Hodge structure and that the Tate conjecture holds for a suitable compactification of $\Gamma\backslash H_2$. If $g\geq 3$, for a congruence subgroup Γ of Γ_g it is difficult to compute the cohomology groups $H^*(\Gamma\backslash H_g,\mathbb{C})$ because $\Gamma\backslash H_g$ is noncompact and highly singular. Therefore in order to study their structure, it is natural to ask if they have pure Hodge structures or mixed Hodge structures.

References

[1] A. N. Andrianov ,*Modular descent and the Saito-Kurokawa conjecture* , Invent. Math., **53**(1979), 267-280.

[2] T. Arakawa and B. Heim ,*Real analytic Jacobi Eisenstein series and Dirichlet series attached to three Jacobi forms*, preprint. (MPI 98-66)

[3] A. Ash, D. Mumford, M. Rapport and Y.-S. Tai, Smooth compactification of locally symmetric varieties, Math. Sci. Press, Brookline, (1975).

[4] R. Berndt, *Zur Arithmetik der elliptischen Funktionenkörper höherer Stufe*, J. reine angew. Math., **326**(1981), 79-94.

[5] R. Bernet, *Meromorphic Funktionen auf Mumfords Kompaktifizierung der universellen elliptischen Kurve N-ter Stufe*, J. reine angew. Math., **326**(1981), 95-103.

[6] R. Bernet, *Shimuras Reziprozitätsgesetz für den Körper der arithmetischen elliptischen Funktionen beliebiger Stufe* , J. reine angew. Math., **343**(1983), 123-145.

[7] R. Bernet, *Die Jacobigruppe und die Wärmeleitungsgleichung* , Math. Z., **191**(1986), 351-361.

[8] R. Bernet, *Some remarks on automorphic forms for the Jacobi group* , IHES/M, (1989).

[9] R. Bernet, *The Continuous Part of* $L^2(\Gamma^J\backslash G^J)$ *for the Jacobi Group*, Abh. Math. Sem. Univ. Hamburg., **60**(1990), 225-248.

[10] R. Bernet, *On Automorphic Forms for the Jacobi Group* , Jb. d. Dt. Math.-Verein., **97**(1995), 1-18.

[11] R. Berndt and S. Böcherer, *Jacobi Forms and Discrete Series Representations of the Jacobi Group*, Math. Z., **204**(1990), 13-44.

[12] F. A. Bogomolov and P. I. Katsylo, *Rationality of some quotient varieties*, Math. USSR Sbornik., **54**(1986), 571-576.

[13] D. Bump, S. Friedberg and J. Hoffstein, *Nonvanishing Theorems for L-functions of Modular Forms and Their Derivatives*, Ann. of Math., **131**(1990), 53-128.

[14] J. L. Cardy, *Operator Content of Two-dimensional Conformally Invariant Theories*, Nuclear Physics B, **270**(1986), 186-204.

[15] H. Clemens, *Double Solids*, Adv. Math., **47**(1983), 107-230.

[16] R. Donagi, *The unirationality of* A_5, Ann. of Math., **119**(1984), 269-307.

[17] J. Dulinski, *A decomposition theorem for Jacobi forms*, Math. Ann., **303**(1995), 473-498.

[18] M. Eichler, *Die Spur der Hecke Operatoren in gewissen Räumen von Jacobischen Modulformen*, Abh. Math. Sem. Univ. Hamburg, **54**(1984), 35-48.

[19] M. Eichler, *Eine neue Klasse von Modulformen und Modulfunktionen*, Abh. Math. Sem. Univ. Hamburg, **55**(1985), 53-68.

[20] M. Eichler, *Eine neue Klasse von Modulformen und Modulfunktionen*, Abh. Math. Sem. Univ. Hamburg, **57**(1987), 57-68.

[21] M. Eichler and D. Zagier, The Theory of Jacobi Forms, Progress in Math., **55**, Birkhäuser, Boston-Basel-Stuttgart, (1985).

[22] G. Faltings and C.-L. Chai, Degeneration of Abelian Varieties, Springer-Verlag, (1990)

[23] A. J. Feingold and I. B. Frenkel, *A Hyperbolic Kac-Moody Algebra and the Theory of Siegel Modular Forms of genus* 2, Math. Ann., **263**(1983), 87-144.

[24] E. Freitag, *Die Kodairadimension von Körpen automorpher Funktionen*, J. reine angew. Math., **296**(1977), 162-170.

[25] E. Freitag, *Thetareihnen mit harmonischen Koeffizienten zur Siegelschen Modulgruppe*, Math. Ann., **254**(1980), 27-51.

[26] E. Freitag, *Holomorphic tensors on subvarieties of the Siegel modular variety*, Birkhäuser, Prog. Math. Boston, **46**(1984), 93-11.

[27] E. Freitag and K. Pommerening, *Reguläre Differentialformen des Körpers der Siegelschen Modulfunktionen*, J. reine angew. Math., **331**(1982), 207-220.

[28] G. Frobenius, *Über die Beziehungen zwischen* 28 *Doppeltangenten einer eben Curve vierte Ordnung* , J. reine angew. Math., **99**(1886), 285-314.

[29] G. van der Geer, *Note on abelian schemes of level three*, Math. Ann., **278**(1987), 401-408.

[30] V. A. Gritsenko, *The action of modular operators on the Fourier-Jacobi coefficients of modular forms*, Math. USSR Sbornik, **74**(1984), 237-268.

[31] B. Gross, W. Kohnen and D. Zagier, *Heegner points and the derivative of L-series II*, Math. Ann., **278**(1987), 497-562.

[32] V. A. Gritsenko and G. K. Sankaran, *Moduli of abelian surfaces with* $(1, p^2)$ *polarization*, preprint.

[33] J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math., **67**(1982), 23-88.

[34] K. Hulek and G. K. Sankaran, *The Kodaira dimension of certain moduli spaces of abelian surfaces* , Compositio Math., **90**(1994), 1-35.

[35] J. Igusa, *A desingularization problem in the theory of Siegel modular functions*, Math. Ann., **168**(1967), 228-260.

[36] C. G. J. Jacobi, *Fundamenta nova theoriae functionum ellipticum*, Königsberg, (1829).

[37] W. Kohnen, *Modular forms of half integral weight on* $\Gamma_0(4)$, Math. Ann., **285**(1980), 249-266.

[38] J. Kollar and F. O. Schreyer, *The moduli of curves is stably rational for* $g \leq 6$, Duke Math. J., **51**(1984), 239-242.

[39] J. Kramer, *A geometrical approach to the theory of Jacobi forms*, Compositio Math., **79**(1991), 1-19.

[40] J. Kramer, *An arithmetic theory of Jacobi forms in higher dimensions* , J. reine angew. Math., **458**(1995), 157-182.

[41] N. V. Kuznetsov, *A new class of identities for the Fourier coefficients of modular forms*, Acta Arith., (1975), 505-519.

[42] H. Maass, *Über eine Spezialschar von Modulformen zweiten Grades*, Invent. Math., **52**(1979), 95-104.

[43] H. Maass, *Über eine Spezialschar von Modulformen zweiten Grades*, Invent. Math., **53**(1979), 249-253.

[44] H. Maass, *Über eine Spezialschar von Modulformen zweiten Grades*, Invent. Math., **53**(1979), 255-265.

[45] D. Mumford, *Hirzebruch's proportionality theorem in the noncompact case*, Invent. Math., **42**(1977), 239-272.

[46] D. Mumford, *On the Kodaira dimension of the Siegel modular variety*, Springer Lecture Note, **997**(1982), 348-375.

[47] A. Murase, *L-functions attached to Jacobi forms of degree n. Part I : The Basic Identity*, J. reine angew. Math., **401**(1989), 122-156.

[48] A. Murase, *L-functions attached to Jacobi forms of degree n. Part II : Functional Equation*, Math. Ann., **290** (1991), 247-276.

[49] A. Murase, *On Explicit Formula for the Whittaker-Shintani Functions on Sp_2.*, Abh. Math. Sem. Univ. Hamburg, **61** (1991), 153-162.

[50] A. Murase and T. Sugano, *Whittaker-Shintani Functions on the Symplectic Group of Fourier-Jacobi Type*, Compositio Math., **79**(1991), 321-349.

[51] Y. Namikawa, Toroidal compactification of Siegel spaces, Springer-Verlag, **812**(1980)

[52] I. Piateski-Sharpiro, Automorphic Functions and the Geometry of Classical Domains, Gordan-Breach, New York, (1966).

[53] B. Riemann, *Zur Theorie der Abel'schen Funktionen für den $p = 3$*, Math. Werke, Teubener, Leipzig, (1876), 456-476.

[54] B. Runge, *Theta functions and Siegel-Jacobi functions*, Acta Math., **175**(1995), 165-196.

[55] I. Satake, *Fock Representations and Theta Functions*, Ann. Math. Studies, **66**(1971), 393-405.

[56] I. Satake, Algebraic structures of symmetric domains, Iwanami Shoten Publ. and Princeton Univ. Press, (1980).

[57] G. Shimura, *On modular forms of half integral weight* , Ann. of Math., **97**(1973), 440-481.

[58] G. Shimura, *On certain reciprocity laws for theta functions and modular forms* , Acta Math., **141**(1979), 35-71.

[59] N.-P. Skoruppa, *Über den Zusammenhang zwischen Jacobi-Formen und Modulformen halbganzen Gewichts, Dissertation* , Universität Bonn, (1984).

[60] Y.-S. Tai, *On the Kodaira dimension of the moduli spaces of abelian varieties*, Invent. Math., **68**(1982), 425-439.

[61] K. Takase, *A note on automorphic forms*, J. reine angew. Math., **409**(1990), 138-171.

[62] K. Takase, *On unitary representations of Jacobi groups* , J. reine angew. Math., **430**(1992), 130-149.

[63] W. Wang, *On the Smooth Compactification of Siegel Spaces* , J. Diff. Geometry, **38**(1993), 351-386.

[64] W. Wang, *On the moduli space of principally polarized abelian varieties*, Contemporary Math., **150**(1993), 361-365.

[65] H. Weber, *Theorie der Abel'schen Funktionen von Geschlecht* 3, Berlin, (1876).

[66] R. Weissauer, *Vektorwertige Siegelsche Modulformen kleine Gewichtes*, J. reine angew. Math., **343**(1983), 184-202.

[67] R. Weissauer, *Untervarietäten der Siegelschen Modulmannigfatigkeiten von allgemeinen Typ*, Math. Ann., **343**(1983), 209-220.

[68] R. Weissauer, *Differentialformen zu Untergruppen der Siegelschen Modulgruppe zweiten Grades*, J. reine angew Math., **391**(1988), 100-156.

[69] T. Yamazaki, *Jacobi forms and a Maass relation for Eisenstein series*, J. Fac. Sci. Univ. Tokyo, **33**(1986), 295-310.

[70] J.-H. Yang, *Harmonic Analysis on the Quotient Spaces of Heisenberg Groups*, Nagoya Math. J., **123**(1991), 103-117.

[71] J.-H. Yang, *Harmonic Analysis on the Quotient Spaces of Heisenberg Groups II* , J. Number Theory, **49**(1)(1994), 63-72.

[72] J.-H. Yang ,*A decomposition theorem on differential polynomials of theta functions of high level*, Japanese J. of Mathematics, The Mathematical Society of Japan, New Series, **22**(1)(1996), 37-49.

[73] J.-H. Yang, *Lattice Representations of the Heisenberg Group* $H_{\mathbb{R}}^{(g,h)}$, Math. Annalen, **317**(2000), 309-323.

[74] J.-H. Yang, *Unitary Representations of the Heisenberg Group* $H_{\mathbb{R}}^{(g,h)}$, Proc. of Workshops in Pure Mathematics on Groups and Algebraic Structures, the Korean Academic Council, **5**, Part I, (1996), 144-159.

[75] J.-H. Yang, *Hermite Differential Operators* , submited.

[76] J.-H. Yang, *Fock Representations*, J. Korean Math. Soc., **34**(1997), 345-370.

[77] J.-H. Yang, *On Theta Functions*, Kyungpook Math. J., **35** (1996), 857-875.

[78] J.-H. Yang, *Geometrical theory of Siegel modular forms*, Proc. Topology and Geometry Research Center, **1**(1990), 73-83.

[79] J.-H. Yang, *Some Results on Jacobi Forms of Higher Degree*, RIMS Symp. on Automorphic Forms and Associated Zeta Functions, RIMS Kôkyû-roku, **805**(1992), 36-50.

[80] J.-H. Yang, *Vanishing theorems on Jacobi forms of higher degree*, J. Korean Math. Soc., **30**(1)(1993), 185-198.

[81] J.-H. Yang, *On Jacobi forms of higher degree (Korean)*, the Notices of the Pyungsan Institute for Mathematical Sciences, **3**(1993), 3-29.

[82] J.-H. Yang, *Stable Jacobi forms*, Proc. of Workshops in Pure Mathematics on Number Theory and Algebra, the Korean Academic Council ; **13**, Part I, (1993), 31-51.

[83] J.-H. Yang, *The Siegel-Jacobi Operator*, Abh. Math. Sem. Univ. Hamburg, **63**(1993), 135-146.

[84] J.-H. Yang, *Remarks on Jacobi forms of higher degree*, Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, edited by Jin-Woo Son and Jae-Hyun Yang, the Pyungsan Institute for Mathematical Sciences, (1993), 33-58.

[85] J.-H. Yang, *Singular Jacobi Forms*, Trans. of American Math. Soc., **347**(6)(1995), 2041-2049.

[86] J.-H. Yang, *Construction of Modular Forms from Jacobi Forms*, Canadian J. of Math., **47**(6)(1995), 1329-1339.

[87] J.-H. Yang, *Kac-Moody Algebras, the Monstrous Moonshine, Jacobi Forms and Infinite Products*, Proceedings of the 1995 Symposium on Number Theory, Geometry and Related Topics, edited by Jin-Woo Son and Jae-Hyun Yang, the Pyungsan Institute for Mathematical Sciences, (May 1996), 13-82.

[88] J.-H. Yang, *Stable Automorphic Forms*, Proceedings of Japan-Korea Joint Seminar on Transcendental Number Theory and Related Topics, (1998), 101-126.

[89] J.-H. Yang, *Invariant metrics on the Siegel-Jacobi space*, preprint (PIMS 99-4), the Pyungsan Institute for Mathematical Sciences, (1999).

[90] J.-H. Yang, *A note on Maass-Jacobi forms* , preprint (PIMS 2000-1), the Pyungsan Institute for Mathematical Sciences, (2000).

[91] J.-H. Yang and Jin Woo Son, *A note on Jacobi forms of higher degree*, J. Korean Math. Soc., **28**(2)(1991), 341-358.

[92] D. Zagier, *Note on the Landweber-Stong Elliptic Genus*, LNM, Springer-Verlag, **1326**(1988), 216-224.

[93] D. Zagier, *Periods of modular forms and Jacobi theta functions*, Invent. Math., **104**(1991), 449-465.

[94] C. Ziegler, *Jacobi Forms of Higher Degree*, Abh. Math. Sem. Univ. Hamburg, **59**(1989), 191-224.

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Erratum : A Geometrical Theory of Jacobi Forms of Higher Degree

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Erratum

In the article *A Geometrical Theory of Jacobi Forms of Higher Degree* by Jae-Hyun Yang [Kyungpook Math. J., **40(2)**(2000), 209-237], the author presents the Laplace-Beltrami operator $\Delta_{g,h}$ of the Siegel-Jacobi space $(H_{g,h},\ ds^2_{g,h})$ given by the formula (10.4) without a proof at the page 227. But the operator $\Delta_{g,h}$ is **not a correct one**.

At the page 227, the formula (10.4) should be replaced by the following **correct formula (10.4)**:

$$
\begin{aligned}
(10.4)\qquad \Delta_{g,h} \;=\; & 4\,\sigma\left(Y\ {}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial Z}\right)+4\,\sigma\left(Y\frac{\partial}{\partial W}\ {}^t\!\left(\frac{\partial}{\partial\overline{W}}\right)\right)\\
&+4\,\sigma\left(VY^{-1}\ {}^tV\ {}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial W}\right)\\
&+4\,\sigma\left(V\ {}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial W}\right)+4\,\sigma\left({}^tV\ {}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial Z}\right).
\end{aligned}
$$

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A NOTE ON A FUNDAMENTAL DOMAIN FOR SIEGEL-JACOBI SPACE

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Abstract. In this paper, we study a fundamental domain for the Siegel-Jacobi space $Sp(g,\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(g,h)} \backslash \mathbb{H}_g \times \mathbb{C}^{(h,g)}$.

1. Introduction

For a given fixed positive integer g, we let

$$\mathbb{H}_g = \{\, \Omega \in \mathbb{C}^{(g,g)} \mid \Omega = {}^t\Omega, \quad \operatorname{Im}\Omega > 0\,\}$$

be the Siegel upper half plane of degree g and let

$$Sp(g,\mathbb{R}) = \{ M \in \mathbb{R}^{(2g,2g)} \mid {}^tMJ_gM = J_g \}$$

be the symplectic group of degree g, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l, ${}^tM$ denotes the transpose matrix of a matrix M and

$$J_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

$Sp(g,\mathbb{R})$ acts on $\mathbb{H}_g$ transitively by

$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{1.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R})$ and $\Omega \in \mathbb{H}_g$. Let Γ_g be the Siegel modular group of degree g. C. L. Siegel [8] found a fundamental domain $\mathcal{F}_g$ for $\Gamma_g \backslash \mathbb{H}_g$ and calculated the volume of $\mathcal{F}_g$. We also refer to [2], [4], [10] for some details on $\mathcal{F}_g$.

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For two positive integers g and h, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(g,h)} = \{\, (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(h,g)},\ \kappa \in \mathbb{R}^{(h,h)},\ \kappa + \mu\, {}^t\lambda \text{ symmetric } \}$$

endowed with the following multiplication law

$$(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') = (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda').$$

We define the semidirect product of $Sp(g, \mathbb{R})$ and $H_{\mathbb{R}}^{(g,h)}$

$$G^J = Sp(g, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$$

endowed with the following multiplication law

$$(M, (\lambda, \mu; \kappa)) \cdot (M', (\lambda', \mu'; \kappa')) = \ (MM', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu'; \kappa + \kappa' + \tilde{\lambda}\, {}^t\mu' - \tilde{\mu}\, {}^t\lambda'))$$

with $M, M' \in Sp(g, \mathbb{R}), (\lambda, \mu; \kappa),\ (\lambda', \mu'; \kappa') \in H_{\mathbb{R}}^{(g,h)}$ and $(\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)M'$. Then G^J acts on $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ transitively by

$$(M, (\lambda, \mu; \kappa)) \cdot (\Omega, Z) = (M \cdot \Omega, (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}), \tag{1.2}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbb{R}),\ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(g,h)}$ and $(\Omega, Z) \in \mathbb{H}_g \times \mathbb{C}^{(h,g)}$. We note that the Jacobi group G^J is *not* a reductive Lie group and also that the space $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ is not a symmetric space. We refer to [11]-[14] and [16] about automorphic forms on G^J and topics related to the content of this paper. From now on, we write $\mathbb{H}_{g,h} := \mathbb{H}_g \times \mathbb{C}^{(h,g)}$.

We let

$$\Gamma_{g,h} := \Gamma_g \ltimes H_{\mathbb{Z}}^{(g,h)}$$

be the discrete subgroup of G^J, where

$$H_{\mathbb{Z}}^{(g,h)} = \{\, (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(g,h)} \mid \lambda, \mu \in \mathbb{Z}^{(h,g)},\ \ \kappa \in \mathbb{Z}^{(h,h)} \,\}.$$

The aim of this paper is to find a fundamental domain for $\Gamma_{g,h} \backslash \mathbb{H}_{g,h}$. This article is organized as follows. In Section 2, we review the Minkowski domain and the Siegel's fundamental domain $\mathcal{F}_g$ roughly. In Section 3, we find a fundamental domain for $\Gamma_{g,h} \backslash \mathbb{H}_{g,h}$ and present Riemannian metrics on the fundamental domain invariant under the action (1.2) of the Jacobi group G^J. In Section 4, we investigate the spectral theory of the Laplacian on the abelian variety A_Ω associated to $\Omega \in \mathcal{F}_g$.

2. Review on a Fundamental Domain $\mathcal{F}_g$ for $\Gamma_g\backslash\mathbb{H}_g$

We let

$$\mathcal{P}_g = \left\{ Y \in \mathbb{R}^{(g,g)} \mid Y = {}^tY > 0 \right\}$$

be an open cone in $\mathbb{R}^N$ with $N = g(g+1)/2$. The general linear group $GL(g,\mathbb{R})$ acts on $\mathcal{P}_g$ transitively by

$$g \circ Y := gY\,{}^tg, \qquad g \in GL(g,\mathbb{R}),\ Y \in \mathcal{P}_g. \tag{2.1}$$

Thus $\mathcal{P}_g$ is a symmetric space diffeomorphic to $GL(g,\mathbb{R})/O(g)$. For a matrix $A \in F^{(k,l)}$ and $B \in F^{(k,l)}$, we write $A[B] = {}^tBAB$ and for a square matrix A, $\sigma(A)$ denotes the trace of A.

The fundamental domain $\mathcal{R}_g$ for $GL(g,\mathbb{Z})\backslash\mathcal{P}_g$ which was found by H. Minkowski [5] is defined as a subset of $\mathcal{P}_g$ consisting of $Y = (y_{ij}) \in \mathcal{P}_g$ satisfying the following conditions (M.1)-(M.2) (cf. [2, p. 191] or [4, p. 123]):

(M.1) $aY\,{}^ta \geq y_{kk}$ for every $a = (a_i) \in \mathbb{Z}^g$ in which $a_k, \cdots, a_g$ are relatively prime for $k = 1, 2, \cdots, g$.

(M.2) $y_{k,k+1} \geq 0$ for $k = 1, \cdots, g-1$.

We say that a point of $\mathcal{R}_g$ is *Minkowski reduced* or simply *M-reduced.* $\mathcal{R}_g$ has the following properties (R1)-(R6):

(R1) For any $Y \in \mathcal{P}_g$, there exist a matrix $A \in GL(g,\mathbb{Z})$ and $R \in \mathcal{R}_g$ such that $Y = R[A]$ (cf. [2, p. 191] or [4, p. 139]). That is,

$$GL(g,\mathbb{Z}) \circ \mathcal{R}_g = \mathcal{P}_g.$$

(R2) $\mathcal{R}_g$ is a convex cone through the origin bounded by a finite number of hyperplanes. $\mathcal{R}_g$ is closed in $\mathcal{P}_g$ (cf. [4, p. 139]).

(R3) If Y and $Y[A]$ lie in $\mathcal{R}_g$ for $A \in GL(g,\mathbb{Z})$ with $A \neq \pm I_g$, then Y lies on the boundary $\partial\mathcal{R}_g$ of $\mathcal{R}_g$. Moreover $\mathcal{R}_g \cap (\mathcal{R}_g[A]) \neq \emptyset$ for only finitely many $A \in GL(g,\mathbb{Z})$ (cf. [4, p. 139]).

(R4) If $Y = (y_{ij})$ is an element of $\mathcal{R}_g$, then

$$y_{11} \leq y_{22} \leq \cdots \leq y_{gg} \quad \text{and} \quad |y_{ij}| < \frac{1}{2} y_{ii} \quad \text{for } 1 \leq i < j \leq g.$$

We refer to [2, p. 192] or [4, pp. 123-124].

Remark. Grenier [1] found another fundamental domain for $GL(g,\mathbb{Z})\backslash\mathcal{P}_g$.

For $Y = (y_{ij}) \in \mathcal{P}_g$, we put

$$dY = (dy_{ij}) \qquad \text{and} \qquad \frac{\partial}{\partial Y} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial y_{ij}} \right).$$

Then we can see easily that

$$ds^2 = \sigma((Y^{-1}dY)^2) \tag{2.2}$$

is a $GL(g,\mathbb{R})$-invariant Riemannian metric on $\mathcal{P}_g$ and its Laplacian is given by

$$\Delta = \sigma\left(\left(Y \frac{\partial}{\partial Y} \right)^2 \right).$$

We also can see that

$$d\mu_g(Y) = (\det Y)^{-\frac{g+1}{2}} \prod_{i \le j} dy_{ij}$$

is a $GL(g,\mathbb{R})$-invariant volume element on $\mathcal{P}_g$. The metric ds^2 on $\mathcal{P}_g$ induces the metric $ds^2_{\mathcal{R}}$ on $\mathcal{R}_g$. Minkowski [5] calculated the volume of $\mathcal{R}_g$ for the volume element $[dY] := \prod_{i \le j} dy_{ij}$ explicitly. Later Siegel [7], [9] computed the volume of $\mathcal{R}_g$ for the volume element $[dY]$ by a simple analytic method and generalized this case to the case of any algebraic number field.

Siegel [8] determined a fundamental domain $\mathcal{F}_g$ for $\Gamma_g \backslash \mathbb{H}_g$. We say that $\Omega = X + iY \in \mathbb{H}_g$ with X, Y real is *Siegel reduced* or *S-reduced* if it has the following three properties:

(S.1) $\det(\mathrm{Im}\,(\gamma \cdot \Omega)) \le \det(\mathrm{Im}\,(\Omega)) \qquad$ for all $\gamma \in \Gamma_g$;

(S.2) $Y = \mathrm{Im}\,\Omega$ is M-reduced, that is, $Y \in \mathcal{R}_g$;

(S.3) $|x_{ij}| \le \frac{1}{2}$ for $1 \le i, j \le g$, where $X = (x_{ij})$.

$\mathcal{F}_g$ is defined as the set of all Siegel reduced points in $\mathbb{H}_g$. Using the highest point method, Siegel proved the following (F1)-(F3) (cf. [2, pp. 194-197] or [4, p. 169]):

(F1) $\Gamma_g \cdot \mathcal{F}_g = \mathbb{H}_g$, i.e., $\mathbb{H}_g = \cup_{\gamma \in \Gamma_g} \gamma \cdot \mathcal{F}_g$.

(F2) $\mathcal{F}_g$ is closed in $\mathbb{H}_g$.

(F3) $\mathcal{F}_g$ is connected and the boundary of $\mathcal{F}_g$ consists of a finite number of hyperplanes.

For $\Omega = (\omega_{ij}) \in \mathbb{H}_g$, we write $\Omega = X + iY$ with $X = (x_{ij})$, $Y = (y_{ij})$ real and $d\Omega = (d\omega_{ij})$. We also put

$$\frac{\partial}{\partial \Omega} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial \omega_{ij}} \right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{\Omega}} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial \overline{\omega}_{ij}} \right).$$

Then

$$ds_*^2 = \sigma(Y^{-1}d\Omega\, Y^{-1}d\overline{\Omega}) \tag{2.3}$$

is a $Sp(g,\mathbb{R})$-invariant Kähler metric on $\mathbb{H}_g$ (cf. [8]) and H. Maass [3] proved that its Laplacian is given by

$$\Delta_* = 4\,\sigma\left(Y\ {}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right). \tag{2.4}$$

And

$$dv_g(\Omega) = (\det Y)^{-(g+1)} \prod_{1\le i\le j\le g} dx_{ij} \prod_{1\le i\le j\le g} dy_{ij} \tag{2.5}$$

is a $Sp(g,\mathbb{R})$-invariant volume element on $\mathbb{H}_g$ (cf. [10, p. 130]). The metric ds_*^2 given by (2.3) induces a metric $ds_{\mathcal{F}}^2$ on $\mathcal{F}_g$.

Siegel [8] computed the volume of $\mathcal{F}_g$

$$\mathrm{vol}\,(\mathcal{F}_g) = 2\prod_{k=1}^{g} \pi^{-k}\Gamma(k)\zeta(2k), \tag{2.6}$$

where $\Gamma(s)$ denotes the Gamma function and $\zeta(s)$ denotes the Riemann zeta function. For instance,

$$\mathrm{vol}\,(\mathcal{F}_1) = \frac{\pi}{3},\quad \mathrm{vol}\,(\mathcal{F}_2) = \frac{\pi^3}{270},\quad \mathrm{vol}\,(\mathcal{F}_3) = \frac{\pi^6}{127575},\quad \mathrm{vol}\,(\mathcal{F}_4) = \frac{\pi^{10}}{200930625}.$$

3. A Fundamental Domain for $\Gamma_{g,h}\backslash\mathbb{H}_{g,h}$

Let E_{kj} be the $h\times g$ matrix with entry 1 where the k-th row and the j-th column meet, and all other entries 0. For an element $\Omega\in\mathbb{H}_g$, we set for brevity

$$F_{kj}(\Omega) := E_{kj}\Omega,\qquad 1\le k\le h,\ 1\le j\le g. \tag{3.1}$$

For each $\Omega\in\mathcal{F}_g$, we define a subset P_Ω of $\mathbb{C}^{(h,g)}$ by

$$P_\Omega = \left\{ \sum_{k=1}^{h}\sum_{j=1}^{g}\lambda_{kj}E_{kj} + \sum_{k=1}^{h}\sum_{j=1}^{g}\mu_{kj}F_{kj}(\Omega)\ \Big|\ 0\le \lambda_{kj},\mu_{kj}\le 1 \right\}.$$

For each $\Omega\in\mathcal{F}_g$, we define the subset D_Ω of $\mathbb{H}_{g,h}$ by

$$D_\Omega := \{\,(\Omega,Z)\in\mathbb{H}_{g,h}\mid Z\in P_\Omega\,\}.$$

We define

$$\mathcal{F}_{g,h} := \cup_{\Omega\in\mathcal{F}_g} D_\Omega.$$

Theorem 3.1. *$\mathcal{F}_{g,h}$ is a fundamental domain for $\Gamma_{g,h}\backslash\mathbb{H}_{g,h}$.*

Proof. Let $(\tilde{\Omega}, \tilde{Z})$ be an arbitrary element of $\mathbb{H}_{g,h}$. We must find an element (Ω, Z) of $\mathcal{F}_{g,h}$ and an element $\gamma^J = (\gamma, (\lambda, \mu; \kappa)) \in \Gamma_{g,h}$ with $\gamma \in \Gamma_g$ such that $\gamma^J \cdot (\Omega, Z) = (\tilde{\Omega}, \tilde{Z})$. Since $\mathcal{F}_g$ is a fundamental domain for $\Gamma_g\backslash\mathbb{H}_g$, there exists an element γ of Γ_g and an element Ω of $\mathcal{F}_g$ such that $\gamma \cdot \Omega = \tilde{\Omega}$. Here Ω is unique up to the boundary of $\mathcal{F}_g$.

We write

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g.$$

It is easy to see that we can find $\lambda, \mu \in \mathbb{Z}^{(h,g)}$ and $Z \in P_\Omega$ satisfying the equation

$$Z + \lambda\Omega + \mu = \tilde{Z}(C\Omega + D).$$

If we take $\gamma^J = (\gamma, (\lambda, \mu; 0)) \in \Gamma_{g,h}$, we see that $\gamma^J \cdot (\Omega, Z) = (\tilde{\Omega}, \tilde{Z})$. Therefore we obtain

$$\mathbb{H}_{g,h} = \cup_{\gamma^J \in \Gamma_{g,h}} \gamma^J \cdot \mathcal{F}_{g,h}.$$

Let (Ω, Z) and $\gamma^J \cdot (\Omega, Z)$ be two elements of $\mathcal{F}_{g,h}$ with $\gamma^J = (\gamma, (\lambda, \mu; \kappa)) \in \Gamma_{g,h}$. Then both Ω and $\gamma \cdot \Omega$ lie in $\mathcal{F}_g$. Therefore both of them either lie in the boundary of $\mathcal{F}_g$ or $\gamma = \pm I_{2g}$. In the case that both Ω and $\gamma \cdot \Omega$ lie in the boundary of $\mathcal{F}_g$, both (Ω, Z) and $\gamma^J \cdot (\Omega, Z)$ lie in the boundary of $\mathcal{F}_{g,h}$. If $\gamma = \pm I_{2g}$, we have

$$Z \in P_\Omega \quad \text{and} \quad \pm (Z + \lambda\Omega + \mu) \in P_\Omega, \quad \lambda, \mu \in \mathbb{Z}^{(h,g)}. \tag{3.2}$$

From the definition of P_Ω and (3.2), we see that either $\lambda = \mu = 0$, $\gamma \neq -I_{2g}$ or both Z and $\pm(Z + \lambda\Omega + \mu)$ lie on the boundary of the parallelepiped P_Ω. Hence either both(Ω, Z) and $\gamma^J \cdot (\Omega, Z)$ lie in the boundary of $\mathcal{F}_{g,h}$ or $\gamma^J = (I_{2g}, (0, 0; \kappa)) \in \Gamma_{g,h}$. Consequently $\mathcal{F}_{g,h}$ is a fundamental domain for $\Gamma_{g,h}\backslash\mathbb{H}_{g,h}$. □

For a coordinate $(\Omega, Z) \in \mathbb{H}_{g,h}$ with $\Omega = (\omega_{\mu\nu}) \in \mathbb{H}_g$ and $Z = (z_{kl}) \in \mathbb{C}^{(h,g)}$, we put

$$\begin{aligned}
\Omega &= X + iY, & X &= (x_{\mu\nu}), & Y &= (y_{\mu\nu}) \text{ real},\\
Z &= U + iV, & U &= (u_{kl}), & V &= (v_{kl}) \text{ real},\\
d\Omega &= (d\omega_{\mu\nu}), & dX &= (dx_{\mu\nu}), & dY &= (dy_{\mu\nu}),\\
dZ &= (dz_{kl}), & dU &= (du_{kl}), & dV &= (dv_{kl}),\\
d\overline{\Omega} &= (d\overline{\omega}_{\mu\nu}), & d\overline{Z} &= (d\bar{z}_{kl}),
\end{aligned}$$

$$\frac{\partial}{\partial\Omega} = \left(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial\omega_{\mu\nu}}\right), \quad \frac{\partial}{\partial\overline{\Omega}} = \left(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial\overline{\omega}_{\mu\nu}}\right),$$

$$\frac{\partial}{\partial Z} = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \cdots & \frac{\partial}{\partial z_{h1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_{1g}} & \cdots & \frac{\partial}{\partial z_{hg}} \end{pmatrix}, \quad \frac{\partial}{\partial\overline{Z}} = \begin{pmatrix} \frac{\partial}{\partial\overline{z}_{11}} & \cdots & \frac{\partial}{\partial\overline{z}_{h1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial\overline{z}_{1g}} & \cdots & \frac{\partial}{\partial\overline{z}_{hg}} \end{pmatrix}.$$

Remark. The following metric

$$\begin{aligned} ds^2_{g,h} = & \,\sigma\left(Y^{-1}d\Omega\, Y^{-1}d\overline{\Omega}\right) + \sigma\left(Y^{-1}\,{}^tV\,V\,Y^{-1}d\Omega\, Y^{-1}d\overline{\Omega}\right) \\ & + \sigma\left(Y^{-1}\,{}^t(dZ)\,d\overline{Z}\right) \\ & - \sigma\left(V\,Y^{-1}d\Omega\, Y^{-1}\,{}^t(d\overline{\Omega}) + V\,Y^{-1}d\overline{\Omega}\, Y^{-1}\,{}^t(dZ)\right) \end{aligned}$$

is a Kähler metric on $\mathbb{H}_{g,h}$ which is invariant under the action (1.2) of the Jacobi group G^J. Its Laplacian is given by

$$\begin{aligned} \Delta_{g,h} = & \,4\,\sigma\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right) + 4\,\sigma\left(Y\,\frac{\partial}{\partial Z}\,{}^t\!\left(\frac{\partial}{\partial\overline{Z}}\right)\right) \\ & + 4\,\sigma\left(VY^{-1}\,{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial Z}\right) \\ & + 4\,\sigma\left(V\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial Z}\right) + 4\sigma\left({}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial\Omega}\right). \end{aligned}$$

The following differential form

$$dv_{g,h} = (\det Y)^{-(g+h+1)}\,[dX]\wedge[dY]\wedge[dU]\wedge[dV]$$

is a G^J-invariant volume element on $\mathbb{H}_{g,h}$, where

$$[dX] = \wedge_{\mu\le\nu}dx_{\mu\nu}, \quad [dY] = \wedge_{\mu\le\nu}dy_{\mu\nu}, \quad [dU] = \wedge_{k,l}du_{kl} \quad \text{and} \quad [dV] = \wedge_{k,l}dv_{kl}.$$

The point is that the invariant metric $ds^2_{g,h}$ and its Laplacian are beautifully expressed in terms of the *trace* form. The proof of the above facts can be found in [15].

4. Spectral Decomposition of $L^2(A_\Omega)$

We fix two positive integers g and h throughout this section.

For an element $\Omega \in \mathbb{H}_g$, we set

$$L_\Omega := \mathbb{Z}^{(h,g)} + \mathbb{Z}^{(h,g)}\Omega$$

We use the notation (3.1). It follows from the positivity of $\operatorname{Im}\Omega$ that the elements $E_{kj},\, F_{kj}(\Omega)\,(1 \leq k \leq h,\ 1 \leq j \leq g)$ of L_Ω are linearly independent over $\mathbb{R}$. Therefore L_Ω is a lattice in $\mathbb{C}^{(h,g)}$ and the set $\{\, E_{kj},\, F_{kj}(\Omega)\,|\, 1 \leq k \leq h,\ 1 \leq j \leq g\,\}$ forms an integral basis of L_Ω. We see easily that if Ω is an element of $\mathbb{H}_g$, the period matrix $\Omega_* := (I_g, \Omega)$ satisfies the Riemann conditions (RC.1) and (RC.2):

(RC.1) $\quad \Omega_* J_g\, {}^t\Omega_* = 0\,;$

(RC.2) $\quad -\frac{1}{i}\Omega_* J_g\, {}^t\overline{\Omega}_* > 0.$

Thus the complex torus $A_\Omega := \mathbb{C}^{(h,g)}/L_\Omega$ is an abelian variety. For more details on A_Ω, we refer to [2] and [6].

It might be interesting to investigate the spectral theory of the Laplacian $\Delta_{g,h}$ on a fundamental domain $\mathcal{F}_{g,h}$. But this work is very complicated and difficult at this moment. It may be that the first step is to develop the spectral theory of the Laplacian Δ_Ω on the abelian variety A_Ω. The second step will be to study the spectral theory of the Laplacian Δ_* (see (2.4)) on the moduli space $\Gamma_g\backslash\mathbb{H}_g$ of principally polarized abelian varieties of dimension g. The final step would be to combine the above steps and more works to develop the spectral theory of the Laplacian $\Delta_{g,h}$ on $\mathcal{F}_{g,h}$. In this section, we deal only with the spectral theory of Δ_Ω on $L^2(A_\Omega)$.

We fix an element $\Omega = X + iY$ of $\mathbb{H}_g$ with $X = \operatorname{Re}\Omega$ and $Y = \operatorname{Im}\Omega$. For a pair (A, B) with $A, B \in \mathbb{Z}^{(h,g)}$, we define the function $E_{\Omega;A,B} : \mathbb{C}^{(h,g)} \longrightarrow \mathbb{C}$ by

$$E_{\Omega;A,B}(Z) = e^{2\pi i\left(\sigma\,({}^tAU\,) + \sigma\,((B-AX)Y^{-1}\,{}^tV)\right)},$$

where $Z = U + iV$ is a variable in $\mathbb{C}^{(h,g)}$ with real U, V.

Lemma 4.1. *For any $A, B \in \mathbb{Z}^{(h,g)}$, the function $E_{\Omega;A,B}$ satisfies the following functional equation*

$$E_{\Omega;A,B}(Z + \lambda\Omega + \mu) = E_{\Omega;A,B}(Z), \quad Z \in \mathbb{C}^{(h,g)}$$

for all $\lambda, \mu \in \mathbb{Z}^{(h,g)}$. Thus $E_{\Omega;A,B}$ can be regarded as a function on A_Ω.

PROOF. We write $\Omega = X + iY$ with real X, Y. For any $\lambda, \mu \in \mathbb{Z}^{(h,g)}$, we have

$$\begin{aligned} E_{\Omega;A,B}(Z + \lambda\Omega + \mu) &= E_{\Omega;A,B}((U + \lambda X + \mu) + i(V + \lambda Y)) \\ &= e^{2\pi i \left\{ \sigma\left({}^tA(U+\lambda X+\mu)\right) + \sigma\left((B-AX)Y^{-1}\, {}^t(V+\lambda Y)\right) \right\}} \\ &= e^{2\pi i \left\{ \sigma\left({}^tAU + {}^tA\lambda X + {}^tA\mu\right) + \sigma\left((B-AX)Y^{-1}\, {}^tV + B\, {}^t\lambda - AX\, {}^t\lambda\right) \right\}} \\ &= e^{2\pi i \left\{ \sigma\left({}^tAU\right) + \sigma\left((B-AX)Y^{-1}\, {}^tV\right) \right\}} \\ &= E_{\Omega;A,B}(Z). \end{aligned}$$

Here we used the fact that ${}^tA\mu$ and $B\,{}^t\lambda$ are integral. $\square$

We use the notations in Section 3.

Lemma 4.2. *The metric*

$$ds^2_\Omega = \sigma\left((Im\,\Omega)^{-1}\, {}^t(dZ)\, d\overline{Z} \right)$$

is a Kähler metric on A_Ω *invariant under the action (1.2) of* $\Gamma^J = Sp(g,\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(h,g)}$ *on* (Ω, Z) *with* Ω *fixed. Its Laplacian* Δ_Ω *of* ds^2_Ω *is given by*

$$\Delta_\Omega = \sigma\left((Im\,\Omega)\, \frac{\partial}{\partial Z}\, {}^t\!\left(\frac{\partial}{\partial \overline{Z}} \right) \right).$$

PROOF. Let $\tilde{\gamma} = (\gamma, (\lambda, \mu; \kappa)) \in \Gamma^J$ with $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{Z})$ and $(\tilde{\Omega}, \tilde{Z}) = \tilde{\gamma} \cdot (\Omega, Z)$ with $\Omega \in \mathbb{H}_g$ fixed. Then according to [4, p. 33],

$$\mathrm{Im}\ \gamma \cdot \Omega = {}^t(C\overline{\Omega} + D)^{-1}\, \mathrm{Im}\ \Omega\, (C\Omega + D)^{-1}$$

and by (1.2),

$$d\tilde{Z} = dZ\, (C\Omega + D)^{-1}.$$

Therefore

$$\begin{aligned} & (\mathrm{Im}\ \tilde{\Omega})^{-1}\, {}^t(d\tilde{Z})\, d\overline{\tilde{Z}} \\ = \;& (C\overline{\Omega} + D)\, (\mathrm{Im}\ \Omega)^{-1}\, {}^t(C\Omega + D)\, {}^t(C\Omega + D)^{-1}\, {}^t(dZ)\, d\overline{Z}\, (C\overline{\Omega} + D)^{-1} \\ = \;& (C\overline{\Omega} + D)\, (\mathrm{Im}\ \Omega)^{-1}\, {}^t(dZ)\, d\overline{Z}\, (C\overline{\Omega} + D)^{-1}. \end{aligned}$$

The metric $ds_{iI_g} = \sigma(dZ\, {}^t(d\overline{Z}))$ at $Z = 0$ is positive definite. Since G^J acts on $\mathbb{H}_{g,h}$ transitively, ds^2_Ω is a Riemannian metric for any $\Omega \in \mathbb{H}_g$. We note that the differential operator Δ_Ω is invariant under the action of Γ^J. In fact, according to (1.2),

$$\frac{\partial}{\partial \tilde{Z}} = (C\Omega + D)\, \frac{\partial}{\partial Z}.$$

Hence if f is a differentiable function on A_Ω, then

$$\begin{aligned}
&\operatorname{Im}\tilde{\Omega}\,\frac{\partial}{\partial\tilde{Z}}\,{}^t\!\left(\frac{\partial f}{\partial\overline{\tilde{Z}}}\right)\\
=\;& {}^t(C\overline{\Omega}+D)^{-1}\,(\operatorname{Im}\Omega)\,(C\Omega+D)^{-1}(C\Omega+D)\,\frac{\partial}{\partial Z}\,{}^t\!\left((C\overline{\Omega}+D)\frac{\partial f}{\partial\overline{Z}}\right)\\
=\;& {}^t(C\overline{\Omega}+D)^{-1}\operatorname{Im}\Omega\,\frac{\partial}{\partial Z}\,{}^t\!\left(\frac{\partial f}{\partial\overline{Z}}\right)\,{}^t(C\overline{\Omega}+D).
\end{aligned}$$

Therefore

$$\sigma\left(\operatorname{Im}\tilde{\Omega}\,\frac{\partial}{\partial\tilde{Z}}\,{}^t\!\left(\frac{\partial}{\partial\overline{\tilde{Z}}}\right)\right)=\sigma\left(\operatorname{Im}\Omega\,\frac{\partial}{\partial Z}\,{}^t\!\left(\frac{\partial f}{\partial\overline{Z}}\right)\right).$$

By the induction on h, we can compute the Laplacian Δ_Ω.

□

We let $L^2(A_\Omega)$ be the space of all functions $f : A_\Omega \longrightarrow \mathbb{C}$ such that

$$||f||_\Omega := \int_{A_\Omega} |f(Z)|^2 dv_\Omega,$$

where dv_Ω is the volume element on A_Ω normalized so that $\int_{A_\Omega} dv_\Omega = 1$. The inner product $(\ ,\ )_\Omega$ on the Hilbert space $L^2(A_\Omega)$ is given by

$$(f,g)_\Omega := \int_{A_\Omega} f(Z)\,\overline{g(Z)}\,dv_\Omega,\quad f,g\in L^2(A_\Omega). \tag{4.1}$$

Theorem 4.3. *The set* $\{\,E_{\Omega;A,B}\mid A,B\in\mathbb{Z}^{(h,g)}\,\}$ *is a complete orthonormal basis for* $L^2(A_\Omega)$. *Moreover we have the following spectral decomposition of* Δ_Ω*:*

$$L^2(A_\Omega)=\oplus_{A,B\in\mathbb{Z}^{(h,g)}}\mathbb{C}\cdot E_{\Omega;A,B}.$$

Proof. Let

$$T=\mathbb{C}^{(h,g)}/(\mathbb{Z}^{(h,g)}\times\mathbb{Z}^{(h,g)})=(\mathbb{R}^{(h,g)}\times\mathbb{R}^{(h,g)})/(\mathbb{Z}^{(h,g)}\times\mathbb{Z}^{(h,g)})$$

be the torus of real dimension $2hg$. The Hilbert space $L^2(T)$ is isomorphic to the $2hg$ tensor product of $L^2(\mathbb{R}/\mathbb{Z})$, where $\mathbb{R}/\mathbb{Z}$ is the one-dimensional real torus. Since $L^2(\mathbb{R}/\mathbb{Z})=\oplus_{n\in\mathbb{Z}}\mathbb{C}\cdot e^{2\pi i n x}$, the Hilbert space $L^2(T)$ is

$$L^2(T)=\oplus_{A,B\in\mathbb{Z}^{(h,g)}}\mathbb{C}\cdot E_{A,B}(W),$$

where $W=P+iQ,\ P,Q\in\mathbb{R}^{(h,g)}$ and

$$E_{A,B}(W):=e^{2\pi i\,\sigma({}^tAP+{}^tBQ)},\quad A,B\in\mathbb{Z}^{(h,g)}.$$

The inner product on $L^2(T)$ is defined by

$$(4.2)\quad (f,g) := \int_0^1 \cdots \int_0^1 f(W)\,\overline{g(W)}\,dp_{11}\cdots dp_{hg}dq_{11}\cdots dq_{hg}, \quad f,g\in L^2(T),$$

where $W = P + iQ \in T$, $P = (p_{kl})$ and $Q = (q_{kl})$. Then we see that the set $\{E_{A,B}(W) \mid A, B \in \mathbb{Z}^{(h,g)}\}$ is a complete orthonormal basis for $L^2(T)$, and each $E_{A,B}(W)$ is an eigenfunction of the standard Laplacian

$$\Delta_T = \sum_{k=}^{h}\sum_{l=1}^{g}\left(\frac{\partial^2}{\partial p_{kl}^2} + \frac{\partial^2}{\partial q_{kl}^2}\right).$$

We define the mapping $\Phi_\Omega : T \longrightarrow A_\Omega$ by

$$(4.3)\qquad \Phi_\Omega(P+iQ) = (P+QX) + i\,QY, \quad P + i\,Q \in T,\ P,Q\in\mathbb{R}^{(h,g)}.$$

This is well defined. We can see that Φ_Ω is a diffeomorphism and that the inverse Φ_Ω^{-1} of Φ_Ω is given by

$$(4.4)\quad \Phi_\Omega^{-1}(U+iV) = (U - VY^{-1}X) + i\,VY^{-1}, \quad U+iV\in A_\Omega,\ U,V\in\mathbb{R}^{(h,g)}.$$

Using (4.4), we can show that for $A, B \in \mathbb{Z}^{(h,g)}$, the function $E_{A,B}(W)$ on T is transformed to the function $E_{\Omega;A,B}$ on A_Ω via the diffeomorphism Φ_Ω. Using (4.2) and the diffeomorphism Φ_Ω, we can choose a normalized volume element dv_Ω on A_Ω and then we get the inner product on $L^2(A_\Omega)$ defined by (4.1). This completes the proof. □

References

[1] D. Grenier, *An analogue of Siegel's ϕ-operator for automorphic forms for $GL(n,\mathbb{Z})$,* Trans. Amer. Math. Soc. **331**, No. 1 (1992), 463-477.

[2] J. Igusa, *Theta Functions,* Springer-Verlag, Berlin-Heidelberg-New York (1971).

[3] H. Maass, *Die Differentialgleichungen in der Theorie der Siegelschen Modulfunktionen,* Math. Ann. **126** (1953), 44–68.

[4] H. Maass, *Siegel modular forms and Dirichlet series,* Lecture Notes in Math. **216**, Springer-Verlag, Berlin-Heidelberg-New York (1971).

[5] H. Minkowski, *Gesammelte Abhandlungen,* Chelsea, New York (1967).

[6] D. Mumford, *Tata Lectures on Theta I,* Progress in Math. **28**, Boston-Basel-Stuttgart (1983).

[7] C. L. Siegel, *The volume of the fundamental domain for some infinite groups,* Transactions of AMS. **39** (1936), 209-218.

[8] C. L. Siegel, *Symplectic geometry,* Amer. J. Math. **65** (1943), 1-86; Academic Press, New York and London (1964); Gesammelte Abhandlungen, no. 41, vol. II, Springer-Verlag (1966), 274-359.

[9] C. L. Siegel, *Zur Bestimmung des Volumens des Fundamental Bereichs der unimodularen Gruppe,* Math. Ann. **137** (1959), 427-432.

[10] C. L. Siegel, *Topics in Complex Function Theory,* Wiley-Interscience, New York, vol. III (1973).

[11] J.-H. Yang, *Remarks on Jacobi forms of higher degree,* Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, the Pyungsan Institute for Mathematical Sciences, Seoul (1993), 33-58.

[12] J.-H. Yang, *Singular Jacobi forms,* Trans. of American Math. Soc. **347**, No. 6 (1995), 2041-2049.

[13] J.-H. Yang, *Construction of vector valued modular forms from Jacobi forms,* Canadian J. of Math. **47** **(6)** (1995), 1329-1339.

[14] J.-H. Yang, *A geometrical theory of Jacobi forms of higher degree,* Proceedings of Symposium on Hodge Theory and Algebraic Geometry (edited by Tadao Oda), Sendai, Japan (1996), 125-147 *or* Kyungpook Math. J. **40**, no. 2 (2000), 209-237.

[15] J.-H. Yang, *Invariant metrics and Laplacians on the Siegel-Jacobi spaces,* arXiv:math.NT/ 0507215 v1.

[16] C. Ziegler, *Jacobi forms of higher degree.* Abh. Math. Sem. Univ. Hamburg **59** (1989), 191-224.

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Invariant metrics and Laplacians on Siegel–Jacobi space ☆

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Abstract

In this paper, we compute Riemannian metrics on the Siegel–Jacobi space which are invariant under the natural action of the Jacobi group explicitly and also provide the Laplacians of these invariant metrics. These are expressed in terms of the trace form.
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1. Introduction

For a given fixed positive integer n, we let

$$\mathbb{H}_n = \left\{ Z \in \mathbb{C}^{(n,n)} \mid Z = {}^tZ,\ \operatorname{Im} Z > 0 \right\}$$

be the Siegel upper half plane of degree n and let

$$Sp(n, \mathbb{R}) = \left\{ M \in \mathbb{R}^{(2n,2n)} \mid {}^tM J_n M = J_n \right\}$$

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be the symplectic group of degree n, where

$$J_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$

We see that $Sp(n,\mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

$$M \cdot Z = (AZ+B)(CZ+D)^{-1}, \tag{1.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $Z \in \mathbb{H}_n$.

For two positive integers n and m, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \{(\lambda,\mu;\kappa) \mid \lambda,\mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\, {}^t\lambda \text{ symmetric}\}$$

endowed with the following multiplication law

$$(\lambda,\mu;\kappa) \circ (\lambda',\mu';\kappa') = (\lambda+\lambda', \mu+\mu'; \kappa+\kappa'+\lambda\, {}^t\mu' - \mu\, {}^t\lambda').$$

We define the semidirect product of $Sp(n,\mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$

$$G^J := Sp(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

endowed with the following multiplication law

$$(M,(\lambda,\mu;\kappa)) \cdot (M',(\lambda',\mu';\kappa')) = (MM', (\tilde{\lambda}+\lambda', \tilde{\mu}+\mu'; \kappa+\kappa'+\tilde{\lambda}\, {}^t\mu' - \tilde{\mu}\, {}^t\lambda'))$$

with $M, M' \in Sp(n,\mathbb{R})$, $(\lambda,\mu;\kappa), (\lambda',\mu';\kappa') \in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde{\lambda},\tilde{\mu}) = (\lambda,\mu)M'$. We call this group G^J the *Jacobi group* of degree n and index m. We have the *natural action* of G^J on $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ defined by

$$(M,(\lambda,\mu;\kappa)) \cdot (Z,W) = (M \cdot Z, (W+\lambda Z+\mu)(CZ+D)^{-1}), \tag{1.2}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$, $(\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(Z,W) \in \mathbb{H}_n \times \mathbb{C}^{(m,n)}$. The homogeneous space $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ is called the *Siegel–Jacobi space* of degree n and index m. We refer to [2,3,6, 7,11,14–21] for more details on materials related to the Siegel–Jacobi space.

For brevity, we write $\mathbb{H}_{n,m} := \mathbb{H}_n \times \mathbb{C}^{(m,n)}$. For a coordinate $(Z,W) \in \mathbb{H}_{n,m}$ with $Z = (z_{\mu\nu}) \in \mathbb{H}_n$ and $W = (w_{kl}) \in \mathbb{C}^{(m,n)}$, we put

$$\begin{aligned}
Z &= X + iY, \quad X = (x_{\mu\nu}), \quad Y = (y_{\mu\nu}) \text{ real},\\
W &= U + iV, \quad U = (u_{kl}), \quad V = (v_{kl}) \text{ real},\\
dZ &= (dz_{\mu\nu}), \quad d\overline{Z} = (d\bar{z}_{\mu\nu}), \quad dY = (dy_{\mu\nu}),\\
dW &= (dw_{kl}), \quad d\overline{W} = (d\overline{w}_{kl}), \quad dV = (dv_{kl}),\\
\frac{\partial}{\partial Z} &= \left(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial z_{\mu\nu}}\right), \qquad \frac{\partial}{\partial \overline{Z}} = \left(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial \bar{z}_{\mu\nu}}\right),
\end{aligned}$$

$$\frac{\partial}{\partial X}=\left(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial x_{\mu\nu}}\right),\qquad \frac{\partial}{\partial Y}=\left(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial y_{\mu\nu}}\right),$$

$$\frac{\partial}{\partial W}=\begin{pmatrix}\frac{\partial}{\partial w_{11}} & \cdots & \frac{\partial}{\partial w_{m1}}\\ \vdots & \ddots & \vdots\\ \frac{\partial}{\partial w_{1n}} & \cdots & \frac{\partial}{\partial w_{mn}}\end{pmatrix},\qquad \frac{\partial}{\partial \overline{W}}=\begin{pmatrix}\frac{\partial}{\partial \overline{w}_{11}} & \cdots & \frac{\partial}{\partial \overline{w}_{m1}}\\ \vdots & \ddots & \vdots\\ \frac{\partial}{\partial \overline{w}_{1n}} & \cdots & \frac{\partial}{\partial \overline{w}_{mn}}\end{pmatrix},$$

$$\frac{\partial}{\partial U}=\begin{pmatrix}\frac{\partial}{\partial u_{11}} & \cdots & \frac{\partial}{\partial u_{m1}}\\ \vdots & \ddots & \vdots\\ \frac{\partial}{\partial u_{1n}} & \cdots & \frac{\partial}{\partial u_{mn}}\end{pmatrix},\qquad \frac{\partial}{\partial V}=\begin{pmatrix}\frac{\partial}{\partial v_{11}} & \cdots & \frac{\partial}{\partial v_{m1}}\\ \vdots & \ddots & \vdots\\ \frac{\partial}{\partial v_{1n}} & \cdots & \frac{\partial}{\partial v_{mn}}\end{pmatrix},$$

where δ_{ij} denotes the Kronecker delta symbol.

C.L. Siegel [12] introduced the symplectic metric ds_n^2 on $\mathbb{H}_n$ invariant under the action (1.1) of $Sp(n,\mathbb{R})$ given by

$$ds_n^2=\sigma\left(Y^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\right) \tag{1.3}$$

and H. Maass [8] proved that the differential operator

$$\Delta_n=\sigma\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial \overline{Z}}\right)\frac{\partial}{\partial Z}\right) \tag{1.4}$$

is the Laplacian of $\mathbb{H}_n$ for the symplectic metric ds_n^2. Here $\sigma(A)$ denotes the trace of a square matrix A.

In this paper, for arbitrary positive integers n and m, we express the G^J-invariant metrics on $\mathbb{H}_n\times\mathbb{C}^{(m,n)}$ and their Laplacians explicitly.

In fact, we prove the following theorems.

Theorem 1.1. *For any two positive real numbers A and B, the following metric*

$$\begin{aligned}ds^2_{n,m;A,B}&=A\sigma\left(Y^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\right)\\&\quad+B\left\{\sigma\left(Y^{-1}\,{}^tV\,VY^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\right)+\sigma\left(Y^{-1}\,{}^t(dW)\,d\overline{W}\right)\right.\\&\quad\left.-\sigma\left(VY^{-1}\,dZ\,Y^{-1}\,{}^t(d\overline{W})\right)-\sigma\left(VY^{-1}\,d\overline{Z}\,Y^{-1}\,{}^t(dW)\right)\right\}\end{aligned}$$

is a Riemannian metric on $\mathbb{H}_{n,m}$ which is invariant under the action (1.2) *of the Jacobi group G^J.*

Theorem 1.2. *For any two positive real numbers A and B, the Laplacian $\Delta_{n,m;A,B}$ of $(\mathbb{H}_{n,m}, ds^2_{n,m;A,B})$ is given by*

$$\begin{aligned}\Delta_{n,m;A,B}&=\frac{4}{A}\left\{\sigma\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial Z}\right)+\sigma\left(VY^{-1}\,{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial W}\right)\right.\\&\quad\left.+\sigma\left(V\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial W}\right)+\sigma\left({}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial Z}\right)\right\}\\&\quad+\frac{4}{B}\sigma\left(Y\frac{\partial}{\partial W}\,{}^t\!\left(\frac{\partial}{\partial\overline{W}}\right)\right).\end{aligned}$$

The following differential form

$$dv = (\det Y)^{-(n+m+1)}[dX] \wedge [dY] \wedge [dU] \wedge [dV]$$

is a G^J-invariant volume element on $\mathbb{H}_{n,m}$, where

$$[dX] = \bigwedge\nolimits_{\mu \leqslant \nu} dx_{\mu\nu}, \qquad [dY] = \bigwedge\nolimits_{\mu \leqslant \nu} dy_{\mu\nu}, \qquad [dU] = \bigwedge\nolimits_{k,l} du_{kl} \quad \text{and} \quad [dV] = \bigwedge\nolimits_{k,l} dv_{kl}.$$

The point is that the invariant metric $ds^2_{n,m;A,B}$ and its Laplacian $\Delta_{n,m;A,B}$ are expressed in terms of the trace form.

For the case $n = m = 1$ and $A = B = 1$, Berndt proved in [1] (cf. [19]) that the metric $ds^2_{1,1}$ on $\mathbb{H} \times \mathbb{C}$ defined by

$$ds^2_{1,1} := ds_{1,1;1,1} = \frac{y + v^2}{y^3}\left(dx^2 + dy^2\right) + \frac{1}{y}\left(du^2 + dv^2\right) - \frac{2v}{y^2}(dx\,du + dy\,dv)$$

is a Riemannian metric on $\mathbb{H} \times \mathbb{C}$ invariant under the action (1.2) of the Jacobi group and its Laplacian $\Delta_{1,1}$ is given by

$$\Delta_{1,1} := \Delta_{1,1;1,1} = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \left(y + v^2\right)\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + 2yv\left(\frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v}\right).$$

Notations. We denote by $\mathbb{R}$ and $\mathbb{C}$ the field of real numbers, and the field of complex numbers, respectively. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. E_n denotes the identity matrix of degree n. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. For a complex matrix A, $\overline{A}$ denotes the complex *conjugate* of A. For $A \in \mathbb{C}^{(k,l)}$ and $B \in \mathbb{C}^{(k,k)}$, we use the abbreviation $B\{A\} = {}^t\overline{A}BA$.

2. Proof of Theorem 1.1

Let $g = (M, (\lambda, \mu; \kappa))$ be an element of G^J with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $(Z, W) \in \mathbb{H}_{n,m}$ with $Z \in \mathbb{H}_n$ and $W \in \mathbb{C}^{(m,n)}$. If we put $(Z_*, W_*) := g \cdot (Z, W)$, then we have

$$Z_* = M \cdot Z = (AZ + B)(CZ + D)^{-1},$$
$$W_* = (W + \lambda Z + \mu)(CZ + D)^{-1}.$$

Thus we obtain

$$dZ_* = dZ\big[(CZ+D)^{-1}\big] = {}^t(CZ+D)^{-1}\, dZ(CZ+D)^{-1} \tag{2.1}$$

and

$$dW_* = dW(CZ+D)^{-1} + \big\{\lambda - (W+\lambda Z+\mu)(CZ+D)^{-1}C\big\}\, dZ(CZ+D)^{-1}. \tag{2.2}$$

Here we used the following facts that

$$d(CZ+D)^{-1} = -(CZ+D)^{-1}C\, dZ(CZ+D)^{-1}$$

and that $(CZ+D)^{-1}C$ is symmetric.

We put

$$Z_* = X_* + iY_*, \quad W_* = U_* + iV_*, \quad X_*, Y_*, U_*, V_* \text{ real}.$$

From [9, p. 33] or [13, p. 128], we know that

$$Y_* = Y\big\{(CZ+D)^{-1}\big\} = {}^t(C\overline{Z}+D)^{-1}Y(CZ+D)^{-1}. \tag{2.3}$$

First of all, we recall that the following matrices

$$t(b) = \begin{pmatrix} E_n & b \\ 0 & E_n \end{pmatrix}, \quad b = {}^tb \text{ real},$$

$$g_0(h) = \begin{pmatrix} {}^th & 0 \\ 0 & h^{-1} \end{pmatrix}, \quad h \in GL(n,\mathbb{R}),$$

$$J_n = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}$$

generate the symplectic group $Sp(n,\mathbb{R})$ (cf. [4,5]). Therefore the following elements $t(b;\lambda,\mu,\kappa)$, $g(h)$ and σ_n of G^J defined by

$$t(b;\lambda,\mu,\kappa) = \left(\begin{pmatrix} E_n & b \\ 0 & E_n \end{pmatrix}, (\lambda,\mu;\kappa)\right), \quad b = {}^tb \text{ real}, \ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)},$$

$$g(h) = \left(\begin{pmatrix} {}^th & 0 \\ 0 & h^{-1} \end{pmatrix}, (0,0;0)\right), \quad h \in GL(n,\mathbb{R}),$$

$$\sigma_n = \left(\begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}, (0,0;0)\right)$$

generate the Jacobi group G^J. So it suffices to prove the invariance of the metric $ds^2_{n,m;A,B}$ under the action of the generators $t(b;\lambda,\mu,\kappa)$, $g(h)$ and σ_n. For brevity, we write

$$\begin{aligned}
(a) &= \sigma\big(Y^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\big),\\
(b) &= \sigma\big(Y^{-1}\,{}^tV\,V\,Y^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\big),\\
(c) &= \sigma\big(Y^{-1}\,{}^t(dW)\,d\overline{W}\big),\\
(d) &= -\sigma\big(V\,Y^{-1}\,dZ\,Y^{-1}\,{}^t(d\overline{W}) + V\,Y^{-1}\,d\overline{Z}\,Y^{-1}\,{}^t(dW)\big)
\end{aligned}$$

and

$$\begin{aligned}
(a)_* &= \sigma\big(Y_*^{-1}\,dZ_*\,Y_*^{-1}\,d\overline{Z}_*\big),\\
(b)_* &= \sigma\big(Y_*^{-1}\,{}^tV_*\,V_*Y_*^{-1}\,dZ_*\,Y_*^{-1}\,d\overline{Z}_*\big),\\
(c)_* &= \sigma\big(Y_*^{-1}\,{}^t(dW_*)\,d\overline{W}_*\big),\\
(d)_* &= -\sigma\big(V_*Y_*^{-1}\,dZ_*\,Y_*^{-1}\,{}^t(d\overline{W}_*) + V_*Y_*^{-1}\,d\overline{Z}_*\,Y_*^{-1}\,{}^t(dW_*)\big).
\end{aligned}$$

Case I. $g = t(b; \lambda, \mu, \kappa)$ with $b = {}^tb$ real and $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}$.

In this case, we have

$$Z_* = Z + b, \quad Y_* = Y, \quad W_* = W + \lambda Z + \mu, \quad V_* = V + \lambda Y$$

and

$$dZ_* = dZ, \quad dW_* = dW + \lambda\, dZ.$$

Therefore

$$\begin{aligned}
(a)_* &= \sigma\big(Y_*^{-1}\,dZ_*\,Y_*^{-1}\,d\overline{Z_*}\big) = \sigma\big(Y^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\big) = (a),\\
(b)_* &= \sigma\big(Y^{-1}\,{}^tV\;V\,Y^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\big) + \sigma\big(Y^{-1}\,{}^tV\lambda\,dZ\,Y^{-1}\,d\overline{Z}\big)\\
&\quad + \sigma\big({}^t\lambda\,V\,Y^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\big) + \sigma\big({}^t\lambda\,\lambda\,dZ\,Y^{-1}\,d\overline{Z}\big),\\
(c)_* &= \sigma\big(Y^{-1}\,{}^t(dW)\,d\overline{W}\big) + \sigma\big(Y^{-1}\,{}^t(dW)\lambda\,d\overline{Z}\big)\\
&\quad + \sigma\big(Y^{-1}\,dZ\,{}^t\lambda\,d\overline{W}\big) + \sigma\big(Y^{-1}\,dZ\,{}^t\lambda\,\lambda\,d\overline{Z}\big)
\end{aligned}$$

and

$$\begin{aligned}
(d)_* &= -\sigma\big(V\,Y^{-1}\,dZ\,Y^{-1}\,{}^t(d\overline{W})\big) - \sigma\big(\lambda\,dZ\,Y^{-1}\,{}^t(d\overline{W})\big)\\
&\quad - \sigma\big(V\,Y^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\,{}^t\lambda\big) - \sigma\big(\lambda\,dZ\,Y^{-1}\,d\overline{Z}\,{}^t\lambda\big)\\
&\quad - \sigma\big(V\,Y^{-1}\,d\overline{Z}\,Y^{-1}\,{}^t(dW)\big) - \sigma\big(\lambda\,d\overline{Z}\,Y^{-1}\,{}^t(dW)\big)\\
&\quad - \sigma\big(V\,Y^{-1}\,d\overline{Z}\,Y^{-1}\,dZ\,{}^t\lambda\big) - \sigma\big(\lambda\,d\overline{Z}\,Y^{-1}\,dZ\,{}^t\lambda\big).
\end{aligned}$$

Thus we see that

$$(a) = (a)_* \quad \text{and} \quad (b) + (c) + (d) = (b)_* + (c)_* + (d)_*.$$

Hence

$$ds^2_{n,m;A,B} = A(a) + B\{(b) + (c) + (d)\}$$

is invariant under the action of $t(B; \lambda, \mu, \kappa)$.

Case II. $g = g(h)$ with $h \in GL(n, \mathbb{R})$.

In this case, we have

$$Z_* = {}^t h\, Z h, \quad Y_* = {}^t h\, Y h, \quad W_* = W h, \quad V_* = V h$$

and

$$dZ_* = {}^t h\, dZ\, h, \quad dW_* = dW\, h.$$

Therefore by an easy computation, we see that each of (a), (b), (c) and (d) is invariant under the action of all $g(h)$ with $h \in GL(n, \mathbb{R})$. Hence the metric $ds^2_{n,m;A,B}$ is invariant under the action of all $g(h)$ with $h \in GL(n, \mathbb{R})$.

Case III. $g = \sigma_n = \left(\begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}, (0, 0; 0)\right)$.

In this case, we have

$$Z_* = -Z^{-1} \quad \text{and} \quad W_* = W Z^{-1}. \tag{2.4}$$

We set

$$\theta_1 := \operatorname{Re} Z^{-1} \quad \text{and} \quad \theta_2 := \operatorname{Im} Z^{-1}.$$

Then θ_1 and θ_2 are symmetric matrices and we have

$$Y_* = -\theta_2 \quad \text{and} \quad V_* := \operatorname{Im} W_* = V\theta_1 + U\theta_2. \tag{2.5}$$

It is easy to see that

$$Y = -Z\theta_2\overline{Z} = -\overline{Z}\theta_2 Z, \tag{2.6}$$

$$\theta_1 Y + \theta_2 X = 0 \tag{2.7}$$

and

$$\theta_1 X - \theta_2 Y = E_n. \tag{2.8}$$

According to (2.6) and (2.7), we obtain

$$X = (-\theta_2)^{-1}\theta_1 Y \quad \text{and} \quad Y^{-1} = \theta_1(-\theta_2)^{-1}\theta_1 - \theta_2. \tag{2.9}$$

From (2.1) and (2.2), we have

$$dZ_* = Z^{-1}\,dZ\,Z^{-1} \tag{2.10}$$

and

$$dW_* = dW\,Z^{-1} - WZ^{-1}\,dZ\,Z^{-1} = \big(dW - WZ^{-1}\,dZ\big)Z^{-1}. \tag{2.11}$$

Therefore we have, according to (2.6) and (2.10),

$$\begin{aligned}(a)_* &= \sigma\big((-\theta_2)^{-1}Z^{-1}\,dZ\,Z^{-1}(-\theta_2)^{-1}\overline{Z}^{-1}\,d\overline{Z}\,\overline{Z}^{-1}\big)\\ &= \sigma\big(Y^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\big) = (a).\end{aligned}$$

According to (2.5)–(2.10), we have

$$\begin{aligned}(b)_* &= \sigma\big((-\theta_2)^{-1}\big(\theta_1\,{}^tV + \theta_2\,{}^tU\big)(V\theta_1 + U\theta_2)(-\theta_2)^{-1}Z^{-1}\,dZ\,Y^{-1}d\overline{Z}\,\overline{Z}^{-1}\big)\\ &= \sigma\big(\big\{{}^tU - (-\theta_2)^{-1}\theta_1\,{}^tV\big\}\big\{U - V\theta_1(-\theta_2)^{-1}\big\}Z^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\,\overline{Z}^{-1}\big)\\ &= \sigma\big(\big\{{}^t\overline{W} + \big(iE_n - (-\theta_2)^{-1}\theta_1\big)\,{}^tV\big\}\big\{W - V\big(iE_n + \theta_1(-\theta_2)^{-1}\big)\big\}Z^{-1}dZ\,Y^{-1}\,d\overline{Z}\,\overline{Z}^{-1}\big),\\ (c)_* &= \sigma\big((-\theta_2)^{-1}\big(Z^{-1}\,{}^t(dW) - Z^{-1}\,dZ\,Z^{-1}\,{}^tW\big)\big(d\overline{W}\,\overline{Z}^{-1} - \overline{W}\,\overline{Z}^{-1}\,d\overline{Z}\,\overline{Z}^{-1}\big)\big)\\ &= \sigma\big((-\theta_2)^{-1}Z^{-1}\,{}^t(dW)\,d\overline{W}\ \overline{Z}^{-1} - (-\theta_2)^{-1}Z^{-1}\,{}^t(dW)\overline{W}\,\overline{Z}^{-1}\,d\overline{Z}\,\overline{Z}^{-1}\\ &\quad - (-\theta_2)^{-1}Z^{-1}\,dZ\,Z^{-1}\,{}^tW d\overline{W}\,\overline{Z}^{-1}\\ &\quad + (-\theta_2)^{-1}Z^{-1}\,dZ\,Z^{-1}\,{}^tW\,\overline{W}\,\overline{Z}^{-1}\,d\overline{Z}\,\overline{Z}^{-1}\big)\\ &= \sigma\big(Y^{-1}\,{}^t(dW)\,d\overline{W}\big) - \sigma\big(Y^{-1}\,{}^t(dW)\overline{W}\,\overline{Z}^{-1}\,d\overline{Z}\big)\\ &\quad - \sigma\big(Y^{-1}\,dZ\,Z^{-1}\,{}^tW\,d\overline{W}\big) + \sigma\big(Y^{-1}\,dZ\,Z^{-1}\,{}^tW\,\overline{W}\,\overline{Z}^{-1}\,d\overline{Z}\big)\end{aligned}$$

and

$$\begin{aligned}(d)_* &= -\sigma\big((V\theta_1 + U\theta_2)(-\theta_2)^{-1}Z^{-1}\,dZ\,Z^{-1}(-\theta_2)^{-1}\big\{\overline{Z}^{-1}\,{}^t(d\overline{W}) - \overline{Z}^{-1}\,d\overline{Z}\,\overline{Z}^{-1}\,{}^t\overline{W}\big\}\big)\\ &\quad - \sigma\big((V\theta_1 + U\theta_2)(-\theta_2)^{-1}\overline{Z}^{-1}\,d\overline{Z}\,\overline{Z}^{-1}(-\theta_2)^{-1}\big\{Z^{-1}\,{}^t(dW) - Z^{-1}\,dZ\,Z^{-1}\,{}^tW\big\}\big)\\ &= -\sigma\big((V\theta_1 + U\theta_2)(-\theta_2)^{-1}Z^{-1}\,dZ\,Y^{-1}\,{}^t(d\overline{W})\big)\\ &\quad + \sigma\big((V\theta_1 + U\theta_2)(-\theta_2)^{-1}Z^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\,\overline{Z}^{-1}\,{}^t\overline{W}\big)\\ &\quad - \sigma\big((V\theta_1 + U\theta_2)(-\theta_2)^{-1}\overline{Z}^{-1}\,d\overline{Z}\,Y^{-1}\,{}^t(dW)\big)\\ &\quad + \sigma\big((V\theta_1 + U\theta_2)(-\theta_2)^{-1}\overline{Z}^{-1}\,d\overline{Z}\,Y^{-1}\,dZ\,Z^{-1}\,{}^tW\big).\end{aligned}$$

Taking the $(dZ, d\overline{W})$-part $\square(Z, \overline{W})$ in $(b)_* + (c)_* + (d)_*$, we have

$$\begin{aligned}\square(Z, \overline{W}) &= -\sigma\big(VY^{-1}\,dZ\,Y^{-1}\,{}^t(d\overline{W})\big) + \sigma\big(Y^{-1}\,dZ\big({}^tW_* - Z^{-1}\,{}^tW\big)\,d\overline{W}\big)\\ &= -\sigma\big(VY^{-1}\,dZ\,Y^{-1}\,{}^t(d\overline{W})\big) \quad \text{because } W_* = WZ^{-1}\ \big(\text{cf. (2.4)}\big).\end{aligned}$$

Similarly, if we take the $(d\overline{Z}, dW)$-part $\square(\overline{Z}, W)$ in $(b)_* + (c)_* + (d)_*$, we have

$$\begin{aligned}\square(\overline{Z}, W) &= -\sigma\big(VY^{-1}\,d\overline{Z}\,Y^{-1}\,{}^t(dW)\big) + \sigma\big(d\overline{Z}\,Y^{-1}\,{}^t(dW)\big(\overline{W_*} - \overline{W}\,\overline{Z}^{-1}\big)\big)\\ &= -\sigma\big(VY^{-1}\,d\overline{Z}\,Y^{-1}\,{}^t(dW)\big) \quad \text{because } W_* = WZ^{-1}.\end{aligned}$$

If we take the $(dW, d\overline{W})$-part $\square(W, \overline{W})$ in $(b)_* + (c)_* + (d)_*$, we have

$$\square(W, \overline{W}) = \sigma\big(Y^{-1}\,{}^t(dW)\,d\overline{W}\big).$$

Finally, if we take the $(dZ, d\overline{Z})$-part $\square(Z, \overline{Z})$ in $(b)_* + (c)_* + (d)_*$, we have

$$\begin{aligned}\square(Z, \overline{Z}) = {}&\sigma\big(\big\{{}^t\overline{W} + \big(iE_n - (-\theta_2)^{-1}\theta_1\big)\,{}^tV\big\}\big\{W - V\big(iE_n + \theta_1(-\theta_2)^{-1}\big)\big\}Z^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\,\overline{Z}^{-1}\big)\\ &+ \sigma\big(Z^{-1}\,{}^tW\,\overline{W}\,\overline{Z}^{-1}\,d\overline{Z}\,Y^{-1}\,dZ\big)\\ &+ \sigma\big({}^t\overline{W}(V\theta_1 + U\theta_2)(-\theta_2)^{-1}Z^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\,\overline{Z}^{-1}\big)\\ &+ \sigma\big({}^tW(V\theta_1 + U\theta_2)(-\theta_2)^{-1}\overline{Z}^{-1}\,d\overline{Z}\,Y^{-1}\,dZ\,Z^{-1}\big).\end{aligned}$$

Since

$$\begin{aligned}(V\theta_1 + U\theta_2)(-\theta_2)^{-1} &= -U + V\theta_1(-\theta_2)^{-1}\\ &= -W + V\big\{iE_n + \theta_1(-\theta_2)^{-1}\big\}\\ &= -\overline{W} - V\big\{iE_n - \theta_1(-\theta_2)^{-1}\big\},\end{aligned}$$

we have

$$\begin{aligned}\square(Z, \overline{Z}) &= \sigma\big(\overline{Z}^{-1}\big\{iE_n - (-\theta_2)^{-1}\theta_1\big\}\,{}^tV\big\{W - V\big(iE_n + \theta_1(-\theta_2)^{-1}\big)\big\}Z^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\big)\\ &\quad - \sigma\big(\overline{Z}^{-1}\big\{iE_n - (-\theta_2)^{-1}\theta_1\big\}\,{}^tV\,WZ^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\big)\\ &= -\sigma\big(\overline{Z}^{-1}\big\{iE_n - (-\theta_2)^{-1}\theta_1\big\}\,{}^tV\,V\big\{iE_n + \theta_1(-\theta_2)^{-1}\big\}Z^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\big).\end{aligned}$$

By the way, according to (2.9), we obtain

$$\begin{aligned}\overline{Z}^{-1}\big\{iE_n - (-\theta_2)^{-1}\theta_1\big\} &= (\theta_1 - i\theta_2)\big\{iE_n - (-\theta_2)^{-1}\theta_1\big\}\\ &= \theta_2 - \theta_1(-\theta_2)^{-1}\theta_1 = -Y^{-1}\end{aligned}$$

and

$$\big\{iE_n + \theta_1(-\theta_2)^{-1}\big\}Z^{-1} = \theta_1(-\theta_2)^{-1}\theta_1 - \theta_2 = Y^{-1}.$$

Therefore

$$\square(Z, \overline{Z}) = \sigma\big(Y^{-1}\,{}^tV\,VY^{-1}\,dZ\,Y^{-1}\,d\overline{Z}\big).$$

Hence $(a) = (a)_*$ and

$$(b)_* + (c)_* + (d)_* = \square(Z, \overline{W}) + \square(\overline{Z}, W) + \square(W, \overline{W}) + \square(Z, \overline{Z})$$
$$= (b) + (c) + (d).$$

This implies that the metric

$$ds^2_{n,m;A,B} = A(a) + B\{(b) + (c) + (d)\}$$

is invariant under the action (1.2) of σ_n.

Consequently $ds^2_{n,m;A,B}$ is invariant under the action (1.2) of the Jacobi group G^J. In particular, for $(Z, W) = (iE_n, 0)$, we have

$$\begin{aligned} ds^2_{n,m;A,B} &= A \cdot \sigma(dZ\,d\overline{Z}) + B \cdot \sigma\big({}^t(dW)\,d\overline{W}\big) \\ &= A\left\{\sum_{\mu=1}^{n}\big(dx^2_{\mu\mu} + dy^2_{\mu\mu}\big) + 2\sum_{1\leqslant\mu<\nu\leqslant n}\big(dx^2_{\mu\nu} + dy^2_{\mu\nu}\big)\right\} \\ &\quad + B\left\{\sum_{1\leqslant k\leqslant m,\,1\leqslant l\leqslant n}\big(du^2_{kl} + dv^2_{kl}\big)\right\}, \end{aligned}$$

which is clearly positive definite. Since G^J acts on $\mathbb{H}_{n,m}$ transitively, $ds^2_{n,m;A,B}$ is positive definite everywhere in $\mathbb{H}_{n,m}$. This completes the proof of Theorem 1.1. □

Remark 2.1. The scalar curvature of the Siegel–Jacobi space $(\mathbb{H}_{n,m}, ds^2_{n,m;A,B})$ is constant because of the transitive group action of G^J on $\mathbb{H}_{n,m}$. In the special case $n = m = 1$ and $A = B = 1$, by a direct computation, we see that the scalar curvature of $(\mathbb{H}_{1,1}, ds^2_{1,1;1,1})$ is -3.

3. Proof of Theorem 1.2

If $(Z_*, W_*) = g \cdot (Z, W)$ with $g = \left(\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right), (\lambda, \mu; \kappa)\right) \in G^J$, we can easily see that

$$\begin{aligned} \frac{\partial}{\partial Z_*} &= (CZ + D)\,{}^t\!\left\{(CZ + D)\frac{\partial}{\partial Z}\right\} \\ &\quad + (CZ + D)\,{}^t\!\left\{\big(C\,{}^tW + C\,{}^t\mu - D\,{}^t\lambda\big)\,{}^t\!\left(\frac{\partial}{\partial W}\right)\right\} \end{aligned} \tag{3.1}$$

and

$$\frac{\partial}{\partial W_*} = (CZ + D)\frac{\partial}{\partial W}. \tag{3.2}$$

For brevity, we put

$$(\alpha) := 4\sigma\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial \overline{Z}}\right)\frac{\partial}{\partial Z}\right),$$
$$(\beta) := 4\sigma\left(Y\frac{\partial}{\partial W}\,{}^t\!\left(\frac{\partial}{\partial \overline{W}}\right)\right),$$

$$(\gamma) := 4\sigma\left(VY^{-1}\,{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial \overline{W}}\right)\frac{\partial}{\partial W}\right),$$
$$(\delta) := 4\sigma\left(V\,{}^t\!\left(Y\frac{\partial}{\partial \overline{Z}}\right)\frac{\partial}{\partial W}\right)$$

and

$$(\epsilon) := 4\sigma\left({}^tV\,{}^t\!\left(Y\frac{\partial}{\partial \overline{W}}\right)\frac{\partial}{\partial Z}\right).$$

We also set

$$(\alpha)_* := 4\sigma\left(Y_*\,{}^t\!\left(Y_*\frac{\partial}{\partial \overline{Z}_*}\right)\frac{\partial}{\partial Z_*}\right),$$
$$(\beta)_* := 4\sigma\left(Y_*\frac{\partial}{\partial W_*}\,{}^t\!\left(\frac{\partial}{\partial \overline{W}_*}\right)\right),$$
$$(\gamma)_* := 4\sigma\left(V_*Y_*^{-1}\,{}^tV_*\,{}^t\!\left(Y_*\frac{\partial}{\partial \overline{W}_*}\right)\frac{\partial}{\partial W_*}\right),$$
$$(\delta)_* := 4\sigma\left(V_*\,{}^t\!\left(Y_*\frac{\partial}{\partial \overline{Z}_*}\right)\frac{\partial}{\partial W_*}\right)$$

and

$$(\epsilon)_* := 4\sigma\left({}^tV_*\,{}^t\!\left(Y_*\frac{\partial}{\partial \overline{W}_*}\right)\frac{\partial}{\partial Z_*}\right).$$

We need the following lemma for the proof of Theorem 1.2. H. Maass [8] observed the following useful fact.

Lemma 3.1.

(a) *Let A be an $n \times k$ matrix and let B be a $k \times n$ matrix. Assume that the entries of A commute with the entries of B. Then $\sigma(AB) = \sigma(BA)$.*

(b) *Let A be an $m \times n$ matrix and B an $n \times l$ matrix. Assume that the entries of A commute with the entries of B. Then ${}^t(AB) = {}^tB\,{}^tA$.*

(c) *Let A, B and C be a $k \times l$, an $n \times m$ and an $m \times l$ matrix, respectively. Assume that the entries of A commute with the entries of B. Then*

$${}^t\big(A\,{}^t(BC)\big) = B\,{}^t\big(A\,{}^tC\big).$$

Proof. The proof follows immediately from the direct computation. □

Now we are ready to prove Theorem 1.2. First of all, we shall prove that $\Delta_{n,m;A,B}$ is invariant under the action of the generators $t(b; \lambda, \mu, \kappa)$, $g(h)$ and σ_n.

Case I. $g = t(b; \lambda, \mu, \kappa) = \left(\begin{pmatrix} E_n & b \\ 0 & E_n \end{pmatrix}, (\lambda, \mu; \kappa)\right)$ with $b = {}^tb$ real.

In this case, we have

$$Y_* = Y, \qquad V_* = V + \lambda Y$$

and

$$\frac{\partial}{\partial Z_*} = \frac{\partial}{\partial Z} - {}^t\left({}^t\lambda\, {}^t\left(\frac{\partial}{\partial W}\right)\right) \quad \text{and} \quad \frac{\partial}{\partial W_*} = \frac{\partial}{\partial W}.$$

Using Lemma 3.1, we obtain

$$\begin{aligned}
(\alpha)_* &= (\alpha) - \sigma\left(\lambda Y\, {}^t\left(Y\frac{\partial}{\partial \overline{Z}}\right)\frac{\partial}{\partial W}\right)\\
&\quad - \sigma\left(Y\, {}^t\lambda\, {}^t\left(Y\frac{\partial}{\partial \overline{W}}\right)\frac{\partial}{\partial Z}\right) + \sigma\left(\lambda Y\, {}^t\lambda\, {}^t\left(Y\frac{\partial}{\partial \overline{W}}\right)\frac{\partial}{\partial W}\right)\\
(\beta)_* &= \sigma\left(Y_*\frac{\partial}{\partial W_*}\, {}^t\left(\frac{\partial}{\partial \overline{W}_*}\right)\right) = (\beta),\\
(\gamma)_* &= (\gamma) + \sigma\left(\lambda\, {}^tV\, {}^t\left(Y\frac{\partial}{\partial \overline{W}}\right)\frac{\partial}{\partial W}\right)\\
&\quad + \sigma\left(V\, {}^t\lambda\, {}^t\left(Y\frac{\partial}{\partial \overline{W}}\right)\frac{\partial}{\partial W}\right) + \sigma\left(\lambda Y\, {}^t\lambda\, {}^t\left(Y\frac{\partial}{\partial \overline{W}}\right)\frac{\partial}{\partial W}\right),\\
(\delta)_* &= (\delta) + \sigma\left(\lambda Y\, {}^t\left(Y\frac{\partial}{\partial \overline{Z}}\right)\frac{\partial}{\partial W}\right)\\
&\quad - \sigma\left(V\, {}^t\lambda\, {}^t\left(Y\frac{\partial}{\partial \overline{W}}\right)\frac{\partial}{\partial W}\right) + \sigma\left(\lambda Y\, {}^t\lambda\, {}^t\left(Y\frac{\partial}{\partial \overline{W}}\right)\frac{\partial}{\partial W}\right)
\end{aligned}$$

and

$$\begin{aligned}
(\epsilon)_* &= (\epsilon) + \sigma\left(Y\, {}^t\lambda\, {}^t\left(Y\frac{\partial}{\partial \overline{W}}\right)\frac{\partial}{\partial Z}\right)\\
&\quad - \sigma\left(\lambda\, {}^tV\, {}^t\left(Y\frac{\partial}{\partial \overline{W}}\right)\frac{\partial}{\partial W}\right) - \sigma\left(\lambda Y\, {}^t\lambda\, {}^t\left(Y\frac{\partial}{\partial \overline{W}}\right)\frac{\partial}{\partial W}\right).
\end{aligned}$$

Thus $(\beta) = (\beta)_*$ and

$$(\alpha) + (\gamma) + (\delta) + (\epsilon) = (\alpha)_* + (\gamma)_* + (\delta)_* + (\epsilon)_*.$$

Hence

$$\Delta_{n,m;A,B} = \frac{4}{B}(\beta) + \frac{4}{A}\bigl\{(\alpha) + (\gamma) + (\delta) + (\epsilon)\bigr\}$$

is invariant under the action of all $t(b; \lambda, \mu, \kappa)$.

Case II. $g = g(h) = \left(\begin{pmatrix} {}^th & 0 \\ 0 & h^{-1} \end{pmatrix}, (0, 0; 0)\right)$ with $h \in GL(n, \mathbb{R})$.

In this case, we have

$$Y_* = {}^t h Y h, \qquad V_* = V h$$

and

$$\frac{\partial}{\partial Z_*} = h^{-1}\,{}^t\!\left(h^{-1}\frac{\partial}{\partial Z}\right), \qquad \frac{\partial}{\partial W_*} = h^{-1}\frac{\partial}{\partial W}.$$

According to Lemma 3.1, we see that each of (α), (β), (γ), (δ) and (ϵ) is invariant under the action of all $g(h)$ with $h \in GL(n,\mathbb{R})$. Therefore $\Delta_{n,m;A,B}$ is invariant under the action of all $g(h)$ with $h \in GL(n,\mathbb{R})$.

Case III. $g = \sigma_n = \left(\begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}, (0,0;0)\right)$.

In this case, we have

$$Z_* = -Z^{-1} \quad \text{and} \quad W_* = W Z^{-1}.$$

We set

$$\theta_1 := \operatorname{Re} Z^{-1} \quad \text{and} \quad \theta_2 := \operatorname{Im} Z^{-1}.$$

Then we obtain the relations (2.5)–(2.9). From (2.6), we have the relation

$$\theta_2 \overline{Z} = -Z^{-1} Y. \tag{3.3}$$

It follows from the relation (2.3) that

$$Y_* = \overline{Z}^{-1} Y Z^{-1} = Z^{-1} Y \overline{Z}^{-1} = -\theta_2. \tag{3.4}$$

From (2.9), we obtain

$$\theta_1 \theta_2^{-1} \theta_1 = -Y^{-1} - \theta_2. \tag{3.5}$$

According to (3.1) and (3.2), we have

$$\frac{\partial}{\partial Z_*} = Z\,{}^t\!\left(Z\frac{\partial}{\partial Z}\right) + Z\,{}^t\!\left({}^t W\,{}^t\!\left(\frac{\partial}{\partial W}\right)\right) \tag{3.6}$$

and

$$\frac{\partial}{\partial W_*} = Z\frac{\partial}{\partial W}. \tag{3.7}$$

From (2.6), (3.3) and Lemma 3.1, we obtain

$$(\alpha)_* = (\alpha) - \sigma\left(U\theta_2\overline{Z}\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial W}\right) - i\sigma\left(V\theta_2\overline{Z}\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial W}\right)$$
$$- \sigma\left(Z\theta_2{}^tU\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial Z}\right) + i\sigma\left(Z\theta_2{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial Z}\right)$$
$$- \sigma\left(W\theta_2{}^tU\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial W}\right) + \sigma\left(W\theta_2{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial W}\right).$$

From the relation (3.4), we see $(\beta)_* = (\beta)$. According to (3.3), (3.5) and Lemma 3.1, we obtain

$$(\gamma)_* = (\gamma) + \sigma\left(\left(V\theta_2{}^tV - V\theta_1{}^tU - U\theta_1{}^tV - U\theta_2{}^tU\right)\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial W}\right).$$

Using the relation (3.3) and Lemma 3.1, we finally obtain

$$(\delta)_* = \sigma\left(V\theta_1\overline{Z}\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial W}\right) + \sigma\left(U\theta_2\overline{Z}\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial W}\right)$$
$$+ \sigma\left(V\theta_1{}^t\overline{W}\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial W}\right) + \sigma\left(U\theta_2{}^t\overline{W}\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial W}\right)$$

and

$$(\epsilon)_* = \sigma\left(Z\theta_1{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial Z}\right) + \sigma\left(Z\theta_2{}^tU\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial Z}\right)$$
$$+ \sigma\left(W\theta_1{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial W}\right) + \sigma\left(W\theta_2{}^tU\,{}^t\!\left(Y\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial W}\right).$$

Using the fact $Z^{-1} = \theta_1 + i\theta_2$, we can show that

$$(\alpha) + (\gamma) + (\delta) + (\epsilon) = (\alpha)_* + (\gamma)_* + (\delta)_* + (\epsilon)_*.$$

Hence

$$\Delta_{n,m;A,B} = \frac{4}{B}(\beta) + \frac{4}{A}\left\{(\alpha) + (\gamma) + (\delta) + (\epsilon)\right\}$$

is invariant under the action of σ_n.

Consequently $\Delta_{n,m;A,B}$ is invariant under the action (1.2) of G^J. In particular, for $(Z, W) = (iE_n, 0)$, the differential operator $\Delta_{n,m;A,B}$ coincides with the Laplacian for the metric $ds^2_{n,m;A,B}$. It follows from the invariance of $\Delta_{n,m;A,B}$ under the action (1.2) and the transitivity of the action of G^J on $\mathbb{H}_{n,m}$ that $\Delta_{n,m;A,B}$ is the Laplacian of $(\mathbb{H}_{n,m}, ds^2_{n,m;A,B})$. The invariance of the differential form dv follows from the fact that the following differential form

$$(\det Y)^{-(n+1)}[dX] \wedge [dY]$$

is invariant under the action (1.1) of $Sp(n,\mathbb{R})$ (cf. [13, p. 130]). □

4. Remark on spectral theory of $\Delta_{n,m;A,B}$ on Siegel–Jacobi space

Before we describe a fundamental domain for the Siegel–Jacobi space, we review the Siegel's fundamental domain for the Siegel upper half plane.

We let

$$\mathcal{P}_n = \{ Y \in \mathbb{R}^{(n,n)} \mid Y = {}^tY > 0 \}$$

be an open cone in $\mathbb{R}^{n(n+1)/2}$. The general linear group $GL(n,\mathbb{R})$ acts on $\mathcal{P}_n$ transitively by

$$h \circ Y := hY\,{}^th, \quad h \in GL(n,\mathbb{R}),\ Y \in \mathcal{P}_n.$$

Thus $\mathcal{P}_n$ is a symmetric space diffeomorphic to $GL(n,\mathbb{R})/O(n)$. We let

$$GL(n,\mathbb{Z}) = \{ h \in GL(n,\mathbb{R}) \mid h \text{ is integral} \}$$

be the discrete subgroup of $GL(n,\mathbb{R})$.

The fundamental domain $\mathcal{R}_n$ for $GL(n,\mathbb{Z}) \setminus \mathcal{P}_n$ which was found by H. Minkowski [10] is defined as a subset of $\mathcal{P}_n$ consisting of $Y = (y_{ij}) \in \mathcal{P}_n$ satisfying the following conditions (M.1)–(M.2) (cf. [9, p. 123]):

(M.1) $aY\,{}^ta \geqslant y_{kk}$ for every $a = (a_i) \in \mathbb{Z}^n$ in which $a_k, \ldots, a_n$ are relatively prime for $k = 1, 2, \ldots, n$.

(M.2) $y_{k,k+1} \geqslant 0$ for $k = 1, \ldots, n-1$.

We say that a point of $\mathcal{R}_n$ is *Minkowski reduced* or simply *M-reduced*.

Siegel [12] determined a fundamental domain $\mathcal{F}_n$ for $\Gamma_n \setminus \mathbb{H}_n$, where $\Gamma_n = Sp(n,\mathbb{Z})$ is the Siegel modular group of degree n. We say that $\Omega = X + iY \in \mathbb{H}_n$ with X, Y real is *Siegel reduced* or *S-reduced* if it has the following three properties:

(S.1) $\det(\operatorname{Im}(\gamma \cdot \Omega)) \leqslant \det(\operatorname{Im}(\Omega))$ for all $\gamma \in \Gamma_n$;

(S.2) $Y = \operatorname{Im} \Omega$ is M-reduced, that is, $Y \in \mathcal{R}_n$;

(S.3) $|x_{ij}| \leqslant \frac{1}{2}$ for $1 \leqslant i, j \leqslant n$, where $X = (x_{ij})$.

$\mathcal{F}_n$ is defined as the set of all Siegel reduced points in $\mathbb{H}_n$. Using the highest point method, Siegel [12] proved the following (F1)–(F3) (cf. [9, p. 169]):

(F1) $\Gamma_n \cdot \mathcal{F}_n = \mathbb{H}_n$, i.e., $\mathbb{H}_n = \bigcup_{\gamma \in \Gamma_n} \gamma \cdot \mathcal{F}_n$.

(F2) $\mathcal{F}_n$ is closed in $\mathbb{H}_n$.

(F3) $\mathcal{F}_n$ is connected and the boundary of $\mathcal{F}_n$ consists of a finite number of hyperplanes.

The metric ds_n^2 given by (1.3) induces a metric $ds_{\mathcal{F}_n}^2$ on $\mathcal{F}_n$. Siegel [12] computed the volume of $\mathcal{F}_n$

$$\operatorname{vol}(\mathcal{F}_n) = 2 \prod_{k=1}^{n} \pi^{-k} \Gamma(k) \zeta(2k),$$

where $\Gamma(s)$ denotes the Gamma function and $\zeta(s)$ denotes the Riemann zeta function. For instance,

$$\mathrm{vol}(\mathcal{F}_1)=\frac{\pi}{3},\quad \mathrm{vol}(\mathcal{F}_2)=\frac{\pi^3}{270},\quad \mathrm{vol}(\mathcal{F}_3)=\frac{\pi^6}{127575},\quad \mathrm{vol}(\mathcal{F}_4)=\frac{\pi^{10}}{200930625}.$$

Let f_{kl} $(1\leqslant k\leqslant m,\ 1\leqslant l\leqslant n)$ be the $m\times n$ matrix with entry 1 where the kth row and the lth column meet, and all other entries 0. For an element $\Omega\in\mathbb{H}_n$, we set for brevity

$$h_{kl}(\Omega):=f_{kl}\Omega,\quad 1\leqslant k\leqslant m,\ 1\leqslant l\leqslant n.$$

For each $\Omega\in\mathcal{F}_n$, we define a subset P_Ω of $\mathbb{C}^{(m,n)}$ by

$$P_\Omega=\left\{\sum_{k=1}^{m}\sum_{j=1}^{n}\lambda_{kl}f_{kl}+\sum_{k=1}^{m}\sum_{j=1}^{n}\mu_{kl}h_{kl}(\Omega)\ \Big|\ 0\leqslant\lambda_{kl},\mu_{kl}\leqslant 1\right\}.$$

For each $\Omega\in\mathcal{F}_n$, we define the subset D_Ω of $\mathbb{H}_n\times\mathbb{C}^{(m,n)}$ by

$$D_\Omega:=\left\{(\Omega,Z)\in\mathbb{H}_n\times\mathbb{C}^{(m,n)}\ \big|\ Z\in P_\Omega\right\}.$$

We define

$$\mathcal{F}_{n,m}:=\bigcup_{\Omega\in\mathcal{F}_n}D_\Omega.$$

Theorem 4.1. *Let*

$$\Gamma_{n,m}:=Sp(n,\mathbb{Z})\ltimes H_{\mathbb{Z}}^{(n,m)}$$

be the discrete subgroup of G^J, where

$$H_{\mathbb{Z}}^{(n,m)}=\left\{(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}\ \big|\ \lambda,\mu,\kappa\ \textit{are integral}\right\}.$$

Then $\mathcal{F}_{n,m}$ is a fundamental domain for $\Gamma_{n,m}\backslash\mathbb{H}_{n,m}$.

Proof. The proof can be found in [20]. □

In the case $n=m=1$, R. Berndt [2] introduced the notion of Maass–Jacobi forms. Now we generalize this notion to the general case.

Definition 4.1. For brevity, we set $\Delta_{n,m}:=\Delta_{n,m;1,1}$ (cf. Theorem 1.2). Let

$$\Gamma_{n,m}:=Sp(n,\mathbb{Z})\ltimes H_{\mathbb{Z}}^{(n,m)}$$

be the discrete subgroup of G^J, where

$$H_{\mathbb{Z}}^{(n,m)}=\left\{(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}\ \big|\ \lambda,\mu,\kappa\ \text{are integral}\right\}.$$

A smooth function $f : \mathbb{H}_{n,m} \to \mathbb{C}$ is called a *Maass–Jacobi form* on $\mathbb{H}_{n,m}$ if f satisfies the following conditions (MJ1)–(MJ3):

(MJ1) f is invariant under $\Gamma_{n,m}$.
(MJ2) f is an eigenfunction of the Laplacian $\Delta_{n,m}$.
(MJ3) f has a polynomial growth, that is, there exist a constant $C > 0$ and a positive integer N such that

$$\left|f(X+iY,Z)\right| \leqslant C\left|p(Y)\right|^N \quad \text{as } \det Y \to \infty,$$

where $p(Y)$ is a polynomial in $Y = (y_{ij})$.

It is natural to propose the following problems.

Problem A. Construct Maass–Jacobi forms.

Problem B. Find all the eigenfunctions of $\Delta_{n,m}$.

We consider the simple case $n = m = 1$. A metric $ds^2_{1,1}$ on $\mathbf{H}_1 \times \mathbb{C}$ given by

$$ds^2_{1,1} = \frac{y+v^2}{y^3}\left(dx^2+dy^2\right) + \frac{1}{y}\left(du^2+dv^2\right) - \frac{2v}{y^2}(dx\,du + dy\,dv)$$

is a G^J-invariant Kähler metric on $\mathbf{H}_1 \times \mathbb{C}$. Its Laplacian $\Delta_{1,1}$ is given by

$$\Delta_{1,1} = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \left(y+v^2\right)\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + 2yv\left(\frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v}\right).$$

We provide some examples of eigenfunctions of $\Delta_{1,1}$.

(1) $h(x,y) = y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|a|y)e^{2\pi i a x}$ ($s \in \mathbb{C}$, $a \neq 0$) with eigenvalue $s(s-1)$. Here

$$K_s(z) := \frac{1}{2}\int_0^\infty \exp\left\{-\frac{z}{2}\left(t+t^{-1}\right)\right\} t^{s-1}\,dt,$$

where $\operatorname{Re} z > 0$.
(2) y^s, $y^s x$, $y^s u$ ($s \in \mathbb{C}$) with eigenvalue $s(s-1)$.
(3) $y^s v$, $y^s uv$, $y^s xv$ with eigenvalue $s(s+1)$.
(4) x, y, u, v, xv, uv with eigenvalue 0.
(5) All Maass wave forms.

We fix two positive integers m and n throughout this section.
For an element $\Omega \in \mathbb{H}_n$, we set

$$L_\Omega := \mathbb{Z}^{(m,n)} + \mathbb{Z}^{(m,n)}\Omega.$$

It follows from the positivity of $\operatorname{Im}\Omega$ that the elements $f_{kl}, h_{kl}(\Omega)$ $(1 \leqslant k \leqslant m,\ 1 \leqslant l \leqslant n)$ of L_Ω are linearly independent over $\mathbb{R}$. Therefore L_Ω is a lattice in $\mathbb{C}^{(m,n)}$ and the set $\{f_{kl}, h_{kl}(\Omega) \mid 1 \leqslant k \leqslant m,\ 1 \leqslant l \leqslant n\}$ forms an integral basis of L_Ω. We see easily that if Ω is an element of $\mathbb{H}_n$, the period matrix $\Omega_* := (I_n, \Omega)$ satisfies the Riemann conditions (RC.1) and (RC.2):

(RC.1) $\Omega_* J_n\, {}^t\Omega_* = 0$;
(RC.2) $-\frac{1}{i}\Omega_* J_n\, {}^t\overline{\Omega}_* > 0$.

Thus the complex torus $A_\Omega := \mathbb{C}^{(m,n)}/L_\Omega$ is an abelian variety.

It might be interesting to investigate the spectral theory of the Laplacian $\Delta_{n,m}$ on a fundamental domain $\mathcal{F}_{n,m}$. But this work is very complicated and difficult at this moment. It may be that the first step is to develop the spectral theory of the Laplacian Δ_Ω on the abelian variety A_Ω. The second step will be to study the spectral theory of the Laplacian Δ_n (see (1.4)) on the moduli space $\Gamma_n \backslash \mathbb{H}_n$ of principally polarized abelian varieties of dimension n. The final step would be to combine the above steps and more works to develop the spectral theory of the Laplacian $\Delta_{n,m}$ on $\mathcal{F}_{n,m}$. *Maass–Jacobi forms* play an important role in the spectral theory of $\Delta_{n,m}$ on $\mathcal{F}_{n,m}$. Here we deal only with the spectral theory Δ_Ω on $L^2(A_\Omega)$.

We fix an element $\Omega = X + iY$ of $\mathbb{H}_n$ with $X = \operatorname{Re}\Omega$ and $Y = \operatorname{Im}\Omega$. For a pair (A, B) with $A, B \in \mathbb{Z}^{(m,n)}$, we define the function $E_{\Omega;A,B} : \mathbb{C}^{(m,n)} \to \mathbb{C}$ by

$$E_{\Omega;A,B}(Z) = e^{2\pi i(\sigma({}^tAU) + \sigma((B-AX)Y^{-1}\,{}^tV))},$$

where $Z = U + iV$ is a variable in $\mathbb{C}^{(m,n)}$ with real U, V.

Lemma 4.1. *For any $A, B \in \mathbb{Z}^{(m,n)}$, the function $E_{\Omega;A,B}$ satisfies the following functional equation*

$$E_{\Omega;A,B}(Z + \lambda\Omega + \mu) = E_{\Omega;A,B}(Z), \quad Z \in \mathbb{C}^{(m,n)},$$

for all $\lambda, \mu \in \mathbb{Z}^{(m,n)}$. Thus $E_{\Omega;A,B}$ can be regarded as a function on A_Ω.

Proof. We write $\Omega = X + iY$ with real X, Y. For any $\lambda, \mu \in \mathbb{Z}^{(m,n)}$, we have

$$\begin{aligned}
E_{\Omega;A,B}(Z + \lambda\Omega + \mu) &= E_{\Omega;A,B}\big((U + \lambda X + \mu) + i(V + \lambda Y)\big) \\
&= e^{2\pi i\{\sigma({}^tA(U+\lambda X+\mu)) + \sigma((B-AX)Y^{-1}\,{}^t(V+\lambda Y))\}} \\
&= e^{2\pi i\{\sigma({}^tAU + {}^tA\lambda X + {}^tA\mu) + \sigma((B-AX)Y^{-1}\,{}^tV + B\,{}^t\lambda - AX\,{}^t\lambda)\}} \\
&= e^{2\pi i\{\sigma({}^tAU) + \sigma((B-AX)Y^{-1}\,{}^tV)\}} \\
&= E_{\Omega;A,B}(Z).
\end{aligned}$$

Here we used the fact that ${}^tA\mu$ and $B\,{}^t\lambda$ are integral. $\square$

Lemma 4.2. *The metric*

$$ds_\Omega^2 = \sigma\big((\operatorname{Im}\Omega)^{-1}\,{}^t(dZ)\,d\overline{Z}\big)$$

is a Kähler metric on A_Ω *invariant under the action* (1.2) *of* $\Gamma^J = Sp(n,\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(m,n)}$ *on* (Ω, Z) *with* Ω *fixed. Its Laplacian* Δ_Ω *of* ds_Ω^2 *is given by*

$$\Delta_\Omega = \sigma\left((\operatorname{Im}\Omega)\frac{\partial}{\partial Z}\,{}^t\!\left(\frac{\partial}{\partial \overline{Z}}\right)\right).$$

Proof. The proof can be found [20]. □

We let $L^2(A_\Omega)$ be the space of all functions $f : A_\Omega \to \mathbb{C}$ such that

$$\|f\|_\Omega := \int_{A_\Omega} |f(Z)|^2\, dv_\Omega,$$

where dv_Ω is the volume element on A_Ω normalized so that $\int_{A_\Omega} dv_\Omega = 1$. The inner product $(\,,\,)_\Omega$ on the Hilbert space $L^2(A_\Omega)$ is given by

$$(f,g)_\Omega := \int_{A_\Omega} f(Z)\overline{g(Z)}\, dv_\Omega, \qquad f, g \in L^2(A_\Omega).$$

Theorem 4.2. *The set* $\{E_{\Omega;A,B} \mid A, B \in \mathbb{Z}^{(m,n)}\}$ *is a complete orthonormal basis for* $L^2(A_\Omega)$. *Moreover we have the following spectral decomposition of* Δ_Ω:

$$L^2(A_\Omega) = \bigoplus_{A,B \in \mathbb{Z}^{(m,n)}} \mathbb{C}\cdot E_{\Omega;A,B}.$$

Proof. The complete proof can be found in [20]. □

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References

[1] R. Berndt, Sur l'arithmétique du corps des fonctions élliptiques de niveau N, in: Séminaire de Théorie de nombres de Paris 1982–1983, Birkhäuser, Boston, 1984, pp. 21–32.
[2] R. Berndt, R. Schmidt, Elements of the Representation Theory of the Jacobi Group, Birkhäuser, 1998.
[3] M. Eichler, D. Zagier, The Theory of Jacobi Forms, Progr. Math., vol. 55, Birkhäuser, Boston, 1985.
[4] E. Freitag, Siegelsche Modulfunktionen, Grundlehren Math. Wiss., Springer-Verlag, 1983.
[5] M. Kashiwara, M. Vergne, On the Segal–Shale–Weil representations and harmonic polynomials, Invent. Math. 44 (1978) 1–47.
[6] J. Kramer, A geometrical approach to the theory of Jacobi forms, Compos. Math. 79 (1991) 1–19.
[7] J. Kramer, An arithmetical theory of Jacobi forms in higher dimensions, J. Reine Angew. Math. 458 (1995) 157–182.
[8] H. Maass, Die Differentialgleichungen in der Theorie der Siegelschen Modulfunktionen, Math. Ann. 126 (1953) 44–68.
[9] H. Maass, Siegel's Modular Forms and Dirichlet Series, Lecture Notes in Math., vol. 216, Springer-Verlag, Berlin, 1971.

[10] H. Minkowski, Gesammelte Abhandlungen, Chelsea, New York, 1967.
[11] B. Runge, Theta functions and Siegel–Jacobi functions, Acta Math. 175 (1995) 165–196.
[12] C.L. Siegel, Symplectic geometry, in: Gesammelte Abhandlungen, Band II, Springer-Verlag, 1966, pp. 274–359.
[13] C.L. Siegel, Topics in Complex Function Theory: Abelian Functions and Modular Functions of Several Variables, vol. III, Wiley–Interscience, 1973.
[14] J.-H. Yang, The Siegel–Jacobi operator, Abh. Math. Sem. Univ. Hamburg 63 (1993) 135–146.
[15] J.-H. Yang, Remarks on Jacobi forms of higher degree, in: Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, Pyungsan Institute for Mathematical Sciences, Seoul, 1993, pp. 33–58.
[16] J.-H. Yang, Singular Jacobi forms, Trans. Amer. Math. Soc. 347 (6) (1995) 2041–2049.
[17] J.-H. Yang, Construction of modular forms from Jacobi forms, Canad. J. Math. 47 (1995) 1329–1339.
[18] J.-H. Yang, A geometrical theory of Jacobi forms of higher degree, in: Tadao Oda (Ed.), Proceedings of Symposium on Hodge Theory and Algebraic Geometry, Sendai, Japan, 1996, pp. 125–147, or Kyungpook Math. J. 40 (2) (2000) 209–237, or arXiv: math.NT/0602267.
[19] J.-H. Yang, A note on Maass–Jacobi forms, Kyungpook Math. J. 43 (2003) 547–566.
[20] J.-H. Yang, A note on a fundamental domain for Siegel–Jacobi space, Houston J. Math. 32 (3) (2006) 701–712.
[21] C. Ziegler, Jacobi forms of higher degree, Abh. Math. Sem. Hamburg 59 (1989) 191–224.

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A PARTIAL CAYLEY TRANSFORM OF SIEGEL–JACOBI DISK

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Abstract. Let $\mathbb{H}_g$ and $\mathbb{D}_g$ be the Siegel upper half plane and the generalized unit disk of degree g respectively. Let $\mathbb{C}^{(h,g)}$ be the Euclidean space of all $h \times g$ complex matrices. We present a partial Cayley transform of the Siegel–Jacobi disk $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ onto the Siegel–Jacobi space $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ which gives a partial bounded realization of $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ by $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$. We prove that the natural actions of the Jacobi group on $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ and $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ are compatible via a partial Cayley transform. A partial Cayley transform plays an important role in computing differential operators on the Siegel–Jacobi disk $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ invariant under the natural action of the Jacobi group on $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ explicitly.

1. Introduction

For a given fixed positive integer g, we let

$$\mathbb{H}_g = \left\{ \Omega \in \mathbb{C}^{(g,g)} \,\middle|\, \Omega = {}^t\Omega, \quad \operatorname{Im}\Omega > 0 \right\}$$

be the Siegel upper half plane of degree g and let

$$Sp(g,\mathbb{R}) = \left\{ M \in \mathbb{R}^{(2g,2g)} \,\middle|\, {}^tMJ_gM = J_g \right\}$$

be the symplectic group of degree g, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l, ${}^tM$ denotes the transpose matrix of a matrix M and

$$J_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

We see that $Sp(g,\mathbb{R})$ acts on $\mathbb{H}_g$ transitively by

$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{1.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R})$ and $\Omega \in \mathbb{H}_g$.

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Let

$$\mathbb{D}_g = \Big\{\, W \in \mathbb{C}^{(g,g)} \mid W = {}^tW,\ I_g - W\overline{W} > 0 \,\Big\}$$

be the generalized unit disk of degree g. The Cayley transform $\Phi : \mathbb{D}_g \longrightarrow \mathbb{H}_g$ defined by

$$\Phi(W) = i\,(I_g + W)(I_g - W)^{-1}, \quad W \in \mathbb{D}_g \tag{1.2}$$

is a biholomorphic mapping of $\mathbb{D}_g$ onto $\mathbb{H}_g$ which gives the bounded realization of $\mathbb{H}_g$ by $\mathbb{D}_g$ (cf. [8, pp. 281–283]). And the action (2.8) of the symplectic group on $\mathbb{D}_g$ is compatible with the action (1.1) via the Cayley transform Φ. A. Korányi and J. Wolf [4] gave a realization of a bounded symmetric domain as a Siegel domain of the third kind investigating a generalized Cayley transform of a bounded symmetric domain that generalizes the Cayley transform Φ of $\mathbb{D}_g$.

For two positive integers g and h, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(g,h)} = \Big\{\, (\lambda, \mu; \kappa) \,\Big|\, \lambda, \mu \in \mathbb{R}^{(h,g)},\ \kappa \in \mathbb{R}^{(h,h)},\ \ \kappa + \mu\,{}^t\lambda \text{ symmetric} \,\Big\}$$

endowed with the following multiplication law

$$(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') = (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda\,{}^t\mu' - \mu\,{}^t\lambda').$$

The Jacobi group G^J is defined as the semidirect product of $Sp(g,\mathbb{R})$ and $H_{\mathbb{R}}^{(g,h)}$

$$G^J = Sp(g,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$$

endowed with the following multiplication law

$$\begin{aligned} &\Big(M, (\lambda, \mu; \kappa)\Big) \cdot \Big(M', (\lambda', \mu'; \kappa')\Big) \\ &= \Big(MM', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu'; \kappa + \kappa' + \tilde{\lambda}\,{}^t\mu' - \tilde{\mu}\,{}^t\lambda')\Big) \end{aligned}$$

with $M, M' \in Sp(g,\mathbb{R}), (\lambda, \mu; \kappa),\ (\lambda', \mu'; \kappa') \in H_{\mathbb{R}}^{(g,h)}$, and $(\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)M'$. Then G^J acts on $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ transitively by

$$\Big(M, (\lambda, \mu; \kappa)\Big) \cdot (\Omega, Z) = \Big(M \cdot \Omega, (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}\Big), \tag{1.3}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R}),\ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(g,h)}$, and $(\Omega, Z) \in \mathbb{H}_g \times \mathbb{C}^{(h,g)}$. In [11, p. 1331], the author presented the natural construction of the action (1.3).

We mention that studying the Siegel–Jacobi space or the Siegel–Jacobi disk associated with the Jacobi group is useful to the study of the universal family of polarized abelian varieties (cf. [12], [14]). The aim of this paper is to present a partial Cayley transform of the Siegel–Jacobi disk $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ onto the Siegel–Jacobi space $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ which gives a partially bounded realization of $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ by $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ and to prove that the natural action of the Jacobi group on $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ and $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ is compatible via a partial Cayley transform. The main reason that we study a partial Cayley transform is that this transform is usefully applied to computing differential operators on the Siegel–Jacobi disk $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ invariant under the action (3.5) of the Jacobi group G^J_* (cf. (3.2)) explicitly.

This paper is organized as follows. In Section 2, we review the Cayley transform of the generalized unit disk $\mathbb{D}_g$ onto the Siegel upper half plane $\mathbb{H}_g$ which gives a bounded realization of $\mathbb{H}_g$ by $\mathbb{D}_g$. In Section 3, we construct a partial Cayley transform of the Siegel–Jacobi disk $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ onto the Siegel–Jacobi space $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ which gives a partially bounded realization of $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ by $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ (cf. (3.6)). We prove that the action (1.3) of the Jacobi group G^J is compatible with the action (3.5) of the Jacobi group G^J_* through a partial Cayley transform (cf. Theorem 3.1). In the final section, we present the canonical automorphic factors of the Jacobi group G^J_*.

NOTATIONS : We denote by $\mathbb{R}$ and $\mathbb{C}$ the field of real numbers, and the field of complex numbers respectively. For a square matrix $A \in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. For $\Omega \in \mathbb{H}_g$, $\mathrm{Re}\,\Omega$ (*resp.* $\mathrm{Im}\,\Omega$) denotes the real (*resp.* imaginary) part of Ω. For a matrix $A \in F^{(k,k)}$ and $B \in F^{(k,l)}$, we write $A[B] = {}^tBAB$. I_n denotes the identity matrix of degree n.

2. The Cayley transform

Let

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} I_g & I_g \\ iI_g & -iI_g \end{pmatrix} \tag{2.1}$$

be the $2g \times 2g$ matrix represented by Φ. Then

$$T^{-1} Sp(g,\mathbb{R})\, T = \left\{ \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \,\Big|\, {}^tP\overline{P} - {}^t\overline{Q}Q = I_g,\ {}^tP\overline{Q} = {}^t\overline{Q}P \right\}. \tag{2.2}$$

Indeed, if $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R})$, then

$$T^{-1}MT = \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \tag{2.3}$$

where

$$P = \frac{1}{2}\Big\{ (A+D) + i\,(B-C) \Big\} \tag{2.4}$$

and

$$Q = \frac{1}{2}\Big\{ (A-D) - i\,(B+C) \Big\}. \tag{2.5}$$

For brevity, we set

$$G_* = T^{-1} Sp(g,\mathbb{R}) T.$$

Then G_* is a subgroup of $SU(g,g)$, where

$$SU(g,g) = \left\{ h \in \mathbb{C}^{(g,g)} \,\big|\, {}^th I_{g,g} \overline{h} = I_{g,g} \right\}, \quad I_{g,g} = \begin{pmatrix} I_g & 0 \\ 0 & -I_g \end{pmatrix}.$$

In the case $g=1$, we observe that

$$T^{-1}Sp(1,\mathbb{R})T = T^{-1}SL_2(\mathbb{R})T = SU(1,1).$$

If $g>1$, then G_* is a *proper* subgroup of $SU(g,g)$. In fact, since ${}^tTJ_gT = -\,i\,J_g$, we get

$$G_* = \Big\{\, h\in SU(g,g)\,\big|\ {}^thJ_gh = J_g \,\Big\} = SU(g,g)\cap Sp(g,\mathbb{C}), \tag{2.6}$$

where

$$Sp(g,\mathbb{C}) = \Big\{\alpha\in\mathbb{C}^{(2g,2g)}\ \big|\ {}^t\alpha\,J_g\,\alpha = J_g\,\Big\}.$$

Let

$$P^+ = \left\{ \begin{pmatrix} I_g & Z \\ 0 & I_g \end{pmatrix} \ \Big|\ Z = {}^tZ\in\mathbb{C}^{(g,g)} \right\}$$

be the P^+-part of the complexification of $G_*\subset SU(g,g)$. We note that the Harish–Chandra decomposition of an element $\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P}\end{pmatrix}$ in G_* is

$$\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P}\end{pmatrix} = \begin{pmatrix} I_g & Q\overline{P}^{-1} \\ 0 & I_g \end{pmatrix}\begin{pmatrix} P-Q\overline{P}^{-1}\overline{Q} & 0 \\ 0 & \overline{P} \end{pmatrix}\begin{pmatrix} I_g & 0 \\ \overline{P}^{-1}\overline{Q} & I_g \end{pmatrix}.$$

For more detail, we refer to [3, p. 155]. Thus the P^+-component of the following element

$$\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P}\end{pmatrix}\cdot\begin{pmatrix} I_g & W \\ 0 & I_g \end{pmatrix},\quad W\in\mathbb{D}_g$$

of the complexification of G_*^J is given by

$$\begin{pmatrix} I_g & (PW+Q)(\overline{Q}W+\overline{P})^{-1} \\ 0 & I_g \end{pmatrix}. \tag{2.7}$$

We note that $Q\overline{P}^{-1}\in\mathbb{D}_g$. We get the Harish-Chandra embedding of $\mathbb{D}_g$ into P^+ (cf. [3, p. 155] or [7, pp. 58–59]). Therefore we see that G_* acts on $\mathbb{D}_g$ transitively by

$$\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P}\end{pmatrix}\cdot W = (PW+Q)(\overline{Q}W+\overline{P})^{-1},\quad \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P}\end{pmatrix}\in G_*,\ W\in\mathbb{D}_g. \tag{2.8}$$

The isotropy subgroup at the origin o is given by

$$K = \left\{ \begin{pmatrix} P & 0 \\ 0 & \overline{P}\end{pmatrix}\ \Big|\ P\in U(g)\ \right\}.$$

Thus G_*/K is biholomorphic to $\mathbb{D}_g$. It is known that the action (1.1) is compatible with the action (2.8) via the Cayley transform Φ (cf. (1.2)). In other words, if $M\in Sp(g,\mathbb{R})$ and $W\in\mathbb{D}_g$, then

$$M\cdot\Phi(W) = \Phi(M_*\cdot W), \tag{2.9}$$

where $M_* = T^{-1}MT\in G_*$. For a proof of Formula (2.9), we refer to the proof of Theorem 3.1.

For $\Omega=(\omega_{ij})\in\mathbb{H}_g$, we write $\Omega=X+iY$ with $X=(x_{ij}),\ Y=(y_{ij})$ real and $d\Omega=(d\omega_{ij})$. We also put

$$\frac{\partial}{\partial\Omega}=\left(\frac{1+\delta_{ij}}{2}\,\frac{\partial}{\partial\omega_{ij}}\right)\qquad\text{and}\qquad\frac{\partial}{\partial\overline{\Omega}}=\left(\frac{1+\delta_{ij}}{2}\,\frac{\partial}{\partial\overline{\omega}_{ij}}\right).$$

Then

$$ds^2=\sigma\Big(Y^{-1}d\Omega\,Y^{-1}d\overline{\Omega}\Big) \tag{2.10}$$

is a $Sp(g,\mathbb{R})$-invariant metric on $\mathbb{H}_g$ (cf. [8]). H. Maass [5] proved that its Laplacian is given by

$$\Delta=\,4\,\sigma\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right). \tag{2.11}$$

For $W=(w_{ij})\in\mathbb{D}_g$, we write $dW=(dw_{ij})$ and $d\overline{W}=(d\overline{w}_{ij})$. We put

$$\frac{\partial}{\partial W}=\left(\frac{1+\delta_{ij}}{2}\,\frac{\partial}{\partial w_{ij}}\right)\qquad\text{and}\qquad\frac{\partial}{\partial\overline{W}}=\left(\frac{1+\delta_{ij}}{2}\,\frac{\partial}{\partial\overline{w}_{ij}}\right).$$

Using the Cayley transform $\Phi:\mathbb{D}_g\longrightarrow\mathbb{H}_g$, H. Maass proved (cf. [5]) that

$$ds_*^2=4\,\sigma\Big((I_g-W\overline{W})^{-1}dW\,(I_g-\overline{W}W)^{-1}d\overline{W}\,\Big) \tag{2.12}$$

is a G_*-invariant Riemannian metric on $\mathbb{D}_g$ and its Laplacian is given by

$$\Delta_*=\,\sigma\left((I_g-W\overline{W})\,{}^t\!\left((I_g-W\overline{W})\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial W}\right). \tag{2.13}$$

3. A partial Cayley transform

In this section, we present a partial Cayley transform of $\mathbb{D}_g\times\mathbb{C}^{(h,g)}$ onto $\mathbb{H}_g\times\mathbb{C}^{(h,g)}$ and prove that the action (1.3) of G^J on $\mathbb{H}_g\times\mathbb{C}^{(h,g)}$ is compatible with the *natural action* (cf. (3.5)) of the Jacobi group G_*^J on $\mathbb{D}_g\times\mathbb{C}^{(h,g)}$ via a partial Cayley transform.

From now on, for brevity we write $\mathbb{H}_{g,h}=\mathbb{H}_g\times\mathbb{C}^{(h,g)}$. We can identify an element $g=\Big(M,(\lambda,\mu;\kappa)\Big)$ of G^J, $M=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in Sp(g,\mathbb{R})$ with the element

$$\begin{pmatrix}A&0&B&A\,{}^t\mu-B\,{}^t\lambda\\ \lambda&I_h&\mu&\kappa\\ C&0&D&C\,{}^t\mu-D\,{}^t\lambda\\ 0&0&0&I_h\end{pmatrix}$$

of $Sp(g+h,\mathbb{R})$. This subgroup plays an important role in investigating the Fourier–Jacobi expansion of a Siegel modular form for $Sp(g+h,\mathbb{R})$ (cf. [6]) and studying the theory of Jacobi forms (cf. [1], [2], [9], [10], [11], [17]).

We set

$$T_*=\frac{1}{\sqrt{2}}\begin{pmatrix}I_{g+h}&I_{g+h}\\ iI_{g+h}&-iI_{g+h}\end{pmatrix}.$$

We now consider the group G^J_* defined by

$$G^J_* = T_*^{-1} G^J T_*.$$

If $g = (M, (\lambda, \mu; \kappa)) \in G^J$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g, \mathbb{R})$, then $T_*^{-1} g T_*$ is given by

$$T_*^{-1} g T_* = \begin{pmatrix} P_* & Q_* \\ \overline{Q}_* & \overline{P}_* \end{pmatrix}, \tag{3.1}$$

where

$$P_* = \begin{pmatrix} P & \frac{1}{2}\Big\{Q\ {}^t(\lambda + i\mu) - P\ {}^t(\lambda - i\mu)\Big\} \\ \frac{1}{2}(\lambda + i\mu) & I_h + i\frac{\kappa}{2} \end{pmatrix},$$

$$Q_* = \begin{pmatrix} Q & \frac{1}{2}\Big\{P\ {}^t(\lambda - i\mu) - Q\ {}^t(\lambda + i\mu)\Big\} \\ \frac{1}{2}(\lambda - i\mu) & -i\frac{\kappa}{2} \end{pmatrix},$$

and P, Q are given by Formulas (2.4) and (2.5). From now on, we write

$$\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \left(\frac{1}{2}(\lambda + i\mu),\, \frac{1}{2}(\lambda - i\mu);\, -i\frac{\kappa}{2} \right) \right) = \begin{pmatrix} P_* & Q_* \\ \overline{Q}_* & \overline{P}_* \end{pmatrix}.$$

In other words, we have the relation

$$T_*^{-1} \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu; \kappa) \right) T_* = \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \left(\frac{1}{2}(\lambda + i\mu),\, \frac{1}{2}(\lambda - i\mu);\, -i\frac{\kappa}{2} \right) \right).$$

Let

$$H^{(g,h)}_{\mathbb{C}} = \Big\{ (\xi, \eta\, ; \zeta) \mid \xi, \eta \in \mathbb{C}^{(h,g)},\ \zeta \in \mathbb{C}^{(h,h)},\ \zeta + \eta\, {}^t\xi \text{ symmetric} \Big\}$$

be the Heisenberg group endowed with the following multiplication

$$(\xi, \eta\, ; \zeta) \circ (\xi', \eta'; \zeta') = (\xi + \xi', \eta + \eta'\, ; \zeta + \zeta' + \xi\, {}^t\eta' - \eta\, {}^t\xi')).$$

We define the semidirect product

$$SL(2g, \mathbb{C}) \ltimes H^{(g,h)}_{\mathbb{C}}$$

endowed with the following multiplication

$$\begin{aligned} &\left(\begin{pmatrix} P & Q \\ R & S \end{pmatrix}, (\xi, \eta\, ; \zeta) \right) \cdot \left(\begin{pmatrix} P' & Q' \\ R' & S' \end{pmatrix}, (\xi', \eta'; \zeta') \right) \\ = &\left(\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} P' & Q' \\ R' & S' \end{pmatrix}, (\tilde{\xi} + \xi', \tilde{\eta} + \eta'; \zeta + \zeta' + \tilde{\xi}\, {}^t\eta' - \tilde{\eta}\, {}^t\xi') \right), \end{aligned}$$

where $\tilde{\xi} = \xi P' + \eta R'$ and $\tilde{\eta} = \xi Q' + \eta S'$.

If we identify $H^{(g,h)}_{\mathbb{R}}$ with the subgroup

$$\Big\{ (\xi, \overline{\xi}; i\kappa) \mid \xi \in \mathbb{C}^{(h,g)},\ \kappa \in \mathbb{R}^{(h,h)} \Big\}$$

of $H^{(g,h)}_{\mathbb{C}}$, we have the following inclusion

$$G^J_* \subset SU(g, g) \ltimes H^{(g,h)}_{\mathbb{R}} \subset SL(2g, \mathbb{C}) \ltimes H^{(g,h)}_{\mathbb{C}}.$$

More precisely, if we recall $G_* = SU(g,g) \cap Sp(g,\mathbb{C})$ (cf. (2.6)), we see that the Jacobi group G_*^J is given by

(3.2)
$$G_*^J = \left\{ \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\xi, \bar{\xi}\,; i\kappa) \right) \;\middle|\; \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*,\ \xi \in \mathbb{C}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)} \right\}.$$

We define the mapping $\Theta : G^J \longrightarrow G_*^J$ by

(3.3)
$$\Theta\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu; \kappa) \right) = \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \left(\frac{1}{2}(\lambda + i\mu),\ \frac{1}{2}(\lambda - i\mu)\,;\, -i\frac{\kappa}{2} \right) \right),$$

where P and Q are given by Formulas (2.4) and (2.5). We can see that if $g_1, g_2 \in G^J$, then $\Theta(g_1 g_2) = \Theta(g_1)\Theta(g_2)$.

According to [13, p. 250], G_*^J is of the Harish-Chandra type (cf. [7, p. 118]). Let

$$g_* = \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\lambda, \mu;\, \kappa) \right)$$

be an element of G_*^J. Since the Harish-Chandra decomposition of an element $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ in $SU(g,g)$ is given by

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} I_g & QS^{-1} \\ 0 & I_g \end{pmatrix} \begin{pmatrix} P - QS^{-1}R & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} I_g & 0 \\ S^{-1}R & I_g \end{pmatrix},$$

the P_*^+-component of the following element

$$g_* \cdot \left(\begin{pmatrix} I_g & W \\ 0 & I_g \end{pmatrix}, (0, \eta; 0) \right), \quad W \in \mathbb{D}_g$$

of $SL(2g,\mathbb{C}) \ltimes H_{\mathbb{C}}^{(g,h)}$ is given by

(3.4)
$$\left(\begin{pmatrix} I_g & (PW+Q)(\overline{Q}W + \overline{P})^{-1} \\ 0 & I_g \end{pmatrix},\ \left(0,\ (\eta + \lambda W + \mu)(\overline{Q}W + \overline{P})^{-1}\,; 0\right) \right).$$

We can identify $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ with the subset

$$\left\{ \left(\begin{pmatrix} I_g & W \\ 0 & I_g \end{pmatrix}, (0, \eta; 0) \right) \;\middle|\; W \in \mathbb{D}_g,\ \eta \in \mathbb{C}^{(h,g)} \right\}$$

of the complexification of G_*^J. Indeed, $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ is embedded into P_*^+ given by

$$P_*^+ = \left\{ \left(\begin{pmatrix} I_g & W \\ 0 & I_g \end{pmatrix}, (0, \eta; 0) \right) \;\middle|\; W = {}^tW \in \mathbb{C}^{(g,g)},\ \eta \in \mathbb{C}^{(h,g)} \right\}.$$

This is a generalization of the Harish-Chandra embedding (cf. [7, p. 119]). Hence G_*^J acts on $\mathbb{D}_g \times \mathbb{C}^{(h,g)}$ transitively by

$$\begin{aligned}(3.5)\qquad &\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P}\end{pmatrix}, (\lambda, \mu; \kappa)\right) \cdot (W, \eta) \\ &= \Big((PW + Q)(\overline{Q}W + \overline{P})^{-1}, (\eta + \lambda W + \mu)(\overline{Q}W + \overline{P})^{-1}\Big).\end{aligned}$$

From now on, for brevity we write $\mathbb{D}_{g,h} = \mathbb{D}_g \times \mathbb{C}^{(h,g)}$. We define the map Φ_* of $\mathbb{D}_{g,h}$ into $\mathbb{H}_{g,h}$ by

$$(3.6)\quad \Phi_*(W, \eta) = \Big(i(I_g + W)(I_g - W)^{-1},\, 2\,i\,\eta\,(I_g - W)^{-1}\Big), \quad (W, \eta) \in \mathbb{D}_{g,h}.$$

We can show that Φ_* is a biholomorphic map of $\mathbb{D}_{g,h}$ onto $\mathbb{H}_{g,h}$ which gives a partial bounded realization of $\mathbb{H}_{g,h}$ by the Siegel–Jacobi disk $\mathbb{D}_{g,h}$. We call this map Φ_* the *partial Cayley transform* of the Siegel–Jacobi disk $\mathbb{D}_{g,h}$.

Theorem 3.1. *The action* (1.3) *of G^J on $\mathbb{H}_{g,h}$ is compatible with the action* (3.5) *of G_*^J on $\mathbb{D}_{g,h}$ through the partial Cayley transform Φ_*. In other words, if $g_0 \in G^J$ and $(W, \eta) \in \mathbb{D}_{g,h}$,*

$$(3.7)\qquad g_0 \cdot \Phi_*(W, \eta) = \Phi_*(g_* \cdot (W, \eta)),$$

where $g_* = T_*^{-1} g_0 T_*$. We observe that Formula (3.7) *generalizes Formula* (2.9). *The inverse of Φ_* is*

$$(3.8)\qquad \Phi_*^{-1}(\Omega, Z) = \Big((\Omega - iI_g)(\Omega + iI_g)^{-1},\, Z(\Omega + iI_g)^{-1}\Big).$$

Proof. Let

$$g_0 = \left(\begin{pmatrix} A & B \\ C & D\end{pmatrix}, (\lambda, \mu; \kappa)\right)$$

be an element of G^J and let $g_* = T_*^{-1} g_0 T_*$. Then

$$g_* = \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P}\end{pmatrix}, \left(\frac{1}{2}(\lambda + i\mu),\, \frac{1}{2}(\lambda - i\mu)\,;\, -i\frac{\kappa}{2}\right)\right),$$

where P and Q are given by Formulas (2.4) and (2.5).

For brevity, we write

$$(\Omega, Z) = \Phi_*(W, \eta) \quad \text{and} \quad (\Omega_*, Z_*) = g_0 \cdot (\Omega, Z).$$

That is,

$$\Omega = i(I_g + W)(I_g - W)^{-1} \quad \text{and} \quad Z = 2\,i\,\eta(I_g - W)^{-1}.$$

Then we get

$$\begin{aligned}\Omega_* &= (A\Omega+B)(C\Omega+D)^{-1}\\ &= \Big\{i\,A(I_g+W)(I_g-W)^{-1}+B\Big\}\Big\{i\,C(I_g+W)(I_g-W)^{-1}+D\Big\}^{-1}\\ &= \Big\{i\,A(I_g+W)+B(I_g-W)\Big\}(I_g-W)^{-1}\\ &\quad\times\Big[\Big\{i\,C(I_g+W)+D(I_g-W)\Big\}(I_g-W)^{-1}\Big]^{-1}\\ &= \Big\{(i\,A-B)W+(i\,A+B)\Big\}\Big\{(i\,C-D)W+(i\,C+D)\Big\}^{-1}\end{aligned}$$

and

$$\begin{aligned}Z_* &= (Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}\\ &= \Big\{2\,i\,\eta\,(I_g-W)^{-1}+i\,\lambda(I_g+W)(I_g-W)^{-1}+\mu\Big\}\\ &\quad\times\Big\{i\,C(I_g+W)(I_g-W)^{-1}+D\Big\}^{-1}\\ &= \Big\{2\,i\,\eta+i\,\lambda\,(I_g+W)+\mu\,(I_g-W)\Big\}(I_g-W)^{-1}\\ &\quad\times\Big[\Big\{i\,C(I_g+W)+D(I_g-W)\Big\}(I_g-W)^{-1}\Big]^{-1}\\ &= \Big\{2\,i\,\eta+(\lambda\,i-\mu)W+\lambda\,i+\mu\Big\}\Big\{(i\,C-D)W+(i\,C+D)\Big\}^{-1}.\end{aligned}$$

On the other hand, we set

$$(W_*,\eta_*)=g_*\cdot(W,\eta)\quad\text{and}\quad(\widehat{\Omega},\widehat{Z})=\Phi_*(W_*,\eta_*).$$

Then

$$W_*=(PW+Q)(\overline{Q}W+\overline{P})^{-1}\quad\text{and}\quad\eta_*=(\eta+\lambda_*W+\mu_*)(\overline{Q}W+\overline{P})^{-1},$$

where $\lambda_*=\frac{1}{2}(\lambda+\,i\,\mu)$ and $\mu_*=\frac{1}{2}(\lambda-\,i\,\mu)$.

According to Formulas (2.4) and (2.5), we get

$$\begin{aligned}\widehat{\Omega} &= i\,(I_g+W_*)(I_g-W_*)^{-1}\\ &= i\Big\{I_g+(PW+Q)(\overline{Q}W+\overline{P})^{-1}\Big\}\Big\{I_g-(PW+Q)(\overline{Q}W+\overline{P})^{-1}\Big\}^{-1}\\ &= i\,(\overline{Q}W+\overline{P}+PW+Q)(\overline{Q}W+\overline{P})^{-1}\\ &\quad\times\Big\{(\overline{Q}W+\overline{P}-PW-Q)(\overline{Q}W+\overline{P})^{-1}\Big\}^{-1}\\ &= i\Big\{(P+\overline{Q}\,)W+\overline{P}+Q\Big\}\Big\{(\overline{Q}-P)W+\overline{P}-Q\Big\}^{-1}\\ &= \Big\{(i\,A-B)W+(i\,A+B)\Big\}\Big\{(i\,C-D)W+(i\,C+D)\Big\}^{-1}.\end{aligned}$$

Therefore $\widehat{\Omega}=\Omega_*$. In fact, this result is the known fact (cf. Formula (2.9)) that the action (1.1) is compatible with the action (2.8) via the Cayley transform

Φ.

$$\begin{aligned}
\widehat{Z} &= 2\,i\,\eta_*(I_g - W_*)^{-1} \\
&= 2\,i\,(\eta + \lambda_* W + \mu_*)(\overline{Q}W + \overline{P})^{-1} \\
&\quad \times \Big\{ I_g - (PW+Q)(\overline{Q}W + \overline{P})^{-1} \Big\}^{-1} \\
&= 2\,i\,(\eta + \lambda_* W + \mu_*)(\overline{Q}W + \overline{P})^{-1} \\
&\quad \times \Big\{ (\overline{Q}W + \overline{P} - PW - Q)(\overline{Q}W + \overline{P})^{-1} \Big\}^{-1} \\
&= 2\,i\,(\eta + \lambda_* W + \mu_*)\Big\{ (\overline{Q} - P)W + \overline{P} - Q \Big\}^{-1}.
\end{aligned}$$

Using Formulas (2.4) and (2.5), we obtain

$$\widehat{Z} = \Big\{ 2\,i\,\eta + (\lambda\,i - \mu)W + \lambda\,i + \mu \Big\}\Big\{ (i\,C - D)W + i\,C + D \Big\}^{-1}.$$

Hence $\widehat{Z} = Z_*$. Consequently we get Formula (3.7). Formula (3.8) follows immediately from a direct computation. □

Remark 3.1. R. Berndt and R. Schmidts (cf. [1, pp. 52–53]) investigated a partial Cayley transform in the case $g = h = 1$.

For a coordinate $(\Omega, Z) \in \mathbb{H}_{g,h}$ with $\Omega = (\omega_{\mu\nu}) \in \mathbb{H}_g$ and $Z = (z_{kl}) \in \mathbb{C}^{(h,g)}$, we put

$$\begin{aligned}
\Omega &= X + iY, \quad X = (x_{\mu\nu}), \quad Y = (y_{\mu\nu}) \ \text{ real}, \\
Z &= U + iV, \quad U = (u_{kl}), \quad V = (v_{kl}) \ \text{ real}, \\
d\Omega &= (d\omega_{\mu\nu}), \quad dX = (dx_{\mu\nu}), \quad dY = (dy_{\mu\nu}), \\
dZ &= (dz_{kl}), \quad dU = (du_{kl}), \quad dV = (dv_{kl}), \\
d\overline{\Omega} &= (d\overline{\omega}_{\mu\nu}), \quad d\overline{Z} = (d\bar{z}_{kl}),
\end{aligned}$$

$$\frac{\partial}{\partial \Omega} = \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial \omega_{\mu\nu}} \right), \quad \frac{\partial}{\partial \overline{\Omega}} = \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial \overline{\omega}_{\mu\nu}} \right),$$

$$\frac{\partial}{\partial Z} = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \cdots & \frac{\partial}{\partial z_{h1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_{1g}} & \cdots & \frac{\partial}{\partial z_{hg}} \end{pmatrix}, \quad \frac{\partial}{\partial \overline{Z}} = \begin{pmatrix} \frac{\partial}{\partial \bar{z}_{11}} & \cdots & \frac{\partial}{\partial \bar{z}_{h1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \bar{z}_{1g}} & \cdots & \frac{\partial}{\partial \bar{z}_{hg}} \end{pmatrix}.$$

Remark 3.2. The author proved in [15] that for any two positive real numbers A and B, the following metric

$$\begin{aligned}
ds^2_{g,h;A,B} &= A\,\sigma\Big(Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega} \Big) \\
&\quad + B \Big\{ \sigma\Big(Y^{-1}\, {}^t V\, V\, Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega} \Big) + \sigma\Big(Y^{-1}\, {}^t(dZ)\, d\overline{Z} \Big) \\
&\quad - \sigma\Big(V\, Y^{-1} d\Omega\, Y^{-1}\, {}^t(d\overline{Z}) \Big) - \sigma\Big(V\, Y^{-1} d\overline{\Omega}\, Y^{-1}\, {}^t(dZ) \Big) \Big\}
\end{aligned} \tag{3.9}$$

is a Riemannian metric on $\mathbb{H}_{g,h}$ which is invariant under the action (1.3) of the Jacobi group G^J and its Laplacian is given by

$$\begin{aligned}\Delta_{n,m;A,B} \;=\;& \frac{4}{A}\left\{\sigma\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right)+\sigma\left(VY^{-1}\,{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial Z}\right)\right.\\ &\left.+\sigma\left(V\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial Z}\right)+\sigma\left({}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial\Omega}\right)\right\}\\ &+\frac{4}{B}\,\sigma\left(Y\,\frac{\partial}{\partial Z}\,{}^t\!\left(\frac{\partial}{\partial\overline{Z}}\right)\right).\end{aligned}\tag{3.10}$$

We observe that Formulas (3.9) and (3.10) generalize Formulas (2.10) and (2.11). The following differential form

$$dv_{g,h} = (\det Y)^{-(g+h+1)}\,[dX]\wedge[dY]\wedge[dU]\wedge[dV]$$

is a G^J-invariant volume element on $\mathbb{H}_{g,h}$, where

$$[dX]=\wedge_{\mu\le\nu}dx_{\mu\nu},\;\; [dY]=\wedge_{\mu\le\nu}dy_{\mu\nu},\;\; [dU]=\wedge_{k,l}du_{kl}\;\;\text{and}\;\;[dV]=\wedge_{k,l}dv_{kl}.$$

Using the partial Cayley transform Φ_* and Theorem 3.1, we can find a G^J_*-invariant Riemannian metric on the Siegel–Jacobi disk $\mathbb{D}_{g,h}$ and its Laplacian explicitly which generalize Formulas (2.12) and (2.13). For more detail, we refer to [16].

4. The canonical automorphic factors

The isotropy subgroup K^J_* at $(0,0)$ under the action (3.5) is

$$K^J_* = \left\{\left(\begin{pmatrix}P & 0\\ 0 & \overline{P}\end{pmatrix},(0,0;\kappa)\right)\;\middle|\; P\in U(g),\; \kappa\in\mathbb{R}^{(h,h)}\right\}.\tag{4.1}$$

The complexification of K^J_* is given by

$$K^J_{*,\mathbb{C}} = \left\{\left(\begin{pmatrix}P & 0\\ 0 & {}^tP^{-1}\end{pmatrix},(0,0;\zeta)\right)\;\middle|\; P\in GL(g,\mathbb{C}),\; \zeta\in\mathbb{C}^{(h,h)}\right\}.\tag{4.2}$$

By a complicated computation, we can show that if

$$g_* = \left(\begin{pmatrix}P & Q\\ \overline{Q} & \overline{P}\end{pmatrix},(\lambda,\mu;\kappa)\right)\tag{4.3}$$

is an element of G^J_*, then the $K^J_{*,\mathbb{C}}$-component of

$$g_*\cdot\left(\begin{pmatrix}I_g & W\\ 0 & I_g\end{pmatrix},(0,\eta;0)\right)$$

is given by

$$\left(\begin{pmatrix}P-(PW+Q)(\overline{Q}W+\overline{P})^{-1}\overline{Q} & 0\\ 0 & \overline{Q}W+\overline{P}\end{pmatrix},(0,0;\kappa_*)\right),\tag{4.4}$$

where

$$\begin{aligned}\kappa_* &= \kappa + \lambda\, {}^t\eta + (\eta + \lambda W + \mu)\, {}^t\lambda \\ &\quad - (\eta + \lambda W + \mu)\, {}^t\overline{Q}\, {}^t(\overline{Q}W + \overline{P})^{-1}\, {}^t(\eta + \lambda W + \mu) \\ &= \kappa + \lambda\, {}^t\eta + (\eta + \lambda W + \mu)\, {}^t\lambda \\ &\quad - (\eta + \lambda W + \mu)(\overline{Q}W + \overline{P})^{-1}\overline{Q}\, {}^t(\eta + \lambda W + \mu).\end{aligned}$$

Here we used the fact that $(\overline{Q}W + \overline{P})^{-1}\overline{Q}$ is symmetric.

For $g_* \in G_*^J$ given by (4.3) with $g_0 = \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*$ and $(W, \eta) \in \mathbb{D}_{g,h}$, we write

$$J(g_*, (W, \eta)) = a(g_*, (W, \eta))\, b(g_0, W), \tag{4.5}$$

where

$$a(g_*, (W, \eta)) = (I_{2g}, (0, 0; \kappa_*)), \quad \text{where } \kappa_* \text{ is given in } (4.4)$$

and

$$b(g_0, W) = \left(\begin{pmatrix} P - (PW + Q)(\overline{Q}W + \overline{P})^{-1}\overline{Q} & 0 \\ 0 & \overline{Q}W + \overline{P} \end{pmatrix}, (0, 0\,; 0) \right).$$

Lemma 4.1. *Let*

$$\rho : GL(g, \mathbb{C}) \longrightarrow GL(V_\rho)$$

be a holomorphic representation of $GL(g, \mathbb{C})$ on a finite dimensional complex vector space V_ρ and $\chi : \mathbb{C}^{(h,h)} \longrightarrow \mathbb{C}^\times$ be a character of the additive group $\mathbb{C}^{(h,h)}$. Then the mapping

$$J_{\chi,\rho} : G_*^J \times \mathbb{D}_{g,h} \longrightarrow GL(V_\rho)$$

defined by

$$J_{\chi,\rho}(g_*, (W, \eta)) = \chi(a(g_*, (W, \eta)))\, \rho(b(g_0, W))$$

is an automorphic factor of G_*^J with respect to χ and ρ.

Proof. We observe that $a(g_*, (W, \eta))$ is a summand of automorphy, i.e.,

$$a(g_1 g_2, (W, \eta)) = a(g_1, g_2 \cdot (W, \eta)) + a(g_2, (W, \eta)),$$

where $g_1, g_2 \in G_*^J$ and $(W, \eta) \in \mathbb{D}_{g,h}$. Together with this fact, the proof follows from the fact that the mapping

$$J_\rho : G_* \times \mathbb{D}_g \longrightarrow GL(V_\rho)$$

defined by

$$J_\rho(g_0, W) := \rho(b(g_0, W)), \quad g_0 \in G_*,\ W \in \mathbb{D}_g$$

is an automorphic factor of G_*. □

Example 4.1. Let $\mathcal{M}$ be a symmetric half-integral semi-positive definite matrix of degree h and let $\rho : GL(g,\mathbb{C}) \longrightarrow GL(V_\rho)$ be a holomorphic representation of $GL(g,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. Then the character

$$\chi_{\mathcal{M}} : \mathbb{C}^{(h,h)} \longrightarrow \mathbb{C}^\times$$

defined by

$$\chi_{\mathcal{M}}(c) = e^{-2\pi i \sigma(\mathcal{M}c)}, \quad c \in \mathbb{C}^{(h,h)}$$

provides the automorphic factor

$$J_{\mathcal{M},\rho} : G_*^J \times \mathbb{D}_{g,h} \longrightarrow GL(V_\rho)$$

defined by

$$J_{\mathcal{M},\rho}(g_*, (W,\eta)) = e^{-2\pi i \sigma(\mathcal{M}\kappa_*)} \rho(\overline{Q}W + \overline{P}),$$

where g_* is an element in G_*^J given by (4.3) and κ_* is given in (4.4). Using $J_{\mathcal{M},\rho}$, we can define the notion of Jacobi forms on $\mathbb{D}_{g,h}$ of index $\mathcal{M}$ with respect to the Siegel modular group $T^{-1}Sp(g,\mathbb{Z})\,T$ (cf. [9], [10], [11]).

Remark 4.1. The P_*^--component of

$$g_* \cdot \left(\begin{pmatrix} I_g & W \\ 0 & I_g \end{pmatrix}, (0,\eta;0) \right)$$

is given by

$$\left(\begin{pmatrix} I_g & 0 \\ (\overline{Q}W + \overline{P})^{-1}\overline{Q} & I_g \end{pmatrix}, \left(\lambda - (\eta + \lambda W + \mu)(\overline{Q}W + \overline{P})^{-1}\overline{Q},\ 0\,;0 \right) \right), \tag{4.6}$$

where

$$P_*^- = \left\{ \left(\begin{pmatrix} I_g & 0 \\ W & I_g \end{pmatrix}, (\xi, 0\,;0) \right) \,\middle|\, W = {}^tW \in \mathbb{C}^{(g,g)},\ \xi \in \mathbb{C}^{(h,g)} \right\}.$$

References

[1] R. Berndt and R. Schmidt, *Elements of the Representation Theory of the Jacobi Group*, Progress in Mathematics, 163. Birkhäuser Verlag, Basel, 1998.

[2] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progress in Math. **55**, Birkhäuser, Boston, Basel and Stuttgart, 1985.

[3] A. W. Knapp, *Representation Theory of Semisimple Groups*, Princeton University Press, Princeton, New Jersey, 1986.

[4] A. Koranyi and J. A. Wolf, *Generalized Cayley transformations of bounded symmetric domains*, Amer. J. Math. **87** (1965), 899–939.

[5] H. Maass, *Die Differentialgleichungen in der Theorie der Siegelschen Modulfunktionen*, Math. Ann. **126** (1953), 44–68.

[6] I. Piateski-Sharpiro, *Automorphic Functions and the Geometry of Classical Domains*, Translated from the Russian. Mathematics and Its Applications, Vol. 8 Gordon and Breach Science Publishers, New York-London-Paris 1969.

[7] I. Satake, *Algebraic structures of symmetric domains*, Kano Memorial Lectures, 4. Iwanami Shoten, Tokyo; Princeton University Press, Princeton, N. J., 1980.

[8] C. L. Siegel, *Symplectic geometry*, Amer. J. Math. **65** (1943), 1–86.

[9] J.-H. Yang, *The Siegel–Jacobi operator*, Abh. Math. Sem. Univ. Hamburg **63** (1993), 135–146.

[10] ______, *Singular Jacobi forms*, Trans. Amer. Math. Soc. **347** (1995), no. 6, 2041–2049.
[11] ______, *Construction of vector valued modular forms from Jacobi forms*, Canad. J. Math. **47** (1995), no. 6, 1329–1339.
[12] ______, *A geometrical theory of Jacobi forms of higher degree*, Kyungpook Math. J. **40** (2000), no. 2, 209–237.
[13] ______, *The method of orbits for real Lie groups*, Kyungpook Math. J. **42** (2002), no. 2, 199–272.
[14] ______, *A note on a fundamental domain for Siegel–Jacobi space*, Houston J. Math. **32** (2006), no. 3, 701–712.
[15] ______, *Invariant metrics and Laplacians on Siegel–Jacobi space*, Journal of Number Theory **127** (2007), 83–102.
[16] ______, *Invariant metrics and Laplacians on Siegel–Jacobi disk*, arXiv:math.NT/0507217 v1 or revised version (2006).
[17] C. Ziegler, *Jacobi forms of higher degree*, Abh. Math. Sem. Univ. Hamburg **59** (1989), 191–224.

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Invariant Metrics and Laplacians on Siegel-Jacobi Disk**

Jae-Hyun YANG*

Abstract Let $\mathbb{D}_n$ be the generalized unit disk of degree n. In this paper, Riemannian metrics on the Siegel-Jacobi disk $\mathbb{D}_n \times \mathbb{C}^{(m,n)}$ which are invariant under the natural action of the Jacobi group are found explicitly and the Laplacians of these invariant metrics are computed explicitly. These are expressed in terms of the trace form.

Keywords Invariant metrics, Siegel-Jacobi disk, Partial Cayley transform
2000 MR Subject Classification 11Fxx, 32F45, 32M10

1 Introduction

For a given fixed positive integer n, we let

$$\mathbb{H}_n = \{\Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega,\ \operatorname{Im}\Omega > 0\}$$

be the Siegel upper half plane of degree n and

$$\operatorname{Sp}(n,\mathbb{R}) = \{M \in \mathbb{R}^{(2n,2n)} \mid {}^tMJ_nM = J_n\}$$

be the symplectic group of degree n, where

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

We see that $\operatorname{Sp}(n,\mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{1.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \operatorname{Sp}(n,\mathbb{R})$ and $\Omega \in \mathbb{H}_n$.

For two positive integers n and m, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \{(\lambda,\mu;\kappa) \mid \lambda,\mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\,{}^t\lambda \text{ symmetric}\}$$

endowed with the following multiplication law

$$(\lambda,\mu;\kappa) \circ (\lambda',\mu';\kappa') = (\lambda+\lambda',\mu+\mu';\kappa+\kappa'+\lambda\,{}^t\mu' - \mu\,{}^t\lambda').$$

We define the semidirect product of $\operatorname{Sp}(n,\mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$ as

$$G^J := \operatorname{Sp}(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

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endowed with the following multiplication law

$$(M,(\lambda,\mu;\kappa))\cdot(M',(\lambda',\mu';\kappa'))=(MM',(\widetilde{\lambda}+\lambda',\widetilde{\mu}+\mu';\kappa+\kappa'+\widetilde{\lambda}\,{}^t\mu'-\widetilde{\mu}\,{}^t\lambda'))$$

with $M,M'\in Sp(n,\mathbb{R})$, $(\lambda,\mu;\kappa),(\lambda',\mu';\kappa')\in H_{\mathbb{R}}^{(n,m)}$ and $(\widetilde{\lambda},\widetilde{\mu})=(\lambda,\mu)M'$. We call this group G^J the Jacobi group of degree n and index m. Then we get the natural action of G^J on $\mathbb{H}_n\times\mathbb{C}^{(m,n)}$ (see [1, 2, 7–9, 15]) defined by

$$(M,(\lambda,\mu;\kappa))\cdot(\Omega,Z)=(M\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}), \tag{1.2}$$

where $M=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in Sp(n,\mathbb{R})$, $(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega,Z)\in\mathbb{H}_n\times\mathbb{C}^{(m,n)}$. We note that the action (1.2) is transitive.

For brevity, we write $\mathbb{H}_{n,m}:=\mathbb{H}_n\times\mathbb{C}^{(m,n)}$. For a coordinate $(\Omega,Z)\in\mathbb{H}_{n,m}$ with $\Omega=(\omega_{\mu\nu})\in\mathbb{H}_n$ and $Z=(z_{kl})\in\mathbb{C}^{(m,n)}$, we put

$$\begin{aligned}
&\Omega=X+iY,\quad X=(x_{\mu\nu}),\quad Y=(y_{\mu\nu})\ \text{real},\\
&Z=U+iV,\quad U=(u_{kl}),\quad V=(v_{kl})\ \text{real},\\
&d\Omega=(d\omega_{\mu\nu}),\quad d\overline{\Omega}=(d\overline{\omega}_{\mu\nu}),\\
&dZ=(dz_{kl}),\quad d\overline{Z}=(d\overline{z}_{kl}),\\
&\frac{\partial}{\partial\Omega}=\Big(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial\omega_{\mu\nu}}\Big),\quad \frac{\partial}{\partial\overline{\Omega}}=\Big(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial\overline{\omega}_{\mu\nu}}\Big),\\
&\frac{\partial}{\partial Z}=\begin{pmatrix}\frac{\partial}{\partial z_{11}}&\cdots&\frac{\partial}{\partial z_{m1}}\\ \vdots&&\vdots\\ \frac{\partial}{\partial z_{1n}}&\cdots&\frac{\partial}{\partial z_{mn}}\end{pmatrix},\quad \frac{\partial}{\partial\overline{Z}}=\begin{pmatrix}\frac{\partial}{\partial\overline{z}_{11}}&\cdots&\frac{\partial}{\partial\overline{z}_{m1}}\\ \vdots&&\vdots\\ \frac{\partial}{\partial\overline{z}_{1n}}&\cdots&\frac{\partial}{\partial\overline{z}_{mn}}\end{pmatrix},
\end{aligned}$$

where δ_{ij} denotes the Kronecker delta symbol.

Siegel [6] introduced the symplectic metric ds_n^2 on $\mathbb{H}_n$ invariant under the action (1.1) of $Sp(n,\mathbb{R})$ given by

$$ds_n^2=\sigma(Y^{-1}d\Omega Y^{-1}d\overline{\Omega}), \tag{1.3}$$

and Maass [4] proved that the differential operator

$$\Delta_n=4\sigma\Big(Y\,{}^t\Big(Y\frac{\partial}{\partial\overline{\Omega}}\Big)\frac{\partial}{\partial\Omega}\Big) \tag{1.4}$$

is the Laplacian of $\mathbb{H}_n$ for the symplectic metric ds_n^2. Here $\sigma(A)$ denotes the trace of a square matrix A.

In [11], the author proved the following theorems.

Theorem 1.1 *For any two positive real numbers A and B, the following metric*

$$\begin{aligned}
ds^2_{n,m;A,B}=&A\sigma(Y^{-1}d\Omega Y^{-1}d\overline{\Omega})+B\{\sigma(Y^{-1}\,{}^tVVY^{-1}d\Omega Y^{-1}d\overline{\Omega})+\sigma(Y^{-1}\,{}^t(dZ)d\overline{Z})\\
&-\sigma(VY^{-1}d\Omega Y^{-1}\,{}^t(d\overline{Z}))-\sigma(VY^{-1}d\overline{\Omega}Y^{-1}\,{}^t(dZ))\}
\end{aligned} \tag{1.5}$$

is a Riemannian metric on $\mathbb{H}_{n,m}$ which is invariant under the action (1.2) *of the Jacobi group G^J.*

Theorem 1.2 *For any two positive real numbers A and B, the Laplacian $\Delta_{n,m;A,B}$ of $(\mathbb{H}_{n,m},ds^2_{n,m;A,B})$ is given by*

$$\begin{aligned}
\Delta_{n,m;A,B}=&\frac{4}{A}\Big\{\sigma\Big(Y\,{}^t\Big(Y\frac{\partial}{\partial\overline{\Omega}}\Big)\frac{\partial}{\partial\Omega}\Big)+\sigma\Big(VY^{-1}\,{}^tV\,{}^t\Big(Y\frac{\partial}{\partial\overline{Z}}\Big)\frac{\partial}{\partial Z}\Big)+\sigma\Big(V\,{}^t\Big(Y\frac{\partial}{\partial\overline{\Omega}}\Big)\frac{\partial}{\partial Z}\Big)\\
&+\sigma\Big({}^tV\,{}^t\Big(Y\frac{\partial}{\partial\overline{Z}}\Big)\frac{\partial}{\partial\Omega}\Big)\Big\}+\frac{4}{B}\sigma\Big(Y\frac{\partial}{\partial Z}\,{}^t\Big(\frac{\partial}{\partial\overline{Z}}\Big)\Big).
\end{aligned} \tag{1.6}$$

Let

$$G_* = \mathrm{SU}(n,n) \cap \mathrm{Sp}(n,\mathbb{C})$$

be the symplectic group and

$$\mathbb{D}_n = \{W \in \mathbb{C}^{(n,n)} \mid W = {}^tW,\ I_n - \overline{W}W > 0\}$$

be the generalized unit disk. Then G_* acts on $\mathbb{D}_n$ transitively by

$$\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \cdot W = (PW + Q)(\overline{Q}W + \overline{P})^{-1},$$

where $\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*$ and $W \in \mathbb{D}_n$. Using the Cayley transform of $\mathbb{D}_n$ onto $\mathbb{H}_n$, we can see that

$$\mathrm{d}s_*^2 = 4\sigma((I_n - W\overline{W})^{-1}\mathrm{d}W(I_n - \overline{W}W)^{-1}\mathrm{d}\overline{W}) \tag{1.7}$$

is a G_*-invariant Kähler metric on $\mathbb{D}_n$ (see [6]) and Maass [4] showed that its Laplacian is given by

$$\Delta_* = \sigma\Big((I_n - W\overline{W})\, {}^t\Big((I_n - W\overline{W})\frac{\partial}{\partial \overline{W}}\Big)\frac{\partial}{\partial W}\Big). \tag{1.8}$$

Let

$$G_*^J = \left\{ \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\xi, \overline{\xi}; \mathrm{i}\kappa) \right) \,\middle|\, \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*,\ \xi \in \mathbb{C}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)} \right\}$$

be the Jacobi group with the following multiplication:

$$\begin{aligned} &\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\xi, \overline{\xi}; \mathrm{i}\kappa) \right) \cdot \left(\begin{pmatrix} P' & Q' \\ \overline{Q'} & \overline{P'} \end{pmatrix}, (\xi', \overline{\xi'}; \mathrm{i}\kappa') \right) \\ &= \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \begin{pmatrix} P' & Q' \\ \overline{Q'} & \overline{P'} \end{pmatrix}, (\widetilde{\xi} + \xi', \widetilde{\overline{\xi}} + \overline{\xi'}; \mathrm{i}\kappa + \mathrm{i}\kappa' + \widetilde{\xi}\,{}^t\overline{\xi'} - \widetilde{\overline{\xi}}\,{}^t\xi') \right), \end{aligned}$$

where $\widetilde{\xi} = \xi P' + \overline{\xi}\,\overline{Q'}$ and $\widetilde{\overline{\xi}} = \xi Q' + \overline{\xi}\,\overline{P'}$. Then we have the natural action of G_*^J on the Siegel-Jacobi disk $\mathbb{D}_n \times \mathbb{C}^{(m,n)}$ (see (2.6)) given by

$$\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\xi, \overline{\xi}; \mathrm{i}\kappa) \right) \cdot (W, \eta) = ((PW + Q)(\overline{Q}W + \overline{P})^{-1}, (\eta + \xi W + \overline{\xi})(\overline{Q}W + \overline{P})^{-1}), \tag{1.9}$$

where $W \in \mathbb{D}_n$ and $\eta \in \mathbb{C}^{(m,n)}$.

For brevity, we write $\mathbb{D}_{n,m} := \mathbb{D}_n \times \mathbb{C}^{(m,n)}$. For a coordinate $(W, \eta) \in \mathbb{D}_{n,m}$ with $W = (w_{\mu\nu}) \in \mathbb{D}_n$ and $\eta = (\eta_{kl}) \in \mathbb{C}^{(m,n)}$, we put

$$\mathrm{d}W = (\mathrm{d}w_{\mu\nu}), \quad \mathrm{d}\overline{W} = (\mathrm{d}\overline{w}_{\mu\nu}), \quad \mathrm{d}\eta = (\mathrm{d}\eta_{kl}), \quad \mathrm{d}\overline{\eta} = (\mathrm{d}\overline{\eta}_{kl})$$

and

$$\frac{\partial}{\partial W} = \Big(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial w_{\mu\nu}}\Big), \qquad \frac{\partial}{\partial \overline{W}} = \Big(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial \overline{w}_{\mu\nu}}\Big),$$

$$\frac{\partial}{\partial \eta} = \begin{pmatrix} \frac{\partial}{\partial \eta_{11}} & \cdots & \frac{\partial}{\partial \eta_{m1}} \\ \vdots & & \vdots \\ \frac{\partial}{\partial \eta_{1n}} & \cdots & \frac{\partial}{\partial \eta_{mn}} \end{pmatrix}, \quad \frac{\partial}{\partial \overline{\eta}} = \begin{pmatrix} \frac{\partial}{\partial \overline{\eta}_{11}} & \cdots & \frac{\partial}{\partial \overline{\eta}_{m1}} \\ \vdots & & \vdots \\ \frac{\partial}{\partial \overline{\eta}_{1n}} & \cdots & \frac{\partial}{\partial \overline{\eta}_{mn}} \end{pmatrix}.$$

In this paper, we find the G_*^J-invariant Riemannian metrics on $\mathbb{D}_{n,m}$ and their Laplacians. In fact, we prove the following theorems.

Theorem 1.3 *For any two positive real numbers A and B, the following metric $d\tilde{s}^2_{n,m;A,B}$ defined by*

$$\begin{aligned} d\tilde{s}^2_{n,m;A,B} = {} & 4A\sigma((I_n - W\overline{W})^{-1}dW(I_n - \overline{W}W)^{-1}d\overline{W}) + 4B\{\sigma((I_n - W\overline{W})^{-1}\,{}^t(d\eta)d\overline{\eta}) \\ & + \sigma((\eta\overline{W} - \overline{\eta})(I_n - W\overline{W})^{-1}dW(I_n - \overline{W}W)^{-1}\,{}^t(d\overline{\eta})) \\ & + \sigma((\overline{\eta}W - \eta)(I_n - \overline{W}W)^{-1}d\overline{W}(I_n - W\overline{W})^{-1}\,{}^t(d\eta)) \\ & - \sigma((I_n - W\overline{W})^{-1}\,{}^t\eta\eta(I_n - \overline{W}W)^{-1}\overline{W}dW(I_n - \overline{W}W)^{-1}d\overline{W}) \\ & - \sigma(W(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\overline{\eta}(I_n - W\overline{W})^{-1}dW(I_n - \overline{W}W)^{-1}d\overline{W}) \\ & + \sigma((I_n - W\overline{W})^{-1}\,{}^t\eta\overline{\eta}(I_n - W\overline{W})^{-1}dW(I_n - \overline{W}W)^{-1}d\overline{W}) \\ & + \sigma((I_n - \overline{W})^{-1}\,{}^t\overline{\eta}\eta\overline{W}(I_n - W\overline{W})^{-1}dW(I_n - \overline{W}W)^{-1}d\overline{W}) \\ & + \sigma((I_n - \overline{W})^{-1}(I_n - W)(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\eta(I_n - \overline{W}W)^{-1}(I_n - \overline{W})(I_n - W)^{-1} \\ & \times dW(I_n - \overline{W}W)^{-1}d\overline{W}) - \sigma((I_n - W\overline{W})^{-1}(I_n - W)(I_n - \overline{W})^{-1}\,{}^t\overline{\eta}\eta \\ & \times (I_n - W)^{-1}dW(I_n - \overline{W}W)^{-1}d\overline{W})\} \end{aligned}$$

is a Riemannian metric on $\mathbb{D}_{n,m}$ which is invariant under the action (1.9) of the Jacobi group G^J_*. Note that if $n = m = 1$ and $A = B = 1$, we get

$$\begin{aligned} \frac{1}{4}d\tilde{s}^2_{1,1;1,1} = {} & \frac{dWd\overline{W}}{(1-|W|^2)^2} + \frac{d\eta d\overline{\eta}}{1-|W|^2} + \frac{(1+|W|^2)|\eta|^2 - \overline{W}\eta^2 - W\overline{\eta}^2}{(1-|W|^2)^3}dWd\overline{W} \\ & + \frac{\eta\overline{W} - \overline{\eta}}{(1-|W|^2)^2}dWd\overline{\eta} + \frac{\overline{\eta}W - \eta}{(1-|W|^2)^2}d\overline{W}d\eta. \end{aligned}$$

Theorem 1.4 *For any two positive real numbers A and B, the Laplacian $\tilde{\Delta}_{n,m;A,B}$ of $(\mathbb{D}_{n,m}, d\tilde{s}^2_{n,m;A,B})$ is given by*

$$\begin{aligned} \tilde{\Delta}_{n,m;A,B} = {} & \frac{1}{A}\Big\{\sigma\Big((I_n - W\overline{W})\ {}^t\Big((I_n - W\overline{W})\frac{\partial}{\partial\overline{W}}\Big)\frac{\partial}{\partial W}\Big) \\ & + \sigma\Big({}^t(\eta - \overline{\eta}W)\ {}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial W}\Big) + \sigma\Big((\overline{\eta} - \eta\overline{W})\ {}^t\Big((I_n - W\overline{W})\frac{\partial}{\partial\overline{W}}\Big)\frac{\partial}{\partial\eta}\Big) \\ & - \sigma\Big(\eta\overline{W}(I_n - W\overline{W})^{-1}\,{}^t\eta\ {}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}\Big) \\ & - \sigma\Big(\overline{\eta}W(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\ {}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}\Big) \\ & + \sigma\Big(\overline{\eta}(I_n - W\overline{W})^{-1}\,{}^t\eta\ {}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}\Big) \\ & + \sigma\Big(\eta\overline{W}W(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\ {}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}\Big)\Big\} \\ & + \frac{1}{B}\sigma\Big((I_n - \overline{W}W)\frac{\partial}{\partial\eta}\ {}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)\Big). \end{aligned}$$

Note that if $n = m = 1$ and $A = B = 1$, we get

$$\begin{aligned} \tilde{\Delta}_{1,1;1,1} = {} & (1-|W|^2)^2\frac{\partial^2}{\partial W\partial\overline{W}} + (1-|W|^2)\frac{\partial^2}{\partial\eta\partial\overline{\eta}} + (1-|W|^2)(\eta - \overline{\eta}W)\frac{\partial^2}{\partial W\partial\overline{\eta}} \\ & + (1-|W|^2)(\overline{\eta} - \eta\overline{W})\frac{\partial^2}{\partial\overline{W}\partial\eta} - (\overline{W}\eta^2 + W\overline{\eta}^2)\frac{\partial^2}{\partial\eta\partial\overline{\eta}} + (1+|W|^2)|\eta|^2\frac{\partial^2}{\partial\eta\partial\overline{\eta}}. \end{aligned}$$

The main ingredients for the proof of Theorems 1.3 and 1.4 are the partial Cayley transform, Theorems 1.1 and 1.2. The paper is organized as follows. In Section 2, we review the partial Cayley transform that was dealt with in [12]. In Section 3, we prove Theorem 1.3. In Section 4, we prove Theorem 1.4. In the final section, we briefly remark the theory of harmonic analysis on the Siegel-Jacobi disk.

We denote by $\mathbb{R}$ and $\mathbb{C}$ the fields of real numbers and the field of complex numbers respectively. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. For $\Omega \in \mathbb{H}_g$, $\operatorname{Re}\Omega$ (resp. $\operatorname{Im}\Omega$) denotes the real (resp. imaginary) part of Ω. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M.

2 A Partial Cayley Transform

In this section, we review the partial Cayley transform (see [12]) of $\mathbb{D}_{n,m}$ onto $\mathbb{H}_{n,m}$ needed for the proof of Theorems 1.3 and 1.4.

We can identify an element $g = (M, (\lambda, \mu; \kappa))$ of G^J, where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{R})$, with the element

$$\begin{pmatrix} A & 0 & B & A\,{}^t\mu - B\,{}^t\lambda \\ \lambda & I_m & \mu & \kappa \\ C & 0 & D & C\,{}^t\mu - D\,{}^t\lambda \\ 0 & 0 & 0 & I_m \end{pmatrix}$$

of $\mathrm{Sp}(m+n, \mathbb{R})$.

Set

$$T_* = \frac{1}{\sqrt{2}} \begin{pmatrix} I_{m+n} & I_{m+n} \\ \mathrm{i}I_{m+n} & -\mathrm{i}I_{m+n} \end{pmatrix}.$$

We now consider the group G_*^J defined by

$$G_*^J := T_*^{-1} G^J T_*.$$

If $g = (M, (\lambda, \mu; \kappa)) \in G^J$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n, \mathbb{R})$, then $T_*^{-1} g T_*$ is given by

$$T_*^{-1} g T_* = \begin{pmatrix} P_* & Q_* \\ \overline{Q}_* & \overline{P}_* \end{pmatrix}, \tag{2.1}$$

where

$$P_* = \begin{pmatrix} P & \frac{1}{2}\{Q\,{}^t(\lambda + \mathrm{i}\mu) - P\,{}^t(\lambda - \mathrm{i}\mu)\} \\ \frac{1}{2}(\lambda + \mathrm{i}\mu) & I_h + \mathrm{i}\frac{\kappa}{2} \end{pmatrix},$$

$$Q_* = \begin{pmatrix} Q & \frac{1}{2}\{P\,{}^t(\lambda - \mathrm{i}\mu) - Q\,{}^t(\lambda + \mathrm{i}\mu)\} \\ \frac{1}{2}(\lambda - \mathrm{i}\mu) & -\mathrm{i}\frac{\kappa}{2} \end{pmatrix},$$

and P, Q are given by the formulas

$$P = \frac{1}{2}\{(A + D) + \mathrm{i}(B - C)\}, \tag{2.2}$$

$$Q = \frac{1}{2}\{(A - D) - \mathrm{i}(B + C)\}. \tag{2.3}$$

From now on, we write

$$\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \left(\frac{1}{2}(\lambda + \mathrm{i}\mu), \frac{1}{2}(\lambda - \mathrm{i}\mu); \mathrm{i}\frac{\kappa}{2}\right)\right) := \begin{pmatrix} P_* & Q_* \\ \overline{Q}_* & \overline{P}_* \end{pmatrix}.$$

In other words, we have the relation

$$T_*^{-1}\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu; \kappa)\right) T_* = \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \left(\frac{1}{2}(\lambda + \mathrm{i}\mu), \frac{1}{2}(\lambda - \mathrm{i}\mu); -\mathrm{i}\frac{\kappa}{2}\right)\right).$$

Let

$$H_{\mathbb{C}}^{(n,m)} := \{(\xi, \eta; \zeta) \mid \xi, \eta \in \mathbb{C}^{(m,n)},\ \zeta \in \mathbb{C}^{(m,m)},\ \zeta + \eta\,{}^t\xi \text{ symmetric}\}$$

be the complex Heisenberg group endowed with the following multiplication:

$$(\xi, \eta; \zeta) \circ (\xi', \eta'; \zeta') := (\xi + \xi', \eta + \eta'; \zeta + \zeta' + \xi\,{}^t\eta' - \eta\,{}^t\xi').$$

We define the semidirect product

$$\mathrm{SL}(2n, \mathbb{C}) \ltimes H_{\mathbb{C}}^{(n,m)}$$

endowed with the following multiplication:

$$\begin{aligned} &\left(\begin{pmatrix} P & Q \\ R & S \end{pmatrix}, (\xi, \eta; \zeta)\right) \cdot \left(\begin{pmatrix} P' & Q' \\ R' & S' \end{pmatrix}, (\xi', \eta'; \zeta')\right) \\ &= \left(\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\begin{pmatrix} P' & Q' \\ R' & S' \end{pmatrix}, (\widetilde{\xi} + \xi', \widetilde{\eta} + \eta'; \zeta + \zeta' + \widetilde{\xi}\,{}^t\eta' - \widetilde{\eta}\,{}^t\xi')\right), \end{aligned}$$

where $\widetilde{\xi} = \xi P' + \eta R'$ and $\widetilde{\eta} = \xi Q' + \eta S'$.

If we identify $H_{\mathbb{R}}^{(n,m)}$ with the subgroup

$$\{(\xi, \overline{\xi}; \mathrm{i}\kappa) \mid \xi \in \mathbb{C}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)}\}$$

of $H_{\mathbb{C}}^{(n,m)}$, we have the following inclusion:

$$G_*^J \subset \mathrm{SU}(n,n) \ltimes H_{\mathbb{R}}^{(n,m)} \subset \mathrm{SL}(2n, \mathbb{C}) \ltimes H_{\mathbb{C}}^{(n,m)}.$$

We define the mapping $\Theta : G^J \to G_*^J$ by

$$\Theta\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu; \kappa)\right) := \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \left(\frac{1}{2}(\lambda + \mathrm{i}\mu), \frac{1}{2}(\lambda - \mathrm{i}\mu); -\mathrm{i}\frac{\kappa}{2}\right)\right), \tag{2.4}$$

where P and Q are given by (2.2) and (2.3). We can see that if $g_1, g_2 \in G^J$, then $\Theta(g_1 g_2) = \Theta(g_1)\Theta(g_2)$.

According to [10, p. 250], G_*^J is of the Harish-Chandra type (see [5, p. 118]). Let

$$g_* = \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\lambda, \mu; \kappa)\right)$$

be an element of G_*^J. Since the Harish-Chandra decomposition of an element $\left(\begin{smallmatrix} P & Q \\ R & S \end{smallmatrix}\right)$ in $\mathrm{SU}(n,n)$ is given by

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} I_n & QS^{-1} \\ 0 & I_n \end{pmatrix}\begin{pmatrix} P - QS^{-1}R & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} I_n & 0 \\ S^{-1}R & I_n \end{pmatrix},$$

the P_*^+-component of the following element

$$g_* \cdot \left(\begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0, \eta; 0) \right), \quad W \in \mathbb{D}_n$$

of $\mathrm{SL}(2n, \mathbb{C}) \ltimes H_{\mathbb{C}}^{(n,m)}$ is given by

$$\left(\begin{pmatrix} I_n & (PW+Q)(\overline{Q}W+\overline{P}\,)^{-1} \\ 0 & I_n \end{pmatrix}, (0, (\eta + \lambda W + \mu)(\overline{Q}W + \overline{P}\,)^{-1}; 0) \right). \tag{2.5}$$

We can identify $\mathbb{D}_{n,m}$ with the subset

$$\left\{ \left(\begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0, \eta; 0) \right) \,\middle|\, W \in \mathbb{D}_n, \ \eta \in \mathbb{C}^{(m,n)} \right\}$$

of the complexification of G_*^J. Indeed, $\mathbb{D}_{n,m}$ is embedded into P_*^+ given by

$$P_*^+ = \left\{ \left(\begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0, \eta; 0) \right) \,\middle|\, W = {}^tW \in \mathbb{C}^{(n,n)}, \ \eta \in \mathbb{C}^{(m,n)} \right\}.$$

This is a generalization of the Harish-Chandra embedding (see [5, p. 119]). Then we get the natural transitive action of G_*^J on $\mathbb{D}_{n,m}$ defined by

$$\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\xi, \overline{\xi}; \mathrm{i}\kappa) \right) \cdot (W, \eta) = ((PW+Q)(\overline{Q}W+\overline{P}\,)^{-1}, (\eta + \xi W + \overline{\xi}\,)(\overline{Q}W+\overline{P}\,)^{-1}), \tag{2.6}$$

where $\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*$, $\xi \in \mathbb{C}^{(m,n)}$, $\kappa \in \mathbb{R}^{(m,m)}$ and $(W, \eta) \in \mathbb{D}_{n,m}$.

The author proved in [12] that the action (1.2) of G^J on $\mathbb{H}_{n,m}$ is compatible with the action (2.6) of G_*^J on $\mathbb{D}_{n,m}$ through a partial Cayley transform $\Phi : \mathbb{D}_{n,m} \to \mathbb{H}_{n,m}$ defined by

$$\Phi(W, \eta) := (\mathrm{i}(I_n + W)(I_n - W)^{-1}, 2\mathrm{i}\eta(I_n - W)^{-1}). \tag{2.7}$$

In other words, if $g_0 \in G^J$ and $(W, \eta) \in \mathbb{D}_{n,m}$, we have

$$g_0 \cdot \Phi(W, \eta) = \Phi(g_* \cdot (W, \eta)), \tag{2.8}$$

where $g_* = T_*^{-1} g_0 T_*$. Φ is a biholomorphic mapping of $\mathbb{D}_{n,m}$ onto $\mathbb{H}_{n,m}$ which gives the partially bounded realization of $\mathbb{H}_{n,m}$ by $\mathbb{D}_{n,m}$. The inverse of Φ is

$$\Phi^{-1}(\Omega, Z) = ((\Omega - \mathrm{i}I_n)(\Omega + \mathrm{i}I_n)^{-1}, Z(\Omega + \mathrm{i}I_n)^{-1}).$$

3 Proof of Theorem 1.3

For $(W, \eta) \in \mathbb{D}_{n,m}$, we write

$$(\Omega, Z) := \Phi(W, \eta).$$

Thus

$$\Omega = \mathrm{i}(I_n + W)(I_n - W)^{-1}, \quad Z = 2\mathrm{i}\eta(I_n - W)^{-1}. \tag{3.1}$$

Since

$$\mathrm{d}(I_n - W)^{-1} = (I_n - W)^{-1} \mathrm{d}W (I_n - W)^{-1},$$
$$I_n + (I_n + W)(I_n - W)^{-1} = 2(I_n - W)^{-1},$$

we get the following formulas from (3.1):

$$Y = \frac{1}{2i}(\Omega - \overline{\Omega}) = (I_n - W)^{-1}(I_n - W\overline{W})(I_n - \overline{W})^{-1}, \tag{3.2}$$

$$V = \frac{1}{2i}(Z - \overline{Z}) = \eta(I_n - W)^{-1} + \overline{\eta}(I_n - \overline{W})^{-1}, \tag{3.3}$$

$$d\Omega = 2i(I_n - W)^{-1}dW(I_n - W)^{-1}, \tag{3.4}$$

$$dZ = 2i\{d\eta + \eta(I_n - W)^{-1}dW\}(I_n - W)^{-1}. \tag{3.5}$$

According to (3.2) and (3.4), we obtain

$$\sigma(Y^{-1}d\Omega Y^{-1}d\overline{\Omega}) = 4\sigma((I_n - W\overline{W})^{-1}dW(I_n - \overline{W}W)^{-1}d\overline{W}). \tag{3.6}$$

From (3.2)–(3.4), we get

$$\sigma(Y^{-1}\,{}^t VVY^{-1}d\Omega Y^{-1}d\overline{\Omega}) = (a) + (b) + (c) + (d), \tag{3.7}$$

where

$$\begin{aligned}
(a) &:= 4\sigma((I_n - W\overline{W})^{-1}\,{}^t\eta\eta(I_n - \overline{W}W)^{-1}(I_n - \overline{W})(I_n - W)^{-1}dW(I_n - \overline{W}W)^{-1}d\overline{W}),\\
(b) &:= 4\sigma((I_n - W\overline{W})^{-1}\,{}^t\eta\overline{\eta}(I_n - W\overline{W})^{-1}dW(I_n - \overline{W}W)^{-1}d\overline{W}),\\
(c) &:= 4\sigma((I_n - \overline{W})^{-1}(I_n - W)(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\eta(I_n - \overline{W}W)^{-1}\\
&\quad \times (I_n - \overline{W})(I_n - W)^{-1}dW(I_n - \overline{W}W)^{-1}d\overline{W}),\\
(d) &:= 4\sigma((I_n - \overline{W})^{-1}(I_n - W)(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\,\overline{\eta}(I_n - W\overline{W})^{-1}dW(I_n - \overline{W}W)^{-1}d\overline{W}).
\end{aligned}$$

According to (3.2) and (3.5), we get

$$\sigma(Y^{-1}\,{}^t(dZ)d\overline{Z}) = (e) + (f) + (g) = (h), \tag{3.8}$$

where

$$\begin{aligned}
(e) &:= 4\sigma((I_n - W\overline{W})^{-1}\,{}^t(d\eta)d\overline{\eta}),\\
(f) &:= 4\sigma((I_n - W\overline{W})^{-1}dW(I_n - W)^{-1}\,{}^t\eta d\overline{\eta}),\\
(g) &:= 4\sigma((I_n - W\overline{W})^{-1}\,{}^t(d\eta)\overline{\eta}(I_n - \overline{W})^{-1}d\overline{W}),\\
(h) &:= 4\sigma((I_n - W\overline{W})^{-1}dW(I_n - W)^{-1}\,{}^t\eta\,\overline{\eta}(I_n - \overline{W})^{-1}d\overline{W}).
\end{aligned}$$

From (3.2)–(3.5), we get

$$-\sigma(VY^{-1}d\Omega Y^{-1}\,{}^t(d\overline{Z})) = (i) + (j) + (k) + (l), \tag{3.9}$$

where

$$\begin{aligned}
(i) &:= -4\sigma(\eta(I_n - W)^{-1}(I_n - \overline{W})(I_n - W\overline{W})^{-1}dW(I_n - \overline{W}W)^{-1}\,{}^t(d\overline{\eta})),\\
(j) &:= -4\sigma((I_n - \overline{W})^{-1}\,{}^t\overline{\eta}\eta(I_n - W)^{-1}(I_n - \overline{W})(I_n - W\overline{W})^{-1}dW(I_n - \overline{W}W)^{-1}d\overline{W}),\\
(k) &:= -4\sigma(\overline{\eta}(I_n - W\overline{W})^{-1}dW(I_n - \overline{W}W)^{-1}\,{}^t(d\overline{\eta})),\\
(l) &:= -4\sigma((I_n - \overline{W})^{-1}\,{}^t\overline{\eta}\,\overline{\eta}(I_n - W\overline{W})^{-1}dW(I_n - \overline{W}W)^{-1}d\overline{W}).
\end{aligned}$$

Conjugating (3.9), we get

$$-\sigma(VY^{-1}d\overline{\Omega}Y^{-1}\,{}^t(dZ)) = (m) + (n) + (o) + (p), \tag{3.10}$$

where

$$\begin{aligned}
(\mathrm{m}) &:= -4\sigma(\overline{\eta}(I_n-\overline{W})^{-1}(I_n-W)(I_n-\overline{W}W)^{-1}d\overline{W}(I_n-W\overline{W})^{-1}\,{}^t(d\eta)),\\
(\mathrm{n}) &:= -4\sigma((I_n-W)^{-1}\,{}^t\eta\overline{\eta}(I_n-\overline{W})^{-1}(I_n-W)(I_n-\overline{W}W)^{-1}d\overline{W}(I_n-W\overline{W})^{-1}dW),\\
(\mathrm{o}) &:= -4\sigma(\eta(I_n-\overline{W}W)^{-1}d\overline{W}(I_n-W\overline{W})^{-1}\,{}^t(d\eta)),\\
(\mathrm{p}) &:= -4\sigma((I_n-W)^{-1}\,{}^t\eta\eta(I_n-\overline{W}W)^{-1}d\overline{W}(I_n-W\overline{W})^{-1}dW).
\end{aligned}$$

If we add (f), (i) and (k), we get

$$(\mathrm{f})+(\mathrm{i})+(\mathrm{k}) = 4\sigma((\eta\overline{W}-\overline{\eta})(I_n-W\overline{W})^{-1}dW(I_n-\overline{W}W)^{-1}\,{}^t(d\overline{\eta})). \tag{3.11}$$

Indeed, transposing the matrix inside (f), we get

$$(\mathrm{f}) = 4\sigma(\eta(I_n-W)^{-1}dW(I_n-\overline{W}W)^{-1}\,{}^t(d\overline{\eta})).$$

Adding (f) and (i) together with (k), we get (3.11) because

$$\begin{aligned}
&(I_n-W)^{-1}-(I_n-W)^{-1}(I_n-\overline{W})(I_n-W\overline{W})^{-1}\\
&=(I_n-W)^{-1}\{(I_n-W\overline{W})-(I_n-\overline{W})\}(I_n-W\overline{W})^{-1}\\
&=(I_n-W)^{-1}(I_n-W)\overline{W}(I_n-W\overline{W})^{-1}=\overline{W}(I_n-W\overline{W})^{-1}.
\end{aligned}$$

If we add formulas (g), (m) and (o), we get

$$(\mathrm{g})+(\mathrm{m})+(\mathrm{o}) = 4\sigma((\overline{\eta}W-\eta)(I_n-\overline{W}W)^{-1}d\overline{W}(I_n-W\overline{W})^{-1}\,{}^t(d\eta)). \tag{3.12}$$

Indeed, we can express (g) as

$$(\mathrm{g}) = 4\sigma(\overline{\eta}(I_n-\overline{W})^{-1}d\overline{W}(I_n-W\overline{W})^{-1}\,{}^t(d\eta)).$$

Adding (g) and (m) together with (o), we get (3.12) because

$$\begin{aligned}
&(I_n-\overline{W})^{-1}-(I_n-\overline{W})^{-1}(I_n-W)(I_n-\overline{W}W)^{-1}\\
&=(I_n-\overline{W})^{-1}\{(I_n-\overline{W}W)-(I_n-W)\}(I_n-\overline{W}W)^{-1}\\
&=(I_n-\overline{W})^{-1}(I_n-\overline{W})W(I_n-\overline{W}W)^{-1}=W(I_n-\overline{W}W)^{-1}.
\end{aligned}$$

If we add (a) and (p), we get

$$(\mathrm{a})+(\mathrm{p}) = -4\sigma((I_n-W\overline{W})^{-1}\,{}^t\eta\eta(I_n-\overline{W}W)^{-1}\overline{W}dW(I_n-\overline{W}W)^{-1}d\overline{W}). \tag{3.13}$$

Indeed, transposing the matrix inside (p), we get

$$(\mathrm{p}) = -4\sigma((I_n-W\overline{W})^{-1}\,{}^t\eta\eta(I_n-W)^{-1}dW(I_n-\overline{W}W)^{-1}d\overline{W}).$$

Adding (a) and (p), we get (3.13) because

$$\begin{aligned}
&(I_n-\overline{W}W)^{-1}(I_n-\overline{W})(I_n-W)^{-1}-(I_n-W)^{-1}\\
&=(I_n-\overline{W}W)^{-1}\{(I_n-\overline{W})-(I_n-\overline{W}W)\}(I_n-W)^{-1}\\
&=(I_n-\overline{W}W)^{-1}(-\overline{W})(I_n-W)(I_n-W)^{-1}=-(I_n-\overline{W}W)^{-1}\overline{W}.
\end{aligned}$$

Adding (d) and (l), we get

$$(\mathrm{d}) + (\mathrm{l}) = -4\sigma(W(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\,\overline{\eta}(I_n - W\overline{W})^{-1}\mathrm{d}W(I_n - \overline{W}W)^{-1}\mathrm{d}\overline{W}) \tag{3.14}$$

because

$$\begin{aligned}&(I_n - \overline{W})^{-1}(I_n - W)(I_n - \overline{W}W)^{-1} - (I_n - \overline{W})^{-1}\\ &= (I_n - \overline{W})^{-1}\{(I_n - W) - (I_n - \overline{W}W)\}(I_n - \overline{W}W)^{-1}\\ &= (I_n - \overline{W})^{-1}(I_n - \overline{W})(-W)(I_n - \overline{W}W)^{-1} = -W(I_n - \overline{W}W)^{-1}.\end{aligned}$$

Adding (h) and (j), we get

$$(\mathrm{h}) + (\mathrm{j}) = 4\sigma((I_n - \overline{W})^{-1}\,{}^t\overline{\eta}\eta\overline{W}(I_n - W\overline{W})^{-1}\mathrm{d}W(I_n - \overline{W}W)^{-1}\mathrm{d}\overline{W}). \tag{3.15}$$

Indeed, transposing the matrix inside (h), we get

$$(\mathrm{h}) = 4\sigma((I_n - \overline{W})^{-1}\,{}^t\overline{\eta}\eta(I_n - W)^{-1}\mathrm{d}W(I_n - \overline{W}W)^{-1}\mathrm{d}\overline{W}).$$

Adding (h) and (j), we get (3.15) because

$$\begin{aligned}&(I_n - W)^{-1} - (I_n - W)^{-1}(I_n - \overline{W})(I_n - W\overline{W})^{-1}\\ &= (I_n - W)^{-1}\{(I_n - W\overline{W}) - (I_n - \overline{W})\}(I_n - W\overline{W})^{-1}\\ &= (I_n - W)^{-1}(I_n - W)\overline{W}(I_n - W\overline{W})^{-1} = \overline{W}(I_n - W\overline{W})^{-1}.\end{aligned}$$

Transposing the matrix inside (n), we get

$$(\mathrm{n}) = -4\sigma((I_n - W\overline{W})^{-1}(I_n - W)(I_n - \overline{W})^{-1}\,{}^t\overline{\eta}\eta(I_n - W)^{-1}\mathrm{d}W(I_n - \overline{W}W)^{-1}\mathrm{d}\overline{W}). \tag{3.16}$$

From (3.7)–(3.16), we obtain

$$\begin{aligned}&\sigma(Y^{-1}\,{}^tVVY^{-1}\mathrm{d}\Omega Y^{-1}\mathrm{d}\overline{\Omega}) + \sigma(Y^{-1}\,{}^t(\mathrm{d}Z)\mathrm{d}\overline{Z})\\ &\quad - \sigma(VY^{-1}\mathrm{d}\Omega Y^{-1}\,{}^t(\mathrm{d}\overline{Z})) - \sigma(VY^{-1}\mathrm{d}\overline{\Omega}Y^{-1}\,{}^t(\mathrm{d}Z))\\ &= (\mathrm{a}) + (\mathrm{b}) + (\mathrm{c}) + (\mathrm{d}) + \cdots + (\mathrm{m}) + (\mathrm{n}) + (\mathrm{o}) + (\mathrm{p})\\ &= 4\sigma((I_n - W\overline{W})^{-1}\,{}^t(\mathrm{d}\eta)\mathrm{d}\overline{\eta}) + 4\sigma((\eta\overline{W} - \overline{\eta})(I_n - W\overline{W})^{-1}\mathrm{d}W(I_n - \overline{W}W)^{-1}\,{}^t(\mathrm{d}\overline{\eta}))\\ &\quad + 4\sigma((\overline{\eta}W - \eta)(I_n - \overline{W}W)^{-1}\mathrm{d}\overline{W}(I_n - W\overline{W})^{-1}\,{}^t(\mathrm{d}\eta))\\ &\quad - 4\sigma((I_n - W\overline{W})^{-1}\,{}^t\eta\eta(I_n - \overline{W}W)^{-1}\overline{W}\mathrm{d}W(I_n - \overline{W}W)^{-1}\mathrm{d}\overline{W})\\ &\quad - 4\sigma(W(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\,\overline{\eta}(I_n - W\overline{W})^{-1}\mathrm{d}W(I_n - \overline{W}W)^{-1}\mathrm{d}\overline{W})\\ &\quad + 4\sigma((I_n - W\overline{W})^{-1}\,{}^t\eta\overline{\eta}(I_n - W\overline{W})^{-1}\mathrm{d}W(I_n - \overline{W}W)^{-1}\mathrm{d}\overline{W})\\ &\quad + 4\sigma((I_n - \overline{W})^{-1}\,{}^t\overline{\eta}\eta\overline{W}(I_n - W\overline{W})^{-1}\mathrm{d}W(I_n - \overline{W}W)^{-1}\mathrm{d}\overline{W})\\ &\quad + 4\sigma((I_n - \overline{W})^{-1}(I_n - W)(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\eta(I_n - \overline{W}W)^{-1}\\ &\quad \times (I_n - \overline{W})(I_n - W)^{-1}\mathrm{d}W(I_n - \overline{W}W)^{-1}\mathrm{d}\overline{W})\\ &\quad - 4\sigma((I_n - W\overline{W})^{-1}(I_n - W)(I_n - \overline{W})^{-1}\,{}^t\overline{\eta}\eta(I_n - W)^{-1}\mathrm{d}W(I_n - \overline{W}W)^{-1}\mathrm{d}\overline{W}).\end{aligned}$$

Consequently, the complete proof follows from the above formula, (3.6), Theorem 1.1 and the fact that the action (1.2) of G^J on $\mathbb{H}_{n,m}$ is compatible with the action (2.6) of G_*^J on $\mathbb{D}_{n,m}$ through the partial Cayley transform.

4 Proof of Theorem 1.4

From (3.1), (3.4) and (3.5), we get

$$\frac{\partial}{\partial \Omega} = \frac{1}{2\mathrm{i}}(I_n - W)\Big[\,{}^t\Big\{(I_n - W)\frac{\partial}{\partial W}\Big\} - {}^t\Big\{{}^t\eta\,{}^t\Big(\frac{\partial}{\partial \eta}\Big)\Big\}\Big], \tag{4.1}$$

$$\frac{\partial}{\partial Z} = \frac{1}{2\mathrm{i}}(I_n - W)\frac{\partial}{\partial \eta}. \tag{4.2}$$

We need the following lemma for the proof of Theorem 1.4. Maass [3] observed the following useful fact.

Lemma 4.1 (a) *Let A be an $m \times n$ matrix and B an $n \times l$ matrix. Assume that the entries of A commute with the entries of B. Then ${}^t(AB) = {}^tB\,{}^tA$.*

(b) *Let A, B and C be a $k \times l$, an $n \times m$ and an $m \times l$ matrix respectively. Assume that the entries of A commute with the entries of B. Then*

$${}^t(A\,{}^t(BC)) = B\,{}^t(A\,{}^tC).$$

Proof The proof follows immediately from a direct computation.

From (3.2), (4.1) and Lemma 4.1, we get the following formula:

$$4\sigma\Big(Y\,{}^t\Big(Y\frac{\partial}{\partial \overline{\Omega}}\Big)\frac{\partial}{\partial \Omega}\Big) = (\alpha) + (\beta) + (\gamma) + (\delta), \tag{4.3}$$

where

$$(\alpha) := \sigma\Big((I_n - W\overline{W})\,{}^t\Big((I_n - W\overline{W})\frac{\partial}{\partial \overline{W}}\Big)\frac{\partial}{\partial W}\Big),$$

$$(\beta) := -\sigma\Big(\eta(I_n - W)^{-1}(I_n - W\overline{W})\,{}^t\Big((I_n - W\overline{W})\frac{\partial}{\partial \overline{W}}\Big)\frac{\partial}{\partial \eta}\Big),$$

$$(\gamma) := -\sigma\Big((I_n - W\overline{W})(I_n - \overline{W})^{-1}\,{}^t\overline{\eta}\,{}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial W}\Big),$$

$$(\delta) := \sigma\Big(\eta(I_n - W)^{-1}(I_n - W\overline{W})(I_n - \overline{W})^{-1}\,{}^t\overline{\eta}\,{}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial \eta}\Big).$$

According to (3.2) and (4.2), we get

$$4\sigma\Big(Y\frac{\partial}{\partial Z}\,{}^t\Big(\frac{\partial}{\partial \overline{Z}}\Big)\Big) = \sigma\Big((I_n - \overline{W}W)\frac{\partial}{\partial \eta}\,{}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)\Big). \tag{4.4}$$

From (3.2), (3.3) and (4.2), we get

$$4\sigma\Big(VY^{-1}\,{}^tV\,{}^t\Big(Y\frac{\partial}{\partial \overline{Z}}\Big)\frac{\partial}{\partial Z}\Big) = (\epsilon) + (\zeta) + (\eta) + (\theta), \tag{4.5}$$

where

$$(\epsilon) := \sigma\Big(\eta(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\,{}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial \eta}\Big),$$

$$(\zeta) := \sigma\Big(\overline{\eta}(I_n - W\overline{W})^{-1}\,{}^t\eta\,{}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial \eta}\Big),$$

$$(\eta) := \sigma\Big(\eta(I_n - W)^{-1}(I_n - \overline{W})(I_n - W\overline{W})^{-1}\,{}^t\eta\,{}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial \eta}\Big),$$

$$(\theta) := \sigma\Big(\overline{\eta}(I_n - \overline{W})^{-1}(I_n - W)(I_n - \overline{W}W)^{-1}\,{}^t\overline{\eta}\,{}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)(I_n - \overline{W}W)\frac{\partial}{\partial \eta}\Big).$$

Using (3.2), (3.3), (4.1), (4.2) and Lemma 4.1, we get

$$4\sigma\Big(V\,{}^t\Big(Y\frac{\partial}{\partial\overline{\Omega}}\Big)\frac{\partial}{\partial Z}\Big) = (\iota)+(\kappa)+(\lambda)+(\mu), \tag{4.6}$$

where

$$(\iota) := \sigma\Big(\overline{\eta}\,{}^t\Big((I_n - W\overline{W})\frac{\partial}{\partial\overline{W}}\Big)\frac{\partial}{\partial\eta}\Big),$$
$$(\kappa) := \sigma\Big(\eta(I_n - W)^{-1}(I_n-\overline{W})\,{}^t\Big((I_n - W\overline{W})\frac{\partial}{\partial\overline{W}}\Big)\frac{\partial}{\partial\eta}\Big),$$
$$(\lambda) := -\sigma\Big(\eta(I_n - W)^{-1}\,{}^t\overline{\eta}\,{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\Big),$$
$$(\mu) := -\sigma\Big(\overline{\eta}(I_n - \overline{W})^{-1}\,{}^t\overline{\eta}\,{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\Big).$$

Similarly, we get

$$4\sigma\Big({}^tV\,{}^t\Big(Y\frac{\partial}{\partial\overline{Z}}\Big)\frac{\partial}{\partial \Omega}\Big) = (\nu)+(\xi)+(o)+(\pi), \tag{4.7}$$

where

$$(\nu) := \sigma\Big({}^t\eta\,{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial W}\Big),$$
$$(\xi) := \sigma\Big((I_n - W)(I_n-\overline{W})^{-1}\,{}^t\overline{\eta}\,{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial W}\Big),$$
$$(o) := -\sigma\Big(\eta(I_n - W)^{-1}\,{}^t\eta\,{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\Big),$$
$$(\pi) := -\sigma\Big(\eta(I_n - \overline{W})^{-1}\,{}^t\overline{\eta}\,{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\Big).$$

Adding (γ), (ν) and (ξ), we get

$$(\gamma)+(\nu)+(\xi) = \sigma\Big({}^t(\eta-\overline{\eta}W)\,{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial W}\Big) \tag{4.8}$$

because

$$-(I_n - W\overline{W})(I_n-\overline{W})^{-1} + (I_n-W)(I_n-\overline{W})^{-1} = -W(I_n-\overline{W})(I_n-\overline{W})^{-1} = -W.$$

Adding (β), (ι) and (κ), we get

$$(\beta)+(\iota)+(\kappa) = \sigma\Big((\overline{\eta}-\eta\overline{W})\,{}^t\Big((I_n-W\overline{W})\frac{\partial}{\partial\overline{W}}\Big)\frac{\partial}{\partial\eta}\Big) \tag{4.9}$$

because

$$-(I_n-W)^{-1}(I_n-W\overline{W}) + (I_n-W)^{-1}(I_n-\overline{W}) = -(I_n-W)^{-1}(I_n-W)\overline{W} = -\overline{W}.$$

If we add (η) and (o), we get

$$(\eta)+(o) = -\sigma\Big(\eta\overline{W}(I_n-W\overline{W})^{-1}\,{}^t\eta\,{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\Big) \tag{4.10}$$

because

$$\begin{aligned}&(I_n-W)^{-1}(I_n-\overline{W})(I_n-W\overline{W})^{-1}-(I_n-W)^{-1}\\ =\,&(I_n-W)^{-1}\{I_n-\overline{W}-(I_n-W\overline{W})\}(I_n-W\overline{W})^{-1}\\ =\,&(I_n-W)^{-1}(I_n-W)(-\overline{W})(I_n-W\overline{W})^{-1}=-\overline{W}(I_n-W\overline{W})^{-1}.\end{aligned}$$

If we add (θ) and (μ), we get

$$(\theta)+(\mu)=-\sigma\Big(\overline{\eta}W(I_n-\overline{W}W)^{-1}\,{}^t\overline{\eta}\;{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\Big) \tag{4.11}$$

because

$$\begin{aligned}&(I_n-\overline{W})^{-1}(I_n-W)(I_n-\overline{W}W)^{-1}-(I_n-\overline{W})^{-1}\\ =\,&(I_n-\overline{W})^{-1}\{I_n-W-(I_n-\overline{W}W)\}(I_n-\overline{W}W)^{-1}\\ =\,&(I_n-\overline{W})^{-1}(I_n-\overline{W})(-W)(I_n-\overline{W}W)^{-1}=-W(I_n-\overline{W}W)^{-1}.\end{aligned}$$

If we add (δ), (ϵ), (λ) and (π), we get

$$(\delta)+(\epsilon)+(\lambda)+(\pi)=\sigma\Big(\eta\overline{W}W(I_n-\overline{W}W)^{-1}\,{}^t\overline{\eta}\;{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\Big) \tag{4.12}$$

because

$$\begin{aligned}&(I_n-W)^{-1}(I_n-W\overline{W})(I_n-\overline{W})^{-1}+(I_n-\overline{W}W)^{-1}-(I_n-W)^{-1}-(I_n-\overline{W})^{-1}\\ =\,&(I_n-W)^{-1}\{(I_n-W\overline{W})-(I_n-\overline{W})\}(I_n-\overline{W})^{-1}+(I_n-\overline{W}W)^{-1}-(I_n-\overline{W})^{-1}\\ =\,&\overline{W}(I_n-\overline{W})^{-1}+(I_n-\overline{W}W)^{-1}-(I_n-\overline{W})^{-1}=-(I_n-\overline{W})(I_n-\overline{W})^{-1}+(I_n-\overline{W}W)^{-1}\\ =\,&-I_n+(I_n-\overline{W}W)^{-1}=\{-(I_n-\overline{W}W)+I_n\}(I_n-\overline{W}W)^{-1}=\overline{W}W(I_n-\overline{W}W)^{-1}.\end{aligned}$$

From (4.3) and (4.5)–(4.12), we obtain

$$\begin{aligned}&\sigma\Big(Y\,{}^t\Big(Y\frac{\partial}{\partial\overline{\Omega}}\Big)\frac{\partial}{\partial\Omega}\Big)+\sigma\Big(VY^{-1}\,{}^tV\,{}^t\Big(Y\frac{\partial}{\partial\overline{Z}}\Big)\frac{\partial}{\partial Z}\Big)\\ &+\sigma\Big(V\,{}^t\Big(Y\frac{\partial}{\partial\overline{\Omega}}\Big)\frac{\partial}{\partial Z}\Big)+\sigma\Big({}^tV\,{}^t\Big(Y\frac{\partial}{\partial\overline{Z}}\Big)\frac{\partial}{\partial\Omega}\Big)\\ =\,&(\alpha)+(\beta)+(\gamma)+(\delta)+\cdots+(\nu)+(\xi)+(o)+(\pi)\\ =\,&\sigma\Big((I_n-W\overline{W})\,{}^t\Big((I_n-W\overline{W})\frac{\partial}{\partial\overline{W}}\Big)\frac{\partial}{\partial W}\Big)+\sigma\Big({}^t(\eta-\overline{\eta}W)\,{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial W}\Big)\\ &+\sigma\Big((\overline{\eta}-\eta\overline{W})\,{}^t\Big((I_n-W\overline{W})\frac{\partial}{\partial\overline{W}}\Big)\frac{\partial}{\partial\eta}\Big)-\sigma\Big(\eta\overline{W}(I_n-W\overline{W})^{-1}\,{}^t\eta\,{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\Big)\\ &-\sigma\Big(\overline{\eta}W(I_n-\overline{W}W)^{-1}\,{}^t\overline{\eta}\,{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\Big)\\ &+\sigma\Big(\overline{\eta}(I_n-W\overline{W})^{-1}\,{}^t\eta\,{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\Big)\\ &+\sigma\Big(\eta\overline{W}W(I_n-\overline{W}W)^{-1}\,{}^t\overline{\eta}\,{}^t\Big(\frac{\partial}{\partial\overline{\eta}}\Big)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\Big).\end{aligned}$$

Consequently, the complete proof follows from (4.4), the above formula, Theorem 1.2 and the fact that the action (1.2) of G^J on $\mathbb{H}_{n,m}$ is compatible with the action (2.6) of G_*^J on $\mathbb{D}_{n,m}$ through the partial Cayley transform.

Remark 4.1 We proved in [11] that the following two differential operators D and $L := \frac{1}{4}\Delta_{n,m;1,1} - D$ on $\mathbb{H}_{n,m}$ defined by

$$D = \sigma\Big(Y \frac{\partial}{\partial Z}\, {}^t\Big(\frac{\partial}{\partial \overline{Z}}\Big)\Big)$$

and

$$\begin{aligned} L = {} & \sigma\Big(Y\, {}^t\Big(Y \frac{\partial}{\partial \overline{\Omega}}\Big) \frac{\partial}{\partial \Omega}\Big) + \sigma\Big(VY^{-1}\, {}^tV\, {}^t\Big(Y \frac{\partial}{\partial \overline{Z}}\Big) \frac{\partial}{\partial Z}\Big) \\ & + \sigma\Big(V\, {}^t\Big(Y \frac{\partial}{\partial \overline{\Omega}}\Big) \frac{\partial}{\partial Z}\Big) + \sigma\Big({}^tV\, {}^t\Big(Y \frac{\partial}{\partial \overline{Z}}\Big) \frac{\partial}{\partial \Omega}\Big) \end{aligned}$$

are invariant under the action (1.2) of G^J. By (4.4) and the proof of Theorem 1.4, we see that the following differential operators $\widetilde{D}$ and $\widetilde{L} := \widetilde{\Delta}_{n,m;1,1} - \widetilde{D}$ on $\mathbb{D}_{n,m}$ defined by

$$\widetilde{D} = \sigma\Big((I_n - \overline{W}W) \frac{\partial}{\partial \eta}\, {}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)\Big)$$

and

$$\begin{aligned} \widetilde{L} = {} & \sigma\Big((I_n - W\overline{W})\, {}^t\Big((I_n - W\overline{W}) \frac{\partial}{\partial \overline{W}}\Big) \frac{\partial}{\partial W}\Big) + \sigma\Big({}^t(\eta - \overline{\eta}W)\, {}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)(I_n - \overline{W}W) \frac{\partial}{\partial W}\Big) \\ & + \sigma\Big((\overline{\eta} - \eta\overline{W})\, {}^t\Big((I_n - W\overline{W}) \frac{\partial}{\partial \overline{W}}\Big) \frac{\partial}{\partial \eta}\Big) - \sigma\Big(\eta\overline{W}(I_n - W\overline{W})^{-1}\, {}^t\eta\, {}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)(I_n - \overline{W}W) \frac{\partial}{\partial \eta}\Big) \\ & - \sigma\Big(\overline{\eta}W(I_n - \overline{W}W)^{-1}\, {}^t\overline{\eta}\, {}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)(I_n - \overline{W}W) \frac{\partial}{\partial \eta}\Big) \\ & + \sigma\Big(\overline{\eta}(I_n - W\overline{W})^{-1}\, {}^t\eta\, {}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)(I_n - \overline{W}W) \frac{\partial}{\partial \eta}\Big) \\ & + \sigma\Big(\eta\overline{W}W(I_n - \overline{W}W)^{-1}\, {}^t\overline{\eta}\, {}^t\Big(\frac{\partial}{\partial \overline{\eta}}\Big)(I_n - \overline{W}W) \frac{\partial}{\partial \eta}\Big) \end{aligned}$$

are invariant under the action (2.6) of G^J_*. Indeed, it is very complicated and difficult at this moment to express the generators of the algebra of all G^J_*-invariant differential operators on $\mathbb{D}_{n,m}$ explicitly. We propose an open problem to find other explicit G^J_*-invariant differential operators on $\mathbb{D}_{n,m}$.

5 Remark on Harmonic Analysis on Siegel-Jacobi Disk

It might be interesting to develop the theory of harmonic analysis on the Siegel-Jacobi disk $\mathbb{D}_{n,m}$. The theory of harmonic analysis on the generalized unit disk $\mathbb{D}_n$ can be done explicitly by the work of Harish-Chandra because $\mathbb{D}_n$ is a symmetric space. However, the Siegel-Jacobi disk $\mathbb{D}_{n,m}$ is not a symmetric space. The work for developing the theory of harmonic analysis on $\mathbb{D}_{n,m}$ explicitly is complicated and difficult at this moment. We observe that this work on $\mathbb{D}_{n,m}$ generalizes the work on the generalized unit disk $\mathbb{D}_n$.

More precisely, if we put $G_* = \mathrm{SU}(n,n) \cap \mathrm{Sp}(n,\mathbb{C})$, then the Jacobi group

$$G^J_* = \left\{ \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\xi, \overline{\xi}; i\kappa) \right) \,\middle|\, \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*,\ \xi \in \mathbb{C}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)} \right\}$$

acts on the Siegel-Jacobi disk $\mathbb{D}_{n,m}$ transitively via the transformation behavior (1.9). It is easily seen that the stabilizer K^J_* of the action (1.9) at the base point $(0,0)$ is given by

$$K^J_* = \left\{ \left(\begin{pmatrix} P & 0 \\ 0 & \overline{P} \end{pmatrix}, (0, 0; i\kappa) \right) \,\middle|\, P \in U(n),\ \kappa \in \mathbb{R}^{(m,m)} \right\}.$$

Therefore, G^J_*/K^J_* is biholomorphic to $\mathbb{D}_{n,m}$ via the correspondence

$$gK^J_* \mapsto g\cdot(0,0), \quad g\in G^J_*.$$

We observe that the Siegel-Jacobi disk $\mathbb{D}_{n,m}$ is not a reductive symmetric space.

Let

$$\Gamma_{n,m} := \mathrm{Sp}(n,\mathbb{Z}) \ltimes H^{(n,m)}_{\mathbb{Z}},$$

where $\mathrm{Sp}(n,\mathbb{Z})$ is the Siegel modular group of degree n and

$$H^{(n,m)}_{\mathbb{Z}} = \{(\lambda,\mu;\kappa)\in H^{(n,m)}_{\mathbb{R}} \mid \lambda,\mu,\kappa \text{ are integral}\}.$$

We set

$$\Gamma^*_{n,m} := T_*^{-1}\Gamma_{n,m}T_*,$$

where T_* was already defined in Section 2. Clearly, the arithmetic subgroup $\Gamma^*_{n,m}$ acts on $\mathbb{D}_{n,m}$ properly continuously. We can describe a fundamental domain $\mathcal{F}^*_{n,m}$ for $\Gamma^*_{n,m}\setminus\mathbb{D}_{n,m}$ explicitly using a partial Cayley transform and a fundamental domain $\mathcal{F}_{n,m}$ for $\Gamma_{n,m}\setminus\mathbb{H}_{n,m}$ which is described explicitly in [13]. The G^J_*-invariant metric $d\widetilde{s}_{n,m;A,B}$ on $\mathbb{D}_{n,m}$ induces a metric on $\mathcal{F}^*_{n,m}$ naturally. It may be interesting to investigate the spectral theory of the Laplacian $\widetilde{\Delta}_{n,m;A,B}$ on a fundamental domain $\mathcal{F}^*_{n,m}$. But this work is very complicated and difficult at this moment.

For instance, we consider the case $n=m=1$ and $A=B=1$. In this case,

$$G^J_* = \left\{ \left(\begin{pmatrix} p & q \\ \overline{q} & \overline{p} \end{pmatrix}, (\xi,\overline{\xi};i\kappa) \right) \,\middle|\, p,q,\xi\in\mathbb{C},\ |p|^2-|q|^2=1,\ \kappa\in\mathbb{R} \right\}$$

and

$$K^J_* = \left\{ \left(\begin{pmatrix} p & 0 \\ 0 & \overline{p} \end{pmatrix}, (0,0;i\kappa) \right) \,\middle|\, p\in\mathbb{C},\ |p|=1,\ \kappa\in\mathbb{R} \right\}.$$

$d\widetilde{s}_{1,1;1,1}$ is a G^J_*-invariant Riemannian metric on $\mathbb{D}_{1,1}=\mathbb{D}_1\times\mathbb{C}$ (see Theorem 1.3) and $\widetilde{\Delta}_{1,1;1,1}$ is its Laplacian. It is well-known that the theory of harmonic analysis on the unit disk $\mathbb{D}_1$ has been well developed explicitly (see [3, pp. 29–72]). I think that so far nobody has investigated the theory of harmonic analysis on $\mathbb{D}_{1,1}$ explicitly. For example, inversion formula, Plancherel formula, Paley-Wiener theorem on $\mathbb{D}_{1,1}$ have not been described explicitly until now. It seems that it is interesting to develop the theory of harmonic analysis on the Siegel-Jacobi disk $\mathbb{D}_{1,1}$ explicitly.

Finally, we mention that it may be interesting to investigate differential operators on $\mathbb{D}_{n,m}$ which are invariant under the natural action (1.9) of the Jacobi group G^J_* in detail (see [14]).

References

[1] Berndt, R. and Schmidt, R., Elements of the Representation Theory of the Jacobi Group, Birkhäuser, Basel, 1998.

[2] Eichler, M. and Zagier, D., The Theory of Jacobi Forms, Progress in Mathematics, **55**, Birkhäuser, Basel, 1985.

[3] Helgason, S., Groups and Geometric Analysis, Academic Press, New York, 1984.

[4] Maass, H., Die differentialgleichungen in der theorie der Siegelschen modulfunktionen, *Math. Ann.*, **126**, 1953, 44–68.

[5] Satake, I., Algebraic Structures of Symmetric Domains, Princeton University Press, Princeton, 1980.

[6] Siegel, C. L., Symplectic geometry, *Amer. J. Math.*, **65**, 1943, 1–86.

[7] Yang, J.-H., The Siegel-Jacobi operator, *Abh. Math. Sem. Univ. Hamburg*, **63**, 1993, 135–146.

[8] Yang, J.-H., Singular Jacobi Forms, *Trans. Amer. Math. Soc.*, **347**(6), 1995, 2041–2049.

[9] Yang, J.-H., Construction of vector valued modular forms from Jacobi forms, *Can. J. Math.*, **47**(6), 1995, 1329–1339.

[10] Yang, J.-H., The method of orbits for real Lie groups, *Kyungpook Math. J.*, **42**(2), 2002, 199–272.

[11] Yang, J.-H., Invariant metrics and Laplacians on Siegel-Jacobi space, *J. Number Theory*, **127**, 2007, 83–102.

[12] Yang, J.-H., A partial Cayley transform for Siegel-Jacobi disk, *J. Korean Math. Soc.*, **45**(3), 2008, 781–794.

[13] Yang, J.-H., A note on a fundamental domain for Siegel-Jacobi space, *Houston J. Math.*, **32**(3), 2006, 701–712.

[14] Yang, J.-H. A note on invariant differential operators on Siegel-Jacobi space, 2006. arXiv:math.NT/0611388

[15] Ziegler, C., Jacobi Forms of Higher Degree, *Abh. Math. Sem. Hamburg*, **59**, 1989, 191–224.

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SECTIONAL CURVATURES OF THE SIEGEL-JACOBI SPACE

Jae-Hyun Yang, Young-Hoon Yong, Su-Na Huh, Jung-Hee Shin, and Gil-Hong Min

Abstract. In this paper, we compute the sectional curvatures and the scalar curvature of the Siegel-Jacobi space $\mathbb{H}_1 \times \mathbb{C}$ of degree 1 and index 1 explicitly.

1. Introduction

For a given fixed positive integer n, we let

$$\mathbb{H}_n := \{\, Z \in \mathbb{C}^{(n,n)} \mid Z = {}^tZ,\ \operatorname{Im} Z > 0\}$$

be the Siegel upper half plane of degree n and let

$$Sp(n,\mathbb{R}) = \{M \in \mathbb{R}^{(2n,2n)} \mid {}^tMJ_nM = J_n\}$$

be the symplectic group of degree n, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l, ${}^tM$ denotes the transpose of a matrix M, $\operatorname{Im} Z$ denotes the imaginary part of Z and

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Here I_n denotes the $n \times n$ identity matrix. It is easy to see that $Sp(n,\mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

$$M\cdot Z := (AZ+B)(CZ+D)^{-1}, \tag{1.1}$$

where $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(n,\mathbb{R})$ and $Z \in \mathbb{H}_n$.

For two positive integers n and m, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} := \{\, (\lambda,\mu;\kappa) \mid \lambda,\mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\, {}^t\lambda \text{ symmetric}\}$$

endowed with the following multiplication law

$$(\lambda,\mu;\kappa) \circ (\lambda',\mu';\kappa') := (\lambda+\lambda',\mu+\mu';\kappa+\kappa'+\lambda\, {}^t\mu' - \mu\, {}^t\lambda').$$

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We define the semidirect product of $Sp(n,\mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$

$$G^J_{n,m} := Sp(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

endowed with the following multiplication law

$$\Big(M,(\lambda,\mu;\kappa)\Big)\cdot\Big(M',(\lambda',\mu';\kappa')\Big) := \Big(MM',(\tilde{\lambda}+\lambda',\tilde{\mu}+\mu';\kappa+\kappa'+\tilde{\lambda}\,{}^t\mu'-\tilde{\mu}\,{}^t\lambda')\Big)$$

with $M, M' \in Sp(n,\mathbb{R}), (\lambda,\mu;\kappa),\ (\lambda',\mu';\kappa') \in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde{\lambda},\tilde{\mu}) := (\lambda,\mu)M'$.

We call this group $G^J_{n,m}$ the *Jacobi group* of degree n and index m. It is easy to see that $G^J_{n,m}$ acts on $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ transitively by

$$\Big(M,(\lambda,\mu;\kappa)\Big)\cdot(Z,W) := \Big(M\cdot Z,\ (W+\lambda Z+\mu)(CZ+D)^{-1}\Big), \tag{1.2}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(Z,W) \in \mathbb{H}_n \times \mathbb{C}^{(m,n)}$.

The homogeneous space $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ is called the *Siegel-Jacobi space* of degree n and index m. We refer to [3, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18] for more details on materials related to the Siegel-Jacobi space.

In [14], the author proved that for any two positive real numbers A and B, the following metric

$$\begin{aligned} ds^2_{n,m;A,B} &= A\,\sigma\Big(Y^{-1}dZ\,Y^{-1}d\overline{Z}\Big) \\ &\quad + B\,\Big\{\sigma\Big(Y^{-1}\,{}^tV\,V\,Y^{-1}dZ\,Y^{-1}d\overline{Z}\Big) + \sigma\Big(Y^{-1}\,{}^t(dW)\,d\overline{W}\Big) \\ &\quad -\sigma\Big(V\,Y^{-1}dZ\,Y^{-1}\,{}^t(d\overline{W})\Big) - \sigma\Big(V\,Y^{-1}d\overline{Z}\,Y^{-1}\,{}^t(dW)\Big)\Big\} \end{aligned} \tag{1.3}$$

is a Riemannian metric on the Siegel-Jacobi space $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ which is invariant under the action (1.2) of the Jacobi group $G^J_{n,m}$, where $Z = X+i\,Y \in \mathbb{H}_n$, $W = U+i\,V \in \mathbb{C}^{(m,n)}$ with $Z=(z_{ij}),\ W=(w_{kl})$ and X,Y,U,V real, we put

$$dZ=(dz_{ij}),\ \ d\overline{Z}=(d\overline{z}_{ij}),\ \ dW=(dw_{kl}),\ \ d\overline{W}=(d\overline{w}_{kl})$$

and $\sigma(A)$ denotes the trace of a square matrix A. Also he computed the Laplace-Beltrami operator of the Siegel-Jacobi space $(\mathbb{H}_n \times \mathbb{C}^{(m,n)}, ds^2_{n,m;A,B})$ explicitly.

In this paper, we consider the case $n=1$ and $m=1$. In this case, we have a Riemannian metric

$$\begin{aligned} &ds^2_{1,1;A,B} \\ &= A\,\frac{dx^2+dy^2}{y^2} + B\,\Big\{\frac{v^2}{y^3}\,(dx^2+dy^2) + \frac{1}{y}\,(du^2+dv^2) - \frac{2v}{y^2}\,(dxdu+dydv)\Big\} \end{aligned} \tag{1.4}$$

on $\mathbb{H}_1 \times \mathbb{C}$ which is invariant under the action (1.2) of the Jacobi group $G^J_{1,1} = SL(2,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(1,1)}$, where $z = x+i\,y \in \mathbb{H}_1$ and $w = u+i\,v \in \mathbb{C}$ with x,y,u,v real coordinates. We also refer to [1] and [4] for the metric (1.4). According to

Theorem 1.2 in [14], we see that the Laplace-Beltrami operator $\Delta_{1,1;A,B}$ of the Siegel-Jacobi space $(\mathbb{H}_1 \times \mathbb{C},\ ds^2_{1,1;A,B})$ is given by

$$\begin{aligned}(1.5)\qquad \Delta_{1,1;A,B} = \frac{1}{A}\Bigg\{ & y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + v^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right) \\ & + 2\,y\,v\left(\frac{\partial^2}{\partial x\partial u}+\frac{\partial^2}{\partial y\partial v}\right)\Bigg\} + \frac{y}{B}\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right).\end{aligned}$$

The purpose of this paper is to compute the sectional curvatures of the Siegel-Jacobi space $(\mathbb{H}_1 \times \mathbb{C},\ ds^2_{1,1;A,B})$ explicitly. We will prove that the scalar curvature $r(p)$ of $(\mathbb{H}_1 \times \mathbb{C},\ ds^2_{1,1;A,B})$ is constant, precisely, $r(p) = -\frac{3}{A}$ for all $p \in \mathbb{H}_1 \times \mathbb{C}$.

This paper is organized as follows. In Section 2, we compute the Christoffel symbols Γ^k_{ij} of the Siegel-Jacobi space $(\mathbb{H}_1 \times \mathbb{C}, ds^2_{1,1;A,B})$ explicitly. In Section 3, we compute the sectional curvatures of the Siegel-Jacobi space $(\mathbb{H}_1 \times \mathbb{C}, ds^2_{1,1;A,B})$ explicitly. We prove that the scalar curvature of the Siegel-Jacobi space $(\mathbb{H}_1 \times \mathbb{C}, ds^2_{1,1;A,B})$ is given by $-\frac{3}{A}$ and that the scalar curvature is independent of the choice of B. In the final section, we discuss the invariant Riemannian metrics of the Siegel-Jacobi disk $\mathbb{D} \times \mathbb{C}$ and their Laplace-Beltrami operators.

Notations: We denote by $\mathbb{R}$ and $\mathbb{C}$ the field of real numbers, and the field of complex numbers respectively. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. I_n denotes the identity matrix of degree n.

2. Preliminaries

For brevity, we write $M := \mathbb{H}_1 \times \mathbb{C}$. Then M is a four dimensional Riemannian manifold with a metric ds^2 given by (1.4). We denote by $C^\infty(M)$ and $\mathcal{X}(M)$ be the algebra of all C^∞ functions on M and the algebra of all C^∞ vector fields on M respectively. It is well known that there exists a uniquely determined Riemannian connection ∇ on M (cf. [2], p. 314). That is, the connection ∇ is a mapping $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$, denoted by $\nabla(X,Y) = \nabla_X Y$ which satisfies the following properties (R1)-(R4): For all $f, g \in C^\infty(M)$ and $X, Y, Z, W \in \mathcal{X}(M)$,

$$\begin{aligned}&(R1)\qquad \nabla_{fX+gY}Z = f(\nabla_X Z) + g(\nabla_Y Z),\\ &(R2)\qquad \nabla_X(fY+gZ) = f(\nabla_X Y) + g(\nabla_X Z) + (Xf)Y + (Xg)Z,\\ &(R3)\qquad [X,Y] = \nabla_X Y - \nabla_Y X \ \text{(symmetry), and}\\ &(R4)\qquad X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),\end{aligned}$$

where $g(Y,Z)$ denoted the inner product determined by the Riemannian metric ds^2 on M.

Now we fix a local cooodinate x,y,u,v with $z=x+i\,y$ and $w=u+i\,v$. Then the smooth vector fields

$$E_1 := \frac{\partial}{\partial x},\quad E_2 := \frac{\partial}{\partial y},\quad E_3 := \frac{\partial}{\partial u}\quad \text{and}\quad E_4 := \frac{\partial}{\partial v}$$

form a local frame fields on M. We recall that the *Christoffel symbols* Γ^k_{ij} $(1 \le i,j,k \le 4)$ are defined by

$$\nabla_{E_i} E_j := \sum_{k=1}^{4} \Gamma^k_{ij}\, E_k, \quad 1 \le i,j \le 4. \tag{2.1}$$

According to (1.4), the matrix form $g=(g_{ij})$ of the metric $ds^2_{1,1;A,B}$ is of the form

$$g=(g_{ij})=\begin{pmatrix} \dfrac{Ay+Bv^2}{y^3} & 0 & -\dfrac{Bv}{y^2} & 0 \\ 0 & \dfrac{Ay+Bv^2}{y^3} & 0 & -\dfrac{Bv}{y^2} \\ -\dfrac{Bv}{y^2} & 0 & \dfrac{B}{y} & 0 \\ 0 & -\dfrac{Bv}{y^2} & 0 & \dfrac{B}{y} \end{pmatrix}. \tag{2.2}$$

Then it is easy to see that $\det(g_{ij}) = A^2B^2\,y^{-6}$ and the inverse matrix $g^{-1} := (g^{ij})$ of $g=(g_{ij})$ is given by

$$g^{-1}=\left(g^{ij}\right)=\begin{pmatrix} \dfrac{y^2}{A} & 0 & \dfrac{yv}{A} & 0 \\ 0 & \dfrac{y^2}{A} & 0 & \dfrac{yv}{A} \\ \dfrac{yv}{A} & 0 & \dfrac{Ay+Bv^2}{AB} & 0 \\ 0 & \dfrac{yv}{A} & 0 & \dfrac{Ay+Bv^2}{AB} \end{pmatrix}. \tag{2.3}$$

Lemma 2.1. *For all* $i,j,k, \Gamma^k_{ij} = \Gamma^k_{ji}$. *The Christoffel symbols* Γ^k_{ij}*'s* $(1 \le i,j,k \le 4)$ *are given by*

$$\Gamma^2_{11} = \frac{2A\,y+Bv^2}{2A\,y^2}, \qquad \Gamma^1_{12}=\Gamma^2_{22} = -\frac{2Ay+Bv^2}{2Ay^2}$$

$$\Gamma^4_{11} = \frac{Bv^3}{2Ay^3}, \qquad \Gamma^3_{12}=\Gamma^4_{22} = -\frac{Bv^3}{2Ay^3}$$

$$\Gamma^1_{14}=\Gamma^1_{23}=\Gamma^2_{24}=\Gamma^4_{33} = \frac{Bv}{2Ay}, \qquad \Gamma^2_{13}=\Gamma^3_{34}=\Gamma^4_{44} = -\frac{Bv}{2Ay}$$

$$\Gamma^4_{13} = \frac{Ay - Bv^2}{2Ay^2}, \qquad \Gamma^3_{14} = \Gamma^3_{23} = \Gamma^4_{24} = -\frac{Ay - Bv^2}{2Ay^2}$$

$$\Gamma^2_{33} = \frac{B}{2A}, \qquad \Gamma^2_{44} = \Gamma^1_{34} = -\frac{B}{2A}$$

and all other $\Gamma^k_{ij} = 0$.

Proof. The first statement follows immediately from the symmetry relation (R3). We recall (cf. [2], p. 318 or [8], p. 210) that

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{s=1}^{4} g^{ks}(E_j g_{si} - E_s g_{ij} + E_i g_{js}) \tag{2.4}$$

for all i, j, k. By an easy computation, we get all Γ^k_{ij}. □

We define the functions

$$h_A := \frac{y^{\frac{3}{2}}}{(Ay + Bv^2)^{\frac{1}{2}}}, \qquad h_B := \frac{\sqrt{B}\,yv}{\sqrt{A}\,(Ay + Bv^2)^{\frac{1}{2}}}, \qquad h_C := \frac{(y + v^2)^{\frac{1}{2}}}{\sqrt{AB}}. \tag{2.5}$$

An easy computation gives the following:

Lemma 2.2.

$$\frac{\partial h_A}{\partial y} = \frac{y^{\frac{1}{2}}(2Ay + 3Bv^2)}{2(Ay + Bv^2)^{\frac{3}{2}}}, \qquad \frac{\partial h_B}{\partial y} = \frac{\sqrt{B}\,v(Ay + 2Bv^2)}{2\sqrt{A}\,(Ay + Bv^2)^{\frac{3}{2}}},$$

$$\frac{\partial h_C}{\partial y} = \frac{\sqrt{A}}{2\sqrt{B}\,(Ay + Bv^2)^{\frac{1}{2}}}, \qquad \frac{\partial h_A}{\partial v} = -\frac{B\,y^{\frac{3}{2}}v}{(Ay + Bv^2)^{\frac{3}{2}}},$$

$$\frac{\partial h_B}{\partial v} = \frac{\sqrt{AB}\,y^2}{(Ay + Bv^2)^{\frac{3}{2}}}, \qquad \frac{\partial h_C}{\partial v} = \frac{\sqrt{B}\,v}{\sqrt{A}\,(Ay + Bv^2)^{\frac{1}{2}}}$$

and

$$\frac{\partial h_A}{\partial x} = \frac{\partial h_B}{\partial x} = \frac{\partial h_C}{\partial x} = \frac{\partial h_A}{\partial u} = \frac{\partial h_B}{\partial u} = \frac{\partial h_C}{\partial u} = 0.$$

Lemma 2.3. *The following frame field* F_1, F_2, F_3, F_4 *defined by*

$$F_1 := h_A E_1, \qquad F_2 := h_A E_2$$
$$F_3 := h_B E_1 + h_C E_3, \qquad F_4 := h_B E_2 + h_C E_4$$

form an orthonomal frame field on M. And they satisfy the following relations

$$[F_1, F_2] = -\frac{y^2(2Ay + 3Bv^2)}{2(Ay + Bv^2)^2}\,E_1, \qquad [F_1, F_3] = 0,$$

$$[F_1, F_4] = -\frac{B\sqrt{B}\,y^{\frac{3}{2}}v^3}{2\sqrt{A}\,(Ay + Bv^2)^2}\,E_1,$$

$$[F_2, F_3] = \frac{\sqrt{B}\,y^{\frac{3}{2}}v(Ay + 2Bv^2)}{2\sqrt{A}\,(Ay + Bv^2)^2}\,E_1 + \frac{\sqrt{A}\,y^{\frac{3}{2}}}{2\sqrt{B}\,(Ay + Bv^2)}\,E_3,$$

$$[F_2, F_4] = \frac{\sqrt{B}\, y^{\frac{3}{2}} v}{2\sqrt{A}\,(Ay + Bv^2)}\, E_2 \;+\; \frac{\sqrt{A}\, y^{\frac{3}{2}}}{2\sqrt{B}\,(Ay + Bv^2)}\, E_4$$

and

$$[F_3, F_4] = -\frac{2A^2y^3 + 3ABy^2v^2 + 2B^2yv^4}{2A(Ay + Bv^2)^2}\, E_1 \;-\; \frac{3Ayv + 2Bv^3}{2A(Ay + Bv^2)}\, E_3.$$

Proof. The first statement follows from the Gram-Schmidt orthogonalization process. The proof of the second statement follows from a direct computation. □

Definition 2.1. Let X and Y be two smooth vector fields on M. The curvature operator $R(X,Y) : \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$ is defined as

$$R(X,Y)Z := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z, \quad Z \in \mathcal{X}(M). \tag{2.6}$$

For a quadruple (X,Y,Z,W) of smooth vector fields on M, we define

$$R(X,Y,Z,W) := g(R(X,Y)Z, W). \tag{2.7}$$

The tensor $R(X,Y,Z,W)$ is called the *Riemann curvature tensor* of M.

3. Sectional curvatures

For any point $p \in M$, we let $\pi_{X,Y}$ be the plane section of tangent space $T_p(M)$ of M at p spanned by two orthonormal tangent vectors X and Y in $T_p(M)$. We recall that the *sectional curvature* $K_p(\pi_{X,Y})$ of $\pi_{X,Y}$ is defined by

$$K_p(\pi_{X,Y}) := -R(X,Y,Z,W) = -g(R(X,Y)Z, W), \tag{3.1}$$

where $R(X,Y,Z,W)$ denotes the Riemann curvature tensor of M. In fact, the sectional curvature $K_p(\pi_{X,Y})$ is independent of the choice of two orthonormal basis of the section $\pi_{X,Y}$.

Theorem 3.1. *For any point $p = (x,y,u,v) \in M$, we let π_{ij} the plane section of $T_p(M)$ spanned by two orthonormal vectors F_{ip} and F_{jp} of $T_p(M)$. Then the sectional curvatures $K_p(\pi_{X,Y})$ are given by*

$$K_p(\pi_{12}) = -\frac{1}{A} + \frac{3B^2v^4}{2A\,(Ay + Bv^2)^2}, \qquad K_p(\pi_{13}) = -\frac{1}{4A},$$

$$K_p(\pi_{14}) = -\frac{1}{4A} + \frac{3AByv^2}{2A\,(Ay + Bv^2)^2}, \qquad K_p(\pi_{23}) = -\frac{1}{4A} + \frac{3AByv^2}{2A\,(Ay + Bv^2)^2},$$

$$K_p(\pi_{24}) = -\frac{1}{4A}, \qquad K_p(\pi_{34}) = \frac{1}{2A} - \frac{3Bv^2(2Ay + Bv^2)}{2A\,(Ay + Bv^2)^2}.$$

Proof. We observe that $K_p(\pi_{ij}) = -g(R(F_{ip}, F_{jp})F_{ip}, F_{jp})$ for $1 \le i,j \le 4$. By a direct computation, we obtain

$$\nabla_{E_1}\nabla_{E_2}E_1 = (\,\Gamma_{11}^2\Gamma_{12}^1 + \Gamma_{12}^3\Gamma_{13}^2\,)\,E_2 + (\,\Gamma_{11}^4\Gamma_{12}^1 + \Gamma_{12}^3\Gamma_{13}^4\,)\,E_4,$$

$$\nabla_{E_1}\nabla_{E_2}E_2 = (\,\Gamma_{12}^1\Gamma_{22}^2 + \Gamma_{14}^1\Gamma_{22}^4\,)\,E_1 + (\,\Gamma_{12}^3\Gamma_{22}^2 + \Gamma_{14}^3\Gamma_{22}^4\,)\,E_3,$$

$$\nabla_{E_1}\nabla_{E_2}E_3 = (\Gamma^2_{11}\Gamma^1_{23} + \Gamma^2_{13}\Gamma^2_{23})\,E_2 + (\Gamma^4_{11}\Gamma^1_{23} + \Gamma^4_{13}\Gamma^3_{23})\,E_4,$$
$$\nabla_{E_1}\nabla_{E_3}E_1 = (\Gamma^1_{12}\Gamma^2_{13} + \Gamma^4_{13}\Gamma^1_{14})\,E_1 + (\Gamma^3_{12}\Gamma^2_{13} + \Gamma^4_{13}\Gamma^3_{14})\,E_3,$$
$$\nabla_{E_1}\nabla_{E_4}E_1 = (\Gamma^2_{11}\Gamma^1_{14} + \Gamma^2_{13}\Gamma^3_{14})\,E_2 + (\Gamma^4_{11}\Gamma^1_{14} + \Gamma^4_{13}\Gamma^3_{14})\,E_4,$$
$$\nabla_{E_1}\nabla_{E_4}E_3 = (\Gamma^2_{11}\Gamma^1_{12} + \Gamma^3_{12}\Gamma^2_{13})\,E_2 + (\Gamma^4_{11}\Gamma^1_{12} + \Gamma^3_{12}\Gamma^4_{13})\,E_4,$$
$$\nabla_{E_2}\nabla_{E_1}E_1 = \left(\Gamma^2_{11}\Gamma^2_{22} + \Gamma^4_{11}\Gamma^2_{24} + \tfrac{\partial\Gamma^2_{11}}{\partial y}\right)E_2 + \left(\Gamma^2_{11}\Gamma^4_{22} + \Gamma^4_{11}\Gamma^4_{24} + \tfrac{\partial\Gamma^4_{11}}{\partial y}\right)E_4,$$
$$\nabla_{E_2}\nabla_{E_1}E_2 = \left(\Gamma^1_{12}\Gamma^1_{12} + \Gamma^3_{12}\Gamma^1_{23} + \tfrac{\partial\Gamma^1_{12}}{\partial y}\right)E_1 + \left(\Gamma^1_{12}\Gamma^3_{12} + \Gamma^3_{12}\Gamma^3_{23} + \tfrac{\partial\Gamma^3_{12}}{\partial y}\right)E_3,$$
$$\nabla_{E_2}\nabla_{E_1}E_3 = \left(\Gamma^2_{13}\Gamma^2_{22} + \Gamma^4_{13}\Gamma^2_{24} + \tfrac{\partial\Gamma^2_{13}}{\partial y}\right)E_2 + \left(\Gamma^2_{13}\Gamma^4_{22} + \Gamma^4_{13}\Gamma^4_{24} + \tfrac{\partial\Gamma^4_{13}}{\partial y}\right)E_4,$$
$$\nabla_{E_2}\nabla_{E_3}E_2 = \left(\Gamma^1_{12}\Gamma^1_{23} + \Gamma^1_{32}\Gamma^3_{23} + \tfrac{\partial\Gamma^1_{23}}{\partial y}\right)E_1 + \left(\Gamma^3_{12}\Gamma^1_{23} + \Gamma^3_{23}\Gamma^3_{23} + \tfrac{\partial\Gamma^3_{23}}{\partial y}\right)E_3,$$
$$\nabla_{E_2}\nabla_{E_3}E_3 = \left(\Gamma^2_{22}\Gamma^2_{33} + \Gamma^2_{24}\Gamma^4_{33} + \tfrac{\partial\Gamma^2_{33}}{\partial y}\right)E_2 + \left(\Gamma^4_{22}\Gamma^2_{33} + \Gamma^4_{24}\Gamma^4_{33} + \tfrac{\partial\Gamma^4_{33}}{\partial y}\right)E_4,$$
$$\nabla_{E_2}\nabla_{E_4}E_2 = \left(\Gamma^2_{22}\Gamma^2_{24} + \Gamma^2_{24}\Gamma^4_{24} + \tfrac{\partial\Gamma^2_{24}}{\partial y}\right)E_2 + \left(\Gamma^4_{22}\Gamma^2_{24} + \Gamma^4_{24}\Gamma^4_{24} + \tfrac{\partial\Gamma^4_{24}}{\partial y}\right)E_4,$$
$$\nabla_{E_3}\nabla_{E_1}E_1 = (\Gamma^2_{11}\Gamma^1_{23} + \Gamma^4_{11}\Gamma^1_{34})\,E_1 + (\Gamma^2_{11}\Gamma^3_{23} + \Gamma^4_{11}\Gamma^3_{34})\,E_3,$$
$$\nabla_{E_3}\nabla_{E_2}E_1 = (\Gamma^1_{12}\Gamma^2_{13} + \Gamma^3_{12}\Gamma^2_{33})\,E_2 + (\Gamma^1_{12}\Gamma^4_{13} + \Gamma^3_{12}\Gamma^4_{33})\,E_4,$$
$$\nabla_{E_3}\nabla_{E_2}E_2 = (\Gamma^2_{22}\Gamma^1_{23} + \Gamma^4_{22}\Gamma^1_{34})\,E_1 + (\Gamma^2_{22}\Gamma^3_{23} + \Gamma^4_{22}\Gamma^3_{34})\,E_3,$$
$$\nabla_{E_3}\nabla_{E_2}E_3 = (\Gamma^2_{13}\Gamma^1_{23} + \Gamma^3_{23}\Gamma^2_{33})\,E_2 + (\Gamma^4_{13}\Gamma^1_{23} + \Gamma^3_{23}\Gamma^4_{33})\,E_4,$$
$$\nabla_{E_3}\nabla_{E_4}E_1 = (\Gamma^2_{13}\Gamma^1_{14} + \Gamma^3_{14}\Gamma^2_{33})\,E_2 + (\Gamma^4_{13}\Gamma^1_{14} + \Gamma^3_{14}\Gamma^4_{33})\,E_4,$$
$$\nabla_{E_3}\nabla_{E_4}E_3 = (\Gamma^2_{13}\Gamma^1_{34} + \Gamma^2_{33}\Gamma^3_{34})\,E_2 + (\Gamma^4_{13}\Gamma^1_{34} + \Gamma^4_{33}\Gamma^3_{34})\,E_4,$$
$$\nabla_{E_4}\nabla_{E_1}E_1 = \left(\Gamma^2_{11}\Gamma^2_{24} + \Gamma^4_{11}\Gamma^2_{44} + \tfrac{\partial\Gamma^2_{11}}{\partial v}\right)E_2 + \left(\Gamma^2_{11}\Gamma^4_{24} + \Gamma^4_{11}\Gamma^4_{44} + \tfrac{\partial\Gamma^4_{11}}{\partial v}\right)E_4,$$
$$\nabla_{E_4}\nabla_{E_1}E_3 = \left(\Gamma^2_{13}\Gamma^2_{24} + \Gamma^4_{13}\Gamma^2_{44} + \tfrac{\partial\Gamma^2_{13}}{\partial v}\right)E_2 + \left(\Gamma^2_{13}\Gamma^4_{24} + \Gamma^4_{13}\Gamma^4_{44} + \tfrac{\partial\Gamma^4_{13}}{\partial v}\right)E_4,$$
$$\nabla_{E_4}\nabla_{E_2}E_2 = \left(\Gamma^2_{22}\Gamma^2_{24} + \Gamma^4_{22}\Gamma^2_{44} + \tfrac{\partial\Gamma^2_{22}}{\partial v}\right)E_2 + \left(\Gamma^2_{22}\Gamma^4_{24} + \Gamma^4_{22}\Gamma^4_{44} + \tfrac{\partial\Gamma^4_{22}}{\partial v}\right)E_4,$$
$$\nabla_{E_4}\nabla_{E_3}E_3 = \left(\Gamma^2_{24}\Gamma^2_{33} + \Gamma^4_{33}\Gamma^2_{44} + \tfrac{\partial\Gamma^2_{33}}{\partial v}\right)E_2 + \left(\Gamma^2_{33}\Gamma^4_{24} + \Gamma^4_{33}\Gamma^4_{44} + \tfrac{\partial\Gamma^4_{33}}{\partial v}\right)E_4.$$

Thus according to Lemma 2.2, Lemma 2.3 and the above formulas, we have

$$\begin{aligned} R(F_1,F_2)F_1 = &-h_A\left\{\left(h_A\tfrac{\partial h_A}{\partial y} + \theta_1\right)\Gamma^2_{11} + h_A^2\tfrac{\partial\Gamma^2_{11}}{\partial y}\right\}E_2 \\ &-h_A\left\{\left(h_A\tfrac{\partial h_A}{\partial y} + \theta_1\right)\Gamma^4_{11} + h_A^2\tfrac{\partial\Gamma^4_{11}}{\partial y}\right\}E_4, \end{aligned}$$

$$\begin{aligned} R(F_1,F_3)F_1 = &\, h_A^2 h_C\left\{\left(\Gamma^1_{14}\Gamma^4_{13} - \Gamma^4_{11}\Gamma^1_{34}\right)\right\}E_1 \\ &+ h_A^2 h_C\left\{\left(\Gamma^3_{12}\Gamma^2_{13} + \Gamma^4_{13}\Gamma^3_{14} - \Gamma^2_{11}\Gamma^3_{32} - \Gamma^4_{11}\Gamma^3_{34}\right)\right\}E_3, \end{aligned}$$

$$R(F_1,F_4)F_1 = h_A\Big\{h_Ah_C\left(\Gamma^2_{13}\Gamma^3_{14} - \Gamma^4_{11}\Gamma^2_{44} - \tfrac{\partial\Gamma^2_{11}}{\partial v}\right)$$
$$- h_Ah_B\tfrac{\partial\Gamma^2_{11}}{\partial y} - \left(h_B\tfrac{\partial h_A}{\partial y} + h_C\tfrac{\partial h_A}{\partial v} + \theta_2\right)\Gamma^2_{11}\Big\}E_2$$
$$+ h_A\Big\{h_Ah_C\left(\Gamma^4_{13}\Gamma^3_{14} + \Gamma^4_{11}\Gamma^1_{14} - \Gamma^2_{11}\Gamma^2_{24} - \Gamma^4_{11}\Gamma^4_{44} - \tfrac{\partial\Gamma^4_{11}}{\partial v}\right)$$
$$- h_Ah_B\tfrac{\partial\Gamma^4_{11}}{\partial y} - \left(h_B\tfrac{\partial h_A}{\partial y} + h_C\tfrac{\partial h_A}{\partial v} + \theta_2\right)\Gamma^4_{11}\Big\}E_4,$$

$$R(F_2,F_3)F_2 = h_A\Big\{h_Ah_C\left(\Gamma^1_{23}\Gamma^3_{23} - \Gamma^4_{22}\Gamma^1_{34} + \tfrac{\partial\Gamma^1_{23}}{\partial y}\right)$$
$$+ \left(h_Ah_B\tfrac{\partial\Gamma^1_{12}}{\partial y} + h_A\tfrac{\partial h_B}{\partial y}\Gamma^1_{12} + h_A\tfrac{\partial h_C}{\partial y}\Gamma^1_{23} - \theta_3\Gamma^1_{12} - \theta_4\Gamma^1_{23}\right)\Big\}E_1$$
$$+ h_A\Big\{h_Ah_C\left(\Gamma^3_{12}\Gamma^1_{23} + \Gamma^3_{23}\Gamma^3_{23} - \Gamma^2_{22}\Gamma^3_{23} - \Gamma^4_{22}\Gamma^3_{34} + \tfrac{\partial\Gamma^3_{23}}{\partial y}\right)$$
$$+ \left(h_Ah_B\tfrac{\partial\Gamma^3_{12}}{\partial y} + h_A\tfrac{\partial h_B}{\partial y}\Gamma^3_{12} + h_A\tfrac{\partial h_C}{\partial y}\Gamma^3_{23} - \theta_3\Gamma^3_{12} - \theta_4\Gamma^3_{23}\right)\Big\}E_3,$$

$$R(F_2,F_4)F_2 = \Big\{h_A^2\,h_C\left(\Gamma^2_{24}\Gamma^4_{24} - \Gamma^4_{22}\Gamma^2_{44} + \tfrac{\partial\Gamma^2_{24}}{\partial y} - \tfrac{\partial\Gamma^2_{22}}{\partial v}\right)$$
$$+ h_A\left(h_A\tfrac{\partial h_B}{\partial y} - h_B\tfrac{\partial h_A}{\partial y} - h_C\tfrac{\partial h_A}{\partial v}\right)\Gamma^2_{22} + h_A^2\tfrac{\partial h_C}{\partial y}\Gamma^2_{24}$$
$$+ h_A\tfrac{\partial h_A}{\partial y}\tfrac{\partial h_B}{\partial y} + h_A\tfrac{\partial h_C}{\partial y}\tfrac{\partial h_A}{\partial v} - h_B\left(\tfrac{\partial h_A}{\partial y}\right)^2 - h_C\tfrac{\partial h_A}{\partial y}\tfrac{\partial h_A}{\partial v}$$
$$- h_A\,\theta_5\,\Gamma^2_{22} - h_A\,\theta_4\,\Gamma^2_{24} - \theta_5\tfrac{\partial h_A}{\partial y} - \theta_4\tfrac{\partial h_A}{\partial v}\Big\}E_2$$
$$+ \Big\{h_A^2\,h_C\left(\Gamma^4_{22}\Gamma^2_{24} + \Gamma^4_{24}\Gamma^4_{24} + \tfrac{\partial\Gamma^4_{24}}{\partial y} - \Gamma^2_{22}\Gamma^4_{24} - \Gamma^4_{22}\Gamma^4_{44}\right.$$
$$\left.- \tfrac{\partial\Gamma^4_{22}}{\partial v}\right) + h_A\left(h_A\tfrac{\partial h_B}{\partial y} - h_B\tfrac{\partial h_A}{\partial y} - h_C\tfrac{\partial h_A}{\partial v}\right)\Gamma^4_{22}$$
$$+ h_A^2\tfrac{\partial h_C}{\partial y}\Gamma^4_{24} - h_A\,\theta_5\,\Gamma^4_{22} - h_A\,\theta_4\,\Gamma^4_{24}\Big\}E_4,$$

$$R(F_3,F_4)F_3 = -\Big\{h_B^2\left(h_B\tfrac{\partial\Gamma^2_{13}}{\partial y} + h_C\tfrac{\partial\Gamma^2_{11}}{\partial v} + 2h_C\tfrac{\partial\Gamma^2_{13}}{\partial y}\right)$$

$$+ h_C^2 \left(h_C \frac{\partial \Gamma_{33}^2}{\partial v} + h_B \frac{\partial \Gamma_{33}^2}{\partial v} + 2h_B \frac{\partial \Gamma_{13}^2}{\partial v} \right) \Bigg\} E_2$$

$$- \Bigg\{ h_B^2 \left(h_B \frac{\partial \Gamma_{11}^4}{\partial y} + h_C \frac{\partial \Gamma_{11}^4}{\partial v} + 2h_C \frac{\partial \Gamma_{13}^4}{\partial y} \right)$$

$$+ h_C^2 \left(h_C \frac{\partial \Gamma_{33}^4}{\partial v} + h_B \frac{\partial \Gamma_{33}^4}{\partial v} + 2h_B \frac{\partial \Gamma_{13}^4}{\partial v} \right) \Bigg\} E_4,$$

where we put

$$\theta_1 := -\frac{y^2\,(2Ay+3Bv^2)}{2\,(Ay+Bv^2)^2},\ \theta_2 := -\frac{B\sqrt{B}\,y^{\frac{3}{2}}v^3}{2\sqrt{A}\,(Ay+Bv^2)^2},\ \theta_3 := \frac{\sqrt{B}\,y^{\frac{3}{2}}v(Ay+2Bv^2)}{2\sqrt{A}\,(Ay+Bv^2)^2}$$

and

$$\theta_4 := \frac{\sqrt{A}\,y^{\frac{3}{2}}}{2\sqrt{B}\,(Ay+Bv^2)}, \qquad \theta_5 := \frac{\sqrt{B}\,y^{\frac{3}{2}}v}{2\sqrt{A}\,(Ay+Bv^2)}.$$

Using (2.2), (2.5), (2.7), Lemma 2.1, Lemma 2.2 and the above formulas, we obtain the above sectional curvatures $K_p(\pi_{ij})$ for $1 \le i \le j \le 4$. □

Theorem 3.2. *The scalar curvature* $r(p)$ *of the Siegel-Jacobi space*

$$(M, ds^2_{1,1;A,B})$$

is

$$r(p) = -\frac{3}{A} \quad \textit{for all } p \in M.$$

Proof. We recall that the scalar curvature $r(p)$ of M is defined as

$$r(p) := \sum_{i,j=1}^{4} R\,(F_{ip}, F_{jp}, F_{jp}, F_{ip})\,, \quad p \in M.$$

We note that the scalar curvature $r(p)$ is independent of the choice of an orthonormal basis of $T_p(M)$. Since the following symmetry relations

$$R(X,Y)Z + R(Y,X)Z = 0$$

hold for all $X, Y, Z \in \mathcal{X}(M)$, we have

$$\begin{aligned} r(p) = \ & -2\Big\{ R\,(F_{1p}, F_{2p}, F_{1p}, F_{2p}) + R\,(F_{1p}, F_{3p}, F_{1p}, F_{3p}) \\ & + R\,(F_{1p}, F_{4p}, F_{1p}, F_{4p}) + R\,(F_{2p}, F_{3p}, F_{2p}, F_{3p}) \\ & + R\,(F_{2p}, F_{4p}, F_{2p}, F_{4p}) + R\,(F_{3p}, F_{4p}, F_{3p}, F_{4p}) \Big\}. \end{aligned}$$

Accoding to Theorem 3.1, we obtain

$$r(p) = -\frac{3}{A}.$$

This completes the proof of the above theorem. □

Remark 3.1. The Poincaré upper half plane $\mathbb{H}_1$ is a two dimensional Riemannian manifold with the Poincaré metric

$$ds_0^2 := \frac{dx^2 + dy^2}{y^2}, \quad z = x + i\,y \in \mathbb{H}_1 \text{ with } x, y \text{ real.}$$

It is easily seen that the Gaussian curvature of $(\mathbb{H}_1,\, ds_0^2)$ is -1 everywhere and $(\mathbb{H}_1,\, ds_0^2)$ is an Einstein manifold. Indeed, if we denote by $S_0(X,Y)$ the Ricci curvature of $(\mathbb{H}_1,\, ds_0^2)$, then we have

$$S_0(X,Y) = -g_0(X,Y) \qquad \text{for all } X, Y \in \mathcal{X}(\mathbb{H}_1),$$

where $g_0(X,Y)$ is the inner product on the tangent bundle $T(\mathbb{H}_1)$ induced by the Poincaré ds_0^2. But the Siegel-Jacobi space $(\mathbb{H}_1 \times \mathbb{C},\, ds^2_{1,1;1,1})$ is not an Einstein manifold. In fact, if we denote by $S(X,Y)$ the Ricci curvature of the Siegel-Jacobi space $(\mathbb{H}_1 \times \mathbb{C},\, ds^2_{1,1;1,1})$, we can see without difficulty that there does not exist a constant c such that

$$S(E_1, E_1) = c\,g(E_1, E_1) = c\,g_{11}.$$

4. Final remarks

Let $\mathbb{D} = \{\zeta \in \mathbb{C} \,|\, |\zeta| < 1\}$ be the unit disk in the complex plane. We let

$$G_*^J := \left\{ \left(\begin{pmatrix} p & q \\ \overline{q} & \overline{p} \end{pmatrix}, (\xi, \overline{\xi}; i\,\kappa) \right) \;\middle|\; \begin{pmatrix} p & q \\ \overline{q} & \overline{p} \end{pmatrix} \in SU(1,1),\ \xi \in \mathbb{C},\ \kappa \in \mathbb{R} \right\}$$

be the Jacobi group equipped with the multiplication law

$$\begin{aligned} &\left(\begin{pmatrix} p & q \\ \overline{q} & \overline{p} \end{pmatrix}, (\xi, \overline{\xi}; i\,\kappa) \right) \cdot \left(\begin{pmatrix} p' & q' \\ \overline{q'} & \overline{p'} \end{pmatrix}, (\xi', \overline{\xi'}; i\,\kappa') \right) \\ = &\left(\begin{pmatrix} p & q \\ \overline{q} & \overline{p} \end{pmatrix} \begin{pmatrix} p' & q' \\ \overline{q'} & \overline{p'} \end{pmatrix},\ (\tilde{\xi} + \xi', \tilde{\theta} + \overline{\xi'};\ i\,\kappa + i\,\kappa' + \tilde{\xi}\,{}^t\overline{\xi'} - \tilde{\theta}\,{}^t\xi') \right), \end{aligned}$$

where $\tilde{\xi} = p'\xi + \overline{q}\,\overline{\xi}$ and $\tilde{\theta} = q'\xi + \overline{p'}\,\overline{\xi}$. Then G_*^J acts on the Siegel-Jacobi disk $\mathbb{D} \times \mathbb{C}$ transitively by

$$\left(\begin{pmatrix} p & q \\ \overline{q} & \overline{p} \end{pmatrix}, (\xi, \overline{\xi}; i\,\kappa) \right) \cdot (\zeta, \eta) = \left(\frac{p\,\zeta + q}{\overline{q}\,\zeta + \overline{p}},\ \frac{\eta + \xi\,\zeta + \overline{\xi}}{\overline{q}\,\zeta + \overline{p}} \right), \tag{4.1}$$

where $\zeta \in \mathbb{D}$ and $\eta \in \mathbb{C}$. According to (1.2), we see that $G_{1,1}^J$ acts on $\mathbb{H}_1 \times \mathbb{C}$ transitively by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu; \kappa) \right) \cdot (z, w) = \left(\frac{az + b}{cz + d},\ \frac{w + \lambda\,z + \mu}{cz + d} \right), \tag{4.2}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R}),\ \lambda, \mu, \kappa \in \mathbb{R},\ z \in \mathbb{H}_1$ and $w \in \mathbb{C}$.

In [15], the author proved that the action (4.1) of G_*^J on the Siegel-Jacobi disk $\mathbb{D} \times \mathbb{C}$ is compatible with the action (4.2) of G_*^J on the Siegel-Jacobi space

$\mathbb{H}_1 \times \mathbb{C}$ via the partial Cayley tranform $\Phi_* : \mathbb{D} \times \mathbb{C} \longrightarrow \mathbb{H}_1 \times \mathbb{C}$ defined by

$$\Phi_*(\zeta,\eta) := \left(\frac{i\,(1+\zeta)}{1-\zeta}, \frac{2\,i\,\eta}{1-\zeta} \right), \qquad (\zeta,\eta) \in \mathbb{D} \times \mathbb{C}. \tag{4.3}$$

Precisely, if $g = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (\lambda,\mu;\kappa) \right) \in G^J_{1,1}$, we put

$$g_* = \left(\begin{pmatrix} p & q \\ \overline{q} & \overline{p} \end{pmatrix}, \left(\frac{1}{2}\,(\lambda + i\,\mu), \frac{1}{2}\,(\lambda - i\,\mu)\,; -i\,\frac{\kappa}{2} \right) \right), \tag{4.4}$$

where

$$p = \frac{1}{2}\,\{\,(a+d) + i\,(b-c)\}$$

and

$$q = \frac{1}{2}\,\{\,(a-d) - i\,(b+c)\}\,.$$

We note that g_* is an element of G^J_*. The compatibility condition means that the following condition

$$g\cdot\Phi_*(\zeta,\eta) = \Phi_*\big(g_*\cdot(\zeta,\eta)\big) \qquad \text{for all } g \in G^J_{1,1} \text{ and } (\zeta,\eta) \in \mathbb{D} \times \mathbb{C} \tag{4.5}$$

holds. Using the compatibility condition (4.5), the author [16] proved that for any two positive real numbers A and B,

$$\begin{aligned} d\tilde{s}^2_{1,1;A,B} = {} & 4\,A\,\frac{d\zeta\,d\overline{\zeta}}{\left(1-|\zeta|^2\right)^2} \\ & + 4\,B \left\{ \frac{d\eta\,d\overline{\eta}}{1-|\zeta|^2} + \frac{(1+|\zeta|^2)\,|\eta|^2 - \overline{\zeta}\,\eta^2 - \zeta\,\overline{\eta}^2}{\left(1-|\zeta|^2\right)^3}\,d\zeta\,d\overline{\zeta} \right. \\ & \left. \qquad + \frac{\eta\,\overline{\zeta} - \overline{\eta}}{\left(1-|\zeta|^2\right)^2}\,d\zeta\,d\overline{\eta} + \frac{\overline{\eta}\,\zeta - \eta}{\left(1-|\zeta|^2\right)^2}\,d\overline{\zeta}\,d\eta \right\} \end{aligned}$$

is a Riemannian metric on the Siegel-Jacobi disk $\mathbb{D} \times \mathbb{C}$ which is invariant under the action (4.1) of G^J_* on $\mathbb{D} \times \mathbb{C}$. According to Theorem 1.4 in [16], we see that the Laplace-Beltrami operator $\widetilde{\Delta}_{1,1;A,B}$ of the Siegel-Jacobi disk $\big(\mathbb{D} \times \mathbb{C},\, d\tilde{s}^2_{1,1;A,B}\big)$ is given by

$$\begin{aligned} \widetilde{\Delta}_{1,1;A,B} = {} & \frac{1}{A} \left\{ \left(1-|\zeta|^2\right)^2 \frac{\partial^2}{\partial\zeta\,\partial\overline{\zeta}} + \left(1-|\zeta|^2\right)(\eta - \overline{\eta}\,\zeta)\,\frac{\partial^2}{\partial\zeta\,\partial\overline{\eta}} \right. \\ & \quad + \left(1-|\zeta|^2\right)(\overline{\eta} - \eta\,\overline{\zeta})\,\frac{\partial^2}{\partial\overline{\zeta}\,\partial\eta} \\ & \quad \left. + \left(|\eta|^2 + |\zeta\,\eta|^2 - \overline{\zeta}\,\eta^2 - \zeta\,\overline{\eta}^2\right) \frac{\partial^2}{\partial\eta\,\partial\overline{\eta}} \right\} \\ & + \frac{1}{B}\left(1-|\zeta|^2\right)\frac{\partial^2}{\partial\eta\,\partial\overline{\eta}}\,. \end{aligned}$$

Theorem 4.1. *The scalar curvature of the Siegel-Jacobi disk* $\big(\mathbb{D}\times\mathbb{C},\ d\tilde{s}^2_{1,1;A,B}\big)$ *is*

$$r(q) = -\frac{3}{A} \qquad \text{for all } q \in \mathbb{D}\times\mathbb{C}.$$

Proof. The proof follows from Theorem 3.2 and the compatibility condition (4.5). □

References

[1] R. Berndt and R. Schmidt, *Elements of the Representation Theory of the Jacobi Group*, Progress in Mathematics, **163**, Birkhäuser, Basel, 1998.

[2] W. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry*, Academic Press, 1975.

[3] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progress in Mathematics, **55**, Birkhäuser, Boston, Basel and Stuttgart, 1985.

[4] E. Kähler, *Mathematische Werke/Mathematical Works*, Edited by R. Berndt and O. Riemenschneider, Walter de Gruyter, Berlin-New York, 2003.

[5] J. Kramer, *A geometrical approach to the theory of Jacobi forms*, Compositio Math. **79** (1991), no. 1, 1–19.

[6] ______, *An arithmetic theory of Jacobi forms in higher dimensions*, J. Reine Angew. Math. **458** (1995), 157–182.

[7] B. Runge, *Theta functions and Siegel-Jacobi forms*, Acta Math. **175** (1995), no. 2, 165–196.

[8] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Publish or Perish, Inc., Berkeley, Vol. II, 1979.

[9] J.-H. Yang, *The Siegel-Jacobi operator*, Abh. Math. Sem. Univ. Hamburg **63** (1993), 135–146.

[10] ______, *Singular Jacobi forms*, Trans. Amer. Math. Soc. **347** (1995), no. 6, 2041–2049.

[11] ______, *Construction of vector valued modular forms from Jacobi forms*, Canadian J. Math. **47** (1995), no. 6, 1329–1339.

[12] ______, *The method of orbits for real Lie groups*, Kyungpook Math. J. **42** (2002), no. 2, 199–272.

[13] ______, *A note on a fundamental domain for Siegel-Jacobi space*, Houston J. Math. **32** (2006), no. 3, 701–712.

[14] ______, *Invariant metrics and Laplacians on Siegel-Jacobi space*, J. Number Theory **127** (2007), no. 1, 83–102.

[15] ______, *A partial Cayley transform for Siegel-Jacobi disk*, J. Korean Math. Soc. **45** (2008), no. 3, 781–794.

[16] ______, *Invariant metrics and Laplacians on Siegel-Jacobi disk*, Chinese Ann. Math. Ser. B **31** (2010), no. 1, 85–100.

[17] ______, *Invariant differential operators on Siegel-Jacobi space*, arXiv:1107.0509 v1 [math.NT] 4 July 2011.

[18] C. Ziegler, *Jacobi forms of higher degree*, Abh. Math. Sem. Univ. Hamburg **59** (1989), 191–224.

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Invariant differential operators on Siegel-Jacobi space and Maass-Jacobi forms

Jae-Hyun Yang

1 Introduction

For a given fixed positive integer n, we let

$$\mathbb{H}_n = \{\, \Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega, \quad \operatorname{Im} \Omega > 0 \,\}$$

be the Siegel upper half plane of degree n and let

$$Sp(n, \mathbb{R}) = \{ M \in \mathbb{R}^{(2n,2n)} \mid {}^tM J_n M = J_n \, \}$$

be the symplectic group of degree n, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l, ${}^tM$ denotes the transpose of a matrix M and

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

$Sp(n, \mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

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$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{1.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $\Omega \in \mathbb{H}_n$.

For two positive integers m and n, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \left\{ (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\, {}^t\lambda \text{ symmetric } \right\}$$

endowed with the following multiplication law

$$\big(\lambda, \mu; \kappa\big) \circ \big(\lambda', \mu'; \kappa'\big) = \big(\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda'\big)$$

with $\big(\lambda, \mu; \kappa\big), \big(\lambda', \mu'; \kappa'\big) \in H_{\mathbb{R}}^{(n,m)}$. We define the *Jacobi group* of degree n and index m that is the semidirect product of $Sp(n, \mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$

$$G^J = Sp(n, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

endowed with the following multiplication law

$$\big(M, (\lambda, \mu; \kappa)\big) \cdot \big(M', (\lambda', \mu'; \kappa'\,)\big) = \big(MM', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu'; \kappa + \kappa' + \tilde{\lambda}\, {}^t\mu' - \tilde{\mu}\, {}^t\lambda'\,)\big)$$

with $M, M' \in Sp(n, \mathbb{R}), (\lambda, \mu; \kappa),\ (\lambda', \mu'; \kappa') \in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)M'$. Then G^J acts on $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ transitively by

$$\big(M, (\lambda, \mu; \kappa)\big) \cdot (\Omega, Z) = \Big(M \cdot \Omega, (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1}\Big), \tag{1.2}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}),\ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega, Z) \in \mathbb{H}_n \times \mathbb{C}^{(m,n)}$. We note that the Jacobi group G^J is *not* a reductive Lie group and the homogeneous space $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ is not a symmetric space. We refer to [2, 7, 22, 23, 24, 25, 27, 28, 29, 30, 31] about automorphic forms on G^J and topics related to the content of this paper. From now on, for brevity we write $\mathbb{H}_{n,m} = \mathbb{H}_n \times \mathbb{C}^{(m,n)}$. The homogeneous space $\mathbb{H}_{n,m}$ is called the *Siegel-Jacobi space* of degree n and index m.

The aim of this survey paper is to present results on differential operators on $\mathbb{H}_{n,m}$ which are invariant under the *natural* action (1.2) of G^J. The study of these invariant differential operators on the Siegel-Jacobi space $\mathbb{H}_{n,m}$ is interesting and important in the aspects of invariant theory, arithmetic and geometry. This article is organized as follows. In Section 2, we review differential operators on $\mathbb{H}_n$ invariant under the action (1.1) of $Sp(n, \mathbb{R})$. In Section 3, we discuss differential operators on $\mathbb{H}_{n,m}$ invariant under the action (1.2) of G^J and propose some natural problems related to invariant differential operators on the Siegel-Jacobi space. We present some results without proofs. In Section 4, we gives some examples of explicit G^J-invariant differential operators on $\mathbb{H}_{n,m}$. In Section 5, we introduce the partial Cayley transform of the Siegel-Jacobi space into the Siegel-Jacobi disk and present some explicit invariant differential operators on the Siegel-Jacobi disk. In Section 6, we present some results in the special case $n = m = 1$ in detail. We give complete solutions of the problems that are proposed in Section 3. In the final section, using these invariant differential operators on the Siegel-Jacobi space, we discuss a notion of Maass-Jacobi forms.

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Notations: We denote by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers respectively. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\mathrm{tr}(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose of a matrix M. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. For a positive integer n, I_n denotes the identity matrix of degree n. For a complex number z, $|z|$ denotes the absolute value of z. For a complex number z, $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ denote the real part of z and the imaginary part of z respectively.

2 Invariant Differential Operators on Siegel Space

For a coordinate $\Omega = (\omega_{ij}) \in \mathbb{H}_n$, we write $\Omega = X + iY$ with $X = (x_{ij})$, $Y = (y_{ij})$ real. We put $d\Omega = \big(d\omega_{ij}\big)$ and $d\overline{\Omega} = \big(d\overline{\omega}_{ij}\big)$. We also put

$$\frac{\partial}{\partial \Omega} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial \omega_{ij}} \right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{\Omega}} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial \overline{\omega}_{ij}} \right).$$

Then for a positive real number A,

$$ds^2_{n;A} = A\, \mathrm{tr}\Big(Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega}\Big) \tag{2.1}$$

is a $Sp(n,\mathbb{R})$-invariant Kähler metric on $\mathbb{H}_n$ (cf. [19, 20]), where $\mathrm{tr}(M)$ denotes the trace of a square matrix M. H. Maass [14] proved that the Laplacian of $ds^2_{n;A}$ is given by

$$\Delta_{n;A} = \frac{4}{A}\,\mathrm{tr}\left(Y\, {}^t\!\left(Y \frac{\partial}{\partial \overline{\Omega}} \right) \frac{\partial}{\partial \Omega} \right). \tag{2.2}$$

And

$$dv_n(\Omega) = (\det Y)^{-(n+1)} \prod_{1\le i\le j\le n} dx_{ij} \prod_{1\le i\le j\le n} dy_{ij}$$

is a $Sp(n,\mathbb{R})$-invariant volume element on $\mathbb{H}_n$ (cf. [20, p. 130]).

For brevity, we write $G = Sp(n,\mathbb{R})$. The isotropy subgroup K at iI_n for the action (1.1) is a maximal compact subgroup of G given by

$$K = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \;\Big|\; A\,{}^tA + B\,{}^tB = I_n,\ A\,{}^tB = B\,{}^tA,\ A, B \in \mathbb{R}^{(n,n)} \right\}.$$

Let $\mathfrak{k}$ be the Lie algebra of K. Then the Lie algebra $\mathfrak{g}$ of G has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{g} = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & -{}^tX_1 \end{pmatrix} \;\Big|\; X_1, X_2, X_3 \in \mathbb{R}^{(n,n)},\ X_2 = {}^tX_2,\ X_3 = {}^tX_3 \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \;\Big|\; {}^tX + X = 0,\ Y = {}^tY \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \;\Big|\; X = {}^tX,\ Y = {}^tY,\ X, Y \in \mathbb{R}^{(n,n)} \right\}.$$

The subspace $\mathfrak{p}$ of $\mathfrak{g}$ may be regarded as the tangent space of $\mathbb{H}_n$ at iI_n. The adjoint representation of G on $\mathfrak{g}$ induces the action of K on $\mathfrak{p}$ given by

$$k \cdot Z = kZ\,{}^tk, \quad k \in K,\ Z \in \mathfrak{p}. \tag{2.3}$$

Let T_n be the vector space of $n \times n$ symmetric complex matrices. We let $\Psi : \mathfrak{p} \longrightarrow T_n$ be the map defined by

$$\Psi\left(\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \right) = X + i\,Y, \quad \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \in \mathfrak{p}. \tag{2.4}$$

We let $\delta : K \longrightarrow U(n)$ be the isomorphism defined by

$$\delta\left(\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \right) = A + i\,B, \quad \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K, \tag{2.5}$$

where $U(n)$ denotes the unitary group of degree n. We identify $\mathfrak{p}$ (resp. K) with T_n (resp. $U(n)$) through the map Ψ (resp. δ). We consider the action of $U(n)$ on T_n defined by

$$h \cdot \omega = h\omega\,{}^th, \quad h \in U(n),\ \omega \in T_n. \tag{2.6}$$

Then the adjoint action (2.3) of K on $\mathfrak{p}$ is compatible with the action (2.6) of $U(n)$ on T_n through the map Ψ. Precisely for any $k \in K$ and $Z \in \mathfrak{p}$, we get

$$\Psi(k\,Z\,{}^tk) = \delta(k)\,\Psi(Z)\,{}^t\delta(k). \tag{2.7}$$

The action (2.6) induces the action of $U(n)$ on the polynomial algebra $\mathrm{Pol}(T_n)$ consisting of polynomials on T_n. We denote by $\mathrm{Pol}(T_n)^{U(n)}$ the subalgebra of $\mathrm{Pol}(T_n)$ consisting of polynomials invariant under the action of $U(n)$. Then we have the so-called Helgason map

$$\Theta_n : \mathrm{Pol}(T_n)^{U(n)} \longrightarrow \mathbb{D}(\mathbb{H}_n) \tag{2.8}$$

of $\mathrm{Pol}(T_n)^{U(n)}$ onto the algebra $\mathbb{D}(\mathbb{H}_n)$ of differential operators on $\mathbb{H}_n$ invariant under the action (1.1) of G. The map Θ_n is a canonical linear bijection but is not an algebra isomorphism. The map Θ_n is described explicitly as follows. We put $N = n(n+1)$. Let $\{\xi_\alpha \mid 1 \leq \alpha \leq N\}$ be a basis of a real vector space $\mathfrak{p}$. If $P \in \mathrm{Pol}(\mathfrak{p})^K$, then

$$\Big(\Theta_n(P)f\Big)(gK) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g\exp\left(\sum_{\alpha=1}^N t_\alpha \xi_\alpha\right)K\right)\right]_{(t_\alpha)=0}, \tag{2.9}$$

where $f \in C^\infty(\mathbb{H}_n)$. We refer to [10, 11] for more detail. In general, it is hard to express $\Phi(P)$ explicitly for a polynomial $P \in \mathrm{Pol}(\mathfrak{p})^K$.

According to the work of Harish-Chandra [8, 9], the algebra $\mathbb{D}(\mathbb{H}_n)$ is generated by n algebraically independent generators and is isomorphic to the commutative algebra $\mathbb{C}[x_1, \cdots, x_n]$ with n indeterminates. We note that n is the real rank of G. Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. It is known that $\mathbb{D}(\mathbb{H}_n)$ is isomorphic to the center of the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$.

Using a classical invariant theory (cf. [12, 21], we can show that $\mathrm{Pol}(T_n)^{U(n)}$ is generated by the following algebraically independent polynomials

$$q_j(\omega) = \mathrm{tr}\Big(\big(\omega\overline{\omega}\big)^j\Big), \quad \omega \in T_n, \quad j = 1, 2, \cdots, n. \tag{2.10}$$

For each j with $1 \leq j \leq n$, the image $\Theta_n(q_j)$ of q_j is an invariant differential operator on $\mathbb{H}_n$ of degree $2j$. The algebra $\mathbb{D}(\mathbb{H}_n)$ is generated by n algebraically independent generators $\Theta_n(q_1), \Theta_n(q_2), \cdots, \Theta_n(q_n)$. In particular,

$$\Theta_n(q_1) = c_1\,\mathrm{tr}\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right) \quad \text{for some constant } c_1. \tag{2.11}$$

We observe that if we take $\omega = x + i\,y \in T_n$ with real x, y, then $q_1(\omega) = q_1(x, y) = \mathrm{tr}\big(x^2 + y^2\big)$ and

$$q_2(\omega) = q_2(x,y) = \operatorname{tr}\Big(\big(x^2+y^2\big)^2 + 2\,x\big(xy - yx\big)y \Big).$$

It is a natural question to express the images $\Theta_n(q_j)$ explicitly for $j = 2, 3, \cdots, n$. We hope that the images $\Theta_n(q_j)$ for $j = 2, 3, \cdots, n$ are expressed in the form of the *trace*.

H. Maass [15] found explicit algebraically independent generators of $\mathbb{D}(\mathbb{H}_n)$. G. Shimura [18] found canonically defined algebraically independent generators of $\mathbb{D}(\mathbb{H}_n)$.

Example 2.1. We consider the case $n = 1$. The algebra $\mathrm{Pol}(T_1)^{U(1)}$ is generated by the polynomial

$$q(\omega) = \omega\,\overline{\omega}, \quad \omega = x + iy \in \mathbb{C} \text{ with } x, y \text{ real}.$$

Using Formula (2.9), we get

$$\Theta_1(q) = 4\,y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Therefore $\mathbb{D}(\mathbb{H}_1) = \mathbb{C}\big[\Theta_1(q)\big]$.

Example 2.2. We consider the case $n = 2$. The algebra $\mathrm{Pol}(T_2)^{U(2)}$ is generated by the polynomial

$$q_1(\omega) = \operatorname{tr}\big(\omega\,\overline{\omega}\big), \quad q_2(\omega) = \operatorname{tr}\Big(\big(\omega\,\overline{\omega}\big)^2\Big), \quad \omega \in T_2.$$

Using Formula (2.9), we may express $\Theta_2(q_1)$ and $\Theta_2(q_2)$ explicitly. $\Theta_2(q_1)$ is expressed by Formula (2.11). The computation of $\Theta_2(q_2)$ might be quite tedious. We leave the detail to the reader. In this case, $\Theta_2(q_2)$ was essentially computed in [5], Proposition 6. Therefore

$$\mathbb{D}(\mathbb{H}_2) = \mathbb{C}\big[\Theta_2(q_1), \Theta_2(q_2)\big].$$

In fact, the center of the universal enveloping algebra $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$ was computed in [5].

3 Invariant Differential Operators on Siegel-Jacobi Space

The stabilizer K^J of G^J at $(iI_n, 0)$ is given by

$$K^J = \Big\{ \big(k, (0,0;\kappa)\big) \,\big|\, k \in K,\ \kappa = {}^t\kappa \in \mathbb{R}^{(m,m)} \Big\}.$$

Therefore $\mathbb{H}_{n,m} \cong G^J/K^J$ is a homogeneous space of *non-reductive type*. The Lie algebra $\mathfrak{g}^J$ of G^J has a decomposition

$$\mathfrak{g}^J = \mathfrak{k}^J + \mathfrak{p}^J,$$

where

$$\mathfrak{g}^J = \Big\{ \big(Z, (P,Q,R)\big) \,\big|\, Z \in \mathfrak{g},\ P,Q \in \mathbb{R}^{(m,n)},\ R = {}^tR \in \mathbb{R}^{(m,m)} \Big\},$$

$$\mathfrak{k}^J = \Big\{ \big(X, (0,0,R)\big) \,\big|\, X \in \mathfrak{k},\ R = {}^tR \in \mathbb{R}^{(m,m)} \Big\},$$

$$\mathfrak{p}^J = \Big\{ \big(Y, (P,Q,0)\big) \,\big|\, Y \in \mathfrak{p},\ P,Q \in \mathbb{R}^{(m,n)} \Big\}.$$

Thus the tangent space of the homogeneous space $\mathbb{H}_{n,m}$ at $(iI_n, 0)$ is identified with $\mathfrak{p}^J$.

If $\alpha = \left(\begin{pmatrix} X_1 & Y_1 \\ Z_1 & -{}^tX_1 \end{pmatrix}, (P_1, Q_1, R_1) \right)$ and $\beta = \left(\begin{pmatrix} X_2 & Y_2 \\ Z_2 & -{}^tX_2 \end{pmatrix}, (P_2, Q_2, R_2) \right)$ are elements of $\mathfrak{g}^J$, then the Lie bracket $[\alpha, \beta]$ of α and β is given by

$$[\alpha,\beta] = \left(\begin{pmatrix} X^* & Y^* \\ Z^* & -{}^tX^* \end{pmatrix}, (P^*, Q^*, R^*) \right), \tag{3.1}$$

where

$$\begin{aligned}
X^* &= X_1X_2 - X_2X_1 + Y_1Z_2 - Y_2Z_1,\\
Y^* &= X_1Y_2 - X_2Y_1 + Y_2\,{}^tX_1 - Y_1\,{}^tX_2,\\
Z^* &= Z_1X_2 - Z_2X_1 + {}^tX_2Z_1 - {}^tX_1Z_2,\\
P^* &= P_1X_2 - P_2X_1 + Q_1Z_2 - Q_2Z_1,\\
Q^* &= P_1Y_2 - P_2Y_1 + Q_2\,{}^tX_1 - Q_1\,{}^tX_2,\\
R^* &= P_1\,{}^tQ_2 - P_2\,{}^tQ_1 + Q_2\,{}^tP_1 - Q_1\,{}^tP_2.
\end{aligned}$$

We recall that T_n denotes the vector space of all $n \times n$ symmetric complex matrices. For brevity, we put $T_{n,m} := T_n \times \mathbb{C}^{(m,n)}$. We define the real linear map $\Phi : \mathfrak{p}^J \longrightarrow T_{n,m}$ by

$$\Phi\left(\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}, (P,Q,0) \right) = \big(X + i\,Y,\ P + i\,Q\big), \tag{3.2}$$

where $\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \in \mathfrak{p}$ and $P,Q \in \mathbb{R}^{(m,n)}$.

Let $S(m,\mathbb{R})$ denote the additive group consisting of all $m\times m$ real symmetric matrices. Now we define the isomorphism $\theta : K^J \longrightarrow U(n)\times S(m,\mathbb{R})$ by

$$\theta(h,(0,0,\kappa)) = (\delta(h),\kappa), \quad h\in K,\ \kappa\in S(m,\mathbb{R}), \tag{3.3}$$

where $\delta : K \longrightarrow U(n)$ is the map defined by (2.5). Identifying $\mathbb{R}^{(m,n)}\times\mathbb{R}^{(m,n)}$ with $\mathbb{C}^{(m,n)}$, we can identify $\mathfrak{p}^J$ with $T_n\times\mathbb{C}^{(m,n)}$.

Theorem 3.1. *The adjoint representation of K^J on $\mathfrak{p}^J$ is compatible with the natural action of $U(n)\times S(m,\mathbb{R})$ on $T_{n,m}$ defined by*

$$(h,\kappa)\cdot(\omega,z) := (h\,\omega\,{}^t h,\ z\,{}^t h) \tag{3.4}$$

through the maps Φ and θ, where $h\in U(n)$, $\kappa\in S(m,\mathbb{R})$, $(\omega,z)\in T_{n,m}$. Precisely, if $k^J\in K^J$ and $\alpha\in\mathfrak{p}^J$, then we have the following equality

$$\Phi\big(Ad(k^J)\alpha\big) = \theta\big(k^J\big)\cdot\Phi(\alpha). \tag{3.5}$$

Here we regard the complex vector space $T_{n,m}$ as a real vector space.

The proof of the above theorem can be found in [13].

We now study the algebra $\mathbb{D}(\mathbb{H}_{n,m})$ of all differential operators on $\mathbb{H}_{n,m}$ invariant under the *natural action* (1.2) of G^J. The action (3.4) induces the action of $U(n)$ on the polynomial algebra $\mathrm{Pol}_{n,m} := \mathrm{Pol}\,(T_{n,m})$. We denote by $\mathrm{Pol}_{n,m}^{U(n)}$ the subalgebra of $\mathrm{Pol}_{n,m}$ consisting of all $U(n)$-invariants. Similarly the adjoint action of K on $\mathfrak{p}^J$ induces the action of K on the polynomial algebra $\mathrm{Pol}\big(\mathfrak{p}^J\big)$. We see that through the identification of $\mathfrak{p}^J$ with $T_{n,m}$, the algebra $\mathrm{Pol}\big(\mathfrak{p}^J\big)$ is isomorphic to $\mathrm{Pol}_{n,m}$.

According to Helgason ([11], p. 287), one obtains the Helgason map

$$\Theta_{n,m} : \mathrm{Pol}_{n,m}^{U(n)} \longrightarrow \mathbb{D}(\mathbb{H}_{n,m})$$

of $\mathrm{Pol}_{n,m}^{U(n)}$ onto $\mathbb{D}(\mathbb{H}_{n,m})$ which is a natural linear bijection but is not an algebra isomorphism. The map $\Theta_{n,m}$ is described explicitly as follows. We put $N_\star = n(n+1)+2mn$. Let $\big\{\eta_\alpha \,\big|\, 1\le\alpha\le N_\star\big\}$ be a basis of $\mathfrak{p}^J$. If $P\in\mathrm{Pol}\big(\mathfrak{p}^J\big)^K = \mathrm{Pol}_{n,m}^{U(n)}$, then

$$\Big(\Theta_{n,m}(P)f\Big)(gK^J) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g\exp\left(\sum_{\alpha=1}^{N_\star} t_\alpha\eta_\alpha\right)K^J\right)\right]_{(t_\alpha)=0}, \tag{3.6}$$

where $g\in G^J$ and $f\in C^\infty(\mathbb{H}_{n,m})$. In general, it is hard to express $\Theta_{n,m}(P)$ explicitly for a polynomial $P\in\mathrm{Pol}\big(\mathfrak{p}^J\big)^K$.

We propose the following natural problems.

Problem 1. Find a complete list of explicit generators of $\mathrm{Pol}_{n,m}^{U(n)}$.

Problem 2. Find all relations among a set of generators of $\mathrm{Pol}_{n,m}^{U(n)}$.

Problem 3. Find an easy or effective way to express explicitly the images of the above invariant polynomials or generators of $\mathrm{Pol}_{n,m}^{U(n)}$ under the Helgason map $\Theta_{n,m}$.

Problem 4. Decompose $\mathrm{Pol}_{n,m}$ into $U(n)$-irreducibles.

Problem 5. Find a complete list of explicit generators of the algebra $\mathbb{D}(\mathbb{H}_{n,m})$ or construct explicit G^J-invariant differential operators on $\mathbb{H}_{n,m}$.

Problem 6. Find all relations among a set of generators of $\mathbb{D}(\mathbb{H}_{n,m})$.

Problem 7. Is $\mathrm{Pol}_{n,m}^{U(n)}$ finitely generated ?

Problem 8. Is $\mathbb{D}(\mathbb{H}_{n,m})$ finitely generated ?

Problem 1 and Problem 7 are solved as follows.

Theorem 3.2. $\mathrm{Pol}_{n,m}^{U(n)}$ *is generated by*

$$
\begin{aligned}
q_j(\omega, z) &= \mathrm{tr}\big((\omega\,\overline{\omega})^{j+1}\big), \quad 0 \le j \le n-1,\\
\alpha_{kp}^{(j)}(\omega, z) &= \mathrm{Re}\,\big(z\,(\overline{\omega}\omega)^j\,{}^t\overline{z}\big)_{kp}, \quad 0 \le j \le n-1,\ 1 \le k \le p \le m,\\
\beta_{lq}^{(j)}(\omega, z) &= \mathrm{Im}\,\big(z\,(\overline{\omega}\omega)^j\,{}^t\overline{z}\big)_{lq}, \quad 0 \le j \le n-1,\ 1 \le l < q \le m,\\
f_{kp}^{(j)}(\omega, z) &= \mathrm{Re}\,(z\,(\overline{\omega}\omega)^j\,\overline{\omega}\,{}^t z)_{kp}, \quad 0 \le j \le n-1,\ 1 \le k \le p \le m,\\
g_{kp}^{(j)}(\omega, z) &= \mathrm{Im}\,(z\,(\overline{\omega}\omega)^j\,\overline{\omega}\,{}^t z\,)_{kp}, \quad 0 \le j \le n-1,\ 1 \le k \le p \le m,
\end{aligned}
$$

where $\omega \in T_n$ *and* $z \in \mathbb{C}^{(m,n)}$.

The proof of Theorem 3.2 can be found in [13]. Here we will not describe the solution of Problem 2 because it is very complicated. The solution of Problem 2 will appear in another paper in the near future.

4 Examples of Explicit G^J-Invariant Differential Operators

In this section we give examples of explicit G^J-invariant differential operators on the Siegel-Jacobi space $\mathbb{H}_{n,m}$.

For $g = \big(M, (\lambda, \mu; \kappa)\big) \in G^J$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$, we set

$$\begin{aligned} \Omega &= X + i\,Y, \quad X = (x_{\mu\nu}),\ Y = (y_{\mu\nu}) \text{ real}, \\ \Omega_* &= M\cdot\Omega = X_* + i\,Y_*, \quad X_*, Y_* \text{ real}, \\ Z_* &= (Z + \lambda\Omega + \mu)(C\Omega + D)^{-1} = U_* + i\,V_*, \quad U_*, V_* \text{ real}. \end{aligned}$$

For a coordinate $(\Omega, Z) \in \mathbb{H}_{n,m}$ with $\Omega = (\omega_{\mu\nu})$ and $Z = (z_{kl})$, we put $d\Omega$, $d\overline{\Omega}$, $\frac{\partial}{\partial\Omega}$, $\frac{\partial}{\partial\overline{\Omega}}$ as before and set

$$\begin{aligned} Z &= U + iV, \quad U = (u_{kl}), \quad V = (v_{kl}) \text{ real}, \\ dZ &= (dz_{kl}), \quad d\overline{Z} = (d\overline{z}_{kl}), \end{aligned}$$

$$\frac{\partial}{\partial Z} = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \cdots & \frac{\partial}{\partial z_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_{1n}} & \cdots & \frac{\partial}{\partial z_{mn}} \end{pmatrix}, \quad \frac{\partial}{\partial \overline{Z}} = \begin{pmatrix} \frac{\partial}{\partial \overline{z}_{11}} & \cdots & \frac{\partial}{\partial \overline{z}_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \overline{z}_{1n}} & \cdots & \frac{\partial}{\partial \overline{z}_{mn}} \end{pmatrix}.$$

The author [29] proved that the following differential operators $\mathbb{M}_1$ and $\mathbb{M}_2$ on $\mathbb{H}_{n,m}$ defined by

$$\mathbb{M}_1 = \mathrm{tr}\left(Y\,\frac{\partial}{\partial Z}\,{}^t\!\left(\frac{\partial}{\partial \overline{Z}}\right)\right) \tag{4.1}$$

and

$$\begin{aligned} \mathbb{M}_2 = {} & \mathrm{tr}\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right) + \mathrm{tr}\left(VY^{-1}\,{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial Z}\right) \\ & + \mathrm{tr}\left(V\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial Z}\right) + \mathrm{tr}\left({}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial\Omega}\right) \end{aligned} \tag{4.2}$$

are invariant under the action (1.2) of G^J.

The authors [13] proved that the following differential operator $\mathbb{K}$ on $\mathbb{H}_{n,m}$ of degree $2n$ defined by

$$\mathbb{M}_3 = \det(Y)\,\det\left(\frac{\partial}{\partial Z}\,{}^t\!\left(\frac{\partial}{\partial\overline{Z}}\right)\right) \tag{4.3}$$

is invariant under the action (1.2) of G^J. Furthermore the authors [13] proved that the following matrix-valued differential operator $\mathbb{T}$ on $\mathbb{H}_{n,m}$ defined by

$$\mathbb{T} = {}^t\!\left(\frac{\partial}{\partial\overline{Z}}\right) Y \frac{\partial}{\partial Z} \tag{4.4}$$

and the following differential operators

$$\mathbb{T}_{kl} = \sum_{i,j=1}^{n} y_{ij} \, \frac{\partial^2}{\partial \overline{z}_{ki} \partial z_{lj}}, \quad 1 \le k, l \le m \tag{4.5}$$

are invariant under the action (1.2) of G^J.

We see that

$$\mathbb{M}_* = [\mathbb{M}_1, \mathbb{M}_2] = \mathbb{M}_1\mathbb{M}_2 - \mathbb{M}_2\mathbb{M}_1$$

is an invariant differential operator of degree three on $\mathbb{H}_{n,m}$ and

$$\mathbb{P}_{kl} = [\mathbb{M}_3, \mathbb{T}_{kl}] = \mathbb{M}_3\mathbb{T}_{kl} - \mathbb{T}_{kl}\mathbb{M}_3, \quad 1 \le k, l \le m$$

is an invariant differential operator of degree $2n+1$ on $\mathbb{H}_{n,m}$.

The author [29] proved that for any two positive real numbers A and B,

$$\begin{aligned} ds^2_{n,m;A,B} = \; & A \, \mathrm{tr}\Big(Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega}\Big) \\ & + B \Big\{ \mathrm{tr}\Big(Y^{-1}\, {}^t V\, V\, Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega}\Big) + \mathrm{tr}\Big(Y^{-1}\, {}^t(dZ)\, d\overline{Z}\Big) \\ & - \mathrm{tr}\Big(V\, Y^{-1} d\Omega\, Y^{-1}\, {}^t(d\overline{Z})\Big) - \mathrm{tr}\Big(V\, Y^{-1} d\overline{\Omega}\, Y^{-1}\, {}^t(dZ)\Big) \Big\} \end{aligned}$$

is a Riemannian metric on $\mathbb{H}_{n,m}$ which is invariant under the action (1.2) of G^J. In fact, $ds^2_{n,m;A,B}$ is a Kähler metric of $\mathbb{H}_{n,m}$. The author [29] proved that for any two positive real numbers A and B, the following differential operator

$$\Delta_{n,m;A,B} = \frac{4}{A}\, \mathbb{M}_2 + \frac{4}{B} \mathbb{M}_1 \tag{4.6}$$

is the Laplacian of the G^J-invariant Riemannian metric $ds^2_{n,m;A,B}$.

We set, for an integer k with $1 \le k \le m$,

$$\frac{\partial}{\partial Z_k} = {}^t\!\left(\frac{\partial}{\partial z_{1k}}, \cdots, \frac{\partial}{\partial z_{nk}} \right)$$

and

$$Y_{+,k} := \frac{\partial}{\partial Z_k}, \quad Y_{-,k} := \frac{\partial}{\partial \overline{Z}_k} Y.$$

We define

$$\begin{aligned}
Y_+ &:= \frac{\partial}{\partial Z}, \qquad Y_- := {}^t\!\left(\frac{\partial}{\partial \overline{Z}}\right) Y, \\
X_+ &:= 2\,i\,\frac{\partial}{\partial \Omega} + i\,Y^{-1}\,{}^tV \frac{\partial}{\partial Z} + {}^t\!\left(i\,Y^{-1}\,{}^tV\,{}^t\!\left(\frac{\partial}{\partial Z}\right)\right), \\
X_- &:= {}^tY_-\,{}^t\big(Y\overline{Y}_+\big), \\
\widetilde{K} &:= 2\,i\,Y\,\frac{\partial}{\partial \Omega} + i\,{}^tV\,{}^t\!\left(\frac{\partial}{\partial Z}\right) + i\,{}^t\!\left(Y^{-1}\,{}^tV\,{}^t\!\left(\frac{\partial}{\partial Z}\right)Y\right),
\end{aligned}$$

and

$$\widetilde{\Lambda} := 2\,i\,Y\,\frac{\partial}{\partial \overline{\Omega}} + i\,{}^tV\,{}^t\!\left(\frac{\partial}{\partial \overline{Z}}\right) + i\,{}^t\!\left(Y^{-1}\,{}^tV\,{}^t\!\left(\frac{\partial}{\partial \overline{Z}}\right)Y\right).$$

Following H. Maass [15] (cf. (2.18)–(2.20)), we put

$$\widetilde{A}^{(1)} = \widetilde{\Lambda}\widetilde{K} + \frac{n+1}{2}\widetilde{K}$$

and define $\widetilde{A}^{(j)}$ $(j = 2, 3, \cdots, n)$ recursively by

$$\begin{aligned}
\widetilde{A}^{(j)} = {} & \widetilde{A}^{(1)}\widetilde{A}^{(j-1)} - \frac{n+1}{2}\,\widetilde{\Lambda}\,\widetilde{A}^{(j-1)} + \frac{1}{2}\,\widetilde{\Lambda}\,\mathrm{tr}\Big(\widetilde{A}^{(j-1)}\Big) \\
& + \frac{1}{2}\,(\Omega - \overline{\Omega})\;{}^t\!\Big\{(\Omega - \overline{\Omega})^{-1}\,{}^t\Big({}^t\widetilde{\Lambda}\,{}^t\widetilde{A}^{(j-1)}\Big)\Big\}.
\end{aligned}$$

For any positive integers j, k, l with $1 \le j \le n,\ 1 \le k, l \le m$, we define

$$\begin{aligned}
\widetilde{H}_j &:= \mathrm{tr}\big(\widetilde{A}^{(j)}\big), \qquad T_{k,l} := \mathrm{tr}\Big({}^tY_{-,k}\,{}^tY_{+,l}\,\widetilde{A}^{(j)}\Big), \\
U_{k,l} &:= \mathrm{tr}\left({}^tY_{-,k}\,Y_{-,l}\,X_+\right), \qquad V_{k,l} := \mathrm{tr}\left(Y_{+,k}\,{}^tY_{+,l}\,X_-\right).
\end{aligned}$$

J. Yang and L. Yin [32] showed that $\widetilde{H}_j$, $T_{k,l}$, $U_{k,l}$ and $V_{k,l}$ are invariant under the action (1.2) of G^J.

5 The Partial Cayley Transform

In this section we discuss a notion of the partial Cayley transform and give examples of explicit G^J-invariant differential operators on the Siegel-Jacobi disk.

Let

$$\mathbb{D}_n = \Big\{ W \in \mathbb{C}^{(n,n)} \mid W = {}^tW,\ I_n - \overline{W}W > 0 \Big\}$$

be the generalized unit disk.

For brevity, we write $\mathbb{D}_{n,m} := \mathbb{D}_n \times \mathbb{C}^{(m,n)}$. This homogeneous space $\mathbb{D}_{n,m}$ is called the *Siegel-Jacobi disk* of degree n and index m. For a coordinate $(W,\eta) \in \mathbb{D}_{n,m}$ with $W = (w_{\mu\nu}) \in \mathbb{D}_n$ and $\eta = (\eta_{kl}) \in \mathbb{C}^{(m,n)}$, we put

$$\begin{aligned} dW &= (dw_{\mu\nu}), \qquad & d\overline{W} &= (d\overline{w}_{\mu\nu}),\\ d\eta &= (d\eta_{kl}), \qquad & d\overline{\eta} &= (d\overline{\eta}_{kl}) \end{aligned}$$

and

$$\frac{\partial}{\partial W} = \left(\frac{1+\delta_{\mu\nu}}{2}\,\frac{\partial}{\partial w_{\mu\nu}} \right), \quad \frac{\partial}{\partial \overline{W}} = \left(\frac{1+\delta_{\mu\nu}}{2}\,\frac{\partial}{\partial \overline{w}_{\mu\nu}} \right),$$

$$\frac{\partial}{\partial \eta} = \begin{pmatrix} \frac{\partial}{\partial \eta_{11}} & \cdots & \frac{\partial}{\partial \eta_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \eta_{1n}} & \cdots & \frac{\partial}{\partial \eta_{mn}} \end{pmatrix}, \quad \frac{\partial}{\partial \overline{\eta}} = \begin{pmatrix} \frac{\partial}{\partial \overline{\eta}_{11}} & \cdots & \frac{\partial}{\partial \overline{\eta}_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \overline{\eta}_{1n}} & \cdots & \frac{\partial}{\partial \overline{\eta}_{mn}} \end{pmatrix}.$$

We can identify an element $g = (M, (\lambda, \mu; \kappa))$ of G^J, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ with the element

$$\begin{pmatrix} A & 0 & B & A\,{}^t\mu - B\,{}^t\lambda \\ \lambda & I_m & \mu & \kappa \\ C & 0 & D & C\,{}^t\mu - D\,{}^t\lambda \\ 0 & 0 & 0 & I_m \end{pmatrix}$$

of $Sp(m+n,\mathbb{R})$.

We set

$$T_* = \frac{1}{\sqrt{2}} \begin{pmatrix} I_{m+n} & I_{m+n} \\ iI_{m+n} & -iI_{m+n} \end{pmatrix}.$$

We now consider the group G^J_* defined by

$$G^J_* := T_*^{-1} G^J T_*.$$

If $g = (M, (\lambda, \mu; \kappa)) \in G^J$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$, then $T_*^{-1} g T_*$ is given by

$$T_*^{-1} g T_* = \begin{pmatrix} P_* & Q_* \\ \overline{Q}_* & \overline{P}_* \end{pmatrix}, \tag{5.1}$$

where

$$P_* = \begin{pmatrix} P & \frac{1}{2}\left\{ Q\,{}^t(\lambda + i\mu) - P\,{}^t(\lambda - i\mu) \right\} \\ \frac{1}{2}(\lambda + i\mu) & I_h + i\frac{\kappa}{2} \end{pmatrix},$$

$$Q_* = \begin{pmatrix} Q & \frac{1}{2}\left\{ P\,{}^t(\lambda - i\mu) - Q\,{}^t(\lambda + i\mu) \right\} \\ \frac{1}{2}(\lambda - i\mu) & -i\frac{\kappa}{2} \end{pmatrix},$$

and P, Q are given by the formulas

$$P=\frac{1}{2}\,\{(A+D)+\,i\,(B-C)\} \tag{5.2}$$

and

$$Q=\frac{1}{2}\,\{(A-D)-\,i\,(B+C)\}\,. \tag{5.3}$$

From now on, we write

$$\left(\begin{pmatrix} P & Q\\ \overline{Q} & \overline{P}\end{pmatrix}, \left(\frac{1}{2}(\lambda+i\mu),\,\frac{1}{2}(\lambda-i\mu);\,-i\frac{\kappa}{2}\right)\right):=\begin{pmatrix} P_* & Q_*\\ \overline{Q}_* & \overline{P}_*\end{pmatrix}.$$

In other words, we have the relation

$$T_*^{-1}\left(\begin{pmatrix} A & B\\ C & D\end{pmatrix}, (\lambda,\mu;\kappa)\right)T_*=\left(\begin{pmatrix} P & Q\\ \overline{Q} & \overline{P}\end{pmatrix}, \left(\frac{1}{2}(\lambda+i\mu),\,\frac{1}{2}(\lambda-i\mu);\,-i\frac{\kappa}{2}\right)\right).$$

Let

$$H_{\mathbb{C}}^{(n,m)}:=\left\{(\xi,\eta\,;\zeta)\mid \xi,\eta\in\mathbb{C}^{(m,n)},\ \zeta\in\mathbb{C}^{(m,m)},\ \zeta+\eta\,{}^t\xi \text{ symmetric}\right\}$$

be the complex Heisenberg group endowed with the following multiplication

$$(\xi,\eta\,;\zeta)\circ(\xi',\eta';\zeta'):=(\xi+\xi',\eta+\eta'\,;\zeta+\zeta'+\xi\,{}^t\eta'-\eta\,{}^t\xi')).$$

We define the semidirect product

$$SL(2n,\mathbb{C})\ltimes H_{\mathbb{C}}^{(n,m)}$$

endowed with the following multiplication

$$\begin{aligned}&\left(\begin{pmatrix} P & Q\\ R & S\end{pmatrix}, (\xi,\eta\,;\zeta)\right)\cdot\left(\begin{pmatrix} P' & Q'\\ R' & S'\end{pmatrix}, (\xi',\eta';\zeta')\right)\\ &=\left(\begin{pmatrix} P & Q\\ R & S\end{pmatrix}\begin{pmatrix} P' & Q'\\ R' & S'\end{pmatrix},\,(\tilde\xi+\xi',\tilde\eta+\eta';\zeta+\zeta'+\tilde\xi\,{}^t\eta'-\tilde\eta\,{}^t\xi')\right),\end{aligned}$$

where $\tilde\xi=\xi P'+\eta R'$ and $\tilde\eta=\xi Q'+\eta S'$.

If we identify $H_{\mathbb{R}}^{(n,m)}$ with the subgroup

$$\left\{(\xi,\overline{\xi};i\kappa)\mid \xi\in\mathbb{C}^{(m,n)},\ \kappa\in\mathbb{R}^{(m,m)}\right\}$$

of $H_{\mathbb{C}}^{(n,m)}$, we have the following inclusion

$$G_*^J\subset SU(n,n)\ltimes H_{\mathbb{R}}^{(n,m)}\subset SL(2n,\mathbb{C})\ltimes H_{\mathbb{C}}^{(n,m)}.$$

We define the mapping $\Theta : G^J \longrightarrow G^J_*$ by

$$\Theta\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda,\mu;\kappa)\right) = \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \left(\frac{1}{2}(\lambda+i\mu),\ \frac{1}{2}(\lambda-i\mu);\ -i\frac{\kappa}{2}\right)\right), \tag{5.4}$$

where P and Q are given by (5.2) and (5.3). We can see that if $g_1, g_2 \in G^J$, then $\Theta(g_1g_2) = \Theta(g_1)\Theta(g_2)$.

According to [26, p. 250], G^J_* is of the Harish-Chandra type (cf. [17, p. 118]). Let

$$g_* = \left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\lambda,\mu;\ \kappa)\right)$$

be an element of G^J_*. Since the Harish-Chandra decomposition of an element $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ in $SU(n,n)$ is given by

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} I_n & QS^{-1} \\ 0 & I_n \end{pmatrix} \begin{pmatrix} P-QS^{-1}R & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} I_n & 0 \\ S^{-1}R & I_n \end{pmatrix},$$

the P^+_*-component of the following element

$$g_* \cdot \left(\begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0,\eta;0)\right), \quad W \in \mathbb{D}_n$$

of $SL(2n,\mathbb{C}) \ltimes H^{(n,m)}_{\mathbb{C}}$ is given by

$$\left(\begin{pmatrix} I_n & (PW+Q)(\overline{Q}W+\overline{P})^{-1} \\ 0 & I_n \end{pmatrix}, \left(0,\ (\eta+\lambda W+\mu)(\overline{Q}W+\overline{P})^{-1};0\right)\right). \tag{5.5}$$

We can identify $\mathbb{D}_{n,m}$ with the subset

$$\left\{ \left(\begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0,\eta;0)\right) \,\middle|\, W \in \mathbb{D}_n,\ \eta \in \mathbb{C}^{(m,n)} \right\}$$

of the complexification of G^J_*. Indeed, $\mathbb{D}_{n,m}$ is embedded into P^+_* given by

$$P^+_* = \left\{ \left(\begin{pmatrix} I_n & W \\ 0 & I_n \end{pmatrix}, (0,\eta;0)\right) \,\middle|\, W = {}^tW \in \mathbb{C}^{(n,n)},\ \eta \in \mathbb{C}^{(m,n)} \right\}.$$

This is a generalization of the Harish-Chandra embedding (cf. [17, p. 119]). Then we get the *natural transitive action* of G^J_* on $\mathbb{D}_{n,m}$ defined by

$$\begin{aligned} &\left(\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, (\xi,\overline{\xi};\ i\kappa)\right) \cdot (W,\eta) \qquad (5.6)\\ &= \left((PW+Q)(\overline{Q}W+\overline{P})^{-1}, (\eta+\xi W+\overline{\xi})(\overline{Q}W+\overline{P})^{-1}\right), \end{aligned}$$

where $\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*,\ \xi \in \mathbb{C}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)}$ and $(W,\eta) \in \mathbb{D}_{n,m}$.

The author [30] proved that the action (1.2) of G^J on $\mathbb{H}_{n,m}$ is compatible with the action (5.6) of G_*^J on $\mathbb{D}_{n,m}$ through the *partial Cayley transform* $\Psi : \mathbb{D}_{n,m} \longrightarrow \mathbb{H}_{n,m}$ defined by

$$\Psi(W,\eta) := \Big(i(I_n + W)(I_n - W)^{-1},\, 2\, i\, \eta\, (I_n - W)^{-1} \Big).$$

In other words, if $g_0 \in G^J$ and $(W,\eta) \in \mathbb{D}_{n,m}$,

$$g_0 \cdot \Psi(W,\eta) = \Psi(g_* \cdot (W,\eta)),$$

where $g_* = T_*^{-1} g_0 T_*$. Ψ is a biholomorphic mapping of $\mathbb{D}_{n,m}$ onto $\mathbb{H}_{n,m}$ which gives the partially bounded realization of $\mathbb{H}_{n,m}$ by $\mathbb{D}_{n,m}$. The inverse of Ψ is

$$\Psi^{-1}(\Omega, Z) = \Big((\Omega - iI_n)(\Omega + iI_n)^{-1},\, Z(\Omega + iI_n)^{-1} \Big).$$

The author [31] proved that for any two positive real numbers A and B, the following metric $d\tilde{s}^2_{n,m;A,B}$ defined by

$$\begin{aligned}
ds^2_{\mathbb{D}_{n,m};A,B} = {} & 4\,A\, \mathrm{tr}\Big((I_n - W\overline{W})^{-1} dW (I_n - \overline{W}W)^{-1} d\overline{W} \Big) \\
& + 4\, B\, \Big\{ \mathrm{tr}\Big((I_n - W\overline{W})^{-1}\, {}^t(d\eta)\, \beta \Big) \\
& \quad + \mathrm{tr}\Big((\eta\overline{W} - \overline{\eta})(I_n - W\overline{W})^{-1} dW (I_n - \overline{W}W)^{-1}\, {}^t(d\overline{\eta}) \Big) \\
& \quad + \mathrm{tr}\Big((\overline{\eta}W - \eta)(I_n - \overline{W}W)^{-1} d\overline{W} (I_n - W\overline{W})^{-1}\, {}^t(d\eta) \Big) \\
& \quad - \mathrm{tr}\Big((I_n - W\overline{W})^{-1}\, {}^t\eta\, \eta\, (I_n - \overline{W}W)^{-1} \overline{W} dW (I_n - \overline{W}W)^{-1} d\overline{W} \Big) \\
& \quad - \mathrm{tr}\Big(W(I_n - \overline{W}W)^{-1}\, {}^t\overline{\eta}\, \overline{\eta}\, (I_n - W\overline{W})^{-1} dW (I_n - \overline{W}W)^{-1} d\overline{W} \Big) \\
& \quad + \mathrm{tr}\Big((I_n - W\overline{W})^{-1}\, {}^t\eta\, \overline{\eta}\, (I_n - W\overline{W})^{-1} dW (I_n - \overline{W}W)^{-1} d\overline{W} \Big) \\
& \quad + \mathrm{tr}\Big((I_n - \overline{W})^{-1}\, {}^t\overline{\eta}\, \eta\, \overline{W}\, (I_n - W\overline{W})^{-1} dW (I_n - \overline{W}W)^{-1} d\overline{W} \Big) \\
& \quad + \mathrm{tr}\Big((I_n - \overline{W})^{-1} (I_n - W)(I_n - \overline{W}W)^{-1}\, {}^t\overline{\eta}\, \eta\, (I_n - \overline{W}W)^{-1} \\
& \qquad\quad \times (I_n - \overline{W})(I_n - W)^{-1} dW (I_n - \overline{W}W)^{-1} d\overline{W} \Big) \\
& \quad - \mathrm{tr}\Big((I_n - W\overline{W})^{-1} (I_n - W)(I_n - \overline{W})^{-1}\, {}^t\overline{\eta}\, \eta\, (I_n - W)^{-1} \\
& \qquad\quad \times dW (I_n - \overline{W}W)^{-1} d\overline{W} \Big) \Big\}
\end{aligned}$$

is a Riemannian metric on $\mathbb{D}_{n,m}$ which is invariant under the action (5.6) of the Jacobi group G_*^J.

The author [31] proved that the following differential operators $\mathbb{S}_1$ and $\mathbb{S}_2$ on $\mathbb{D}_{n,m}$ defined by

$$\mathbb{S}_1 = \sigma\left((I_n - \overline{W}W)\frac{\partial}{\partial\eta}\,{}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)\right)$$

and

$$\begin{aligned}\mathbb{S}_2 = &\ \mathrm{tr}\left((I_n - W\overline{W})\,{}^t\!\left((I_n - W\overline{W})\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial W}\right)\\ &+\mathrm{tr}\left({}^t(\eta-\overline{\eta}W)\,{}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n-\overline{W}W)\frac{\partial}{\partial W}\right)\\ &+\mathrm{tr}\left((\overline{\eta}-\eta\overline{W})\,{}^t\!\left((I_n - W\overline{W})\frac{\partial}{\partial\overline{W}}\right)\frac{\partial}{\partial\eta}\right)\\ &-\mathrm{tr}\left(\eta\overline{W}(I_n-W\overline{W})^{-1}\,{}^t\eta\,{}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\right)\\ &-\mathrm{tr}\left(\overline{\eta}W(I_n-\overline{W}W)^{-1}\,{}^t\overline{\eta}\,{}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\right)\\ &+\mathrm{tr}\left(\overline{\eta}(I_n-W\overline{W})^{-1}\,{}^t\eta\,{}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\right)\\ &+\mathrm{tr}\left(\eta\overline{W}W(I_n-\overline{W}W)^{-1}\,{}^t\overline{\eta}\,{}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n-\overline{W}W)\frac{\partial}{\partial\eta}\right)\end{aligned}$$

are invariant under the action (5.6) of G_*^J. The author also proved that

$$\Delta_{\mathbb{D}_{n,m};A,B} := \frac{1}{A}\,\mathbb{S}_2 + \frac{1}{B}\,\mathbb{S}_1 \tag{5.7}$$

is the Laplacian of the invariant metric $ds^2_{\mathbb{D}_{n,m};A,B}$ on $\mathbb{D}_{n,m}$ (cf. [31]).

The authors [13] proved that the following differential operator on $\mathbb{D}_{n,m}$ defined by

$$\mathbb{S}_3 = \det(I_n - \overline{W}W)\,\det\left(\frac{\partial}{\partial\eta}\,{}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)\right)$$

is invariant under the action (5.6) of G_*^J on $\mathbb{D}_{n,m}$. Furthermore the authors [13] proved that the following matrix-valued differential operator on $\mathbb{D}_{n,m}$ defined by

$$\mathbb{J} := {}^t\!\left(\frac{\partial}{\partial\overline{\eta}}\right)(I_n - \overline{W}W)\frac{\partial}{\partial\eta}$$

and each (k,l)-entry $\mathbb{J}_{kl}$ of $\mathbb{J}$ given by

$$\mathbb{J}_{kl} = \sum_{i,j=1}^{n} \left(\delta_{ij} - \sum_{r=1}^{n} \overline{w}_{ir}\, w_{jr} \right) \frac{\partial^2}{\partial \overline{\eta}_{ki} \partial \eta_{lj}}, \quad 1 \le k, l \le m$$

are invariant under the action (5.6) of G_*^J on $\mathbb{D}_{n,m}$.

$$\mathbb{S}_* = [\mathbb{S}_1, \mathbb{S}_2] = \mathbb{S}_1\mathbb{S}_2 - \mathbb{S}_2\mathbb{S}_1$$

is an invariant differential operator of degree three on $\mathbb{D}_{n,m}$ and

$$\mathbb{Q}_{kl} = [\mathbb{S}_3, \mathbb{J}_{kl}] = \mathbb{S}_3\mathbb{J}_{kl} - \mathbb{J}_{kl}\mathbb{S}_3, \quad 1 \le k, l \le m$$

is an invariant differential operator of degree $2n+1$ on $\mathbb{D}_{n,m}$.

Indeed it is very complicated and difficult at this moment to express the generators of the algebra of all G_*^J-invariant differential operators on $\mathbb{D}_{n,m}$ explicitly.

6 Invariant Differential Operators on the Siegel-Jacobi Space of Lowest Dimension

We consider the case $n = m = 1$. For a coordinate (w, ξ) in $T_{1,1} = \mathbb{C} \times \mathbb{C}$, we write $w = r + i\,s,\ \xi = \zeta + i\,\eta \in \mathbb{C},\ r, s, \zeta, \eta$ real. The author [27] proved that the algebra $\mathrm{Pol}_{1,1}^{U(1)}$ is generated by

$$\begin{aligned}
q(w,\xi) &= \frac{1}{4}\, w\,\overline{w} = \frac{1}{4}\big(r^2 + s^2\big),\\
\alpha(w,\xi) &= \xi\,\overline{\xi} = \zeta^2 + \eta^2,\\
\phi(w,\xi) &= \frac{1}{2}\,\mathrm{Re}\,\big(\xi^2\overline{w}\big) = \frac{1}{2}\,r\big(\zeta^2 - \eta^2\big) + s\,\zeta\eta,\\
\psi(w,\xi) &= \frac{1}{2}\,\mathrm{Im}\,\big(\xi^2\overline{w}\big) = \frac{1}{2}\,s\big(\eta^2 - \zeta^2\big) + r\,\zeta\eta.
\end{aligned}$$

In [27], using Formula (3.6) the author calculated explicitly the images

$$D_1 = \Theta_{1,1}(q), \quad D_2 = \Theta_{1,1}(\alpha), \quad D_3 = \Theta_{1,1}(\phi) \quad \text{and} \quad D_4 = \Theta_{1,1}(\psi)$$

of q, ξ, ϕ and ψ under the Helgason map $\Theta_{1,1}$. We can show that the algebra $\mathbb{D}(\mathbb{H}_{1,1})$ is generated by the following differential operators

$$\begin{aligned} D_1 =& y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \\ & + 2\, y\, v \left(\frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v} \right), \end{aligned}$$

$$D_2 = y \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right),$$

$$\begin{aligned} D_3 =& y^2 \frac{\partial}{\partial y} \left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) - 2y^2 \frac{\partial^3}{\partial x \partial u \partial v} \\ & - \left(v \frac{\partial}{\partial v} + 1 \right) D_2 \end{aligned}$$

and

$$\begin{aligned} D_4 =& y^2 \frac{\partial}{\partial x} \left(\frac{\partial^2}{\partial v^2} - \frac{\partial^2}{\partial u^2} \right) - 2\, y^2 \frac{\partial^3}{\partial y \partial u \partial v} \\ & - v \frac{\partial}{\partial u} D_2, \end{aligned}$$

where $\tau = x + iy$ and $z = u + iv$ with real variables x, y, u, v. Moreover, we have

$$\begin{aligned} D_1 D_2 - D_2 D_1 =& 2\, y^2 \frac{\partial}{\partial y} \left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) \\ & - 4\, y^2 \frac{\partial^3}{\partial x \partial u \partial v} - 2 \left(v \frac{\partial}{\partial v} D_2 + D_2 \right). \end{aligned}$$

In particular, the algebra $\mathbb{D}(\mathbb{H}_{1,1})$ is not commutative. We refer to [2, 6, 27] for more detail.

Recently the authors [13] proved the following results.

Theorem 6.3. *We have the following relation*

$$\phi^2 + \psi^2 = q\, \alpha^2.$$

This relation exhausts all the relations among the generators q, α, ϕ and ψ of $\mathrm{Pol}_{1,1}^{U(1)}$.

Theorem 6.4. *We have the following relations*

(*a*) $[D_1, D_2] = 2D_3$

(*b*) $[D_1, D_3] = 2D_1 D_2 - 2D_3$

(*c*) $[D_2, D_3] = -D_2^2$

(*d*) $[D_4, D_1] = 0$

(*e*) $[D_4, D_2] = 0$

(*f*) $[D_4, D_3] = 0$

(*g*) $D_3^2 + D_4^2 = D_2 D_1 D_2$

These seven relations exhaust all the relations among the generators D_1, D_2, D_3 *and* D_4 *of* $\mathbb{D}(\mathbb{H}_{1,1})$.

Theorem 6.5. *The action of* $U(1)$ *on* $\mathrm{Pol}_{1,1}$ *is not multiplicity-free.*

Finally we see that for the case when $n = m = 1$, the eight problems proposed in Section 3 are completely solved.

Remark 1. According to Theorem 6.4, we see that D_4 is a generator of the center of $\mathbb{D}(\mathbb{H}_{1,1})$. We observe that the Lapalcian

$$\Delta_{1,1;A,B} = \frac{4}{A} D_1 + \frac{4}{B} D_2 \qquad \text{(see (4.6))}$$

of $(\mathbb{H}_{1,1}, ds^2_{1,1;A,B})$ does not belong to the center of $\mathbb{D}(\mathbb{H}_{1,1})$.

Remark 2. When $n = 1$ and m is an arbitrary integer, Conley and Raum [6] found the $2m^2 + m + 1$ explicit generators of $\mathbb{D}(\mathbb{H}_{1,m})$ and the explicit one generator of the center of $\mathbb{D}(\mathbb{H}_{1,m})$. They also found the generators of the center of the universal enveloping algebra of $\mathfrak{U}(\mathfrak{g}^J)$ of the Jacobi Lie algebra $\mathfrak{g}^J$. The number of generators of the center of $\mathfrak{U}(\mathfrak{g}^J)$ is $1 + \frac{m(m+1)}{2}$.

7 Remarks on Maass-Jacobi Forms

Using G^J-invariant differential operators on the Siegel-Jacobi space, we introduce a notion of Maass-Jacobi forms.

Definition 1. Let

$$\Gamma_{n,m} := Sp(n, \mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(n,m)}$$

be the discrete subgroup of G^J, where

$$H_{\mathbb{Z}}^{(n,m)} = \left\{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)} \mid \lambda, \mu, \kappa \text{ are integral } \right\}.$$

A smooth function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is called a Maass-Jacobi form on $\mathbb{H}_{n,m}$ if f satisfies the following conditions (MJ1)-(MJ3) :

(MJ1) f is invariant under $\Gamma_{n,m}$.

(MJ2) f is an eigenfunction of the Laplacian $\Delta_{n,m;A,B}$ (cf. Formula (4.6)).
(MJ3) f has a polynomial growth, that is, there exist a constant $C > 0$ and a positive integer N such that

$$|f(X + i\,Y, Z)| \leq C\,|p(Y)|^N \quad \text{as } \det Y \longrightarrow \infty,$$

where $p(Y)$ is a polynomial in $Y = (y_{ij})$.

Remark 3. Let $\mathbb{D}_*$ be a commutative subalgebra of $\mathbb{D}(\mathbb{H}_{n,m})$ containing the Laplacian $\Delta_{n,m;A,B}$. We say that a smooth function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is a Maass-Jacobi form with respect to $\mathbb{D}_*$ if f satisfies the conditions $(MJ1)$, $(MJ2)_*$ and $(MJ3)$: the condition $(MJ2)_*$ is given by

$(MJ2)_*$ f is an eigenfunction of any invariant differential operator in $\mathbb{D}_*$.

Remark 4. Erik Balslev [1] developed the spectral theory of $\Delta_{1,1;1,1}$ on $\mathbb{H}_{1,1}$ to prove that the set of all eigenvalues of $\Delta_{1,1;1,1}$ satisfies the Weyl law.

It is natural to propose the following problems.

Problem A : Find all the eigenfunctions of $\Delta_{n,m;A,B}$.

Problem B : Construct Maass-Jacobi forms.

If we find a *nice* eigenfunction ϕ of the Laplacian $\Delta_{n,m;A,B}$, we can construct a Maass-Jacobi form f_ϕ on $\mathbb{H}_{n,m}$ in the usual way defined by

$$f_\phi(\Omega, Z) := \sum_{\gamma \in \Gamma^\infty_{n,m} \backslash \Gamma_{n,m}} \phi\big(\gamma \cdot (\Omega, Z)\big),$$

where

$$\Gamma^\infty_{n,m} = \left\{ \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu; \kappa) \right) \in \Gamma_{n,m} \,\Big|\, C = 0 \right\}$$

is a subgroup of $\Gamma_{n,m}$.

We consider the simple case when $n = m = 1$ and $A = B = 1$. A metric $ds^2_{1,1;1,1}$ on $\mathbb{H}_{1,1}$ given by

$$\begin{aligned} ds^2_{1,1;1,1} = & \frac{y + v^2}{y^3}\,(\,dx^2 + dy^2\,) + \frac{1}{y}\,(\,du^2 + dv^2\,) \\ & - \frac{2v}{y^2}\,(\,dx\,du + dy\,dv\,) \end{aligned}$$

is a G^J-invariant Kähler metric on $\mathbb{H}_{1,1}$. Its Laplacian $\Delta_{1,1;1,1}$ is given by

$$\begin{aligned}\Delta_{1,1;1,1} \,&= y^2\,\left(\frac{\partial^2}{\partial x^2}\,+\,\frac{\partial^2}{\partial y^2}\right)\\&\quad+\,(\,y\,+\,v^2\,)\left(\frac{\partial^2}{\partial u^2}\,+\,\frac{\partial^2}{\partial v^2}\right)\\&\quad+\,2\,y\,v\,\left(\frac{\partial^2}{\partial x\partial u}\,+\,\frac{\partial^2}{\partial y\partial v}\right).\end{aligned}$$

We provide some examples of eigenfunctions of $\Delta_{1,1;1,1}$.

(a) $h(x,y)=y^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|a|y)\,e^{2\pi i a x}$ $(s\in\mathbb{C},\ a\neq 0\,)$ with eigenvalue $s(s-1)$. Here

$$K_s(z):=\frac{1}{2}\int_0^\infty \exp\left\{-\frac{z}{2}(t+t^{-1})\right\}\,t^{s-1}\,dt,$$

where $\mathrm{Re}\,z>0$.

(b) $y^s,\ y^s x,\ y^s u$ $(s\in\mathbb{C})$ with eigenvalue $s(s-1)$.

(c) $y^s v,\ y^s uv,\ y^s xv$ with eigenvalue $s(s+1)$.

(d) $x,\ y,\ u,\ v,\ xv,\ uv$ with eigenvalue 0.

(e) All Maass wave forms.

Let ρ be a rational representation of $GL(n,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M}\in\mathbb{R}^{(m,m)}$ be a symmetric half-integral semi-positive definite matrix of degree m. Let $C^\infty(\mathbb{H}_{n,m},V_\rho)$ be the algebra of all C^∞ functions on $\mathbb{H}_{n,m}$ with values in V_ρ. We define the $|_{\rho,\mathcal{M}}$-slash action of G^J on $C^\infty(\mathbb{H}_{n,m},V_\rho)$ as follows: If $f\in C^\infty(\mathbb{H}_{n,m},V_\rho)$,

$$\begin{aligned}&f|_{\rho,\mathcal{M}}[(M,(\lambda,\mu;\kappa))](\Omega,Z)\\:=\ &e^{-2\pi i\,\mathrm{tr}(\mathcal{M}[Z+\lambda\Omega+\mu](C\Omega+D)^{-1}C)}\cdot e^{2\pi i\,\mathrm{tr}(\mathcal{M}(\lambda\Omega\,{}^t\lambda\,+\,2\lambda\,{}^tZ\,+\,\kappa\,+\,\mu\,{}^t\lambda))}\\&\times\rho(C\Omega+D)^{-1}f(M\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}),\end{aligned}$$

where $\begin{pmatrix}A&B\\C&D\end{pmatrix}\in Sp(n,\mathbb{R})$ and $(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$. We recall the Siegel's notation $\alpha[\beta]={}^t\beta\alpha\beta$ for suitable matrices α and β. We define $\mathbb{D}_{\rho,\mathcal{M}}$ to be the algebra of all differential operators D on $\mathbb{H}_{n,m}$ satisfying the following condition

$$(Df)|_{\rho,\mathcal{M}}[g]=\,D(f|_{\rho,\mathcal{M}}[g])$$

for all $f\in C^\infty(\mathbb{H}_{n,m},V_\rho)$ and for all $g\in G^J$. We denote by $\mathcal{Z}_{\rho,\mathcal{M}}$ the center of $\mathbb{D}_{\rho,\mathcal{M}}$.

We define another notion of Maass-Jacobi forms as follows.

Definition 2. A vector-valued smooth function $\phi : \mathbb{H}_{n,m} \longrightarrow V_\rho$ is called a Maass-Jacobi form on $\mathbb{H}_{n,m}$ of type ρ and index $\mathcal{M}$ if it satisfies the following conditions $(MJ1)_{\rho,\mathcal{M}}$, $(MJ2)_{\rho,\mathcal{M}}$ and $(MJ3)_{\rho,\mathcal{M}}$:

$(MJ1)_{\rho,\mathcal{M}}$ $\phi|_{\rho,\mathcal{M}}[\gamma] = \phi$ for all $\gamma \in \Gamma_{n,m}$.

$(MJ2)_{\rho,\mathcal{M}}$ f is an eigenfunction of all differential operators in the center $\mathcal{Z}_{\rho,\mathcal{M}}$ of $\mathbb{D}_{\rho,\mathcal{M}}$.

$(MJ3)_{\rho,\mathcal{M}}$ f has a growth condition

$$\phi(\Omega, Z) = O\Big(e^{a \det Y} \cdot e^{2\pi \mathrm{tr}(\mathcal{M}[V]Y^{-1})}\Big)$$

as $\det Y \longrightarrow \infty$ for some $a > 0$.

The case when $n = 1,\ m = 1$ and $\rho = \det^k (k = 0, 1, 2, \cdots)$ was studied by R. Berndt and R. Schmidt [2], A. Pitale [16] and K. Bringmann and O. Richter [4]. The case when $n = 1,\ m =$arbitrary and $\rho = \det^k (k = 1, 2, \cdots)$ was investigated by C. Conley and M. Raum [6]. In [6] the authors proved that the center $\mathcal{Z}_{\det^k,\mathcal{M}}$ of $\mathbb{D}_{\det^k,\mathcal{M}}$ is the polynomial algebra with one generator $\mathcal{C}^{k,\mathcal{M}}$, the so-called *Casimir* operator which is a $|_{\det^k,\mathcal{M}}$–slash invariant differential operator of degree three for the case when $n = m = 1$ or of degree four for the case when $n = 1,\ m \geq 2$. Bringmann and Richter [4] considered the Poincaré series $\mathcal{P}^{(n,r)}_{k,\mathcal{M},s}$ (the case when $n = m = 1$) that is a *harmonic* Maass-Jacobi form in the sense of Definition 2 and investigated its Fourier expansion and its Fourier coefficients. Here the *harmonicity* of $\mathcal{P}^{(n,r)}_{k,\mathcal{M},s}$ means that $\mathcal{C}^{k,\mathcal{M}}\mathcal{P}^{(n,r)}_{k,\mathcal{M},s} = 0$, i.e., $\mathcal{P}^{(n,r)}_{k,\mathcal{M},s}$ is an eigenfunction of $\mathcal{C}^{k,\mathcal{M}}$ with zero eigenvalue. Conley and Raum [6] generalized the results in [16] and [4] to the case when $n = 1$ and m is arbitrary.

Remark 5. In [3], Bringmann, Conley and Richter proved that the center of the algebra of differential operators invariant under the action of the Jacobi group over a complex quadratic field is generated by two Casimir operators of degree three. They also introduce an analogue of Kohnen's plus space for modular forms of half-integral weight over $K = \mathbb{Q}(i)$, and provide a lift from it to the space of Jacobi forms over K.

References

1. E. Balslev, Spectral theory of the Laplacian on the modular Jacobi group manifold, preprint, Aarhus University (2012).
2. R. Berndt and R. Schmidt, *Elements of the Representation Theory of the Jacobi Group*, Progress in Mathematics, **163**, Birkhäuser, Basel, 1998.
3. K. Bringmann, C. Conley and O. K. Richter, *Jacobi forms over complex quadratic fields via the cubic Casimier operators*, preprint.

4. K. Bringmann and O. K. Richter, *Zagier-type dualities and lifting maps for harmonic Maass-Jacobi forms*, Advances in Math. **225** (2010), 2298-2315.
5. D. Bump and Y. J. Choie, *Derivatives of modular forms of negative weight*, Pure Appl. Math. Q. **2** (2006), no. 1, 111-133.
6. C. Conley and M. Raum, *Harmonic Maass-Jacobi forms of degree 1 with higher rank indices*, arXiv:1012.289/v1 [math.NT] 13 Dec 2010.
7. M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progress in Mathematics, **55**, Birkhäuser, Boston, Basel and Stuttgart, 1985.
8. Harish-Chandra, *Representations of a semisimple Lie group on a Banach space. I.,* Trans. Amer. Math. Soc. **75** (1953), 185-243.
9. Harish-Chandra, *The characters of semisimple Lie groups,* Trans. Amer. Math. Soc. **83** (1956), 98-163.
10. S. Helgason, *Differential operators on homogeneous spaces,* Acta Math. **102** (1959), 239-299.
11. S. Helgason, *Groups and geometric analysis,* Academic Press, New York (1984).
12. R. Howe, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond,* The Schur lectures (1992) (Tel Aviv), Israel Math. Conf. Proceedings, **vol. 8** (1995), 1-182.
13. M. Itoh, H. Ochiai and J.-H. Yang, Invariant differential operators on Siegel-Jacobi space, in preparation.
14. H. Maass, *Die Differentialgleichungen in der Theorie der Siegelschen Modulfunktionen,* Math. Ann. **126** (1953), 44–68.
15. H. Maass, *Siegel modular forms and Dirichlet series,* Lecture Notes in Math., **vol. 216**, Springer-Verlag, Berlin-Heidelberg-New York (1971).
16. A. Pitale, *Jacobi Maass forms*, Abh. Math. Sem. Univ. Hamburg **79** (2009), 87-111.
17. I. Satake, *Algebraic Structures of Symmetric Domains,* Kano Memorial Lectures 4, Iwanami Shoton, Publishers and Princeton University Press (1980).
18. G. Shimura, *Invariant differential operators on hermitian symmetric spaces,* Ann. Math. **132** (1990), 237-272.
19. C. L. Siegel, *Symplectic Geometry,* Amer. J. Math. **65** (1943), 1-86; Academic Press, New York and London (1964); Gesammelte Abhandlungen, no. 41, vol. II, Springer-Verlag (1966), 274-359.
20. C. L. Siegel, *Topics in Complex Function Theory: Abelian Functions and Modular Functions of Several Variables*, vol. III, Wiley-Interscience, 1973.
21. H. Weyl, *The classical groups: Their invariants and representations,* Princeton Univ. Press, Princeton, New Jersey, second edition (1946).
22. J.-H. Yang, *The Siegel-Jacobi Operator*, Abh. Math. Sem. Univ. Hamburg **63** (1993), 135–146.
23. J.-H. Yang, *Singular Jacobi Forms,* Trans. Amer. Math. Soc. **347 (6)** (1995), 2041-2049.
24. J.-H. Yang, *Construction of vector valued modular forms from Jacobi forms,* Canad. J. Math. **47 (6)** (1995), 1329-1339.
25. J.-H. Yang, *A geometrical theory of Jacobi forms of higher degree,* Proceedings of Symposium on Hodge Theory and Algebraic Geometry (edited by Tadao Oda), Sendai, Japan (1996), 125-147 *or* Kyungpook Math. J. **40 (2)** (2000), 209-237.
26. J.-H. Yang, *The Method of Orbits for Real Lie Groups,* Kyungpook Math. J. **42 (2)** (2002), 199-272.
27. J.-H. Yang, *A note on Maass-Jacobi forms,* Kyungpook Math. J. **43 (4)** (2003), 547–566.
28. J.-H. Yang, *A note on a fundamental domain for Siegel-Jacobi space,* Houston J. Math. **32 (3)** (2006), 701–712.
29. J.-H. Yang, *Invariant metrics and Laplacians on Siegel-Jacobi space,* Journal of Number Theory **127** (2007), 83–102.
30. J.-H. Yang, *A partial Cayley transform for Siegel-Jacobi disk,* J. Korean Math. Soc. **45**, No. 3 (2008), 781-794.

31. J.-H. Yang, *Invariant metrics and Laplacians on Siegel-Jacobi disk,* Chinese Annals of Mathematics, Vol. 31B(1), 2010, 85-100.
32. J. Yang and L. Yin, Derivation of Jacobi forms from connections, arXiv:1301.1156v1 [math.NT] 7 Jan 2013.

[C] Theory of the Heisenberg Group $H_R^{(n,m)}$

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HARMONIC ANALYSIS ON THE QUOTIENT SPACES OF HEISENBERG GROUPS

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A certain nilpotent Lie group plays an important role in the study of the foundations of quantum mechanics ([Wey]) and of the theory of theta series (see [C], [I] and [Wei]). This work shows how theta series are applied to decompose the natural unitary representation of a Heisenberg group.

For any positive integers g and h, we consider the Heisenberg group

$$H_R^{(g,h)} := \{[(\lambda, \mu), \kappa] \mid \lambda, \mu \in R^{(h,g)},\ \kappa \in R^{(h,h)},\ \kappa + \mu\, {}^t\lambda \text{ symmetric}\}$$

endowed with the following multiplication law

$$[(\lambda, \mu), \kappa] \circ [(\lambda', \mu'), \kappa'] = [(\lambda + \lambda', \mu + \mu'), \kappa + \kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda'] .$$

The mapping

$$H_R^{(g,h)} \ni [(\lambda, \mu), \kappa] \longrightarrow \begin{pmatrix} E_g & 0 & 0 & {}^t\mu \\ \lambda & E_h & \mu & \kappa \\ 0 & 0 & E_g & -{}^t\lambda \\ 0 & 0 & 0 & E_h \end{pmatrix}$$

defines an embedding of $H_R^{(g,h)}$ into the symplectic group $Sp(g + h, R)$. We refer to [Z] for the motivation of the study of this Heisenberg group $H_R^{(g,h)}$. $H_Z^{(g,h)}$ denotes the discrete subgroup of $H_R^{(g,h)}$ consisting of integral elements, and $L^2(H_Z^{(g,h)}\backslash H_R^{(g,h)})$ is the L^2-space of the quotient space $H_Z^{(g,h)}\backslash H_R^{(g,h)}$ with respect to the invariant measure

$$d\lambda_{11} \cdots d\lambda_{h,g-1} d\lambda_{hg} d\mu_{11} \cdots d\mu_{h,g-1} d\mu_{hg} d\kappa_{11} d\kappa_{12} \cdots d\kappa_{h-1,h} d\kappa_{hh}.$$

We have the natural unitary representation ρ on $L^2(H_Z^{(g,h)}\backslash H_R^{(g,h)})$ given by

$$\rho([(\lambda', \mu'), \kappa'])\phi([(\lambda, \mu), \kappa]) = \phi([(\lambda, \mu), \kappa] \circ [(\lambda', \mu'), \kappa']).$$

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The Stone-von Neumann theorem says that an irreducible representation ρ of $H_R^{(g,h)}$ is characterized by a real symmetric matrix $c \in R^{(h,h)}$ $(c \neq 0)$ such that

$$\rho_c([(0,0),\kappa]) = \exp\{\pi i \sigma(c\kappa)\}I, \qquad \kappa = {}^t\kappa \in R^{(h,h)},$$

where I denotes the identity mapping of the representation space. If $c = 0$, then it is characterized by a pair $(k, m) \in R^{(h,g)} \times R^{(h,g)}$ such that

$$\rho_{k,m}([(\lambda,\mu),\kappa]) = \exp\{2\pi i \sigma(k\,{}^t\lambda + m\,{}^t\mu)\}I.$$

But only the irreducible representations $\rho_{\mathcal{M}}$ with $\mathcal{M} = {}^t\mathcal{M}$ even integral and $\rho_{k,m}$ $(k, m \in Z^{(h,g)})$ could occur in the right regular representation ρ in $L^2(H_Z^{(g,h)} \backslash H_R^{(g,h)})$.

In this article, we decompose the right regular representation ρ. The real analytic functions defined in (1.5) play an important role in decomposing the right regular representation ρ.

NOTATIONS. We denote Z, R and C the ring of integers, the field of real numbers and the field of complex numbers respectively. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. E_g denotes the identity matrix of degree g. $\sigma(A)$ denotes the trace of a square matrix A.

$$\begin{aligned}
Z_{\geq 0}^{(h,g)} &= \{J = (J_{kl}) \in Z^{(h,g)} \mid J_{kl} \geq 0 \text{ for all } k, l\}, \\
|J| &= \sum_{k,l} J_{kl}, \\
J \pm \varepsilon_{kl} &= (J_{11}, \cdots, J_{kl} \pm 1, \cdots, J_{hg}), \\
(\lambda + N + A)^J &= (\lambda_{11} + N_{11} + A_{11})^{J_{11}} \cdots (\lambda_{hg} + N_{hg} + A_{hg})^{J_{hg}}.
\end{aligned}$$

§1. Theta series

Let H_g be the Siegel upper half plane of degree g. We fix an element $\Omega \in H_g$ once and for all. Let $\mathcal{M}$ be a positive definite, symmetric even integral matrix of degree h. A holomorphic function $f\colon C^{(h,g)} \to C$ satisfying the functional equation

$$f(W + \lambda\Omega + \mu) = \exp\{-\pi i \sigma(\mathcal{M}(\lambda\Omega\,{}^t\lambda + 2\lambda\,{}^tW))\}f(W) \tag{1.1}$$

for all $\lambda, \mu \in Z^{(h,g)}$ is called a *theta series* of *level* $\mathcal{M}$ with respect to Ω. The set $T_{\mathcal{M}}(\Omega)$ of all theta series of level $\mathcal{M}$ with respect to Ω is a vector space of dimension $(\det \mathcal{M})^g$ with a basis consisting of theta series

(1.2)

$$\vartheta^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega, W) := \sum_{N \in Z^{(h,g)}} \exp\{\pi i\sigma\{\mathscr{M}((N+A)\Omega\,{}^t(N+A) + 2W\,{}^t(N+A))\}\},$$

where A runs over a complete system of representatives of the cosets $\mathscr{M}^{-1}Z^{(h,g)}/Z^{(h,g)}$.

DEFINITION 1.1. A function φ: $C^{(h,g)} \times C^{(h,g)} \to C$ is called an *auxiliary theta series* of *level* $\mathscr{M}$ with respect to Ω if it satisfies the following conditions (i) and (ii):

(i) $\varphi(U, W)$ is a polynomial in W whose coefficients are entire functions,

(ii) $\varphi(U + \lambda, W + \lambda\Omega + \mu) = \exp\{-\pi i(\mathscr{M}(\lambda\Omega\,{}^t\lambda + 2\lambda\,{}^tW))\}\varphi(U, W)$ for all $(\lambda, \mu) \in Z^{(h,g)} \times Z^{(h,g)}$.

The space $\Theta_\Omega^{(\mathscr{M})}$ of all auxiliary theta series of level $\mathscr{M}$ with respect to Ω has a basis consisting of the following functions:

(1.3)

$$\vartheta_J^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega \mid \lambda, \mu + \lambda\Omega) := \sum_{N \in Z^{(h,g)}} (\lambda + N + A)^J \times \exp\{\pi i\sigma(\mathscr{M}((N+A)\Omega\,{}^t(N+A) + (\mu + \lambda\Omega)\,{}^t(N+A)))\}.$$

where A (resp. J) runs over the cosets $\mathscr{M}^{-1}Z^{(h,g)}/Z^{(h,g)}$ (resp. $Z_{\geq 0}^{(h,g)}$).

DEFINITION 1.2. A real analytic function φ: $R^{(h,g)} \times R^{(h,g)} \to C$ is called a *mixed theta series* of *level* $\mathscr{M}$ with respect to Ω if φ satisfies the following conditions (1) and (2):

(1) $\varphi(\lambda, \mu)$ is a polynomial in λ whose coefficients are entire functions in complex variables $Z = \mu + \lambda\Omega$;

(2) $\varphi(\lambda + \tilde{\lambda}, \mu + \tilde{\mu}) = \exp\{-\pi i\sigma(\mathscr{M}(\tilde{\lambda}\Omega\,{}^t\tilde{\lambda} + 2(\mu + \lambda\Omega)\,{}^t\tilde{\lambda}))\}\varphi(\lambda, \mu)$ for all $(\tilde{\lambda}, \tilde{\mu}) \in Z^{(h,g)} \times Z^{(h,g)}$.

If $A \in \mathscr{M}^{-1}Z^{(h,g)}/Z^{(h,g)}$ and $J \in Z_{\geq 0}^{(h,g)}$,

(1.4)

$$\vartheta_J^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega \mid \lambda, \mu + \lambda\Omega) := \sum_{N \in Z^{(h,g)}} (\lambda + N + A)^J \times \exp\{\pi i\sigma(\mathscr{M}((N+A)\Omega\,{}^t(N+A) + 2(\mu + \lambda\Omega)\,{}^t(N+A)))\}$$

is a mixed theta series of level $\mathscr{M}$.

Now for a positive definite symmetric even integral matrix $\mathscr{M}$ of degree h, we define a function on $H_R^{(g,h)}$.

(1.5)

$$\Phi_J^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega \mid [(\lambda, \mu), \kappa]) := \exp\{\pi i\sigma(\mathscr{M}(\kappa - \lambda\,{}^t\mu))\} \sum_{N \in Z^{(h,g)}} (\lambda + N + A)^J \times \exp\{\pi i\sigma(\mathscr{M}(\lambda + N + A)\Omega\,{}^t(\lambda + N + A) + 2(\lambda + N + A)\,{}^t\mu))\},$$

where $A \in \mathcal{M}^{-1}Z^{(h,g)}/Z^{(h,g)}$.

PROPOSITION 1.3.

$$(1.6)\qquad \Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega \mid [(\lambda, \mu), \kappa]) = \exp\{2\pi i\sigma(\mathcal{M}\mu\, {}^tA)\}\Phi_J^{(\mathcal{M})}\begin{bmatrix} 0 \\ 0 \end{bmatrix}(\Omega \mid [(\lambda, \mu), \kappa] \circ [(A, 0), 0]).$$

$$(1.7)\qquad \Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega \mid [(\tilde\lambda, \tilde\mu), \tilde\kappa] \circ [(\lambda, \mu), \kappa]) = \Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega \mid [(\lambda, \mu), \kappa]).$$
$$([(\tilde\lambda, \tilde\mu), \tilde\kappa] \in H_Z^{(h,g)},\ [(\lambda, \mu), \kappa] \in H_R^{(h,g)},\ A \in \mathcal{M}^{-1}Z^{(h,g)}/Z^{(h,g)}).$$

Proof.

$$\begin{aligned} &\Phi_J^{(\mathcal{M})}\begin{bmatrix} 0 \\ 0 \end{bmatrix}(\Omega \mid [(\lambda + A, \mu), \kappa - \mu\, {}^tA]) \\ &= \exp\{\pi i\sigma(\mathcal{M}(\kappa - \mu\, {}^tA - (\lambda + A)\, {}^t\mu))\} \sum_{N \in Z^{(h,g)}} (\lambda + A + N)^J \\ &\quad \times \exp\{\pi i\sigma(\mathcal{M}((\lambda + A + N)\Omega\, {}^t(\lambda + N + A) + 2(\lambda + N + A)\, {}^t\mu))\} \\ &= \exp\{-2\pi i\sigma(\mathcal{M}\mu\, {}^tA)\}\Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega \mid [(\lambda, \mu), \kappa]). \end{aligned}$$

On the other hand, if $[(\tilde\lambda, \tilde\mu), \tilde\kappa] \in H_Z^{(h,g)}$,

$$\begin{aligned} &\Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega \mid [(\tilde\lambda, \tilde\mu), \tilde\kappa] \circ [(\lambda, \mu), \kappa]) \\ &= \exp\{\pi i\sigma(\mathcal{M}(\tilde\kappa + \kappa + \tilde\lambda\, {}^t\mu - \tilde\mu\, {}^t\lambda - (\tilde\lambda + \lambda)\, {}^t(\tilde\mu + \mu)))\} \sum_{N \in Z^{(h,g)}} (\tilde\lambda + \lambda + N + A)^J \\ &\quad \times \exp\{\pi i\sigma((\tilde\lambda + \lambda + N + A)\Omega\, {}^t(\tilde\lambda + \lambda + N + A) + 2(\tilde\lambda + \lambda + N + A)\, {}^t(\tilde\mu + \mu))\} \\ &= \Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega \mid [(\lambda, \mu), \kappa]). \end{aligned}$$

Here in the last equality we used the facts that $\sigma(\mathcal{M}(\tilde\kappa - {}^t\tilde\lambda\tilde\kappa)) \in 2Z$ and $\sigma(\mathcal{M}A\, {}^t\tilde\mu) \in Z$. q.e.d.

Remark. Proposition 1.3 implies that $\Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega \mid [(\lambda, \mu), \kappa])\,(J \in Z_{\geq 0}^{(h,g)})$ are real analytic functions on the quotient space $H_Z^{(g,h)} \backslash H_R^{(g,h)}$.

The following matrices

$$X_{kl}^0 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{kl}^0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad 1 \leq k \leq l \leq h,$$

$$\hat{X}_{ij} := \begin{pmatrix} 0 & 0 & 0 & {}^tE_{ij} \\ 0 & 0 & E_{ij} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad 1 \le i \le h, \quad 1 \le j \le g,$$

$$X_{ij} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ E_{ij} & 0 & 0 & 0 \\ 0 & 0 & 0 & -{}^tE_{ij} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad 1 \le i \le h, \quad 1 \le j \le g$$

form a basis of the Lie algebra $\mathscr{H}_R^{(g,h)}$ of the Heisenberg group $H_R^{(g,h)}$. Here E_{kl}^0 $(k \ne l)$ and $h \times h$ symmetric matrix with entry 1/2 where the k-th (or l-th) row and the l-th (or k-th) column meet, all other entries 0, E_{kk}^0 is an $h \times h$ diagonal matrice with the k-th diagonal entry 1 and all other entries 0 and E_{ij} is an $h \times g$ matrix with entry 1 where the i-th row and the j-th column meet, all other entries 0. By an easy calculation, we see that the following vector fields

$$D_{kl}^0 = \frac{\partial}{\partial \kappa_{kl}}, \qquad 1 \le k \le l \le h,$$

$$D_{mp} = \frac{\partial}{\partial \lambda_{mp}} - \left(\sum_{k=1}^{m} \mu_{kp} \frac{\partial}{\partial \kappa_{km}} + \sum_{k=m+1}^{h} \mu_{kp} \frac{\partial}{\partial \kappa_{mk}} \right),$$

$$\hat{D}_{mp} = \frac{\partial}{\partial \mu_{mp}} + \left(\sum_{k=1}^{m} \lambda_{kp} \frac{\partial}{\partial \kappa_{km}} + \sum_{k=m+1}^{h} \lambda_{kp} \frac{\partial}{\partial \kappa_{mk}} \right),$$

form a basis for the Lie algebra of left invariant vector fields on $H_R^{(g,h)}$.

THEOREM 1.

$$D_{kl}^0 \Phi_J^{(\mathcal{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega \,|\, [(\lambda, \mu), \kappa]) = \pi i \mathcal{M}_{kl} \Phi_J^{(\mathcal{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega \,|\, [(\lambda, \mu), \kappa]), \tag{1.8}$$

$$\hat{D}_{mp} \Phi_J^{(\mathcal{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega \,|\, [(\lambda, \mu), \kappa]) = 2\pi i \sum_{l=1}^{h} \mathcal{M}_{ml} \Phi_{J+\epsilon_{lp}}^{(\mathcal{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega \,|\, [(\lambda, \mu), \kappa]), \tag{1.9}$$

$$\begin{aligned} D_{mp} \Phi_J^{(\mathcal{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega \,|\, [(\lambda, \mu), \kappa]) &= 2\pi i \sum_{l=1}^{h} \sum_{q=1}^{g} \mathcal{M}_{lm} \Omega_{pq} \Phi_{J+\epsilon_{lq}}^{(\mathcal{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega \,|\, [(\lambda, \mu), \kappa]) \\ &\quad + J_{mp} \Phi^{(\mathcal{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega \,|\, [(\lambda, \mu), \kappa]). \end{aligned} \tag{1.10}$$

$$(1 \le k \le l \le h, \; 1 \le m \le h, \; 1 \le p \le g)$$

Proof. (1.8) follows immediately from the definition of $\Phi_J^{(\mathcal{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega \,|\, [(\lambda, \mu), \kappa])$.

$$\hat{D}_{mp}\Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda, \mu), \kappa])$$

$$= -\pi i \sum_{l=1}^{h} \mathcal{M}_{ml}\lambda_{lp}\Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda, \mu), \kappa])$$

$$+ 2\pi i\{\pi i\sigma(\mathcal{M}(\kappa - \lambda\,{}^t\mu))\} \sum_{N\in Z^{(h,g)}} (\lambda + N + A)^J \sum_{l=1}^{h} \mathcal{M}_{ml}(\lambda + N + A)_{lp}$$

$$\times \exp\{\pi i\sigma(\mathcal{M}((\lambda + N + A)\Omega\,{}^t(\lambda + N + A) + 2(\lambda + N + A)\,{}^t\mu))\}$$

$$+ \pi i \sum_{l=1}^{h} \mathcal{M}_{ml}\lambda_{lp}\Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda, \mu), \kappa])$$

$$= 2\pi i \sum_{l=1}^{h} \mathcal{M}_{ml}\Phi_{J+\varepsilon_{lp}}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda, \mu), \kappa])\,.$$

We compute

$$\frac{\partial}{\partial\lambda_{mp}}\Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda, \mu), \kappa])$$

$$= -\pi i \sum_{k=1}^{h} \mathcal{M}_{km}\mu_{kp}\Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda, \mu), \kappa])$$

$$+ 2\pi i \sum_{k=1}^{h}\sum_{q=1}^{g} \mathcal{M}_{km}\Omega_{pq}\Phi_{J+\varepsilon_{kq}}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda, \mu), \kappa])$$

$$+ J_{mp}\Phi_{J-\varepsilon_{mp}}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda, \mu), \kappa])$$

$$+ 2\pi i \sum_{k=1}^{h} \mathcal{M}_{km}\mu_{kp}\Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda, \mu), \kappa])\,.$$

Therefore we obtain (1.8) and (1.10). q.e.d.

COROLLARY 1.4.

$$\left(D_{mp} - \sum_{q=1}^{g} \Omega_{pq}\hat{D}_{mq}\right)\Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda, \mu), \kappa]) = J_{mp}\Phi_{J-\varepsilon_{mp}}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda, \mu), \kappa])\,.$$

Let $H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$ *be the completion of the vector space spanned by* $\Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda, \mu), \kappa])$ $(J \in Z_{\geq}^{(h,g)})$ *and let* $\overline{H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}}$ *be the complex conjugate of* $H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$.

THEOREM 2. $H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$ *and* $\overline{H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}}$ *are irreducible invariant subspaces of* $L^2(H_Z^{(h,g)}\backslash H_R^{(h,g)})$ *with respect to the right regular representation* ρ. *In addition, we have*

$$H_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix} = \exp\{2\pi i\sigma(\mathcal{M}\mu\,{}^t A)\}H_{\Omega}^{(\mathcal{M})}\begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$\rho([(0,0),\tilde{\kappa}])\phi = \exp\{\pi i\sigma(\mathcal{M}\tilde{\kappa})\}\phi \qquad \left(\phi \in H_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}\right),$$

$$\rho([(0,0),\tilde{\kappa}])\bar{\phi} = \exp\{-\pi i\sigma(\mathcal{M}\tilde{\kappa})\}\bar{\phi} \qquad \left(\bar{\phi} \in \overline{H_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}}\right).$$

Proof. It follows from Theorem 1, Proposition 1.3 and the definition of $\Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])$. q.e.d.

§ 2. Proof of the Main Theorem

We fix an element $\Omega \in H_g$ once and for all. We introduce a system of complex coordinates with respect to Ω:

$$Z = \mu + \lambda\Omega\,, \quad \bar{Z} = \mu + \lambda\bar{\Omega}\,, \quad \lambda, \mu \text{ real}\,. \tag{2.1}$$

We set

$$dZ = \begin{pmatrix} dZ_{11} & \cdots & dZ_{1g} \\ \vdots & \ddots & \vdots \\ dZ_{h1} & \cdots & dZ_{hg} \end{pmatrix}, \qquad \frac{\partial}{\partial Z} = \begin{pmatrix} \dfrac{\partial}{\partial Z_{11}} & \cdots & \dfrac{\partial}{\partial Z_{h1}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial}{\partial Z_{1g}} & \cdots & \dfrac{\partial}{\partial Z_{hg}} \end{pmatrix}.$$

Then an easy computation yields

$$\frac{\partial}{\partial\lambda} = \Omega\frac{\partial}{\partial Z} + \bar{\Omega}\frac{\partial}{\partial\bar{Z}}\,,$$

$$\frac{\partial}{\partial\mu} = \frac{\partial}{\partial Z} + \frac{\partial}{\partial\bar{Z}}\,.$$

Thus we obtain the following

$$\frac{\partial}{\partial\bar{Z}} = \frac{i}{2}(\operatorname{Im}\Omega)^{-1}\left(\frac{\partial}{\partial\lambda} - \Omega\frac{\partial}{\partial\mu}\right). \tag{2.2}$$

LEMMA 2.1.

$$\Phi_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])$$
$$= \exp\{\pi i\sigma(\mathcal{M}(\lambda\Omega\,{}^t\lambda + \lambda\,{}^t\mu + \kappa))\}\vartheta_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,\lambda, \mu + \lambda\Omega)\,.$$

Proof. It follows immediately from (1.4) and (1.5).

LEMMA 2.2. *Let $\Phi([(\lambda,\mu),\kappa])$ be a real analytic function on $H_Z^{(g,h)}\backslash H_R^{(g,h)}$ such that*

i) $\exp\{-\pi i\sigma(\mathcal{M}\kappa)\}\Phi([(\lambda,\mu),\kappa])$ *is independent of* κ,

ii) $(D_{mp}-\sum_{q=1}^{g}\Omega_{pq}\hat{D}_{mq})\Phi=0$ *for all* $1\le m\le h$ *and* $1\le p\le g$, *where* $\mathcal{M}$ *is a positive definite symmetric even integral matrix of degree h. Let*

$$\Psi(\lambda,\mu)=\exp\{-\pi i\sigma(\mathcal{M}(\lambda\Omega\,{}^t\lambda+\lambda\,{}^t\lambda+\kappa))\}\Phi([(\lambda,\mu),\kappa])\,. \tag{2.3}$$

Then $\Psi(\lambda,\mu)$ is a mixed theta function of level $\mathcal{M}$ in $Z=\mu+\lambda\Omega$ with respect to Ω.

Proof. By the assumption (i), we have

$$\begin{aligned}
&\Psi(\lambda+\tilde{\lambda},\mu+\tilde{\mu})\\
&\quad=\exp\{-\pi i\sigma(\mathcal{M}((\lambda+\tilde{\lambda})\Omega\,{}^t(\lambda+\tilde{\lambda})+(\lambda+\tilde{\lambda})\,{}^t(\mu+\tilde{\mu})+\kappa+\tilde{\kappa}+\tilde{\lambda}\,{}^t\mu-\tilde{\mu}\,{}^t\lambda))\}\\
&\qquad\qquad\Phi([(\tilde{\lambda},\tilde{\mu}),\tilde{\kappa}]\circ[(\lambda,\mu),\kappa])\\
&\qquad\qquad\quad=\exp\{-\pi i\sigma(\mathcal{M}(\tilde{\lambda}\Omega\,{}^t\tilde{\lambda}+2(\mu+\lambda\Omega)\,{}^t\tilde{\lambda}))\}\Psi(\lambda,\mu)\,,
\end{aligned}$$

where $[(\tilde{\lambda},\tilde{\mu}),\tilde{\kappa}]\in H_Z^{(g,h)}$. In the last equality, we used the facts that $\sigma(\mathcal{M}(\tilde{\kappa}+\tilde{\lambda}\,{}^t\tilde{\mu}))\in 2Z$ because $\tilde{\kappa}+\tilde{\mu}\,{}^t\tilde{\lambda}$ is symmetric. This implies that $\Psi(\lambda,\mu)$ satisfies the condition (2) in Definition 1.2. Now we must show that $\Psi(\lambda,\mu)$ is holomorphic in $Z=\mu+\lambda\Omega$, that is,

$$\frac{\partial\Psi}{\partial\bar{Z}}=0\,,\qquad Z=\mu+\lambda\Omega\,. \tag{2.4}$$

By (2.2) the equation (2.4) is equivalent to the equation

$$\Big(\frac{\partial}{\partial\lambda_{mp}}-\sum_{q=1}^{g}\Omega_{pq}\frac{\partial}{\partial\mu_{mq}}\Big)\Psi(\lambda,\mu)=0\,,\quad 1\le m\le h\,,\quad 1\le p\le g\,. \tag{2.5}$$

But according to (1.9) and (1.10), we have

$$\frac{\partial}{\partial\lambda_{mp}}-\sum_{q=1}^{g}\Omega_{pq}\frac{\partial}{\partial\mu_{mq}}=D_{mp}-\sum_{q=1}^{g}\Omega_{pq}\hat{D}_{mq}+P\,,$$

where

$$\begin{aligned}
P=\sum_{k=1}^{m}\mu_{kp}D_{km}^{0}+\sum_{k=m+1}^{h}\mu_{kp}D_{mk}^{0}-\sum_{k=1}^{m}\sum_{q=1}^{g}\Omega_{pq}\lambda_{kq}D_{km}^{0}\\
-\sum_{k=m+1}^{h}\sum_{q=1}^{g}\Omega_{pq}\lambda_{kq}D_{mk}^{0}\,.
\end{aligned}$$

We observe that $P\cdot\Psi(\lambda,\mu)=0$ because $\Psi(\lambda,\mu)$ is independent of κ by the assumption (i). We let

$$f([(\lambda, \mu), \kappa]) = \exp\{-\pi i\sigma(\mathscr{M}(\lambda\Omega\,{}^t\lambda + \lambda\,{}^t\mu + \kappa))\}\,.$$

Then $\Psi(\lambda, \mu) = f([(\lambda, \mu), \kappa])\Phi([(\lambda, \mu), \kappa])$. Then in order to show that $\Psi(\lambda, \mu)$ is holomorphic in the complex variables $Z = \mu + \lambda\Omega$ with respect to Ω, by the assumption (ii), it suffices to show the following:

$$\left(D_{mp} - \sum_{q=1}^{g} \Omega_{pq}\hat{D}_{mq}\right) f([(\lambda, \mu), \kappa]) = 0\,. \tag{2.6}$$

By an easy computation, we obtain (2.6). This completes the proof of Lemma 2.2. q.e.d.

The Stone-von Neumann theorem says that an irreducible representation ρ_c of $H_R^{(g,h)}$ is characterized by a real symmetric matrix $c \in R^{(h,h)}$ $(c \neq 0)$ such that

$$\rho_c([(\lambda, \mu), \kappa]) = \exp\{\pi i\sigma(c\kappa)\}I\,, \qquad \kappa = {}^t\kappa \in R^{(h,h)}\,, \tag{2.7}$$

where I denotes the identity map of the representation space. If $c = 0$, it is characterized by a pair $(k, m) \in R^{(h,g)} \times R^{(h,g)}$ such that

$$\rho_{k,m}([(\lambda, \mu), \kappa]) = \exp\{2\pi i\sigma(k\,{}^t\lambda + m\,{}^t\mu)\}I\,. \tag{2.8}$$

If $\Phi \in L^2(H_Z^{(g,h)} \backslash H_R^{(g,h)})$ and $\tilde{\kappa} = {}^t\tilde{\kappa} \in Z^{(h,h)}$, then

$$\begin{aligned}\Phi([(\lambda, \mu), \kappa]) &= \Phi([(0, 0), \tilde{\kappa}] \circ [(\lambda, \mu), \kappa]) \\ &= \Phi([(\lambda, \mu), \kappa] \circ [(0, 0), \tilde{\kappa}]) \\ &= \rho_c([(0, 0), \tilde{\kappa}])\Phi([(\lambda, \mu), \kappa]) \\ &= \exp\{\pi i\sigma(c\tilde{\kappa})\}\Phi([(\lambda, \mu), \kappa])\,.\end{aligned}$$

Thus if $c \neq 0$, $\sigma(c\tilde{\kappa}) \in 2Z$ for all $\tilde{\kappa} = {}^t\tilde{\kappa} \in Z^{(h,h)}$. It means that ${}^tc = c = (c_{ij})$ must be even integral, that is, all diagonal elements c_{ii} $(1 \leq i \leq h)$ are even integers and all c_{ij} $(i \neq j)$ are integers. If $c = 0$, $\sigma(k\,{}^t\lambda + m\,{}^t\mu) \in Z$ for all $\lambda, \mu \in Z^{(h,g)}$ and hence $k, m \in Z^{(h,g)}$. Therefore only the irreducible representation $\rho_{\mathscr{M}}$ with $\mathscr{M} = {}^t\mathscr{M}$ even integral and $\rho_{k,m}$ $(k, m \in Z^{(h,g)})$ could occur in the right regular representation ρ in $L^2(H_Z^{(g,h)} \backslash H_R^{(g,h)})$.

Now we prove

MAIN THEOREM. *Let $\mathscr{N} \neq 0$ be an even integral matrix of degree h which is neither positive nor negative definite. Let $R(\mathscr{N})$ be the sum of irreducible representations $\rho_{\mathscr{N}}$ which occur in the right regular representation ρ of $H_R^{(g,h)}$. Let $H_\Omega^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$ be defined in Theorem 2 for a positive definite even integral matrix $\mathscr{M} > 0$. Then the decomposition of the right regular representation ρ is given by*

$$L^2(H_Z^{(g,h)}\backslash H_R^{(g,h)}) = \bigoplus_{\mathcal{M},A} H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix} \oplus \overline{\Big(\bigoplus_{\mathcal{M},A} H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}\Big)} \oplus \Big(\bigoplus_{\mathcal{N}} R(\mathcal{N})\Big)$$
$$\oplus \Big(\bigoplus_{(k,m)\in Z^{(h,g)}} C \exp\{2\pi i\sigma(k\,{}^t\lambda + m\,{}^t\mu)\}\Big).$$

where $\mathcal{M}$ (resp. $\mathcal{N}$) *runs over the set of all positive definite symmetric, even integral matrices of degree h* (resp. *the set of all even integral nonzero matrices of degree h which are neither positive nor negative definite*) *and A runs over a complete system of representatives of the cosets* $\mathcal{M}^{-1}Z^{(h,g)}/Z^{(h,g)}$. $H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$ *and* $\overline{H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}}$ *are irreducible invariant subspaces of* $L^2(H_Z^{(g,h)}\backslash H_R^{(g,h)})$ *such that*

$$\rho([(0,0),\tilde{\kappa}])\phi([(\lambda,\mu),\kappa]) = \exp\{\pi i\sigma(\mathcal{M}\tilde{\kappa})\}\phi([(\lambda,\mu),\kappa]),$$
$$\rho([(0,0),\tilde{\kappa}])\overline{\phi([(\lambda,\mu),\kappa])} = \exp\{-\pi i\sigma(\mathcal{M}\tilde{\kappa})\}\overline{\phi([(\lambda,\mu),\kappa])}$$

for all $\phi \in H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$. *And we have*

$$H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix} = \exp\{2\pi i\sigma(\mathcal{M}\mu\,{}^tA)\}H_\Omega^{(\mathcal{M})}\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This result generalizes that of H. Morikawa ([M]).

Proof. Let $\mathcal{A}$ be the space of real analytic functions on $L^2(H_Z^{(g,h)}\backslash H_R^{(g,h)})$. Since $\mathcal{A}$ is dense in $L^2(H_Z^{(g,h)}\backslash H_R^{(g,h)})$ and $\mathcal{A}$ is invariant under ρ, it suffices to decompose $\mathcal{A}$. Let W be an irreducible invariant subspace of $\mathcal{A}$ such that $\rho([(0,0),\tilde{\kappa}])w = \exp\{2\pi i\sigma(\mathcal{M}\tilde{\kappa})\}w$ for all $w \in W$, where $\mathcal{M} = {}^t\mathcal{M}$ is a positive definite even integral matrix of degree h. Then W is isomorphic to $H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix} \cap \mathcal{A}$ for some $A \in \mathcal{M}^{-1}Z^{(h,g)}/Z^{(h,g)}$ and $\Omega \in H_g$. Since $H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix} \cap \mathcal{A}$ contains an element $\Phi_0^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])$ (see Corollary 1.4) satisfying

$$\Big(D_{mp} - \sum_{q=1}^{g} \Omega_{pq}\hat{D}_{mq}\Big)\Phi_0^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa]) = 0$$

for all $1 \le m \le h$, $1 \le p \le g$, there exists an element $\Phi_0([(\lambda,\mu),\kappa])$ in W such that

$$\Big(D_{mp} - \sum_{q=1}^{g} \Omega_{pq}\hat{D}_{mq}\Big)\Phi_0([(\lambda,\mu),\kappa]) = 0$$

for all $1 \le m \le h$, $1 \le p \le g$. On the other hand, we have

$$\begin{aligned}\Phi_0([(\lambda,\mu),\kappa]) &= \rho([(0,0),\kappa])\Phi_0([(\lambda,\mu),0])\\ &= \exp\{\pi i\sigma(\mathcal{M}\kappa)\}\Phi_0([(\lambda,\mu),0])\,.\end{aligned}$$

Therefore $\Phi_0([(\lambda,\mu),\kappa])$ satisfies the conditions of Lemma 2. Thus we have

$$\begin{aligned}\Phi_0([(\lambda,\mu),\kappa]) &= \exp\{\pi i\sigma(\mathcal{M}(\lambda\Omega\,{}^t\lambda+\lambda\,{}^t\mu+\kappa))\}\sum_{A,J}\alpha_{AJ}\vartheta_J^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}(\Omega\,|\,\lambda,\mu+\lambda\Omega)\\ &= \sum_{A,J}\alpha_{AJ}\Phi_J^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa]) \qquad \text{(by Lemma 2.1)}\,,\end{aligned}$$

where A (resp. J) runs over $\mathcal{M}^{-1}Z^{(h,g)}/Z^{(h,g)}$ (resp. $Z_{\geq 0}^{(h,g)}$). Hence $\Phi_0\in\bigoplus_A H_\Omega^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}$. By the way, since W is spanned by $D_{kl}^0\Phi_0$, $D_{mp}\Phi_0$ and $\hat{D}_{mp}\Phi_0$, we have $W\subset\bigoplus_A H_\Omega^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}$. So $W=H_\Omega^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}\cap\mathcal{A}$ for some $A\in\mathcal{M}^{-1}Z^{(h,g)}/Z^{(h,g)}$. Similarly, $\overline{W}=\overline{H_\Omega^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}}\cap\mathcal{A}$. Clearly for each $(k,m)\in Z^{(h,g)}\times Z^{(h,g)}$,

$$W_{k,m} := C\exp\{2\pi i(k\,{}^t\lambda+m\,{}^t\mu)\}$$

is a one dimensional irreducible invariant subspace of $L^2(H_Z^{(h,g)}\backslash H_R^{(g,h)})$. The latter part of the above theorem is the restatement of Theorem 2. This completes the main theorem. q.e.d.

COROLLARY. *For even integral matrix $\mathcal{M}={}^t\mathcal{M}>0$ of degree h, the multiplicity $m_{\mathcal{M}}$ of $\rho_{\mathcal{M}}$ in ρ is given by*

$$m_{\mathcal{M}} = (\det\mathcal{M})^g\,.$$

CONJECTURE. *For any even integral matrix $\mathcal{N}\neq 0$ of degree h which is neither positive nor negative definite, the multiplicity $m_{\mathcal{N}}$ of $\rho_{\mathcal{N}}$ in ρ is a zero, that is, $R(\mathcal{N})$ vanishes.*

§ 3. Schrödinger representations

Let $\Omega\in H_g$ and let $\mathcal{M}={}^t\mathcal{M}$ be a positive definite even integral matrix of degree h. We set $\Omega=\Omega_1+i\Omega_2$ ($\Omega_1,\Omega_2\in R^{(g,g)}$). Let $L^2(R^{(h,g)},\mu_{\Omega_2}^{(\mathcal{M})})$ be the L^2-space of $R^{(h,g)}$ with respect to the measure

$$\mu_{\Omega_2}^{(\mathcal{M})}(d\xi) = \exp\{-2\pi\sigma(\mathcal{M}\xi\Omega_2\,{}^t\xi)\}d\xi\,.$$

It is easy to show that the transformation $f(\xi)\mapsto\exp\{\pi i\sigma(\mathcal{M}\xi\Omega_2\,{}^t\xi)\}f(\xi)$ of $L^2(R^{(h,g)},\mu_{\Omega_2}^{(\mathcal{M})})$ into $L^2(R^{(h,g)},d\xi)$ is an isomorphism. Since the set $\{\xi^J\,|\,J\in Z_{\geq 0}^{(h,g)}\}$ is a basis of $L^2(R^{(h,g)},\mu_{\Omega_2}^{(\mathcal{M})})$, the set $\{\exp(\pi i\sigma(\mathcal{M}\xi\Omega\,{}^t\xi))\xi^J\,|\,J\in Z_{\geq 0}^{(h,g)}\}$ is a basis of $L^2(R^{(h,g)},d\xi)$.

LEMMA 3.1.

$$\left\langle \Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa]),\ \Phi_K^{(\tilde{\mathcal{M}})}\begin{bmatrix} \tilde{A} \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])\right\rangle$$

$$= \int_{H_Z^{(g,h)}\backslash H_R^{(g,h)}} \Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])\cdot\overline{\Phi_K^{(\tilde{\mathcal{M}})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[\lambda,\mu),\kappa])}\,d\lambda d\mu d\kappa$$

$$= \begin{cases} \int_{R^{(h,g)}} y^{J+K}\exp\{-2\pi\sigma(\mathcal{M}y\Omega_2\,{}^t y)\}dy & \text{if } \mathcal{M}=\tilde{\mathcal{M}},\ A\equiv\tilde{A}\ (\mathrm{mod}\,\mathcal{M}), \\ 0, & \text{otherwise}. \end{cases}$$

It is easy to prove the above lemma and so we omit its proof. According to the above argument and Lemma 3.1, we obtain the following:

LEMMA 3.2. *The transformation of* $L^2(R^{(h,g)}, \mu_{\Omega_2}^{(\mathcal{M})})$ *onto* $H_\Omega^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$ *given by*

$$\xi^J \longmapsto \Phi_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa]),\qquad J\in Z_{\geq 0}^{(h,g)} \tag{3.1}$$

is an isomorphism of Hilbert spaces.

Now we define a unitary representation of $H_R^{(h,g)}$ on $L^2(R^{(h,g)}, d\xi)$ by

$$U_{\mathcal{M}}([(\lambda,\mu),\kappa])f(\xi) = \exp\{-\pi i\sigma(\mathcal{M}(\kappa+\mu\,{}^t\lambda+2\mu\,{}^t\xi))\}f(\xi+\lambda), \tag{3.2}$$

where $[(\lambda,\mu),\kappa]\in H_R^{(g,h)}$ and $f\in L^2(R^{(h,g)}, d\xi)$. $U_{\mathcal{M}}$ is called the *Schrödinger representation* of $H_R^{(h,g)}$ *of index* $\mathcal{M}$.

PROPOSITION 3.3. *If we set* $f_J(\xi)=\exp\{\pi i\sigma(\mathcal{M}\xi\Omega\,{}^t\xi)\}\xi^J$ $(J\in Z_{\geq 0}^{(h,g)})$, *we have*

$$dU_{\mathcal{M}}(D_{kl}^0)f_J(\xi) = -\pi i\mathcal{M}_{kl}f_J(\xi),\qquad 1\leq k\leq l\leq h. \tag{3.3}$$

$$dU_{\mathcal{M}}(D_{mp})f_J(\xi) = 2\pi i\sum_{l=1}^{h}\sum_{q=1}^{g}\mathcal{M}_{ml}\Omega_{pq}f_{J+\varepsilon_{lq}}(\xi) + J_{mp}f_{J-\varepsilon_{mp}}(\xi). \tag{3.4}$$

$$dU_{\mathcal{M}}(\hat{D}_{mp})f_J(\xi) = -\pi i\sum_{l=1}^{h}\mathcal{M}_{ml}f_{J+\varepsilon_{lp}}(\xi). \tag{3.4}$$

Proof.

$$\begin{aligned} dU_{\mathcal{M}}(D_{kl}^0)f_J(\xi) &= \frac{d}{dt}\Big|_{t=0} U_{\mathcal{M}}(\exp(tX_{kl}^0))f_J(\xi) \\ &= \frac{d}{dt}\Big|_{t=0} U_{\mathcal{M}}([(0,0),tE_{kl}^0])_J(\xi) \\ &= \lim_{t\to 0}\frac{\exp\{-\pi i\sigma(t\mathcal{M}E_{kl}^0)\}-I}{t}f_J(\xi) \\ &= -\pi i\mathcal{M}_{kl}f_J(\xi). \end{aligned}$$

$$\begin{aligned} dU_{\mathscr{M}}(D_{mp})f_J(\xi) &= \frac{d}{dt}\Big|_{t=0} U_{\mathscr{M}}(\exp(tX_{mp}))f_J(\xi) \\ &= \frac{d}{dt}\Big|_{t=0} U_{\mathscr{M}}([(tE_{mp}, 0), 0])f_J(\xi) \\ &= \frac{d}{dt}\Big|_{t=0} \exp\{\pi i\sigma(\mathscr{M}(\xi + {}^tE_{mp})\Omega\, {}^t(\xi + tE_{mp}))\}(\xi + tE_{mp})^J \\ &= 2\pi i \sum_{l=1}^{h}\sum_{q=1}^{g} \mathscr{M}_{ml}\Omega_{pq} f_{J+\varepsilon_{lq}}(\xi) + J_{mp} f_{J-\varepsilon_{mp}}(\xi)\,. \end{aligned}$$

Finally,

$$\begin{aligned} dU_{\mathscr{M}}(\hat{D}_{mp})f_J(\xi) &= \frac{d}{dt}\Big|_{t=0} U_{\mathscr{M}}(\exp(t\hat{X}_{mp}))f_J(\xi) \\ &= \frac{d}{dt}\Big|_{t=0} U_{\mathscr{M}}([(0, tE_{mp}), 0])f_J(\xi) \\ &= \lim_{t\to 0} \frac{\exp\{-2\pi i\sigma(t\mathscr{M}E_{mp}\,{}^t\xi)\} - I}{t} f_J(\xi) \\ &= -\pi i \sum_{l=1}^{h} \mathscr{M}_{ml} f_{J+\varepsilon_{lp}}(\xi)\,. \end{aligned}$$

q.e.d.

THEOREM 3. *Let* $\Phi_{\Omega}^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$ *be the transform of* $L^2(R^{(h,g)}, d\xi)$ *onto* $H_{\Omega}^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$ *defined by*

$$\Phi_{\Omega}^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\exp(\pi i\sigma(\mathscr{M}\xi\Omega\,{}^t\xi))\xi^J)([(\lambda, \mu), \kappa]) = \Phi_J^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda, \mu), \kappa])\,, \qquad J \in Z_{\geq 0}^{(h,g)}\,. \tag{3.6}$$

Then $\Phi_{\Omega}^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$ *is an isomorphism of the Hilbert space* $L^2(R^{(h,g)}, d\xi)$ *onto the Hilbert space* $H_{\Omega}^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$ *such that*

$$\hat{\rho}([(\lambda, \mu), \kappa]) \circ \Phi_{\Omega}^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix} = \Phi_{\Omega}^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix} \circ U_{\mathscr{M}}([(\lambda, \mu), \kappa])\,, \tag{3.7}$$

$$\Phi_{\Omega}^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix} = \exp\{2\pi i\sigma(\mathscr{M}A\,{}^t\mu)\}\rho([(A, 0), 0])\Phi_{\Omega}^{(\mathscr{M})}\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{3.8}$$

where $\hat{\rho}$ *is the unitary representation of* $H_R^{(g,h)}$ *on* $H_{\Omega}^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$ *defined by*

$$\hat{\rho}([(\lambda, \mu), \kappa])\phi = \rho([(\lambda, -\mu), -\kappa])\phi\,, \qquad \phi \in H_{\Omega}^{(\mathscr{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}.$$

Proof. For brevity, we set $f_J(\xi) = \exp\{\pi i\sigma(\mathcal{M}\xi\Omega\,{}^t\xi)\}\xi^J$ $(J \in Z_{\geq 0}^{(h,g)})$. Using Proposition 3.3, we obtain

$$\begin{aligned}
&\Phi_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(dU_{\mathcal{M}}(-\,D_{kl}^0)(f_J(\xi)))([(\lambda,\mu),\kappa]) \\
&= \pi i \mathcal{M}_{kl}\Phi_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(f_J(\xi))([(\lambda,\mu),\kappa]) \\
&= \pi i \mathcal{M}_{kl}\Phi_{J}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa]) \\
&= d\rho(D_{kl}^0)\left\{\Phi_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(f_J(\xi))[(\lambda,\mu),\kappa]\right\}.
\end{aligned}$$

$$\begin{aligned}
&\Phi_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(dU_{\mathcal{M}}(D_{mp})(f_J(\xi)))([(\lambda,\mu),\kappa]) \\
&= 2\pi i \sum_{l=1}^{h}\sum_{q=1}^{g} \mathcal{M}_{ml}\Omega_{pq}\Phi_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(f_{J+\varepsilon_{lq}}(\xi))([(\lambda,\mu),\kappa]) \\
&\quad + J_{mp}\Phi_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(f_{J-\varepsilon_{mp}}(\xi))([(\lambda,\mu),\kappa]) \\
&= 2\pi i \sum_{l=1}^{h}\sum_{q=1}^{g} \mathcal{M}_{ml}\Omega_{pq}\Phi_{J+\varepsilon_{lq}}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa]) \\
&\quad + J_{mp}\Phi_{J-\varepsilon_{mp}}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa]) \\
&= d\rho(D_{mp})\Phi_{J}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa]) \\
&= d\rho(D_{mp})\left\{\Phi_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(f_J(\xi))([(\lambda,\mu),\kappa])\right\}.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
&\Phi_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(dU_{\mathcal{M}}(-\,\hat{D}_{mp}(f_J(\xi)))([(\lambda,\mu),\kappa]) \\
&= \pi i \sum_{l=1}^{h} \mathcal{M}_{ml}\Phi_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(f_{J+\varepsilon_{lp}}(\xi))([(\lambda,\mu),\kappa]) \\
&= \pi i \sum_{l=1}^{h} \mathcal{M}_{ml}\Phi_{J+\varepsilon_{lp}}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa]) \\
&= d\rho(\hat{D}_{mp})\Phi_{J}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa]) \\
&= d\rho(\hat{D}_{mp})\left\{\Phi_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(f_J(\xi))([(\lambda,\mu),\kappa])\right\},
\end{aligned}$$

where $1 \leq k \leq l \leq h$, $1 \leq p \leq g$. The last statement is obvious. q.e.d.

Remark 3.4. Theorem 3 means that the unitary representation $\hat{\rho}$ of $H_{\mathbb{R}}^{(g,h)}$ on $H_{\Omega}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}$ is equivalent to the Schrödinger representation $U_{\mathcal{M}}$ of index $\mathcal{M}$. Thus the Schrödinger representation $U_{\mathcal{M}}$ is irreducible.

References

[C] P. Cartier, Quantum Mechanical Commutation Relations and Theta Functions, Proc. of Symposia in Pure Mathematics, A.M.S., **9** (1966), 361–383.

[I] J. Igusa, Theta functions, Springer-Verlag (1972).

[M] H. Morikawa, Some results on harmonic analysis on compact quotients of Heisenberg groups, Nagoya Math. J., **99** (1985), 45–62.

[T] M. Taylor, Noncommutative Harmonic Analysis, Math. Surveys and Monographs, Amer. Math. Soc., No. **22** (1986).

[Wei] A. Weil, Sur certains groupes d'operateurs unitaires, Acta Math., **113** (1964), 143–211.

[Wey] H. Weyl, The theory of groups and quantum mechanics, Dover Publications, New York (1950).

[Z] C. Ziegler, Jacobi Forms of Higher Degree, Abh. Math. Sem. Univ. Hamburg, **59** (1989), 191–224.

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Harmonic Analysis on the Quotient Spaces of Heisenberg Groups, II

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This article is a continuation of a previous article by the author [Harmonic analysis on the quotient spaces of Heisenberg groups, *Nagoya Math. J.* **123** (1991), 103–117]. In this article, we construct an orthonormal basis of the irreducible invariant component $H_{\Omega}^{(\mathcal{M})}\left[\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right]$ of the Hilbert space $L^2(H_R^{(g,h)}\backslash H_R^{(g,h)})$ in the previous article and also construct a nonholomorphic modular form of half integral weight using the Hermite functions. © 1994 Academic Press, Inc.

1. INTRODUCTION

This article is a continuation of a previous article by the author. In [Y], we showed that the vector space $H_{\Omega}^{(\mathcal{M})}\left[\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right]$ is an irreducible invariant subspace of the Hilbert space $L^2(H_Z^{(g,h)}\backslash H_R^{(g,h)})$ with respect to the right regular representation of the Heisenberg group $H_R^{(g,h)}$ (see Section 3 for the precise definition). In this article, we construct an orthonormal basis for the vector space $H_{\Omega}^{(\mathcal{M})}\left[\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right]$ using the *Hermite polynomials.* Hermite polynomials arise from the problem of a quantum harmonic oscillator in one dimension. They are solutions of the *confluent hypergeometric equation.* Thus Hermite polynomials can be expressed in terms of the hypergeometric functions (see (2.25a) and (2.25b)). Using the *Hermite functions*, we construct a nonholomorphic modular form of half integral weight. This implies that the hypergeometric functions are related to the theory of automorphic forms.

In Section 2, we review Hermite polynomials and Hermite functions. We collect their properties to be used in the following sections. In Section 3, we construct an orthonormal basis for the vector space $H_{\Omega}^{(\mathcal{M})}\left[\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right]$ using Hermite polynomials. In Section 4, we prove that the theta series $\vartheta_J(\Omega)$ (see (4.14)) obtained by using the Hermite function is a nonholomorphic modular form of half integral weight. In fact, this result is a generalization of that of Vigneras [V].

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Notations. We denote Z, R, and C the ring of integers, the field of real numbers, and the field of complex numbers, respectively. $F^{(k,l)}$ denotes the set of all $k\times l$ matrices with entries in a commutative ring F. E_g denotes the identity matrix of degree g. $\sigma(A)$ denotes the trace of a square matrix A. For $A\in F^{(k,l)}$ and $B\in F^{(k,k)}$, we set $B[A]={}^tABA$. For a real number α, $[\alpha]$ denotes the greatest integer not exceeding α. We denote by H_g the Siegel upper half plane of degree g.

$$Z_{\geqslant 0}^{(h,g)}=\{J=(J_{kl})\in Z^{(h,g)}\mid J_{kl}\geqslant 0 \text{ for all } k,l\},$$

$$|J|=\sum_{k,l}J_{kl},$$

$$(\lambda+N+A)^J=(\lambda_{11}+N_{11}+A_{11})^{J_{11}}\cdots(\lambda_{hg}+N_{hg}+A_{hg})^{J_{hg}}.$$

2. The Hermite Functions

In this section, we collect some properties of the Hermite polynomials and the Hermite functions to be used in the following sections.

The *Hermite polynomials* $H_n(x)$ $(n=0,1,2,\ldots)$ in one variable x are defined by the generating functions

$$e^{-(t^2-2xt)}:=\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n. \tag{2.1}$$

The Hermite polynomial $H_n(x)$ is a solution of the differential equation, the so-called *Hermite equation*

$$y''-2xy'+2ny=0. \tag{2.2}$$

There are several ways to represent the Hermite polynomial (cf. [S]). Indeed,

$$H_n(x)=\sum_{k=0}^{[n/2]}\frac{(-1)^k\,n!}{k!(n-2k)!}(2x)^{n-2k},\qquad n=0,1,2,\ldots. \tag{2.3}$$

$$H_n(x)=(-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})\qquad \text{(the Rodrigues formula)} \tag{2.4}$$

$$H_n(x)=\frac{i^n}{2\pi^{1/2}}\int_{-\infty}^{\infty}t^n e^{-(1/4)(t+2ix)^2}\,dt \tag{2.5}$$

$$H_n(x)=\frac{n!}{2\pi i}\oint_C\frac{e^{-(t^2-2xt)}}{t^{n+1}}\,dt, \tag{2.6}$$

where the contour C encloses the origin.

LEMMA 2.1. *We have the recursion formulas*

$$H_n'(x) = 2nH_{n-1}(x), \qquad n \geqslant 1. \tag{2.7}$$

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0, \qquad n \geqslant 1. \tag{2.8}$$

Proof. These formulas follows immediately from (2.6). Q.E.D.

LEMMA 2.2.

$$x^n = \frac{1}{2^n} \sum_{k=0}^{[n/2]} \frac{n!}{k!\,(n-k)!} H_{n-2k}(x). \tag{2.9}$$

Proof. (2.9) follows from (2.3). Q.E.D.

The Hermite polynomials satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} H_m(x)\, H_n(x)\, e^{-x^2}\, dx = 2^n n!\, \pi^{1/2} \delta_{mn}, \tag{2.10}$$

where δ_{mn} denotes the Kronecker delta symbol. We set

$$u_n(x) := 2^{-n/2}(n!)^{-1/2}\, \pi^{-1/4} e^{-x^2/2} H_n(x), \qquad n = 0, 1, 2, \ldots. \tag{2.11}$$

Then according to (2.10), we have the orthonormality relation

$$\int_{-\infty}^{\infty} u_m(x)\, u_n(x)\, dx = \delta_{mn}. \tag{2.12}$$

We see easily from (2.7) and (2.8) that $u_n(x)$ is a solution of the differential equation

$$y'' - (x^2 - (2n+1))\, y = 0. \tag{2.13}$$

We set

$$c_n := (-1)^n\, (n!)^{1/2}\, 2^{n-1/4} \pi^{n/2}, \qquad n = 0, 1, 2, \ldots. \tag{2.14}$$

Now we define the *Hermite function* $\mathscr{H}_n(x)$ by

$$e^{-2\pi(x+t)^2} e^{\pi x^2} = \sum_{n=0}^{\infty} c_n \mathscr{H}_n(x) \frac{t^n}{n!}. \tag{2.15}$$

Then we see easily from (2.1) that $\mathscr{H}_n(x)$ is given by

$$\mathscr{H}_n(x) = c_n^{-1} (2\pi)^{n/2}\, e^{-\pi x^2} H_n(-\sqrt{2\pi}\, x), \qquad n = 0, 1, 2, \ldots. \tag{2.16}$$

Using the recursion formulas (2.7) and (2.8), we easily obtain

LEMMA 2.3. *For any positive integer* $n \in Z^+$, we have

$$c_{n+1}\mathscr{H}_{n+1}(x)+4\pi c_n x\mathscr{H}_n(x)+4\pi n c_{n-1}\mathscr{H}_{n-1}(x)=0. \tag{2.17}$$

$$\mathscr{H}'_n(x)-2\pi x\mathscr{H}_n(x)=\frac{c_{n+1}}{c_n}\mathscr{H}_{n+1}(x). \tag{2.18}$$

An easy computation and (2.13) *yields*

LEMMA 2.4.

$$\mathscr{H}_n(-x)=(-1)^n\mathscr{H}(x). \tag{2.19}$$

$$\hat{\mathscr{H}}_n(x)=(-i)^n\mathscr{H}_n(x). \tag{2.20}$$

$$\mathscr{H}''_n(x)-4\pi^2x^2\mathscr{H}_n(x)=-4\pi(n+\tfrac{1}{2})\mathscr{H}_n(x). \tag{2.21}$$

Here $\hat{f}(x)$ denotes the Fourier transform of $f(x)$. That is,

$$\hat{f}(x):=\int_{-\infty}^{\infty} f(y)\,e^{-2\pi xy}\,dy.$$

Thus $\mathscr{H}_n(x)$ is an eigenfunction of the differential operator $L=d^2/dx^2-4\pi^2x^2$. We set $p_n(x):=\mathscr{H}_n(x)\,e^{\pi x^2}$. Let $E:=x(d/dx)$ be the Euler operator. Then (2.21) is equivalent to

$$\Delta p_n(x)=4\pi(E-n)\,p_n(x), \tag{2.22}$$

where $\Delta=d^2/dx^2$ denotes the Laplacian operator on the real line. We set $h_n(x):=\mathscr{H}_n(x)\,e^{-\pi x^2}$. Then (2.21) is equivalent to

$$\Delta h_n(x)=-4\pi(E+n+1)\,h(x). \tag{2.23}$$

According to (2.10) and (2.16), the Hermite functions $\mathscr{H}_n(x)$ $(n=0, 1, \ldots)$ satisfy the orthonormality relation

$$\int_{-\infty}^{\infty}\mathscr{H}_m(x)\,\mathscr{H}_n(x)\,dx=\delta_{mn}. \tag{2.24}$$

In the introduction, we mentioned that the Hermite polynomials are solutions of the hypergeometric equations. Indeed, Hermite polynomials are expressed in terms of hypergeometric functions (cf. [S], p. 97). Precisely,

$$H_n(x)=\frac{n!\,(-1)^{-n/2}}{(n/2)!}\,{}_1F_1\left(-\frac{n}{2};\frac{1}{2};x^2\right)\quad\text{for even } n \tag{2.25a}$$

and

$$H_n(x)=\frac{2n!\,(-1)^{(1-n)/2}}{((n-1)/2)!}\,x\,{}_1F_1\left(-\frac{n-1}{2};\frac{3}{2};x^2\right)\quad\text{for odd } n. \tag{2.25b}$$

3. An Orthonormal Basis of $H_\Omega^{(\mathcal{M})}\left[\begin{smallmatrix}A\\0\end{smallmatrix}\right]$

For any positive integers g and h, we consider the Heisenberg group

$$H_R^{(g,h)} := \{[(\lambda, \mu), \kappa] \mid \lambda, \mu \in R^{(h,g)}, \kappa \in R^{(h,h)}, \kappa + \mu {}^t\lambda \text{ symmetric}\}$$

endowed with the following multiplication law

$$[(\lambda, \mu), \kappa] \circ [(\lambda', \mu'), \kappa'] = [(\lambda + \lambda', \mu + \mu'), \kappa + \kappa' + \lambda {}^t\mu' - \mu {}^t\lambda'].$$

We denote by $H_Z^{(g,h)}$ the discrete subgroup of $H_R^{(g,h)}$ consisting of integral elements.

From now on, we fix an element Ω of the Siegel upper half plane H_g of degree g. Now for a positive definite symmetric even integral matrix $\mathcal{M}$ of degree h and $J \in Z_{\geqslant 0}^{(h,g)}$, we define a function on $H_R^{(g,h)}$

$$\Phi_J^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}(\Omega \mid [(\lambda, \mu), \kappa]) := e^{\pi i \sigma(\mathcal{M}(\kappa - \lambda {}^t\mu))}$$

$$\times \sum_{N \in Z^{(h,g)}} (\lambda + N + A)^J e^{\pi i \sigma\{\mathcal{M}((\lambda + N + A)\Omega {}^t(\lambda + N + A) + 2(\lambda + N + A){}^t\mu)\}}, \tag{3.1}$$

where $A \in \mathcal{M}^{-1}Z^{(h,g)}/Z^{(h,g)}$.

Let $H_\Omega^{(\mathcal{M})}\left[\begin{smallmatrix}A\\0\end{smallmatrix}\right]$ be the completion of the vector space spanned by $\Phi_J^{(\mathcal{M})}\left[\begin{smallmatrix}A\\0\end{smallmatrix}\right](\Omega \mid [(\lambda, \mu), \kappa])(J \in Z_{\geqslant 0}^{(h,g)})$. Then by Theorem 2 in [Y], $H_\Omega^{(\mathcal{M})}\left[\begin{smallmatrix}A\\0\end{smallmatrix}\right]$ is an irreducible invariant subspace of $L^2(H_Z^{(h,g)} \backslash H_R^{(h,g)})$ with respect to the right regular representation of the Heisenberg group $H_R^{(g,h)}$.

Now we will construct an orthonormal basis for $H_\Omega^{(\mathcal{M})}\left[\begin{smallmatrix}A\\0\end{smallmatrix}\right]$ using the Hermite polynomials. For $J = (J_{kl}) \in Z_{\geqslant 0}^{(h,g)}$ and $x = (x_{kl}) \in R^{(h,g)}$, we define the *Hermite polynomial* $H_J(x)$ in several variables

$$H_J(x) := H_{J_{11}}(x_{11}) H_{J_{12}}(x_{12}) \cdots H_{J_{hg}}(x_{hg}). \tag{3.2}$$

Then according to (2.10), Hermite polynomials $H_J(x)$ $(J \in Z_{\geqslant 0}^{(h,g)})$ satisfy the following orthogonality relation

$$\int_{R^{(h,g)}} H_J(x) H_K(x) e^{-\sigma(x {}^t x)} dx = \begin{cases} 2^{|J|} J! \, \pi^{hg/2} & \text{if } J = K \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

We set $Y = \operatorname{Im} \Omega = (1/2i)(\Omega - \bar{\Omega})$. Since Y is positive definite, we may define the unique square root $Y^{1/2}$. Let $\mathcal{M}$ be a positive definite, symmetric even integral matrix of degree h and let $\mathcal{M}^{1/2}$ be its unique square root. Then by an easy computation we see that the functions

$$H_J(\sqrt{2\pi}\, \mathcal{M}^{1/2} x Y^{1/2}), \quad J \in Z_{\geqslant 0}^{(h,g)}$$

satisfy the orthogonality relation

$$\int_{R^{(h,g)}} H_J(\sqrt{2\pi}\,\mathcal{M}^{1/2}xY^{1/2})\,H_K(\sqrt{2\pi}\,\mathcal{M}^{1/2}xY^{1/2})\,e^{-2\pi\sigma(\mathcal{M}[x]Y)}\,dx$$

$$=\begin{cases}2^{|J|-hg/2}J!\,(\det\mathcal{M})^{-g/2}\,(\det Y)^{-h/2} & \text{if } J=K\\ 0 & \text{otherwise.}\end{cases} \tag{3.4}$$

We define

$$H_J^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa]) := 2^{hg/4-|J|/2}(J!)^{-1/2}\,(\det\mathcal{M})^{g/4}\,(\det Y)^{h/4}$$

$$\times e^{\pi i\sigma(\mathcal{M}(\kappa-\lambda'\mu))}\sum_{N\in Z^{(h,g)}} H_J(\sqrt{2\pi}\,\mathcal{M}^{1/2}(\lambda+N+A)\,Y^{1/2})$$

$$\times e^{\pi i\sigma\{\mathcal{M}((\lambda+N+A)\Omega'(\lambda+N+A)+2(\lambda+N+A)'\mu)\}}, \tag{3.5}$$

where $A\in\mathcal{M}^{-1}Z^{(h,g)}/Z^{(h,g)}$.

LEMMA 3.1. *The functions* $H_J^{(\mathcal{M})}[{}^A_0](\Omega\,|\,[\lambda,\mu),\kappa])$ $(J\in J_{\geqslant 0}^{(h,g)})$ *satisfy the orthonormality relation*

$$\int_{R^{(h,g)}} H_J^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])\,\overline{H_K^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}(\Omega\,|\,[(\lambda,\mu),\kappa])}\,d\lambda$$

$$=\begin{cases}1 & \text{if } J=K\\ 0 & \text{otherwise}\end{cases} \tag{3.6}$$

Proof. It follows easily from (3.4) and Lemma 3.1 in [Y]. Q.E.D.

From Lemma 2.2, we obtain

THEOREM 1. *The functions* $H_J^{(\mathcal{M})}[{}^A_0](\Omega\,|\,[(\lambda,\mu),\kappa])$ $(J\in Z_{\geqslant 0}^{(h,g)})$ *form an orthonormal basis for* $H_\Omega^{(\mathcal{M})}[{}^A_0]$.

4. THETA SERIES ASSOCIATED TO INDEFINITE QUADRATIC FORMS

Let $q(\xi)$ be an indefinite quadratic form on R^h $(h\in Z^+)$ of signature (s,t) with $s+t=h$. Let L be a lattice in R^h such that $q(L)\subset Z$. The bilinear form $\langle\,,\rangle$ on R^h associated to the quadratic form $q(\xi)$ is given by $\langle\xi,\eta\rangle := q(\xi+\eta)-q(\xi)-q(\eta)$ $(\xi,\eta\in R^h)$. We recall that the dual L^* of a lattice L relative to $q(\xi)$ is defined by

$$L^* := \{a\in R^h\,|\,\langle a,k\rangle\in Z \text{ for all } k\in L\}.$$

We choose a basis $\{e_1, ..., e_h\}$ for the real vector space R^h such that for the coordinate $\xi = {}^t(\xi_1, ..., \xi_h) \in R^h$ with respect to this basis $\{e_1, ..., e_h\}$

$$q(\xi) = \tfrac{1}{2}(\xi_1^2 + \cdots + \xi_s^2 - \xi_{s+1}^2 - \cdots - \xi_h^2).$$

For $x = (x_1, ..., x_h) \in R^h \times \cdots \times R^h = R^{(h,g)}$ with the column vectors $x_i = {}^t(x_{1i}, x_{2i}, ..., x_{h_i})$ $(1 \leqslant i \leqslant g)$, we define

$$\tilde{q}(x) := \tfrac{1}{2}(x_{11}^2 + \cdots + x_{sg}^2 - x_{s+1,1}^2 - \cdots - x_{hg}^2),$$
$$\tilde{q}_+(x) := \tfrac{1}{2}(x_{11}^2 + \cdots + x_{hg}^2).$$

For $J = (J_{kl}) \in Z_{\geqslant 0}^{(h,g)}$, $\lambda \in R$, and $a = (a_{kl}) \in R^{(h,g)}$, we define

$$J! = J_{11}! \cdots J_{hg}!, \qquad \lambda^J = \lambda^{|J|}, \qquad a^J = a_{11}^{J_{11}} \cdots a_{hg}^{J_{hg}},$$
$$\varepsilon(J) = J_{11} + \cdots + J_{sg} - J_{s+1,1} - \cdots - J_{hg}.$$

For any $J \in Z_{\geqslant 0}^{(h,g)}$, we set

$$c_J := (-1)^J (J!)^{1/2}\, 2^{J-1/4} \pi^{J/2}. \tag{4.1}$$

We define the *Hermite functions* $\mathscr{H}_J(x)$ $(J \in Z_{\geqslant 0}^{(h,g)})$ in several variables by the relation

$$e^{-4\pi\tilde{q}_+(x+t)} \cdot e^{2\pi\tilde{q}_+(x)} = \sum_{J \in Z_{\geqslant 0}^{(h,g)}} c_J \mathscr{H}_J(x) \frac{t^J}{J!}. \tag{4.2}$$

For a function f on $R^{(h,g)}$, we define the Fourier transform by

$$\hat{f}(x) := \int_{R^{(h,g)}} f(y)\, e^{-2\pi i \langle x, y \rangle}\, dy, \qquad x \in R^{(h,g)},$$

where $\langle x, y \rangle = x_{11} y_{11} + \cdots + x_{sg} y_{sg} - x_{s+1,1} y_{s+1,1} - \cdots - x_{hg} y_{hg}$ for $x = (x_{kl})$, $y = (y_{kl}) \in R^{(h,g)}$ and dy is the normalized Haar measure so that $\mathrm{vol}(L^g) := \mathrm{vol}(R^{(h,g)}/L^g) = 1$.

LEMMA 4.1. *For $J = (J_{kl}) \in J_{\geqslant 0}^{(h,g)}$ and $x = (x_{kl}) \in R^{(h,g)}$, we have*

$$\mathscr{H}_J(x) = \mathscr{H}_{J_{11}}(x_{11})\, \mathscr{H}_{J_{12}}(x_{12}) \cdots \mathscr{H}_{J_{hg}}(x_{hg}). \tag{4.3}$$

$$\mathscr{H}_J(-x) = (-1)^J\, \mathscr{H}_J(x). \tag{4.4}$$

$$\hat{\mathscr{H}}_J(x) = \hat{\mathscr{H}}_{J_{11}}(x_{11}) \cdots \hat{\mathscr{H}}_{J_{sg}}(x_{sg}) \times \hat{\mathscr{H}}_{J_{s+1,1}}(-x_{s+1,1}) \cdots \hat{\mathscr{H}}_{J_{hg}}(-x_{hg}). \tag{4.5}$$

$$\hat{\mathscr{H}}_J(x) = (-1)^{J_{11} + \cdots + J_{sg}}\, i^{|J|} \mathscr{H}_J(x). \tag{4.6}$$

Proof. (4.3) follows easily from (2.15). (4.4) follows immediately from (2.19). (4.5) follows from (4.3) and the definition of the Fourier transform. (4.6) follows from (2.19), (2.20), and (4.5). Q.E.D.

LEMMA 4.2. *Let* $\Delta=\sum_{k=1}^{s}\sum_{l=1}^{g}(\partial^2/\partial x_{kl}^2)-\sum_{k=s+1}^{h}\sum_{l=1}^{g}(\partial^2/\partial x_{kl}^2)$ *be the Laplacian on* $R^{(h,g)}$ *associated with the quadratic form* $\tilde{q}(x)$. *Then we have*

$$(\Delta-8\pi^2\tilde{q}(x))\,\mathscr{H}_J(x)=-4\pi\left(\varepsilon(J)+\frac{(s-t)\,g}{2}\right)\mathscr{H}_J(x). \tag{4.7}$$

Proof. It follows immediately from (2.21). Q.E.D.

LEMMA 4.3. *Let* $E:=\sum_{k=1}^{h}\sum_{l=1}^{g}x_{kl}(\partial/\partial x_{kl})$ *be the Euler operator on* $R^{(h,g)}$. *We set* $P_J(x)=\mathscr{H}_J(x)\,e^{2\pi\tilde{q}(x)}$ $(J\in Z_{\geqslant 0}^{(h,g)})$. *Then we have*

$$\Delta P_J(x)=4\pi(E-\varepsilon(J)+\mu)\,P_J(x),\qquad \mu=(h-s)\,g. \tag{4.8}$$

Proof. (4.8) follows from (2.22), (2.23), and (4.7). Q.E.D.

For the present time being, we fix an element $\Omega=X+iY\in H_g$. We define the function $f_{J,\Omega}(x)$ on $R^{(h,g)}$ by

$$f_{J,\Omega}(x):=(\det Y)^{-\lambda/2}\,\mathscr{H}_J(xY^{1/2})\,e^{2\pi i\sigma(Q[x]X)},\qquad x\in R^{(h,g)}, \tag{4.9}$$

where $\lambda=\varepsilon(J)-\mu$. Here $2Q=\mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$ is the symmetric matrix of degree h associated with the quadratic form $2q(x)$.

LEMMA 4.4. *For any* $\Omega\in H_g$, $J\in Z_{\geqslant 0}^{(h,g)}$, *and* $x\in R^h$, *we have*

$$\hat{f}_{J,-\Omega^{-1}}(x)=(-i)^a(\det\Omega)^{\lambda+h/2}f_{J,\Omega}(x),\qquad a=\varepsilon(J)+\left(\lambda+\frac{h}{2}\right)g. \tag{4.10}$$

Proof. It suffices to show (4.10) for $\Omega=iY$, $Y>0$. Then we have

$$\begin{aligned}
\hat{f}_{J,iY^{-1}}(x)&=(\det Y)^{\lambda/2}\int_{R^{(h,g)}}\mathscr{H}_J(\xi Y^{-1/2})\,e^{-2\pi i\langle x,\xi\rangle}\,d\xi\\
&=(\det Y)^{\lambda/2+h/2}\int_{R^{(h,g)}}\mathscr{H}_J(\xi)\,e^{-2\pi i\langle x,\xi Y^{1/2}\rangle}\,d\xi\\
&=(\det Y)^{\lambda/2+h/2}\int_{R^{(h,g)}}\mathscr{H}_J(\xi)\,e^{-2\pi i\langle xY^{1/2},\xi\rangle}\,d\xi\\
&=(\det Y)^{\lambda/2+h/2}\,\hat{\mathscr{H}}_J(xY^{1/2})\\
&=(-1)^{J_{11}+\cdots+J_{sg}}\,i^{|J|}(\det Y)^{\lambda/2+h/2}\,\mathscr{H}_J(xY^{1/2})\qquad\text{by (4.6)}\\
&=(-1)^{J_{11}+\cdots+J_{sg}}\,i^{|J|}(\det Y)^{\lambda+h/2}f_{J,iY}(x)\qquad\text{by (4.9).}
\end{aligned}$$

By an easy calculation, we obtain the desired result (4.10). Q.E.D.

For any $\alpha \in (L^*)^g$ and $J \in Z^{(h,g)}_{\geqslant 0}$, we define the *theta series* on H_g by

$$\vartheta_{\alpha,J}(\Omega) := \sum_{x \in L^g + \alpha} f_{J,\Omega}(x), \qquad \Omega = X + iY \in H_g. \tag{4.11}$$

It is known that the Siegel modular group $\Gamma_g := \mathrm{Sp}(g, Z)$ is generated by

$$\begin{pmatrix} E_g & S \\ 0 & E_g \end{pmatrix}, \qquad S = {}^tS \text{ integral} \qquad \text{and} \qquad \begin{pmatrix} 0 & -E_g \\ E_g & 0 \end{pmatrix}.$$

Therefore in order to investigate the transformation behaviour of the theta series $\vartheta_{\alpha,J}(\Omega)$ $(\Omega \in H_g)$ for the action of the Siegel modular group, it suffices to investigate the transformation law of $\vartheta_{\alpha,J}(\Omega)$ under the two actions $\Omega \mapsto \Omega + S$ with $S = {}^tS$ integral and $\Omega \mapsto -\Omega^{-1}$ $(\Omega \in H_g)$.

If $S = {}^tS$ is a symmetric integral matrix of degree g, by an easy computation we have

$$\vartheta_{\alpha,J}(\Omega + S) = e^{2\pi i \sigma(Q[\alpha]S)} \vartheta_{\alpha,J}(\Omega). \tag{4.12}$$

The Poisson formula says that for a function f on $R^h \times \cdots \times R^h = R^{(h,g)}$

$$\sum_{\alpha \in L^g} f(\alpha) = \sum_{\alpha \in (L^*)^g} \hat{f}(\alpha).$$

THEOREM 2. *Let $a = \varepsilon(J) + (\lambda + h/2)g$. Then for any $\alpha \in (L^*)^g$, we have the transformation law*

$$\vartheta_{\alpha,J}(-\Omega^{-1}) = (-i)^a (\det \Omega)^{\lambda + h/2} \sum_{\substack{k \in (L^*)^g \\ k (\mathrm{mod}\ L^g)}} e^{2\pi i \langle \alpha, k \rangle} \vartheta_{k,J}(\Omega). \tag{4.13}$$

Proof. Using the Poisson formula, we obtain

$$\vartheta_{\alpha,J}(-\Omega^{-1}) = \sum_{\substack{k \in (L^*)^g \\ k (\mathrm{mod}\ L^g)}} e^{2\pi i \langle \alpha, k \rangle} \sum_{x \in L^g + k} \hat{f}_{J, -\Omega^{-1}}(x).$$

Therefore by (4.10), we obtain (4.13). Q.E.D.

We now define the theta series

$$\vartheta_J(\Omega) = (\det Y)^{-\lambda/2} \sum_{\alpha \in L^g} \mathscr{H}_J(\alpha Y^{1/2})\, e^{2\pi\sigma(Q[\alpha]Y)} e^{2\pi i \sigma(Q[\alpha]\Omega)}, \tag{4.14}$$

where $\Omega = X + iY \in H_g$. According to (4.12) and Theorem 2, we have

$$\vartheta_J(\Omega + S) = \vartheta_J(\Omega), \qquad \text{for any integral} \quad S = {}^tS \in Z^{(g,g)}. \tag{4.15}$$

$$\vartheta_J(-\Omega^{-1}) = (-i)^a (\det \Omega)^{\lambda + h/2} \sum_{\substack{k \in (L^*)^g \\ k (\mathrm{mod}\, L^g)}} \vartheta_{k,J}(\Omega). \tag{4.16}$$

Therefore we obtain

THEOREM 3. *We assume that a lattice L is self-dual with respect to the quadratic form $q(x)$, that is, $L = L^*$. Then the theta series $\vartheta_J(\Omega)$ is a nonholomorphic modular form of weight $\lambda + h/2$ with respect to a certain congruence subgroup of Γ_g. Its level is the same as that of the quadratic form $\tilde{q}(x)$ on $R^{(h,g)}$.*

Remark. In [F], using the pluriharmonic forms, Freitag constructed the vector-valued theta series of a certain type and proved that this theta series is a vector-valued modular form of half integral weight with respect to a certain congruence subgroup.

REFERENCES

[F] E. FREITAG, The transformation formalism of vector valued theta functions with respect to the Siegel modular group, *J. Indian Math. Soc.* **52** (1987), 185–207.

[S] J. B. SEABORN, "Hypergeometric Functions and Their Applications," Texts in Applied Mathematics, Vol. 8, Springer-Verlag, New York/Berlin, 1991.

[V] M. F. VIGNERAS, Series theta des formes quadratiques indefinies, *in* "International Summer School on Modular Functions of One Variable, Bonn," Lecture Notes in Mathematics, Vol. 627, Springer-Verlag, New York/Berlin, 1976.

[Y] J.-H. YANG, Harmonic analysis on the quotient spaces of Heisenberg groups, *Nagoya Math. J.* **123** (1991), 103–117.

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A decomposition theorem on differential polynomials of theta functions of high level

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Let h and g be two positive integers. We fix an element Ω of the Siegel upper half plane

$$H_g := \left\{ Z \in \mathbb{C}^{(g,g)} \mid Z = {}^tZ, \ \ \operatorname{Im} Z > 0 \right\}$$

of degree g once and for all. Let $\mathcal{M}$ be positive symmetric, even integral matrix of degree h. An entire function f on $\mathbb{C}^{(h,g)}$ satisfying the transformation behaviour

$$f(W + \xi\Omega + \eta) = \exp\left\{-\pi i \sigma(\mathcal{M}(\xi\Omega\,{}^t\xi + 2W\,{}^t\xi))\right\} f(W)$$

for all $W \in \mathbb{C}^{(h,g)}$ and $(\xi, \eta) \in \mathbb{Z}^{(h,g)} \times \mathbb{Z}^{(h,g)}$ is called a *theta function of level* $\mathcal{M}$ with respect to Ω. The set $T_{\mathcal{M}}(\Omega)$ of all theta functions of level $\mathcal{M}$ with respect to Ω forms a complex vector space of dimension $(\det \mathcal{M})^g$ with a canonical basis consisting of theta series

$$\begin{aligned}&\vartheta^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega|W)\\ &\quad := \sum_{N \in \mathbb{Z}^{(h,g)}} \exp\left\{\pi i \sigma(\mathcal{M}((N+A)\Omega\,{}^t(N+A) + 2W\,{}^t(N+A)))\right\},\end{aligned}$$

where A runs over a complete system of representatives of the cosets $\mathcal{L}_{\mathcal{M}} := \mathcal{M}^{-1}\mathbb{Z}^{(h,g)}/\mathbb{Z}^{(h,g)}$.

We let

$$T(\Omega) := \sum_{\mathcal{M}} T_{\mathcal{M}}(\Omega)$$

be the graded algebra of theta functions, where $\mathcal{M} = (\mathcal{M}_{kl})$ $(1 \le k, l \le h)$ runs over the set $\mathbb{M}(h)$ of all positive symmetric, even integral $h \times h$ matrices with $\mathcal{M}_{kl} \neq 0$ for all k, l.

In this paper we prove the following decomposition theorem:

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The algebra of differential polynomials of theta functions has a canonical basis

$$\left\{ \left(\frac{\partial}{\partial W}\right)^J \vartheta^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega|W) \,\middle|\, J \in \mathbb{Z}_{\geq 0}^{(h,g)},\ A \in \mathcal{L}_{\mathcal{M}},\ \mathcal{M} \in \mathbb{M}(h) \right\},$$

i.e., any differential polynomials of theta functions can be expressed uniquely as a linear combination of $\left(\frac{\partial}{\partial W}\right)^J \vartheta^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega|W)$ $(J \in \mathbb{Z}_{\geq 0}^{(h,g)},\ A \in \mathcal{L}_{\mathcal{M}},\ \mathcal{M} \in \mathbb{M}(h))$ with constant coefficients depending only on Ω.

The key idea is quite similiar to that of making transvectants in the classical invariant theory (cf. [M1], [M2]). However the Lie algebra is the Heisenberg Lie algebra instead of sl_2. The graded algebra $T(\Omega)$ of theta functions is embedded in the graded algebra $A(\Omega)$ of auxiliary theta functions in (Z,W) with $Z, W \in \mathbb{C}^{(h,g)}$ with respect to Ω satisfying the following conditions:

1^0. A realization $\{\mathcal{E}_{kl},\, \mathcal{D}_{ma},\, \Delta_{nb} \,|\, 1 \leq k,l,m,n \leq h,\ 1 \leq a,b \leq g\}$ (cf. see section 2 for detail) of the Heisenberg Lie algebra acts on $A(\Omega)$ as derivations.

2^0. $T(\Omega)$ is the subalgebra consisting of all the elements $\varphi \in A(\Omega)$ such that $\Delta_{nb}\varphi = 0$ for all $1 \leq n \leq h$, $1 \leq b \leq g$.

3^0. The set

$$\left\{ \Delta^J \vartheta^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega|W) \,\middle|\, J \in \mathbb{Z}_{\geq 0}^{(h,g)},\ A \in \mathcal{L}_{\mathcal{M}},\ \mathcal{M} \in \mathbb{M}(h) \right\}$$

forms a canonical basis of $A(\Omega)$.

4^0. The mapping

$$\Delta^J \vartheta^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega|W) \longmapsto \left(\frac{\partial}{\partial W}\right)^J \vartheta^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega|W)$$

$(J \in \mathbb{Z}_{\geq 0}^{(h,g)},\ A \in \mathcal{L}_{\mathcal{M}}$ and $\mathcal{M} \in \mathbb{M}(h))$ induces an algebra isomorphism of $A(\Omega)$ onto the algebra of differential polynomials of theta functions.

NOTATIONS. We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers, and the field of complex numbers, respectively. The symbol ":=" means that the expression on the right is the definition of that on the left. We denote by $\mathbb{Z}^+$ the set of all positive integers. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. For $A \in F^{(k,k)}$, $\sigma(A)$ denotes the trace of A. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. For a positive symmetric, even integral matrix $\mathcal{M}$ of degree h, $\mathcal{L}_{\mathcal{M}}$ denotes a complete system of representatives of the cosets

$\mathcal{M}^{-1}\mathbb{Z}^{(h,g)}/\mathbb{Z}^{(h,g)}$.

$$\begin{aligned}
\mathbb{Z}^{(h,g)}_{\geq 0} &= \left\{ J = (J_{ka}) \in \mathbb{Z}^{(h,g)} \mid J_{ka} \geq 0 \text{ for all } k, a \right\}, \\
|J| &= \sum_{k,a} J_{k,a}, \\
J \pm \epsilon_{ka} &= (J_{11}, \ldots, J_{ka} \pm 1, \ldots, J_{hg}), \\
J! &= J_{11}! \cdots J_{ka}! \cdots J_{hg}!.
\end{aligned}$$

For $J = (J_{ka}) \in \mathbb{Z}^{(h,g)}_{\geq 0}$, $Z = (Z_{ka})$ and $W = (W_{ka})$, we set

$$Z^J = Z_{11}^{J_{11}} \cdots Z_{hg}^{J_{hg}}, \quad W^J = W_{11}^{J_{11}} \cdots W_{hg}^{J_{hg}}.$$

For $J = (J_{ka}) \in \mathbb{Z}^{(h,g)}_{\geq 0}$, we put

$$\left(\frac{\partial}{\partial W}\right)^J = \left(\frac{\partial}{\partial W_{11}}\right)^{J_{11}} \cdots \left(\frac{\partial}{\partial W_{hg}}\right)^{J_{hg}}.$$

1. Auxiliary theta functions

We fix an element Ω of H_g once and for all. Let $\mathcal{M}$ be a positive symmetric, even integral matrix of degree h. An *auxiliary theta function of level* $\mathcal{M}$ with respect to Ω means a function $\varphi(Z, W)$ in complex variables $(Z, W) \in \mathbb{C}^{(h,g)} \times \mathbb{C}^{(h,g)}$ such that

(a) $\varphi(Z, W)$ is a polynomial in complex variables $Z = (Z_{ka})$ whose coefficients are entire functions in $W = (W_{ka})$, and

(b) for all $(\xi, \eta) \in \mathbb{Z}^{(h,g)} \times \mathbb{Z}^{(h,g)}$ and $(Z, W) \in \mathbb{C}^{(h,g)} \times \mathbb{C}^{(h,g)}$,

$$\varphi(Z + \xi, W + \xi\Omega + \eta) = \exp\left\{-\pi i \sigma(\mathcal{M}(\xi\Omega\,{}^t\xi + 2W\,{}^t\xi))\right\} \varphi(Z, W)$$

holds.

Let $A_{\mathcal{M}}(\Omega)$ be the vector space of auxiliary theta functions of level $\mathcal{M}$ with respect to Ω. We let

$$A(\Omega) := \sum_{\mathcal{M}} A_{\mathcal{M}}(\Omega)$$

the graded algebra of auxiliary theta functions, where $\mathcal{M} = (\mathcal{M}_{kl})$ $(1 \leq k, l \leq h)$ runs over the set $\mathbb{M}(h)$ of all positive symmetric, even integral $h \times h$ matrices such that $\mathcal{M}_{kl} \neq 0$ for all k, l. We note that $A(\Omega)$ contains the graded algebra $T(\Omega)$ as the subalgebra of polynomials of degree zero in Z.

We define the auxiliary theta series

$$\begin{aligned}&\widetilde{\vartheta}_J^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}(\Omega|Z,W)\\ &= (2\pi i)^{|J|}\sum_{N\in\mathbb{Z}^{(h,g)}}\prod_{k=1}^{h}\prod_{a=1}^{g}\left(\sum_{l=1}^{h}\mathcal{M}_{kl}(Z+N+A)_{la}\right)^{J_{ka}}\\ &\quad\times \exp\left\{\pi i\sigma(\mathcal{M}((N+A)\Omega\,{}^t(N+A)+2W\,{}^t(N+A)))\right\},\end{aligned}\tag{1.1}$$

where $J=(J_{ka})\in\mathbb{Z}^{(h,g)}_{\geq 0}$, $\mathcal{M}=(\mathcal{M}_{kl})\in\mathbb{M}(h)$, $A\in\mathcal{L}_{\mathcal{M}}$ and $(Z+N+A)_{la}=Z_{la}+N_{la}+A_{la}$.

LEMMA 1.1. *For each* $J\in\mathbb{Z}^{(h,g)}_{\geq 0}$ *and* $\mathcal{M}\in\mathbb{M}(h)$, *we have*

$$\begin{aligned}&\widetilde{\vartheta}_J^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}(\Omega|Z+\xi,W+\xi\Omega+\eta)\\ &=\exp\left\{-\pi i\sigma(\mathcal{M}(\xi\Omega\,{}^t\xi+2W\,{}^t\xi))\right\}\widetilde{\vartheta}_J^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}(\Omega|Z,W),\end{aligned}$$

where $A\in\mathcal{L}_{\mathcal{M}}$ *and* $(\xi,\eta)\in\mathbb{Z}^{(h,g)}\times\mathbb{Z}^{(h,g)}$. *In particular,* $\widetilde{\vartheta}_J^{(\mathcal{M})}\begin{bmatrix}A\\0\end{bmatrix}(\Omega|Z,W)\in A_{\mathcal{M}}(\Omega)$.

PROOF. We observe that

$$\begin{aligned}&(N+A)\Omega\,{}^t(N+A)+2(W+\xi\Omega+\eta)\,{}^t(N+A)\\ &=(N+\xi+A)\Omega\,{}^t(N+\xi+A)+2W\,{}^t(N+\xi+A)-(N+A)\Omega\,{}^t\xi\\ &\quad+\xi\Omega\,{}^t(N+A)+2\eta\,{}^t(N+A)-(\xi\Omega\,{}^t\xi+2W\,{}^t\xi).\end{aligned}$$

In addition, it is easy to see that

$$\sigma(\mathcal{M}(N+A)\Omega\,{}^t\xi)=\sigma(\mathcal{M}\xi\Omega\,{}^t(N+A))$$

and

$$\sigma(\mathcal{M}\eta\,{}^t(N+A))=\sigma(\mathcal{M}\eta\,{}^tN)+\sigma(\mathcal{M}A\,{}^t\eta)\in\mathbb{Z}\quad(\text{because } A\in\mathcal{L}_{\mathcal{M}}).$$

The proof follows immediately from these facts. □

For any $k,a\in\mathbb{Z}^+$ with $1\leq k\leq h$, $1\leq a\leq g$ and $\mathcal{M}=(\mathcal{M}_{kl})\in\mathbb{M}(h)$, we put

$$\partial(\mathcal{M},Z,W)_{ka}:=2\pi i\sum_{l=1}^{h}\mathcal{M}_{kl}Z_{la}+\frac{\partial}{\partial W_{ka}}.\tag{1.2}$$

For each $J = (J_{ka}) \in \mathbb{Z}_{\geq 0}^{(h,g)}$, we put

$$(1.3) \qquad \partial(\mathcal{M}, Z, W)^J := \partial(\mathcal{M}, Z, W)_{11}^{J_{11}} \cdots \partial(\mathcal{M}, Z, W)_{ka}^{J_{ka}} \cdots \partial(\mathcal{M}, Z, W)_{hg}^{J_{hg}}.$$

Then we obtain the following.

LEMMA 1.2. *For each $J \in \mathbb{Z}_{\geq 0}^{(h,g)}$, $\mathcal{M} \in \mathbb{M}(h)$ and $A \in \mathcal{L}_{\mathcal{M}}$, we have*

$$(1.4) \qquad \widetilde{\vartheta}_J^{(\mathcal{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|Z, W) = \partial(\mathcal{M}, Z, W)^J \vartheta^{(\mathcal{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|W).$$

PROOF. It is easy to compute that if $\mathcal{M} = (\mathcal{M}_{kl}) \in \mathbb{M}(h)$,

$$\begin{aligned} &\partial(\mathcal{M}, Z, W)_{ka} \vartheta^{(\mathcal{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|W) \\ &= 2\pi i \sum_{N \in \mathbb{Z}^{(h,g)}} \left(\sum_{l=1}^{h} \mathcal{M}_{kl} (Z + N + A)_{la} \right) \\ &\quad \times \exp\left\{ 2\pi i \sigma(\mathcal{M}((N + A)\Omega\, {}^t(N + A) + 2W\, {}^t(N + A))) \right\}. \end{aligned}$$

The proof follows immediately from the fact that if $J = (J_{ka}) \in \mathbb{Z}_{\geq 0}^{(h,g)}$ and $\mathcal{M} = (\mathcal{M}_{kl}) \in \mathbb{M}(h)$,

$$\begin{aligned} &\partial(\mathcal{M}, Z, W)_{ka}^{J_{ka}} \vartheta^{(\mathcal{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|W) \\ &= (2\pi i)^{J_{ka}} \sum_{N \in \mathbb{Z}^{(h,g)}} \left(\sum_{l=1}^{h} \mathcal{M}_{kl} (Z + N + A)_{la} \right)^{J_{ka}} \\ &\quad \times \exp\left\{ \pi i \sigma(\mathcal{M}((N + A)\Omega\, {}^t(N + A) + 2W\, {}^t(N + A))) \right\}. \end{aligned}$$

□

THEOREM 1. *For a fixed $\mathcal{M} \in \mathbb{M}(h)$, the set*

$$\left\{ \widetilde{\vartheta}_J^{(\mathcal{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|Z, W) \,\middle|\, J \in \mathbb{Z}_{\geq 0}^{(h,g)},\ A \in \mathcal{L}_{\mathcal{M}} \right\}$$

is a basis of the vector space $A_{\mathcal{M}}(\Omega)$ of auxiliary theta functions of level $\mathcal{M}$ with respect to Ω.

PROOF. According to Lemma 1.1, the functions $\widetilde{\vartheta}_J^{(\mathcal{M})} \begin{bmatrix} A \\ 0 \end{bmatrix} (\Omega|Z, W)$ ($J \in \mathbb{Z}_{\geq 0}^{(h,g)}$ and $A \in \mathcal{L}_{\mathcal{M}}$) are contained in $A_{\mathcal{M}}(\Omega)$ and it is obvious that they are linearly independent. We give an ordering $\prec$ on $\mathbb{Z}_{\geq 0}^{(h,g)}$ as follows. For $J, K \in \mathbb{Z}_{\geq 0}^{(h,g)}$, we write $J \prec K$ if $|J| \leq |K|$. We say that Z^K has higher degree in Z than Z^J if $J \prec K$. Now we let $\varphi(Z,W) = \sum_J Z^J f_J(W)$ be an element of $A_{\mathcal{M}}(\Omega)$ and let $Z^K f_K(W)$ be one of the terms with highest degree K in Z. Since $\varphi \in A_{\mathcal{M}}(\Omega)$, we obtain for each $\xi, \eta \in \mathbb{Z}^{(h,g)}$

$$\begin{aligned}&\sum_J (Z+\xi)^J f_J(W+\xi\Omega+\eta)\\ &\qquad = \exp\left\{-\pi i \sigma(\mathcal{M}(\xi\Omega\, {}^t\xi + 2W\, {}^t\xi))\right\} \sum_J Z^J f_J(W).\end{aligned}$$

Comparing the coefficients of Z^K, we get

$$f_K(W+\xi\Omega+\eta) = \exp\left\{-\pi i \sigma(\mathcal{M}(\xi\Omega\, {}^t\xi + 2W\, {}^t\xi))\right\} f_K(W)$$

for each $(\xi,\eta) \in \mathbb{Z}^{(h,g)} \times \mathbb{Z}^{(h,g)}$. Thus $f_K \in T_{\mathcal{M}}(\Omega)$ and so we obtain

$$f_K(W) = \sum_{\alpha=1}^{(\det \mathcal{M})^g} c_\alpha\, \vartheta^{(\mathcal{M})} \begin{bmatrix} A_\alpha \\ 0 \end{bmatrix} (\Omega|W),$$

where $c_\alpha \in \mathbb{C}$ and $A_\alpha \in \mathcal{L}_{\mathcal{M}}$. Therefore for suitable constants $d_\alpha \in \mathbb{C}$ ($\alpha = 1, \ldots, (\det \mathcal{M})^g$), the function

$$\varphi(Z,W) - \sum_{\alpha=1}^{(\det \mathcal{M})^g} d_\alpha\, \widetilde{\vartheta}_K^{(\mathcal{M})} \begin{bmatrix} A_\alpha \\ 0 \end{bmatrix} (\Omega|Z,W)$$

is an element of $A_{\mathcal{M}}(\Omega)$ without Z^K-term and all the new terms are of lower degree than K in Z. Continuing this process successively, we can express $\varphi(Z,W)$ as a linear combination of auxiliary theta functions $\widetilde{\vartheta}_J^{(\mathcal{M})} \begin{bmatrix} A_\alpha \\ 0 \end{bmatrix} (\Omega|Z,W)$ ($J \in \mathbb{Z}_{\geq 0}^{(h,g)}$, $A_\alpha \in \mathcal{L}_{\mathcal{M}}$). □

2. A realization of Heisenberg Lie algebra

For each $\mathcal{M} \in \mathbb{M}(h)$, we let

$$\sigma^{(\mathcal{M})} : A(\Omega) \longrightarrow A_{\mathcal{M}}(\Omega)$$

be the projection operator of $A(\Omega)$ onto $A_{\mathcal{M}}(\Omega)$. We define the differential operators

$$\mathcal{E}_{kl} := \sum_{\mathcal{M} \in \mathbb{M}(h)} \mathcal{M}_{kl}\, \sigma^{(\mathcal{M})}, \quad \mathcal{M} = (\mathcal{M}_{kl}),$$

$$\mathcal{D}_{ma} := \sum_{\mathcal{M}\in\mathbb{M}(h)} \frac{1}{2\pi i}\frac{\partial}{\partial Z_{ma}} \circ \sigma^{(\mathcal{M})},$$
$$\Delta_{nb} := \sum_{\mathcal{M}\in\mathbb{M}(h)} \partial(\mathcal{M}, Z, W)_{nb} \circ \sigma^{(\mathcal{M})},$$

where $1 \leq k, l, m, n \leq h$ and $1 \leq a, b \leq g$.

For $J = (J_{ka}) \in \mathbb{Z}_{\geq 0}^{(h,g)}$, we put

$$\mathcal{D}^J := \mathcal{D}_{11}^{J_{11}} \cdots \mathcal{D}_{hg}^{J_{hg}}, \quad \Delta^J := \Delta_{11}^{J_{11}} \cdots \Delta_{hg}^{J_{hg}}.$$

PROPOSITION 2.1. *Let* $\mathcal{M} = (\mathcal{M}_{kl}) \in \mathbb{M}(h)$ *and* $J = (J_{ka}) \in \mathbb{Z}_{\geq 0}^{(h,g)}$. *Then we have*

$$\mathcal{D}_{ma}\widetilde{\vartheta}_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega|Z, W) = \sum_{l=1}^{h} \mathcal{M}_{ml} J_{la}\, \widetilde{\vartheta}_{J-\epsilon_{la}}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega|Z, W), \tag{2.1}$$

$$\Delta_{nb}\widetilde{\vartheta}_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega|Z, W) = \widetilde{\vartheta}_{J+\epsilon_{nb}}^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega|Z, W), \tag{2.2}$$

$$\widetilde{\vartheta}_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega|Z, W) = \Delta^J \vartheta^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega|W), \tag{2.3}$$

where $A \in \mathcal{L}_{\mathcal{M}}$, $1 \leq m, n \leq h$ *and* $1 \leq a, b \leq g$.

PROOF. (2.1) follows from a direct application of $\mathcal{D}_{ma}$ to (1.1). (2.2) and (2.3) follow immediately from Lemma 1.2, (1.4). □

PROPOSITION 2.2. $\mathcal{E}_{kl}$, $\mathcal{D}_{ma}$, Δ_{nb} $(1 \leq k, l, m, n \leq h,\ 1 \leq a, b \leq g)$ *are derivations of* $A(\Omega)$ *such that*

$$[\mathcal{E}_{kl}, \mathcal{D}_{ma}] = [\mathcal{E}_{kl}, \Delta_{nb}] = [\mathcal{D}_{ma}, \mathcal{D}_{nb}] = [\Delta_{ma}, \Delta_{nb}] = 0,$$
$$[\mathcal{D}_{ma}, \Delta_{nb}] = \delta_{ab}\, \mathcal{E}_{mn}.$$

PROOF. According to Proposition 2.1, $\mathcal{E}_{kl}$, $\mathcal{D}_{ma}$, Δ_{nb} $(1 \leq k, l, m, n \leq h,\ 1 \leq a, b \leq g)$ map $A(\Omega)$ into itself. Since $A(\Omega) = \sum_{\mathcal{M}\in\mathbb{M}(h)} A_{\mathcal{M}}(\Omega)$ is a graded algebra, $\mathcal{E}_{kl}$, $\mathcal{D}_{mb}$, Δ_{nb} $(1 \leq k, l, m, n \leq h,\ 1 \leq a, b \leq g)$ are derivations of $A(\Omega)$. An easy calculation yields the above commutation relations. □

REMARK 2.3. According to Proposition 2.2, $\{\mathcal{E}_{kl}, \mathcal{D}_{ma}, \Delta_{nb} \mid 1 \leq k, l, m, n \leq h,\ 1 \leq a, b \leq g\}$ is a realization of the Heisenberg Lie algebra acting on $A(\Omega)$ as derivations (cf. [Y1], [Y2]).

PROPOSITION 2.4. *The graded algebra $T(\Omega)$ of theta functions with respect to Ω is the subalgebra of $A(\Omega)$ consisting of φ in $A(\Omega)$ such that $\mathcal{D}_{ma}\varphi = 0$ $(1 \le m \le h$, $1 \le a \le g)$.*

PROOF. If $\varphi \in T(\Omega)$, φ does not contain any variables Z_{ma} and $\mathcal{D}_{ma}\varphi = 0$ for any $m, a \in \mathbb{Z}^+$ with $1 \le m \le h$, $1 \le a \le g$.

Conversely, we assume that

$$\mathcal{D}_{ma}\left(\sum_{J,\alpha,\mathcal{M}} c_{J,\alpha,\mathcal{M}}\, \widetilde{\vartheta}_J^{(\mathcal{M})}\begin{bmatrix} A_{\alpha,\mathcal{M}} \\ 0 \end{bmatrix}(\Omega|Z,W)\right) = 0,$$

where $A_{\alpha,\mathcal{M}} \in \mathcal{L}_{\mathcal{M}}$, $1 \le m \le h$ and $1 \le a \le g$. Then by (2.1), we get

$$(*) \qquad \sum_{J,\alpha,\mathcal{M}} \sum_{l=1}^{h} \mathcal{M}_{ml} J_{la}\, c_{J,\alpha,\mathcal{M}}\, \widetilde{\vartheta}_{J-\epsilon_{la}}^{(\mathcal{M})}\begin{bmatrix} A_{\alpha,\mathcal{M}} \\ 0 \end{bmatrix}(\Omega|Z,W) = 0.$$

Since $\mathcal{M} = (\mathcal{M}_{kl}) \in \mathbb{M}(h)$, $\mathcal{M}_{kl} \neq 0$ for all k, l with $1 \le k, l \le h$. Therefore if $J \neq (0, \dots, 0)$, we get $c_{J,\alpha,\mathcal{M}} = 0$ from the condition (*). Hence this completes the proof. □

THEOREM 2. *$A(\Omega)$ has the direct sum decomposition*

$$(2.4) \qquad A(\Omega) = \sum_{J \in \mathbb{Z}_{\ge 0}^{(h,g)}} \Delta^J T(\Omega) = \sum_{J \in \mathbb{Z}_{\ge 0}^{(h,g)}} \sum_{\mathcal{M} \in \mathbb{M}(h)} \Delta^J T_{\mathcal{M}}(\Omega)$$

such that Δ^J induces a vector space isomorphism of $T_{\mathcal{M}}(\Omega)$ onto $\Delta^J T_{\mathcal{M}}(\Omega)$.

PROOF. The proof follows from (2.3) and the fact that

$$\left\{ \widetilde{\vartheta}_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega|Z,W) \,\middle|\, J \in \mathbb{Z}_{\ge 0}^{(h,g)},\ A \in \mathcal{L}_{\mathcal{M}},\ \mathcal{M} \in \mathbb{M}(h) \right\},$$

$$\left\{ \widetilde{\vartheta}_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega|Z,W) \,\middle|\, A \in \mathcal{L}_{\mathcal{M}} \right\}$$

and

$$\left\{ \vartheta^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega|W) \,\middle|\, A \in \mathcal{L}_{\mathcal{M}} \right\}$$

are the bases of $A(\Omega)$, $\Delta^J T_{\mathcal{M}}(\Omega)$ and $T_{\mathcal{M}}(\Omega)$ respectively. □

REMARK 2.5. We may express the inverse mapping of $\Delta^J : T_{\mathcal{M}}(\Omega) \longrightarrow \Delta^J T_{\mathcal{M}}(\Omega)$ in terms of $\mathcal{D}_{ma}$, Δ_{nb} $(1 \le m, n \le h,\ 1 \le a, b \le g)$. The expression is very complicated and so we omit it.

3. Decomposition theorem on differential polynomials of theta functions of high level

In this section, we prove the algebra isomorphism theorem.

THEOREM 3. *The replacement*

$$\Delta^J \varphi(W) \longrightarrow \left(\frac{\partial}{\partial W}\right)^J \varphi(W) \quad (J \in \mathbb{Z}_{\geq 0}^{(h,g)},\ \varphi \in T(\Omega))$$

induces a $T(\Omega)$*-algebra isomorphism of* $A(\Omega)$ *onto the algebra*

$$\mathbb{C}\left[\cdots, \left(\frac{\partial}{\partial W}\right)^J \vartheta^{(\mathcal{M})}\begin{bmatrix} A_{\alpha,\mathcal{M}} \\ 0 \end{bmatrix}(\Omega|W), \cdots\right], \quad A_{\alpha,\mathcal{M}} \in \mathcal{L}_{\mathcal{M}},\ \mathcal{M} \in \mathbb{M}(h)$$

of differential polynomials of theta functions, namely

(1) $G\left(\cdots, \Delta^J \vartheta^{(\mathcal{M})}\begin{bmatrix} A_{\alpha,\mathcal{M}} \\ 0 \end{bmatrix}(\Omega|W), \cdots\right) = 0$

if and only if $G\left(\cdots, \left(\frac{\partial}{\partial W}\right)^J \vartheta^{(\mathcal{M})}\begin{bmatrix} A_{\alpha,\mathcal{M}} \\ 0 \end{bmatrix}(\Omega|W), \cdots\right) = 0,$

(2) $G\left(\cdots, \Delta^J \vartheta^{(\mathcal{M})}\begin{bmatrix} A_{\alpha,\mathcal{M}} \\ 0 \end{bmatrix}(\Omega|W), \cdots\right)$

$$= G\left(\cdots, \left(\frac{\partial}{\partial W}\right)^J \vartheta^{(\mathcal{M})}\begin{bmatrix} A_{\alpha,\mathcal{M}} \\ 0 \end{bmatrix}(\Omega|W), \cdots\right)$$

if and only if $G\left(\cdots, \left(\frac{\partial}{\partial W}\right)^J \vartheta^{(\mathcal{M})}\begin{bmatrix} A_{\alpha,\mathcal{M}} \\ 0 \end{bmatrix}(\Omega|W), \cdots\right) \in T(\Omega).$

PROOF. It is enough to assume that $G\left(\cdots, \Delta^J \vartheta^{(\mathcal{M})}\begin{bmatrix} A_{\alpha,\mathcal{M}} \\ 0 \end{bmatrix}(\Omega|W), \cdots\right)$ belongs to $A_{\mathcal{M}}(\Omega)$ for some $\mathcal{M} \in \mathbb{M}(h)$. Suppose

$$G\left(\cdots, \Delta^J \vartheta^{(\mathcal{M})}\begin{bmatrix} A_{\alpha,\mathcal{M}} \\ 0 \end{bmatrix}(\Omega|W), \cdots\right) = 0.$$

By putting $Z = 0$, we obtain

$$G\left(\cdots, \left(\frac{\partial}{\partial W}\right)^J \vartheta^{(\mathcal{M})}\begin{bmatrix} A_{\alpha,\mathcal{M}} \\ 0 \end{bmatrix}(\Omega|W), \cdots\right) = 0.$$

Conversely, we suppose that $G\left(\cdots, \left(\frac{\partial}{\partial W}\right)^J \vartheta^{(\mathcal{M})}\begin{bmatrix} A_{\alpha,\mathcal{M}} \\ 0 \end{bmatrix}(\Omega|W), \cdots\right) = 0.$

According to Theorem 2, we may write

$$G\left(\cdots,\Delta^J\vartheta^{(\mathcal{M})}\begin{bmatrix}A_{\alpha,\mathcal{M}}\\0\end{bmatrix}(\Omega|W),\cdots\right)=\sum_K \Delta^K\phi_K(W), \tag{3.1}$$

where $\phi_K\in T_{\mathcal{M}}(\Omega)$. Then we have

$$\begin{aligned}\sum_K\left(\frac{\partial}{\partial W}\right)^K\phi_K(W)&=G\left(\cdots,\Delta^J\vartheta^{(\mathcal{M})}\begin{bmatrix}A_{\alpha,\mathcal{M}}\\0\end{bmatrix}(\Omega|W),\cdots\right)\Big|_{Z=0}\\&=G\left(\cdots,\left(\frac{\partial}{\partial W}\right)^J\vartheta^{(\mathcal{M})}\begin{bmatrix}A_{\alpha,\mathcal{M}}\\0\end{bmatrix}(\Omega|W),\cdots\right)=0.\end{aligned}$$

Therefore it suffices to show $\phi_K(W)=0$ under the condition

$$\sum_K\left(\frac{\partial}{\partial W}\right)^K\phi_K(W)=0\quad\text{and}\quad\phi_K\in T_{\mathcal{M}}(\Omega).$$

For each $\xi\in\mathbb{Z}^{(h,g)}$, we get

$$\phi_K(W+\xi\Omega)=\exp\left\{-\pi i\sigma(\mathcal{M}(\xi\Omega\,{}^t\xi+2W\,{}^t\xi))\right\}\phi_K(W)$$

and

$$\begin{aligned}&\sum_K\left(\frac{\partial}{\partial W}\right)^K\phi_K(W+\xi\Omega)\\&=\sum_K\left(\frac{\partial}{\partial W}\right)^K\left[\exp\left\{-\pi i\sigma(\mathcal{M}(\xi\Omega\,{}^t\xi+2W\,{}^t\xi))\right\}\phi_K(W)\right]\\&=\exp\left\{-\pi i\sigma(\mathcal{M}(\xi\Omega\,{}^t\xi+2W\,{}^t\xi))\right\}\\&\quad\times\sum_K\sum_P\binom{K}{P}(-2\pi i\,\mathcal{M}\xi)^P\left(\frac{\partial}{\partial W}\right)^{K-P}\phi_K(W).\end{aligned}$$

Here if $K=(K_{ma})$ and $P=(P_{ma})$ in $\mathbb{Z}_{\geq 0}^{(h,g)}$, we put

$$\binom{K}{P}:=\binom{K_{11}}{P_{11}}\cdots\binom{K_{ma}}{P_{ma}}\cdots\binom{K_{hg}}{P_{hg}}$$

and if $\mathcal{M}=(\mathcal{M}_{kl})$ and $\xi=(\xi_{ma})\in\mathbb{Z}^{(h,g)}$, we put

$$(-2\pi i\,\mathcal{M}\xi)^P:=\left(-2\pi i\sum_{l=1}^h\mathcal{M}_{1l}\xi_{l1}\right)^{P_{11}}\cdots\left(-2\pi i\sum_{l=1}^h\mathcal{M}_{hl}\xi_{lg}\right)^{P_{hg}}.$$

Thus we have

$$\sum_K \sum_P \binom{K}{P} (-2\pi i\, \mathcal{M}\xi)^P \left(\frac{\partial}{\partial W}\right)^{K-P} \phi_K(W) = 0 \tag{3.2}$$

for all $\xi \in \mathbb{Z}^{(h,g)}$. Let K_0 be one of maximal K in the above sum (3.2). Then the coefficients of ξ^{K_0} in the polynomial relation (3.2) in ξ is given by $C(\mathcal{M})\,\phi_{K_0}(W)$ with nonzero constant $C(\mathcal{M}) \neq 0$. Thus we get $\phi_{K_0}(W) = 0$. Continuing this process successively, we have $\phi_K = 0$ for all K appearing in the sum (3.1). Hence from (3.1), we have

$$G\left(\cdots, \Delta^J \vartheta^{(\mathcal{M})} \begin{bmatrix} A_{\alpha,\mathcal{M}} \\ 0 \end{bmatrix} (\Omega|W), \cdots\right) = 0.$$

We assume that

$$\begin{aligned} &G\left(\cdots, \Delta^J \vartheta^{(\mathcal{M})} \begin{bmatrix} A_{\alpha,\mathcal{M}} \\ 0 \end{bmatrix} (\Omega|W), \cdots\right) \\ &= G\left(\cdots, \left(\frac{\partial}{\partial W}\right)^J \vartheta^{(\mathcal{M})} \begin{bmatrix} A_{\alpha,\mathcal{M}} \\ 0 \end{bmatrix} (\Omega|W), \cdots\right). \end{aligned}$$

Then we have, for any $(\xi, \eta) \in \mathbb{Z}^{(h,g)} \times \mathbb{Z}^{(h,g)}$,

$$\begin{aligned} &G\left(\cdots, \left(\frac{\partial}{\partial W}\right)^J \vartheta^{(\mathcal{M})} \begin{bmatrix} A_{\alpha,\mathcal{M}} \\ 0 \end{bmatrix} (\Omega|W), \cdots\right)\Bigg|_{W \mapsto W+\xi\Omega+\eta} \\ &= G\left(\cdots, \Delta^J \vartheta^{(\mathcal{M})} \begin{bmatrix} A_{\alpha,\mathcal{M}} \\ 0 \end{bmatrix} (\Omega|W), \cdots\right)\Bigg|_{(Z,W)\mapsto(Z+\xi, W+\xi\Omega+\eta)} \\ &= \exp\left\{-\pi i \sigma(\mathcal{M}(\xi\Omega\, {}^t\xi + 2W\, {}^t\xi))\right\} G\left(\cdots, \Delta^J \vartheta^{(\mathcal{M})} \begin{bmatrix} A_{\alpha,\mathcal{M}} \\ 0 \end{bmatrix} (\Omega|W), \cdots\right) \\ &= \exp\left\{-\pi i \sigma(\mathcal{M}(\xi\Omega\, {}^t\xi + 2W\, {}^t\xi))\right\} G\left(\cdots, \left(\frac{\partial}{\partial W}\right)^J \vartheta^{(\mathcal{M})} \begin{bmatrix} A_{\alpha,\mathcal{M}} \\ 0 \end{bmatrix} (\Omega|W), \cdots\right). \end{aligned}$$

Therefore we obtain

$$G\left(\cdots, \left(\frac{\partial}{\partial W}\right)^J \vartheta^{(\mathcal{M})} \begin{bmatrix} A_{\alpha,\mathcal{M}} \\ 0 \end{bmatrix} (\Omega|W), \cdots\right) \in T_{\mathcal{M}}(\Omega).$$

Conversely, we assume that

$$G\left(\cdots, \left(\frac{\partial}{\partial W}\right)^J \vartheta^{(\mathcal{M})} \begin{bmatrix} A_{\alpha,\mathcal{M}} \\ 0 \end{bmatrix} (\Omega|W), \cdots\right) \in T_{\mathcal{M}}(\Omega).$$

Applying (1) to

$$F\left(\cdots,\Delta^J\vartheta^{(\mathcal{M})}\begin{bmatrix}A_{\alpha,\mathcal{M}}\\0\end{bmatrix}(\Omega|W),\cdots\right)$$
$$=G\left(\cdots,\Delta^J\vartheta^{(\mathcal{M})}\begin{bmatrix}A_{\alpha,\mathcal{M}}\\0\end{bmatrix}(\Omega|W),\cdots\right)$$
$$-G\left(\cdots,\left(\frac{\partial}{\partial W}\right)^J\vartheta^{(\mathcal{M})}\begin{bmatrix}A_{\alpha,\mathcal{M}}\\0\end{bmatrix}(\Omega|W),\cdots\right),$$

we obtain

$$F\left(\cdots,\Delta^J\vartheta^{(\mathcal{M})}\begin{bmatrix}A_{\alpha,\mathcal{M}}\\0\end{bmatrix}(\Omega|W),\cdots\right)=0.$$

Hence we get

$$G\left(\cdots,\Delta^J\vartheta^{(\mathcal{M})}\begin{bmatrix}A_{\alpha,\mathcal{M}}\\0\end{bmatrix}(\Omega|W),\cdots\right)$$
$$=G\left(\cdots,\left(\frac{\partial}{\partial W}\right)^J\vartheta^{(\mathcal{M})}\begin{bmatrix}A_{\alpha,\mathcal{M}}\\0\end{bmatrix}(\Omega|W),\cdots\right).$$

□

Combining Theorem 2 and Theorem 3, we obtain the decomposition theorem.

THEOREM 4. *The algebra* $\mathbb{C}\left[\cdots,\left(\frac{\partial}{\partial W}\right)^J\vartheta^{(\mathcal{M})}\begin{bmatrix}A_{\alpha,\mathcal{M}}\\0\end{bmatrix}(\Omega|W),\cdots\right]$ *of differential polynomials of theta functions has a canonical linear basis*

$$\left\{\left(\frac{\partial}{\partial W}\right)^J\vartheta^{(\mathcal{M})}\begin{bmatrix}A_{\alpha,\mathcal{M}}\\0\end{bmatrix}(\Omega|W)\,\middle|\,J\in\mathbb{Z}^{(h,g)}_{\geq 0},\ A_{\alpha,\mathcal{M}}\in\mathcal{L}_{\mathcal{M}},\ \mathcal{M}\in\mathbb{M}(h)\right\},\tag{3.3}$$

namely, differential polynomials of theta functions are uniquely expressed as linear combinations of (3.3) *with constant coefficients depending only on* Ω.

REMARK 3.1. In [M3], Morikawa proved the decomposition theorem on differential polynomials of theta functions in the case that $h=1$. He investigated the graded algebras of theta functions and of auxiliary theta functions:

$$\Theta_0(\Omega)=\sum_{n=1}^{\infty}\Theta_0^{(n)}(\Omega),\qquad \Theta(\Omega)=\sum_{n=1}^{\infty}\Theta^{(n)}(\Omega),$$

where $\Theta_0^{(n)}(\Omega)$ (resp. $\Theta^{(n)}(\Omega)$) denotes the vector space of theta functions (resp. auxiliary theta functions) of level n with respect to Ω. In this paper, when $h=1$, we investigated the following graded algebras

$$T(\Omega)=\sum_{n=1}^{\infty}\Theta_0^{(2n)}(\Omega),\qquad A(\Omega)=\sum_{n=1}^{\infty}\Theta^{(2n)}(\Omega)$$

of theta functions and of auxiliary theta functions of *even level* with respect to Ω.

References

[M1] H. Morikawa, Some analytic and geometric applications of the invariant theoretic method, Nagoya Math. J., **80** (1980), 1–47.

[M2] H. Morikawa On Poisson brackets of semi-invariants, Manifolds and Lie groups, Progress in Math., Birkhäuser, 1981, pp. 267–281.

[M3] H. Morikawa, A decomposition theorem on differential polynomials of theta functions, Nagoya Math. J., **96** (1984), 113–126.

[Y1] J.-H. Yang, Harmonic analysis on the quotient spaces of Heisenberg groups, Nagoya Math. J., **123** (1991), 103–117.

[Y2] J.-H. Yang, Harmonic analysis on the quotient spaces of Heisenberg groups, II, J. Number Theory, **49**(1) (1994), 63–72.

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FOCK REPRESENTATIONS OF THE HEISENBERG GROUP $H_{\mathbb{R}}^{(g,h)}$

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ABSTRACT. In this paper, we introduce the Fock representation $U^{F,\mathcal{M}}$ of the Heisenberg group $H_{\mathbb{R}}^{(g,h)}$ associated with a positive definite symmetric half-integral matrix $\mathcal{M}$ of degree h and prove that $U^{F,\mathcal{M}}$ is unitarily equivalent to the Schrödinger representation of index $\mathcal{M}$.

1. Introduction

For any positive integers g and h, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(g,h)} := \Big\{ (\lambda,\mu,\kappa) \mid \lambda,\mu \in \mathbb{R}^{(h,g)},\ \kappa \in \mathbb{R}^{(h,h)},\ \kappa + \mu\,{}^t\lambda \text{ symmetric } \Big\}$$

endowed with the following multiplication law

$$(1.1)\quad (\lambda,\mu,\kappa)\circ(\lambda',\mu',\kappa') := (\lambda+\lambda',\mu+\mu',\kappa+\kappa'+\lambda\,{}^t\mu'-\mu\,{}^t\lambda').$$

The Heisenberg group $H_{\mathbb{R}}^{(g,h)}$ is embedded in the symplectic group $Sp(g+h,\mathbb{R})$ via the mapping

$$H_{\mathbb{R}}^{(g,h)} \ni (\lambda,\mu,\kappa) \longmapsto \begin{pmatrix} E_g & 0 & 0 & {}^t\mu \\ \lambda & E_h & \mu & \kappa \\ 0 & 0 & E_g & -{}^t\lambda \\ 0 & 0 & 0 & E_h \end{pmatrix} \in Sp(g+h,\mathbb{R}).$$

This Heisenberg group is a 2-step nilpotent Lie group and is important in the study of toroidal compactification of Siegel moduli spaces. In

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fact, $H_{\mathbb{R}}^{(g,h)}$ is obtained as the unipotent radical of the parabolic subgroup of $Sp(g+h,\mathbb{R})$ associated with the rational boundary component F_g (cf. [4] p. 21).

The purpose of this article is to study the Fock representation of the Heisenberg group $H_{\mathbb{R}}^{(g,h)}$ associated with a positive definite symmetric half-integral matrix of degree h. This paper is organized as follows. In section two, we review the Schrödinger representation $U(\sigma_c)$ of $H_{\mathbb{R}}^{(g,h)}$ associated with a real symmetric matrix c of degree h. In section three, we construct the *Fock representation* $U^{F,\mathcal{M}}$ of the Heisenberg group $H_{\mathbb{R}}^{(g,h)}$ associated with a positive definite symmetric half-integral matrix $\mathcal{M}$ of degree h and prove that $U^{F,\mathcal{M}}$ is unitarily equivalent to the Schrödinger representation $U(\sigma_{\mathcal{M}})$ of index $\mathcal{M}$. For more results on the Heisenberg group $H_{\mathbb{R}}^{(g,h)}$, we refer to [5]-[11].

NOTATIONS. We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers, and the field of complex numbers respectively. $\mathbb{C}_1^\times$ denotes the multiplicative group consisting of all complex numbers z with $|z|=1$. $Sp(g,\mathbb{R})$ denotes the symplectic group of degree g. The symbol ":=" means that the expression on the right is the definition of that on the left. We denote by $\mathbb{Z}^+$ the set of all positive integers. $F^{(k,l)}$ denotes the set of all $k\times l$ matrices with entries in a commutative ring F. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. For $A \in F^{(k,k)}$, $\sigma(A)$ denotes the trace of A. E_k denotes the identity matrix of degree k. For a positive integer n, $Sym\,(n,K)$ denotes the vector space consisting of all symmetric $n\times n$ matrices with entries in a field K.

$$\begin{aligned}
\mathbb{Z}_{\geq 0}^{(h,g)} &= \left\{ J=(J_{ka}) \in \mathbb{Z}^{(h,g)} \mid J_{ka} \geq 0 \ \text{ for all } k,a \right\}, \\
|J| &= \sum_{k,a} J_{k,a}, \\
J \pm \epsilon_{kl} &= (J_{11},\cdots,J_{ka}\pm 1,\cdots,J_{hg}), \\
J! &= J_{11}!\cdots J_{ka}!\cdots J_{hg}!.
\end{aligned}$$

For $\xi=(\xi_{ka})\in\mathbb{R}^{(h,g)}$ or $\mathbb{C}^{(h,g)}$ and $J=(J_{ka})\in\mathbb{Z}_{\geq 0}^{(h,g)}$, we denote

$$\xi^J = \xi_{11}^{J_{11}}\,\xi_{12}^{J_{12}}\cdots\xi_{ka}^{J_{ka}}\cdots\xi_{hg}^{J_{hg}}.$$

2. Schrödinger representations

First of all, we observe that $H_{\mathbb{R}}^{(g,h)}$ is a 2-step nilpotent Lie group. It is easy to see that the inverse of an element $(\lambda,\mu,\kappa)\in H_{\mathbb{R}}^{(g,h)}$ is given by

$$(\lambda,\mu,\kappa)^{-1}=(-\lambda,-\mu,-\kappa+\lambda\,{}^t\mu-\mu\,{}^t\lambda).$$

Now we put

$$(2.1)\qquad [\lambda,\mu,\kappa]:=(0,\mu,\kappa)\circ(\lambda,0,0)=(\lambda,\mu,\kappa-\mu\,{}^t\lambda).$$

Then $H_{\mathbb{R}}^{(g,h)}$ may be regarded as a group equipped with the following multiplication

$$(2.2)\quad [\lambda,\mu,\kappa]\diamond[\lambda_0,\mu_0,\kappa_0]:=[\lambda+\lambda_0,\mu+\mu_0,\kappa+\kappa_0+\lambda\,{}^t\mu_0+\mu_0\,{}^t\lambda].$$

The inverse of $[\lambda,\mu,\kappa]\in H_{\mathbb{R}}^{(g,h)}$ is given by

$$[\lambda,\mu,\kappa]^{-1}=[-\lambda,-\mu,-\kappa+\lambda\,{}^t\mu+\mu\,{}^t\lambda].$$

We set

$$(2.3)\qquad K:=\left\{\,[0,\mu,\kappa]\in H_{\mathbb{R}}^{(g,h)}\;\middle|\;\mu\in\mathbb{R}^{(h,g)},\ \kappa={}^t\kappa\in\mathbb{R}^{(h,h)}\,\right\}.$$

Then K is a commutative normal subgroup of $H_{\mathbb{R}}^{(g,h)}$. Let $\hat{K}$ be the Pontrajagin dual of K, i.e., the commutative group consisting of all unitary characters of K. Then $\hat{K}$ is isomorphic to the additive group $\mathbb{R}^{(h,g)}\times Sym\,(h,\mathbb{R})$ via

$$(2.4)\quad <a,\hat{a}>:=e^{2\pi i\sigma(\hat{\mu}\,{}^t\mu+\hat{\kappa}\kappa)},\quad a=[0,\mu,\kappa]\in K,\ \hat{a}=(\hat{\mu},\hat{\kappa})\in\hat{K}.$$

We put

$$(2.5)\qquad S:=\left\{\,[\lambda,0,0]\in H_{\mathbb{R}}^{(g,h)}\;\middle|\;\lambda\in\mathbb{R}^{(h,g)}\,\right\}\cong\mathbb{R}^{(h,g)}.$$

Then S acts on K as follows:

$$(2.6)\qquad \alpha_\lambda([0,\mu,\kappa]):=[0,\mu,\kappa+\lambda\,{}^t\mu+\mu\,{}^t\lambda],\quad [\lambda,0,0]\in S.$$

It is easy to see that the Heisenberg group $\left(H_{\mathbb{R}}^{(g,h)},\diamond\right)$ is isomorphic to the semidirect product $S \ltimes K$ of S and K whose multiplication is given by

$$(\lambda,a)\cdot(\lambda_0,a_0):=(\lambda+\lambda_0,a+\alpha_\lambda(a_0)),\quad \lambda,\lambda_0\in S,\ a,a_0\in K.$$

On the other hand, S acts on $\hat{K}$ by

$$\alpha_\lambda^*(\hat{a}):=(\hat{\mu}+2\hat{\kappa}\lambda,\hat{\kappa}),\quad [\lambda,0,0]\in S,\quad a=(\hat{\mu},\hat{\kappa})\in\hat{K}. \tag{2.7}$$

Then we have the relation $<\alpha_\lambda(a),\hat{a}>=<a,\alpha_\lambda^*(\hat{a})>$ for all $a\in K$ and $\hat{a}\in\hat{K}$.

We have two types of S-orbits in $\hat{K}$.

TYPE I. Let $\hat{\kappa}\in Sym\,(h,\mathbb{R})$ with $\hat{\kappa}\neq 0$. The S-orbit of $\hat{a}(\hat{\kappa}):=(0,\hat{\kappa})\in\hat{K}$ is given by

$$\hat{\mathcal{O}}_{\hat{\kappa}}:=\left\{(2\hat{\kappa}\lambda,\hat{\kappa})\in\hat{K}\;\middle|\;\lambda\in\mathbb{R}^{(h,g)}\right\}\cong\mathbb{R}^{(h,g)}. \tag{2.8}$$

TYPE II. Let $\hat{y}\in\mathbb{R}^{(h,g)}$. The S-orbit $\hat{\mathcal{O}}_{\hat{y}}$ of $\hat{a}(\hat{y}):=(\hat{y},0)$ is given by

$$\hat{\mathcal{O}}_{\hat{y}}:=\{(\hat{y},0)\}=\hat{a}(\hat{y}). \tag{2.9}$$

We have

$$\hat{K}=\left(\bigcup_{\hat{\kappa}\in Sym(h,\mathbb{R})}\hat{\mathcal{O}}_{\hat{\kappa}}\right)\bigcup\left(\bigcup_{\hat{y}\in\mathbb{R}^{(h,g)}}\hat{\mathcal{O}}_{\hat{y}}\right)$$

as a set. The stabilizer $S_{\hat{\kappa}}$ of S at $\hat{a}(\hat{\kappa})=(0,\hat{\kappa})$ is given by

$$S_{\hat{\kappa}}=\{0\}. \tag{2.10}$$

And the stabilizer $S_{\hat{y}}$ of S at $\hat{a}(\hat{y})=(\hat{y},0)$ is given by

$$S_{\hat{y}}=\left\{[\lambda,0,0]\;\middle|\;\lambda\in\mathbb{R}^{(h,g)}\right\}=S\cong\mathbb{R}^{(h,g)}. \tag{2.11}$$

From now on, we set $G:=H_{\mathbb{R}}^{(g,h)}$ for brevity. K is a closed, commutative normal subgroup of G. Since $(\lambda,\mu,\kappa)=(0,\mu,\kappa+\mu\,{}^t\lambda)\circ(\lambda,0,0)$

for $(\lambda, \mu, \kappa) \in G$, the homogeneous space $X := K\backslash G$ is identified with $\mathbb{R}^{(h,g)}$ via

$$Kg = K \circ (\lambda, 0, 0) \longmapsto \lambda, \quad g = (\lambda, \mu, \kappa) \in G.$$

We observe that G acts on X by

$$(2.12) \qquad (Kg) \cdot g_0 := K(\lambda + \lambda_0, 0, 0) = \lambda + \lambda_0,$$

where $g = (\lambda, \mu, \kappa) \in G$ and $g_0 = (\lambda_0, \mu_0, \kappa_0) \in G$.

If $g = (\lambda, \mu, \kappa) \in G$, we have

$$(2.13) \qquad k_g = (0, \mu, \kappa + \mu\, {}^t\lambda), \quad s_g = (\lambda, 0, 0)$$

in the Mackey decomposition of $g = k_g \circ s_g$ (cf. [3]). Thus if $g_0 = (\lambda_0, \mu_0, \kappa_0) \in G$, then we have

$$(2.14) \qquad s_g \circ g_0 = (\lambda, 0, 0) \circ (\lambda_0, \mu_0, \kappa_0) = (\lambda + \lambda_0, \mu_0, \kappa_0 + \lambda\, {}^t\mu_0)$$

and so

$$(2.15) \qquad k_{s_g \circ g_0} = (0, \mu_0, \kappa_0 + \mu_0\, {}^t\lambda_0 + \lambda\, {}^t\mu_0 + \mu_0\, {}^t\lambda).$$

For a real symmetric matrix $c = {}^tc \in \mathbb{R}^{(h,h)}$ with $c \neq 0$, we consider the one-dimensional unitary representation σ_c of K defined by

$$(2.16) \qquad \sigma_c((0, \mu, \kappa)) := e^{2\pi i \sigma(c\kappa)}\, I, \quad (0, \mu, \kappa) \in K,$$

where I denotes the identity mapping. Then the induced representation $U(\sigma_c) := \mathrm{Ind}_K^G\, \sigma_c$ of G induced from σ_c is realized in the Hilbert space $\mathcal{H}_{\sigma_c} = L^2(X, d\dot{g}, \mathbb{C}) \cong L^2\left(\mathbb{R}^{(h,g)}, d\xi\right)$ as follows. If $g_0 = (\lambda_0, \mu_0, \kappa_0) \in G$ and $x = Kg \in X$ with $g = (\lambda, \mu, \kappa) \in G$, we have

$$(2.17) \qquad \left(U_{g_0}(\sigma_c) f\right)(x) = \sigma_c\left(k_{s_g \circ g_0}\right)\left(f(x g_0)\right), \quad f \in \mathcal{H}_{\sigma_c}.$$

It follows from (2.15) that

$$(2.18) \qquad \left(U_{g_0}(\sigma_c) f\right)(\lambda) = e^{2\pi i \sigma\{c(\kappa_0 + \mu_0\, {}^t\lambda_0 + 2\lambda\, {}^t\mu_0)\}}\, f(\lambda + \lambda_0).$$

Here we identified $x = Kg$ (resp. $xg_0 = Kgg_0$) with λ (resp. $\lambda+\lambda_0$). The induced representation $U(\sigma_c)$ is called the *Schrödinger representation* of G associated with σ_c. Thus $U(\sigma_c)$ is a monomial representation.

Now we denote by $\mathcal{H}^{\sigma_c}$ the Hilbert space consisting of all functions $\phi : G \longrightarrow \mathbb{C}$ which satisfy the following conditions:

(1) $\phi(g)$ is measurable with respect to dg.

(2) $\phi((0,\mu,\kappa)\circ g)) = e^{2\pi i\sigma(c\kappa)}\phi(g)$ for all $g \in G$.

(3) $\| \phi \|^2 := \int_X |\phi(g)|^2\, d\dot{g} < \infty, \quad \dot{g} = Kg$,

where dg (resp. $d\dot{g}$) is a G-invariant measure on G (resp. $X = K\backslash G$). The inner product $(\,,\,)$ on $\mathcal{H}^{\sigma_c}$ is given by

$$(\phi_1,\phi_2) := \int_G \phi_1(g)\,\overline{\phi_2(g)}\,dg, \quad \phi_1,\ \phi_2 \in \mathcal{H}^{\sigma_c}.$$

We observe that the mapping $\Phi_c : \mathcal{H}_{\sigma_c} \longrightarrow \mathcal{H}^{\sigma_c}$ defined by

(2.19)
$$(\Phi_c(f))(g) := e^{2\pi i\sigma\{c(\kappa+\mu\,{}^t\lambda)\}}\,f(\lambda), \quad f \in \mathcal{H}_{\sigma_c},\ g = (\lambda,\mu,\kappa) \in G$$

is an isomorphism of Hilbert spaces. The inverse $\Psi_c : \mathcal{H}^{\sigma_c} \longrightarrow \mathcal{H}_{\sigma_c}$ of Φ_c is given by

(2.20)
$$(\Psi_c(\phi))(\lambda) := \phi((\lambda,0,0)), \quad \phi \in \mathcal{H}^{\sigma_c},\ \lambda \in \mathbb{R}^{(h,g)}.$$

The Schrödinger representation $U(\sigma_c)$ of G on $\mathcal{H}^{\sigma_c}$ is given by

(2.21)
$$(U_{g_0}(\sigma_c)\phi)(g) = e^{2\pi i\sigma\{c(\kappa_0+\mu_0\,{}^t\lambda_0+\lambda\,{}^t\mu_0-\lambda_0\,{}^t\mu)\}}\,\phi((\lambda_0,0,0)\circ g),$$

where $g_0 = (\lambda_0,\mu_0,\kappa_0)$, $g = (\lambda,\mu,\kappa) \in G$ and $\phi \in \mathcal{H}^{\sigma_c}$. (2.21) can be expressed as follows.

(2.22)
$$(U_{g_0}(\sigma_c)\phi)(g) = e^{2\pi i\sigma\{c(\kappa_0+\kappa+\mu_0\,{}^t\lambda_0+\mu\,{}^t\lambda+2\lambda\,{}^t\mu_0)\}}\,\phi((\lambda_0+\lambda,0,0)).$$

THEOREM 2.1. *Let c be a positive symmetric half-integral matrix of degree h. Then the Schrödinger representation $U(\sigma_c)$ of G is irreducible.*

Proof. The proof can be found in [5], Theorem 3. □

3. Fock representations

We consider the vector space $V^{(h,g)} := \mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)}$. We put

$$P_{ka} := (E_{ka}, 0), \quad Q_{lb} := (0, E_{lb}), \tag{3.1}$$

where $1 \leq k, l \leq h$ and $1 \leq a, b \leq g$. Then the set $\{P_{ka},\, Q_{ka}\}$ forms a basis for $V^{(h,g)}$. We define the alternating bilinear form $\mathbf{A} : V^{(h,g)} \times V^{(h,g)} \longrightarrow \mathbb{R}$ by

$$\mathbf{A}\left((\lambda_0, \mu_0), (\lambda, \mu)\right) := \sigma(\lambda_0\, {}^t\mu - \mu_0\, {}^t\lambda), \quad (\lambda_0, \mu_0),\, (\lambda, \mu) \in V^{(h,g)}. \tag{3.2}$$

Then we have

$$\mathbf{A}(P_{ka}, P_{lb}) = \mathbf{A}(Q_{ka}, Q_{lb}) = 0, \quad \mathbf{A}(P_{ka}, Q_{lb}) = \delta_{ab}\, \delta_{kl}, \tag{3.3}$$

where $1 \leq k, l \leq h$ and $1 \leq a, b \leq g$. Any element $v \in V^{(h,g)}$ can be written uniquely as

$$v = \sum_{k,a} x_{ka} P_{ka} + \sum_{l,b} y_{lb} Q_{lb}, \quad x_{ka},\, y_{lb} \in \mathbb{R}. \tag{3.4}$$

From now on, for brevity, we write $V := V^{(h,g)}$ and $v = xP + yQ$ instead of (3.4). Then it is easy to see that the endomorphism $J : V \longrightarrow V$ defined by

$$J(xP + yQ) := -yP + xQ, \quad xP + yQ \in V \tag{3.5}$$

is a complex structure on V which is compatible with the alternating bilinear form $\mathbf{A}$. This means that J is an endomorphism of V satisfying the following conditions:

(J1) $J^2 = -I$ on V.

(J2) $\mathbf{A}(Jv_0, Jv) = \mathbf{A}(v_0, v)$ for all $v_0, v \in V$.

(J3) $\mathbf{A}(v, Jv) > 0$ for all $v \in V$ with $v \neq 0$.

Now we let $V_{\mathbb{C}} = V + iV$ be the complexification of V, where $i = \sqrt{-1}$. For an element $w = v_1 + iv_2 \in V_{\mathbb{C}}$ with $v_1, v_2 \in V$, we put

$$\bar{w} := v_1 - iv_2. \tag{3.6}$$

Let $\mathbf{A}_{\mathbb{C}}$ be the complex bilinear form on $V_{\mathbb{C}}$ extending $\mathbf{A}$ and let $J_{\mathbb{C}}$ be the complex linear map of $V_{\mathbb{C}}$ extending J. Since $J_{\mathbb{C}}^2 = -I$, $J_{\mathbb{C}}$ has the only eigenvalues $\pm i$. We denote by V^+ (resp. V^-) the eigenspace of $V_{\mathbb{C}}$ corresponding to the eigenvalues i (resp. $-i$). Thus $V_{\mathbb{C}} = V^+ + V^-$. Since

$$J_{\mathbb{C}}(P_{ka} \pm iQ_{ka}) = \mp i(P_{ka} \pm iQ_{ka}),$$

we have

$$\text{(3.7)} \qquad V^+ = \sum_{k,a} \mathbb{C}\,(P_{ka} - iQ_{ka}), \quad V^- = \sum_{k,a} \mathbb{C}\,(P_{ka} + iQ_{ka}).$$

Let

$$\text{(3.8)} \qquad V_* := \sum_{k,a} \mathbb{C}\,P_{ka}, \quad 1 \le k \le h, \;\; 1 \le a \le g$$

be the subspace of $V_{\mathbb{C}}$ as a $\mathbb{C}$-vector space. It is easy to see that V_* is isomorphic to V as $\mathbb{R}$-vector spaces via the isomorphism $T : V \longrightarrow V_*$ defined by

$$\text{(3.9)} \qquad T(P_{ka}) := P_{ka}, \quad T(Q_{lb}) := iP_{lb}.$$

We define the complex linear map $J_* : V_* \longrightarrow V_*$ by $J_*(P_{ka}) = iP_{ka}$ for $1 \le k \le h$, $1 \le a \le g$. Then J_* is compatible with J, that is, $T \circ J = J_* \circ T$. It is easily seen that there exists a unique hermitian form $\mathbf{H}$ on V_* with $\operatorname{Im}\mathbf{H} = \mathbf{A}$. Indeed, $\mathbf{H}$ is given by

$$\text{(3.10)} \qquad \mathbf{H}(v, w) = \mathbf{A}(v, J_* w) + i\mathbf{A}(v, w), \quad v, w \in V_*.$$

For $v = \sum_{k,a} z_{ka} P_{ka} \in V_*$ with $z_{ka} = x_{ka} + iy_{ka}$ $(x_{ka}, y_{ka} \in \mathbb{R})$, for brevity we write $v = zP$. For two elements $v = zP$ and $v' = z'P$ in V_*, $\mathbf{H}(v, v') = \sum_{k,a} \overline{z_{ka}}\, z'_{ka}$.

We observe that

$$V_{\mathbb{C}} = \sum_{k,a} \mathbb{C}\,P_{ka} + \sum_{l,b} \mathbb{C}\,Q_{lb} = V^+ + V^- \supset V^{\pm}.$$

For $w = z^0 P + z^1 Q \in V_{\mathbb{C}}$, we put

$$w = w^+ + w^-, \quad w^+ := z^+(P - iQ), \quad w^- := z^-(P + iQ).$$

The relations among z^0, z^1, z^+, z^- are given by

$$(3.11) \qquad z^{\pm} = \frac{1}{2}(z^0 \pm i z^1), \quad z^0 = z^+ + z^-, \quad z^1 = i(z^- - z^+).$$

Precisely, (3.11) implies that

$$z^{\pm}_{ka} = \frac{1}{2}(z^0_{ka} \pm i z^1_{ka}), \quad z^0_{ka} = z^+_{ka} + z^-_{ka}, \quad z^1_{ka} = i(z^-_{ka} - z^+_{ka}),$$

where $1 \le k \le h$ and $1 \le a \le g$. It is easy to see that

$$(3.12) \quad \mathbf{A}_{\mathbb{C}}(w^-, w^+) = -2i \sum_{k,a} z^-_{ka} z^+_{ka} = -\frac{i}{2} \sum_{k,a} \left\{ (z^0_{ka})^2 + (z^1_{ka})^2 \right\}.$$

Let

$$G_{\mathbb{C}} := \left\{ (z^0, z^1, a) \mid z^0, z^1 \in \mathbb{C}^{(h,g)},\ a \in \mathbb{C}^{(h,h)},\ a + z^1\, {}^t z^0 \text{ symmetric} \right\}$$

be the complexification of the real Heisenberg group $G := H^{(h,g)}_{\mathbb{R}}$. Analogously in the real case, the multiplication on $G_{\mathbb{C}}$ is given by (1.1). If $w = z^0 P + z^1 Q := \sum_{k,a} z^0_{ka} P_{ka} + \sum_{l,b} z^1_{lb} Q_{lb}$, we identify z^0, z^1 with the $h \times g$ matrices respectively:

$$z^0 := \begin{pmatrix} z^0_{11} & z^0_{12} & \cdots & z^0_{1g} \\ z^0_{21} & z^0_{22} & \cdots & z^0_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ z^0_{h1} & z^0_{h2} & \cdots & z^0_{hg} \end{pmatrix}, \quad z^1 := \begin{pmatrix} z^1_{11} & z^1_{12} & \cdots & z^1_{1g} \\ z^1_{21} & z^1_{22} & \cdots & z^1_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ z^1_{h1} & z^1_{h2} & \cdots & z^1_{hg} \end{pmatrix}.$$

That is, we identify $w = z^0 P + z^1 Q \in V_{\mathbb{C}}$ with $(z^0, z^1) \in \mathbb{C}^{(h,g)} \times \mathbb{C}^{(h,g)}$. If $w = z^0 P + z^1 Q$, $\hat{w} = \hat{z^0} P + \hat{z^1} Q \in V_{\mathbb{C}}$, then

$$(3.13) \quad (w, a) \circ (\hat{w}, \hat{a}) = (w + \hat{w}, a + \hat{a} + z^0\, {}^t\hat{z}^1 - z^1\, {}^t\hat{z}^0), \quad a, \hat{a} \in \mathbb{C}^{(h,h)}.$$

From now on, for brevity we put

$$(3.14) \qquad R^+ := P - iQ, \qquad R^- := P + iQ.$$

If $w = z^+ R^+ + z^- R^-$, $\hat{w} = \hat{z}^+ R^+ + \hat{z}^- R^- \in V_{\mathbb{C}}$, by an easy computation, we have

$$(w, a) \circ (\hat{w}, \hat{a}) = (\tilde{w}, a + \hat{a} + 2i(z^+ \, {}^t\hat{z}^- - z^- \, {}^t\hat{z}^+)) \tag{3.15}$$

with

$$\tilde{w} = (z^+ + \hat{z}^+) R^+ + (z^- + \hat{z}^-) R^-.$$

Here we identified z^+, z^- with $h \times g$ matrices

$$z^+ := \begin{pmatrix} z^+_{11} & z^+_{12} & \cdots & z^+_{1g} \\ z^+_{21} & z^+_{22} & \cdots & z^+_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ z^+_{h1} & z^+_{h2} & \cdots & z^+_{hg} \end{pmatrix}, \quad z^- := \begin{pmatrix} z^-_{11} & z^-_{12} & \cdots & z^-_{1g} \\ z^-_{21} & z^-_{22} & \cdots & z^-_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ z^-_{h1} & z^-_{h2} & \cdots & z^-_{hg} \end{pmatrix}.$$

It is easy to see that

$$P_{\mathbb{C}} := \left\{ (w^-, a) \in G_{\mathbb{C}} \,|\, w^- \in V^-, \;\; a \in \mathbb{C}^{(h,h)} \right\} \tag{3.16}$$

is a commutative subgroup of $G_{\mathbb{C}}$ and

$$G \cap P_{\mathbb{C}} = \mathcal{Z}, \quad G_{\mathbb{C}} = G \circ P_{\mathbb{C}},$$

where $\mathcal{Z} := \left\{ (0, 0, \kappa) \in G \,|\, \kappa = {}^t\kappa \in \mathbb{R}^{(h,h)} \right\} \cong Sym\,(h, \mathbb{R})$ is the center of G. Moreover,

$$P_{\mathbb{C}} \backslash G_{\mathbb{C}} \cong V^+ \cong \mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)} \cong \mathcal{Z} \backslash G. \tag{3.17}$$

For $c = {}^tc \in \mathrm{Sym}\,(h, \mathbb{R})$ with $c > 0$, we let $\delta_c : P_{\mathbb{C}} \longrightarrow \mathbb{C}^\times$ be a quasi-character of $P_{\mathbb{C}}$ defined by

$$\delta_c((w^-, a)) := e^{2\pi i \sigma(ca)}, \quad (w^-, a) \in P_{\mathbb{C}}. \tag{3.18}$$

Let

$$U^{F,c} := \mathrm{Ind}_{P_{\mathbb{C}}}^{G_{\mathbb{C}}} \delta_c$$

be the representation of $G_{\mathbb{C}}$ induced from a quasi-character δ_c of $P_{\mathbb{C}}$. Then $U^{F,c}$ is realized in the Hilbert space $\mathcal{H}^{F,c}$ consisting of all holomorphic functions $\psi : G_{\mathbb{C}} \longrightarrow \mathbb{C}$ satisfying the following conditions:

(F1) $\psi((w^-,a)\circ g)=\delta_c((w^-,a))\psi(g)=e^{2\pi i\sigma(ca)}\,\psi(g)$ for all (w^-,a) $\in P_{\mathbb{C}}$ and $g\in G_{\mathbb{C}}$.

(F2) $\int_{\mathcal{Z}\backslash G}|\psi(\dot g)|^2\,d\dot g<\infty$.

The inner product $<,>_{F,c}$ on $\mathcal{H}^{F,c}$ is given by

$$<\psi_1,\psi_2>_{F,c}:=\int_{\mathcal{Z}\backslash G}\psi_1(\dot g)\,\overline{\psi_2(\dot g)}\,d\dot g,\quad \psi_1,\psi_2\in\mathcal{H}^{F,c},\ \dot g=\mathcal{Z}g.$$

$U^{F,c}$ is realized by the right regular representation of $G_{\mathbb{C}}$ on $\mathcal{H}^{F,c}$:

$$\left(U^{F,c}(g_0)\psi\right)(g)=\psi(gg_0),\quad \psi\in\mathcal{H}^{F,c},\ g_0,g\in G_{\mathbb{C}}. \tag{3.19}$$

Now we will show that $U^{F,c}$ is realized as a representation of G in the Fock space. The Fock space $\mathcal{H}_{F,c}$ is the Hilbert space consisting of all holomorphic functions $f:\mathbb{C}^{(h,g)}\cong V_*\longrightarrow\mathbb{C}$ satisfying the condition

$$\|\,f\,\|^2_{F,c}:=\int_{\mathbb{C}^{(h,g)}}|f(W)|^2\,e^{-2\pi\sigma(cW\,{}^t\overline{W})}\,dW<\infty.$$

The inner product $(\,,\,)_{F,c}$ on $\mathcal{H}_{F,c}$ is given by

$$(f_1,f_2)_{F,c}:=\int_{\mathbb{C}^{(h,g)}}f_1(W)\,\overline{f_2(W)}\,e^{-2\pi\sigma(cW\,{}^t\overline{W})}\,dW,\quad f_1,f_2\in\mathcal{H}_{F,c}.$$

LEMMA 3.1. *The mapping* $\Lambda:\mathcal{H}_{F,c}\longrightarrow\mathcal{H}^{F,c}$, $\Lambda_f:=\Lambda(f)\,(f\in\mathcal{H}_{F,c})$ *defined by*

$$\Lambda_f((z^0P+z^1Q,a)):=e^{2\pi i\sigma\{c(a+2iz^-\,{}^tz^+)\}}\,f(2z^+) \tag{3.20}$$

is an isometry of $\mathcal{H}_{F,c}$ *onto* $\mathcal{H}^{F,c}$, *where* $2z^\pm=z^0\pm iz^1$ (cf. (3.11)). *The inverse* $\Delta:\mathcal{H}^{F,c}\longrightarrow\mathcal{H}_{F,c}$, $\Delta_\psi:=\Delta(\psi)\,(\psi\in\mathcal{H}^{F,c})$ *is given by*

$$\Delta_\psi(W):=\psi\left(\frac{1}{2}WR^+\right),\quad W\in\mathbb{C}^{(h,g)}, \tag{3.21}$$

where $R^\pm=P\mp iQ$ *(cf. (3.14)).*

Proof. First we observe that for $w = z^0P + z^1Q = z^+R^+ + z^-R^- \in V_{\mathbb{C}}$,

$$(w,a) = (z^-R^-, a + 2iz^-\,{}^tz^+) \circ (z^+R^+, 0).$$

Thus if $\psi \in \mathcal{H}^{F,c}$ and $w = z^0P + z^1Q = z^+R^+ + z^-R^-$, by (F1),

$$\psi((w,a)) = e^{2\pi i\sigma\{c(a+2iz^-\,{}^tz^+)\}}\,\psi((z^+R^+,0)). \tag{3.22}$$

Let $W = x + iy \in \mathbb{C}^{(h,g)}$ with $x, y \in \mathbb{R}^{(h,g)}$. Then

$$xP + yQ = z^+R^+ + z^-R^-, \quad 2z^\pm = x \pm iy.$$

So $z^-\,{}^tz^+ = \frac{1}{4}W\,{}^t\overline{W}$. According to (3.22), if $\psi \in \mathcal{H}^{F,c}$, we have

$$\psi((xP+yQ,0)) = e^{-\pi\sigma(cW\,{}^t\overline{W})}\,\psi((\frac{1}{2}WR^+,0)).$$

Thus we get

$$|\psi((xP+yQ,0))|^2 = e^{-2\pi\sigma(cW\,{}^t\overline{W})}\left|\psi((\frac{1}{2}WR^+,0))\right|^2.$$

Therefore

$$\int_{Z\backslash G} |\psi(\dot{g})|^2\,d\dot{g} = \int_{\mathbb{C}^{(h,g)}} e^{-2\pi\sigma(cW\,{}^t\overline{W})}\,\left|\Delta_\psi(W)\right|^2 dW < \infty.$$

It is easy to see that Δ is the inverse of Λ. Hence we obtain the desired results. □

LEMMA 3.2. *The representation $U^{F,c}$ is realized as a representation of G in the Fock space $\mathcal{H}_{F,c}$ as follows. If $g = (\lambda P+\mu Q, \kappa) = (\lambda, \mu, \kappa) \in G$ and $f \in \mathcal{H}_{F,c}$, then*

$$\left(U^{F,c}(g)f\right)(W) = e^{2\pi i\sigma(c\kappa)}\,e^{-\pi\sigma\{c(\zeta\,{}^t\bar{\zeta}+2W\,{}^t\bar{\zeta})\}}\,f(W+\zeta), \quad W \in \mathbb{C}^{(h,g)}, \tag{3.23}$$

where $\zeta = \lambda + i\mu$.

Proof.

$$\begin{aligned}\left(U^{F,c}(g)f\right)(W) &= \left(\Delta(U^{F,c}(g)(\Lambda_f))\right)(W)\\ &= \left(U^{F,c}(g)(\Lambda_f)\right)\left(\frac{1}{2}WR^+\right)\\ &= \Lambda_f\left(\left(\frac{1}{2}WR^+,0\right)\circ g\right)\\ &= \Lambda_f\left(\left(\frac{1}{2}W,-\frac{i}{2}W,0\right)\circ(\lambda,\mu,\kappa)\right)\\ &= \Lambda_f\left((\lambda+\frac{1}{2}W)P+(\mu-\frac{i}{2}W)Q,\kappa+\frac{1}{2}W\,{}^t\mu+\frac{i}{2}W\,{}^t\lambda\right)\\ &= e^{2\pi i\sigma\{c(\kappa+\frac{i}{2}W\,{}^t\bar{\zeta}+\frac{i}{2}\bar{\zeta}\,{}^tW+\frac{i}{2}\bar{\zeta}\,{}^t\zeta)\}}\,f(W+\zeta) \qquad (*)\\ &= e^{2\pi i\sigma(c\kappa)}\cdot e^{-\pi\sigma\{c(\zeta\,{}^t\bar{\zeta}+W\,{}^t\bar{\zeta})\}}\,f(W+\zeta),\end{aligned}$$

where $\zeta=\lambda+i\mu$. In (*), we used (3.20) and the facts that $2iz^{-}\,{}^tz^{+}=\frac{i}{2}(\overline{W}\,{}^t\zeta+\overline{W}\,{}^tW)$ and $2z^{+}=W+\zeta$. □

DEFINITION 3.3. The induced representation $U^{F,c}$ of G in the Fock space $\mathcal{H}_{F,c}$ is called the *Fock representation* of G.

Let $W=U+iV\in\mathbb{C}^{(h,g)}$ with $U,V\in\mathbb{R}^{(h,g)}$. If $U=(u_{ka})$, $V=(v_{lb})$ are coordinates in $\mathbb{C}^{(h,g)}$, we put

$$dU=du_{11}du_{12}\cdots du_{hg},\quad dV=dv_{11}dv_{12}\cdots dv_{hg}$$

and $dW=dUdV$. And we set

$$d\mu(W):=e^{-\pi\sigma(W\,{}^t\overline{W})}\,dW. \tag{3.24}$$

Let f be a holomorphic function on $\mathbb{C}^{(h,g)}$. Then $f(W)$ has the Taylor expansion

$$f(z)=\sum_{J\in\mathbb{Z}_{\geq 0}^{(h,g)}}a_JW^J,\quad W=(w_{ka})\in\mathbb{C}^{(h,g)},$$

where $J=(J_{ka})\in J\in\mathbb{Z}_{\geq 0}^{(h,g)}$ and $W^J:=w_{11}^{J_{11}}w_{12}^{J_{12}}\cdots w_{hg}^{J_{hg}}$.

We set $|W|_\infty := \max_{k,a}(|w_{ka}|)$. Then by an easy computation, we have

$$\begin{aligned}\int_{\mathbb{C}^{(h,g)}} |f(W)|^2\, d\mu(W) &= \lim_{r\to\infty}\int_{|W|_\infty\le r} |f(W)|^2 d\mu(W)\\ &= \lim_{r\to\infty}\sum_{J,K} a_J\overline{a_K}\int_{|W|_\infty\le r} W^J\overline{W^K}\, d\mu(W)\\ &= \sum_J |a_J|^2\pi^{-|J|}J!,\end{aligned}$$

where J runs over $J\in\mathbb{Z}_{\ge 0}^{(h,g)}$.

Let $\mathcal{H}_{h,g}$ be the Hilbert space consisting of all holomorphic functions $f:\mathbb{C}^{(h,g)}\longrightarrow\mathbb{C}$ satisfying the condition

$$\|f\|^2 := \int_{\mathbb{C}^{(h,g)}} |f(W)|^2\, d\mu(W) < \infty. \tag{3.25}$$

The inner product $(\ ,\ )$ on $\mathcal{H}_{h,g}$ is given by

$$(f_1,f_2) := \int_{\mathbb{C}^{(h,g)}} f_1(W)\,\overline{f_2(W)}\, d\mu(W),\quad f_1,f_2\in\mathcal{H}_{h,g}.$$

Thus we have

LEMMA 3.4. *Let $f\in\mathcal{H}_{h,g}$ and let $f(W)=\sum_J a_J W^J$ be the Taylor expansion of f. Then*

$$\|f\|^2 = \sum_{J\in\mathbb{Z}_{\ge 0}^{(h,g)}} |a_J|^2\pi^{-|J|}J!.$$

For each $J\in\mathbb{Z}_{\ge 0}^{(h,g)}$, we define the holomorphic function $\Phi_J(W)$ on $\mathbb{C}^{(h,g)}$ by

$$\Phi_J(W) := (J!)^{-\frac{1}{2}}\left(\pi^{\frac{1}{2}}W\right)^J,\quad W\in\mathbb{C}^{(h,g)}. \tag{3.26}$$

Then

$$(\Phi_J,\Phi_K) = \begin{cases} 1 & \text{if } J=K\\ 0 & \text{otherwise.}\end{cases} \tag{3.27}$$

It is easy to see that the set $\left\{ \Phi_J \,|\, J \in \mathbb{Z}_{\geq 0}^{(h,g)} \right\}$ forms a complete orthonormal system in $\mathcal{H}_{h,g}$. By the Schwarz inequality, for any $f \in \mathcal{H}_{h,g}$, we have

$$|f(W)| \leq e^{\frac{\pi}{2}\sigma(W\,{}^t\overline{W})}\,\|f\|,\quad W \in \mathbb{C}^{(h,g)}. \tag{3.28}$$

Consequently, the norm convergence in $\mathcal{H}_{h,g}$ implies the uniform convergence on any bounded subset of $\mathbb{C}^{(h,g)}$. We observe that for a fixed $W' \in \mathbb{C}^{(h,g)}$, the holomorphic function $W \longrightarrow e^{\pi\sigma(W\,{}^t\overline{W'})}$ admits the following Taylor expansion

$$e^{\pi\sigma(W\,{}^t\overline{W'})} = \sum_{J \in \mathbb{Z}_{\geq 0}^{(h,g)}} \Phi_J(W)\,\Phi_J(\overline{W'}). \tag{3.29}$$

From (3.29), we obtain

$$\Phi_J(\overline{W'}) = (J!)^{-\frac{1}{2}} \int_{\mathbb{C}^{(h,g)}} e^{\pi\sigma(W\,{}^t\overline{W'})} \left(\pi^{\frac{1}{2}}\overline{W}\right)^J d\mu(W). \tag{3.30}$$

Thus if $f \in \mathcal{H}_{h,g}$, we get

$$\begin{aligned}\left(f(W),\, e^{\pi\sigma(W\,{}^t\overline{W'})}\right) &= \left(f,\, \sum_J \Phi_J(\overline{W'})\,\Phi_J(\cdot)\right) \\ &= \sum_J \Phi_J(W')\,(f, \Phi_J) \\ &= f(W').\end{aligned}$$

Hence $e^{\pi\sigma(W\,{}^t\overline{W'})}$ is the reproducing kernel for $\mathcal{H}_{h,g}$ in the sense that for any $f \in \mathcal{H}_{h,g}$,

$$f(W) = \int_{\mathbb{C}^{(h,g)}} e^{\pi\sigma(W\,{}^t\overline{W'})}\, f(W')\, d\mu(W'). \tag{3.31}$$

We set

$$\kappa(W, W') := e^{\pi\sigma(W\,{}^t\overline{W'})},\quad W, W' \in \mathbb{C}^{(h,g)}. \tag{3.32}$$

Obviously $\kappa(W,W') = \overline{\kappa(W',W)}$. (3.31) may be written as

$$f(W) = \int_{\mathbb{C}^{(h,g)}} \kappa(W,W')\, f(W')\, d\mu(W'), \quad f \in \mathcal{H}_{h,g}. \tag{3.33}$$

Let $\mathcal{M}$ be a positive definite, symmetric half-integral matrix of degree h. We define the measure

$$d\mu_{\mathcal{M}}(W) := e^{-2\pi\sigma(\mathcal{M}W\,{}^t\overline{W})}\, dW. \tag{3.34}$$

We recall the *Fock* space $\mathcal{H}_{F,\mathcal{M}}$ consisting of all holomorphic functions $f : \mathbb{C}^{(h,g)} \longrightarrow \mathbb{C}$ that satisfy the condition

$$\|f\|^2_{\mathcal{M}} := \|f\|^2_{F,\mathcal{M}} := \int_{\mathbb{C}^{(h,g)}} |f(W)|^2\, d\mu_{\mathcal{M}}(W) < \infty. \tag{3.35}$$

The inner product $(\ ,\ )_{\mathcal{M}} := (\ ,\ )_{F,\mathcal{M}}$ on $\mathcal{H}_{F,\mathcal{M}}$ is given by

$$(f_1, f_2)_{\mathcal{M}} := \int_{\mathbb{C}^{(h,g)}} f_1(W)\, \overline{f_2(W)}\, d\mu_{\mathcal{M}}(W), \quad f_1, f_2 \in \mathcal{H}_{F,\mathcal{M}}.$$

LEMMA 3.5. *Let $f \in \mathcal{H}_{F,\mathcal{M}}$ and let $g(W) := f\left((2\mathcal{M})^{-\frac{1}{2}}W\right)$ be the holomorphic function on $\mathbb{C}^{(h,g)}$. We let*

$$g(W) = \sum_{J \in \mathbb{Z}^{(h,g)}_{\geq 0}} a_{\mathcal{M},J}\, W^J$$

be the Taylor expansion of $g(W)$. Then we have

$$\|f\|^2_{\mathcal{M}} = (f,f)_{\mathcal{M}} = 2^{-g}(\det \mathcal{M})^{-g} \sum_{J \in \mathbb{Z}^{(h,g)}_{\geq 0}} |a_{\mathcal{M},J}|^2\, \pi^{-|J|} J!.$$

Proof. Let $\mathcal{M}^{\frac{1}{2}}$ be the unique positive definite symmetric matrix of degree h such that $\left(\mathcal{M}^{\frac{1}{2}}\right)^2 = \mathcal{M}$. We put $\tilde{W} := \sqrt{2}\mathcal{M}^{\frac{1}{2}}W$. Obviously $d\tilde{W} = 2^g\,(\det \mathcal{M})^g dW$. Thus for $f \in \mathcal{H}_{F,\mathcal{M}}$, we have

$$\begin{aligned}
(f,f)_{\mathcal{M}} &= \int_{\mathbb{C}^{(h,g)}} |f(W)|^2\, d\mu_{\mathcal{M}}(W) \\
&= 2^{-g}(\det \mathcal{M})^{-g} \int_{\mathbb{C}^{(h,g)}} |g(W)|^2\, d\mu(W) \\
&= 2^{-g}(\det \mathcal{M})^{-g} \sum_{J \in \mathbb{Z}^{(h,g)}_{\geq 0}} |a_{\mathcal{M},J}|^2 \pi^{-|J|} J! \quad \text{(by Lemma 3.4)}
\end{aligned}$$

Proof. Let $\mathcal{M}^{\frac{1}{2}}$ be the unique positive definite symmetric matrix of degree h such that $\left(\mathcal{M}^{\frac{1}{2}}\right)^2 = \mathcal{M}$. We put $\tilde{W} := \sqrt{2}\mathcal{M}^{\frac{1}{2}}W$. Obviously $d\tilde{W} = 2^g\,(det\,\mathcal{M})^g dW$. Thus for $f \in \mathcal{H}_{F,\mathcal{M}}$, we have

$$\begin{aligned}(f,f)_{\mathcal{M}} &= \int_{\mathbb{C}^{(h,g)}} |f(W)|^2\, d\mu_{\mathcal{M}}(W)\\ &= 2^{-g}(det\,\mathcal{M})^{-g} \int_{\mathbb{C}^{(h,g)}} |g(W)|^2\, d\mu(W)\\ &= 2^{-g}(det\,\mathcal{M})^{-g} \sum_{J\in\mathbb{Z}_{\geq 0}^{(h,g)}} |a_{\mathcal{M},J}|^2 \pi^{-|J|} J! \quad \text{(by Lemma 3.4)}\end{aligned}$$

□

For each $J \in \mathbb{Z}_{\geq 0}^{(h,g)}$, we put

(3.36)

$$\Phi_{\mathcal{M},J}(W) := 2^{\frac{g}{2}}\,(det\,\mathcal{M}\,)^{\frac{g}{2}}\,(J!)^{-\frac{1}{2}}\left((2\pi\mathcal{M})^{\frac{1}{2}}W\right)^J, \quad W \in \mathbb{C}^{(h,g)}.$$

LEMMA 3.6. *The set* $\left\{\Phi_{\mathcal{M},J} \mid J \in \mathbb{Z}_{\geq 0}^{(h,g)}\right\}$ *is a complete orthonormal system in* $\mathcal{H}_{F,\mathcal{M}}$.

Proof. For $J, K \in \mathbb{Z}_{\geq 0}^{(h,g)}$, we have

$$\begin{aligned}(\Phi_{\mathcal{M},J},\Phi_{\mathcal{M},K})_{\mathcal{M}} &= 2^g(det\,\mathcal{M})^g(J!)^{-\frac{1}{2}}(K!)^{-\frac{1}{2}}\\ &\quad\times \int_{\mathbb{C}^{(h,g)}} \left((2\pi\mathcal{M})^{\frac{1}{2}}W\right)^J \left((2\pi\mathcal{M})^{\frac{1}{2}}\overline{W}\right)^K d\mu_{\mathcal{M}}(W)\\ &= (J!)^{-\frac{1}{2}}(K!)^{-\frac{1}{2}} \int_{\mathbb{C}^{(h,g)}} (\pi^{\frac{1}{2}}W)^J\overline{(\pi^{\frac{1}{2}}W)^K}\, d\mu(W)\\ &= (\Phi_J,\Phi_K).\end{aligned}$$

By (3.27), we have

$$(3.37) \qquad (\Phi_{\mathcal{M},J},\Phi_{\mathcal{M},K})_{\mathcal{M}} = \begin{cases} 1 & \text{if } J = K\\ 0 & \text{otherwise.}\end{cases}$$

We leave the proof of the completeness to the reader. □

We observe that for a fixed $W' \in \mathbb{C}^{(h,g)}$, the holomorphic function $W \longrightarrow e^{\pi\sigma(\mathcal{M}W\,{}^t\overline{W'})}$ admits the following Taylor expansion

$$e^{\pi\sigma(\mathcal{M}W\,{}^t\overline{W'})} = \sum_{J\in\mathbb{Z}_{\geq 0}^{(h,g)}} \Phi_{\mathcal{M},J}(W)\,\Phi_{\mathcal{M},J}(\overline{W'}). \tag{3.38}$$

If $f \in \mathcal{H}_{F,\mathcal{M}}$, we have

$$\begin{aligned}\left(f(W),\, e^{\pi\sigma(\mathcal{M}W\,{}^t\overline{W'})}\right)_{\mathcal{M}} &= \sum_{J\in\mathbb{Z}_{\geq 0}^{(h,g)}} (f,\Phi_{\mathcal{M},J})_{\mathcal{M}}\,\Phi_{\mathcal{M},J}(W') \\ &= f(W').\end{aligned}$$

Hence $e^{\pi\sigma(\mathcal{M}W\,{}^t\overline{W'})}$ is the reproducing kernel for $\mathcal{H}_{F,\mathcal{M}}$ in the sense that

$$f(W) = \int_{\mathbb{C}^{(h,g)}} f(W')\,e^{\pi\sigma(\mathcal{M}W\,{}^t\overline{W'})}\,d\mu_{\mathcal{M}}(W'). \tag{3.39}$$

For $U \in \mathbb{R}^{(h,g)}$ and $W \in \mathbb{C}^{(h,g)}$, we put

$$k(U,W) := e^{2\pi\sigma(-U\,{}^tU + \frac{1}{2}W\,{}^tW + 2iU\,{}^tW)}. \tag{3.40}$$

Then we have the following lemma.

LEMMA 3.7.

$$\int_{\mathbb{R}^{(h,g)}} k(U,W)\,\overline{k(U,W')}\,dU = e^{2\pi\sigma(W\,{}^tW')}.$$

Proof. We put

$$\mathcal{I}(W,W') := \int_{\mathbb{R}^{(h,g)}} k(U,W)\,\overline{k(U,W')}\,dU.$$

Then we have

$$\begin{aligned}\mathcal{I}(W,W') &= e^{\pi\sigma(W\,{}^tW + \overline{W'}\,{}^t\overline{W'})} \int_{\mathbb{R}^{(h,g)}} e^{-4\pi\sigma(U\,{}^tU)}\, e^{4\pi i\sigma\{U\,{}^t(W-\overline{W'})\}}\,dU \\ &= e^{\pi\sigma(W\,{}^tW + \overline{W'}\,{}^t\overline{W'})} \cdot \prod_{k,a} \int_{\mathbb{R}} e^{-4\pi\{u_{ka}^2 - iu_{ka}(w_{ka} - \overline{w'_{ka}})\}}\,du_{ka},\end{aligned}$$

where $W = (w_{ka})$, $W' = (w'_{ka}) \in \mathbb{C}^{(h,g)}$ and $U = (u_{ka}) \in \mathbb{R}^{(h,g)}$. It is easy to show that

$$\int_{\mathbb{R}} e^{-4\pi\{u_{ka}^2 - iu_{ka}(w_{ka} - \overline{w'_{ka}})\}}\, du_{ka} = e^{-\pi(w_{ka} - \overline{w'_{ka}})^2}.$$

Thus we get

$$\begin{aligned}\mathcal{I}(W, W') &= e^{\pi\sigma(W\,{}^tW + \overline{W'}\,{}^t\overline{W'})} \cdot e^{-\pi\sum_{k,a}(w_{ka} - \overline{w'_{ka}})^2} \\ &= e^{2\pi\sum_{k,a} w_{ka}\overline{w'_{ka}}} \\ &= e^{2\pi\sigma(W\,{}^t\overline{W'})}.\end{aligned}$$

For $U \in \mathbb{R}^{(h,g)}$ and $W \in \mathbb{C}^{(h,g)}$, we put

(3.41) $$k_{\mathcal{M}}(U, W) := e^{2\pi\sigma\{\mathcal{M}(-U\,{}^tU - \frac{1}{2}W\,{}^tW + 2U\,{}^tW)\}}.$$ □

LEMMA 3.8. *Let $\mathcal{M}$ be a positive definite, symmetric half-integral matrix of degree h. Then we have*

(3.42) $$k_{\mathcal{M}}(U, W) = k(\mathcal{M}^{\frac{1}{2}}U,\ -i\mathcal{M}^{\frac{1}{2}}W)$$

and

(3.43) $$\int_{\mathbb{R}^{(h,g)}} k_{\mathcal{M}}(U, W)\,\overline{k_{\mathcal{M}}(U, W')}\, dU = (\det \mathcal{M})^{-\frac{g}{2}} \cdot e^{2\pi\sigma(\mathcal{M}W\,{}^t\overline{W'})}.$$

Proof. The formula (3.42) follows immediately from a straightforward computation. We put

$$\mathcal{I}_{\mathcal{M}}(W, W') := \int_{\mathbb{R}^{(h,g)}} k_{\mathcal{M}}(U, W)\,\overline{k_{\mathcal{M}}(U, W')}\, dU.$$

Using (3.42), we have

$$\begin{aligned}\mathcal{I}_{\mathcal{M}}(W, W') &= \int_{\mathbb{R}^{(h,g)}} k\left(\mathcal{M}^{\frac{1}{2}}U,\ -i\mathcal{M}^{\frac{1}{2}}W\right) \cdot \overline{k\left(\mathcal{M}^{\frac{1}{2}}U, -i\mathcal{M}^{\frac{1}{2}}W'\right)}\, dU \\ &= (\det \mathcal{M})^{-\frac{g}{2}} \int_{\mathbb{R}^{(h,g)}} k\left(U, -i\mathcal{M}^{\frac{1}{2}}W\right) \cdot \overline{k\left(U, -i\mathcal{M}^{\frac{1}{2}}W'\right)}\, dU \\ &= (\det \mathcal{M})^{-\frac{g}{2}} \cdot e^{2\pi\sigma(\mathcal{M}W\,{}^t\overline{W'})} \qquad \text{(by Lemma 3.7)}\end{aligned}$$ □

We recall that the Fock representation $U^{F,\mathcal{M}}$ of the real Heisenberg group G in $\mathcal{H}_{F,\mathcal{M}}$ (cf. (3.23)) is given by

(3.44) $$\left(U^{F,\mathcal{M}}(g)f\right)(W) = e^{2\pi i\sigma(\mathcal{M}\kappa)} \cdot e^{-\pi\sigma\{\mathcal{M}(\zeta\,{}^t\bar{\zeta} + 2W\,{}^t\bar{\zeta})\}}\, f(W + \zeta),$$

where $g = (\lambda, \mu, \kappa) \in G$, $f \in \mathcal{H}_{F,\mathcal{M}}$ and $\zeta = \lambda + i\mu \in \mathbb{C}^{(h,g)}$.

LEMMA 3.9. *The Fock representation $U^{F,\mathcal{M}}$ of G in $\mathcal{H}_{F,\mathcal{M}}$ is unitary.*

Proof. For brevity, we put $U_{g,f}(W) := \left(U^{F,\mathcal{M}}(g)f\right)(W)$ for $g = (\lambda, \mu, \kappa) \in G$ and $f \in \mathcal{H}_{F,\mathcal{M}}$. Then we have

$$\begin{aligned}(U_{g,f}, U_{g,f})_{\mathcal{M}} &= \|U_{g,f}\|^2_{\mathcal{M}}\\ &= \int_{\mathbb{C}^{(h,g)}} U_{g,f}(W)\,\overline{U_{g,f}(W)}\,d\mu_{\mathcal{M}}(W)\\ &= \int_{\mathbb{C}^{(h,g)}} e^{-\pi\sigma\{\mathcal{M}(\zeta\,{}^t\bar{\zeta}+2W\,{}^t\bar{\zeta}+\bar{\zeta}\,{}^t\zeta+2\overline{W}\,{}^tW+2W\,{}^t\overline{W})\}}\,|f(W+\zeta)|^2\,dW\\ &= \int_{\mathbb{C}^{(h,g)}} |f(W)|^2\,d\mu_{\mathcal{M}}(W)\\ &= (f,f)_{\mathcal{M}} = \|f\|^2_{\mathcal{M}}.\end{aligned}$$

□

We recall that the Schrödinger representation $U^{S,\mathcal{M}} := U(\sigma_{\mathcal{M}})$ of the real Heisenberg group G in the Hilbert space $\mathcal{H}_{S,\mathcal{M}} \cong L^2\left(\mathbb{R}^{(h,g)}, d\xi\right)$ (cf. (2.17) or (2.18)) is given by

$$\left(U^{S,\mathcal{M}}(g)f\right)(\xi) = e^{2\pi i\sigma\{\mathcal{M}(\kappa+\mu\,{}^t\lambda+2\mu\,{}^t\xi)\}}\,f(\xi+\lambda), \tag{3.45}$$

where $g = (\lambda, \mu, \kappa) \in G$, $f \in \mathcal{H}_{S,\mathcal{M}}$ and $\xi \in \mathbb{R}^{(h,g)}$. In order to emphasize $\mathcal{M}$, sometimes we call $U^{S,\mathcal{M}}$ the Schrödinger representation of G of *index* $\mathcal{M}$. The inner product $(\ ,\ )_{S,\mathcal{M}}$ on $\mathcal{H}_{S,\mathcal{M}}$ is given by

$$(f_1, f_2)_{S,\mathcal{M}} := \int_{\mathbb{R}^{(h,g)}} f_1(U)\,\overline{f_2(U)}\,dU, \quad f_1, f_2 \in \mathcal{H}_{S,\mathcal{M}}.$$

And we define the norm $\|\ \|_{S,\mathcal{M}}$ on $\mathcal{H}_{S,\mathcal{M}}$ by

$$\|f\|^2_{S,\mathcal{M}} := \int_{\mathbb{R}^{(h,g)}} |f(U)|^2\,dU, \quad f \in \mathcal{H}_{S,\mathcal{M}}.$$

THEOREM 3.10. *The Fock representation $(U^{F,\mathcal{M}}, \mathcal{H}_{F,\mathcal{M}})$ of G is untarily equivalent to the Schrödinger representation $(U^{S,\mathcal{M}}, \mathcal{H}_{S,\mathcal{M}})$ of G of index $\mathcal{M}$. Therefore the Fock representation $U_{F,\mathcal{M}}$ is irreducible. The intertwining unitary isometry $I_{\mathcal{M}} : \mathcal{H}_{S,\mathcal{M}} \longrightarrow \mathcal{H}^{F,\mathcal{M}}$ is given by*

$$(I_{\mathcal{M}}f)(W) := \int_{\mathbb{R}^{(h,g)}} k_{\mathcal{M}}(\xi, W)\,f(\xi)\,d\xi, \tag{3.46}$$

where $f \in \mathcal{H}_{S,\mathcal{M}} = L^2\left(\mathbb{R}^{(h,g)}, d\xi\right)$, $W \in \mathbb{C}^{(h,g)}$ and $k_{\mathcal{M}}(\xi, W)$ is a function on $\mathbb{R}^{(h,g)} \times \mathbb{C}^{(h,g)}$ defined by (3.41).

Proof. For any $f \in \mathcal{H}_{S,\mathcal{M}} = L^2\left(\mathbb{R}^{(h,g)},\, d\xi\right)$, we define

$$(I_{\mathcal{M}} f)(W) := \int_{\mathbb{R}^{(h,g)}} k_{\mathcal{M}}(\xi, W)\, f(\xi)\, d\xi, \quad W \in \mathbb{C}^{(h,g)}.$$

Now we will show the following (I1), (I2) and (I3):

(I1) The image of $\mathcal{H}_{S,\mathcal{M}}$ under $I_{\mathcal{M}}$ is contained in $\mathcal{H}_{F,\mathcal{M}}$.

(I2) $I_{\mathcal{M}}$ preserves the norms, i.e., $\|f\|_{S,\mathcal{M}} = \|I_{\mathcal{M}} f\|_{\mathcal{M}}$.

(I3) $I_{\mathcal{M}}$ is a bijective operator of $\mathcal{H}_{S,\mathcal{M}}$ onto $\mathcal{H}_{F,\mathcal{M}}$.

Before we prove (I1), (I2) and (I3), we prove the following lemma. □

LEMMA 3.11. *For a fixed $U \in \mathbb{R}^{(h,g)}$, we consider the Taylor expansion*

$$k_{\mathcal{M}}(U, W) = \sum_{J \in \mathbb{Z}_{\geq 0}^{(h,g)}} h_J(U)\, \Phi_{\mathcal{M},J}(W), \quad W \in \mathbb{C}^{(h,g)} \tag{3.47}$$

of the holomorphic function $k_{\mathcal{M}}(U, \cdot\,)$ on $\mathbb{C}^{(h,g)}$. Then the set $\left\{h_J \mid J \in \mathbb{Z}_{\geq 0}^{(h,g)}\right\}$ forms a complete orthonormal system in $L^2\left(\mathbb{R}^{(h,g)},\, d\xi\right)$. Moreover, for a fixed $W \in \mathbb{C}^{(h,g)}$, (3.47) is the Fourier expansion of $k_{\mathcal{M}}(\,\cdot\,, W)$ with respect to this orthonormal system $\left\{ h_J \mid J \in \mathbb{Z}_{\geq 0}^{(h,g)} \right\}$.

Proof. Following Igusa [2], pp. 33-34, we can prove it. The detail will be left to the reader.

If $f \in \mathcal{H}_{S,\mathcal{M}}$, then by the Schwarz inequality and lemma 3.8, (3.43), we have

$$\begin{aligned} |\,(I_{\mathcal{M}} f)(W)| &\leq \left(\int_{\mathbb{R}^{(h,g)}} |k_{\mathcal{M}}(U,W)|^2\, dU \right)^{\frac{1}{2}} \cdot \left(\int_{\mathbb{R}^{(h,g)}} |f(U)|^2\, dU \right)^{\frac{1}{2}} \\ &= (\det \mathcal{M})^{-\frac{g}{4}} \cdot e^{\pi\sigma(\mathcal{M} W\, {}^t\overline{W})}\, \|f\|_{S,\mathcal{M}}. \end{aligned}$$

Thus the above integral $(I_{\mathcal{M}} f)(W)$ converges uniformly on any compact subset of $\mathbb{C}^{(h,g)}$ and hence $(I_{\mathcal{M}} f)(W)$ is holomorphic in $\mathbb{C}^{(h,g)}$. And according to lemma 6.11, we get

$$\begin{aligned} (I_{\mathcal{M}} f)(W) &= \sum_{J \in \mathbb{Z}_{\geq 0}^{(h,g)}} \int_{\mathbb{R}^{(h,g)}} h_J(U)\, f(U)\, \Phi_{\mathcal{M},J}(W)\, dU \\ &= \sum_{J \in \mathbb{Z}_{\geq 0}^{(h,g)}} (h_J,\, \bar{f})_{S,\mathcal{M}}\, \Phi_{\mathcal{M},J}(W). \end{aligned}$$

Therefore we get

$$\begin{aligned}
\|I_{\mathcal{M}}f\|^2_{F,\mathcal{M}} &= \int_{\mathbb{C}^{(h,g)}} |I_{\mathcal{M}}f(W)|^2\, d\mu_{\mathcal{M}}(W) \\
&= \sum_{J,\,K\in\mathbb{Z}^{(h,g)}_{\geq 0}} (h_J,\, \bar{f})_{S,\mathcal{M}} \cdot \overline{(h_K,\, \bar{f})} \\
&\quad \int_{\mathbb{C}^{(h,g)}} \Phi_{\mathcal{M},J}(W)\, \overline{\Phi_{\mathcal{M},K}(W)}\, d\mu_{\mathcal{M}}(W) \\
&= \sum_{J\in\mathbb{Z}^{(h,g)}_{\geq 0}} |(h_j,\, \bar{f})_{S,\mathcal{M}}|^2 \qquad \text{(by (3.37))} \\
&= \|f\|^2_{S,\mathcal{M}} < \infty.
\end{aligned}$$

This proves (I1) and (I2). It is easy to see that $I_{\mathcal{M}}\overline{h_J} = \Phi_{\mathcal{M},J}$ for all $J \in \mathbb{Z}^{(h,g)}_{\geq 0}$. Since the set $\left\{ \Phi_{\mathcal{M},J} \mid J \in \mathbb{Z}^{(h,g)}_{\geq 0} \right\}$ forms a complete orthonormal system of $\mathcal{H}_{F,\mathcal{M}}$, $I_{\mathcal{M}}$ is surjective. Obviously the injectivity of $I_{\mathcal{M}}$ follows immediately from the fact that $I_{\mathcal{M}}\overline{h_J} = \Phi_{\mathcal{M},J}$ for all $J \in \mathbb{Z}^{(h,g)}_{\geq 0}$. This proves (I3).

On the other hand, we let $f \in \mathcal{H}_{S,\mathcal{M}}$ and $g = (\lambda, \mu, \kappa) \in G$. We put $\zeta = \lambda + i\mu$. Then we get

$$\begin{aligned}
&\left(U^{F,\mathcal{M}}(g)(I_{\mathcal{M}}f) \right)(W) \\
&= e^{2\pi i\sigma(\mathcal{M}\kappa)} \cdot e^{-\pi\sigma\{\mathcal{M}(\zeta\,{}^t\bar{\zeta}+2W\,{}^t\bar{\zeta})\}}\, (I_{\mathcal{M}}f)(W+\zeta) \quad (\text{by (3.44)}) \\
&= e^{2\pi i\sigma(\mathcal{M}\kappa)} \cdot e^{-\pi\sigma\{\mathcal{M}(\zeta\,{}^t\bar{\zeta}+2W\,{}^t\bar{\zeta})\}} \int_{\mathbb{R}^{(h,g)}} k_{\mathcal{M}}(U, W+\zeta)\, f(U)\, dU.
\end{aligned}$$

We define the function $A : \mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)} \longrightarrow \mathbb{C}$ by

$$A_{\mathcal{M}}(U,W) := \sigma\left\{ \mathcal{M}\left(-U\,{}^tU - \frac{W\,{}^tW}{2} + 2U\,{}^tW \right) \right\}. \tag{3.48}$$

Obviously $\kappa_{\mathcal{M}}(U,W) = e^{2\pi A_{\mathcal{M}}(U,W)}$ for $U \in \mathbb{R}^{(h,g)}$ and $W \in \mathbb{C}^{(h,g)}$.

By an easy computation, we get

$$A_{\mathcal{M}}(U, W+\zeta) - A(U-\lambda, W) = \sigma\left\{ \mathcal{M}\left(\frac{\zeta\,{}^t\bar{\zeta}}{2} + W\,{}^t\bar{\zeta} - i\lambda\,{}^t\mu + 2iU\,{}^t\mu \right) \right\}.$$

Therefore we get

$$\begin{aligned}
&k_{\mathcal{M}}(U, W+\zeta) \\
&= e^{2\pi A_{\mathcal{M}}(U-\lambda, W)} \cdot e^{2\pi\sigma\left\{\mathcal{M}\left(\frac{1}{2}\zeta\,{}^t\bar{\zeta}+W\,{}^t\bar{\zeta}-i\lambda\,{}^t\mu+2iU\,{}^t\mu\right)\right\}} \\
&= k_{\mathcal{M}}(U-\lambda, W) \cdot e^{2\pi\sigma\left\{\mathcal{M}\left(\frac{1}{2}\zeta\,{}^t\bar{\zeta}+W\,{}^t\bar{\zeta}-i\lambda\,{}^t\mu+2iU\,{}^t\mu\right)\right\}}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\left(U^{F,\mathcal{M}}(g)(I_{\mathcal{M}}f)\right)(W) \\
&= \int_{\mathbb{R}^{(h,g)}} e^{2\pi i\sigma\{\mathcal{M}(\kappa+2U\,{}^t\mu-\lambda\,{}^t\mu)\}}\, k_{\mathcal{M}}(U-\lambda, W)\, f(U)\, dU \\
&= \int_{\mathbb{R}^{(h,g)}} e^{2\pi i\sigma\{\mathcal{M}(\kappa+2\lambda\,{}^t\mu+2U\,{}^t\mu-\lambda\,{}^t\mu)\}}\, k_{\mathcal{M}}(U, W)\, f(U+\lambda)\, dU \\
&= \int_{\mathbb{R}^{(h,g)}} e^{2\pi i\sigma\{\mathcal{M}(\kappa+2U\,{}^t\mu+\lambda\,{}^t\mu)\}}\, k_{\mathcal{M}}(U, W)\, f(U+\lambda)\, dU \\
&= \int_{\mathbb{R}^{(h,g)}} k_{\mathcal{M}}(U, W)\, \left(U^{S,\mathcal{M}}(g)f\right)(U)\, dU \qquad \text{(by (3.45))} \\
&= \left(I_{\mathcal{M}}\left(U^{S,\mathcal{M}}(g)f\right)\right)(W).
\end{aligned}$$

So far we proved that $U^{F,\mathcal{M}} \circ I_{\mathcal{M}} = I_{\mathcal{M}} \circ U^{S,\mathcal{M}}(g)$ for all $g \in G$. That is, the unitary isometry $I_{\mathcal{M}}$ of $\mathcal{H}_{S,\mathcal{M}}$ onto $\mathcal{H}_{F,\mathcal{M}}$ is the intertwining operator. This completes the proof. □

The infinitesimal representation $dU^{F,\mathcal{M}}$ associated to the Fock representation $U^{F,\mathcal{M}}$ is given as follows.

PROPOSITION 3.12. *Let $\mathcal{M}$ be as before. We put*

$$\mathcal{M} = (\mathcal{M}_{kl}), \quad (2\pi\mathcal{M})^{\frac{1}{2}} = (\tau_{kl}),$$

where $\tau_{kl} \in \mathbb{R}$ and $1 \le k, l \le h$. For each $J = (J_{ka}) \in \mathbb{Z}_{\ge 0}^{(h,g)}$ and $W = (W_{ka}) \in \mathbb{C}^{(h,g)}$, we have

$$\text{(3.49)} \quad dU^{F,\mathcal{M}}(D_{kl}^0)\, \Phi_{\mathcal{M},J}(W) = 2\pi i\, \mathcal{M}_{kl}\, \Phi_{\mathcal{M},J}(W), \quad 1 \le k \le l \le h.$$

(3.50)

$$dU^{F,\mathcal{M}}(D_{ka})\,\Phi_{\mathcal{M},J}(W) = -2\pi\left(\sum_{m=1}^{h}\mathcal{M}_{mk}W_{ma}\right)\Phi_{\mathcal{M},J}(W) + \sum_{m=1}^{h}\tau_{mk}J_{ma}^{\frac{1}{2}}\,\Phi_{\mathcal{M},J-\epsilon_{ma}}(W).$$

(3.51)

$$dU^{F,\mathcal{M}}(\hat{D}_{lb}\,\Phi_{\mathcal{M},J}(W) = 2\pi i\left(\sum_{m=1}^{h}\mathcal{M}_{ml}W_{mb}\right)\Phi_{\mathcal{M},J}(W) + i\sum_{m=1}^{h}\tau_{ml}J_{mb}^{\frac{1}{2}}\,\Phi_{\mathcal{M},J-\epsilon_{lb}}(W).$$

Proof. We put $E_{kl}^0 = \frac{1}{2}(E_{kl}+E_{lk})$, where $1 \leq k \leq l \leq h$.

$$\begin{aligned}
dU^{F,\mathcal{M}}(D_{kl}^0)\,\Phi_{\mathcal{M},J}(W) &= \frac{d}{dt}\Big|_{t=0} U^{F,\mathcal{M}}(\exp tX_{kl}^0)\,\Phi_{\mathcal{M},J}(W)\\
&= \frac{d}{dt}\Big|_{t=0} U^{F,\mathcal{M}}\left((0,0,tE_{kl}^0)\right)\,\Phi_{\mathcal{M},J}(W)\\
&= \lim_{t\to 0}\frac{e^{2\pi i\sigma(t\mathcal{M}E_{kl}^0)}-I}{t}\,\Phi_{\mathcal{M},J}(W)\\
&= \lim_{t\to 0}\frac{e^{2\pi it\mathcal{M}_{kl}}-I}{t}\,\Phi_{\mathcal{M},J}(W)\\
&= 2\pi i\,\mathcal{M}_{kl}\,\Phi_{\mathcal{M},J}(W).
\end{aligned}$$

And we have

$$\begin{aligned}
&dU^{F,\mathcal{M}}(D_{ka})\,\Phi_{\mathcal{M},J}(W)\\
&= \frac{d}{dt}\Big|_{t=0} U^{F,\mathcal{M}}(\exp tX_{ka})\,\Phi_{\mathcal{M},J}(W)\\
&= \frac{d}{dt}\Big|_{t=0} U^{F,\mathcal{M}}\left((tE_{ka},0,0)\right)\Phi_{\mathcal{M},J}(W)\\
&= \frac{d}{dt}\Big|_{t=0} e^{-\pi t^2\sigma(\mathcal{M}E_{ka}\,{}^tE_{ka})-2\pi t\sigma(\mathcal{M}W\,{}^tE_{ka})}\,\Phi_{\mathcal{M},J}(W+tE_{ka})
\end{aligned}$$

$$= -2\pi \left(\sum_{m=1}^{h} \mathcal{M}_{mk} W_{ma} \right) \Phi_{\mathcal{M},J}(W)$$
$$+ \left. \frac{d}{dt} \right|_{t=0} \Phi_{\mathcal{M},J}(W + tE_{ka})$$
$$= -2\pi \left(\sum_{m=1}^{h} \mathcal{M}_{mk} W_{ma} \right) \Phi_{\mathcal{M},J}(W)$$
$$+ \sum_{m=1}^{h} \tau_{mk} \, J_{ma}^{\frac{1}{2}} \, \Phi_{\mathcal{M},J-\epsilon_{ma}}(W).$$

Finally,

$$dU^{F,\mathcal{M}}(\hat{D}_{lb}) \, \Phi_{\mathcal{M},J}(W)$$
$$= \left. \frac{d}{dt} \right|_{t=0} U^{F,\mathcal{M}}(\exp t\hat{X}_{lb}) \Phi_{\mathcal{M},J}(W)$$
$$= \left. \frac{d}{dt} \right|_{t=0} U^{F,\mathcal{M}}\left((0, tE_{lb}, 0)\right) \Phi_{\mathcal{M},J}(W)$$
$$= \left. \frac{d}{dt} \right|_{t=0} e^{-\pi t^2 \sigma(\mathcal{M} E_{lb} \, {}^t E_{lb}) + 2\pi i t \, \sigma(\mathcal{M} W \, {}^t E_{lb})} \, \Phi_{\mathcal{M},J}(W + itE_{lb})$$
$$= 2\pi i \left(\sum_{m=1}^{h} \mathcal{M}_{ml} W_{mb} \right) \Phi_{\mathcal{M},J}(W)$$
$$+ \left. \frac{d}{dt} \right|_{t=0} \Phi_{\mathcal{M},J}(W + itE_{lb})$$
$$= 2\pi i \left(\sum_{m=1}^{h} \mathcal{M}_{ml} W_{mb} \right) \Phi_{\mathcal{M},J}(W)$$
$$+ \, i \sum_{m=1}^{h} \tau_{ml} \, J_{mb}^{\frac{1}{2}} \, \Phi_{\mathcal{M},J-\epsilon_{mb}}(W).$$

□

References

[1] P. Cartier, *Quantum Mechanical Commutation Relations and Theta Functions*, Proc. of Symposia in Pure Mathematics **9** (1966), 361-383.

[2] J. Igusa, *Theta functions*, Springer-Verlag, New York/Berlin, 1972.
[3] G. W. Mackey, *Induced Representations of Locally Compact Groups I*, Ann. of Math. **55** (1952), 101-139.
[4] Y. Namikawa, *Toroidal Compactification of Siegel Spaces, Lecture Notes in Math.,* **812**, Springer-Verlag, New York/Berlin, 1980.
[5] J.-H. Yang, *Harmonic Analysis on the Quotient Spaces of Heisenberg Groups*, Nagoya Math. J. **123** (1991), 103-117.
[6] ———, *Harmonic Analysis on the Quotient Spaces of Heisenberg Groups, II*, J. Number Theory **49** (1994), 63-72.
[7] ———, *A decomposition theorem on differential polynomials of theta functions of high level*, Japanese J. of Mathematics, The Mathematical Society of Japan, New Series **22** (1996), 37-49.
[8] ———, *On theta functions*, Kyungpook Math. J. **35** (1996), 857-875.
[9] ———, *Lattice representations of Heisenberg groups*, preprint, MPI97-16 (1997).
[10] ———, *Unitary representations of the Heisenberg group* $H_{\mathbb{R}}^{(g,h)}$, Proc. of the Workshops in Pure Mathematics on Groups and Algebraic Structures, the Korean Academic Council; **5** (1995), 144-159.
[11] ———, *Hermite differential operators*, preprint (1995).

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Lattice representations of Heisenberg groups

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1. Introduction

For any positive integers g and h, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(g,h)} := \left\{ (\lambda, \mu, \kappa) \mid \lambda, \mu \in \mathbb{R}^{(h,g)},\ \kappa \in \mathbb{R}^{(h,h)},\ \kappa + \mu\, {}^t\lambda \text{ symmetric} \right\}.$$

Recall that the multiplication law is

$$(\lambda, \mu, \kappa) \circ (\lambda', \mu', \kappa') := (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda').$$

Here $\mathbb{R}^{(h,g)}$ (resp. $\mathbb{R}^{(h,h)}$) denotes the set of all $h \times g$ (resp. $h \times h$) real matrices.

The Heisenberg group $H_{\mathbb{R}}^{(g,h)}$ is embedded into the symplectic group $Sp(g + h, \mathbb{R})$ via the mapping

$$H_{\mathbb{R}}^{(g,h)} \ni (\lambda, \mu, \kappa) \longmapsto \begin{pmatrix} E_g & 0 & 0 & {}^t\mu \\ \lambda & E_h & \mu & \kappa \\ 0 & 0 & E_g & -{}^t\lambda \\ 0 & 0 & 0 & E_h \end{pmatrix} \in Sp(g + h, \mathbb{R}).$$

This Heisenberg group is a 2-step nilpotent Lie group and is important in the study of toroidal compactifications of Siegel moduli spaces. In fact, $H_{\mathbb{R}}^{(g,h)}$ is obtained as the unipotent radical of the parabolic subgroup of $Sp(g + h, \mathbb{R})$ associated with the rational boundary component F_g (cf. [F-C] p. 123 or [N] p. 21). For the motivation of the study of this Heisenberg group we refer to [Y4]-[Y8] and [Z]. We refer to [Y1]-[Y3] for more results on $H_{\mathbb{R}}^{(g,h)}$.

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In [C], P. Cartier stated without proof that for $h = 1$, the lattice representation of $H_{\mathbb{R}}^{(g,1)}$ associated to the lattice L is unitarily equivalent to the direct sum of $[L^* : L]^{\frac{1}{2}}$ copies of the Schrödinger representation of $H_{\mathbb{R}}^{(g,1)}$, where L^* is the dual lattice of L with respect to a certain nondegenerate alternating bilinear form. R. Berndt proved the above fact for the case $h = 1$ in his lecture notes [B]. In this paper, we give a complete proof of Cartier's theorem for $H_{\mathbb{R}}^{(g,h)}$.

Main Theorem. Let $\mathcal{M}$ be a positive definite, symmetric half-integral matrix of degree h and L be a self-dual lattice in $\mathbb{C}^{(h,g)}$. Then the lattice representation $\pi_{\mathcal{M}}$ of $H_{\mathbb{R}}^{(g,h)}$ associated with L and $\mathcal{M}$ is unitarily equivalent to the direct sum of $(\det 2\mathcal{M})^g$ copies of the Schrödinger representation of $H_{\mathbb{R}}^{(g,h)}$. For more details, we refer to Sect. 3.

The paper is organized as follows. In Sect. 2, we review the Schrödinger representations of the Heisenberg group $H_{\mathbb{R}}^{(g,h)}$. In Sect. 3, we prove the main theorem. In the final section, we provide a relation between lattice representations and theta functions.

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Notations. We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers, and the field of complex numbers respectively. The symbol $\mathbb{C}_1^{\times}$ denotes the multiplicative group consisting of all complex numbers z with $|z| = 1$, and the symbol $Sp(g, \mathbb{R})$ the symplectic group of degree g, H_g the Siegel upper half plane of degree g. The symbol ":=" means that the expression on the right hand side is the definition of that on the left. We denote by $\mathbb{Z}^+$ the set of all positive integers, by $F^{(k,l)}$ the set of all $k \times l$ matrices with entries in a commutative ring F. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. For $A \in F^{(k,k)}$, $\sigma(A)$ denotes the trace of A. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. We denote the identity matrix of degree k by E_k. For a positive integer n, $\mathrm{Symm}\,(n, K)$ denotes the vector space consisting of all symmetric $n \times n$ matrices with entries in a field K.

2. Schrödinger representations

First of all, we observe that $H_{\mathbb{R}}^{(g,h)}$ is a 2-step nilpotent Lie group. It is easy to see that the inverse of an element $(\lambda, \mu, \kappa) \in H_{\mathbb{R}}^{(g,h)}$ is given by

$$(\lambda, \mu, \kappa)^{-1} = (-\lambda, -\mu, -\kappa + \lambda\, {}^t\mu - \mu\, {}^t\lambda).$$

Now we set

(2.1) $$[\lambda, \mu, \kappa] := (0, \mu, \kappa) \circ (\lambda, 0, 0) = (\lambda, \mu, \kappa - \mu\, {}^t\lambda).$$

Then $H_{\mathbb{R}}^{(g,h)}$ may be regarded as a group equipped with the following multiplication

(2.2) $$[\lambda, \mu, \kappa] \diamond [\lambda_0, \mu_0, \kappa_0] := [\lambda + \lambda_0, \mu + \mu_0, \kappa + \kappa_0 + \lambda\, {}^t\mu_0 + \mu_0\, {}^t\lambda].$$

The inverse of $[\lambda, \mu, \kappa] \in H_{\mathbb{R}}^{(g,h)}$ is given by

$$[\lambda, \mu, \kappa]^{-1} = [-\lambda, -\mu, -\kappa + \lambda\, {}^t\mu + \mu\, {}^t\lambda].$$

We set

(2.3) $$K := \left\{ [0, \mu, \kappa] \in H_{\mathbb{R}}^{(g,h)} \;\middle|\; \mu \in \mathbb{R}^{(h,g)},\ \kappa = {}^t\kappa \in \mathbb{R}^{(h,h)} \right\}.$$

Then K is a commutative normal subgroup of $H_{\mathbb{R}}^{(g,h)}$. Let $\hat{K}$ be the Pontrajagin dual of K, i.e., the commutative group consisting of all unitary characters of K. Then $\hat{K}$ is isomorphic to the additive group $\mathbb{R}^{(h,g)} \times \mathrm{Symm}\,(h, \mathbb{R})$ via

(2.4) $$< a, \hat{a} > := e^{2\pi i \sigma(\hat{\mu}\, {}^t\mu + \hat{\kappa}\kappa)}, \quad a = [0, \mu, \kappa] \in K,\ \hat{a} = (\hat{\mu}, \hat{\kappa}) \in \hat{K}.$$

We put

(2.5) $$S := \left\{ [\lambda, 0, 0] \in H_{\mathbb{R}}^{(g,h)} \;\middle|\; \lambda \in \mathbb{R}^{(h,g)} \right\} \cong \mathbb{R}^{(h,g)}.$$

Then S acts on K as follows:

(2.6) $$\alpha_\lambda([0, \mu, \kappa]) := [0, \mu, \kappa + \lambda\, {}^t\mu + \mu\, {}^t\lambda], \quad [\lambda, 0, 0] \in S.$$

It is easy to see that the Heisenberg group $\left(H_{\mathbb{R}}^{(g,h)}, \diamond\right)$ is isomorphic to the semidirect product $S \ltimes K$ of S and K whose multiplication is given by

$$(\lambda, a) \cdot (\lambda_0, a_0) := (\lambda + \lambda_0, a + \alpha_\lambda(a_0)), \quad \lambda, \lambda_0 \in S,\ a, a_0 \in K.$$

On the other hand, S acts on $\hat{K}$ by

(2.7) $$\alpha_\lambda^*(\hat{a}) := (\hat{\mu} + 2\hat{\kappa}\lambda, \hat{\kappa}), \quad [\lambda, 0, 0] \in S, \quad a = (\hat{\mu}, \hat{\kappa}) \in \hat{K}.$$

Then, we have the relation $< \alpha_\lambda(a), \hat{a} > = < a, \alpha_\lambda^*(\hat{a}) >$ for all $a \in K$ and $\hat{a} \in \hat{K}$.

We have two types of S-orbits in $\hat{K}$.

TYPE I. Let $\hat{\kappa} \in \mathrm{Symm}\,(h, \mathbb{R})$ with $\hat{\kappa} \neq 0$. The S-orbit of $\hat{a}(\hat{\kappa}) := (0, \hat{\kappa}) \in \hat{K}$ is given by

(2.8) $$\hat{\mathcal{O}}_{\hat{\kappa}} := \left\{ (2\hat{\kappa}\lambda, \hat{\kappa}) \in \hat{K} \;\middle|\; \lambda \in \mathbb{R}^{(h,g)} \right\} \cong \mathbb{R}^{(h,g)}.$$

TYPE II. Let $\hat{y} \in \mathbb{R}^{(h,g)}$. The S-orbit $\hat{\mathcal{O}}_{\hat{y}}$ of $\hat{a}(\hat{y}) := (\hat{y}, 0)$ is given by

$$\hat{\mathcal{O}}_{\hat{y}} := \left\{ (\hat{y}, 0) \right\} = \hat{a}(\hat{y}). \tag{2.9}$$

We have

$$\hat{K} = \left(\bigcup_{\hat{\kappa} \in \mathrm{Symm}(h,\mathbb{R})} \hat{\mathcal{O}}_{\hat{\kappa}} \right) \bigcup \left(\bigcup_{\hat{y} \in \mathbb{R}^{(h,g)}} \hat{\mathcal{O}}_{\hat{y}} \right)$$

as a set. The stabilizer $S_{\hat{\kappa}}$ of S at $\hat{a}(\hat{\kappa}) = (0, \hat{\kappa})$ is given by

$$S_{\hat{\kappa}} = \{0\}. \tag{2.10}$$

And the stabilizer $S_{\hat{y}}$ of S at $\hat{a}(\hat{y}) = (\hat{y}, 0)$ is given by

$$S_{\hat{y}} = \left\{ [\lambda, 0, 0] \,\middle|\, \lambda \in \mathbb{R}^{(h,g)} \right\} = S \cong \mathbb{R}^{(h,g)}. \tag{2.11}$$

From now on, we set $G := H_{\mathbb{R}}^{(g,h)}$ for brevity. It is known that K is a closed, commutative normal subgroup of G. Since $(\lambda, \mu, \kappa) = (0, \mu, \kappa + \mu\, {}^t\lambda) \circ (\lambda, 0, 0)$ for $(\lambda, \mu, \kappa) \in G$, the homogeneous space $X := K\backslash G$ can be identified with $\mathbb{R}^{(h,g)}$ via

$$Kg = K \circ (\lambda, 0, 0) \longmapsto \lambda, \quad g = (\lambda, \mu, \kappa) \in G.$$

We observe that G acts on X by

$$(Kg) \cdot g_0 := K\,(\lambda + \lambda_0, 0, 0) = \lambda + \lambda_0, \tag{2.12}$$

where $g = (\lambda, \mu, \kappa) \in G$ and $g_0 = (\lambda_0, \mu_0, \kappa_0) \in G$.

If $g = (\lambda, \mu, \kappa) \in G$, we have

$$k_g = (0, \mu, \kappa + \mu\, {}^t\lambda), \quad s_g = (\lambda, 0, 0) \tag{2.13}$$

in the Mackey decomposition of $g = k_g \circ s_g$ (cf. [M]). Thus if $g_0 = (\lambda_0, \mu_0, \kappa_0) \in G$, then we have

$$s_g \circ g_0 = (\lambda, 0, 0) \circ (\lambda_0, \mu_0, \kappa_0) = (\lambda + \lambda_0, \mu_0, \kappa_0 + \lambda\, {}^t\mu_0) \tag{2.14}$$

and so

$$k_{s_g \circ g_0} = (0, \mu_0, \kappa_0 + \mu_0\, {}^t\lambda_0 + \lambda\, {}^t\mu_0 + \mu_0\, {}^t\lambda). \tag{2.15}$$

For a real symmetric matrix $c = {}^t c \in \mathbb{R}^{(h,h)}$ with $c \neq 0$, we consider the one-dimensional unitary representation σ_c of K defined by

$$\sigma_c\,((0, \mu, \kappa)) := e^{2\pi i \sigma(c\kappa)}\, I, \quad (0, \mu, \kappa) \in K, \tag{2.16}$$

where I denotes the identity mapping. Then the induced representation $U(\sigma_c) := \mathrm{Ind}_K^G\, \sigma_c$ of G induced from σ_c is realized in the Hilbert space $\mathcal{H}_{\sigma_c} = L^2(X, d\dot{g},$

$\mathbb{C}) \cong L^2\left(\mathbb{R}^{(h,g)}, d\xi\right)$ as follows. If $g_0 = (\lambda_0, \mu_0, \kappa_0) \in G$ and $x = Kg \in X$ with $g = (\lambda, \mu, \kappa) \in G$, we have

$$(2.17) \qquad \left(U_{g_0}(\sigma_c)f\right)(x) = \sigma_c\left(k_{s_g \circ g_0}\right)\left(f(xg_0)\right), \quad f \in \mathcal{H}_{\sigma_c}.$$

It follows from (2.15) that

$$(2.18) \qquad \left(U_{g_0}(\sigma_c)f\right)(\lambda) = e^{2\pi i \sigma\{c(\kappa_0 + \mu_0\, {}^t\lambda_0 + 2\lambda\, {}^t\mu_0)\}} f(\lambda + \lambda_0).$$

Here, we identified $x = Kg$ (resp. $xg_0 = Kgg_0$) with λ (resp. $\lambda + \lambda_0$). The induced representation $U(\sigma_c)$ is called the *Schrödinger representation* of G associated with σ_c. Thus $U(\sigma_c)$ is a monomial representation.

Now, we denote by $\mathcal{H}^{\sigma_c}$ the Hilbert space consisting of all functions $\phi : G \longrightarrow \mathbb{C}$ which satisfy the following conditions:

(1) $\phi(g)$ is measurable with respect to dg,
(2) $\phi\left((0, \mu, \kappa) \circ g)\right) = e^{2\pi i \sigma(c\kappa)}\phi(g)$ for all $g \in G$,
(3) $\| \phi \|^2 := \int_X |\phi(g)|^2 d\dot{g} < \infty, \quad \dot{g} = Kg$,

where dg (resp. $d\dot{g}$) is a G-invariant measure on G (resp. $X = K\backslash G$). The inner product $(\, , \,)$ on $\mathcal{H}^{\sigma_c}$ is given by

$$(\phi_1, \phi_2) := \int_G \phi_1(g)\, \overline{\phi_2(g)}\, dg \quad \text{for } \phi_1, \phi_2 \in \mathcal{H}^{\sigma_c}.$$

We observe that the mapping $\Phi_c : \mathcal{H}_{\sigma_c} \longrightarrow \mathcal{H}^{\sigma_c}$ defined by

$$(2.19) \quad \left(\Phi_c(f)\right)(g) := e^{2\pi i \sigma\{c(\kappa + \mu\, {}^t\lambda)\}} f(\lambda), \quad f \in \mathcal{H}_{\sigma_c}, \; g = (\lambda, \mu, \kappa) \in G$$

is an isomorphism of Hilbert spaces. The inverse $\Psi_c : \mathcal{H}^{\sigma_c} \longrightarrow \mathcal{H}_{\sigma_c}$ of Φ_c is given by

$$(2.20) \qquad \left(\Psi_c(\phi)\right)(\lambda) := \phi((\lambda, 0, 0)), \quad \phi \in \mathcal{H}^{\sigma_c}, \; \lambda \in \mathbb{R}^{(h,g)}.$$

The Schrödinger representation $U(\sigma_c)$ of G on $\mathcal{H}^{\sigma_c}$ is given by

$$(2.21) \qquad \left(U_{g_0}(\sigma_c)\phi\right)(g) = e^{2\pi i \sigma\{c(\kappa_0 + \mu_0\, {}^t\lambda_0 + \lambda\, {}^t\mu_0 - \lambda_0\, {}^t\mu)\}}\, \phi\left((\lambda_0, 0, 0) \circ g\right),$$

where $g_0 = (\lambda_0, \mu_0, \kappa_0)$, $g = (\lambda, \mu, \kappa) \in G$ and $\phi \in \mathcal{H}^{\sigma_c}$. (2.21) can be expressed as follows.

$$(2.22) \quad \left(U_{g_0}(\sigma_c)\phi\right)(g) = e^{2\pi i \sigma\{c(\kappa_0 + \kappa + \mu_0\, {}^t\lambda_0 + \mu\, {}^t\lambda + 2\lambda\, {}^t\mu_0)\}}\, \phi((\lambda_0 + \lambda, 0, 0)).$$

Theorem 2.1. Let c be a positive symmetric half-integral matrix of degree h. Then the Schrödinger representation $U(\sigma_c)$ of G is irreducible.

Proof. The proof can be found in [Y1], theorem 3. □

3. Proof of the Main Theorem

Let $L := \mathbb{Z}^{(h,g)} \times \mathbb{Z}^{(h,g)}$ be the lattice in the vector space $V \cong \mathbb{C}^{(h,g)}$. Let B be an alternating bilinear form on V such that $B(L, L) \subset \mathbb{Z}$, that is, $\mathbb{Z}$-valued on $L \times L$. The dual L_B^* of L with respect to B is defined by

$$L_B^* := \{\, v \in V \mid B(v, l) \in \mathbb{Z} \text{ for all } l \in L \,\}.$$

Then $L \subset L_B^*$. If B is nondegenerate, L_B^* is also a lattice in V, called the *dual lattice* of L. In case B is nondegenerate, there exist a $\mathbb{Z}$-basis $\{\, \xi_{11}, \xi_{12}, \cdots, \xi_{hg}, \eta_{11}, \eta_{12}, \cdots, \eta_{hg} \,\}$ of L and a set $\{\, e_{11}, e_{12}, \cdots, e_{hg} \,\}$ of positive integers such that $e_{11}|e_{12},\ e_{12}|e_{13}, \cdots, e_{h,g-1}|e_{hg}$ for which

$$\begin{pmatrix} B(\xi_{ka}, \xi_{lb}) & B(\xi_{ka}, \eta_{lb}) \\ B(\eta_{ka}, \xi_{lb}) & B(\eta_{ka}, \eta_{lb}) \end{pmatrix} = \begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix},$$

where $1 \leq k, l \leq h,\ 1 \leq a, b \leq g$ and $e := \operatorname{diag}(e_{11}, e_{12}, \cdots, e_{hg})$ is the diagonal matrix of degree hg with entries $e_{11}, e_{12}, \cdots, e_{hg}$. It is well known that $[L_B^* : L] = (\det e)^2 = (e_{11}e_{12} \cdots e_{hg})^2$ (cf. [I] p. 72). The number $\det e$ is called the *Pfaffian* of B.

Now, we consider the following subgroups of G:

(3.1) $$\Gamma_L := \left\{ (\lambda, \mu, \kappa) \in G \mid (\lambda, \mu) \in L,\ \kappa \in \mathbb{R}^{(h,h)} \right\}$$

and

(3.2) $$\Gamma_{L_B^*} := \left\{ (\lambda, \mu, \kappa) \in G \mid (\lambda, \mu) \in L_B^*,\ \kappa \in \mathbb{R}^{(h,h)} \right\}.$$

Then both Γ_L and $\Gamma_{L_B^*}$ are normal subgroups of G. We set

(3.3) $$\mathcal{Z}_0 := \left\{ (0, 0, \kappa) \in G \mid \kappa = {}^t\kappa \in \mathbb{Z}^{(h,h)} \text{ integral} \right\}.$$

It is easy to show that

$$\Gamma_{L_B^*} = \left\{ g \in G \mid g\gamma g^{-1}\gamma^{-1} \in \mathcal{Z}_0 \text{ for all } \gamma \in \Gamma_L \right\}.$$

We define

(3.4) $$Y_L := \left\{ \varphi \in \operatorname{Hom}(\Gamma_L, \mathbb{C}_1^\times) \mid \varphi \text{ is trivial on } \mathcal{Z}_0 \right\}$$

and

(3.5) $$Y_{L,S} := \left\{ \varphi \in Y_L \mid \varphi(\kappa) = e^{2\pi i \sigma(S\kappa)} \text{ for all } \kappa = {}^t\kappa \in \mathbb{R}^{(h,h)} \right\}$$

for each symmetric real matrix S of degree h. We observe that, if S is not half-integral, then $Y_L = \emptyset$ and so $Y_{L,S} = \emptyset$. It is clear that, if S is symmetric half-integral, then $Y_{L,S}$ is not empty.

Thus we have

$$Y_L = \cup_{\mathcal{M}} Y_{L,\mathcal{M}}, \tag{3.6}$$

where $\mathcal{M}$ runs through the set of all symmetric half-integral matrices of degree h.

Lemma 3.1. Let $\mathcal{M}$ be a symmetric half-integral matrix of degree h with $\mathcal{M} \neq 0$. Then any element φ of $Y_{L,\mathcal{M}}$ is of the form $\varphi_{\mathcal{M},q}$. Here $\varphi_{\mathcal{M},q}$ is the character of Γ_L defined by

$$\varphi_{\mathcal{M},q}((l,\kappa)) := e^{2\pi i \sigma(\mathcal{M}\kappa)} \cdot e^{\pi i q(l)} \quad \text{for } (l,\kappa) \in \Gamma_L, \tag{3.7}$$

where $q : L \longrightarrow \mathbb{R}/2\mathbb{Z} \cong [0,2)$ is a function on L satisfying the following condition:

$$q(l_0 + l_1) \equiv q(l_0) + q(l_1) - 2\sigma\{\mathcal{M}(\lambda_0\, {}^t\mu_1 - \mu_0\, {}^t\lambda_1)\} \pmod 2 \tag{3.8}$$

for all $l_0 = (\lambda_0, \mu_0) \in L$ and $l_1 = (\lambda_1, \mu_1) \in L$.

Proof. (3.8) follows immediately from the fact that $\varphi_{\mathcal{M},q}$ is a character of Γ_L. It is obvious that any element of $Y_{L,\mathcal{M}}$ is of the form $\varphi_{\mathcal{M},q}$. □

Lemma 3.2. An element of $Y_{L,0}$ is of the form $\varphi_{k,l}$ $(k, l \in \mathbb{R}^{(h,g)})$. Here $\varphi_{k,l}$ is the character of Γ_L defined by

$$\varphi_{k,l}(\gamma) := e^{2\pi i \sigma(k\, {}^t\lambda + l\, {}^t\mu)}, \quad \gamma = (\lambda, \mu, \kappa) \in \Gamma_L. \tag{3.9}$$

Proof. It is easy to prove and so we omit the proof. □

Lemma 3.3. Let $\mathcal{M}$ be a nonsingular symmetric half-integral matrix of degree h. Let $\varphi_{\mathcal{M},q_1}$ and $\varphi_{\mathcal{M},q_2}$ be the characters of Γ_L defined by (3.7). The character φ of Γ_L defined by $\varphi := \varphi_{\mathcal{M},q_1} \cdot \varphi^{-1}_{\mathcal{M},q_2}$ is an element of $Y_{L,0}$.

Proof. It follows from the existence of an element $g = (\mathcal{M}^{-1}\lambda, \mathcal{M}^{-1}\mu, 0) \in G$ with $(\lambda, \mu) \in V$ such that

$$\varphi_{\mathcal{M},q_1}(\gamma) = \varphi_{\mathcal{M},q_2}(g\gamma g^{-1}) \quad \text{for all } \gamma \in \Gamma_L.$$

□

For a unitary character $\varphi_{\mathcal{M},q}$ of Γ_L defined by (3.7), we let

$$\pi_{\mathcal{M},q} := \mathrm{Ind}_{\Gamma_L}^{G}\, \varphi_{\mathcal{M},q} \tag{3.10}$$

be the representation of G induced from $\varphi_{\mathcal{M},q}$. Let $\mathcal{H}_{\mathcal{M},q}$ be the Hilbert space consisting of all measurable functions $\phi : G \longrightarrow \mathbb{C}$ satisfying

(L1) $\phi(\gamma g) = \varphi_{\mathcal{M},q}(\gamma)\, \phi(g)$ for all $\gamma \in \Gamma_L$ and $g \in G$.
(L2) $\|\phi\|^2_{\mathcal{M},q} := \int_{\Gamma_L \backslash G} |\phi(\bar{g})|^2\, d\bar{g} < \infty, \quad \bar{g} = \Gamma_L g.$

The induced representation $\pi_{\mathcal{M},q}$ is realized in $\mathcal{H}_{\mathcal{M},q}$ as follows:

$$\left(\pi_{\mathcal{M},q}(g_0)\phi\right)(g) := \phi(gg_0), \quad g_0, g \in G, \ \phi \in \mathcal{H}_{\mathcal{M},q}. \tag{3.11}$$

The representation $\pi_{\mathcal{M},q}$ is called the *lattice representation* of G associated with the lattice L.

Main Theorem. Let $\mathcal{M}$ be a positive definite, symmetric half integral matrix of degree h. Let $\varphi_{\mathcal{M}}$ be the character of Γ_L defined by $\varphi_{\mathcal{M}}((\lambda,\mu,\kappa)) := e^{2\pi i\sigma(\mathcal{M}\kappa)}$ for all $(\lambda,\mu,\kappa) \in \Gamma_L$. Then the lattice representation

$$\pi_{\mathcal{M}} := \operatorname{Ind}_{\Gamma_L}^G \varphi_{\mathcal{M}}$$

induced from the character $\varphi_{\mathcal{M}}$ is unitarily equivalent to the direct sum

$$\bigoplus U(\sigma_{\mathcal{M}}) := \bigoplus \operatorname{Ind}_K^G \sigma_{\mathcal{M}} \quad (\,(\det 2\mathcal{M}\,)^g\text{-copies}\,)$$

of the Schrödinger representation $\operatorname{Ind}_K^G \sigma_{\mathcal{M}}$.

Proof. We first recall that the induced representation $\pi_{\mathcal{M}}$ is realized in the Hilbert space $\mathcal{H}_{\mathcal{M}}$ consisting of all measurable functions $\phi : G \longrightarrow \mathbb{C}$ satisfying the conditions

$$\phi((\lambda_0,\mu_0,\kappa_0)\circ g) = e^{2\pi i\sigma(\mathcal{M}\kappa_0)}\,\phi(g), \quad (\lambda_0,\mu_0,\kappa_0) \in \Gamma_L, \ g \in G \tag{3.13}$$

and

$$\|\phi\|^2_{\pi,\mathcal{M}} := \int_{\Gamma_L\backslash G} |\phi(\bar{g})|^2\, d\bar{g} < \infty, \quad \bar{g} = \Gamma_L \circ g. \tag{3.14}$$

Now, we write

$$g_0 = [\lambda_0,\mu_0,\kappa_0] \in \Gamma_L \ \ and \ \ g = [\lambda,\mu,\kappa] \in G.$$

For $\phi \in \mathcal{H}_{\mathcal{M}}$, we have

$$\phi(g_0 \diamond g) = \phi([\lambda_0+\lambda, \mu_0+\mu, \kappa_0+\kappa+\lambda_0\,{}^t\mu + \mu\,{}^t\lambda_0]). \tag{3.15}$$

On the other hand, we get

$$\begin{aligned}\phi(g_0 \diamond g) &= \phi((\lambda_0,\mu_0,\kappa_0-\mu_0\,{}^t\lambda_0)\circ g)\\ &= e^{2\pi i\sigma\{\mathcal{M}(\kappa_0-\mu_0\,{}^t\lambda_0)\}}\,\phi(g)\\ &= e^{2\pi i\sigma(\mathcal{M}\kappa_0)}\,\phi(g) \quad (\text{because } \sigma(\mathcal{M}\mu_0\,{}^t\lambda_0) \in \mathbb{Z}\,)\end{aligned}$$

Thus, putting $\kappa' := \kappa_0 + \lambda_0\,{}^t\mu + \mu\,{}^t\lambda_0$, we get

$$\phi([\lambda_0+\lambda, \mu_0+\mu, \kappa+\kappa']) = e^{2\pi i\sigma(\mathcal{M}\kappa')}\cdot e^{-4\pi i\sigma(\mathcal{M}\lambda_0\,{}^t\mu)}\,\phi([\lambda,\mu,\kappa]). \tag{3.16}$$

Putting $\lambda_0 = \kappa_0 = 0$ in (3.16), we have

(3.17) $\phi([\lambda, \mu+\mu_0, \kappa]) = \phi([\lambda, \mu, \kappa])$ for all $\mu_0 \in \mathbb{Z}^{(h,g)}$ and $[\lambda, \mu, \kappa] \in G$.

Therefore if we fix λ and κ, ϕ is periodic in μ with respect to the lattice $\mathbb{Z}^{(h,g)}$ in $\mathbb{R}^{(h,g)}$. We note that

$$\phi([\lambda, \mu, \kappa]) = \phi([0, 0, \kappa] \diamond [\lambda, \mu, 0]) = e^{2\pi i\sigma(\mathcal{M}\kappa)}\, \phi([\lambda, \mu, 0])$$

for $[\lambda, \mu, \kappa] \in G$. Hence, ϕ admits a Fourier expansion in μ :

$$\phi([\lambda, \mu, \kappa]) = e^{2\pi i\sigma(\mathcal{M}\kappa)} \sum_{N\in\mathbb{Z}^{(h,g)}} c_N(\lambda)\, e^{2\pi i\sigma(N\,{}^t\mu)}. \tag{3.18}$$

If $\lambda_0 \in \mathbb{Z}^{(h,g)}$, then we have

$$\begin{aligned}\phi([\lambda+\lambda_0, \mu, \kappa]) &= e^{2\pi i\sigma(\mathcal{M}\kappa)} \sum_{N\in\mathbb{Z}^{(h,g)}} c_N(\lambda+\lambda_0)\, e^{2\pi i\sigma(N\,{}^t\mu)}\\ &= e^{-4\pi i\sigma(\mathcal{M}\lambda_0\,{}^t\mu)}\, \phi([\lambda, \mu, \kappa]) \qquad (\text{by } (3.16))\\ &= e^{-4\pi i\sigma(\mathcal{M}\lambda_0\,{}^t\mu)}\, e^{2\pi i\sigma(\mathcal{M}\kappa)} \sum_{N\in\mathbb{Z}^{(h,g)}} c_N(\lambda)\, e^{2\pi i\sigma(N\,{}^t\mu)},\\ &= e^{2\pi i\sigma(\mathcal{M}\kappa)} \sum_{N\in\mathbb{Z}^{(h,g)}} c_N(\lambda)\, e^{2\pi i\sigma\{(N-2\mathcal{M}\lambda_0)\,{}^t\mu\}}. \quad (\text{by } (3.18))\end{aligned}$$

So we get

$$\begin{aligned}&\sum_{N\in\mathbb{Z}^{(h,g)}} c_N(\lambda+\lambda_0)\, e^{2\pi i\sigma(N\,{}^t\mu)}\\ &= \sum_{N\in\mathbb{Z}^{(h,g)}} c_N(\lambda)\, e^{2\pi i\sigma\{(N-2\mathcal{M}\lambda_0)\,{}^t\mu\}}\\ &= \sum_{N\in\mathbb{Z}^{(h,g)}} c_{N+2\mathcal{M}\lambda_0}(\lambda)\, e^{2\pi i\sigma(N\,{}^t\mu)}.\end{aligned}$$

Hence, we get

$$c_N(\lambda+\lambda_0) = c_{N+2\mathcal{M}\lambda_0}(\lambda) \quad \text{for all } \lambda_0 \in \mathbb{Z}^{(h,g)} \text{ and } \lambda \in \mathbb{R}^{(h,g)}. \tag{3.19}$$

Consequently, it is enough to know only the coefficients $c_\alpha(\lambda)$ for the representatives α in $\mathbb{Z}^{(h,g)}$ modulo $2\mathcal{M}$. It is obvious that the number of all such α's is $(\det 2\mathcal{M})^g$. We denote by $\mathcal{J}$ a complete system of such representatives in $\mathbb{Z}^{(h,g)}$ modulo $2\mathcal{M}$.

Then, we have

$$\begin{aligned}\phi([\lambda,\mu,\kappa]) = e^{2\pi i\sigma(\mathcal{M}\kappa)}\Big\{ &\sum_{N\in\mathbb{Z}^{(h,g)}} c_{\alpha+2\mathcal{M}N}(\lambda)\, e^{2\pi i\sigma\{(\alpha+2\mathcal{M}N)\,{}^t\mu\}} \\ &+ \sum_{N\in\mathbb{Z}^{(h,g)}} c_{\beta+2\mathcal{M}N}(\lambda)\, e^{2\pi i\sigma\{(\beta+2\mathcal{M}N)\,{}^t\mu\}} \\ &\cdot \\ &\cdot \\ &+ \sum_{N\in\mathbb{Z}^{(h,g)}} c_{\gamma+2\mathcal{M}N}(\lambda)\, e^{2\pi i\{(\gamma+2\mathcal{M}N)\,{}^t\mu\}}\Big\},\end{aligned}$$

where $\{\alpha,\beta,\cdots,\gamma\}$ denotes the complete system $\mathcal{J}$.

For each $\alpha\in\mathcal{J}$, we denote by $\mathcal{H}_{\mathcal{M},\alpha}$ the Hilbert space consisting of Fourier expansions

$$e^{2\pi i\sigma(\mathcal{M}\kappa)}\sum_{N\in\mathbb{Z}^{(h,g)}} c_{\alpha+2\mathcal{M}N}(\lambda)\, e^{2\pi i\sigma\{(\alpha+2\mathcal{M}N)\,{}^t\mu\}},\quad (\lambda,\mu,\kappa)\in G,$$

where $c_N(\lambda)$ denotes the coefficients of the Fourier expansion (3.18) of $\phi\in\mathcal{H}_{\mathcal{M}}$ and ϕ runs over the set $\{\phi\in\pi_{\mathcal{M}}\}$. It is easy to see that $\mathcal{H}_{\mathcal{M},\alpha}$ is invariant under $\pi_{\mathcal{M}}$. We denote the restriction of $\pi_{\mathcal{M}}$ to $\mathcal{H}_{\mathcal{M},\alpha}$ by $\pi_{\mathcal{M},\alpha}$. Then we have

$$\pi_{\mathcal{M}} = \bigoplus_{\alpha\in\mathcal{J}} \pi_{\mathcal{M},\alpha}. \tag{3.20}$$

Let $\phi_\alpha\in\pi_{\mathcal{M},\alpha}$. Then for $[\lambda,\mu,\kappa]\in G$, we get

$$\phi_\alpha([\lambda,\mu,\kappa]) = e^{2\pi i\sigma(\mathcal{M}\kappa)}\sum_{N\in\mathbb{Z}^{(h,g)}} c_{\alpha+2\mathcal{M}N}(\lambda)\, e^{2\pi i\sigma\{(\alpha+2\mathcal{M}N)\,{}^t\mu\}}. \tag{3.21}$$

We put

$$I_\lambda := \overbrace{[0,1]\times[0,1]\times\cdots\times[0,1]}^{(h\times g)\text{-times}} \subset \left\{[\lambda,0,0]\,|\,\lambda\in\mathbb{R}^{(h,g)}\right\}$$

and

$$I_\mu := \overbrace{[0,1]\times[0,1]\times\cdots\times[0,1]}^{(h\times g)\text{-times}} \subset \left\{[0,\mu,0]\,|\,\mu\in\mathbb{R}^{(h,g)}\right\}.$$

Then, we obtain

$$\int_{I_\mu}\phi_\alpha([\lambda,\mu,\kappa])\, e^{-2\pi i\sigma(\alpha\,{}^t\mu)}\, d\mu = e^{2\pi i\sigma(\mathcal{M}\kappa)}\, c_\alpha(\lambda),\quad \alpha\in\mathcal{J}. \tag{3.22}$$

Since $\Gamma_L\backslash G \cong I_\lambda \times I_\mu$, we get

$$\begin{aligned}
\|\phi_\alpha\|^2_{\pi,\mathcal{M},\alpha} :&= \|\phi_\alpha\|^2_{\pi,\mathcal{M}} = \int_{\Gamma_L\backslash G} |\phi_\alpha(\bar{g})|^2\, d\bar{g} \\
&= \int_{I_\lambda}\int_{I_\mu} |\phi_\alpha(\bar{g})|^2\, d\lambda d\mu \\
&= \int_{I_\lambda\times I_\mu} \left| \sum_{N\in\mathbb{Z}^{(h,g)}} c_{\alpha+2\mathcal{M}N}(\lambda)\, e^{2\pi i\sigma\{(\alpha+2\mathcal{M}N)\,{}^t\mu\}} \right|^2 d\lambda d\mu \\
&= \int_{I_\lambda} \sum_{N\in\mathbb{Z}^{(h,g)}} |c_{\alpha+2\mathcal{M}N}(\lambda)|^2\, d\lambda \\
&= \int_{I_\lambda} \sum_{N\in\mathbb{Z}^{(h,g)}} |c_\alpha(\lambda+N)|^2\, d\lambda \qquad (\text{by } (3.19)) \\
&= \int_{\mathbb{R}^{(h,g)}} |c_\alpha(\lambda)|^2\, d\lambda.
\end{aligned}$$

Since $\phi_\alpha \in \pi_{\mathcal{M},\alpha}$, $\|\phi_\alpha\|_{\pi,\mathcal{M},\alpha} < \infty$ and so $c_\alpha(\lambda) \in L^2\left(\mathbb{R}^{(h,g)}, d\xi\right)$ for all $\alpha \in \mathcal{J}$.

For each $\alpha \in \mathcal{J}$, we define the mapping $\vartheta_{\mathcal{M},\alpha}$ on $L^2\left(\mathbb{R}^{(h,g)}, d\xi\right)$ by

$$(3.23)\quad (\vartheta_{\mathcal{M},\alpha} f)([\lambda,\mu,\kappa]) := e^{2\pi i\sigma(\mathcal{M}\kappa)} \sum_{N\in\mathbb{Z}^{(h,g)}} f(\lambda+N)\, e^{2\pi i\sigma\{(\alpha+2\mathcal{M}N)\,{}^t\mu\}},$$

where $f \in L^2\left(\mathbb{R}^{(h,g)}, d\xi\right)$ and $[\lambda,\mu,\kappa] \in G$.

Lemma 3.4. For each $\alpha \in \mathcal{J}$, the image of $L^2\left(\mathbb{R}^{(h,g)}, d\xi\right)$ under $\vartheta_{\mathcal{M},\alpha}$ is contained in $\mathcal{H}_{\mathcal{M},\alpha}$. Moreover, the mapping $\vartheta_{\mathcal{M},\alpha}$ is a one-to-one unitary operator of $L^2\left(\mathbb{R}^{(h,g)}, d\xi\right)$ onto $\mathcal{H}_{\mathcal{M},\alpha}$ preserving the norms. In other words, the mapping

$$\vartheta_{\mathcal{M},\alpha} : L^2\left(\mathbb{R}^{(h,g)}, d\xi\right) \longrightarrow \mathcal{H}_{\mathcal{M},\alpha}$$

is an isometry.

Proof. We already showed that $\vartheta_{\mathcal{M},\alpha}$ preserves the norms. First, we observe that if $(\lambda_0,\mu_0,\kappa_0) \in \Gamma_L$ and $g = [\lambda,\mu,\kappa] \in G$,

$$\begin{aligned}
(\lambda_0,\mu_0,\kappa_0)\circ g &= [\lambda_0,\mu_0,\kappa_0+\mu_0\,{}^t\lambda_0] \diamond [\lambda,\mu,\kappa] \\
&= [\lambda_0+\lambda,\ \mu_0+\mu,\ \kappa+\kappa_0+\mu_0\,{}^t\lambda_0+\lambda_0\,{}^t\mu+\mu\,{}^t\lambda_0].
\end{aligned}$$

Thus we get

$$\begin{aligned}
&(\vartheta_{\mathcal{M},\alpha} f)((\lambda_0,\mu_0,\kappa_0)\circ g)\\
&= e^{2\pi i\sigma\{\mathcal{M}(\kappa+\kappa_0+\mu_0\,{}^t\lambda_0+\lambda_0\,{}^t\mu+\mu\,{}^t\lambda_0)\}} \sum_{N\in\mathbb{Z}^{(h,g)}} f(\lambda+\lambda_0+N)\, e^{2\pi i\{(\alpha+2\mathcal{M}N)\,{}^t(\mu_0+\mu)\}}\\
&= e^{2\pi i\sigma(\mathcal{M}\kappa_0)}\cdot e^{2\pi i\sigma(\mathcal{M}\kappa)}\cdot e^{2\pi i\sigma(\alpha\,{}^t\mu_0)} \sum_{N\in\mathbb{Z}^{(h,g)}} f(\lambda+N)\, e^{2\pi i\sigma\{(\alpha+2\mathcal{M}N)\,{}^t\mu\}}\\
&= e^{2\pi i\sigma(\mathcal{M}\kappa_0)}\,(\vartheta_{\mathcal{M},\alpha} f)(g).
\end{aligned}$$

Here, in the above equalities we used the facts that $2\sigma(\mathcal{M}N\,{}^t\mu_0)\in\mathbb{Z}$ and $\alpha\,{}^t\mu_0\in\mathbb{Z}$. It is easy to show that

$$\int_{\Gamma_L\backslash G} |\vartheta_{\mathcal{M},\alpha} f(\bar{g})|^2\, d\bar{g} = \int_{\mathbb{R}^{(h,g)}} |f(\lambda)|^2\, d\lambda = \|f\|_2^2 < \infty.$$

This completes the proof of Lemma 3.4.

Finally, it is easy to show that for each $\alpha\in\mathcal{J}$, the mapping $\vartheta_{\mathcal{M},\alpha}$ intertwines the Schrödinger representation $\big(U(\sigma_{\mathcal{M}}), L^2(\mathbb{R}^{(h,g)}, d\xi)\big)$ and the representation $(\pi_{\mathcal{M},\alpha},\mathcal{H}_{\mathcal{M},\alpha})$. Therefore, by Lemma 3.4, for each $\alpha\in\mathcal{J}$, $\pi_{\mathcal{M},\alpha}$ is unitarily equivalent to $U(\sigma_{\mathcal{M}})$ and so $\pi_{\mathcal{M},\alpha}$ is an irreducible unitary representation of G. According to (3.20), the induced representation $\pi_{\mathcal{M}}$ is unitarily equivalent to

$$\bigoplus U(\sigma_{\mathcal{M}}) \quad (\,(\det 2\mathcal{M})^g\text{-copies}\,).$$

This completes the proof of the Main Theorem. □

4. Relation of lattice representations to theta functions

In this section, we state the connection between lattice representations and theta functions. As before, we write $V=\mathbb{R}^{(h,g)}\times\mathbb{R}^{(h,g)}\cong\mathbb{C}^{(h,g)}$, $L=\mathbb{Z}^{(h,g)}\times\mathbb{Z}^{(h,g)}$ and $\mathcal{M}$ is a positive symmetric half-integral matrix of degree h. The function $q_{\mathcal{M}}: L\longrightarrow\mathbb{R}/2\mathbb{Z}=[0,2)$ defined by

$$(4.1)\qquad q_{\mathcal{M}}((\xi,\eta)) := 2\,\sigma(\mathcal{M}\xi\,{}^t\eta),\quad (\xi,\eta)\in L$$

satisfies Condition (3.8). We let $\varphi_{\mathcal{M},q_{\mathcal{M}}}:\Gamma_L\longrightarrow\mathbb{C}_1^\times$ be the character of Γ_L defined by

$$\varphi_{\mathcal{M},q_{\mathcal{M}}}((l,\kappa)) = e^{2\pi i\,\sigma(\mathcal{M}\kappa)}\, e^{\pi i\, q_{\mathcal{M}}(l)},\quad (l,\kappa)\in\Gamma_L.$$

We denote by $\mathcal{H}_{\mathcal{M},q_{\mathcal{M}}}$ the Hilbert space consisting of measurable functions $\phi: G\longrightarrow\mathbb{C}$ which satisfy Condition (4.2) and Condition (4.3):

(4.2) $\phi((l,\kappa)\circ g) = \varphi_{\mathcal{M},q_{\mathcal{M}}}((l,\kappa))\,\phi(g)$ for all $(l,\kappa)\in\Gamma_L$ and $g\in G$.

(4.3) $$\int_{\Gamma_L\backslash G} \|\phi(\dot{g})\|^2 \, d\dot{g} \, < \, \infty, \quad \dot{g} = \Gamma_L \circ g.$$

Then the lattice representation

$$\pi_{\mathcal{M},q_\mathcal{M}} := \operatorname{Ind}_{\Gamma_L}^G \varphi_{\mathcal{M},q_\mathcal{M}}$$

of G induced from the character $\varphi_{\mathcal{M},q_\mathcal{M}}$ is realized in $\mathcal{H}_{\mathcal{M},q_\mathcal{M}}$ as

$$\left(\pi_{\mathcal{M},q_\mathcal{M}}(g_0)\,\phi\right)(g) \, = \, \phi(gg_0), \quad g_0, g \in G, \ \phi \in \mathcal{H}_{\mathcal{M},q_\mathcal{M}}.$$

Let $\mathbf{H}_{\mathcal{M},q_\mathcal{M}}$ be the vector space consisting of measurable functions $F : V \longrightarrow \mathbb{C}$ satisfying Conditions (4.4) and (4.5).

(4.4) $$F(\lambda + \xi, \mu + \eta) \, = \, e^{2\pi i\,\sigma\{\mathcal{M}(\xi\,{}^t\eta + \lambda\,{}^t\eta - \mu\,{}^t\xi)\}}\, F(\lambda, \mu)$$

for all $(\lambda, \mu) \in V$ and $(\xi, \eta) \in L$.

(4.5) $$\int_{L\backslash V} \|F(\dot{v})\|^2 \, d\dot{v} \, = \, \int_{I_\lambda \times I_\mu} \|F(\lambda, \mu)\|^2 \, d\lambda d\mu \, < \, \infty.$$

Given $\phi \in \mathcal{H}_{\mathcal{M},q_\mathcal{M}}$ and a fixed element $\Omega \in H_g$, we put

(4.6) $E_\phi(\lambda, \mu) := \phi((\lambda, \mu, 0)), \quad \lambda, \mu \in \mathbb{R}^{(h,g)}$,

(4.7) $F_\phi(\lambda, \mu) := \phi([\lambda, \mu, 0]), \quad \lambda, \mu \in \mathbb{R}^{(h,g)}$,

(4.8) $F_{\Omega,\phi}(\lambda, \mu) := e^{-2\pi i\,\sigma(\mathcal{M}\lambda\Omega\,{}^t\lambda)}\, F_\phi(\lambda, \mu), \quad \lambda, \mu \in \mathbb{R}^{(h,g)}$.

In addition, we put for $W = \lambda\Omega + \mu \in \mathbb{C}^{(h,g)}$,

(4.9) $$\vartheta_{\Omega,\phi}(W) \, = \, \vartheta_{\Omega,\phi}(\lambda\Omega + \mu) \, := \, F_{\Omega,\phi}(\lambda, \mu).$$

We observe that E_ϕ, F_ϕ and $F_{\Omega,\phi}$ are functions defined on V and $\vartheta_{\Omega,\phi}$ is a function defined on $\mathbb{C}^{(h,g)}$.

Proposition 4.1. If $\phi \in \mathcal{H}_{\mathcal{M},q_\mathcal{M}}$, $(\xi, \eta) \in L$ and $(\lambda, \mu) \in V$, then we have the formulas

(4.10) $$E_\phi(\lambda + \xi, \mu + \eta) \, = \, e^{2\pi i\,\sigma\{\mathcal{M}(\xi\,{}^t\eta + \lambda\,{}^t\eta - \mu\,{}^t\xi)\}}\, E_\phi(\lambda, \mu).$$

(4.11) $$F_\phi(\lambda + \xi, \mu + \eta) \, = \, e^{-4\pi i\,\sigma(\mathcal{M}\xi\,{}^t\mu)}\, F_\phi(\lambda, \mu).$$

(4.12) $$F_{\Omega,\phi}(\lambda + \xi, \mu + \eta) \, = \, e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi + 2\lambda\Omega\,{}^t\xi + 2\mu\,{}^t\xi)\}}\, F_{\Omega,\phi}(\lambda, \mu).$$

If $W = \lambda\Omega + \eta \in \mathbb{C}^{(h,g)}$, then we have

(4.13) $$\vartheta_{\Omega,\phi}(W + \xi\Omega + \eta) \, = \, e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi + 2W\,{}^t\xi)\}}\, \vartheta_{\Omega,\phi}(W).$$

Moreover, F_ϕ is an element of $\mathbf{H}_{\mathcal{M},q_\mathcal{M}}$.

Proof. We note that

$$(\lambda+\xi,\,\mu+\eta,0)=(\xi,\eta,-\xi\,{}^t\mu+\eta\,{}^t\lambda)\circ(\lambda,\mu,0).$$

Thus we have

$$\begin{aligned}E_\phi(\lambda+\xi,\,\mu+\eta)&=\phi((\lambda+\xi,\,\mu+\eta,0))\\&=\phi((\xi,\eta,\,-\xi\,{}^t\mu+\eta\,{}^t\lambda)\circ(\lambda,\mu,0))\\&=e^{2\pi i\sigma\{\mathcal{M}(\xi\,{}^t\eta+\lambda\,{}^t\eta-\mu\,{}^t\xi)\}}\,\phi((\lambda,\mu,0))\\&=e^{2\pi i\sigma\{\mathcal{M}(\xi\,{}^t\eta+\lambda\,{}^t\eta-\mu\,{}^t\xi)\}}\,E_\phi(\lambda,\mu).\end{aligned}$$

This proves Formula (4.10). We observe that

$$[\lambda+\xi,\,\mu+\eta,0]=(\xi,\eta,\,-\xi\,{}^t\mu-\mu\,{}^t\xi-\eta\,{}^t\xi)\circ[\lambda,\mu,0].$$

Thus we have

$$\begin{aligned}F_\phi(\lambda+\xi,\,\mu+\eta)&=\phi([\lambda+\xi,\,\mu+\eta,0])\\&=e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\,{}^t\mu+\mu\,{}^t\xi+\eta\,{}^t\xi)\}}\\&\quad\times e^{2\pi i\,\sigma(\mathcal{M}\xi\,{}^t\eta)}\,\phi([\lambda,\mu,0])\\&=e^{-4\pi i\,\sigma(\mathcal{M}\xi\,{}^t\mu)}\,\phi([\lambda,\mu,0])\\&=e^{-4\pi i\,\sigma(\mathcal{M}\xi\,{}^t\mu)}\,F_\phi(\lambda,\mu).\end{aligned}$$

This proves Formula (4.11). According to (4.11), we have

$$\begin{aligned}F_{\Omega,\phi}(\lambda+\xi,\,\mu+\eta)&=e^{-2\pi i\,\sigma\{\mathcal{M}(\lambda+\xi)\Omega\,{}^t(\lambda+\xi)\}}\,F_\phi(\lambda+\xi,\,\mu+\eta)\\&=e^{-2\pi i\,\sigma\{\mathcal{M}(\lambda+\xi)\Omega\,{}^t(\lambda+\xi)\}}\\&\quad\times e^{-4\pi i\,\sigma(\mathcal{M}\xi\,{}^t\mu)}\,F_\phi(\lambda,\mu)\\&=e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi+2\lambda\Omega\,{}^t\xi+2\mu\,{}^t\xi)\}}\\&\quad\times e^{-2\pi i\,\sigma(\mathcal{M}\lambda\Omega\,{}^t\lambda)}\,F_\phi(\lambda,\mu)\\&=e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi+2\lambda\Omega\,{}^t\xi+2\mu\,{}^t\xi)\}}\,F_{\Omega,\phi}(\lambda,\mu).\end{aligned}$$

This proves Formula (4.12). Formula (4.13) follows immediately from Formula (4.12). Indeed, if $W=\lambda\Omega+\mu$ with $\lambda,\mu\in\mathbb{R}^{(h,g)}$, we have

$$\begin{aligned}\vartheta_{\Omega,\phi}(W+\xi\Omega+\eta)&=F_{\Omega,\phi}(\lambda+\xi,\,\mu+\eta)\\&=e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi+2(\lambda\Omega+\mu)\,{}^t\xi)\}}\,F_{\Omega,\phi}(\lambda,\mu)\\&=e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi+2W\,{}^t\xi)\}}\,\vartheta_{\Omega,\phi}(W).\end{aligned}$$

□

Remark 4.2. The function $\vartheta_{\Omega,\phi}(W)$ is a theta function of level $2\mathcal{M}$ with respect to Ω if $\vartheta_{\Omega,\phi}$ is holomorphic. For any $\phi \in \mathcal{H}_{\mathcal{M},q_{\mathcal{M}}}$, the function $\vartheta_{\Omega,\phi}$ satisfies the well known transformation law of a theta function. In this sense, the lattice representation $(\pi_{\mathcal{M},q_{\mathcal{M}}}, \mathcal{H}_{\mathcal{M},q_{\mathcal{M}}})$ is closely related to theta functions.

References

[B] R. Berndt, Darstellungen der Heisenberggruppe und Thetafunktionen. Vorlesungsausarbeitung, Hamburg, 1988

[C] P. Cartier, Quantum Mechanical Commutation Relations and Theta Functions, Proc. of Symp. Pure Mathematics, **9** Amer. Math. Soc., 1966, pp. 361–383

[F-C] G. Faltings and C.-L. Chai, Degeneration of Abelian Varieties, EMG, Band **22**, Springer-Verlag, New York/Berlin, 1990

[I] J. Igusa, Theta functions, Springer-Verlag, New York/Berlin, 1972

[M] G.W. Mackey, Induced Representations of Locally Compact Groups I, Ann. of Math. **55** (1952), 101–139

[N] Y. Namikawa, Toroidal Compactification of Siegel Spaces, Lect. Notes in Math. 812, Springer-Verlag, New York/Berlin, 1980

[Y1] J.-H. Yang, Harmonic Analysis on the Quotient Spaces of Heisenberg Groups, Nagoya Math. J. **123** (1991), 103–117

[Y2] J.-H. Yang, Harmonic Analysis on the Quotient Spaces of Heisenberg Groups, II, J. Number Theory **49** (1994), 63–72

[Y3] J.-H. Yang, A decomposition theorem on differential polynomials of theta functions of high level, Japanese J. Math., Math. Soc. Japan, New Series **22** (1996), 37–49

[Y4] J.-H. Yang, The Siegel-Jacobi Operator, Abh. Math. Sem. Univ. Hamburg **63** (1993), 135–146

[Y5] J.-H. Yang, Remarks on Jacobi forms of higher degree, Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, PIMS (1993), 33–58

[Y6] J.-H. Yang, Singular Jacobi Forms, Trans. Amer. Math. Soc. **347** (1995), 2041–2049

[Y7] J.-H. Yang, Construction of Vector-Valued Modular Forms from Jacobi Forms, Canadian J. Math. **47** (1995), 1329–1339

[Y8] J.-H. Yang, A geometrical theory of Jacobi forms of higher degree, Proc. of Symposium on Hodge Theory and Algebraic Geometry (edited by Tadao Oda), Sendai, Japan (1996), 125–147

[Z] C. Ziegler, Jacobi Forms of Higher Degree, Abh. Math. Sem. Univ. Hamburg **59** (1989), 191–224

[D] Theory of Maass-Jacobi Forms

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A Note on Maass-Jacobi Forms

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Abstract. In this paper, we introduce the notion of Maass-Jacobi forms and investigate some properties of these new automorphic forms. We also characterize these automorphic forms in several ways.

1. Introduction

We let $SL_{2,1}(\mathbb{R}) = SL(2,\mathbb{R}) \ltimes \mathbb{R}^{(1,2)}$ be the semi-direct product of the special linear group $SL(2,\mathbb{R})$ of degree 2 and the commutative group $\mathbb{R}^{(1,2)}$ equipped with the following multiplication law

$$(1.1) \qquad (g,\alpha) * (h,\beta) = (gh,\, \alpha\, {}^t h^{-1} + \beta), \quad g,h \in SL(2,\mathbb{R}), \quad \alpha,\beta \in \mathbb{R}^{(1,2)},$$

where $\mathbb{R}^{(1,2)}$ denotes the set of all 1×2 real matrices. We let

$$SL_{2,1}(\mathbb{Z}) = SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^{(1,2)}$$

be the discrete subgroup of $SL_{2,1}(\mathbb{R})$ and $K = SO(2)$ the special orthogonal group of degree 2.

Throughout this paper, for brevity we put

$$G = SL_{2,1}(\mathbb{R}), \quad \Gamma_1 = SL(2,\mathbb{Z}) \quad \text{and} \quad \Gamma = SL_{2,1}(\mathbb{Z}).$$

Let $\mathbb{H}$ be the Poincaré upper half plane. Then G acts on $\mathbb{H} \times \mathbb{C}$ transitively by

$$(1.2) \qquad (g,\alpha) \circ (\tau, z) = ((d\tau - c)(-b\tau + a)^{-1}, (z + \alpha_1 \tau + \alpha_2)(-b\tau + a)^{-1}),$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^{(1,2)}$ and $(\tau, z) \in \mathbb{H} \times \mathbb{C}$. We observe that K is the stabilizer of this action (1.2) at the origin $(i, 0)$. $\mathbb{H} \times \mathbb{C}$ may be identified with the homogeneous space G/K in a natural way.

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The aim of this paper is to define the notion of Maass-Jacobi forms generalizing that of Maass wave forms and study some properties of these new automorphic forms. For the convenience of the reader, we review Maass wave forms. For $s \in \mathbb{C}$, we denote by $W_s(\Gamma_1)$ the vector space of all smooth bounded functions $f : SL(2,\mathbb{R}) \longrightarrow \mathbb{C}$ satisfying the following conditions (a) and (b):

(a) $f(\gamma g k) = f(g)$ for all $\gamma \in \Gamma_1$, $g \in SL(2,\mathbb{R})$ and $k \in K$.

(b) $\Delta_0 f = \frac{1-s^2}{4} f$,

where $\Delta_0 = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - y \frac{\partial^2}{\partial x \partial \theta} + \frac{5}{4} \frac{\partial^2}{\partial \theta^2}$ is the Laplace-Beltrami operator associated to the $SL(2,\mathbb{R})$-invariant Riemannian metric

$$ds_0^2 = \frac{1}{y^2}(dx^2 + dy^2) + \left(d\theta + \frac{dx}{2y} \right)^2$$

on $SL(2,\mathbb{R})$ whose coordinates x, y, θ ($x \in \mathbb{R}$, $y > 0$, $0 \le \theta < 2\pi$) are given by

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad g \in SL(2,\mathbb{R})$$

by means of the Iwasawa decomposition of $SL(2,\mathbb{R})$. The elements in $W_s(\Gamma_1)$ are called *Maass wave forms*. It is well known that $W_s(\Gamma_1)$ is nontrivial for infinitely many values of s. For more detail, we refer to [6], [9], [13], [17] and [20].

The paper is organized as follows. In Section 2, we calculate the algebra of all invariant differential operators under the action (1.2) of G on $\mathbb{H} \times \mathbb{C}$ completely. In addition, we provide a G-invariant Riemannian metric on $\mathbb{H} \times \mathbb{C}$ and compute its Laplace-Beltrami operator. In Section 3, using the above Laplace-Beltrami operator, we introduce a concept of Maass-Jacobi forms generalizing that of Maass wave forms. We characterize Maass-Jacobi forms as smooth functions on G or $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ satisfying a certain invariance property, where $S\mathcal{P}_2$ denotes the symmetric space consisting of all 2×2 positive symmetric real matrices Y with $\det Y = 1$. In Section 4, we find the unitary dual of G and present some properties of G. In Section 5, we describe the decomposition of the Hilbert space $L^2(\Gamma\backslash G)$. In the final section, we make some comments on the Fourier expansion of Maass-Jacobi forms.

Notations. We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers and the field of complex numbers respectively. $\mathbb{Z}^+$ denotes the set of all positive integers. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix A, $\sigma(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose of M. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. We denote the identity matrix of degree n by E_n. $\mathbb{H}$ denotes the Poincaré upper-half plane.

2. Invariant Differential Operators on $\mathbb{H} \times \mathbb{C}$

We recall that $\mathcal{SP}_2$ is the symmetric space consisting of all 2×2 positive symmetric real matrices Y with $\det Y = 1$. Then G acts on $\mathcal{SP}_2 \times \mathbb{R}^{(1,2)}$ transitively by

$$(g,\alpha)\cdot(Y,V) = (\,gY\,{}^tg,\ (V+\alpha)\,{}^tg\,), \tag{2.1}$$

where $g\in SL(2,\mathbb{R}),\ \alpha\in\mathbb{R}^{(1,2)},\ Y\in\mathcal{SP}_2$ and $V\in\mathbb{R}^{(1,2)}$. It is easy to see that K is a maximal compact subgroup of G stabilizing the origin $(E_2,0)$. Thus $\mathcal{SP}_n\times\mathbb{R}^{(m,n)}$ may be identified with the homogeneous space G/K as follows:

$$G/K \ni (g,\alpha)K \longmapsto (g,\alpha)\cdot(E_2,0)\in \mathcal{SP}_2\times\mathbb{R}^{(1,2)}, \tag{2.2}$$

where $g\in SL(2,\mathbb{R})$ and $\alpha\in\mathbb{R}^{(1,2)}$.

We know that $SL(2,\mathbb{R})$ acts on $\mathbb{H}$ transitively by

$$g<\tau>=(a\tau+b)(c\tau+d)^{-1},\quad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{R}),\quad \tau\in\mathbb{H}.$$

Now we observe that the action (1.2) of G on $\mathbb{H}\times\mathbb{C}$ may be rewritten as

$$(g,\alpha)\circ(\tau,z) = (\,{}^tg^{-1}<\tau>,\ (z+\alpha_1\tau+\alpha_2)(-b\tau+a)^{-1}\,),$$

where $g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{R}),\ \alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^{(1,2)}$, and $(\tau,z)\in\mathbb{H}\times\mathbb{C}$. Since the action (1.2) is transitive and K is the stabilizer of this action at the origin $(i,0)$, $\mathbb{H}\times\mathbb{C}$ can be identified with the homogeneous space G/K as follows:

$$G/K\ni(g,\alpha)K\longmapsto(g,\alpha)\circ(i,0). \tag{2.3}$$

We see that we can express an element Y of $\mathcal{SP}_2$ uniquely as

$$Y=\begin{pmatrix} y^{-1} & 0\\ 0 & y\end{pmatrix}\left[\begin{pmatrix} 1 & -x\\ 0 & 1\end{pmatrix}\right]=\begin{pmatrix} y^{-1} & -xy^{-1}\\ -xy^{-1} & x^2y^{-1}+y\end{pmatrix} \tag{2.4}$$

with $x,y\in\mathbb{R}$ and $y>0$.

Lemma 2.1. *We define the mapping* $T:\mathcal{SP}_2\times\mathbb{R}^{(1,2)}\longrightarrow\mathbb{H}\times\mathbb{C}$ *by*

$$T(Y,V) = (x+iy,\ v_1(x+iy)+v_2\,), \tag{2.5}$$

where Y is of the form (2.4) and $V=(v_1,v_2)\in\mathbb{R}^{(1,2)}$. Then the mapping T is a bijection which is compatible with the above two actions (1.2) and (2.1).

For any $Y\in\mathcal{SP}_2$ of the form (2.4), we put

$$g_Y=\begin{pmatrix} 1 & 0\\ -x & 1\end{pmatrix}\begin{pmatrix} y^{-1/2} & 0\\ 0 & y^{1/2}\end{pmatrix}=\begin{pmatrix} y^{-1/2} & 0\\ -xy^{-1/2} & y^{1/2}\end{pmatrix}. \tag{2.6}$$

and

$$\alpha_{Y,V} = V\, {}^t g_Y^{-1}. \tag{2.7}$$

Then we have

$$T(Y,V) = (\, g_Y,\, \alpha_{Y,V}\,) \circ (i,0). \tag{2.8}$$

Proof. It is easy to prove the lemma. So we leave the proof to the reader. □

Now we give a complete description of the algebra $\mathbb{D}(\mathbb{H}\times\mathbb{C})$ of all differential operators on $\mathbb{H}\times\mathbb{C}$ invariant under the action (1.2) of G. First we note that the Lie algebra $\mathfrak{g}$ of G is given by $\mathfrak{g} = \left\{ (X,Z) \mid X \in \mathbb{R}^{(2,2)},\ \sigma(X)=0,\ Z\in\mathbb{R}^{(1,2)} \right\}$ equipped with the following Lie bracket

$$[(X_1,Z_1),\,(X_2,Z_2)] = (\,[X_1,X_2]_0,\, Z_2\,{}^tX_1 - Z_1\,{}^tX_2\,),$$

where $[X_1,X_2]_0 = X_1X_2 - X_2X_1$ denotes the usual matrix bracket and (X_1,Z_1), $(X_2, Z_2)\in\mathfrak{g}$. And $\mathfrak{g}$ has the following decomposition

$$\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p} \qquad (\,\text{direct sum}\,),$$

where $\mathfrak{k} = \left\{ (X,0)\in\mathfrak{g} \mid X = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix},\ \ x\in\mathbb{R} \right\}$ and $\mathfrak{p} = \left\{ (X,Z)\in\mathfrak{g} \mid X = {}^tX\in\mathbb{R}^{(2,2)},\ \sigma(X)=0,\ Z\in\mathbb{R}^{(1,2)} \right\}$.

We observe that $\mathfrak{k}$ is the Lie algebra of K and that we have the following relations

$$[\mathfrak{k},\,\mathfrak{k}]\subset\mathfrak{k} \qquad \text{and} \qquad [\mathfrak{k},\,\mathfrak{p}]\subset\mathfrak{p}.$$

Thus the coset space $G/K\cong\mathbb{H}\times\mathbb{C}$ is a *reductive* homogeneous space in the sense of [12], p. 284. It is easy to see that the adjoint action Ad of K on $\mathfrak{p}$ is given by

$$\mathrm{Ad}\,(k)((X,Z)) \;=\; (\,kX\,{}^tk,\ Z\,{}^tk\,), \tag{2.9}$$

where $k\in K$ and $(X,Z)\in\mathfrak{p}$ with $X={}^tX$, $\sigma(X)=0$. The action (2.9) extends uniquely to the action ρ of K on the polynomial algebra $\mathrm{Pol}\,(\mathfrak{p})$ of $\mathfrak{p}$ given by

$$\rho\,:\,K \longrightarrow \mathrm{Aut}\,(\,\mathrm{Pol}(\mathfrak{p}\,)). \tag{2.10}$$

Let $\mathrm{Pol}\,(\mathfrak{p})^K$ be the subalgebra of $\mathrm{Pol}\,(\mathfrak{p})$ consisting of all invariants of the action ρ of K. Then according to [12], Theorem 4.9, p. 287, there exists a canonical linear bijection $\lambda\,(\,P\longmapsto D_{\lambda(P)})$ of $\mathrm{Pol}\,(\mathfrak{p})^K$ onto $\mathbb{D}(\mathbb{H}\times\mathbb{C})$. Indeed, if $(\xi_k)\,(\,1\le k\le 4\,)$ is any basis of $\mathfrak{p}$ and $P\in\mathrm{Pol}\,(\mathfrak{p})^K$, then

$$\left(D_{\lambda(P)}f\right)(\tilde g\circ(i,0)) \;=\; \left[P\left(\frac{\partial}{\partial t_k}\right) f((\tilde g * \exp\,(\sum_{k=1}^4 t_k\xi_k))\circ(i,0)) \right]_{(t_k)=0}, \tag{2.11}$$

where $\tilde{g} \in G$ and $f \in C^{\infty}(\mathbb{H} \times \mathbb{C})$.

We put

$$e_1 = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, (0,0) \right), \quad e_2 = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, (0,0) \right)$$

and

$$f_1 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (1,0) \right), \quad f_2 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (0,1) \right).$$

Then e_1, e_2, f_1, f_2 form a basis of $\mathfrak{p}$. We write for coordinates (X, Z) by

$$X = \begin{pmatrix} x & y \\ y & -x \end{pmatrix} \quad \text{and} \quad Z = (z_1, z_2)$$

with real variables x, y, z_1 and z_2.

Lemma 2.2. *The following polynomials*

$$\begin{aligned} P(X,Z) &= \frac{1}{8}\,\sigma(X^2) = \frac{1}{4}\,(x^2+y^2), \\ \xi(X,Z) &= Z\,{}^tZ = z_1^2 + z_2^2, \\ P_1(X,Z) &= -\frac{1}{2}\,ZX\,{}^tZ = \frac{1}{2}\,(z_2^2 - z_1^2)x - z_1 z_2 y \quad \textit{and} \\ P_2(X,Z) &= \frac{1}{2}\,(z_2^2 - z_1^2)\,y + z_1 z_2\, x \end{aligned}$$

are algebraically independent generators of $Pol(\mathfrak{p})^K$.

Proof. We leave the proof of the above lemma to the reader. □

Now we are ready to compute the G-invariant differential operators D, Ψ, D_1 and D_2 corresponding to the K-invariants P, ξ, P_1 and P_2 respectively under the canonical linear bijection (2.11). For real variables $t = (t_1, t_2)$ and $s = (s_1, s_2)$, we have

$$\exp\left(t_1 e_1 + t_2 e_2 + s_1 f_1 + s_2 f_2 \right) = \left(\begin{pmatrix} a_1(t,s) & a_3(t,s) \\ a_3(t,s) a_2(t,s) & \end{pmatrix},\ (b_1(t,s),\, b_2(t,s)) \right),$$

where

$$\begin{aligned} a_1(t,s) &= 1 + t_1 + \frac{1}{2!}\,(t_1^2 + t_2^2) + \frac{1}{3!}\,t_1(t_1^2+t_2^2) + \frac{1}{4!}\,(t_1^2+t_2^2)^2 + \cdots \\ a_2(t,s) &= 1 - t_1 + \frac{1}{2!}\,(t_1^2 + t_2^2) - \frac{1}{3!}\,t_1(t_1^2+t_2^2) + \frac{1}{4!}\,(t_1^2+t_2^2)^2 - \cdots, \\ a_3(t,s) &= t_2 + \frac{1}{3!}\,t_2(t_1^2 + t_2^2) + \frac{1}{5!}\,t_2(t_1^2+t_2^2)^2 + \cdots, \\ b_1(t,s) &= s_1 - \frac{1}{2!}\,(s_1t_1 + s_2t_2) + \frac{1}{3!}\,s_1(t_1^2+t_2^2) - \frac{1}{4!}\,(s_1t_1+s_2t_2)(t_1^2+t_2^2) + \cdots, \\ b_2(t,s) &= s_2 - \frac{1}{2!}\,(s_1t_2 - s_2t_1) + \frac{1}{3!}\,s_2(t_1^2+t_2^2) - \frac{1}{4!}\,(s_1t_2-s_2t_1)(t_1^2+t_2^2) + \cdots. \end{aligned}$$

For brevity, we write $a_j,\, b_k$ for $a_j(t,s),\, b_k(t,s)$ $(\, j=1,2,3,\ k=1,2\,)$ respectively. We now fix an element $(g,\alpha)\in G$ and write

$$g\,=\,\begin{pmatrix} g_1 & g_{12} \\ g_{21} & g_2 \end{pmatrix}\in SL(2,\mathbb{R})\quad\text{and}\quad \alpha\,=\,(\alpha_1,\alpha_2)\in\mathbb{R}^{(1,2)}.$$

We put $(\,\tau(t,s),\, z(t,s)\,)=\,(\,(g,\alpha)*\exp\,(\,t_1e_1\,+\,t_2e_2\,+\,s_1f_1\,+\,s_2f_2\,))\circ(i,0)$ with $\tau(t,s)=\,x(t,s)\,+\,i\,y(t,s)$ and $z(t,s)=\,u(t,s)\,+\,i\,v(t,s)$.
Here $x(t,s)$, $y(t,s)$, $u(t,s)$ and $v(t,s)$ are real. By an easy calculation, we obtain

$$\begin{aligned} x(t,s)\,&=\,-(\,\tilde a\tilde c+\tilde b\tilde d\,)\,(\,\tilde a^2+\tilde b^2\,)^{-1},\\ y(t,s)\,&=\,(\,\tilde a^2+\tilde b^2\,)^{-1},\\ u(t,s)\,&=\,(\,\tilde a\,\tilde\alpha_2-\tilde b\tilde\alpha_1\,)\,(\,\tilde a^2+\tilde b^2\,)^{-1},\\ v(t,s)\,&=\,(\,\tilde a\,\tilde\alpha_1+\tilde b\,\tilde\alpha_2\,)\,(\,\tilde a^2+\tilde b^2\,)^{-1}, \end{aligned}$$

where $\tilde a=\,g_1a_1\,+\,g_{12}a_3,\ \ \tilde b=\,g_1a_3\,+\,g_{12}a_2,\ \tilde c=\,g_{21}a_1\,+\,g_2a_3,\ \ \tilde d=\,g_{21}a_3\,+\,g_2a_2,$ $\tilde\alpha_1=\,\alpha_1a_2\,-\,\alpha_2a_3\,+\,b_1,\ \tilde\alpha_2=\,-\alpha_1a_3\,+\,\alpha_2a_1\,+\,b_2.$
By an easy calculation, at $t=s=0$, we have

$$\begin{aligned} \frac{\partial x}{\partial t_1}\,&=\,4\,g_1\,g_{12}\,(\,g_1^2\,+\,g_{12}^2\,)^{-2},\\ \frac{\partial y}{\partial t_1}\,&=\,-2\,(\,g_1^2\,-\,g_{12}^2\,)\,(\,g_1^2\,+\,g_{12}^2\,)^{-2},\\ \frac{\partial u}{\partial t_1}\,&=\,4\,g_1\,g_{12}\,(\,g_1\,\alpha_1\,+\,g_{12}\,\alpha_2\,)\,(\,g_1^2\,+\,g_{12}^2\,)^{-2},\\ \frac{\partial v}{\partial t_1}\,&=\,-\,2\,(\,g_1\,\alpha_1\,+\,g_{12}\,\alpha_2\,)\,(\,g_1^2\,-\,g_{12}^2\,)\,(\,g_1^2\,+\,g_{12}^2\,)^{2},\\ \frac{\partial^2 x}{\partial t_1^2}\,&=\,-\,16\,g_1\,g_{12}\,(\,g_1^2\,-\,g_{12}^2\,)\,(\,g_1^2\,+\,g_{12}^2\,)^{-3},\\ \frac{\partial^2 y}{\partial t_1^2}\,&=\,8\,(\,g_1^2\,-\,g_{12}^2\,)^2\,(\,g_1^2\,+\,g_{12}^2\,)^{-3}\,-\,4\,(\,g_1^2\,+\,g_{12}^2\,)^{-1},\\ \frac{\partial^2 u}{\partial t_1^2}\,&=\,-\,16\,g_1\,g_{12}\,(\,g_1\,\alpha_1\,+\,g_{12}\,\alpha_2\,)\,(\,g_1^2\,-\,g_{12}^2\,)\,(\,g_1^2\,+\,g_{12}^2\,)^{-3},\\ \frac{\partial^2 v}{\partial t_1^2}\,&=\,4\,(\,g_1\,\alpha_1\,+\,g_{12}\,\alpha_2\,)\,(\,g_1^4\,+\,g_{12}^4\,-\,6\,g_1^2\,g_{12}^2\,)\,(\,g_1^2\,+\,g_{12}^2\,)^{-3} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial x}{\partial t_2}\,&=\,-\,2\,(\,g_1^2\,-\,g_{12}^2\,)\,(\,g_1^2\,+\,g_{12}^2\,)^{-2},\\ \frac{\partial y}{\partial t_2}\,&=\,-\,4\,g_1\,g_{12}\,(\,g_1^2\,+\,g_{12}^2\,)^{-2},\\ \frac{\partial u}{\partial t_2}\,&=\,-\,2\,(\,g_1\,\alpha_1\,+\,g_{12}\,\alpha_2\,)\,(\,g_1^2\,-\,g_{12}^2\,)\,(\,g_1^2\,+\,g_{12}^2\,)^{-2}, \end{aligned}$$

$$\frac{\partial v}{\partial t_2} = -4\, g_1\, g_{12}\, (\, g_1\, \alpha_1 \,+\, g_{12}\, \alpha_2\,)\, (\, g_1^2 \,+\, g_{12}^2\,)^{-2},$$
$$\frac{\partial^2 x}{\partial t_2^2} = 16\, g_1\, g_{12}\, (\, g_1^2 \,-\, g_{12}^2\,)\, (\, g_1^2 \,+\, g_{12}^2\,)^{-3},$$
$$\frac{\partial^2 y}{\partial t_2^2} = 32\, g_1^2\, g_{12}^2\, (\, g_1^2 \,+\, g_{12}^2\,)^{-3} \,-\, 4\, (\, g_1^2 \,+\, g_{12}^2\,)^{-1},$$
$$\frac{\partial^2 u}{\partial t_2^2} = 16\, g_1\, g_{12}\, (\, g_1\, \alpha_1 \,+\, g_{12}\, \alpha_2\,)\, (\, g_1^2 \,-\, g_{12}^2\,)\, (\, g_1^2 \,+\, g_{12}^2\,)^{-3},$$
$$\frac{\partial^2 v}{\partial t_2^2} = -4\, (\, g_1\, \alpha_1 \,+\, g_{12}\, \alpha_2\,)\, (\, g_1^4 \,+\, g_2^4 \,-\, 6\, g_1\, g_{12}^2\,)\, (\, g_1^2 \,+\, g_{12}^2\,)^{-3}.$$

We note that $\tilde{a}\tilde{d} - \tilde{b}\tilde{c} = 1$, $a_1 a_2 - a_3^2 = 1$ and $g_1 g_2 - g_{12} g_{21} = 1$.

Using the above facts and applying the chain rule, we can easily compute the differential operators D, Ψ, D_1 and D_2. It is known that the images of generators P, ξ, P_1 and P_2 under λ are generators of $\mathbb{D}(\mathbb{H} \times \mathbb{C})$ (cf. [11]).

Summarizing, we have the following.

Theorem 2.3. *The algebra $\mathbb{D}(\mathbb{H} \times \mathbb{C})$ is generated by the following differential operators*

(2.12)
$$D = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + 2\, y\, v \left(\frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v} \right),$$

(2.13)
$$\Psi = y \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right),$$

(2.14)
$$D_1 = 2\, y^2 \frac{\partial^3}{\partial x \partial u \partial v} - y^2 \frac{\partial}{\partial y} \left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) + \left(v \frac{\partial}{\partial v} + 1 \right) \Psi$$

and

(2.15)
$$D_2 = y^2 \frac{\partial}{\partial x} \left(\frac{\partial^2}{\partial v^2} - \frac{\partial^2}{\partial u^2} \right) - 2\, y^2 \frac{\partial^3}{\partial y \partial u \partial v} - v \frac{\partial}{\partial u} \Psi,$$

where $\tau = x + iy$ and $z = u + iv$ with real variables x, y, u, v. Moreover, we have

$$[D,\, \Psi] = D\Psi - \Psi D = 2\, y^2 \frac{\partial}{\partial y} \left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) - 4\, y^2 \frac{\partial^3}{\partial x \partial u \partial v} - 2 \left(v \frac{\partial}{\partial v} \Psi + \Psi \right).$$

In particular, the algebra $\mathbb{D}(\mathbb{H}\times\mathbb{C})$ *is not commutative. Thus the homogeneous space* $\mathbb{H}\times\mathbb{C}$ *is not weakly symmetric in the sense of A. Selberg* ([19]).

Now we provide a natural G-invariant Kähler metric on $\mathbb{H}\times\mathbb{C}$.

Proposition 2.4. *The Riemannian metric* ds^2 *on* $\mathbb{H}\times\mathbb{C}$ *defined by*

$$ds^2 = \frac{y+v^2}{y^3}\,(\,dx^2+dy^2\,) + \frac{1}{y}\,(\,du^2+dv^2\,) - \frac{2v}{y^2}\,(\,dx\,du + dy\,dv\,)$$

is invariant under the action (1.2) *of* G *and is a Kähler metric on* $\mathbb{H}\times\mathbb{C}$. *The Laplace-Beltrami operator* Δ *of the Riemannian space* $(\,\mathbb{H}\times\mathbb{C},\, ds^2\,)$ *is given by*

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + (\,y+v^2\,)\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right) + 2\,y\,v\left(\frac{\partial^2}{\partial x\partial u}+\frac{\partial^2}{\partial y\partial v}\right).$$

That is, $\Delta = D+\Psi$.

Proof. For $Y\in S\mathcal{P}_2$ of the form (2.4) and $(v_1,v_2)\in\mathbb{R}^{(1,2)}$, it is easy to see that

$$dY = \begin{pmatrix} -\,y^{-2}\,dy & -\,y^{-1}\,dx + x\,y^{-2}\,dy \\ -\,y^{-1}\,dx + x\,y^{-2}\,dy & 2\,x\,y^{-1}\,dx + (\,1 - x^2\,y^{-2}\,)\,dy \end{pmatrix}$$

and $dV=(dv_1,dv_2)$. Then we can show that the following metric $d\tilde{s}^2$ on $S\mathcal{P}_2\times\mathbb{R}^{(1,2)}$ defined by

$$d\tilde{s}^2 = \frac{dx^2+dy^2}{y^2} + \frac{1}{y}\,\{\,(\,x^2+y^2\,)\,dv_1^2 + 2\,x\,dv_1\,dv_2 + dv_2^2\,\}$$

is invariant under the action (2.1) of G. Indeed, since

$$Y^{-1} = \begin{pmatrix} y + x^2\,y^{-1} & x\,y^{-1} \\ x\,y^{-1} & y^{-1}\end{pmatrix},$$

we can easily show that $d\tilde{s}^2 = \frac{1}{2}\,\sigma(Y^{-1}dYY^{-1}dY) + dV\,Y^{-1}\,{}^t(dV)$.
For an element $(\,g,\,\alpha\,)\in G$ with $g\in SL(2,\mathbb{R})$ and $\alpha\in\mathbb{R}^{(1,2)}$, we put

$$(\,Y^*,\,V^*\,) = (\,g,\,\alpha\,)\cdot(Y,V) = (\,gY\,{}^tg,\,(V+\alpha)\,{}^tg\,).$$

Since $Y^* = gY\,{}^tg$ and $V^* = (\,V+\alpha\,)\,{}^tg$, we get $dY^* = g\,dY\,{}^tg$ and $V^* = (\,V+\alpha\,)\,{}^tg$.
Therefore by a simple calculation, we can show that

$$\begin{aligned}&\sigma\left(\,Y^{*-1}\,dY^*\,Y^{*-1}\,dY^*\,\right) + dV^*\,Y^{*-1}\,{}^t(dV^*)\\ &= \sigma(\,Y^{-1}dY\,Y^{-1}\,dY\,) + dV\,Y^{-1}\,{}^t(dV).\end{aligned}$$

Hence the metric $d\tilde{s}^2$ is invariant under the action (2.1) of G.

Using this fact and Lemma 2.1, we can prove that the metric ds^2 in the above theorem is invariant under the action (1.2). Since the matrix form (g_{ij}) of the metric ds^2 is given by

$$(g_{ij}) = \begin{pmatrix} (y+v^2)y^{-3} & 0 & -v\,y^{-2} & 0 \\ 0 & (y+v^2)y^{-3} & 0 & -v\,y^{-2} \\ -v\,y^{-2} & 0 & y^{-1} & 0 \\ 0 & -v\,y^{-2} & 0 & y^{-1} \end{pmatrix}$$

and $\det(g_{ij}) = y^{-6}$, the inverse matrix (g^{ij}) of (g_{ij}) is easily obtained by

$$(g^{ij}) = \begin{pmatrix} y^2 & 0 & y\,v & 0 \\ 0 & y^2 & 0 & y\,v \\ y\,v & 0 & y+v^2 & 0 \\ 0 & y\,v & 0 & y+v^2 \end{pmatrix}.$$

Now it is easily shown that $D+\Psi$ is the Laplace-Beltrami operator of $(\mathbb{H}\times\mathbb{C},\, ds^2)$.
□

Remark 2.5. We can show that for any two positive real numbers α and β, the following metric

$$ds^2_{\alpha,\beta} = \alpha\,\frac{dx^2+dy^2}{y^2} + \beta\,\frac{v^2(dx^2+dy^2)+y^2(du^2+dv^2)-2\,yv\,(dx\,du+dy\,dv)}{y^3}$$

is also a Riemannian metric on $\mathbb{H}\times\mathbb{C}$ which is invariant under the action (1.2) of G. In fact, we can see that the two-parameter family of $ds^2_{\alpha,\beta}\,(\alpha>0,\ \beta>0)$ provides a complete family of Riemannian metrics on $\mathbb{H}\times\mathbb{C}$ invariant under the action of (1.2) of G. It can be easily seen that the Laplace-Beltrami operator $\Delta_{\alpha,\beta}$ of $ds^2_{\alpha,\beta}$ is given by

$$\begin{aligned}\Delta_{\alpha,\beta} &= \frac{1}{\alpha}\,y^2\,\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big) + \left(\frac{y}{\beta}+\frac{v^2}{\alpha}\right)\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right) \\ &\quad + \frac{2\,yv}{\alpha}\left(\frac{\partial^2}{\partial x\partial u}+\frac{\partial^2}{\partial y\partial v}\right) \\ &= \frac{1}{\alpha}\,D + \frac{1}{\beta}\,\Psi.\end{aligned}$$

Remark 2.6. By a tedious computation, we see that the scalar curvature of $(\mathbb{H}\times\mathbb{C},\, ds^2)$ is -3.

We want to propose the following problem to be studied in the future.

Problem 2.7. Find all the eigenfunctions of Δ.

We will give some examples of eigenfunctions of Δ.

(1) $h(x,y) = y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|a|y)\, e^{2\pi i a x} \quad (s \in \mathbb{C},\ a \neq 0\,)$ with eigenvalue $s(s-1)$, where

$$K_s(z) := \frac{1}{2}\int_0^\infty \exp\left\{-\frac{z}{2}(t+t^{-1})\right\} t^{s-1}\,dt, \quad \mathrm{Re}\, z > 0. \tag{2.16}$$

(2) $y^s,\ y^s x,\ y^s u$ $(s \in \mathbb{C})$ with eigenvalue $s(s-1)$.

(3) $y^s v,\ y^s uv,\ y^s xv$ with eigenvalue $s(s+1)$.

(4) $x,\ y,\ u,\ v,\ xv,\ uv$ with eigenvalue 0.

(5) All Maass wave forms.

3. Maass-Jacobi forms

Let Δ be the Laplace-Beltrami operator of the Riemannian metric ds^2 on $\mathbb{H}\times\mathbb{C}$ defined in Proposition 2.4. Using this operator, we define the notion of Maass-Jacobi forms.

Definition 3.1. A smooth bounded function $f : \mathbb{H}\times\mathbb{C} \longrightarrow \mathbb{C}$ is called a *Maass-Jacobi form* if it satisfies the following conditions (MJ1)-(MJ3):

(MJ1) $f(\tilde{\gamma}\circ(\tau,z)) = f(\tau,z)$ for all $\tilde{\gamma}\in\Gamma$ and $(\tau,z)\in\mathbb{H}\times\mathbb{C}$.

(MJ2) f is an eigenfunction of the Laplace-Beltrami operator Δ.

(MJ3) f has a polynomial growth, that is, f fulfills a boundedness condition.

For a complex number $\lambda\in\mathbb{C}$, we denote by $MJ(\Gamma,\lambda)$ the vector space of all Maass-Jacobi forms f such that $\Delta f = \lambda f$. We note that, since $\Delta f = \lambda f$ is an elliptic partial differential equation, Maass-Jacobi forms are real analytic (see [8]). Professor Berndt kindly informed me that he also considered such automorphic forms in ([1]) (also see [4], p.82).

Let $f \in MJ(\Gamma,\lambda)$ be a Maass-Jacobi form with eigenvalue λ. Then it is easy to see that the function $\phi_f : G \longrightarrow \mathbb{C}$ defined by

$$\phi_f(g,\alpha) = f((g,\alpha)\circ(i,0)), \quad (g,\alpha)\in G \tag{3.1}$$

satisfies the following conditions $(\mathrm{MJ1})^0$-$(\mathrm{MJ3})^0$:

$(\mathrm{MJ1})^0$ $\phi_f(\gamma x k) = \phi_f(x)$ for all $\gamma\in\Gamma,\ x\in G$ and $k\in K$.

$(\mathrm{MJ2})^0$ ϕ_f is an eigenfunction of the Laplace-Beltrami operator Δ_0 of $(G,\ ds_0^2)$, where ds_0^2 is a G-invariant Riemannian metric on G induced by $(\mathbb{H}\times\mathbb{C},\ ds^2)$.

$(\mathrm{MJ3})^0$ ϕ_f has a suitable polynomial growth (cf. [5]).

For any right K-invariant function $\phi : G \longrightarrow \mathbb{C}$ on G, we define the function $f_\phi : \mathbb{H} \times \mathbb{C} \longrightarrow \mathbb{C}$ by

$$f_\phi(\tau, z) = \phi(g, \alpha), \quad (\tau, z) \in \mathbb{H} \times \mathbb{C}, \tag{3.2}$$

where (g, α) is an element of G such that $(g, \alpha) \circ (i, 0) = (\tau, z)$. Obviously it is well defined because (3.2) is independent of the choice of $(g, \alpha) \in G$ such that $(g, \alpha) \circ (i, 0) = (\tau, z)$. It is easy to see that if ϕ is a smooth bounded function on G satisfying the conditions (MJ1)^0-(MJ3)^0, then the function f_ϕ defined by (3.2) is a Maass-Jacobi form.

Now we characterize Maass-Jacobi forms as smooth eigenfunctions on $S\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ satisfying a certain invariance property.

Proposition 3.2. *Let $f : \mathbb{H} \times \mathbb{C} \longrightarrow \mathbb{C}$ be a nonzero Maass-Jacobi form in $MJ(\Gamma, \lambda)$. Then the function $h_f : S\mathcal{P}_2 \times \mathbb{R}^{(1,2)} \longrightarrow \mathbb{C}$ defined by*

$$h_f(Y, V) = f((g,\, V\,{}^t g^{-1}) \circ (i, 0)) \quad \text{for some } g \in SL(2, \mathbb{R}) \text{ with } Y = g\,{}^t g \tag{3.3}$$

satisfies the following conditions:

(MJ1)* $h_f(\gamma Y\,{}^t\gamma,\, (V+\delta)\,{}^t\gamma) = h_f(Y, V)$ *for all* $(\gamma, \delta) \in \Gamma$ *with* $\gamma \in SL(2, \mathbb{Z})$ *and* $\delta \in \mathbb{Z}^{(1,2)}$.

(MJ2)* *h_f is an eigenfunction of the Laplace-Beltrami operator $\tilde{\Delta}$ on the homogeneous space $(S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}, d\tilde{s}^2)$, where $d\tilde{s}^2$ is the G-invariant Riemannian metric on $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ induced from $d\tilde{s}^2$.*

(MJ3)* *h_f has a suitable polynomial growth.*

Here if (Y, V) is a coordinate of $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ given in Lemma 2.1, then the G-invariant Riemannian metric $d\tilde{s}^2$ and its Laplace-Beltrami operator $\tilde{\Delta}$ on $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ are given by

$$d\tilde{s}^2 = \frac{1}{y^2}(dx^2 + dy^2) + \frac{1}{y}\left\{(x^2 + y^2)dv_1^2 + 2x dv_1 dv_2 + dv_2^2\right\}$$

and

$$\tilde{\Delta} = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{y}\left\{\frac{\partial^2}{\partial v_1^2} - 2x\frac{\partial^2}{\partial v_1 \partial v_2} + (x^2 + y^2)\frac{\partial^2}{\partial v_2^2}\right\}.$$

Conversely, if h is a smooth bounded function on $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ satisfying the above conditions (MJ1)*-(MJ3)*, *then the function $f_h : \mathbb{H} \times \mathbb{C} \longrightarrow \mathbb{C}$ defined by*

$$f_h(\tau, z) = h(g\,{}^t g,\, \alpha\,{}^t g) \tag{3.4}$$

for some $(g, \alpha) \in G$ with $(g, \alpha) \circ (i, 0) = (\tau, z)$ is a Maass-Jacobi form on $\mathbb{H} \times \mathbb{C}$.

Proof. First of all, we note that h_f is well defined because (3.3) is independent

of the choice of g with $Y = g\,{}^tg$. If $(\gamma,\delta) \in \Gamma$ with $\gamma \in \Gamma_1,\ \delta \in \mathbb{Z}^{(1,2)}$ and $(Y,V) \in \mathcal{SP}_2 \times \mathbb{R}^{(1,2)}$ with $Y = g\,{}^tg$ for some $g \in SL(2,\mathbb{R})$, then the element $g_\gamma := \gamma g$ satisfies $\gamma Y\,{}^t\gamma = \gamma g\,{}^t(\gamma g)$.
Thus according to the definition of h_f, for all $(\gamma,\delta) \in \Gamma$ and $(Y,V) \in \mathcal{SP}_n \times \mathbb{R}^{(m,n)}$, we have

$$\begin{aligned} h_f(\gamma Y\,{}^t\gamma,\,(V+\delta)\,{}^t\gamma) &= f((\gamma g,\,(V+\delta)\,{}^t\gamma\,{}^t(\gamma g)^{-1}) \circ (i,0)) \\ &= f((\gamma g,\,(V+\delta)\,{}^tg^{-1}) \circ (i,0)) \\ &= f(((\gamma,\delta) * (g,\,V\,{}^tg^{-1})) \circ (i,0)) \\ &= f((g,\,V\,{}^tg^{-1}) \circ (i,0)) \quad (\text{because } f \text{ is } \Gamma-\text{invariant}) \\ &= h_f(Y,V). \end{aligned}$$

Therefore this proves the condition (MJ1)*. $d\tilde{s}^2$ and $\tilde{\Delta}$ are obtained from Lemma 2.1 and Proposition 2.3. Hence h_f is an eigenfunction of $\tilde{\Delta}$. Clearly h_f satisfies the condition (MJ3)*.

Conversely we note that f_h is well defined because (3.4) is independent of the choice of $(g,\alpha) \in G$ with $(g,\alpha) \circ (i,0) = (\tau,z)$. If $\tilde{\gamma} = (\gamma,\delta) \in \Gamma$ and $(\tau,z) \in \mathbb{H} \times \mathbb{C}$ with $(g,\alpha) \circ (i,0) = (\tau,z)$, then we have

$$\begin{aligned} f_h(\tilde{\gamma} \circ (\tau,z)) &= f_h(\tilde{\gamma} \circ ((g,\alpha) \circ (i,0))) \\ &= f_h((\tilde{\gamma} * (g,\alpha)) \circ (i,0)) \\ &= f_h((\gamma g,\,\delta\,{}^tg^{-1} + \alpha) \circ (i,0)) \\ &= h((\gamma g)\,{}^t(\gamma g),\,(\delta\,{}^tg^{-1} + \alpha)\,{}^t(\gamma g)) \\ &= h((\gamma(g\,{}^tg)\,{}^t\gamma,\,(\delta + \alpha\,{}^tg)\,{}^t\gamma) \\ &= h(g\,{}^tg,\,\alpha\,{}^tg) \\ &= f_h((g,\alpha) \circ (i,0)) = f_h(\tau,z). \end{aligned}$$

Thus f_h satisfies the condition (MJ1). It is easy to see that f_h satisfies the conditions (MJ2) and (MJ3). □

Definition 3.3. A smooth bounded function on G or $\mathcal{SP}_2 \times \mathbb{R}^{(1,2)}$ is also called a *Maass-Jacobi form* if it satisfies the conditions (MJ1)^0-(MJ3)^0 or (MJ1)*-(MJ3)*.

Remark 3.4. We note that Maass wave forms are special ones of Maass-Jacobi forms. Thus the number of λ's with $MJ(\Gamma,\lambda) \neq 0$ is infinite.

Theorem 3.5. *For any complex number $\lambda \in \mathbb{C}$, the vector space $MJ(\Gamma,\lambda)$ is finite dimensional.*

Proof. The proof follows from [10], Theorem 1, p. 8 and [5], p. 191. □

4. On the group $SL_{2,1}(\mathbb{R})$

For brevity, we set $H = \mathbb{R}^{(1,2)}$. Then we have the split exact sequence

$$0 \longrightarrow H \longrightarrow G \longrightarrow SL(2,\mathbb{R}) \longrightarrow 1.$$

We see that the unitary dual $\hat{H}$ of H is isomorphic to $\mathbb{R}^2$. The unitary character $\chi_{(\lambda,\mu)}$ of H corresponding to $(\lambda,\mu) \in \mathbb{R}^2$ is given by

$$\chi_{(\lambda,\mu)}(x,y) = e^{2\pi i(\lambda x+\mu y)}, \qquad (x,y) \in H.$$

G acts on H by conjugation and hence this action induces the action of G on $\hat{H}$ as follows.

$$G \times \hat{H} \longrightarrow \hat{H}, \quad (g,\chi) \mapsto \chi^g, \quad g \in G,\ \chi \in \hat{H}, \tag{4.1}$$

where the character χ^g is defined by $\chi^g(a) = \chi(gag^{-1}),\ \ a \in H$.
If $g = (g_0, \alpha) \in G$ with $g_0 \in SL(2,\mathbb{R})$ and $\alpha\ inH$, it is easy to check that for each $(\lambda,\mu) \in \mathbb{R}^2$,

$$\chi^g_{(\lambda,\mu)} = \chi_{(\lambda,\mu)g_0}. \tag{4.2}$$

We see easily from (4.2) that the G-orbits in $\hat{H} \cong \mathbb{R}^2$ consist of two orbits Ω_0, Ω_1 given by

$$\Omega_0 = \{(0,0)\}, \quad \Omega_1 = \mathbb{R}^2 - \{(0,0)\}.$$

We observe that Ω_0 is the G-orbit of $(0,0)$ and Ω_1 is the G-orbit of any element $(\lambda,\mu) \neq 0$.

Now we choose the element $\delta = \chi_{(1,0)}$ of $\hat{H}$. That is, $\delta(x,y) = e^{2\pi i x}$ for all $(x,y) \in \mathbb{R}^2$. It is easy to check that the stabilizer of $\chi_{(0,0)}$ is G and the stabilizer G_δ of δ is given by

$$G_\delta = \left\{ \left(\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \alpha \right) \,\Big|\, c \in \mathbb{R},\ \alpha \in \mathbb{R}^{(1,2)} \right\}.$$

We see that H is regularly embedded. This means that for every G-orbit Ω in $\hat{H}$ and for every $\sigma \in \Omega$ with stabilizer G_σ of σ, the canonical bijection $G_\sigma \backslash G \longrightarrow \Omega$ is a homeomorphism.

According to G. Mackey ([18]), we obtain

Theorem 4.1. *The irreducible unitary representations of G are the following:*

(a) *The irreducible unitary representations π, where the restriction of π to H is trivial and the restriction of π to $SL(2,\mathbb{R})$ is an irreducible unitary representation of $SL(2,\mathbb{R})$. For the unitary dual of $SL(2,\mathbb{R})$, we refer to* [7] *or* [15], p. 123.

(b) *The representations* $\pi_{(r)} = Ind_{G_\delta}^G \sigma_r$ $(r \in \mathbb{R})$ *induced from the unitary character* σ_r *of* G_δ *defined by*

$$\sigma_r\left(\left(\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, (\lambda, \mu)\right)\right) = \delta(rc+\lambda) = e^{2\pi i(rc+\lambda)}, \quad c, \lambda, \mu \in \mathbb{R}.$$

Proof. The proof of the above theorem can be found in [22], p. 850. □

We put

$$W_1 = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, (0,0)\right), \quad W_2 = \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, (0,0)\right), \quad W_3 = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, (0,0)\right)$$

and

$$W_4 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (1,0)\right), \qquad W_5 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (0,1)\right).$$

Clearly $W_1, \cdots, W_5$ form a basis of $\mathfrak{g}$.

Lemma 4.2. *We have the following relations.*

$$[W_1, W_2] = W_3, \quad [W_3, W_1] = 2W_1, \quad [W_3, W_2] = -2W_2,$$

$$[W_1, W_4] = 0, \quad [W_1, W_5] = -W_4, \quad [W_2, W_4] = W_5, \quad [W_2, W_5] = 0,$$

$$[W_3, W_4] = W_4, \quad [W_3, W_5] = -W_5, \quad [W_4, W_5] = 0.$$

Proof. The proof follows from an easy computation. □

Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexfication of $\mathfrak{g}$. We put

$$\mathfrak{k}_{\mathbb{C}} = \mathbb{C}\,(W_1 - W_2), \qquad \mathfrak{p}_{\pm} = \mathbb{C}\,(W_3 \pm i(W_1 + W_2)).$$

Then we have

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_+ + \mathfrak{p}_-, \quad [\mathfrak{k}_{\mathbb{C}}, \mathfrak{p}_{\pm}] \subset \mathfrak{p}_{\pm}, \quad \mathfrak{p}_- = \overline{\mathfrak{p}_+}.$$

We note that $\mathfrak{k}_{\mathbb{C}}$ is the complexification of the Lie algebra $\mathfrak{k}$ of K.

We set $\mathfrak{a} = \mathbb{R}\,W_3$. By Lemma 4.2, the roots of $\mathfrak{g}$ relative to $\mathfrak{a}$ are given by $\pm e$, $\pm 2e$, where e is the linear functional $e : \mathfrak{a} \longrightarrow \mathbb{C}$ defined by $e(W_3) = 1$. The set $\Sigma^+ = \{e, 2e\}$ is the set of positive roots of $\mathfrak{g}$ relative to $\mathfrak{a}$. We recall that for a root α, the root space $\mathfrak{g}_\alpha$ is defined by

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}.$$

Then we see easily that

$$\mathfrak{g}_e = \mathbb{R}\,W_4, \quad \mathfrak{g}_{-e} = \mathbb{R}\,W_5, \quad \mathfrak{g}_{2e} = \mathbb{R}\,W_1, \quad \mathfrak{g}_{-2e} = \mathbb{R}\,W_2$$

and

$$\mathfrak{g} = \mathfrak{g}_{-2e} \oplus \mathfrak{g}_{-e} \oplus \mathfrak{a} \oplus \mathfrak{g}_e \oplus \mathfrak{g}_{2e}.$$

Proposition 4.3. *The Killing form B of $\mathfrak{g}$ is given by*

$$B((X_1, Z_1), (X_2, Z_2)) = 5\,\sigma(X_1 X_2), \tag{4.3}$$

where $(X_1, Z_1), (X_2, Z_2) \in \mathfrak{g}$ with $X_1, X_2 \in \mathfrak{sl}(2, \mathbb{R})$ and $Z_1, Z_2 \in \mathbb{R}^{(1,2)}$. Hence the Killing form is highly nondegenerate. The adjoint representation Ad of G is given by

$$Ad((g, \alpha))(X, Z) = (gXg^{-1},\, (Z - \alpha\, {}^tX)\, {}^tg), \tag{4.4}$$

where $(g, \alpha) \in G$ with $g \in SL(2, \mathbb{R})$, $\alpha \in \mathbb{R}^{(1,2)}$ and $(X, Z) \in \mathfrak{g}$ with $X \in \mathfrak{sl}(2, \mathbb{R})$, $Z \in \mathbb{R}^{(1,2)}$.

Proof. The proof follows immediately from a direct computation. □

An Iwasawa decomposition of the group G is given by

$$G = NAK, \tag{4.5}$$

where

$$N = \left\{ \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, a \right) \in G \,\middle|\, x \in \mathbb{R},\ a \in \mathbb{R}^{(1,2)} \right\}$$

and

$$A = \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, 0 \right) \in G \,\middle|\, a > 0 \right\}.$$

An Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ of G is given by

$$\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k},$$

where

$$\mathfrak{n} = \left\{ \left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}, Z \right) \in \mathfrak{g} \,\middle|\, x \in \mathbb{R},\ Z \in \mathbb{R}^{(1,2)} \right\}$$

and

$$\mathfrak{a} = \left\{ \left(\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}, 0 \right) \in \mathfrak{g} \,\middle|\, x \in \mathbb{R} \right\}.$$

In fact, $\mathfrak{a}$ is the Lie algebra of A and $\mathfrak{n}$ is the Lie algebra of N.

Now we compute the Lie derivatives for functions on G explicitly. We define the differential operators $L_k,\ R_k\ (1 \le k \le 5)$ on G by

$$L_k f(\tilde{g}) = \frac{d}{dt}\bigg|_{t=0} f(\tilde{g} * \exp tW_k)$$

and

$$R_k f(\tilde{g}) = \frac{d}{dt}\Big|_{t=0} f(\exp tW_k * \tilde{g}),$$

where $f \in C^\infty(G)$ and $\tilde{g} \in G$.

By an easy calculation, we get

$$\exp tW_1 = \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, (0,0) \right), \quad \exp tW_2 = \left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, (0,0) \right)$$

$$\exp tW_3 = \left(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, (0,0) \right), \quad \exp tW_4 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (t,0) \right)$$

and

$$\exp tW_5 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (0,t) \right).$$

Now we use the following coordinates (g, α) in G given by

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{4.6}$$

and

$$\alpha = (\alpha_1, \alpha_2), \tag{4.7}$$

where $x, \alpha_1, \alpha_2 \in \mathbb{R},\ y > 0$ and $0 \le \theta < 2\pi$. By an easy computation, we have

$$L_1 = y\cos 2\theta \frac{\partial}{\partial x} + y\sin 2\theta \frac{\partial}{\partial y} + \sin^2\theta \frac{\partial}{\partial\theta} - \alpha_2 \frac{\partial}{\partial\alpha_1},$$

$$L_2 = y\cos 2\theta \frac{\partial}{\partial x} + y\sin 2\theta \frac{\partial}{\partial y} - \cos^2\theta \frac{\partial}{\partial\theta} - \alpha_1 \frac{\partial}{\partial\alpha_2},$$

$$L_3 = -2y\sin 2\theta \frac{\partial}{\partial x} + 2y\cos 2\theta \frac{\partial}{\partial y} + \sin 2\theta \frac{\partial}{\partial\theta} - \alpha_1 \frac{\partial}{\partial\alpha_1} + \alpha_2 \frac{\partial}{\partial\alpha_2},$$

$$L_4 = \frac{\partial}{\partial\alpha_1},$$

$$L_5 = \frac{\partial}{\partial\alpha_2},$$

$$R_1 = \frac{\partial}{\partial x},$$

$$R_2 = (y^2 - x^2)\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y} - y\frac{\partial}{\partial\theta},$$

$$R_3 = 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y},$$

$$R_4 = y^{-1/2}\cos\theta \frac{\partial}{\partial\alpha_1} + y^{-1/2}\sin\theta \frac{\partial}{\partial\alpha_2},$$

$$R_5 = -y^{-1/2}(x\cos\theta + y\sin\theta)\frac{\partial}{\partial\alpha_1} + y^{-1/2}(y\cos\theta - x\sin\theta)\frac{\partial}{\partial\alpha_2}.$$

In fact, the calculation for L_3 and R_5 can be found in [22], p. 837-839.

We define the differential operators $\mathbb{L}_j\,(\,1 \leq j \leq 5\,)$ on $\mathbb{H} \times \mathbb{C}$ by

$$\mathbb{L}_j f(\tau, z) = \frac{d}{dt}\bigg|_{t=0} f(\exp tW_j \circ (\tau, z)), \quad 1 \leq j \leq 5,$$

where $f \in C^\infty(\mathbb{H} \times \mathbb{C})$. Using the coordinates $\tau = x + iy$ and $z = u + iv$ with x, y, u, v real and $y > 0$, we can easily compute the explicit formulas for $\mathbb{L}_j$'s. They are given by

$$\begin{aligned}
\mathbb{L}_1 &= (x^2 - y^2)\,\frac{\partial}{\partial x} + 2xy\,\frac{\partial}{\partial y} + (xu - yv)\,\frac{\partial}{\partial u} + (yu + xv)\,\frac{\partial}{\partial v},\\
\mathbb{L}_2 &= -\,\frac{\partial}{\partial x},\\
\mathbb{L}_3 &= -2x\,\frac{\partial}{\partial x} - 2y\,\frac{\partial}{\partial y} - u\,\frac{\partial}{\partial u} - v\,\frac{\partial}{\partial v},\\
\mathbb{L}_4 &= x\,\frac{\partial}{\partial u} + y\frac{\partial}{\partial v},\\
\mathbb{L}_5 &= \frac{\partial}{\partial u}.
\end{aligned}$$

5. The decomposition of $L^2(\Gamma\backslash G)$

Let R be the right regular representation of G on the Hilbert space $L^2(\Gamma\backslash G)$. We set $G_1 = SL(2,\mathbb{R})$. Then the decomposition of R is given by

$$L^2(\Gamma\backslash G) = L^2_{\text{disc}}(\Gamma_1\backslash G_1) \bigoplus L^2_{\text{cont}}(\Gamma_1\backslash G_1) \bigoplus \int_{-\infty}^{\infty} \mathcal{H}_{(r)}\, dr, \tag{5.1}$$

where $L^2_{\text{disc}}(\Gamma_1\backslash G_1)$ (resp. $L^2_{\text{cont}}(\Gamma_1\backslash G_1)$) is the discrete (resp. continuous) part of $L^2(\Gamma_1\backslash G_1)$ (cf. [14], [15]) and $\mathcal{H}_{(r)}$ is the representation space of $\pi_{(r)}$ (cf. Theorem 4.1. (b)).

We recall the result of Rolf Berndt (cf. [2], [3], [4]). Let $H_{\mathbb{R}}^{(1,1)}$ denote the Heisenberg group which is $\mathbb{R}^3$ as a set and is equipped with the following multiplication

$$(\lambda, \mu, \kappa)\,(\lambda', \mu', \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\mu' - \mu\lambda').$$

We let $G^J = SL(2,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(1,1)}$ be the semidirect product of $SL(2,\mathbb{R})$ and $H_{\mathbb{R}}^{(1,1)}$, called the Jacobi group whose multiplication law is given by

$$(M, (\lambda, \mu, \kappa)) \cdot (M', (\lambda', \mu', \kappa')) = \,(MM', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu', \kappa + \kappa' + \tilde{\lambda}\mu' - \tilde{\mu}\lambda'))$$

with $M, M' \in SL(2,\mathbb{R}), (\lambda,\mu,\kappa), (\lambda',\mu',\kappa') \in H_{\mathbb{R}}^{(1,1)}$ and $(\tilde{\lambda},\tilde{\mu}) = (\lambda,\mu)M'$. Obviously the center $Z(G^J)$ of G^J is given by $\{(0,0,\kappa)\,|\,\kappa\in\mathbb{R}\}$. We denote

$$H_{\mathbb{Z}}^{(1,1)} = \{\,(\lambda,\mu,\kappa)\in H_{\mathbb{R}}^{(1,1)}\mid \lambda,\mu,\kappa \text{ integral}\,\}.$$

We set

$$\Gamma^J = SL(2,\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(1,1)}, \quad K^J = K \times Z(G^J).$$

R. Berndt proved that the decomposition of the right regular representation R^J of G^J in $L^2(\Gamma^J\backslash G^J)$ is given by

$$L^2(\Gamma^J\backslash G^J) = \left(\bigoplus_{m,n\in\mathbb{Z}} \mathcal{H}_{m,n}\right) \bigoplus \left(\bigoplus_{\nu=\pm\frac{1}{2}} \int_{\substack{\mathrm{Re}\, s=0\\ \mathrm{Im}\, s>0}} \mathcal{H}_{m,s,\nu}\, ds\right), \tag{5.2}$$

where the $\mathcal{H}_{m,n}$ is the irreducible unitary representation isomorphic to the discrete series $\pi_{m,k}^{\pm}$ or the principal series $\pi_{m,s,\nu}$, and the $\mathcal{H}_{m,s,\nu}$ is the representation space of $\pi_{m,s,\nu}$ (cf. [4], p. 47-48). For more detail on the decomposition of $L^2(\Gamma^J\backslash G^J)$, we refer to [4], p. 75-103.

Since $\mathbb{H}\times\mathbb{C} = K^J\backslash G^J = K\backslash G$, the space of the Hilbert space $L^2(\Gamma\backslash(\mathbb{H}\times\mathbb{C}))$ consists of K^J-fixed elements in $L^2(\Gamma^J\backslash G^J)$ or K-fixed elements in $L^2(\Gamma\backslash G)$. Hence we obtain the spectral decomposition of $L^2(\Gamma\backslash(\mathbb{H}\times\mathbb{C}))$ for the Laplacian Δ or $\Delta_{\alpha,\beta}$ (cf. Proposition 2.4 or Remark 2.5).

6. Remarks on Fourier expansions of Maass-Jacobi forms

We let $f:\mathbb{H}\times\mathbb{C}\longrightarrow\mathbb{C}$ be a Maass-Jacobi form with $\Delta f = \lambda f$. Then f satisfies the following invariance relations

$$f(\tau+n,\,z) = f(\tau,z) \quad \text{for all } n\in\mathbb{Z} \tag{6.1}$$

and

$$f(\tau,\,z+n_1\tau+n_2) = f(\tau,z) \quad \text{for all } n_1,\,n_2\in\mathbb{Z}. \tag{6.2}$$

Therefore f is a smooth function on $\mathbb{H}\times\mathbb{C}$ which is periodic in x and u with period 1. So f has the following Fourier series

$$f(\tau,z) = \sum_{n\in\mathbb{Z}}\sum_{r\in\mathbb{Z}} c_{n,r}(y,v)\, e^{2\pi i(nx+ru)}. \tag{6.3}$$

For two fixed integers n and r, we have to calculate the function $c_{n,r}(y,v)$. For brevity, we put $F(y,v) = c_{n,r}(y,v)$. Then F satisfies the following differential equation

$$\left[y^2\frac{\partial^2}{\partial y^2} + (y+v^2)\frac{\partial^2}{\partial v^2} + 2yv\frac{\partial^2}{\partial y\partial v} - \left\{(ay+bv)^2 + b^2y + \lambda\right\}\right] F = 0. \tag{6.4}$$

Here $a = 2\pi n$ and $b = 2\pi r$ are constant. We note that the function $u(y) = y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|y)$ satisfies the differential equation (6.4) with $\lambda = s(s-1)$. Here $K_s(z)$ is the K-Bessel function defined by (2.16) (see Lebedev [16] or Watson [21]). The problem is that if there exist solutions of the differential equation (6.4), we have to find their solutions explicitly.

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References

[1] R. Berndt, *Some Differential Operators in the Theory of Jacobi Forms*, IHES/M/84/10.

[2] R. Berndt, *The Continuous Part of* $L^2(\Gamma^J\backslash G^J)$ *for the Jacobi Group* G^J, Abh. Math. Sem. Univ. Hamburg, **60**(1990), 225-248.

[3] R. Berndt and S. Böcherer, *Jacobi Forms and Discrete Series Representations of the Jacobi Group*, Math. Z., **204**(1990), 13-44.

[4] R. Berndt and R. Schmidt, Elements of the Representation Theory of the Jacobi Group, Birkhäuser, **163**(1998).

[5] A. Borel and H. Jacquet, *Automorphic forms and automorphic representations*, Proc. Symposia in Pure Math., **XXXIII(Part 1)**(1979), 189-202.

[6] D. Bump, Automorphic Forms and Representations, Cambridge University Press, (1997).

[7] R. W. Donley, *Irreducible Representations of* $SL(2,\mathbb{R})$, Proceedings of Symposia in Pure Mathematics on Representation Theory and Automorphic Forms, American Math. Soc., **61**(1997), 51-59.

[8] P. R. Garabedian, Partial Differential Equations, Wiley, New York, (1964).

[9] S. Gelbart, Automorphic forms on adele groups, Annals of Math. Studies, Princeton Univ. Press, **83**(1975).

[10] Harish-Chandra, Automorphic forms on semi-simple Lie groups, Notes by J.G.M. Mars, Lecture Notes in Math., Springer-Verlag, Berlin-Heidelberg-New York, **62**(1968).

[11] S. Helgason, *Differential operators on homogeneous spaces*, Acta Math., **102**(1959), 239-299.

[12] S. Helgason, Groups and geometric analysis, Academic Press, (1984).

[13] H. Iwaniec, Introduction to the spectral theory of automorphic forms, Biblioteca de la Revista Mathemática Iberoamericana, Madrid, (1995).

[14] T. Kubota, Elementary Theory of Eisenstein Series, John Wiley and Sons, New York, (1973).

[15] S. Lang, $SL_2(\mathbb{R})$, Springer-Verlag, (1985).

[16] N. N. Lebedev, Special Functions and their Applications, Dover, New York, (1972).

[17] H. Maass, *Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichlescher Reihen durch Funktionalgleichung*, Math. Ann., **121**(1949), 141-183.

[18] G. Mackey, *Unitary Representations of Group Extensions 1*, Acta Math., **99**(1958), 265-311.

[19] A. Selberg, *Hamonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc., **20**(1956), 47-87.

[20] A. Terras, Harmonic analysis on symmetric spaces and applications I, Springer-Verlag, (1985).

[21] G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, London, (1962).

[22] J.-H. Yang, *On the group* $SL(2,\mathbb{R}) \ltimes \mathbb{R}^{(m,2)}$, J. Korean Math. Soc., **40(5)**, 831-867.

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초청논문

마쓰-야코비 형식에 관한 연구

양재현

요 약. 마쓰 형식을 일반화하는 마쓰-야코비 형식의 개념을 소개하고 이 형식의 성질을 연구한다. 그리고 마쓰-야코비 형식의 연구와 관련된 중요한 문제들을 제시한다.

머리말

아래의 Lie 군

$$SL_{2,1}(\mathbb{R}) := SL(2,\mathbb{R}) \ltimes \mathbb{R}^{(1,2)}$$

는 특별선형군 $SL(2,\mathbb{R})$과 가환군 $\mathbb{R}^{(1,2)}$과의 반직접곱이다. 여기서 $\mathbb{R}^{(1,2)}$는 1×2 실행렬들의 집합을 나타내고 있다. $SL_{2,1}(\mathbb{R})$에서의 곱은

$$(g,\alpha)\cdot(h,\beta) = (gh,\ \alpha\,{}^t h^{-1} + \beta) \tag{0.1}$$

(단, $g,h \in SL(2,\mathbb{R})$, $(\alpha,\beta) \in \mathbb{R}^{(1,2)}$) 으로 주어진다. 그리고

$$SL_{2,1}(\mathbb{Z}) := SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^{(1,2)}$$

를 $SL_{2,1}(\mathbb{R})$의 이산 부분군이라 하자.

$\mathbb{H}$를 Poincaré 상반평면이라 하면 Lie 군 $SL_{2,1}(\mathbb{R})$은 등질공간 $\mathbb{H}\times\mathbb{C}$ 상에서

$$(g,\alpha)\circ(\tau,z) = ((d\tau-c)(-b\tau+a)^{-1}, (z+\alpha_1\tau+\alpha_2)(-b\tau+a)^{-1}) \tag{0.2}$$

와 같이 추이적으로 작용한다. 여기서 $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$, $\alpha = (\alpha_1,\alpha_2) \in \mathbb{R}^{(1,2)}$이고 $(\tau,z) \in \mathbb{H}\times\mathbb{C}$이다.

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본 논문은 마쓰 형식을 일반화하는 마쓰-야코비 형식의 개념을 소개하고 이 형식의 여러 성질을 연구한다. 또한 마쓰-야코비 형식의 연구와 관련된 중요한 문제들을 제시한다.

제 1 절에서는 마쓰 형식의 연구와 관련된 여러 결과들과 문제들을 살펴본다. 제 2 절에서는 (0.2)의 작용에 불변인 $\mathbb{H}\times\mathbb{C}$ 상의 미분작용소들을 구체적으로 계산한다. $\mathbb{H}\times\mathbb{C}$ 상의 불변 Riemann 계량을 구한 후 이 계량의 라플라스 작용소를 구체적으로 계산한다. 제 3 절에서는 $\mathbb{H}\times\mathbb{C}$ 상에서 마쓰 형식을 일반화하는 마쓰-야코비 형식의 개념을 정의하고 이 형식의 여러 성질을 연구한다. 제 4 절에서는 $\mathbb{H}\times\mathbb{C}$ 상의 라플라스 작용소의 고유함수를 사용하여 $\mathbb{H}\times\mathbb{C}$ 상에 Eisenstein 급수를 형식적으로 구성한 후 마쓰-야코비 형식의 구성법에 관하여 논한다. 제 5 절에서는 야코비 군의 표현에 관하여 간략하게 논한다. 제 6 절에서는 마쓰-야코비 형식의 연구와 관련된 문제들을 제시한다.

기호: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ 는 각각 정수환, 유리수체, 실수체, 복소수체 등을 나타낸다. $\mathbb{H}$ 는 Poincaré 상반평면을 나타낸다. $\mathbb{Z}^+$ 는 자연수들의 집합을 나타낸다. " $:=$ " 는 오른쪽의 내용이 왼쪽 수식의 정의이다라는 의미를 나타내는 기호이다. 가환환 F 에 대하여 $F^{(k,l)}$ 는 $k\times l$ F-행렬을 나타낸다. 임의의 행렬 $M\in F^{(k,l)}$ 과 $B\in F^{(k,k)}$ 에 대하여 ${}^tM$ 은 M 의 전치행렬을 나타내고 $B[M]:={}^tMBM$ 를 나타낸다. I_n은 $n\times n$ 단위행렬이다. 정방행렬 A 에 대하여 $\sigma(A)$ 는 A 의 대각합(trace)을 나타낸다.

제 1 절 마쓰 형식(Maass forms)

이 절에서는 지난 50여년 동안 마쓰형식에 관하여 얻어졌던 여러 결과들을 소개한다.

$\mathbb{H}=\{\tau\in\mathbb{C}\,|\,\mathrm{Im}\,\tau>0\}$ 를 Poincaré 상반 평면이라 하자. 그러면 특별 선형군 $SL(2,\mathbb{R})$은 $\mathbb{H}$상에서

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\langle\tau\rangle := (a\tau+b)(c\tau+d) \tag{1.1}$$

(단, $a,b,c,d\in\mathbb{R},\ ad-bc=1$) 와 같이 추이적으로(transitively)작용한다. $\mathbb{H}$ 상의 좌표를 $\tau=x+iy$ $(x\in\mathbb{R},\ y>0)$ 이라 놓으면 $\mathbb{H}$ 상의 계량

$$ds^2=\frac{1}{y^2}(dx^2+dy^2)=\frac{dz\,d\bar{z}}{y^2} \tag{1.2}$$

는 작용 (1.1)에 불변인 Kaehler 계량이다. 이것을 $\mathbb{H}$ 상의 Poincaré 계량이라고 한다. Poincaré 계량 ds^2의 라플라스 작용소 Δ는

$$\Delta=y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) \tag{1.3}$$

으로 주어지며 작용 (1.1)에 불변인 미분작용소이다.

정의 1.1. $\mathbb{H}$상의 복소함수 $f : \mathbb{H} \to \mathbb{C}$ 가 있어 아래의 성질 $(M1)$-$(M3)$을 만족할 때 f 를 **마쓰 형식**(Maass form 또는 Maass waveform)이라고 부른다.

$(M1)$ f는 Δ의 고유함수이다.

$(M2)$ 임의의 $\gamma \in SL(2,\mathbb{Z})$ 에 대하여 $f(\gamma\langle\tau\rangle) = f(\tau)$ 이다.

$(M3)$ f는 무한점에서 기껏해야 다항식적인 증가성을 지니고 있다. 다시 말하면, 적당한 상수 C와 상수 k가 존재하여, 변수 x에 균등하게 $y \to \infty$일 때 $|f(x+iy)| \leq C\, y^k$ 이다.

$(M1)$에 의하여 $\Delta f = \lambda f$ (단, λ 는 상수)는 타원 편미분 방정식이므로 마쓰 형식 f는 실해석적(real analytic)이다. (참고문헌 : [11]) 기호편의상

$$\Gamma = SL(2,\mathbb{Z}), \quad \Gamma_\infty = \left\{ \begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix} \in \Gamma \right\}$$

이라 표기한다. 주어진 복소수 $s \in \mathbb{C}$ 에대하여 Eisenstein 급수 $E_s(\tau)$를 형식적으로(formally)

$$E_s(\tau) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\mathrm{Im}\, \gamma\langle\tau\rangle)^s, \quad \tau \in \mathbb{H} \tag{1.4}$$

이라 정의한다. 여기서 $\Gamma_\infty = \{\gamma \in \Gamma \mid \gamma\langle\infty\rangle = \infty\}$ 임을 유의하고 임의의 $\gamma \in \Gamma_\infty$에 대하여 $\mathrm{Im}\,\gamma\langle\tau\rangle = \mathrm{Im}\,\tau$임을 쉽게 알수 있다. 급수 (1.4)는 s의 실수 부분 $\mathrm{Re}\, s$가 1보다 클 때에만 수렴한다는 사실을 어렵지 않게 보일수 있다. 실제로 $\mathrm{Re}\, s > 1$ 이면 Eisenstein 급수 $E_s(\tau)$는 고유값이 $s(s-1)$ 인 마쓰 형식임을 알 수 있다. 구체적으로 기술하면

$$\Delta E_s(\tau) = s(s-1) E_s(\tau). \tag{1.5}$$

$$E_s(\gamma\langle\tau\rangle) = E_s(\tau), \quad \gamma \in \Gamma. \tag{1.6}$$

$$E_s(\tau) \sim y^s \quad y \to \infty. \tag{1.7}$$

또한 $E_s(\tau)$가 s의 함수로 간주되었을 때 $E_s(\tau)$는 전복소평면 $\mathbb{C}$상으로 해석적으로 접속이 가능하며, 이 해석접속 (analytic continuation) $E_s(\tau)$는 $s=1$에서 단순극(simple pole)을 갖는 유리형(meromorphic)함수이다. $s=1$에서의 $E_s(\tau)$의 유수(留數; residue)는 $3/\pi$이며 이 값은 기본영역 $\Gamma\backslash\mathbb{H}$의 부피와 같음이 알려져 있다. 그러므로 $E_s(\tau)$는 $\Gamma\backslash\mathbb{H}$의 기하학적인 성질과 밀접한 관계가 있음을 알 수 있다. $E_s(\tau)$는 $L^1(\Gamma\backslash\mathbb{H})$의 원소이지만 $L^2(\Gamma\backslash\mathbb{H})$의 원소가 아님을 유의하기 바란다.

이제부터는

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1} dt, \qquad z \in \mathbb{C}, \quad \mathrm{Re}\, z > 0$$

으로 정의되는 감마함수이고

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \mathrm{Re}\ s > 1$$

을 Riemann 제타함수라고 하자. $\Gamma(z)$ 와 $\zeta(s)$ 는 전복소평면 $\mathbb{C}$ 상으로의 해석접속을 지니고 있다는 사실은 잘 알려져 있다. 이들의 해석접속도 역시 같은 기호 $\Gamma(z)$ 와 $\zeta(s)$ 로 표기하기로 한다. K-Bessel함수 $K_s(z)$는

$$K_s(z) = \frac{1}{2}\int_0^{\infty} \exp\left[-\frac{z}{2}(t+t^{-1})\right] \cdot t^{s-1}dt, \qquad \mathrm{Re}\ z > 0$$

으로 정의한다. 그러면 $\mathrm{Re}\, s > 0$ 인 복소수 $s \in \mathbb{C}$를 고정하면 $z \to 0$ 일 때, $K_s(z) \sim 2^{s-1}\Gamma(s)z^{-s}$ 이고 $z \to \infty$ 일 때는 $K_s(z) \sim (\pi/(2z))^{\frac{1}{2}} \cdot e^{-z}$ 임을 알 수 있다. (참고문헌 : [20], [32])

Eisenstein 급수 $E_s(\tau)$는

(1.8) $$\Lambda(s)E_s(\tau) = \Lambda(1-s)E_{1-s}(\tau), \quad s \in \mathbb{C},\ \tau \in \mathbb{H}$$

으로 주어지는 함수방정식을 만족한다. 여기서

$$\Lambda(s) := \pi^{-s}\Gamma(s)\,\zeta(2s).$$

함수 $f : \mathbb{H} \to \mathbb{C}$ 가 고유값 $s(s-1)$ 을 갖는 마쓰 형식이라고 하면 f는

(1.9) $$f(x+iy) = ay^s + by^{1-s} + \sum_{n\neq 0} a_n y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|y) \cdot e^{2\pi inx}$$

와 같은 형태로 주어지는 푸리에 전개를 갖는다는 사실을 어렵지 않게 증명할 수 있다.

정리 1.2. *함수 $E_s^*(\tau) := \Lambda(s)E_s(\tau)$ 의 푸리에 전개는*

(1.10) $$\begin{aligned} E_s^*(x+iy) = &\ y^s\Lambda(s) + y^{1-s}\Lambda(1-s) \\ &+ 2\sum_{n\neq 0} |n|^{s-1}\sigma_{1-2s}(n) y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|y) \cdot e^{2\pi inx} \end{aligned}$$

으로 주어진다. 단, $\sigma_s(n) = \sum_{0<d|n} d^s$ 는 약수함수(division function)이다.

이제 복소수 $\lambda \in \mathbb{C}$ 에 대하여 고유값 λ를 갖는 마쓰 형식들로 이루어진 벡터공간을 $M(\Gamma, \lambda)$ 로 표기하기로 한다. 마쓰 형식 $f \in M(\Gamma, \lambda)$ 의

푸리에 전개 (1.9)에서 이의 상수항이 0 일 때 f 를 **첨점 형식**(cusp form) 이라 한다. 즉 f 가 첨점 형식이면

$$\int_0^1 f(x+iy)\,dx = ay^s + by^{1-s} = 0$$

이다. 고유값 λ 를 지니는 첨점 형식들의 벡터공간을 $M_0(\Gamma,\lambda)$ 로 표기하기로 한다. 첨점 형식 f 의 푸리에 전개 (1.9)에서의 K-Bessel함수 $K_s(y)$ 의 점근적 성향과 기본 영역 $\Gamma\backslash\mathbb{H}$ 의 부피가 $3/\pi$ 이라는 사실로부터 f 는 $L^2(\Gamma\backslash\mathbb{H})$ 의 원소임을 알 수 있다. 애석하게도 첨점 형식이 무수히 많다는 사실은 알려져 있지만 구체적으로 구성된 첨점 형식은 아직까지도 알려져 있는 것이 없다.

정리 1.3. (*a*) Re $s > \frac{1}{2}$ *이고* $s \notin [\frac{1}{2},1]$ *이면* $M(\Gamma, s(s-1)) = \mathbb{C}E_s$ *이다.*

(*b*) $M(\Gamma,0)=\mathbb{C}$.

(*c*) Re $s = \frac{1}{2}$ *또는* $s \in [\frac{1}{2},1]$ *이면*

$$M(\Gamma, s(s-1)) = \mathbb{C}E_s \oplus M_0(\Gamma, s(s-1)).$$

(*d*) $s \in [\frac{1}{2},1)$ *이면* $M_0(\Gamma, s(s-1)) = 0$ *이다.*

(*e*) *집합* $\{s \in \mathbb{C} \mid M_0(\Gamma, s(s-1)) \neq 0\}$ *는 무한집합이다.*

정리 1.4. $M_0(\Gamma, s(s-1))$ *의 원소인 첨점 형식* f *의 푸리에 전개가*

$$f(z) = \sum_{n\neq 0} a_n\, y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|y)\, e^{2\pi i n x}$$

으로 주어진다고 하자. 그러면

$$|a_n| < c\,|n|^{1/2}$$

이다. 단, c *는* n *에 의존하지 않는 어떤 상수이다.*

정리 1.5. (*a*) $M_0(\Gamma,\lambda) \neq 0$ *이면* $\lambda < -\frac{3}{2}\pi^2$ *이다.*

(*b*) $M_0(\Gamma,\lambda)$ *는 유한차원의 벡터공간이다.*

복소수 $s \in \mathbb{C}$ *에 대하여*

$$M_0^e(\Gamma, s(s-1)) = \{f \in M_0(\Gamma, s(s-1)) \mid f(-\bar{\tau}) = f(\tau),\ \tau \in \mathbb{H}\}$$

이라 하고

$$S_{\Gamma,e} = \{s \in \mathbb{C} \mid M_0^e(\Gamma, s(s-1)) \neq 0\}$$

이라 정의한다. 이 때 $M_0^e(\Gamma, s(s-1))$ *의 원소를 첨점 우형식*(even cusp form)*이라고 한다. 실제로* $S_{\Gamma,e}$ *는 무한집합이라는 사실을 알 수 있다.*

$\mathbb{R}^- := (-\infty, 0]$, $\mathbb{C}' := \mathbb{C} - \mathbb{R}^-$ 이라 놓고 함수 $\chi_s : \mathbb{C}' \longrightarrow \mathbb{C}$ (단, $s \in \mathbb{C}$) 와 $E : \mathbb{C} - 2\pi\mathbb{Z} \longrightarrow \mathbb{C}$ 를 각각

$$\chi_s(z) := z^s, \qquad E(z) := \frac{1}{e^z - 1}$$

이라 정의한다. $\mathbb{R}^+ := (0, \infty)$, $\mathcal{R} := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ 이라 두고, 복소수 $s \in \mathbb{C}$ 와 $f \in C(\mathbb{R}^+)$ ($\mathbb{R}^+$상의 연속함수들의 집합)에 대하여 $\mathbb{R}^+$상에서 f 의 제 2의 Hankel 변환 $\mathcal{H}_s f$ 는

$$\mathcal{H}_s f(y) := \int_0^\infty f(x) \left(\frac{x}{y}\right)^{s-\frac{1}{2}} J_{2s-1}(2\sqrt{yx})dx, \quad y \in \mathbb{R}^+ \tag{1.11}$$

으로 정의된다. 단, J_ν 는 위수(位數;order) ν인 Bessel 함수이다. 복소수 $s \in \mathbb{C}$에 대하여

$$\mathcal{E}_s^0 = \{f \in C(\mathbb{C}) \mid \chi_{-1} f \in L^\infty(\mathcal{R}) \cap L^2(i\mathbb{R}) \cap L^1(\mathbb{R}^+), \mathcal{H}_s(E \cdot f) = f\}$$

이라 정의한다. J. B. Lewis [21] 는 Re $s > 0$ 인 임의의 $s \in \mathbb{C}$ 에 대하여 벡터공간 $M_0^e(\Gamma, s(s-1))$ 에서 $\mathcal{E}_s^0$ 으로의 자연스런 전단사 사상 $\mathcal{U}_s$가 존재한다는 사실을 증명하였다. 실제로 사상 $\mathcal{U}_s : M_0^e(\Gamma, s(s-1)) \longrightarrow \mathcal{E}_s^0$ 는

$$\mathcal{U}_s(f)(z) := z^{1-s}(1 - e^{-z}) \int_0^\infty \sqrt{zy}\, J_{s-\frac{1}{2}}(zy) f(iy)\, dy \tag{1.12}$$

으로 주어진다. 다시 말하면 Lewis는 첨점 우형식을 어떤 성질을 만족하는 $\mathbb{C}$ 상의 복소해석적 함수로 특징화 (characterization)하였다.

함수 $f : \mathbb{H} \longrightarrow \mathbb{C}$ 가 고유값 $r(r-1)$ 을 갖는 마쓰 형식이라 하자. 그러면 (1.9)에 의하여 f 의 푸리에 전개는

$$f(\tau) = ay^r + by^{1-r} + \sum_{n \neq 0} a_n\, y^{\frac{1}{2}}\, K_{r-\frac{1}{2}}(2\pi|n|y)\, e^{2\pi i n x} \tag{1.13}$$

으로 주어진다. 이 때 f 의 Mellin변환

$$W_f(s) := \int_0^\infty \{f(iy) - a\, y^r - b\, y^{1-r}\}\, y^{s-1}\, dy \tag{1.14}$$

은 K-Bessel 함수의 적분 공식에 의하여

$$W_f(s) = 2^{-\frac{s}{2}} \pi^{-(s+\frac{1}{2})} \Gamma\left(\frac{s-r+1}{2}\right) \Gamma\left(\frac{s+r}{2}\right) \sum_{n \neq 0} a_n |n|^{-(s+\frac{1}{2})} \tag{1.15}$$

으로 주어지는 등식을 만족함을 보일수 있다. H. Maass [24] 는 마쓰 형식과 이에 대응되는 Dirichlet 급수와의 관계가 아래와 같이 주어짐을 증명하였다.

정리 1.6. 함수 $f:\mathbb{H}\to\mathbb{C}$ 가 푸리에 전개 (1.13)을 갖는다고 하자. 그러면 f 가 $M(\Gamma, r(r-1))$ 의 원소이기 위한 필요충분조건은 아래의 두 조건 (ㄱ) 과 (ㄴ) 의 성질을 만족하는 것이다.

(ㄱ) Dirichlet급수

$$W_f(s)-a\left(\frac{1}{s-r}+\frac{1}{-s-r}\right)-b\left(\frac{1}{s+r-1}+\frac{1}{-s+r-1}\right)$$

이 수직대(vertical strip)에서 유계인 전해석적(entire) 함수이고 함수 방정식 $W_f(s)=W_f(-s)$ 을 만족한다.

(ㄴ) 2차 Dirichlet 급수

$$W_{f_x}(s):=2^{-\frac{s}{2}}\pi^{-(s+\frac{1}{2})}\Gamma\left(\frac{s-r+1}{2}\right)\Gamma\left(\frac{s+r}{2}\right)\cdot\sum_{n\neq 0}(2\pi n)a_n|n|^{-(s+\frac{1}{2})}$$

도 역시 (ㄱ) 과 유사한 성질을 갖는다. 여기서 f_x는 변수 x 에 대한 f 의 미분함수를 나타낸다.

정의 1.7. 마쓰 형식 $f\in M(\Gamma, r(r-1))$ 에 대하여 Hecke 작용소 $T_n\,(n=1,2,3,\cdots)$ 을

$$T_nf(\tau):=n^{-\frac{1}{2}}\sum_{\substack{ad=n,\ d>0\\ b \bmod d}} f\left(\frac{a\tau+b}{d}\right) \tag{1.16}$$

으로 정의한다. 두 개의 마쓰 형식 $f,g\in M(\Gamma, r(r-1))$ 에 대하여 Petersson 내적을 형식적으로

$$(f,g):=\int_{\Gamma\backslash\mathbb{H}} f(\tau)\,\overline{g(\tau)}\,\frac{dxdy}{y^2}$$

으로 정의한다.

Hecke 작용소는 아래와 같은 성질을 지니고 있다.

정리 1.8. (1) $T_n(M(\Gamma, r(r-1))\subset M(\Gamma, r(r-1))$ 이고

$$T_n(M_0(\Gamma, r(r-1))\subset M_0(\Gamma, r(r-1))$$

이다.

(2) 마쓰 형식 $f\in M(\Gamma, r(r-1))$ 가 푸리에 급수 (1.13)을 갖는다고 하자. 그러면 마쓰 형식 $T_nf\,(n=1,2,3,\cdots)$ 의 푸리에 급수는

$$T_nf(\tau)=c\,y^r+d\,y^{1-r}+\sum_{m\neq 0}c_m\,y^{\frac{1}{2}}\,K_{r-\frac{1}{2}}(2\pi|m|y)\,e^{2\pi imx}$$

으로 주어진다. 여기서

$$c=n^{1/2-r}\sigma_{2r-1}(n)\,a,\qquad d=n^{r-1/2}\sigma_{1-2r}(n)\,b$$

이고

$$c_m = \sum_{d|(m,n)} c_{mn/d^2}$$

이다.

(3) Hecke 작용소 $T_n\,(n = 1, 2, 3, \cdots)$ 에 의하여 생성되는 Hecke 대수는 가환대수이다. 그리고 T_n 은

$$T_m T_n = T_n T_m = \sum_{d|(m,n)} T_{mn/d^2}$$

의 관계를 만족한다. 여기서 (m, n)은 m 과 n 의 최대공약수를 나타내고 있다.

(4) 첨점형식 $f, g \in M_0(\Gamma, r(r-1))$ 에 대하여 항상 $(T_n f, g) = (f, T_n g)$ (단, $n = 1, 2, 3, \cdots$). 그러므로 Hecke 작용소는 벡터공간 $M_0(\Gamma, r(r-1))$ 상에서 동시에 대각화가 가능하다.

(5) 마쓰 형식 $f \in M_0(\Gamma, r(r-1))$ 가 푸리에 전개 (1.13)을 가지며 모든 Hecke 작용소 $T_n\,(n = 1, 2, 3, \cdots)$ 의 고유함수라고 가정하자. 그리고 $T_n f = \lambda_n f\,(n = 1, 2, 3, \cdots)$ 이라 가정하자. 그러면

$$a_n = \lambda_{|n|} a_{n/|n|}$$

의 관계가 성립하고 f와 연관되어 있는 Dirichlet 급수 $L_f(s)$ 와 $L_{f_x}(s)$ 는

$$\begin{aligned} L_f(s - 1/2) &:= \sum_{m \neq 0} a_m |m|^{-s} = (a_1 + a_{-1}) \prod_p (1 - u_p p^{-s} + p^{-2s})^{-1} \\ L_{f_x}(s + 1/2) &:= \sum_{m \neq 0} (2\pi i m) a_m |m|^{-s-1} \\ &= 2\pi i (c_1 - c_{-1}) \prod_p (1 - u_p p^{-s} + p^{-2s})^{-1} \end{aligned}$$

으로 주어지는 Euler 곱을 갖는다. 여기서 p 는 모든 소수들의 집합 위를 움직인다. 역으로 상기와 같은 Euler 곱을 갖는 Dirichlet 급수 L_f 와 L_{f_x} 에 대응되는 마쓰 형식 $f \in M(\Gamma, r(r-1))$ 은 모든 Hecke 작용소의 고유함수이다.

모듈라 곡선 $X(\Gamma) := \Gamma\backslash\mathbb{H}$ 상의 라플라스 작용소 $-\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ 는 $L^2(X(\Gamma))$ 상의 음이 아닌 자기수반 작용소 Δ_Γ 로 확장된다. 이제 힐베르트 공간 $L^2(X(\Gamma))$ 상에서의 작용소 Δ_Γ 의 스펙트럼 분해에 관하여 설명하겠다.

우선 $L^2(X(\Gamma))$ 상에 내적

$$(1.17) \qquad (f, g) := \int_{X(\Gamma)} f(\tau)\, \overline{g(\tau)}\, \frac{dxdy}{y^2}, \qquad f, g \in L^2(X(\Gamma))$$

을 정의한다. 그러면 임의의 $f \in L^2(X(\Gamma))$ 는

$$f(\tau) = \sum_{n \geq 0} (f, u_n) u_n + \frac{1}{4\pi i} \int_{\mathrm{Re}\ s = \frac{1}{2}} (f, E_s)\, E_s(\tau)\, ds \tag{1.18}$$

와 같이 분해된다. 여기서 $u_0 = \sqrt{\frac{3}{\pi}}$, $\{u_n \mid n = 1, 2, 3, \cdots\}$ 은 첨점 형식들의 집합으로 $(u_m, u_n) = \delta_{mn}$, $\Delta_\Gamma u_n = \lambda_n u_n$, $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ 을 만족하고 $\{u_0, u_1, u_2, \cdots\}$ 은 $L^2(X(\Gamma))$ 의 이산부분 공간의 정규 직교기저이다. (1.18)로 주어지는 분해를 **Roelcke-Selberg 분해**라고 한다.

$$L^2_{(1)}(X(\Gamma)) := \{f \in L^2(X(\Gamma)) \mid f(-\bar{\tau}) = -f(\tau),\ \tau \in \mathbb{H}\},$$

$$L^2_{(2)}(X(\Gamma)) := \{f \in L^2(X(\Gamma)) \mid f(-\bar{\tau}) = f(\tau),\ \tau \in \mathbb{H}\}$$

이라 놓으면

$$L^2(X(\Gamma)) = L^2_{(1)}(X(\Gamma)) \bigoplus L^2_{(2)}(X(\Gamma)) \qquad \text{(direct sum)}$$

이 된다. 작용소 Δ_Γ의 이산 스펙트럼의 (상수가 아닌) 고유함수들에 의하여 생성되는 $L^2(X(\Gamma))$ 의 닫힌 부분공간을 $L^2_{\mathrm{cusp}}(X(\Gamma))$ 로 표기하고 Δ_Γ의 연속 스펙트럼의 고유함수들에 의하여 생성되는 $L^2(X(\Gamma))$ 의 부분공간을 $L^2_{\mathrm{cont}}(X(\Gamma))$ 로 표기한다. 실제로 $L^2_{\mathrm{cusp}}(X(\Gamma))$ 는 Γ 에 대한 첨점 형식들의 집합이다. 그러면 힐베르트공간 $L^2(X(\Gamma))$ 는

$$L^2(X(\Gamma)) = L^2_{\mathrm{cusp}}(X(\Gamma)) \oplus \mathbb{C} \oplus L^2_{\mathrm{cont}}(X(\Gamma)) \quad \text{(direct sum)}$$

으로 분해된다. Eisenstein 급수 $E_s(\tau)$ 는 $E_s(-\bar{\tau}) = E_s(\tau)$ 의 성질을 만족하므로 $L^2_{\mathrm{cont}}(X(\Gamma))$ 는 $L^2_{(2)}(X(\Gamma))$ 의 부분집합이다.

$$R(\tau) := E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)\, e^{2\pi i \tau}$$

$$Q(\tau) := E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)\, e^{2\pi i \tau}$$

$$P(\tau) := E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)\, e^{2\pi i \tau}$$

를 각각 무게(weight)가 $6, 4, 2$ 인 해석적 Eisenstein 급수라고 두고 함수 $S : \mathbb{H} \longrightarrow \mathbb{C}$ 를

$$S(\tau) := P(\tau) - 3\pi \cdot \mathrm{Im}\, \tau, \qquad \tau \in \mathbb{H}$$

이라 정의한다. 그리고 음이 아닌 정수 k 에 대하여 함수 $e_k : \mathbb{H} \longrightarrow \mathbb{C}$ 를

$$e_k(\tau) := (\mathrm{Im}\ \tau)^k R(\tau)^{k_1} Q(\tau)^{k_2} S(\tau)^{k_3}\, \overline{R(\tau)^{m_1} Q(\tau)^{m_2} S(\tau)^{m_3}},$$

$$k = 6k_1 + 4k_2 + 2k_3 = 6m_1 + 4m_2 + 2m_3,$$

k_j, m_j $(j = 1, 2, 3)$ 는 음이 아닌 정수 와 같이 정의한다. $e_0 = 1$ 임을 유의하라. 집합 $\{e_k \mid k$ 는 음이 아닌 정수$\}$에 의하여 생성되는 복소벡터공간을 M_E 이라고 표기한다. 또 $L^2(X(\Gamma))$ 안에서의 작용소 Δ_Γ 의 정의역을 $D(\Delta_\Gamma)$ 이라 표기한다. 그리고

$$M_E^D := M_E \cap D(\Delta_\Gamma)$$

이라 정의한다. A. Venkov [31]는

$$\overline{M_E^D} = L^2_{(2)}(X(\Gamma))$$

임을 증명하였다. 여기서 $\overline{M_E^D}$ 는 $L^2_{(2)}(X(\Gamma))$ 안에서의 M_E^D의 폐포(closure)를 나타내고 있다.

수론에서 심오하고 중요한 업적 중의 하나인 $\Gamma := SL(2, \mathbb{R})$ 에 관한 Selberg 대각합 공식(trace formula)에 관하여 간략하게 설명하겠다. Γ 의 원소들은 다음과 같이 분류된다.

(Γ1) $I_2, -I_2$.

(Γ2) 포물원(parabolic elements): Jordan 형식이 $\pm\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ (단, $a \neq 0$) 의 형태로 주어지는 Γ 의 원소들임.

(Γ3) 타원원(elliptic elememnts): Jordan 형식이 $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ (단, $a \notin \mathbb{R}$, $|a| = 1$) 의 형태로 주어지는 원소들임.

(Γ4) 쌍곡원(hyperbolic elememnts): Jordan 형식이 $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ (단, $a \in \mathbb{R}$, $a \neq 0, \pm 1$) 의 형태로 주어지는 원소들임.

$\gamma \in \Gamma$ 에 대하여 집합 $\{\gamma\} := \{\gamma_1\gamma\gamma_1^{-1} \mid \gamma_1 \in \Gamma\}$ 를 Γ 안에서 γ 의 공액류(concugacy class)라고 부르며 집합 $\Gamma_\gamma := \{\gamma_1 \in \Gamma \mid \gamma_1\gamma\gamma_1^{-1} = \gamma\}$ 을 Γ 안에서 γ 의 중심화군 (centralizer)이라고 한다. 쌍곡원 $\gamma \in \Gamma$ 의 Jordan 형식이 $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ (단, $a \in \mathbb{R}$, $a \neq 0, \pm 1$)으로 주어질 때 $N\gamma := a^2$ 를 γ 의 노옴(norm)이라고 일컫는다. 쌍곡원 γ 의 중심화군 Γ_γ 은 무한 순환군 $\Gamma_\gamma = \langle \gamma_0 \rangle$ 임을 쉽게 알 수 있다. 이 때 Γ_γ 의 생성원 γ_0 를 원시 쌍곡원 (primitive hyperbolic element)이라고 한다.

편의상 $G = SL(2, \mathbb{R})$, $K = SO(2)$ 이라 두자. 그러면 대칭공간 G/K는 $\mathbb{H}$ 와 겹복소해석적(biholomorphic)임을 유의한다. G 상의 함수로서 컴팩트 받침함수인 동시에 K-겹불변(K-bi-invariant)인 C^∞-함수들의 집합을 $C_c^\infty(K\backslash G/K)$ 으로 표기한다. $C_c^\infty(K\backslash G/K)$의 원소 ϕ 에 대

하여 ϕ의 Helgason 변환 $\hat{\phi}(s)$를

$$\hat{\phi}(s) = \int_{\mathbb{H}} \phi(\tau)\, y^{\bar{s}}\, \frac{dxdy}{y^2}, \qquad s \in \mathbb{C} \tag{1.19}$$

으로 정의한다. $(0,\infty)$ 상의 함수 Φ를 Harish-Chandra 변환

$$T_{\mathrm{HC}}\phi(y) := y^{-\frac{1}{2}} \int_{-\infty}^{\infty} \phi(x+iy)\, dx = \Phi(\log y) \tag{1.20}$$

에 의하여 결정되는 함수라고 정의한다.

정리 1.9 (Selberg [28]). *함수 $\phi : \mathbb{H} \longrightarrow \mathbb{C}$가 $C_c^\infty(K\backslash G/K)$의 원소라고 하자. 그러면*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \hat{\phi}(s_n) \\
= \;& \frac{Area(\Gamma\backslash\mathbb{H})}{4\pi} \int_{r\in\mathbb{R}} \hat{\phi}\left(\frac{1}{2}+ir\right) r \tanh \pi r\, dr \\
&+ \sum_{\substack{\{\gamma_0\} \\ \text{원시 쌍곡원}}} \sum_{k=1}^{\infty} \frac{\log N\gamma_0}{N\gamma_0^{\frac{k}{2}} - N\gamma_0^{-\frac{k}{2}}}\, \Phi(k \log N\gamma_0) \\
&+ \int_{r\in\mathbb{R}} \left(\frac{1}{4} + \frac{1}{3\sqrt{3}}(e^{\frac{\pi r}{3}} + e^{-\frac{\pi r}{3}})\right) \hat{\phi}\left(\frac{1}{2}+ir\right)(e^{\pi r}+e^{-\pi r})^{-1}\, dr \\
&- \Phi(0) \log(2\pi) + \frac{1}{\pi}\int_{r\in\mathbb{R}} \hat{\phi}\left(\frac{1}{2}+ir\right) \frac{\zeta'}{\zeta}(-2ir)\, dr
\end{aligned}$$

의 공식을 얻는다. 여기서 $s_0 = 0$이고 $s_n\,(n = 1,2,3,\cdots)$은 $\Delta u_n = s_n(s_n-1)\,u_n\,(s_n \in \frac{1}{2} + i\mathbb{R})$에 의하여 주어지는 복소수이다. (여기서, $u_n\,(n=1,2,3,\cdots)$은 (1.18)에서 주어지는 첨점 형식들이다.)

Selberg의 고유값 가설(Selberg's Eigenvalue Conjecture)에 관하여 간략하게 언급하겠다. 자연수 N에 대하여

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid a \equiv d \equiv 1,\; b \equiv c \equiv 0 \pmod N \right\}$$

이라 정의한다. $\Gamma(N)$을 주합동 부분군(a principal congruence subgroup)이라 일컫는다. Γ_*가 Γ의 부분군으로 어떤 자연수 N이 존재하여 $\Gamma(N) \subset \Gamma_* \subset \Gamma$이고 $[\Gamma:\Gamma_*] < \infty$ (유한지수; finite index)일 때 Γ_*를 Γ의 합동 부분군이라 한다. 1965년에 Selberg는 그의 논문 [29]에서 아래의 가설을 제기하였다.

Selberg의 고유값 가설. Γ_* 를 Γ 의 합동 부분군이라하고 $0 = \lambda_0 < \lambda_1(\Gamma_*) < \lambda_2(\Gamma_*) < \cdots$ 를 모듈라 곡선 $\Gamma_*\backslash\mathbb{H}$ 상의 라플라스 작용소 $-\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$의 이산 스펙트럼이라고 하자. 그러면

$$\lambda_1(\Gamma_*) \geqq \frac{1}{4}$$

이다.

임의의 자연수 N 에 대하여 모듈라 곡선

$$X(N) := \Gamma(N)\backslash\mathbb{H}$$

이라 표기한다. 1949년에 H. Maass는 그의 논문 [24]에서 $\lambda = \frac{1}{4}$ 는 어떤 자연수 N 이 존재하여 모듈라 곡선 $X(N)$ 상에서 $-\Delta$ 의 고유값이 될 수 있다는 사실을 보였다. 1965년에 Selberg는 [29]에서 임의의 합동 부분군 Γ_* 에 대하여 $\lambda_1(\Gamma_*) \geq \frac{3}{16} = 0.1875$ 임을 증명하였고, 1978년에는 Gelbart와 Jacquet는 논문 [13]에서 Selberg의 증명의 방법과는 완전히 다른 방법으로 $\lambda_1(\Gamma_*) > \frac{3}{16}$ 임을 증명하였다. 1995년에 Luo, Rudnick와 Sarnak는 그들의 논문 [22]에서 $\lambda_1(\Gamma_*) \geq \frac{121}{784} = 0.2181\cdots$ 임을 증명하였다. Selberg의 고유값 가설은 Ramanujan 가설과 밀접한 관계가 있다. (참고문헌 : [23])

Γ_* 를 Γ 의 합동 부분군이라 하자. 모듈라 곡선 $X(\Gamma_*)$ 상의 비유크리드 작용소 $-\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ 는 $L^2(X(\Gamma_*))$ 상의 음이 아닌 자기수반 작용소 $\Delta(\Gamma_*)$ 으로 자연스럽게 확장이 된다. $X(\Gamma_*)$ 는 비긴밀 공간이므로 작용소 $\Delta(\Gamma_*)$ 는 이산 스펙트럼뿐만아니라 연속 스펙트럼도 갖는다. $\Delta(\Gamma_*)$ 의 이산 스펙트럼을 $0 = \lambda_0 < \lambda_1(\Gamma_*) < \lambda_2(\Gamma_*) < \lambda_3(\Gamma_*) < \cdots$ 와 같이 크기 순서로 표기한다. Zograf는 논문 [39]에서 Selberg 정리 (즉, $\lambda_1(\Gamma_*) \geq \frac{3}{16}$) 와 스펙트럴 이론을 사용하여

$$g(\Gamma_*) + 1 > \frac{[\Gamma : \Gamma_*]}{128} \tag{1.21}$$

의 관계가 성립함을 증명하였다. 여기서 $g(\Gamma*)$ 는 모듈라 곡선 $X(\Gamma_*)$ 의 종수(genus)를 나타내고 있다. 게다가 그는 주어진 종수 g 를 갖는 Γ 의 합동 부분군들의 개수는 유한임을 증명하였다.

Sarnak의 흥미로운 survey 논문, "*On cusp forms*" [26]의 내용을 언급하겠다. G 가 반단순 Lie 군이고 K 를 G 의 최대 긴밀부분군이라고 하자. 그리고 $X = G/K$ 를 비긴밀 타입의 대칭공간이라 하자. Γ_* 를 G 안의 비균등격자(a non-uniform lattice)라고 하자. Γ_* 가 G 의 이산 부분군으로 $\Gamma_*\backslash X$ 의 부피는 유한이고 $\Gamma_*\backslash X$가 비긴밀 공간일 때 Γ_* 를 G 의 비균등격자라고 일컫는다. X 상의 첨점 형식을 아래와 같이 G 상의 함수로 간주할 수 있다. G 상의 함수 $\phi : G \longrightarrow \mathbb{C}$ 가 첨점 형식이란 $L^2(\Gamma\backslash X)$ 의

원소이고 X 상의 불변 미분작용소들의 대수의 동시적 고유함수이며 임의의 첨점 부분군(a cuspidal subgroup) N (단, $N \neq G$) 에 대하여

$$\int_{(N\cap\Gamma_*)\backslash N} \phi(nx)dn = 0 \tag{1.22}$$

일 때를 말한다. Δ_X 를 X 상의 라플라스 스펙트럼이라 하고 $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ 를 Δ_X 의 이산 스펙트럼이라 하자. 또한 $\phi_1, \phi_2, \cdots$ 를 이산 스펙트럼 $\lambda_1, \lambda_2, \cdots$ 에 대응하는 고유함수라고 하자. 다시 말하면,

$$\phi_j \in L^2(\Gamma_*\backslash X), \qquad \Delta\phi_j + \lambda_j\phi_j = 0, \quad j = 1, 2, 3, \cdots.$$

양의 실수 R 에 대하여

$$N_{\text{cusp}}(R) := \#\left\{\phi_j \mid \lambda_j \leqq \sqrt{R}\right\}$$

이라 정의한다. 여기서 기호 $\#A$ 는 집합 A 의 개수를 나타낸다. H. Donnelly 는 그의 논문 [9]에서

$$\limsup_{R\to\infty} \frac{N_{\text{cusp}}(R)}{R^d} \leq \frac{(4\pi)^{-d/2}\text{vol}(\Gamma_*\backslash X)}{\Gamma(\frac{d}{2}+1)} =: c(\Gamma_*) \tag{1.23}$$

으로 주어지는 부등식을 발견하였다. 여기서 d 는 X 의 차원이다. Γ 의 첨점 스펙트럼의 밀도(density) $\delta(\Gamma_*)$를

$$\delta(\Gamma_*) := \frac{1}{c(\Gamma_*)} \limsup_{R\to\infty} \frac{N_{\text{cusp}}(R)}{R^d}$$

이라 정의한다. $\delta(\Gamma_*) = 1$ 일 때 Γ_*를 *본질적으로 첨점적*(essentially cuspidal)이라고 한다. Selberg는 다음의 결과를 얻었다.

정리 1.10 (Selberg). *Γ_* 를 $\Gamma := SL(2, \mathbb{Z})$ 의 합동부분군이라 하자. 그러면*

$$\begin{aligned} N_{cusp}(R) &= \frac{vol(\Gamma_*\backslash\mathbb{H})}{4\pi} R^2 + O(R\log R) \\ &= c(\Gamma_*)\,R^2 + O(R\log R) \end{aligned} \tag{1.24}$$

의 관계식이 성립한다. 따라서 Γ_* 는 본질적으로 첨점적이다.

상기의 증명은 Selberg 대각합 공식으로부터 얻어진다.

I. Efrat 는 그의 박사학위논문 [10]에서 아래와 같은 사실을 증명하였다.

정리 1.11 (Efrat). *$G = (SL(2, \mathbb{R}))^n$ (단, $n > 1$ 인 자연수)이고 Γ_* 를 G 의 기약인 비균등격자라고 하자. 그러면*

$$N_{cusp}(R) = c(\Gamma_*)\,R^{2n} + O(R^{2n-1}(\log R)^{-1}) \tag{1.25}$$

이다. 따라서 Γ_* 는 본질적으로 첨점적이다.

Sarnak과 Efrat는 $G = SL(3,\mathbb{R}), \Gamma_* = SL(3,(Z))$인 경우에는

$$N_{cusp}(R) = c(\Gamma_*)\, R^5 + O(R^4) \tag{1.26}$$

임을 증명하였다. 따라서 $\Gamma_* = SL(3,\mathbb{Z})$는 본질적으로 첨점적이다. 그래서 Sarnak는 (1.25)와 (1.26)의 결과를 근거로 하여 아래의 가설을 제기하였다.

가설. rank$(X) > 1$ 이면 임의의 기약인 비균등격자는 본질적으로 첨점적이다.

나아가 Sarnak는 Γ_*의 산수성(arithmetcity)과 본질적 첨점성과의 관계를 논하였다. 흥미로운 문제임에는 틀림없다.

마쓰 형식에 관한 참고 저서로 [7], [12], [17], [18], [30]을 추천한다.

제 2 절 $\mathbb{H}\times\mathbb{C}$ 상의 불변 미분 작용소

이제부터는

$$G = SL_{2,1}(\mathbb{R}),\ \ K = SO(2),\ \ \Gamma_1 = SL(2,\mathbb{Z}),$$

$$\Gamma_{1,2} \ = \ SL_{2,1}(\mathbb{Z})$$

이라 두자. 그리고

$$S\mathcal{P}_2 := \{\, Y \in \mathbb{R}^{(2,2)} \mid Y = {}^t Y > 0 \,\}$$

이라 하자. 그러면 G는 $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ 상에서

$$(g,\alpha)\cdot(Y,V) := (gY\,{}^tg,\ (V+\alpha)\,{}^tg) \tag{2.1}$$

와 같이 추이적으로 작용한다. 여기서 $g \in SL(2,\mathbb{Z})$, $\alpha \in \mathbb{R}^{(1,2)}$, $Y \in S\mathcal{P}_2$, $V \in \mathbb{R}^{(1,2)}$ 이다. K는 $(I_2,0)$을 고정시키는 최대 기밀부분군이므로 $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$는 등질공간 G/K와 아래에 의하여 미분동상적이다.

$$G/K \ni (g,\alpha)K \longmapsto (g,\alpha)\cdot(I_2,0) \in S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}. \tag{2.2}$$

$SL_2(\mathbb{R})$은 $\mathbb{H}$ 상에서

$$g\langle\tau\rangle := (a\tau+b)(c\tau+d)^{-1},\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}),\ \tau\in\mathbb{H}$$

와 같이 추이적으로 작용함을 상기하자. 그러면 G의 $\mathbb{H}\times\mathbb{C}$의 작용 (0.2)는

$$(g,\alpha)\circ(\tau,z) = ({}^tg^{-1}\langle\tau\rangle,\ (z+\alpha\tau+\alpha_2)(-b\tau+a)^{-1})$$

으로 쓸수있다. 단 $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z)$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^{(1,2)}$ 이고 $(\tau, z) \in \mathbb{H} \times \mathbb{C}$ 이다. $\mathbb{H} \times \mathbb{C}$ 는 등질공간 G/K 와 미분동상적이다. 구체적으로 설명하면

$$G/K \ni (g,\alpha)K \longmapsto (g,\alpha) \circ (i,0), \quad (g,\alpha) \in G \tag{2.3}$$

$\mathcal{SP}_2$의 임의의 원소 Y 는

$$Y = \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix} \left[\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} y^{-1} & -xy^{-1} \\ -xy^{-1} & x^2y^{-1} + y \end{pmatrix} \tag{2.4}$$

의 형태로 유일하게 나타낼 수 있음을 유의하라. 단 $x, y \in \mathbb{R}$ 이고 $y > 0$ 이다.

보조정리 2.1. *사상* $T : \mathcal{SP}_2 \times \mathbb{R}^{(1,2)} \to \mathbb{H} \times \mathbb{C}$ 를

$$T(Y,V) := (x+iy,\ v_1(x+iy)+v_2) \tag{2.5}$$

으로 *정의한다. 단* Y 는 (2.4)*의 형태이고* $V = (v_1, v_2) \in \mathbb{R}^{(1,2)}$ *이다. 그러면* T 는 (0.2)*와* (2.1)*의 두 작용과 양립을 하는 전단사 사상이다. 실제로* $\mathcal{SP}_2 \times \mathbb{C}$ *와* $\mathbb{H} \times \mathbb{C}$ 는 *복소다양체이며* T 는 *겹해석적*(*biholomorphic*) *사상이다.*

증명. 상기의 보조정리의 증명은 독자들에게 남겨두겠다.

이제 G의 작용 (1.2)에 불변인 $\mathbb{H} \times \mathbb{C}$ 상의 미분작용소들을 구하여 보자. G의 Lie 대수 $\mathfrak{g}$ 는

$$\mathfrak{g} = \left\{ (X,Z) \,\middle|\, X \in \mathbb{R}^{(2,2)},\ \sigma(X) = 0,\ Z \in \mathbb{R}^{(1,2)} \right\}$$

으로 주어지고 $\mathfrak{g}$ 상의 Lie 괄호는

$$[(X_1,Z_1),(X_2,Z_2)] = (X_1X_2 - X_2X_1,\ Z_2\,{}^tX_1 - Z_1\,{}^tX_2)$$

주어진다.

$$\mathfrak{k} = \left\{ (X,0) \in g \,\middle|\, X = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix},\ x \in \mathbb{R} \right\}$$

$$\mathfrak{p} = \left\{ (X,Z) \in \mathfrak{g} \mid X = {}^tX \in \mathbb{R}^{(2,2)},\ \sigma(X) = 0,\ Z \in \mathbb{R}^{(1,2)} \right\}$$

이라 두자. 그러면 $\mathfrak{k}$ 는 K 의 Lie 대수이며 $\mathfrak{g}$ 는

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \quad \text{(direct sum)}$$

와 같이 분해된다. $[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}$, $\quad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$ 이므로 등질공간 $\mathbb{H} \times \mathbb{C} = G/K$ 는 reductive 등질공간이다. (참고문헌 : [16]의 281 쪽). $\mathfrak{p}$ 상의 K 의 수반작용 (adjoint action) Ad 는

$$\mathrm{Ad}(k)((X,Z)) = (kX\,{}^tk,\ Z\,{}^tk), \quad k \in K,\ (X,Z) \in \mathfrak{p} \tag{2.6}$$

으로 주어진다는 사실을 쉽게 알수 있다. (2.6)은 유일하게 $\mathfrak{p}$의 다항식 대수 Pol($\mathfrak{p}$)상의 K의 작용 ρ로 확장된다. Pol$(\mathfrak{p})^K$를 작용 ρ에 불변하는 Pol($\mathfrak{p}$)의 원소들로 이루어지는 부분대수라고 하자. $\mathbb{D}(\mathbb{H}\times\mathbb{C})$를 G의 작용 (0.2)에 불변인 $\mathbb{H}\times\mathbb{C}$상의 미분작용소들로 이루어지는 대수라고 하자. 그러면 [16], 정리 4.9에 의하여 자연스런 선형 전단사 사상

$$D_{\lambda(\cdot)} : \mathrm{Pol}(\mathfrak{p})^K \to \mathbb{D}(\mathbb{H}\times\mathbb{C}), \quad P \longmapsto D_{\lambda(P)}$$

가 존재한다. 구체적으로 설명하면 (ξ_k) (단, $k = 1,2,3,4$) 가 $\mathfrak{p}$의 기저이고 $P \in \mathrm{Pol}(\mathfrak{p})^K$ 이면

(2.7)

$$\left(D_{\lambda(P)}f\right)(\tilde{g}\circ(i,0)) = \left[P\left(\frac{\partial}{\partial t_k}\right) f((\tilde{g} * \exp(\sum_{k=1}^{4} t_k\xi_k))\circ(i,0))\right]_{(t_k)=0}$$

으로 주어진다. 단, $\tilde{g}\in G$이고 $f\in C^\infty(\mathbb{H}\times\mathbb{C})$ 이다.

$$e_1 = \left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, (0,0)\right) \quad e_2 = \left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, (0,0)\right)$$
$$f_1 = \left(\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, (1,0)\right) \quad f_2 = \left(\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, (0,1)\right)$$

이라 두자. 그러면 e_1, e_2, f_1, f_2는 $\mathfrak{p}$의 기저이다. $\mathfrak{p}$의 원소 (X, Z)에

$$X = \begin{pmatrix}x & y\\ y & -x\end{pmatrix}, \quad Z = (z_1, z_2)$$

와 같은 좌표계를 도입한다. 불변이론으로부터 아래의 결과를 얻는다. □

보조정리 2.2. *아래의 다항식 P, ξ, P_1, P_2는 $Pol(\mathfrak{p})^K$의 대수적 독립(algebraically independent)인 생성원들이다. 단*

$$\begin{aligned}
P(X,Z) &= \frac{1}{8}\sigma(X^2) = \frac{1}{4}(x^2+y^2),\\
\xi(X,Z) &= Z\,{}^tZ = z_1^2 + z_2^2,\\
P_1(X,Z) &= -\frac{1}{2}ZX\,{}^tZ = \frac{1}{2}(z_2^2 - z_1^2)x - z_1z_2y,\\
P_2(X,Z) &= \frac{1}{2}(z_2^2 - z_1^2)y + z_1z_2x.
\end{aligned}$$

증명. 상기의 증명을 독자에게 남겨 두겠다.

공식 (2.7)을 사용하여 P, ξ, P_1, P_2 에 대응하는 $\mathbb{H}\times\mathbb{C}$ 상의 불변미분 작용소를 계산을 한다. 실변수 $t=(t_1,t_2), s=(s_1,s_2)$ 에 대하여

$$\begin{aligned}&\exp(t_1e_1+t_2e_2+s_1f_1+s_2f_2)\\ =&\left(\begin{pmatrix} a_1(t,s) & a_3(t,s)\\ a_3(t,s) & a_2(t,s)\end{pmatrix},\ (b_1(t,s),\, b_2(t,s))\right)\end{aligned}$$

을 얻는다. 단,

$$\begin{aligned}
a_1(t,s) &= 1+t_1+\frac{1}{2!}(t_1^2+t_2^2)+\frac{1}{3!}t_1(t_1^2+t_2^2)\\
&\quad +\frac{1}{4!}(t_1^2+t_2^2)^2+\cdots\\
a_2(t,s) &= 1-t_1+\frac{1}{2!}(t_1^2+t_2^2)-\frac{1}{3!}t_1(t_1^2+t_2^2)\\
&\quad +\frac{1}{4!}(t_1^2+t_2^2)^2-\cdots,\\
a_3(t,s) &= t_2+\frac{1}{3!}t_2(t_1^2+t_2^2)+\frac{1}{5!}t_2(t_1^2+t_2^2)^2+\cdots,\\
b_1(t,s) &= s_1-\frac{1}{2!}(s_1t_1+s_2t_2)+\frac{1}{3!}s_1(t_1^2+t_2^2)\\
&\quad -\frac{1}{4!}(s_1t_1+s_2t_2)(t_1^2+t_2^2)+\cdots,\\
b_2(t,s) &= s_2-\frac{1}{2!}(s_1t_2-s_2t_1)+\frac{1}{3!}s_2(t_1^2+t_2^2)\\
&\quad -\frac{1}{4!}(s_1t_2-s_2t_1)(t_1^2+t_2^2)+\cdots.
\end{aligned}$$

기호편의상 $a_j(t,s), b_k(t,s)$ 를 간단히 a_j, b_k 로 표기한다. G 의 원소 (g,α) 를 고정시킨후

$$g=\begin{pmatrix} g_1 & g_{12}\\ g_{21} & g_2\end{pmatrix}\in SL(2,\mathbb{Z}),\qquad \alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^{(1,2)}$$

이라 표기한다. 그리고

$$(\tau(t,s),z(t,s))=((g,\alpha)*\exp(t_1e_1+t_2e_2+s_1f_1+s_2f_2))\circ(i,0)$$

이라 놓고

$$\tau(t,s)=x(t,s)+i\,y(t,s)\quad\text{and}\quad z(t,s)=u(t,s)+i\,v(t,s)$$

이라 두자. 여기서 $x(t,s), y(t,s), u(t,s), v(t,s)$ 는 실함수이다. 지루한 계산에 의하여

$$\begin{aligned} x(t,s) &= -(\tilde{a}\tilde{c}+\tilde{b}\tilde{d})(\tilde{a}^2+\tilde{b}^2)^{-1}, \\ y(t,s) &= (\tilde{a}^2+\tilde{b}^2)^{-1}, \\ u(t,s) &= (\tilde{a}\,\tilde{\alpha}_2-\tilde{b}\tilde{\alpha}_1)(\tilde{a}^2+\tilde{b}^2)^{-1}, \\ v(t,s) &= (\tilde{a}\,\tilde{\alpha}_1+\tilde{b}\,\tilde{\alpha}_2)(\tilde{a}^2+\tilde{b}^2)^{-1} \end{aligned}$$

을 얻는다. 단,

$$\begin{aligned} \tilde{a} &= g_1a_1 + g_{12}a_3, \\ \tilde{b} &= g_1a_3 + g_{12}a_2, \\ \tilde{c} &= g_{21}a_1 + g_2a_3, \\ \tilde{d} &= g_{21}a_3 + g_2a_2, \\ \tilde{\alpha}_1 &= \alpha_1a_2 - \alpha_2a_3 + b_1, \\ \tilde{\alpha}_2 &= -\alpha_1a_3 + \alpha_2a_1 + b_2 \end{aligned}$$

이다. $t=s=0$ 을 대입하면

$$\begin{aligned} \tfrac{\partial x}{\partial t_1} &= 4\,g_1\,g_{12}\,(g_1^2 + g_{12}^2)^{-2}, \\ \tfrac{\partial y}{\partial t_1} &= -2\,(g_1^2 - g_{12}^2)(g_1^2 + g_{12}^2)^{-2}, \\ \tfrac{\partial u}{\partial t_1} &= 4\,g_1\,g_{12}\,(g_1\,\alpha_1 + g_{12}\,\alpha_2)(g_1^2 + g_{12}^2)^{-2}, \\ \tfrac{\partial v}{\partial t_1} &= -2\,(g_1\,\alpha_1 + g_{12}\,\alpha_2)(g_1^2 - g_{12}^2)(g_1^2 + g_{12}^2)^{2}, \\ \tfrac{\partial^2 x}{\partial t_1^2} &= -16\,g_1\,g_{12}\,(g_1^2 - g_{12}^2)(g_1^2 + g_{12}^2)^{-3}, \\ \tfrac{\partial^2 y}{\partial t_1^2} &= 8\,(g_1^2 - g_{12}^2)^2(g_1^2 + g_{12}^2)^{-3} - 4\,(g_1^2 + g_{12}^2)^{-1}, \\ \tfrac{\partial^2 u}{\partial t_1^2} &= -16\,g_1\,g_{12}\,(g_1\,\alpha_1 + g_{12}\,\alpha_2)(g_1^2 - g_{12}^2)(g_1^2 + g_{12}^2)^{-3}, \\ \tfrac{\partial^2 v}{\partial t_1^2} &= 4\,(g_1\,\alpha_1 + g_{12}\,\alpha_2)(g_1^4 + g_{12}^4 - 6\,g_1^2\,g_{12}^2)(g_1^2 + g_{12}^2)^{-3}, \\ \tfrac{\partial x}{\partial t_2} &= -2\,(g_1^2 - g_{12}^2)(g_1^2 + g_{12}^2)^{-2}, \\ \tfrac{\partial y}{\partial t_2} &= -4\,g_1\,g_{12}\,(g_1^2 + g_{12}^2)^{-2}, \\ \tfrac{\partial u}{\partial t_2} &= -2\,(g_1\,\alpha_1 + g_{12}\,\alpha_2)(g_1^2 - g_{12}^2)(g_1^2 + g_{12}^2)^{-2}, \\ \tfrac{\partial v}{\partial t_2} &= -4\,g_1\,g_{12}\,(g_1\,\alpha_1 + g_{12}\,\alpha_2)(g_1^2 + g_{12}^2)^{-2}, \\ \tfrac{\partial^2 x}{\partial t_2^2} &= 16\,g_1\,g_{12}\,(g_1^2 - g_{12}^2)(g_1^2 + g_{12}^2)^{-3}, \\ \tfrac{\partial^2 y}{\partial t_2^2} &= 32\,g_1^2\,g_{12}^2\,(g_1^2 + g_{12}^2)^{-3} - 4\,(g_1^2 + g_{12}^2)^{-1}, \end{aligned}$$

$$\frac{\partial^2 u}{\partial t_2^2} = 16\, g_1\, g_{12}\,(\, g_1\, \alpha_1 \,+\, g_{12}\, \alpha_2\,)\,(\, g_1^2 \,-\, g_{12}^2\,)\,(\, g_1^2 \,+\, g_{12}^2\,)^{-3},$$

$$\frac{\partial^2 v}{\partial t_2^2} = -\,4\,(\, g_1\, \alpha_1 \,+\, g_{12}\, \alpha_2\,)\,(\, g_1^4 \,+\, g_2^4 \,-\, 6\, g_1\, g_{12}^2\,)\,(\, g_1^2 \,+\, g_{12}^2\,)^{-3}$$

을 얻는다. 그리고 $\tilde{a}\tilde{d} - \tilde{b}\tilde{c} = 1, \quad a_1 a_2 - a_3^2 = 1, g_1 g_2 - g_{12} g_{21} = 1$ 임을 유의하라. 상기의 사실들을 종합하여 아래와 같은 정리를 얻는다. □

정리 2.3. 공식 (2.11)에 의하여 P, ξ, P_1, P_2 에 대응되는 $\mathbb{H} \times \mathbb{C}$ 상의 불변 미분작용소를 각각 D, Ψ, D_1, D_2 라고 하자. 그러면

$$D = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + 2\, y\, v \left(\frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v} \right), \tag{2.8}$$

$$\Psi = y \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right), \tag{2.9}$$

$$D_1 = 2\, y^2 \frac{\partial^3}{\partial x \partial u \partial v} - y^2 \frac{\partial}{\partial y} \left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) + \left(v \frac{\partial}{\partial v} + 1 \right) \Psi, \tag{2.10}$$

$$D_2 = y^2 \frac{\partial}{\partial x} \left(\frac{\partial^2}{\partial v^2} - \frac{\partial^2}{\partial u^2} \right) - 2\, y^2 \frac{\partial^3}{\partial y \partial u \partial v} - v \frac{\partial}{\partial u} \Psi \tag{2.11}$$

이다. 여기서 $\tau = x + iy,\ z = u + iv$ 이고 x, y, u, v 는 실변수이다. 게다가

$$[D, \Psi] = 2\, y^2 \frac{\partial}{\partial y} \left(\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) - 4\, y^2 \frac{\partial^3}{\partial x \partial u \partial v} - 2 \left(v \frac{\partial}{\partial v} \Psi + \Psi \right)$$

인 관계가 성립한다. 따라서 $\mathbb{D}(\mathbb{H} \times \mathbb{C})$ 는 가환대수가 아니다. 또 A. Selberg [28]의 의미에서 weakly symmetric 공간이 아니다.

정리 2.4. $\mathbb{H} \times \mathbb{C}$ 상에 주어지는 Riemann 계량

$$ds^2 = \frac{y + v^2}{y^3} (\, dx^2 + dy^2\,) + \frac{1}{y} (\, du^2 + dv^2\,) - \frac{2v}{y^2} (\, dx\, du + dy\, dv\,)$$

은 G의 작용 (0.2)에 불변인 Kaehler계량이다. Riemann 공간 $(\mathbb{H} \times \mathbb{C}, ds^2)$ 의 라플라스 작용소 □ 는

$$\square = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + (\, y + v^2\,) \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right)$$

$$+2yv\left(\frac{\partial^2}{\partial x\partial u}+\frac{\partial^2}{\partial y\partial v}\right)$$

으로 주어진다. 즉 $\Box = D + \Psi$.

증명. [35] 의 Proposition 2.4 를 참고하길 바란다. □

도움말 2.5. 임의의 두 양수 α, β에 대하여

$$ds^2_{\alpha,\beta} = \alpha\,\frac{dx^2\,+\,dy^2}{y^2} + \beta\,\frac{v^2(dx^2\,+\,dy^2)\,+\,y^2(du^2\,+\,dv^2)\,-\,2\,yv\,(dx\,du\,+\,dy\,dv)}{y^3}$$

는 G의 작용 (0.2)에 불변인 $\mathbb{H}\times\mathbb{C}$ 상의 Kaehler계량이다. Riemann공간$(\mathbb{H}\times\mathbb{C}, ds^2_{\alpha,\beta})$ 의 라플라스 작용소 $\Box_{\alpha,\beta}$ 는

$$\begin{aligned}\Box_{\alpha,\beta} &= \frac{1}{\alpha}\,y^2\,(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})+\left(\frac{y}{\beta}+\frac{v^2}{\alpha}\right)\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)\\ &\quad +\frac{2\,yv}{\alpha}\left(\frac{\partial^2}{\partial x\partial u}+\frac{\partial^2}{\partial y\partial v}\right)\\ &= \frac{1}{\alpha}\,D+\frac{1}{\beta}\,\Psi\end{aligned}$$

으로 주어진다.

도움말 2.6. 지루한 계산으로 $(\mathbb{H}\times\mathbb{C}, ds^2)$ 의 스칼라 곡률이 -3 임을 알수 있다.

$(\mathbb{H}\times\mathbb{C}, ds^2)$ 의 라플라스 작용소 $\Box$의 고유함수를 구하는 문제는 흥미롭다

문제 2.7. $\Box$의 고유함수들을 구하여라.

아래에 $\Box$의 고유함수들을 예로 들겠다

(ㄱ) 함수 $h(x,y) = y^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|a|y)\,e^{2\pi iax}\quad (s\in\mathbb{C},\ a\neq 0)$ 는 고유값이 $s(s-1)$ 인 고유함수이다.
(ㄴ) y^s, y^sx, $y^su\,(s\in\mathbb{C})$ 는 고유값이 $s(s-1)$ 인 고유함수이다.
(ㄷ) y^sv, y^suv, $y^sxv\,(s\in\mathbb{C})$ 는 고유값이 $s(s+1)$ 인 고유함수이다.
(ㄹ) x, y, u, v, xu, uv 는 고유값이 0 인 고유함수이다.
(ㅁ) 모든 마쓰 형식들은 $\Box$의 고유함수이다.

제 3 절 마쓰-야코비 형식

이 절에서는 □는 정리 2.4에서 정의된 $\mathbb{H}\times\mathbb{C}$ 상의 계량 ds^2의 라플라스 작용소를 나타내고 $G = SL_{2,1}(\mathbb{R})$ 이다.

정의 3.1. $\mathbb{H}\times\mathbb{C}$ 상에서 유계이고 매끄러운 함수 $f:\mathbb{H}\times\mathbb{C}\longrightarrow\mathbb{C}$가 아래의 성질 (MJ1)-(MJ3)를 만족할때 함수 f를 *마쓰-야코비* 형식이라고 한다.

(MJ1) 임의의 $\gamma\in\Gamma_{1,2}$에 대하여 $f(\gamma\circ(\tau,z)) = f(\tau,z),\quad (\tau,z)\in\mathbb{H}\times\mathbb{C}$.

(MJ2) f는 □의 고유함수이다.

(MJ3) f는 무한점에서 기껏해야 다항식적인 증가성을 지니고 있다.

복소수 $\lambda\in\mathbb{C}$에 대하여 $\square f = \lambda f$의 성질을 만족하는 마쓰-야코비 형식들로 이루어진 벡터 공간을 $MJ(\Gamma_{1,2},\lambda)$로 표기한다. $\square f = \lambda f$가 타원적 편미분 방정식이므로 f는 실해석적이다.

f가 $MJ(\Gamma_{1,2},\lambda)$의 원소라고 하면

$$\phi_f(g,\alpha) = f((g,\alpha)\circ(i,0)),\quad (g,\alpha)\in G \tag{3.1}$$

으로 정의되는 함수 $\phi_f : G\to\mathbb{C}$는 아래의 성질$(MJ1)_* - (MJ3)_*$를 만족한다.

$(MJ1)_*$ 임의의 $\gamma\in\Gamma_{1,2}$, $x\in\Gamma$, $k\in K$에 대하여 $\phi_f(\gamma xk) = \phi_f(x)$이다.

$(MJ2)_*$ ϕ_f는 (G, ds_0^2)상의 라플라스 작용소 □의 고유함수이다. 단 ds_0^2는 $(\mathbb{H}\times\mathbb{C}, ds^2)$으로부터 유도되는 G상의 불변계량이다.

$(MJ3)_*$ ϕ_f는 적합한 다항식적인 증가성을 지니고 있다. (참고문헌: [5])

오른쪽 K-불변인 함수 $\phi : G\to\mathbb{C}$ (즉, 임의의 $x\in G$, $k\in K$에 대하여 $f_\phi(xk) = \phi(x)$ 인 함수) 가 주어져 있을때 함수 $f_\phi : \mathbb{H}\times\mathbb{C}\to\mathbb{C}$를

$$f_\phi(\tau,z) = \phi(g,\alpha),\quad (\tau,z)\in\mathbb{H}\times\mathbb{C} \tag{3.2}$$

으로 정의한다. 단 (g,α)는 $(g,\alpha)\circ(i,0) = (\tau,z)$ 인 성질을 만족하는 G의 원소이다. ϕ가 G상에서 유계이며 매끄러운 함수이면 f_ϕ가 마쓰-야코비 형식이라는 사실을 쉽게 알 수 있다. 이제 $MJ(\Gamma_{1,2},\lambda)$의 원소인 마쓰-야코비 형식 f를 비대칭공간 $S\mathcal{P}_2\times\mathbb{R}^{(1,2)}$ 상의 함수로 특징화하여 보자. 함수 $h_f : S\mathcal{P}_2\times\mathbb{R}^{(1,2)}\to\mathbb{C}$를

$$h_f(Y,V) = f((g, V\,{}^tg^{-1})\circ(i,0)) \tag{3.3}$$

으로 정의한다. 여기서 $(Y,V)\in S\mathcal{P}_2\times\mathbb{R}^{(1,2)}$ 이고 g는 $Y = g\,{}^tg$ 인 G의 원소이다. 그러면 h_f는 아래의 성질 $(MJ1)^*$-$(MJ3)^*$를 만족한다.

$(MJ1)^*$ 임의의 $\gamma \in SL(2,\mathbb{Z})$, $\delta \in \mathbb{Z}^{(1,2)}$ 에 대하여 $h_f(\gamma Y\, {}^t\gamma, (V+\delta)\, {}^t\gamma) = h_f(Y,V)$ 이다.

$(MJ2)^*$ h_f 는 등질공간$(\mathcal{SP}_2\times\mathbb{R}^{(1,2)}, ds_*^2)$ 상의 라플라스 작용소 $\Box_*$ 의 고유함수이다.

$(MJ3)^*$ h_f 는 적합한 다항식적인 증가성을 지니고 있다.

$(MJ2)^*$ 에서의 ds_*^2 과 $\Box_*$ 은 아래와 같이 주어진다. (Y,X) 가 보조정리 2.1 에서 주어진 좌표계이면

$$ds_*^2 = \frac{1}{y^2}(dx^2+dy^2) + \frac{1}{y}\{(x^2+y^2)\,dv_1 + 2\,x\,dv_1\,dv_2 + dv_2^2\}$$

이고

$$\Box_* = y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) + \frac{1}{y}\left\{\frac{\partial^2}{\partial v_1} - 2x\frac{\partial^2}{\partial v_1\partial v_2} + (x^2+y^2)\frac{\partial^2}{\partial v_2^2}\right\}$$

이다.

도움말 3.2. 마쓰형식은 마쓰-야코비 형식이므로 집합

$$\{\lambda\in\mathbb{C} \mid MJ(\Gamma_{1,2},\lambda)\neq 0\}$$

은 무한집합이다.

정리 3.3. *임의의 복소수 $\lambda\in\mathbb{C}$ 에 대하여 $MJ(\Gamma_{1,2},\lambda)$ 는 유한차원의 벡터공간이다.*

증명. Harish-Chandra 의 강의 노트 [14] 의 정리 1, 8쪽과 [5] 의 191쪽의 결과로부터 상기의 정리를 증명할 수 있다.

f 를 $MJ(\Gamma_{1,2},\lambda)$ 의 원소인 마쓰-야코비 형식이라고 하자. 그러면 f 는

$$f(\tau+n,z) = f(\tau,z), \quad \forall n\in\mathbb{Z} \tag{3.4}$$

와

$$f(\tau,z+n_1\tau+n_2) = f(\tau,z), \quad \forall n_1,n_2\in\mathbb{Z} \tag{3.5}$$

의 변환을 만족한다. $\tau = x+iy,\ z = u+iv\,(x,y,u,v$는 실변수$)$ 으로 두면 f 는 실변수 x 와 u 의 함수로서 주기가 각각 1인 함수이므로

$$f(\tau,z) = \sum_{n,r\in\mathbb{Z}} c_{n,r}(y,v)\,e^{2\pi i(nx+ru)} \tag{3.6}$$

와 같은 푸리에 전개를 얻는다. 주어진 두 정수 n, r 에 대하여 푸리에 계수 $\varphi = c_{n,r}(y,v)$ 에 관하여 논하여 보자. 푸리에 계수 $c_{n,r}(y,0)$ 는 미

분 방정식

(3.7)

$$\left[y^2\frac{\partial^2}{\partial y^2}+(y+v^2)\frac{\partial^2}{\partial v^2}+2\,y\,v\frac{\partial^2}{\partial y\partial v}-\{(ay+bv)^2+b^2y+\lambda\}\right]\varphi=0$$

을 만족한다. 단, $a=2\pi n$, $b=2\pi r$ 인 상수이다. □

도움말 3.4. 함수 $u(y)=y^{1/2}K_{s-\frac{1}{2}}(2\pi|n|y)$ 는 미분방정식 (3.7)의 해이며 $\lambda=s(s-1)$ 임을 알수 있다. 여기서 $K_s(z)$ 는 K-Bessel 함수이다. (참고문헌 : [20], [32])

제 4 절 형식적 Eisenstein급수

이제 (1.4)에서 정의된 Eisenstein 급수 $E_s(\tau)$ 처럼 비대칭 공간 $\mathbb{H}\times\mathbb{C}$ 상에 이와 유사한 급수를 정의하여 마쓰-야코비 형식의 구성에 관한 문제를 논하여 보자.

(4.1)
$$\Gamma_{1,2}^{\infty}:=\left\{\left(\begin{pmatrix}\pm1 & m\\ 0 & \pm1\end{pmatrix},(0,n,\kappa)\right)\;\middle|\; m,n,\kappa\in\mathbb{Z}\right\}$$

를 $\Gamma_{1,2}:=SL_{2,1}(\mathbb{Z})$ 의 부분군이라 하자. 그리고

$$\gamma=\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix},(\lambda,\mu,\kappa)\right)\in\Gamma_{1,2}$$

이고 $(\tau,z)\in\mathbb{H}\times\mathbb{C}$ 일때 $(\tau_\gamma,z_\gamma):=\gamma\circ(\tau,z)$ 이라고 두자. 즉

$$\tau_\gamma=(a\tau+b)(c\tau+d)^{-1},\quad z_\gamma=(z+\lambda\tau+\mu)(c\tau+d)^{-1}$$

이다. $\gamma\in\Gamma_{2,1}^{\infty}$ 이면

(4.2)
$$\operatorname{Im}\tau_\gamma=\operatorname{Im}\tau,\quad \operatorname{Im}z_\gamma=\operatorname{Im}z$$

임을 유의하라. 역으로 $\gamma\in\Gamma_{1,2}$ 이고 (4.2)를 만족하면 γ 는 $\Gamma_{1,2}^{\infty}$ 의 원소이다. 주어진 복소수 $s\in\mathbb{C}$ 에 관하여 형식적 Eisenstein 급수 $E_s(\tau,z)$ 를

(4.3)
$$E_s(\tau,z):=\sum_{\gamma\in\Gamma_{1,2}^{\infty}\backslash\Gamma_{1,2}}(\operatorname{Im}\tau_\gamma)^s\cdot\operatorname{Im}z_\gamma$$

으로 정의한다. 그러면 $E_s(\tau,z)$ 는 형식적으로

(4.4)
$$E_s(\gamma\circ(\tau,z))=E_s(\tau,z),\quad \forall\,\gamma\in\Gamma_{1,2}$$

의 성질과

(4.5)
$$\Box E_s(\tau,z)=s(s+1)E_s(\tau,z)$$

의 성질을 만족한다는 사실을 쉽게 알수 있다. 여기서 □는 정리 2.4에서 정의된 $\mathbb{H}\times\mathbb{C}$ 상의 계량 ds^2 의 라플라스 작용소를 나타내고 있다. 그

러나 임의의 $s \in \mathbb{C}$에 대하여 $E_s(\tau, z)$는 변수 z의 허수부분때문에 수렴하지 않는다 그래서 $E_s(\tau, z)$를 적절하게 변형하여(4.3)과 (4.4)의 성질을 만족하고 다항식적인 증가성을 지니는 급수를 구성할 필요가 있다. 실제로 마쓰 형식인 Eisenstein 급수 $E_s(\tau)$는 말할 것도 없이 마쓰-야코비 형식이지만 변수 $z = u + iv$가 관여하지 않는다. 제 1 절에서 언급하였지만 마쓰 형식중에서도 첨점 형식이 무수히 많다는 것은 알려져 있지만 이의 구체적인 구성은 아직까지도 알려져 있지 않다.

두 변수 τ와 z가 모두 관여하는 마쓰-야코비 형식의 구체적인 구성에 관한 문제를 제기하고 싶다

제 5 절 야코비 군의 표현

3차원 Heisenberg 군

$$H_{\mathbb{R}} := \{\, (\lambda, \mu, \kappa) \mid \lambda, \mu, \kappa \in \mathbb{R} \,\}$$

상의 곱은

$$(\lambda, \mu, \kappa) \circ (\lambda', \mu', \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\mu' - \lambda'\mu)$$

으로 주어진다. 3차원의 특별 선형군 $SL(2, \mathbb{R})$과 Heisenberg 군 $H_{\mathbb{R}}$의 반직접곱 $G^J := SL(2, \mathbb{R}) \ltimes H_{\mathbb{R}}$은 6 차원의 Lie 군이며 야코비 군이라 불린다.

$$H_{\mathbb{Z}} := \{\, (\lambda, \mu, \kappa) \in H_{\mathbb{R}} \mid \lambda, \mu, \kappa \in \mathbb{Z} \,\}$$

이라 두고

$$\Gamma^J := SL(2, \mathbb{Z}) \ltimes H_{\mathbb{Z}}$$

을 G^J의 이산 부분군이라 하자. $Z(G^J)$를 G^J의 중심군(center)이라고 할 때 $K^J = SO(2) \times Z(G^J)$ 이라 하자.

이제 G^J의 유니터리 쌍대(unitary dual)를 상세하게 구하여 보자. 우선 $SL(2, \mathbb{R})$의 유니터리 쌍대를 상기한다. 이 절에서는 기호 편의상 $G = SL(2, \mathbb{R})$ 이라 두자. 그리고

$$M = \{\pm I_2\}, \quad A = \left\{ \begin{pmatrix} |a| & 0 \\ 0 & |a|^{-1} \end{pmatrix} \,\middle|\, a \in \mathbb{R},\ a \neq 0 \right\},$$

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\}, \quad \bar{N} = \left\{ \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \,\middle|\, y \in \mathbb{R} \right\}$$

이라 두자. a가 0 이 아닌 실수이고 $\epsilon = \mathrm{sgn}\,(\mathrm{a})$ 이라 하면

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \begin{pmatrix} |a| & 0 \\ 0 & |a|^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}$$

임을 쉽게 알 수 있다. 이 분해는 유일하게 정해지며 $\bar{N}MAN$ 은 G 의 조밀한 열린 부분분집합이다. 이 절에서 당분간 G 의 원소 g 를

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a,b,c,d \in \mathbb{R}, \quad ad - bc = 1$$

이라 쓰기로 하자. G 의 기약인 유니터리 표현은 아래와 같이 분류된다.

(ㄱ) Φ_n ($n \geq 0$ 인 정수)은 $(n+1)$ 차원의 기약 표현임:

Φ_n 의 표현공간 $\mathcal{F}_n$은 차수가 n 이하인 $\mathbb{C}$ 상의 다항식들의 벡터공간이며 Φ_n 은

$$(\Phi_n(g)f)(z) = (-bz+d)^n f\left(\frac{az-c}{-bz+d}\right), \quad f \in \mathcal{F}_n$$

으로 주어진다.

(ㄴ) 주조성렬 (principal series) $\mathcal{P}^{+,i\alpha}$ 와 $\mathcal{P}^{-,i\alpha}$ $(\alpha \in \mathbb{R})$:

먼저 실수 $\alpha \in \mathbb{R}$ 을 고정시킨다. $\mathcal{P}^{\pm,i\alpha}$ 의 표현공간은 $L^2(\mathbb{R})$ 이며 이 표현은

$$(\mathcal{P}^{\epsilon,i\alpha}(g)f)(x) = \begin{cases} |-bx+d|^{-1-i\alpha} f\left(\frac{az-c}{-bz+d}\right), & \text{if } \epsilon = +, \\ \operatorname{sgn}(-bx+d)\,|-bx+d|^{-1-i\alpha} f\left(\frac{az-c}{-bz+d}\right), & \text{if } \epsilon = - \end{cases}$$

으로 주어진다. $\mathcal{P}^{-,0}$ 를 제외하고는 나머지 $\mathcal{P}^{\epsilon,i\alpha}$ 는 기약이며

$$\mathcal{P}^{+,i\alpha} \cong \mathcal{P}^{+,-i\alpha}, \qquad \mathcal{P}^{-,i\alpha} \cong \mathcal{P}^{-,-i\alpha}$$

이다. 여기서 $\cong$ 는 유니터리 동치 (unitary equivalence)를 나타낸다. 실제로

$$\mathcal{P}^{\pm,i\alpha} = \operatorname{Ind}_{MAN}(\sigma \otimes e^{i\alpha} \otimes 1)$$

이다. 단, $e^{i\alpha}$ 는

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \longmapsto e^{i\alpha t}$$

로 주어지는 A 의 지표이다.

(ㄷ) 보계열 (complementary series) $\mathcal{C}^s$ $(0 < s < 1)$:

$\mathcal{C}^s$ 의 표현공간 $\mathcal{C}(s)$ 는

$$C(s) = \left\{ f : \mathbb{R} \longrightarrow \mathbb{C} \;\middle|\; \|f\|_s^2 := \int_{\mathbb{R}^2} \frac{f(x)\overline{f(y)}}{|x-y|^{1-s}} dxdy < \infty \right\}$$

으로 주어지며

$$(\mathcal{C}^s(g)f) = |-bx+d|^{-1-s} f\left(\frac{az-c}{-bz+d}\right), \quad f \in \mathcal{C}(s)$$

이다. $\mathcal{C}^s$ 는 기약인 유니터리 표현이다.

(ㄹ) 이산표현 (discrete series) $\mathbb{D}_n^+$ 과 $\mathbb{D}_n^-$ $(n \geq 2,\ n \in \mathbb{Z}^+)$:

$n \geq 2$ 인 자연수 n을 고정시키자. $\tau \in \mathbb{H}$에 대하여 $\tau = x + iy$ $(x, y \in \mathbb{R})$ 이라 표기하기로 한다.

$$\|f\|^2 := \int_{\mathbb{R}^2} |f(x+iy)|^2 y^{n-2} dxdy < \infty$$

의 성질을 만족하는 $\mathbb{H}$ 의 해석적 함수 $f : \mathbb{H} \longrightarrow \mathbb{C}$ 들로 이루어진 힐베르트 공간을 $L^2_{n,+}(\mathbb{H})$ 이라 표기한다. $\mathbb{D}_n^+$ 는

$$(\mathbb{D}_n^+(g)f)(z) = (-bx+d)^n f\left(\frac{az-c}{-bz+d}\right), \quad f \in L^2_{n,+}(\mathbb{H})$$

으로 주어진다. $L^2_{n,+}(\mathbb{H})$ 의 복소공액공간을 $L^2_{n,-}(\mathbb{H})$ 이라 표기하면 $\mathbb{D}_n^-$ 는

$$(\mathbb{D}_n^-(g)f)(z) = \overline{(-bx+d)^n} f\left(\frac{az-c}{-bz+d}\right), \quad f \in L^2_{n,-}(\mathbb{H})$$

으로 주어진다.

(ㅁ) $\mathbb{D}_1^+$ 와 $\mathbb{D}_1^-$ (이산표현의 극한):

$L^2_{1,+}(\mathbb{H})$ 은

$$\|f\|^2_\infty = \sup_{y>0} \int_\infty^\infty |f(x+iy)|^2 dx$$

의 성질을 만족하는 $\mathbb{H}$ 상의 해석적 함수들의 벡터공간이며 $\mathbb{D}_1^+$ 의 표현공간이다. $\mathbb{D}_1^+$ 의 표현은 (ㄹ)에서 $n = 1$ 을 대입하여 얻어진다. $L^2_{1,-}(\mathbb{H})$ 는 $L^2_{1,+}(\mathbb{H})$ 의 복소공액공간으로 $\mathbb{D}_1^-$ 의 표현공간이다. $\mathbb{D}_1^-$ 의 표현은 (ㄹ)에서 $n = 1$ 을 대입하여 얻어진다.

(ㅂ) 비유니터리 주조성렬 $\mathcal{P}^{\epsilon,w}$ $(\epsilon = \pm,\ w \in \mathbb{C})$:

복소수 $w \in \mathbb{C}$에 대하여 힐베르트 공간 $L^2(\mathbb{R}, (1+x^2)^{\mathrm{Re}\, w} dx)$ 이 $\mathcal{P}^{\epsilon,w}$ 의 표현공간이다. $\mathcal{P}^{\epsilon,w}$ 는

$$(\mathcal{P}^{\epsilon,w}(g)f)(x) = \begin{cases} |-bx+d|^{1-w} f\left(\frac{az-c}{-bz+d}\right), & \text{if } \epsilon = +, \\ \mathrm{sgn}(-bx+d)\,|-bx+d|^{1-w} f\left(\frac{az-c}{-bz+d}\right), & \text{if } \epsilon = - \end{cases}$$

으로 주어진다. $w \notin i\mathbb{R}$ 이면 $\mathcal{P}^{\epsilon,w}$ 는 유니터리 표현이 아니다. 만약에 $w \in (0,1)$ 이면 노옴을 변형한 후에 $\mathcal{P}^{\epsilon,w}$ 를 유니터리 표현으로 만들 수 있다.

(ㅅ) 자명한 표현

G의 유니터리 쌍대에 관한 보다 자세한 내용은 [8], [19] 을 참고하길 바란다.

Heisenberg 군 $H_{\mathbb{R}}$ 의 Schrödinger 표현 $U_m\,(m \in \mathbb{R})$ 은

$$(U_m(\lambda,\mu,\kappa)f)(x) = e^{2\pi i m\{\kappa+(2x+\lambda)\mu\}}f(x+\lambda), \quad f \in L^2(\mathbb{R})$$

으로 주어진다. 상세한 것은 논문 [34]의 (2.18), 313 쪽을 참고하길 바란다. U_m 은 이의 중심지표(central character)가 $\sigma_m(\kappa) := e^{2\pi i m\kappa}\,(\kappa \in \mathbb{R})$ 으로 주어지는 $H_{\mathbb{R}}$ 의 기약 유니터리 표현이다. G 는 $H_{\mathbb{R}}$ 상에서

$$g \star (\lambda,\mu,\kappa) := g(\lambda,\mu,\kappa)g^{-1} = ((\lambda,\mu)g^{-1},\kappa), \quad g \in G$$

와 같이 작용한다. 특히, 임의의 $\kappa \in \mathbb{R}$ 에 대하여 $g \star (0,0,\kappa) = (0,0,\kappa)$ 임을 유의한다. 임의의 실수 $m \in \mathbb{R}$ 과 $g \in G$ 에 대하여 $U_m^{[g]}$ 를

$$U_m^{[g]}(h) := U_m(ghg^{-1}), \quad h \in H_{\mathbb{R}}$$

으로 정의한다. 그러면, U_m 과 $U_m^{[g]}$ 는 동일한 중심지표 σ_m 을 지니므로 Stone-von Neumann 정리에 의하여 $U_m \cong U_m^{[g]}$ 이다. 다시 말하면 유니터리 선형 가역연산자 $\Phi_{W,m}(g) : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ 가 존재하여

$$U_m^{[g]}(h) = \Phi_{W,m}(g)U_m(h)\Phi_{W,m}(g)^{-1}, \quad h \in H_{\mathbb{R}}$$

인 관계를 만족한다. Schur 의 보조정리에 의하여

$$\Phi_{W,m}(g_1g_2) = c_m(g_1,g_2)\Phi_{W,m}(g_1)\Phi_{W,m}(g_2), \quad g_1, g_2 \in G$$

의 성질을 만족시키는 사상 $c_m : G \times G \longrightarrow U(1)$ 이 존재한다. 여기서 $U(1) := \{\, z \in \mathbb{C} \mid |z| = 1 \,\}$ 이다. 실은 사상 c_m 은 cocycle 조건을 만족한다. $\tilde{G}$ 를 승법인자(multiplier) c_m 을 지니는 metaplectic 군이라 하자. 즉, 집합인 관점에서는 $\tilde{G} = G \times \{\pm 1\}$ 이고 $\tilde{G}$ 상의 곱은

$$(g,\epsilon)\cdot(g',\epsilon') := (gg', c_m(g,g')\epsilon\epsilon')$$

으로 주어진다. $L^2(\mathbb{R})$ 상에서의 $\tilde{G}$ 의 표현 $\pi_W^{[m]} : \tilde{G} \longrightarrow GL(L^2(\mathbb{R}))$ 을

$$\pi_W^{[m]}((g,\epsilon)) := \Phi_{W,m}(g)\epsilon, \quad (g,\epsilon) \in \tilde{G}$$

으로 정의한다. $\pi_W^{[m]}$ 을 G 의 *Weil* 표현이라고 일컫는다. $\pi_W^{[m]}$ 은

$$\pi_W^{[m]} = \pi_W^{[m],+} \oplus \pi_W^{[m],-}$$

으로 분해된다. 여기서, $\pi_W^{[m],+}$ 와 $\pi_W^{[m],-}$ 은 $\tilde{G}$ 의 기약표현이다.

$\pi_{SW}^{[m]} : G^J \longrightarrow GL(L^2(\mathbb{R}))$ 을

$$\pi_{SW}^{[m]}(hg) := U_m(h)\pi_W^{[m]}(g), \quad g \in G,\ h \in H_{\mathbb{R}}$$

으로 정의한다. 그러면 $\pi_{SW}^{[m]}$ 은 G^J 의 사영표현(projective representation)이 되며 자연스럽게 $\tilde{G}^J := \tilde{G} \ltimes H_{\mathbb{R}}$ 의 표현으로 확장된다는 사실

을 알 수 있다. 그래서 $\pi_{SW}^{[m]}$ 을 야코비 군 G^J 의 *Schrödinger-Weil* 표현이라 일컫는다.

변수 x, y, z, p, q, r 에 대하여

$$G(x,y,z,p,q,r) := \begin{pmatrix} x & 0 & y+z & q \\ p & 0 & q & r \\ y-z & 0 & -x & -p \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

이라 두자. 그리고

$$\begin{aligned} \hat{Z} &= -i\,G(0,0,1,0,0,0), \\ \hat{Z}_0 &= -i\,G(0,0,0,0,0,1), \\ X_+ &= \tfrac{1}{2}\,G(1,i,0,0,0,0), \\ X_- &= \tfrac{1}{2}\,G(1,-i,0,0,0,0), \\ Y_+ &= \tfrac{1}{2}\,G(0,0,0,1,i,0), \\ Y_- &= \tfrac{1}{2}\,G(0,0,0,1,-i,0). \end{aligned}$$

이라 놓자. G^J 의 Lie 대수 $\mathfrak{g}^J$ 의 복소화를 $\mathfrak{g}^J_{\mathbb{C}} = \mathfrak{g}^J \otimes \mathbb{C}$ 이라 표기한다. $\hat{Z}$, $\hat{Z}_0$, $X_\pm$, $Y_\pm$ 은 $\mathfrak{g}^J_{\mathbb{C}}$ 의 원소이며

$$[\hat{Z}_0, \mathfrak{g}^J_{\mathbb{C}}] = 0, \quad [\hat{Z}, Y_\pm] = \pm Y_\pm, \quad [\hat{Z}, X_\pm] = \pm 2X_\pm$$

와 같은 교환관계를 만족한다. 그러므로 (π, V) 가 $\mathfrak{g}^J_{\mathbb{C}}$ 의 기약표현이면 이의 표현공간 V 는 $V = \sum_{k\in\mathbb{Z}} V_k$ 으로 분해되며

$$\pi(\hat{Z}_0)V_k = \mu V_k, \quad \pi(\hat{Z})V_k = \rho_k V_k, \quad \pi(Y_\pm)V_k \subseteq V_{k\pm1}, \quad \pi(X_\pm)V_k \subseteq V_{k\pm2}$$

의 성질을 만족한다. 여기서 μ 와 ρ_k 는 표현 π에 의하여 결정되는 복소수이다.

Berndt 와 Schmidt 는 [4]에서 $m > 0$ 일 때 $\pi_{SW}^{[m]}$ 의 무한소 표현(infinitesimal representation)은 벡터공간 $V^+ = \sum_{j=0}^{\infty} \mathbb{C}\, v_j$ 상에서 아래와 같이 작용하는 $\mathfrak{g}^J_{\mathbb{C}}$ 의 최저 무게표현(lowest weight representation)임을 보였다. 구체적으로 기술하면

$$\hat{Z}_0 v_j = \mu v_j, \quad Y_+ v_j = v_{j+1}, \quad Y_- v_j = -\mu j v_{j-1},$$

$$\hat{Z} v_j = (j + \frac{1}{2})\, v_j, \quad X_+ v_j = -\frac{1}{2\mu}\, v_{j+2}, \quad X_- v_j = \frac{\mu}{2}\, j(j-1) v_{j-2}$$

이다. 여기서 $\mu = 2\pi m$ 이고 $v_{-1} = v_{-2} = 0$ 이다. $m < 0$ 일 때 $\pi_{SW}^{[m]}$ 의 무한소 표현은 벡터공간 $V^- = \sum_{j=0}^{\infty} \mathbb{C}\, v_{-j}$ 상에서 다음과 같이 작용하는 $\mathfrak{g}^J_{\mathbb{C}}$ 의 최고 무게표현임을 보였다. 구체적으로 적으면

$$\hat{Z}_0 v_{-j} = \mu v_{-j}, \quad Y_- v_{-j} = v_{-(j+1)}, \quad Y_+ v_{-j} = -\mu j v_{-(j-1)},$$

$$\hat{Z}v_{-j} = -(j+\frac{1}{2})\,v_{-j},\quad X_{-}v_{-j} = \frac{1}{2\mu}\,v_{-(j+2)},\quad X_{+}v_{-j} = -\frac{\mu}{2}\,j(j-1)v_{-(j-2)}$$

이다. 단, $v_1 = v_2 = 0$ 이다.

(ㅇ) 주조성렬 $\pi_{\alpha,\nu}\,(\alpha \in \mathbb{C} \setminus \{\mathbb{Z}+\frac{1}{2}\},\ \nu = \frac{1}{2})$:

$\pi_{\alpha,\nu}$ 의 무한소 표현은 이의 표현공간 $W_{\alpha,\nu} = \sum_{j\in 2\mathbb{Z}+\nu+\frac{1}{2}} \mathbb{C}\,w_j$ 안에서

$$\hat{Z}w_l = \left(l-\frac{1}{2}\right)w_l,\quad X_{\pm}w_l = \frac{1}{2}\left(\alpha+1\pm\left(l-\frac{1}{2}\right)\right)w_{l\pm 2}$$

의 조건에 의하여 결정된다.

(ㅈ) 이산표현 $\pi^{\pm}_{k_0}\,(k_0 \in \mathbb{Z}+\frac{1}{2})$:

$\pi^{\pm}_{k_0}$ 의 무한소 표현은 이의 표현공간 $W^{\pm}_{k_0} = \sum_{l\in 2\mathbb{Z}^{+}\cup\{0\}} \mathbb{C}\,w_{\pm l}$ 안에서

$$\hat{Z}w_{\pm l} = \pm(k_0+l)\,w_{\pm l},$$
$$X_{\pm}w_{\pm l} = w_{\pm(l+2)},$$
$$X_{\mp}w_{\pm l} = -\frac{l}{2}\left(k_0+\frac{l}{2}-1\right)w_{\pm(l-2)}$$

의 조건에 의하여 결정된다. Berndt와 Schmidt [4]는 G^J 의 기약인 유니터리 표현을 아래와 같이 구체적으로 분류를 하였다.

(J1) $\pi|_G$ (π 의 G 상으로의 제한)이 G 의 기약인 유니터리 표현이고 $\pi|_{H_{\mathbb{R}}}$ 은 $H_{\mathbb{R}}$ 의 자명한 표현인 G^J 의 표현 π:

(J2) 유도표현 $\mathrm{Ind}^{G^J}_{G^J_{\psi}}\tau$: 여기서 $\psi : H_{\mathbb{R}} \longrightarrow U(1)$ 는 $\psi(\lambda,\mu,\kappa) = e^{2\pi i\lambda}$ 으로 정의되는 $H_{\mathbb{R}}$ 의 지표이고 G^J_{ψ} 는

$$G^J_{\psi} = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} h \,\middle|\, c \in \mathbb{R},\ h \in H_{\mathbb{R}} \right\}$$

으로 주어지는 G^J 의 부분군이다.

(J3) G^J 의 주조성렬:

$$\pi_{m,\alpha,\nu} = \pi^{[m]}_{SW} \otimes \pi_{\alpha,\nu},\quad m \in \mathbb{R}^{\times},\ \alpha \in i\mathbb{R},\ \nu = \pm\frac{1}{2}.$$

(J4) G^J 의 보계열:

$$\pi_{m,\alpha,\nu} = \pi^{[m]}_{SW} \otimes \pi_{\alpha,\nu},\quad m \in \mathbb{R}^{\times},\ \alpha \in \mathbb{R},\ \alpha^2 < \frac{1}{4},\ \nu = \pm\frac{1}{2}.$$

(J5) G^J 의 양부호 이산표현 :

$$\pi^+_{m,k} = \pi^{[m]}_{SW} \otimes \pi^+_{k-\frac{1}{2}}, \quad m \in \mathbb{R}^\times,\ k \in \mathbb{Z}^+.$$

(J6) G^J 의 음부호 이산표현 :

$$\pi^-_{m,k} = \pi^{[m]}_{SW} \otimes \pi^-_{k-\frac{1}{2}}, \quad m \in \mathbb{R}^\times,\ k \in \mathbb{Z}^+.$$

상기의 표현들 사이에서

$$\pi_{m,\alpha,\nu} \cong \pi_{m,-\alpha,\nu}$$

만이 유니터리 동치관계가 있고 나머지는 서로가 모두 유니터리 동치가 아니다.

이제 부터는 기호 편의상, $\Gamma = SL(2,\mathbb{Z})$ 이라 두자. R 을 힐베르트 공간 $L^2(\Gamma\backslash G)$ 상의 G 의 오른쪽 정칙표현이라 하자. 다시 말하면

$$R(g)f(x) = f(xg), \qquad g \in G,\ x \in \Gamma\backslash G,\ f \in L^2(\Gamma\backslash G)$$

이다. 그러면 R 은

$$L^2(\Gamma\backslash G) = L^2_{\text{cusp}}(\Gamma\backslash G) \oplus L^2_{\text{res}}(\Gamma\backslash G) \oplus L^2_{\text{cont}}(\Gamma\backslash G)$$

와 같이 분해된다. 여기서 $L^2_{\text{cusp}}(\Gamma\backslash G)$ 은 R 의 첨점적 부분이며, $L^2_{\text{res}}(\Gamma\backslash G)$ 은 R 의 잉여부분이고 은 $L^2_{\text{cont}}(\Gamma\backslash G)$ 는 R 의 연속부분이다. 이에 관한 상세한 설명은 참고문헌 [18], [19]에 기술되어 있으므로 참고하길 바란다.

R^J 를 힐베르트 공간 $L^2(\Gamma^J\backslash G^J)$ 상의 G^J 의 오른쪽 정칙표현이라 하자. 그러면 R^J 는

$$L^2(\Gamma^J\backslash G^J) = \left(\bigoplus_{m,n\in\mathbb{Z}} \mathcal{H}_{m,n} \right) \bigoplus \left(\bigoplus_{\nu=\pm\frac{1}{2}} \int_{\substack{\text{Re}\, s=0 \\ \text{Im}\, s>0}} \mathcal{H}_{m,s,\nu}\, ds \right)$$

와 같이 분해된다는 사실이 Berndt 에 의하여 증명되었다. (참고문헌 : [2], [3], [4]) 여기서 $\mathcal{H}_{m,n}$ 은 이산표현 $\pi^\pm_{m,k}$ 또는 주조성렬 $\pi_{m,s,\nu}$ 와 동치인 표현공간이며 $\mathcal{H}_{m,s,\nu}$ 는 주조성렬 $\pi_{m,s,\nu}$ 의 표현공간이다. $\mathbb{H}\times\mathbb{C} = K^J\backslash G^J$ 이므로 힐베르트 공간 $L^2(\Gamma_{1,2}\backslash(\mathbb{H}\times\mathbb{C}))$ 의 원소는 $L^2(\Gamma^J\backslash G^J)$ 안에 있는 K^J-고정 (K^J-fixed) 원소이다. 그러므로 $L^2(\Gamma_{1,2}\backslash(\mathbb{H}\times\mathbb{C}))$ 에서 라플라스 작용소 $\square$에 관한 스펙트럴 분해를 얻는다.

제 6 절 끝맺음 말

마지막 절에서는 마쓰-야코비 형식의 연구와 관련된 여러 문제들을 제시하겠다.

문제 1. 정리 2.4에서 언급된 $\mathbb{H}\times\mathbb{C}$ 상에서 라플라스 작용소 $\square$의 고유함수들을 구체적으로 구하여라. 그리고 $\square$의 스펙트럼에 관하여 연구 및 조사하여라.

문제 2. 마쓰-야코비 형식을 구체적으로 구성하여라. 특히 첨점 마쓰-야코비 형식의 구성법에 관하여 연구하여라. 제 1 절에서 언급하였지만 심지어 첨점 마쓰 형식의 구체적인 구성법도 아직 알려져 있지 않다.

문제 3. 마쓰-야코비 형식의 푸리에 계수를 구체적으로 기술하여라. 다시 말하면, 미분방정식 (3.7)의 해를 구하여라.

문제 4. 야코비 군의 대각합 공식을 구체적으로 기술하고 연구하여라.

문제 5. 야코비 군의 표현과 쌍대 수반궤적(coadjoint orbit)과의 연관성을 여러 각도에서 상세하게 연구하여라. 이 문제와 관련된 참고문헌으로 [36], [37]을 추천한다.

문제 6. 마쓰-야코비 형식 상에서 Hecke 작용소의 이론을 전개하여라.

문제 7. Selberg의 고유값 가설을 $\square$의 스펙트럼과 연관하여 연구 및 조사하여라. 이 문제는 Ramanujan 가설과 아주 밀접하게 연관되어 있다.

$$\mathbb{H}_n = \left\{ \Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega,\ \operatorname{Im}\Omega > 0 \right\}$$

을 Siegel 상반평면이라 하자. 심플렉틱 군 $Sp(n,\mathbb{R})$는 $\mathbb{H}_n$ 상에서

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \langle\Omega\rangle = (A\Omega + B)(C\Omega + D)^{-1} \tag{6.1}$$

와 같이 추이적으로 작용한다. 여기서, A, B는

$${}^tAC = {}^tCA, \quad {}^tBD = {}^tDB, \quad {}^tAD - {}^tCB = I_n \tag{6.2}$$

의 성질을 만족하는 $n\times n$ 실행렬이다.

자연수 m과 n에 대하여

$$H_{\mathbb{R}}^{(n,m)} = \{ (\lambda,\mu,\kappa) \mid \lambda,\mu \in \mathbb{R}^{(m,n)},\ \ \kappa + \mu\,{}^t\lambda \text{ symmetric} \}$$

을 Heisenberg 군이라 하자. 이 Heisenberg 군에서의 곱은

$$(\lambda,\mu,\kappa)\circ(\lambda',\mu',\kappa')=(\lambda+\lambda',\mu+\mu',\kappa+\kappa'+\lambda\,{}^t\mu'-\mu\,{}^t\lambda')$$

으로 주어진다. $Sp(n,\mathbb{R})$ 과 $H_{\mathbb{R}}^{(n,m)}$ 의 반직접곱인 야코비 군

$$G^J_{n,m}=Sp(n,\mathbb{R})\ltimes H_{\mathbb{R}}^{(n,m)}$$

은 등질공간 $\mathbb{H}_n\times\mathbb{C}^{(m,n)}$ 상에서

$$(6.3)\qquad (M,(\lambda,\mu,\kappa))\cdot(\Omega,Z)=(M\langle\Omega\rangle,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}),$$

와 같이 추이적으로 작용한다. 여기서, $M=\begin{pmatrix}A&B\\C&D\end{pmatrix}$ 은 (6.2) 의 조건을 만족하는 $Sp(n,\mathbb{R})$ 의 원소이다. 논문 [36] 은 야코비 군 $G^J_{n,m}$ 의 표현에 관하여 부분적으로 다루고 있다.

$\Omega\in\mathbb{H}_n$ 의 좌표를 $\Omega=(\omega_{\mu\nu})$, $\omega_{\mu\nu}=x_{\mu\nu}+iy_{\mu\nu}$ ($x_{\mu\nu}$, $y_{\mu\nu}$ 은 실수) 이라 놓자. 그리고

$$\begin{aligned}\Omega&=X+iY, & X&=(x_{\mu\nu}), & Y&=(y_{\mu\nu})\ \text{real},\\ d\Omega&=(d\omega_{\mu\nu}), & dX&=(dx_{\mu\nu}), & dY&=(dy_{\mu\nu}),\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial\Omega}&=\left(\tfrac{1+\delta_{\mu\nu}}{2}\,\tfrac{\partial}{\partial\omega_{\mu\nu}}\right), & \frac{\partial}{\partial\overline{\Omega}}&=\left(\tfrac{1+\delta_{\mu\nu}}{2}\,\tfrac{\partial}{\partial\overline{\omega}_{\mu\nu}}\right),\\ \frac{\partial}{\partial X}&=\left(\tfrac{1+\delta_{\mu\nu}}{2}\,\tfrac{\partial}{\partial x_{\mu\nu}}\right), & \frac{\partial}{\partial Y}&=\left(\tfrac{1+\delta_{\mu\nu}}{2}\,\tfrac{\partial}{\partial y_{\mu\nu}}\right),\end{aligned}$$

이라 두자. 그러면 아래의 n 개의 미분작용소

$$\mathbb{D}_k:=4\,\sigma\left(\left(Y\frac{\partial}{\partial\Omega}\,Y\frac{\partial}{\partial\overline{\Omega}}\right)^k\right),\quad k=1,2,\cdots,n$$

은 작용 (6.1) 에 불변인 미분작용소들의 대수를 생성할 뿐만 아니라 대수적으로 독립이다. 또한

$$ds_n^2=\sigma\left(Y^{-1}d\Omega\,Y^{-1}d\overline{\Omega}\right)$$

은 작용 (6.1) 에 불변인 Riemann 계량이고

$$\Delta_n=4\,\sigma\left(Y\frac{\partial}{\partial\Omega}\,Y\frac{\partial}{\partial\overline{\Omega}}\right)$$

은 계량 ds_n^2 의 라플라스 연산자임을 유의하여라. 본인은 논문 [38] 에서 작용 (6.3) 에 불변인 Riemann 계량과 이의 라플라스 연산자를 구체적으로 계산하였다. 또한 작용 (6.3) 에 불변인 $\mathbb{H}_n\times\mathbb{C}^{(m,n)}$ 상의 미분작용소를 계산하였다.

기호 편의상

$$\Gamma_n := Sp(n,\mathbb{Z}), \quad \Gamma_{n,m} := Sp(n,\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(n,m)}, \quad \mathbb{H}_{n,m} := \mathbb{H}_n \times \mathbb{C}^{(m,n)}$$

이라 두자. 그러면 $\Gamma_{n,m}\backslash\mathbb{H}_{n,m}$ 은 Siegel 모듈라 다양체 $\Gamma_n\backslash\mathbb{H}_n$ 상의 화이버 속 (fiber bundle)이며 이의 화이버는 mn 차원의 아벨다양체(abelian variety)임을 유의하라.

문제 8. 작용 (6.3) 에 불변인 $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ 상의 미분작용소들의 대수를 구체적으로 계산하여라. 그리고 이 대수의 생성원(generators)과 이 생성원들의 대수적 관계를 구하여라.

$m = n = 1$ 인 특별한 경우를 생각하여 보자. 그러면 $\Gamma_{1,2}\backslash(\mathbb{H} \times \mathbb{C})$ 는 $\Gamma_1\backslash\mathbb{H}$ 상의 화이버 속이며 이의 화이버는 타원곡선이다. 가령, $[\tau] \in \Gamma_1\backslash\mathbb{H}$ 이면 $[\tau]$ 상의 화이버는 $E_\tau := \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$ 이다.

문제 9. $L^2(\Gamma_{1,2}\backslash(\mathbb{H}\times\mathbb{C}))$ 의 스펙트럴 분해를 구체적으로 구하고 $L^2(\Gamma\backslash\mathbb{H})$ 의 스펙트럴 분해와의 연관성을 연구 및 조사하여라.

참고 문헌

[1] R. Berndt, *On Automorphic Forms for the Jacobi Group*, Jb.d. Math.-Verein. **97** (1995), 1–18.

[2] ———, *The Continuous Part of $L^2(\Gamma^J\backslash G^J)$ for the Jacobi Group G^J*, Abh. Math. Sem. Univ. Hamburg **60** (1990), 225–248.

[3] R. Berndt and S. Böcherer, *Jacobi Forms and Discrete Series Representations of the Jacobi Group*, Math. Z. **204** (1990), 13–44.

[4] R. Berndt and R. Schmidt, *Elements of the Representation Theory of the Jacobi Group*, Birkhäuser **163** (1998).

[5] A. Borel and H. Jacquet, *Automorphic forms and automorphic representations*, Proc. Symposia in Pure Math. XXXIII, Part 1 (1979), 189–202.

[6] R. W. Bruggeman, *Fourier coefficient of cusp forms*, Invent. Math. **45** (1978), 1–18.

[7] D. Bump, *Automorphic Forms and Representations*, Cambridge University Press, 1997.

[8] R. W. Donley *Irreducible Representations of $SL(2,\mathbb{R})$*, Proceedings of Symposia in Pure Mathematics on Representation Theory and Automorphic Forms, American Math. Soc. **61** (1997), 51–59.

[9] H. Donnelly, *On the cuspidal spectrum for finite volume symmetric spaces*,J. Diff. Geometry **17** (1982), 239–253.

[10] I. Efrat, *Cusp forms and higher rank*, Thesis, New York Univerity, 1983.

[11] P. R. Garabedian, *Partial Differential Equations*, Wiley, New York, 1964.

[12] S. Gelbart, *Automorphic forms on adele groups*, Annals of Math. Studies, Princeton Univ. Press, No. 83, 1975.

[13] S. Gelbart and H. Jacquet, *A relation between automorphic representations on $GL(2)$ and $GL(3)$*, Ann. Ecole Norm. Sup. **11** (1978), 471–552.

[14] Harish-Chandra, *Automorphic forms on semi-simple Lie groups*, Notes by J.G.M. Mars, Lecture Notes in Math., vol. 62, Springer-Verlag, Berlin-Heidelberg-New York, 1968.
[15] S. Helgason, *Differential operators on homogeneous spaces*, Acta Math. **102** (1959), 239–299.
[16] ———, *Groups and geometric analysis*, Academic Press, 1984.
[17] H. Iwaniec, *Introduction to the spectral theory of automorphic forms*, Biblioteca de la Revista Mathemática Iberoamericana, Madrid, 1995.
[18] T. Kubota, *Elementary Theory of Eisenstein Series*, John Wiley and Sons, New York, 1973.
[19] S. Lang, $SL_2(\mathbb{R})$, Springer-Verlag, 1985.
[20] N. N. Lebedev, *Special Functions and their Applications*, Dover, New York, 1972.
[21] J. B. Lewis, *Spaces of holomorphic functions equivalent to the even Maass cusp forms*, Invent. Math. **127** (1997), 271–306.
[22] W. Luo, Z. Rudnick and P. Sarnak, *On Selberg's eigenvalue conjecture*, Geom. Funct. Anal. **5, no. 2** (1995), 387–401.
[23] ———, *On the generalized Ramanujan eigenvalue conjecture for $GL(n)$*, Automorphic forms,automorphic representations and arithmetic, Proc. Sympos. Pure Math., Ameri. Math. Soc. **66, Part 2** (1999), 301–310.
[24] H. Maass, *Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichlescher Reihen durch Funktionalgleichung*, Math. Ann. **121** (1949), 141–183.
[25] G. Mackey, *Unitary Representations of Group Extensions 1*, Acta Math. **99** (1958), 265–311.
[26] P. Sarnak, *On cusp forms*, Contemp. Math., Ameri. Math. Soc. **53** (1986), 393–407.
[27] ———, *Selberg's Eigenvalue Conjecture*, Notices of the Americam Math. Soc. **42** (1995), no. 11, 1272–1277.
[28] A. Selberg, *Hamonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. **20** (1956), 47–87.
[29] ———, *On the estimation of Fourier coefficients of modular forms*, Proc. Symos. Pure Math., American Math. Soc., vol. III (1965), 1–15.
[30] A. Terras, *Harmonic analysis on symmetric spaces and applications I*, Springer-Verlag, 1985.
[31] A. Venkov, *Approximation of Maass forms by analytic modular forms (Russian)*, Algebra i Analiz **6**, no. 6 (1994), 51–64 ; translation in St. Petersburg Math. J. **6** (1995), no. 6, 1167–1177.
[32] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, London, 1962.
[33] J.-H. Yang, *Invariant differential operators on $\mathcal{SP}_2 \times \mathbb{R}^{(1,2)}$*, MPI 97-17, Max-Planck Institut für Mathematik (Bonn), 1997.
[34] ———, *Lattice representations of Heisenberg groups*, Math. Ann. **317** (2000), 309–323.
[35] ———, *Maass-Jacobi forms*, submitted for publication, 2001.
[36] ———, *The Method of Orbits for Real Lie Groups*, Kyungpook Math. Journal **42** (2002), no. 2, 199–272.

[37] ———, *The orbit method for the Jacobi group*, preprint, 2003.
[38] ———, *Invariant metrics on the Siegel-Jacobi space*, preprint, 2003.
[39] P. Zograf, *A spectral proof of Rademacher's conjecture for congruence subgroups of the modular group*, J. reine und angewandte Math. **414** (1991), 113–116.

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A Note on Maass-Jacobi Forms II

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Abstract. This article is a continuation of the paper [21]. In this paper we deal with Maass-Jacobi forms on the Siegel-Jacobi space $\mathbb{H} \times \mathbb{C}^m$, where $\mathbb{H}$ denotes the Poincaré upper half plane and m is any positive integer.

1. Introduction

This article is a continuation of the paper [21]. Recently A. Pitale [14], K. Bringmann and O. Richter [4], and C. Conley and M. Raum [5] defined another notion of Maass-Jacobi forms and studied some properties of Maass-Jacobi forms. In [4], [14] and [21], the authors considered the case $n = m = 1$ and in [5], the authors dealt with the case $n = 1$ and m is arbitrary. In this paper, we consider mainly the case $n = 1$ and m is an arbitrary positive integer.

This paper is organized as follows. In Section 2, we give some useful geometric properties of the Siegel-Jacobi space $\mathbb{H} \times \mathbb{C}^m$. We study the invariant metrics, their Laplacians, a fundamental domain, geodesics, the scalar curvature and invariant differential forms on $\mathbb{H} \times \mathbb{C}^m$. In Section 3 we describe the center of the universal enveloping algebra of the complexfied Jacobi Lie algebra. This work is due to Conley and Raum [5]. In Section 4, we present some interesting and important results on invariant differential operators on the Siegel-Jacobi space $\mathbb{H} \times \mathbb{C}^m$. In Section 5, we discuss the notion of Maass-Jacobi forms introduced by J.-H. Yang [21]. Maass-Jacobi forms play an important role in the spectral theory of the Laplace operator on a fundamental domain for the Siegel-Jacobi space $\mathbb{H} \times \mathbb{C}^m$. In Section 6, we discuss the notion of Maass-Jacobi forms introduced by A. Pitale [14], Bringman-Richter [4] and Conley-Raum [5]. We describe the results obtained in [4] and [5]. More precisely the authors of [4] and [5] obtained an explicit Fourier expansion of

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the Poincaré series that is an example of harmonic Maass-Jacobi form. In Section 7, we discuss skew-holomorphic Jacobi forms introduced by N.-P. Skoruppa [18]. We describe the relation between cuspidal harmonic Maass-Jacobi forms and cuspidal skew-holomorphic Jacobi forms via the lowering operator $D_-^{(\mathcal{M})}$ (cf. (7.3)) In Section 8, we briefly review some results on covariant differential operators on the Siegel-Jacobi space $\mathbb{H}\times\mathbb{C}^m$ obtained by Conley and Raum [5]. In the final section we briefly mention two notions of Maass-Jacobi forms on the Siegel-Jacobi space $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ for the general case $n > 1$ and $m > 1$. Here $\mathbb{H}_n$ denotes the Siegel upper half plane of degree n. We present some natural problems related to the study of Maass-Jacobi forms.

Notations: We denote by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers respectively. $\mathbb{R}^\times$ denotes the set of all nonzero real numbers. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\mathrm{tr}(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. For a complex matrix A, $\overline{A}$ denotes the complex *conjugate* of A. For $A \in \mathbb{C}^{(k,l)}$ and $B \in \mathbb{C}^{(k,k)}$, we use the abbreviation $B\{A\} = {}^t\overline{A}BA$. For a positive integer n, I_n denotes the identity matrix of degree n. For a positive integer m and a commutative ring F, we denote by $S(m, F)$ the space of all $m \times m$ symmetric matrices with entries in F. For a complex number z, $|z|$ denotes the absolute value of z. For a complex number z, $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ denote the real part of z and the imaginary part of z respectively.

2. Geometric properties of the Siegel-Jacobi space $\mathbb{H} \times \mathbb{C}^m$

We fix a positive integer m throughout this paper and let

$$\mathbb{H} = \{\, \tau \in \mathbb{C} \mid \mathrm{Im}\, \tau > 0 \,\}$$

be the Poincaré upper half plane. Let $G = SL_2(\mathbb{R})$ be the special linear group of degree 2 and let

$$H_{\mathbb{R}}^{(m)} = \{\, (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^m,\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\, {}^t\lambda \text{ symmetric} \,\}$$

be the Heisenberg group endowed with the following multiplication law

$$\big(\lambda, \mu; \kappa\big) \circ \big(\lambda', \mu'; \kappa'\big) = \big(\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda'\big)$$

with $\big(\lambda, \mu; \kappa\big), \big(\lambda', \mu'; \kappa'\big) \in H_{\mathbb{R}}^{(m)}$. We define the semidirect product of $SL_2(\mathbb{R})$ and $H_{\mathbb{R}}^{(m)}$

$$G^J = SL_2(\mathbb{R}) \ltimes H_{\mathbb{R}}^{(m)}$$

endowed with the following multiplication law

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot\big(M',(\lambda',\mu';\kappa')\big) = \big(MM',(\tilde\lambda+\lambda',\tilde\mu+\mu';\kappa+\kappa'+\tilde\lambda\,{}^t\mu'-\tilde\mu\,{}^t\lambda')\big)$$

with $M, M' \in SL_2(\mathbb{R}), (\lambda,\mu;\kappa),\ (\lambda',\mu';\kappa') \in H_{\mathbb{R}}^{(m)}$ and $(\tilde\lambda,\tilde\mu)=(\lambda,\mu)M'$. Then G^J acts on the Siegel-Jacobi space $\mathbb{H}\times\mathbb{C}^m$ of degree 1 and index m transitively by

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot(\tau,z) = \Big((a\tau+b)(c\tau+d)^{-1},(z+\lambda\tau+\mu)(c\tau+d)^{-1}\Big), \tag{2.1}$$

where $M=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{R}),\ (\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(m)}$, $\tau\in\mathbb{H}$ and $z={}^t(z_1,z_2,\cdots,z_m)\in\mathbb{C}^m$ with $z_i\in\mathbb{C}\,(1\le i\le m)$. We note that the Jacobi group G^J is *not* a reductive Lie group and that the homogeneous space $\mathbb{H}\times\mathbb{C}^m$ is not a symmetric space.

For a coordinate $(\tau,z)\in\mathbb{H}\times\mathbb{C}^n$, we write $\tau=x+i\,y$ with x real and $y>0$, and

$$z={}^t(z_1,z_2,\cdots,z_m),\quad z_j=u_j+i\,v_j,\quad u_j,\,v_j \text{ real},\quad i=1,2,\cdots,m.$$

According to [23], for any two positive real numbers A and B, the following metric given by

$$\begin{aligned} ds^2_{m;A,B} &= \frac{1}{y^3}\left(A\,y+B\sum_{j=1}^m v_j^2\right)d\tau d\overline{\tau}\\ &\quad+\frac{B}{y^2}\left\{y\sum_{j=1}^m dz_j d\overline{z}_j-\sum_{j=1}^m v_j(d\tau d\overline{z}_j+d\overline{\tau}dz_j)\right\}\\ &= \frac{1}{y^3}\left(A\,y+B\sum_{j=1}^m v_j^2\right)(dx^2+dy^2)\\ &\quad+\frac{B}{y^2}\left\{y\sum_{j=1}^m(du_j^2+dv_j^2)-2\sum_{j=1}^m v_j(dx du_j+dy dv_j)\right\} \end{aligned} \tag{2.2}$$

is a Kähler metric on $\mathbb{H}\times\mathbb{C}^m$ invariant under the action (2.1) of G^J.

We put

$$M_1 := \operatorname{tr}\left(y\frac{\partial}{\partial z}\,{}^t\!\left(\frac{\partial}{\partial\overline{z}}\right)\right)=y\sum_{j=1}^m\frac{\partial^2}{\partial z_j\partial\overline{z}_j}=\frac{y}{4}\left(\frac{\partial}{\partial u_j^2}+\frac{\partial}{\partial v_j^2}\right) \tag{2.3}$$

and

$$\begin{aligned}
(2.4)\quad M_2: \;&=\; y^2\frac{\partial^2}{\partial\tau\partial\overline{\tau}}+\sum_{a,b=1}^m v_av_b\,\frac{\partial^2}{\partial z_a\partial\overline{z}_b}+y\sum_{j=1}^m v_j\left(\frac{\partial^2}{\partial\tau\partial\overline{z}_j}+\frac{\partial^2}{\partial\overline{\tau}\partial z_j}\right)\\
&=\; \frac{1}{4}\left\{y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\sum_{a=1}^m v_a^2\left(\frac{\partial^2}{\partial u_a^2}+\frac{\partial^2}{\partial v_a^2}\right)\right\}\\
&\quad+\frac{1}{2}\sum_{1\le a<b\le m} v_av_b\left(\frac{\partial^2}{\partial u_a\partial u_b}+\frac{\partial^2}{\partial v_a\partial v_b}\right).\\
&\quad+\frac{y}{2}\sum_{j=1}^m v_j\,\left(\frac{\partial^2}{\partial x\partial u_j}+\frac{\partial^2}{\partial y\partial v_j}\right).
\end{aligned}$$

Then M_1 and M_2 are differential operators on $\mathbb{H}\times\mathbb{C}^m$ invariant under the action (2.1). The author [23] proved that

$$(2.5)\qquad \Delta_{m;A,B}:=\frac{4}{B}\,M_1+\frac{4}{A}\,M_2$$

is the Laplacian of $(\mathbb{H}\times\mathbb{C}^m, ds^2_{m;A,B})$. Furthermore the following $2(m+1)$-differential form

$$(2.6)\qquad dv=dx\wedge dy\wedge du_1\wedge\cdots\wedge du_m\wedge dv_1\wedge\cdots\wedge dv_m$$

is a G^J-invariant volume element on the Siegel-Jacobi space $\mathbb{H}\times\mathbb{C}^m$.

Let K^J be the stabilizer of G^J at $(i,0)$. Then

$$K^J=\left\{\left(\begin{pmatrix} a & -b\\ b & a\end{pmatrix},(0,0,R)\right)\;\Big|\; a^2+b^2=1,\; a,b\in\mathbb{R},\; R={}^tR\in\mathbb{R}^{(m,m)}\right\}.$$

Thus G^J/K^J is diffeomorphic to $\mathbb{H}\times\mathbb{C}^m$ via

$$gK^J\longmapsto g\cdot(i,0)=\left(\frac{a\,i+b}{c\,i+d},\frac{\lambda\,i+\mu}{c\,i+d}\right),$$

where $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{R})$ and $(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(m)}$. The Siegel-Jacobi space $\mathbb{H}\times\mathbb{C}^m$ is a homogeneous space which is not symmetric. Let $\mathfrak{k}^J$ be the Lie algebra of K^J. Then the Lie algebra $\mathfrak{g}^J$ of G^J has the Cartan decomposition

$$(2.7)\qquad \mathfrak{g}^J=\mathfrak{k}^J+\mathfrak{p}^J,$$

where

$$\begin{aligned}
\mathfrak{g}^J &= \left\{\left(\begin{pmatrix} x & y\\ z & -x\end{pmatrix},(P,Q,R)\right)\;\Big|\; x,y,z\in\mathbb{R},\; P,Q\in\mathbb{R}^m,\; R={}^tR\in\mathbb{R}^{(m,m)}\right\},\\
\mathfrak{k}^J &= \left\{\left(\begin{pmatrix} 0 & x\\ -x & 0\end{pmatrix},(0,0,R)\right)\;\Big|\; x\in\mathbb{R},\; R={}^tR\in\mathbb{R}^{(m,m)}\right\},\\
\mathfrak{p}^J &= \left\{\left(\begin{pmatrix} x & y\\ y & -x\end{pmatrix},(P,Q,0)\right)\;\Big|\; x,y\in\mathbb{R},\; P,Q\in\mathbb{R}^m\right\}.
\end{aligned}$$

Lemma 2.1. *We have the relations*

$$[\mathfrak{k}^J, \mathfrak{k}^J] \subset \mathfrak{k}^J \quad and \quad [\mathfrak{k}^J, \mathfrak{p}^J] \subset \mathfrak{p}^J. \tag{2.8}$$

Proof. The Lie bracket operation on $\mathfrak{g}^J$ is given by

$$[(X_1, (P_1, Q_1, R_1)), (X_2, (P_2, Q_2, R_2))] = (X^*, (P^*, Q^*, R^*)), \tag{2.9}$$

where $X_1, X_2 \in \mathfrak{sl}_2(\mathbb{R}),\ P_1, Q_1, P_2, Q_2 \in \mathbb{R}^m,\ R_1 = {}^tR_1,\ R_2 = {}^tR_2 \in \mathbb{R}^{(m,m)}$,

$$\begin{aligned} X^* &= [X_1, X_2] = X_1X_2 - X_2X_1, \\ (P^*, Q^*) &= (P_1, Q_1)X_2 - (P_2, Q_2)X_1, \\ R^* &= P_1\,{}^tQ_2 - P_2\,{}^tQ_1 + Q_2\,{}^tP_1 - Q_1\,{}^tP_2. \end{aligned}$$

The relations (2.8) follow immediately from Formula (2.9). □

Remark 2.1. The relation

$$[\mathfrak{p}^J, \mathfrak{p}^J] \subset \mathfrak{k}^J$$

does not hold.

The vector space $\mathfrak{p}^J$ can be regarded as the tangent space of the Siegel-Jacobi space $\mathbb{H} \times \mathbb{C}^m \cong G^J/K^J$ at $(i, 0)$. We define a complex structure I^J on the tangent space $\mathfrak{p}^J$ of $\mathbb{H} \times \mathbb{C}^m \cong G^J/K^J$ at $(i, 0)$ by

$$I^J\left(\begin{pmatrix} x & y \\ y & -x \end{pmatrix}, (P, Q, 0)\right) = \left(\begin{pmatrix} y & -x \\ -x & -y \end{pmatrix}, (Q, -P, 0)\right). \tag{2.10}$$

Let

$$\mathfrak{p} = \left\{ \begin{pmatrix} x & y \\ y & -x \end{pmatrix} \in \mathbb{R}^{(2,2)} \,\middle|\, x, y \in \mathbb{R} \right\}$$

be the real vector space of dimension 2. Identifying $\mathfrak{p}$ with $\mathbb{C}$ via

$$\begin{pmatrix} x & y \\ y & -x \end{pmatrix} \longmapsto x + i\,y \in \mathbb{C}$$

and identifying $\mathbb{R}^m \times \mathbb{R}^m$ with $\mathbb{C}^m$ via

$$(P, Q) \longmapsto Q + i\,P, \quad P, Q \in \mathbb{R}^m,$$

we may regard the complex structure I^J as a real linear map on $\mathbb{C} \times \mathbb{C}^m$ defined by

$$I^J(x + i\,y, Q + i\,P) = (-y + i\,x, -P + i\,Q), \quad x + i\,y \in \mathbb{C},\ Q + i\,P \in \mathbb{C}^m. \tag{2.11}$$

Clearly I^J extends complex linearly on the complexification $\mathfrak{p}^J_{\mathbb{C}} = \mathfrak{p}^J \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathfrak{p}^J$. Then $\mathfrak{p}^J_{\mathbb{C}}$ has a decomposition

$$\mathfrak{p}^J_{\mathbb{C}} = \mathfrak{p}^J_+ \oplus \mathfrak{p}^J_-, \tag{2.12}$$

where $\mathfrak{p}_+^J$ (resp. $\mathfrak{p}_-^J$) denotes the $(+i)$-eigenspace (resp. $(-i)$-eigenspace) of I^J. Precisely, both $\mathfrak{p}_+^J$ and $\mathfrak{p}_-^J$ are given by

$$\mathfrak{p}_+^J = \left\{ \left(\begin{pmatrix} x & i\,x \\ i\,x & -x \end{pmatrix}, (P, iP, 0) \right) \,\Big|\, x \in \mathbb{C},\ P \in \mathbb{C}^m \right\}$$

and

$$\mathfrak{p}_-^J = \left\{ \left(\begin{pmatrix} x & -i\,x \\ -i\,x & -x \end{pmatrix}, (P, -iP, 0) \right) \,\Big|\, x \in \mathbb{C},\ P \in \mathbb{C}^m \right\}.$$

Proposition 2.1. *Fix an element* $g = \big(M, (\lambda, \mu; \kappa)\big) \in G^J$ *with* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ *and* $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(m)}$. *We let* $(\tau_*, z_*) = g \cdot (\tau, z)$. *Let*

$$\mathbb{F}_g : \mathbb{H} \times \mathbb{C}^m \longrightarrow \mathbb{H} \times \mathbb{C}^m$$

be the biholomorphic mapping defined by the action (2.1) of g. Then the differential mapping

$$d\mathbb{F}_g : T_{(\tau,z)}\big(\mathbb{H} \times \mathbb{C}^m\big) \longrightarrow T_{(\tau_*,z_*)}\big(\mathbb{H} \times \mathbb{C}^m\big)$$

is given by

$$(w, \xi) \longmapsto \big(w(g), \xi(g)\big), \quad w \in \mathbb{C},\ \xi \in \mathbb{C}^m \tag{2.13}$$

with

$$w(g) = \frac{w}{(c\,\tau + d)^2} \qquad \text{and} \qquad \xi(g) = \frac{\xi}{c\,\tau + d} + \frac{w(d\,\lambda - c\,\mu - c\,z)}{(c\,\tau + d)^2}.$$

Here we identified $\mathfrak{p}^J$ *with* $\mathbb{C} \times \mathbb{C}^m$.

Proof. Let $\alpha(t) = \big(\tau(t), z(t)\big)$ $(-\epsilon < t < \epsilon,\ \epsilon > 0)$ be a smooth curve in $\mathbb{H} \times \mathbb{C}^m$ passing through $\alpha(0) = (\tau, z)$ with $\alpha'(0) = (w, \xi) \in T_{(\tau,z)}\big(\mathbb{H} \times \mathbb{C}^m\big)$. Then

$$\begin{aligned} \chi(t) :=\ & g \cdot \alpha(t) = \big(\tau(g;t), z(g;t)\big) \\ =\ & \left(\frac{a\,\tau(t) + b}{c\,\tau(t) + d}, \frac{z(t) + \lambda\,\tau(t) + \mu}{c\,\tau(t) + d} \right) \end{aligned}$$

is a smooth curve in $\mathbb{H} \times \mathbb{C}^m$ passing through $\chi(0) = (\tau_*, z_*)$. Then by an easy computation, we see that

$$\tau'(g;0) = \frac{\partial}{\partial t}\Big|_{t=0} \tau(g;t) = \frac{\tau'(0)}{(c\,\tau + d)^2} = \frac{w}{(c\,\tau + d)^2}$$

and

$$z'(g;0) = \frac{\partial}{\partial t}\Big|_{t=0} z(g;t) = \frac{\xi}{c\,\tau + d} + \frac{w(d\,\lambda - c\,\mu - c\,z)}{(c\,\tau + d)^2}.$$

□

Let $\Gamma_1 := SL_2(\mathbb{Z})$ be the elliptic modular group. We let

$$\Gamma_{1,m} := \Gamma_1 \ltimes H_{\mathbb{Z}}^{(m)}$$

be the arithmetic subgroup of G^J, where

$$H_{\mathbb{Z}}^{(m)} := \left\{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(m)} \mid \lambda, \mu, \kappa \textit{ are integral} \right\}$$

is a discrete subgroup of $H_{\mathbb{R}}^{(m)}$. Let $E_k := {}^t(0, \cdots, 1, 0, \cdots, 0)$ $(1 \leq k \leq m)$ be the $m \times 1$ matrix with the $(k,1)$-th entry 1 and other entries 0. For an element $\tau \in \mathbb{H}$, we set for brevity

$$F_k(\tau) := \tau E_k, \quad 1 \leq k \leq m.$$

Let

$$\mathcal{F} := \left\{ \tau \in \mathbb{H} \,\middle|\, |\tau| \geq 1, \ |\mathrm{Re}\, \tau| \leq 1/2 \right\}$$

be a fundamental domain for $\Gamma_1 \backslash \mathbb{H}$. We refer to [16], pp. 78-79 for more detail. For each $\tau \in \mathcal{F}$, we define the subset P_τ of $\mathbb{C}^m$ by

$$P_\tau := \left\{ \sum_{k=1}^{m} \lambda_k E_k + \sum_{k=1}^{m} \mu_k F_k(\tau) \,\middle|\, 0 \leq \lambda_k, \mu_k \leq 1 \right\}.$$

For each $\tau \in \mathcal{F}$, we define the subset $\mathcal{D}_\tau$ of $\mathbb{H} \times \mathbb{C}^m$ by

$$\mathcal{D}_\tau := \{ (\tau, z) \in \mathbb{H} \times \mathbb{C}^m \mid z \in P_\tau \}.$$

Theorem 2.1. *The following subset*

$$\mathcal{F}_{[m]} := \bigcup_{\tau \in \mathcal{F}} \mathcal{D}_\tau \tag{2.14}$$

is a fundamental domain for $\Gamma_{1,m} \backslash (\mathbb{H} \times \mathbb{C}^m)$ *with respect to the action (2.1).*

Proof. Let (τ_*, z_*) be an arbitrary element of $\mathbb{H} \times \mathbb{C}^m$. We must find an element (τ, z) of $\mathcal{F}_{[m]}$ and $\gamma_* = (\gamma, (\lambda, \mu; \kappa)) \in \Gamma_{1,m}$ with $\gamma \in \Gamma_1 = SL_2(\mathbb{Z})$ such that $\gamma_* \cdot (\tau, z) = (\tau_*, z_*)$. Since $\mathcal{F}$ is a fundamental domain for $\Gamma_1 \backslash \mathbb{H}$, there is an element γ of Γ_1 and an element $\tau \in \mathcal{F}$ such that $\tau_* = \gamma \cdot \tau$. Here τ is unique up to the boundary of $\mathcal{F}$. We write

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 = SL_2(\mathbb{Z}).$$

We can find $\lambda, \mu \in \mathbb{Z}^m$ and $z \in P_\tau$ satisfying the equation

$$z + \lambda\, \tau + \mu = z_*(x\, \tau + d).$$

If we take $\gamma_* = (\gamma, (\lambda, \mu; 0)) \in \Gamma_{1,m}$, we see that $\gamma_* \cdot (\tau, z) = (\tau_*, z_*)$. Therefore

$$\mathbb{H} \times \mathbb{C}^m = \bigcup_{\gamma_* \in \Gamma_{1,m}} \gamma_* \cdot \mathcal{F}_{[m]}.$$

Let (τ, z) and $\gamma_* \cdot (\tau, z)$ be two elements of $\mathcal{F}_{[m]}$ with $\gamma_* = (\gamma, (\lambda, \mu; \kappa)) \in \Gamma_{1,m}$ with $\gamma \in \Gamma_1$. Then both τ and $\gamma \cdot \tau$ lie in $\mathcal{F}$. Therefore both of them either lie in the boundary of $\mathcal{F}$ or $\gamma = \pm I_2$. In the case that both τ and $\gamma \cdot \tau$ lie in the boundary of $\mathcal{F}$, both (τ, z) and $\gamma_* \cdot (\tau, z)$ lie in the boundary of $\mathcal{F}_{[m]}$. If $\gamma = \pm I_2$, we get

$$z \in P_\tau \qquad \text{and} \qquad \pm\,(z + \lambda\,\tau + \mu) \in P_\tau. \tag{2.15}$$

From the definition of P_τ and (2.16), we see that either $\lambda = \mu = 0,\ \gamma \neq -I_2$ or both z and $\pm(z + \lambda\,\tau + \mu)$ lie on the boundary of the parallelepiped P_τ. Hence either both (τ, z) and $\gamma_* \cdot (\tau, z)$ lie in the boundary of $\mathcal{F}_{[m]}$ or $\gamma_* = (I_2, (0, 0; \kappa)) \in \Gamma_{1,m}$. Consequently $\mathcal{F}_{[m]}$ is a fundamental domain for $\Gamma_{1,m} \backslash (\mathbb{H} \times \mathbb{C}^m)$ with respect to the action (2.1). □

Now we consider the Siegel-Jacobi space $\mathbb{H}_{1,1} := \mathbb{H} \times \mathbb{C}$ endowed with the Riemannian metric (cf. (2.2))

$$ds^2_{1;1,1} = \frac{y + v^2}{y^3}\,(dx^2 + dy^2) + \frac{1}{y}\,(du^2 + dv^2) - \frac{2v}{y^2}\,(dx\,du + dy\,dv),$$

where $\tau = x + i\,y$ with $x,\ y > 0$ real and $z = u + i\,v$ with u, v real are coordinates in $\mathbb{H}_{1,1}$. Then

$$E_1 := \frac{\partial}{\partial x}, \quad E_2 := \frac{\partial}{\partial y}, \quad E_3 := \frac{\partial}{\partial u}, \quad E_4 := \frac{\partial}{\partial v}$$

form a local frame field on $\mathbb{H}_{1,1}$. Let Γ^k_{ij} $(i, j, k = 1, 2, 3, 4)$ be the Christoffel symbols for the Riemannian connection ∇ determined uniquely by the Riemannian metric $ds^2_{1;1,1}$. That is,

$$\nabla_{E_i} E_j = \sum_{k=1}^{4} \Gamma^k_{ij}\, E_k, \qquad i, j = 1, 2, 3, 4.$$

Lemma 2.2. *For all $i, j, k = 1, 2, 3, 4$, $\Gamma^k_{ij} = \Gamma^k_{ji}$. The Christoffel symbols Γ^k_{ij}'s $(1 \leq i, j, k \leq 4)$ are given by*

$$\begin{aligned}
\Gamma^2_{11} &= \frac{2\,y + y^2}{2\,y^2}, \qquad \Gamma^1_{12} = \Gamma^2_{22} = -\,\frac{2\,y + v^2}{2\,y^2},\\
\Gamma^4_{11} &= \frac{v^3}{2\,y^3}, \qquad \Gamma^3_{12} = \Gamma^4_{22} = -\,\frac{v^3}{2\,y^3},\\
\Gamma^1_{14} &= \Gamma^1_{23} = \Gamma^2_{24} = \Gamma^4_{33} = \frac{v}{2\,y},\\
\Gamma^2_{13} &= \Gamma^3_{34} = \Gamma^4_{44} = -\,\frac{v}{2\,y}, \qquad \Gamma^4_{13} = \frac{y - v^2}{2\,y^2},\\
\Gamma^3_{14} &= \Gamma^3_{23} = \Gamma^4_{24} = -\,\frac{y - v^2}{2\,y^2}, \quad \Gamma^2_{33} = \frac{1}{2}, \quad \Gamma^1_{34} = \Gamma^2_{44} = -\,\frac{1}{2}
\end{aligned}$$

and all other $\Gamma_{ij}^k = 0$.

Proof. It is easy to prove the above lemma. We leave the proof to the reader. □

Proposition 2.2. *Let* $\gamma(t) = \big(x(t) + i\,y(t), u(t) + i\,v(t)\big)$ *be a smooth curve in* $\mathbb{H}_{1,1}$. *For brevity we write*

$$\begin{aligned} \ddot{x} &= \frac{d^2x}{dt^2}, \quad \ddot{y} = \frac{d^2y}{dt^2}, \quad \ddot{u} = \frac{d^2u}{dt^2}, \quad \ddot{v} = \frac{d^2v}{dt^2}, \\ \dot{x} &= \frac{dx}{dt}, \quad \dot{y} = \frac{dy}{dt}, \quad \dot{u} = \frac{du}{dt}, \quad \dot{v} = \frac{dv}{dt}. \end{aligned}$$

Then the curve $\gamma(t)$ *is a geodesic in* $\mathbb{H}_{1,1}$ *with respect to the metric* $ds^2_{1;1,1}$ *if and only if it satisfies the following four differential equations*

$$\ddot{x} - \frac{2\,y + y^2}{2\,y^2}\,\dot{x}\,\dot{y} + \frac{v}{y}\,\dot{x}\,\dot{v} + \frac{v}{y}\,\dot{y}\,\dot{u} - \dot{u}\,\dot{v} = 0 \tag{2.16}$$

$$\ddot{y} + \frac{2\,y + y^2}{2\,y^2}\,\dot{x}^2 - \frac{2\,y + y^2}{2\,y^2}\,\dot{y}^2 + \frac{1}{2}\,\dot{u}^2 - \frac{1}{2}\,\dot{v}^2 - \frac{v}{y}\,\dot{x}\,\dot{u} + \frac{v}{y}\,\dot{y}\,\dot{v} = 0 \tag{2.17}$$

$$\ddot{u} - \frac{v^3}{y^3}\,\dot{x}\,\dot{y} - \frac{y - v^2}{y^2}\,\dot{x}\,\dot{v} - \frac{y - v^2}{y^2}\,\dot{y}\,\dot{u} - \frac{v}{y}\,\dot{u}\,\dot{v} = 0 \tag{2.18}$$

$$\ddot{v} + \frac{v^3}{2y^3}\,\dot{x}^2 - \frac{v^3}{2y^3}\,\dot{y}^2 + \frac{v}{2y}\,\dot{u}^2 - \frac{v}{2y}\,\dot{v}^2 + \frac{y - v^2}{y^2}\,\dot{x}\,\dot{u} - \frac{y - v^2}{y^2}\,\dot{y}\,\dot{v} = 0 \tag{2.19}$$

Proof. Using Lemma 2.2 and the geodesic equations, we obtain the above equations. □

Remark 2.2. If $u = v = 0$, the equations (2.16)-(2.19) reduce to the following two equations

$$\ddot{x} - \frac{2}{y}\,\dot{x}\,\dot{y} = 0 \tag{2.20}$$

and

$$\ddot{y} + \frac{1}{y}\,\dot{x}^2 - \frac{1}{y}\,\dot{y}^2 = 0. \tag{2.21}$$

Thus these two equations (2.20) and (2.21) give geodesics in the Poincaré upper half plane $\mathbb{H}$ which are circles perpendicular to the x-axis or straight lines perpendicular to the x-axis. Therefore the curve $\gamma(t) = \big(x(t) + i\,y(t), 0\big)$ $(-\infty < t < \infty)$ such that $\alpha(t) = x(t) + i\,y(t)$ is a geodesic in $\mathbb{H}$ is a geodesic in $\mathbb{H}_{1,1}$ with respect to the metric $ds^2_{1;1,1}$.

Proposition 2.3. *Let $\gamma(t)$ be a geodesic in $\mathbb{H}_{1,1}$ joining two points $\gamma(0)=(\tau_1,0)$ and $\gamma(1)=(\tau_2,0)$ such that $\gamma(t)$ is contained in the subset $\{(\tau,0)\in\mathbb{H}_{1,1}\mid \tau\in\mathbb{H}\}$. Then the length ρ of the geodesic segment between $\gamma(0)=(\tau_1,0)$ and $\gamma(1)=(\tau_2,0)$ is given by*

$$\rho=\log\frac{1+R^{1/2}}{1-R^{1/2}}, \tag{2.22}$$

where $R:=R(\tau_1,\tau_2)$ is the cross-ratio of τ_1 and τ_2 defined by

$$R(\tau_1,\tau_2):=\frac{\tau_1-\tau_2}{\tau_1-\overline{\tau}_2}\cdot\frac{\overline{\tau}_1-\overline{\tau}_2}{\overline{\tau}_1-\tau_2}.$$

Proof. By remark 2.2, the length ρ is equal to the length ρ_0 of the geodesic in $\mathbb{H}$ joining τ_1 and τ_2 with respect to the Poincaré metric

$$ds^2=\frac{dx^2+dy^2}{y^2}.$$

It is well known that ρ_0 is given by the formula (2.22). We refer to [17] for the general case. □

Proposition 2.4. *Let (τ_1,z_1) and (τ_2,z_2) be two points in the Siegel-Jacobi space $\mathbb{H}\times\mathbb{C}^m$. Then there exists an element $g=\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix},(\lambda,\mu;\kappa)\right)\in G^J$ such that*

$$g\cdot(\tau_1,z_1)=(i,0)\qquad\text{and}\qquad g\cdot(\tau_2,z_2)=\left(i\,\delta,\ \frac{z_2+\lambda\,\tau_2+\mu}{c\,\tau_2+d}\right)$$

with $\delta>0$. Therefore the length of the geodesic joining (τ_1,z_1) to (τ_2,z_2) with respect to the Riemannian metric $ds^2_{m;A,B}$ is equal to that of the geodesic joining $(i,0)$ to $\left(i\,\delta,\frac{z_2+\lambda\tau_2+\mu}{c\,\tau_2+d}\right)$ with respect to the metric $ds^2_{m;A,B}$.

Proof. We see that there is an element $h=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{R})$ such that

$$h\cdot\tau_1=\frac{a\,\tau_1+b}{c\,\tau_1+d}=i\qquad\text{and}\qquad h\cdot\tau_2=\frac{a\,\tau_2+b}{c\,\tau_2+d}=i\,\delta$$

with $\delta>0$. We take

$$\lambda=-\frac{\operatorname{Im}z_1}{\operatorname{Im}\tau_1}\qquad\text{and}\qquad \mu=-\operatorname{Re}z_1+\frac{\operatorname{Re}\tau_1\cdot\operatorname{Im}z_1}{\operatorname{Im}\tau_1}$$

We easily see that the element

$$g=\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix},(\lambda,\mu;\kappa)\right)\in G^J$$

satisfies the condition

$$g\cdot(\tau_1,z_1)=(i,0)\qquad\text{and}\qquad g\cdot(\tau_2,z_2)=\left(i\,\delta,\,\frac{z_2+\lambda\,\tau_2+\mu}{c\,\tau_2+d}\right)$$

with $\delta>0$.

For each fixed element $g\in G^J$, according to the G^J-invariance of the metric $ds^2_{m;A,B}$, the map $\mathbb{F}_g$ of $\mathbb{H}\times\mathbb{C}^m$ defined by the action (2.1) of g is an isometry of $\mathbb{H}\times\mathbb{C}^m$ with respect to the metric $ds^2_{m;A,B}$. Consequently we obtain the second statement. □

Proposition 2.5. *The scalar curvature* $r(p)$ *of the Siegel-Jacobi space* $\left(\mathbb{H}_{1,1},\,ds^2_{1;1,1}\right)$ *is* -3 *for each point* p *of* $\mathbb{H}_{1,1}$.

Proof. Using Lemma 2.2, we obtain the scalar curvature $r(p)=-3$ for each point p of $\mathbb{H}_{1,1}$ by a tedious computation. □

Now we study differential forms on $\mathbb{H}\times\mathbb{C}^m$ invariant under the action (2.1) of $\Gamma_{1,m}$.

Proposition 2.6. *(a) Assume that*

$$\alpha=f(\tau,z)\,d\tau\,+\,\sum_{k=1}^{m}\phi_k(\tau,z)\,dz_k$$

is a differential 1*-form on* $\mathbb{H}\times\mathbb{C}^m$ *invariant under the action (2.1) of* $\Gamma_{1,m}$*. Then the functions* f *and* ϕ_k $(k=1,2,\cdots,m)$ *satisfy the following conditions*

$$f(\gamma\cdot(\tau,z))=(c\,\tau+d)^2f(\tau,z)\,+\,(c\,\tau+d)\sum_{k=1}^{m}(c\,z_k+c\,\mu_k-d\,\lambda_k)\,\phi_k(\tau,z)\tag{2.23}$$

and

$$\phi_k(\gamma\cdot(\tau,z))=(c\,\tau+d)\,\phi_k(\tau,z),\qquad k=1,2,\cdots,m\tag{2.24}$$

for all $\gamma=\left(\begin{pmatrix}a&b\\c&d\end{pmatrix},(\lambda,\mu;\kappa)\right)\in\Gamma_{1,m}$ *with* $\lambda={}^t(\lambda_1,\cdots,\lambda_m)\in\mathbb{Z}^m$ *and* $\mu={}^t(\mu_1,\cdots,\mu_m)\in\mathbb{Z}^m$.

(b) Let

$$\eta=\,d\tau\wedge dz_1\wedge dz_2\wedge\cdots\wedge dz_m$$

be a differential $(m+1)$*-form on* $\mathbb{H}\times\mathbb{C}^m$*. Assume that*

$$\theta=g(\tau,z)\,\eta^{\otimes\ell},\qquad \ell=1,2,3,\cdots,$$

is a differential $\ell(m+1)$*-form on* $\mathbb{H}\times\mathbb{C}^m$ *invariant under the action (2.1) of* $\Gamma_{1,m}$*. Then the function* g *satisfies the following condition*

$$g(\gamma\cdot(\tau,z))=(c\,\tau+d)^{\ell(m+2)}g(\tau,z)\tag{2.25}$$

for all $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu; \kappa) \right) \in \Gamma_{1,m}$.

(c) For $k = 1, 2, \cdots, m$, *we let*

$$\widetilde{\omega}_k = (-1)^{m-k}\, d\tau \wedge dz_1 \wedge \cdots \wedge dz_{k-1} \wedge \widehat{dz_k} \wedge dz_{k+1} \wedge \cdots \wedge dz_m$$

be a differential m-form on $\mathbb{H} \times \mathbb{C}^m$. *Assume that*

$$\beta = \sum_{k=1}^{m} a_k(\tau, z)\, \widetilde{\omega}_k \, + \, (-1)^m \, b(\tau, z)\, dz_1 \wedge \cdots \wedge dz_m$$

is a differential m-form on $\mathbb{H} \times \mathbb{C}^m$ *invariant under the action (2.1) of* $\Gamma_{1,m}$. *Then the functions* $a(\tau, z)$ *and* b_k $(k = 1, 2, \cdots, m)$ *satisfy the following conditions*

$$a_k(\gamma \cdot (\tau, z)) = (c\,\tau + d)^{m+1}\, a_k(\tau, z) - (c\,\tau + d)^m (c\, z_k + c\, \mu_k - d\, \lambda_k)\, b(\tau, z) \tag{2.26}$$

for $k - 1, 2, \cdots, m$ *and*

$$b(\gamma \cdot (\tau, z)) = (c\,\tau + d)^m\, b(\tau, z) \tag{2.27}$$

for all $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu; \kappa) \right) \in \Gamma_{1,m}$ *with* $\lambda = {}^t(\lambda_1, \cdots, \lambda_m) \in \mathbb{Z}^m$ *and* $\mu = {}^t(\mu_1, \cdots, \mu_m) \in \mathbb{Z}^m$.

Proof. For $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu; \kappa) \right) \in \Gamma_{1,m}$ with $\lambda = {}^t(\lambda_1, \cdots, \lambda_m) \in \mathbb{Z}^m$ and $\mu = {}^t(\mu_1, \cdots, \mu_m) \in \mathbb{Z}^m$ and $(\tau, z) \in \mathbb{H} \times \mathbb{C}^m$ with $z = {}^t(z_1, \cdots, z_m) \in \mathbb{C}^m$, we set $(\tau^*, z^*) = \gamma \cdot (\tau, z)$. In other words,

$$\tau^* = \frac{a\,\tau + b}{c\,\tau + d}, \qquad z_k^* = \frac{z_k + \lambda_k\, \tau + \mu_k}{c\,\tau + d}, \qquad k = 1, 2, \cdots, m.$$

Then we have

$$d\tau^* = \frac{d\tau}{(c\,\tau + d)^2} \tag{2.28}$$

and

$$dz_k^* = \left\{ \frac{\lambda_k}{c\,\tau + d} - \frac{c\,(z_k + \lambda_k\,\tau + \mu_k)}{(c\,\tau + d)^2} \right\} d\tau \, + \, \frac{dz_k}{c\,\tau + d}, \qquad k = 1, 2, \cdots, m. \tag{2.29}$$

Using the formulas (2.28) and (2.29), we obtain the desired results (a), (b) and (c). □

3. The center of the universal enveloping algebra of $\mathfrak{g}^J$

In this section we describe the center of the universal enveloping algebra of the complexication of the Jacobi Lie algebra $\mathfrak{g}^J$ explicitly.

Let $\mathfrak{g}^J_{\mathbb{C}}$ be the complexification of the Jacobi Lie algebra $\mathfrak{g}^J$. We put the 2×2 matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $\{H,E,F\}$ is a basis of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Let $\epsilon_{ij}\,(1\le i\le m,\ j=1,2)$ be the $m\times 2$ matrices whose (i,j)-th entry is 1 and whose other entries are zero, and let E_{kl} be the $m\times m$ elementary matrix whose (k,l)-th entry is 1 and whose other entries are zero. We set $e_i := \epsilon_{i1},\ f_i := \epsilon_{i2}\ (1\le i\le m)$ and

$$R_{kl} := \frac{1}{2}\,(E_{kl} + E_{ji}), \quad R_{kl} = R_{lk}, \quad 1\le k,l\le m.$$

Then $\{H,E,F,e_i,f_i,R_{kl}\,|\,1\le i\le m,\ 1\le k\le l\le m\}$ is a basis for $\mathfrak{g}^J_{\mathbb{C}}$. It is easily seen that

$$\mathcal{Z}_m := \left\{(0,(0,0,R))\in\mathfrak{g}^J_{\mathbb{C}}\;|\;R = {}^tR\in\mathbb{C}^{(m,m)}\right\}$$

is the center of $\mathfrak{g}^J_{\mathbb{C}}$.

Lemma 3.1. *We have the following.*

(1) $[H,E]=2\,E, \quad [H,F]=-2\,F, \quad [E,F]=H.$

(2) $[H,e_i]=-e_i, \quad [H,f_i]=f_i, \quad 1\le i\le m.$

(3) $[E,e_i]=f_i, \quad [E,f_i]=0, \quad 1\le i\le m.$

(4) $[F,e_i]=0, \quad [F,f_i]=-e_i, \quad 1\le i\le m.$

(5) $[e_i,f_j]=2\,R_{ij}, \quad 1\le i,j\le m.$

Proof. The proof follows immediately from the fact that

$$\begin{aligned} (3.1)\quad & [(X_1,(P_1,Q_1,R_1)),(X_2,(P_2,Q_2,R_2))] \\ = \quad & \Big([X_1,X_2],\big((P_1,Q_1)X_2-(P_2,Q_2)X_1,P_1\,{}^tQ_2-P_2\,{}^tQ_1+Q_2\,{}^tP_1-Q_1\,{}^tP_2\big)\Big), \end{aligned}$$

where $X_1,X_2\in\mathfrak{sl}_2(\mathbb{C})$, $[X_1,X_2]=X_1X_2-X_2X_1$, $P_i,Q_i\in\mathbb{C}^{(m,1)}\,(i=1,2)$, $R_1,R_2\in\mathbb{C}^{(m,m)}$ with $R_1={}^tR_1$ and $R_2={}^tR_2$. □

Formally we put

$$e := \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{pmatrix}, \qquad f := \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix},$$

and

$$R := \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1m} \\ R_{21} & R_{22} & \cdots & R_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ R_{m1} & R_{m2} & \cdots & R_{mm} \end{pmatrix}, \quad R_{kl} = R_{lk}, \quad 1 \leq k, l \leq m.$$

Theorem 3.1. *The center* $\mathcal{Z}_m(\mathfrak{g}_{\mathbb{C}}^J)$ *of the universal enveloping algebra* $\mathcal{U}(\mathfrak{g}_{\mathbb{C}}^J)$ *of* $\mathfrak{g}_{\mathbb{C}}^J$ *is given by*

$$\mathcal{Z}_m(\mathfrak{g}_{\mathbb{C}}^J) = \mathbb{C}[\Omega_m, R_{kl} \mid 1 \leq k \leq l \leq m].$$

That is, $\mathcal{Z}_m(\mathfrak{g}_{\mathbb{C}}^J)$ *is a polynomial algebra on* $1 + \frac{m(m+1)}{2}$ *generators* Ω_m, R_{kl} $(1 \leq k \leq l \leq m)$. *Here*

$$\begin{aligned} \Omega_m : \;\; = \;\; & \det R \left\{ H^2 - (m+2)H + 4\,EF \right\} \\ & + \det R \left\{ E\,{}^t e R^{-1} e - {}^t f R^{-1} f F - \left(H - \frac{m+3}{2} \right) {}^t f R^{-1} e \right\} \\ & + \det R \left\{ \frac{1}{4}\, {}^t f \left({}^t f R^{-1} e \right) R^{-1} e - \frac{1}{4} \left({}^t e R^{-1} f \right) \left({}^t e R^{-1} e \right) \right\} \end{aligned}$$

is a Casimir operator of $\mathcal{U}(\mathfrak{g}^J\mathbb{C})$ *of degree* $m+2$.

Proof. Using the method computing the center of the universal enveloping algebra of a certain class of semidirect sum Lie algebras invented by Campoamer-Stursburg and Low [6] (cf. [2], [15]), Conley and Raum [5] proved the above theorem. We refer to [5] for the detail. □

Let $\gamma : G^J \times (\mathbb{H} \times \mathbb{C}^m) \longrightarrow \mathbb{C}^\times$ be a scalar cocycle with respect to the action (2.1). This means that γ is a smooth function satisfying the cocycle condition

$$\gamma(g_1 g_2, (\tau, z)) = \gamma(g_1, g_2 \cdot (\tau, z))\, \gamma(g_2, (\tau, z)) \tag{3.2}$$

for all $g_1, g_2 \in G^J$ and $(\tau, z) \in \mathbb{H} \times \mathbb{C}^m$. Then we get the map

$$\widehat{\gamma}(g) : G^J \longrightarrow C^\infty(\mathbb{H} \times \mathbb{C}^m)$$

defined by

$$\widehat{\gamma}(g)(\tau, z) := \gamma(g, (\tau, z)), \quad g \in G^J,\ (\tau, z) \in \mathbb{H} \times \mathbb{C}^m.$$

Then we obtain the right action $|_\gamma$ of G^J on $C^\infty(\mathbb{H} \times \mathbb{C}^m)$ defined by

$$(g \cdot f)(\tau, z) := \left(f|_\gamma [g^{-1}] \right)(\tau, z) := \gamma(g^{-1}, (\tau, z)) f(g^{-1} \cdot (\tau, z)), \tag{3.3}$$

where $g \in G^J$, $f \in C^\infty(\mathbb{H} \times \mathbb{C}^m)$ and $(\tau, z) \in \mathbb{H} \times \mathbb{C}^m$.

We note that the differential $d\widehat{\gamma}$ of $\widehat{\gamma}$ at the identity is given by

$$d\widehat{\gamma}(Y)(\tau, z) = \frac{d}{dt}\Big|_{t=0} \gamma(\exp(tY), (\tau, z)).$$

Therefore we have the differential right action $|_\gamma$ of $\mathfrak{g}^J_\mathbb{C}$ on $C^\infty(\mathbb{H}\times\mathbb{C}^m)$ defined by

$$(3.4)\qquad (\phi|_\gamma[Y])(\tau,z):\ =\ \frac{d}{dt}\Big|_{t=0}\left(\gamma(\exp(tY),(\tau,z))\phi(\exp(tY)\cdot(\tau,z))\right)$$

$$(3.5)\qquad =\ \gamma(Y,(\tau,z))\phi(\tau,z)\,+\,\frac{d}{dt}\Big|_{t=0}\phi(\exp(tY),(\tau,z)),$$

where $Y\in\mathfrak{g}^J_\mathbb{C}$ and $\phi\in C^\infty(\mathbb{H}\times\mathbb{C}^m)$. The action (3.4) extends to $\mathcal{U}(\mathfrak{g}^J_\mathbb{C})$ as usual, and elements of $\mathcal{U}(\mathfrak{g}^J_\mathbb{C})$ of order r act by differential operators of order $\le r$.

Let $\mathbb{D}_\gamma$ be the algebra of all differential operators D on $\mathbb{H}\times\mathbb{C}^m$ satisfying the following condition

$$(3.6)\qquad (D\phi)|_\gamma[g]=D\big(\phi|_\gamma[g]\big)$$

for all $g\in G^J$ and for all $\phi\in C^\infty(\mathbb{H}\times\mathbb{C}^m)$. Since G^J is connected, $\mathbb{D}_\gamma$ is the algebra of all differential operators $\mathbb{D}_\gamma$on $\mathbb{H}\times\mathbb{C}^m$ commuting with the $|_\gamma$-action of $\mathfrak{g}^J_\mathbb{C}$. In particular, the action $|_\gamma$ maps the center $\mathcal{Z}_m(\mathfrak{g}^J_\mathbb{C})$ of $\mathcal{U}(\mathfrak{g}^J_\mathbb{C})$ into the center $\mathcal{Z}_m(\mathbb{D}_\gamma)$ of $\mathbb{D}_\gamma$.

Throughout this section we let $\mathcal{M}$ be a positive definite half-integral symmetric matrix of degree m and let $k\in\mathbb{Z}^+$. We let $\gamma_{k,\mathcal{M}}:G^J\times(\mathbb{H}\times\mathbb{C}^m)\longrightarrow\mathbb{C}^\times$ be the canonical automorphic factor for G^J on $\mathbb{H}\times\mathbb{C}^m$ defined by

$$\gamma_{k,\mathcal{M}}\big((M,(\lambda,\mu;\kappa)),(\tau,z)\big):$$
$$(3.7)\qquad =\ (c\tau+d)^k\,e^{2\pi i\,\mathcal{M}[z+\lambda\tau+\mu]\,c\,(c\tau+d)^{-1}}\;e^{-2\pi i\,\mathrm{tr}\left(\mathcal{M}(\tau\lambda\,{}^t\lambda+2\,\lambda\,{}^tz+\kappa+\mu\,{}^t\lambda)\right)},$$

where $(M,(\lambda,\mu;\kappa))\in G^J$ with $M=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{R})$, $(\lambda,\mu;\kappa)\in H^{(m)}_\mathbb{R}$ and $(\tau,z)\in\mathbb{H}\times\mathbb{C}^m$.

For brevity we write

$$\partial_\tau:\ =\ \frac{\partial}{\partial\tau}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\,\frac{\partial}{\partial y}\right),\qquad \partial_{\overline{\tau}}:=\frac{\partial}{\partial\overline{\tau}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\,\frac{\partial}{\partial y}\right),$$
$$\partial_{z_j}:\ =\ \frac{\partial}{\partial z_j}=\frac{1}{2}\left(\frac{\partial}{\partial u_j}-i\,\frac{\partial}{\partial v_j}\right),\quad 1\le j\le m,$$
$$\partial_{\overline{z}_j}:\ =\ \frac{\partial}{\partial \overline{z}_j}=\frac{1}{2}\left(\frac{\partial}{\partial u_j}+i\,\frac{\partial}{\partial v_j}\right),\quad 1\le j\le m,$$
$$\partial_z:\ =\ {}^t\!\big(\partial_{z_1},\partial_{z_2},\cdots,\partial_{z_m}\big),\quad \partial_{\overline{z}}:={}^t\!\big(\partial_{\overline{z}_1},\partial_{\overline{z}_2},\cdots,\partial_{\overline{z}_m}\big).$$

Lemma 3.2. *Let $\mathcal{M}$ and k be as above. We set $\widetilde{\mathcal{M}}:=2\pi i\,\mathcal{M}$. Then we have the following:*

$$(3.8)\qquad |_{\gamma_{k,\mathcal{M}}}[E]\ =\ 2\,\mathrm{Re}\,(\partial_\tau),$$

$$(3.9)\qquad |_{\gamma_{k,\mathcal{M}}}[F]\ =\ -2\,\mathrm{Re}\big(\tau\,(\tau\,\partial_\tau+{}^tz\,\partial_z)\big)-k\,\tau-\widetilde{\mathcal{M}}[z],$$

$$(3.10)\qquad |_{\gamma_{k,\mathcal{M}}}[H]\ =\ 2\,\mathrm{Re}\big(2\,\tau\,\partial_\tau+{}^tz\,\partial_z\big)+k,$$

$$(3.11)\qquad |_{\gamma_{k,\mathcal{M}}}[(0,(P,Q,R))]\ =\ 2\,\mathrm{Re}\big({}^t(P\,\tau+Q)\,\partial_z\big)+2\,{}^tP\widetilde{\mathcal{M}}\,z+\mathrm{tr}\big(R\widetilde{\mathcal{M}}\big).$$

Proof. We observe that if $(X,(P,Q,R)) \in \mathfrak{g}_{\mathbb{C}}^J$ with $X \in \mathfrak{sl}_2(\mathbb{C}),\ P,Q \in \mathbb{C}^{(m,1)}$ and $R = {}^tR \in \mathbb{C}^{(m,m)}$, then

$$\exp\big((X,(P,Q,R))\big) = \Big(\exp(X), \big((P,Q)\,g(X), R-(P,Q)\,h(X)\,{}^t(-Q,P)\big)\Big), \tag{3.12}$$

where

$$\exp(t) := \sum_{n=0}^{\infty}\frac{t^n}{n!},\quad g(t) := \frac{e^t-1}{t}\quad \text{and}\quad h(t) := \frac{e^t-1-t}{t}.$$

Using the formula (3.12) we easily obtain the formulas (3.8)-(3.11). □.

Theorem 3.2.

$$|_{\gamma_{k,\,\mathcal{M}}}[\Omega_m] = \det\big(\widetilde{\mathcal{M}}\big)\ \big\{k(k-m-2)\ -\ 2\,\mathcal{C}^{k,\,\mathcal{M}}\big\}, \tag{3.13}$$

where

$$\begin{aligned}\mathcal{C}^{k,\,\mathcal{M}}:\ =\ & -8\,y^2\partial_\tau\partial_{\overline{\tau}} + 4\,i\left(k-\frac{m}{2}\right)y\,\partial_{\overline{\tau}}\\ & +2\,y^2\Big(\partial_{\overline{\tau}}\,\widetilde{\mathcal{M}}^{-1}[\partial_z] + \partial_\tau\,\widetilde{\mathcal{M}}^{-1}[\partial_{\overline{z}}]\Big) - 8\,y\,\partial_\tau\,{}^tv\,\partial_{\overline{z}}\\ & -\frac{1}{2}\,y^2\left\{\widetilde{\mathcal{M}}^{-1}[\partial_{\overline{z}}]\,\widetilde{\mathcal{M}}^{-1}[\partial_z] - {}^t\big(\partial_{\overline{z}}\widetilde{\mathcal{M}}^{-1}\partial_z\big)^2\right\} + 2\,y\,\big({}^tv\,\partial_{\overline{z}}\big)\,{}^t\partial_z\widetilde{\mathcal{M}}^{-1}\partial_u\\ & -\frac{i}{2}\,(2k-m+1)\,y\,{}^t\partial_{\overline{z}}\,\widetilde{\mathcal{M}}^{-1}\partial_u + 2\,{}^tv\,\big({}^tv\,\partial_{\overline{z}}\big)\partial_{\overline{z}} + i\,(2k-m-1)\,{}^tv\,\partial_{\overline{z}}.\end{aligned}$$

The operator $\mathcal{C}^{k,\,\mathcal{M}}$ generates the image of the $|_{\gamma_{k,\mathcal{M}}}$-action of the center $\mathcal{Z}_m(\mathfrak{g}_{\mathbb{C}}^J)$. In particular, $\mathcal{C}^{k,\,\mathcal{M}}$ is an element of the center of $\mathbb{D}_{\gamma_{k,\mathcal{M}}}$.

Proof. We write $\widetilde{\mathcal{M}} = \big(\widetilde{\mathcal{M}}_{pq}\big)$. According to (3.11), we have the relation $|_{\gamma_{k,\mathcal{M}}}[R_{pq}] = \widetilde{\mathcal{M}}_{pq}$ for all $1 \le p \le q \le m$. The proof follows from Theorem 3.1. and Lemma 3.2. □

4. Invariant differential operators on $\mathbb{H}\times\mathbb{C}^m$

For brevity we put

$$T_{1,m} := \mathbb{C}\times\mathbb{C}^m.$$

We define the real linear map $\Phi_m : \mathfrak{p}^J \longrightarrow T_{1,m}$ by

$$\Phi_m\left(\begin{pmatrix} x & y\\ y & -x\end{pmatrix}, (P,Q,0)\right) = (x+i\,y, P+i\,Q), \tag{4.1}$$

where $\left(\begin{pmatrix} x & y\\ y & -x\end{pmatrix}, (P,Q,0)\right) \in \mathfrak{p}^J$. Obviously Φ_m is a real linear isomorphism of $\mathfrak{p}^J$ onto $T_{1,m}$.

Let $S(m,\mathbb{R})$ denote the additive group consisting of all $m\times m$ real symmetric matrices. We define the group isomorphism $\theta_m : K^J \longrightarrow U(1)\times S(m,\mathbb{R})$ by

$$\theta_m\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, (0,0;\kappa)\right) = (a+i\,b,\kappa), \tag{4.2}$$

where $\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, (0,0;\kappa)\right) \in K^J$.

Theorem 4.1. *The adjoint representation Ad of K^J on $\mathfrak{p}^J$ is compatible with the natural action of $U(1)\times S(m,\mathbb{R})$ on $T_{1,m}=\mathbb{C}\times\mathbb{C}^m$ defined by*

$$(h,\kappa)\cdot(w,\xi) := (h^2 w, h\,\xi),\quad h\in U(1),\ \kappa\in S(m,\mathbb{R}),\ w\in\mathbb{C},\ \xi\in\mathbb{C}^m \tag{4.3}$$

through the map Φ_m and θ_m. Precisely if $k^J\in K^J$ and $\alpha\in\mathfrak{p}^J$, then we have the following equality

$$\Phi_m\big(Ad(k^J)\alpha\big) = \theta_m(k^J)\cdot\Phi_m(\alpha). \tag{4.4}$$

We recall that we identified $\mathfrak{p}^J$ with $\mathbb{C}\times\mathbb{C}^m$.

Proof. We refer to [26] for the proof. □

The action (4.3) induces the action of $U(1)$ on the polynomial algebra $\mathrm{Pol}_{[m]} := \mathrm{Pol}(T_{1,m})$. We denote by $\mathrm{Pol}_{[m]}^{U(1)}$ the subalgebra of $\mathrm{Pol}_{[m]}$ consisting of $U(1)$-invariants. We let $\mathbb{D}\big(\mathbb{H}\times\mathbb{C}^m\big)$ be the algebra of all differential operators invariant under the action (2.1) of G^J. According to [7], one gets a canonical linear bijection

$$\Theta_{[m]} : \mathrm{Pol}_{[m]}^{U(1)} \longrightarrow \mathbb{D}\big(\mathbb{H}\times\mathbb{C}^m\big) \tag{4.5}$$

of $\mathrm{Pol}_{[m]}^{U(1)}$ onto $\mathbb{D}\big(\mathbb{H}\times\mathbb{C}^m\big)$. But $\Theta_{[m]}$ is not multiplicative. The map $\Theta_{[m]}$ is described explicitly as follows. Let $\big\{\eta_\alpha \mid 1\le\alpha\le 2(m+1)\big\}$ be a basis of $\mathfrak{p}^J$. If $P\in\mathrm{Pol}_{[m]}^{U(1)}$, then

$$\Big(\Theta_{[m]}(P)f\Big)(gK^J) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g\exp\left(\sum_{\alpha=1}^{2(m+1)} t_\alpha\eta_\alpha\right)K^J\right)\right]_{(t_\alpha)=0}, \tag{4.6}$$

where $g\in G^J$ and $f\in C^\infty(\mathbb{H}\times\mathbb{C}^m)$.

Theorem 4.2. $\mathrm{Pol}_{[m]}^{U(1)}$ *is generated by*

$$q(w,\xi) = \mathrm{tr}\big(w\,\overline{w}\big), \tag{4.7}$$

$$\alpha_{kp}(w,\xi) = \mathrm{Re}\,\big(\xi\,{}^t\overline{\xi}\big)_{kp},\quad 1\le k\le p\le m, \tag{4.8}$$

$$\beta_{lq}(w,\xi) = \mathrm{Im}\,\big(\xi\,{}^t\overline{\xi}\big)_{lq},\quad 1\le l< q\le m, \tag{4.9}$$

$$f_{kp}(w,\xi) = \mathrm{Re}\,(\overline{w}\,\xi\,{}^t\xi)_{kp},\quad 1\le k\le p\le m, \tag{4.10}$$

$$g_{kp}(w,\xi) = \mathrm{Im}\,(\overline{w}\,\xi\,{}^t\xi)_{kp},\quad 1\le k\le p\le m, \tag{4.11}$$

where $w \in \mathbb{C}$ and $\xi \in \mathbb{C}^m$.

Proof. We refer to [9] or [26] for the general case. □

We let

$$w = r + i\,s \in \mathbb{C} \quad \text{and} \quad \xi = {}^t(\xi_1, \cdots, \xi_m) \in \mathbb{C}^m \text{ with } \xi_k = \zeta_k + i\,\eta_k,\ 1 \le k \le m,$$

where $r, s, \zeta_1, \eta_1, \cdots, \zeta_m, \eta_m$ are real. The invariants q, α_{kp}, β_{lq}, f_{kp} and g_{kp} are expressed in terms of $r, s, \zeta_k, \eta_l\ (1 \le k, l \le m)$ as follows:

$$\begin{aligned}
q(w,\xi) &= r^2 + s^2, \\
\alpha_{kp}(w,\xi) &= \zeta_k\zeta_p + \eta_k\eta_p, \quad 1 \le k \le p \le m, \\
\beta_{lq}(w,\xi) &= \zeta_q\eta_l - \zeta_l\eta_q, \quad 1 \le l < q \le m, \\
f_{kp}(w,\xi) &= r(\zeta_k\zeta_p - \eta_k\eta_p) + s(\zeta_k\eta_p + \eta_k\zeta_p), \quad 1 \le k \le p \le m, \\
g_{kp}(w,\xi) &= r(\zeta_k\eta_p + \eta_k\zeta_p) - s(\zeta_k\zeta_p - \eta_k\eta_p), \quad 1 \le k \le p \le m.
\end{aligned}$$

Theorem 4.3. *The $\frac{m(m+1)}{2}$ relations*

$$f_{kp}^2 + g_{kp}^2 = q\,\alpha_{kk}\,\alpha_{pp}, \quad 1 \le k \le p \le m \tag{4.12}$$

exhaust all the relations among a complete set of generators q, α_{kp}, β_{lq}, f_{kp} and g_{kp} of $\mathrm{Pol}_{[m]}^{U(1)}$ with $1 \le k \le p \le m$ and $1 \le l < q \le m$.

Theorem 4.4. *The action of $U(1)$ on $Pol_{1,m}$ is not multiplicity-free. In fact, if*

$$\mathrm{Pol}_{[m]} = \sum_{\sigma \in \widehat{U(1)}} m_\sigma\,\sigma,$$

then $m_\sigma = \infty$.

For the proofs of the above theorems we refer to [26].

We consider the case $m = 1$. For a coordinate (w, ξ) in $T_{1,1}$, we write $w = r + i\,s,\ \xi = \zeta + i\,\eta,\ r, s, \zeta, \eta$ real. The author [21] proved that the algebra $\mathrm{Pol}_{[1]}^{U(1)}$ is generated by

$$\begin{aligned}
q(w,\xi) &= \frac{1}{4}\,w\,\overline{w} = \frac{1}{4}\big(r^2 + s^2\big), \\
\alpha(w,\xi) &= \xi\,\overline{\xi} = \zeta^2 + \eta^2, \\
\phi(w,\xi) &= \frac{1}{2}\,\mathrm{Re}\,\big(\xi^2\overline{w}\big) = \frac{1}{2}\,r\big(\zeta^2 - \eta^2\big) + s\,\zeta\eta, \\
\psi(w,\xi) &= \frac{1}{2}\,\mathrm{Im}\,\big(\xi^2\overline{w}\big) = \frac{1}{2}\,s\big(\eta^2 - \zeta^2\big) + r\,\zeta\eta.
\end{aligned}$$

In [21], using Formula (3.6) the author calculated explicitly the images

$$D_1 = \Theta_{[1]}(q), \quad D_2 = \Theta_{[1]}(\alpha), \quad D_3 = \Theta_{[1]}(\phi) \quad \text{and} \quad D_4 = \Theta_{[1]}(\psi)$$

of q, α, ϕ and ψ under the Halgason map $\Theta_{[1]}$. We can show that the algebra $\mathbb{D}(\mathbb{H}\times\mathbb{C})$ is generated by the following differential operators

$$\begin{aligned} D_1 =& y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+v^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right) \\ &+2\,y\,v\left(\frac{\partial^2}{\partial x\partial u}+\frac{\partial^2}{\partial y\partial v}\right), \\ D_2 =& y\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right), \\ D_3 =& y^2\,\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial u^2}-\frac{\partial^2}{\partial v^2}\right)-2y^2\,\frac{\partial^3}{\partial x\partial u\partial v} \\ &-\left(v\,\frac{\partial}{\partial v}+1\right)D_2 \end{aligned}$$

and

$$\begin{aligned} D_4 =& y^2\,\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial v^2}-\frac{\partial^2}{\partial u^2}\right)-2\,y^2\,\frac{\partial^3}{\partial y\partial u\partial v} \\ &-v\,\frac{\partial}{\partial u}D_2, \end{aligned}$$

where $\tau = x+iy$ and $z = u+iv$ with real variables x, y, u, v. Moreover, we have

$$\begin{aligned} D_1D_2-D_2D_1 =& \,2\,y^2\,\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial u^2}-\frac{\partial^2}{\partial v^2}\right) \\ &-4\,y^2\,\frac{\partial^3}{\partial x\partial u\partial v}-2\left(v\,\frac{\partial}{\partial v}D_2+D_2\right). \end{aligned}$$

In particular, the algebra $\mathbb{D}(\mathbb{H}\times\mathbb{C})$ is not commutative. We refer to [1, 21] for more detail.

Recently Hiroyuki Ochiai [13] (see also [1]) proved the following result.

Theorem 4.5. *We have the following relations*

(*a*) $[D_1, D_2] = 2D_3$

(*b*) $[D_1, D_3] = 2D_1D_2 - 2D_3$

(*c*) $[D_2, D_3] = -D_2^2$

(*d*) $[D_4, D_1] = 0$

(*e*) $[D_4, D_2] = 0$

(*f*) $[D_4, D_3] = 0$

(g) $D_3^2 + D_4^2 = D_2 D_1 D_2$

These seven relations exhaust all the relations among the generators D_1, D_2, D_3 *and* D_4 *of* $\mathbb{D}(\mathbb{H} \times \mathbb{C})$.

Remark 4.1. According to Theorem 4.5, we see that D_4 is a generator of the center of $\mathbb{D}(\mathbb{H} \times \mathbb{C})$. We observe that the Lapalcian

$$\Delta_{1;A,B} = \frac{4}{A} D_1 + \frac{4}{B} D_2 \qquad \text{(see (2.5))}$$

of $(\mathbb{H} \times \mathbb{C}, ds^2_{1;A,B})$ does not belong to the center of $\mathbb{D}(\mathbb{H} \times \mathbb{C})$.

5. Maass-Jacobi Forms due to Yang

Using G^J-invariant differential operators on the Siegel-Jacobi space, we introduce a notion of Maass-Jacobi forms.

Definition 5.1. Let

$$\Gamma_{1,m} := SL_2(\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(m)}$$

be the discrete subgroup of G^J, where

$$H_{\mathbb{Z}}^{(m)} = \left\{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(m)} \mid \lambda, \mu, \kappa \text{ are integral } \right\}.$$

A smooth function $f : \mathbb{H} \times \mathbb{C}^m \longrightarrow \mathbb{C}$ is called a Maass-Jacobi form on $\mathbb{H} \times \mathbb{C}^m$ if f satisfies the following conditions (MJ1)-(MJ3) :

(MJ1) f is invariant under $\Gamma_{1,m}$.
(MJ2) f is an eigenfunction of the Laplacian $\Delta_{m;A,B}$ (cf. Formula (2.5)).
(MJ3) f has a polynomial growth, that is, there exist a constant $C > 0$ and a positive integer N such that

$$|f(x+iy,z)| \leq C\,|p(y)|^N \quad \text{as } y \longrightarrow \infty,$$

where $p(y)$ is a polynomial in y.

Remark 5.1. Let $\mathbb{D}_*$ be a commutative subalgebra of $\mathbb{D}(\mathbb{H} \times \mathbb{C}^m)$ containing the Laplacian $\Delta_{m;A,B}$. We say that a smooth function $f : \mathbb{H} \times \mathbb{C}^m \longrightarrow \mathbb{C}$ is a Maass-Jacobi form with respect to $\mathbb{D}_*$ if f satisfies the conditions $(MJ1)$, $(MJ2)_*$ and $(MJ3)$: the condition $(MJ2)_*$ is given by

$(MJ2)_*$ f is an eigenfunction of any invariant differential operator in $\mathbb{D}_*$.

It is natural to propose the following problems.

Problem A : Find all the eigenfunctions of $\Delta_{m;A,B}$.

Problem B : Construct Maass-Jacobi forms.

Problem C : Develop the spectral theory of the Laplacian $\Delta_{m;A,B}$ on a fundamental domain for the Siegel-Jacobi space $\mathbb{H}\times\mathbb{C}^m$ with respect to $\Gamma_{1,m}$.

If we find a *nice* eigenfunction ϕ of the Laplacian $\Delta_{m;A,B}$, we can construct a Maass-Jacobi form f_ϕ on $\mathbb{H}\times\mathbb{C}^m$ in the usual way defined by

$$f_\phi(\tau,z) := \sum_{\gamma\in\Gamma_{1,m}^\infty\backslash\Gamma_{1,m}} \phi(\gamma\cdot(\tau,z)), \tag{5.1}$$

where

$$\Gamma_{1,m}^\infty = \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda,\mu;\kappa) \right) \in \Gamma_{1,m} \;\middle|\; c=0 \right\}$$

is a subgroup of $\Gamma_{1,m}$.

We consider the simple case $m=1$ and $A=B=1$. We take a coordinate $(\tau,z)\in\mathbb{H}\times\mathbb{C}$ with $\tau=x+iy,\ x\in\mathbb{R},\ y>0$ and $z=u+iv,\ u,v$ real. A metric $ds^2_{1;1,1}$ on $\mathbb{H}\times\mathbb{C}$ given by

$$\begin{aligned} ds^2_{1;1,1} &= \frac{y+v^2}{y^3}\,(\,dx^2+dy^2\,) + \frac{1}{y}\,(\,du^2+dv^2\,) \\ &\quad - \frac{2v}{y^2}\,(\,dx\,du + dy\,dv\,) \end{aligned}$$

is a G^J-invariant Kähler metric on $\mathbb{H}\times\mathbb{C}$. Its Laplacian $\Delta_{1;1,1}$ is given by

$$\begin{aligned} \Delta_{1;1,1} &= y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) \\ &\quad + (\,y+v^2\,)\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right) \\ &\quad + 2\,y\,v\left(\frac{\partial^2}{\partial x\partial u}+\frac{\partial^2}{\partial y\partial v}\right). \end{aligned}$$

We provide some examples of eigenfunctions of $\Delta_{1;1,1}$.

(1) $h(x,y)=y^{\frac12}K_{s-\frac12}(2\pi|a|y)\,e^{2\pi iax}$ $(s\in\mathbb{C},\ a\neq 0\,)$ with eigenvalue $s(s-1)$. Here

$$K_s(z) := \frac12\int_0^\infty \exp\left\{-\frac{z}{2}(t+t^{-1})\right\} t^{s-1}\,dt, \tag{5.2}$$

where $\mathrm{Re}\,z>0$.

(2) $y^s,\ y^s x,\ y^s u$ $(s\in\mathbb{C})$ with eigenvalue $s(s-1)$.

(3) $y^s v,\ y^s uv,\ y^s xv$ with eigenvalue $s(s+1)$.

(4) $x,\ y,\ u,\ v,\ xv,\ uv$ with eigenvalue 0.

(5) All Maass wave forms.

We let $f;\mathbb{H}\times\mathbb{C}\longrightarrow\mathbb{C}$ be a Maass-Jacobi form with $\Delta_{1;1,1}f=\Lambda f$. Then f satisfies the following invariance relations

$$f(\tau+n,z)=f(\tau,z)\qquad\text{for all } n\in\mathbb{Z}$$

and

$$f(\tau,z+n_1\tau+n_2)=f(\tau,z)\qquad\text{for all } n_1,n_2\in\mathbb{Z}.$$

Therefore f is a smooth function on $\mathbb{H}\times\mathbb{C}$ which is periodic in x and u with period 1. So f has the following Fourier series

$$f(\tau,z)=\sum_{n\in\mathbb{Z}}\sum_{r\in\mathbb{Z}}c_{n,r}(y,v)\,e^{2\pi i\,(nx+ru)}. \tag{5.3}$$

For two fixed integers n and r, for brevity, we set $\varphi(y,v)=c_{n,r}(y,v)$. Then φ satisfies the following differential equation

$$\left[y^2\frac{\partial^2}{\partial y^2}+(y+v^2)\frac{\partial^2}{\partial v^2}+2\,yv\,\frac{\partial^2}{\partial y\partial v}-\left\{(A\,y+B\,v)^2+B^2\,y+\Lambda\right\}\right]\varphi=0, \tag{5.4}$$

where $A=2\pi n$ and $B=2\pi r$ are constants. We note that the function $\phi(y)=y^{\frac{1}{2}}\,K_{s-\frac{1}{2}}(2\pi|n|y)$ satisfies the the differential equation (5.4) with $\Lambda=s(s-1)$. Here $K_s(z)$ is the K-Bessel function defined by (5.2) (cf. [10], [19]).

6. Maass-Jacobi forms due to Pitale, Bringmann et al

We fix a positive integer m. Let $\mathcal{M}$ be a symmetric half-integral semi-positive definite matrix of degree m. Let $C^\infty(\mathbb{H}\times\mathbb{C}^m)$ be the algebra of all C^∞-functions on $\mathbb{H}\times\mathbb{C}^m$. For any nonnegative integer $k\in\mathbb{Z}$, we define the $|_{k,\mathcal{M}}$-slash action of G^J on $C^\infty(\mathbb{H}\times\mathbb{C}^m)$ as follows: If $f\in C^\infty(\mathbb{H}\times\mathbb{C}^m)$, and $(M,(\lambda,\mu;\kappa))\in G^J$ with $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{R})$ and $(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(m)}$,

$$\begin{aligned}&\big(f|_{k,\mathcal{M}}[(M,(\lambda,\mu;\kappa))]\big)(\tau,z):\\ =\;&(c\tau+d)^{-k}\,e^{-2\pi i\,\mathcal{M}[z+\lambda\tau+\mu]\,c\,(c\tau+d)^{-1}}\\ &\times e^{2\pi i\,\mathrm{tr}\left(\mathcal{M}(\tau\lambda\,{}^t\lambda+2\,\lambda\,{}^tz+\kappa+\mu\,{}^t\lambda)\right)}\,f\left(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\mu}{c\tau+d}\right),\end{aligned} \tag{6.1}$$

where $\tau\in\mathbb{H}$ and $z\in\mathbb{C}^m$. We recall the Siegel's notation $\alpha[\beta]={}^t\beta\alpha\beta$ for suitable matrices α and β. Let $\mathbb{D}_{k,\mathcal{M}}$ be the algebra of all differential operators D on $\mathbb{H}\times\mathbb{C}^m$ satisfying the following condition

$$(Df)|_{k,\mathcal{M}}[g]=D\big(f|_{k,\mathcal{M}}[g]\big) \tag{6.2}$$

for all $f\in C^\infty(\mathbb{H}\times\mathbb{C}^m)$ and for all $g\in G^J$. We recall the arithmetic subgroup $\Gamma_{1,m}$ of G^J defined by

$$\Gamma_{1,m}:=SL_2(\mathbb{Z})\ltimes H_{\mathbb{Z}}^{(m)}.$$

Definition 6.1. Let $\mathcal{C}^{k,\mathcal{M}}$ be the Casimir operator defined in Theorem 3.2. A smooth function $\phi : \mathbb{H} \times \mathbb{C}^m \longrightarrow \mathbb{C}$ is called a Maass-Jacobi form of weight k and index $\mathcal{M}$ if it satisfies the following conditions:

(MJ1*) $\phi|_{k,\mathcal{M}}[\gamma] = \phi$ for all $\gamma \in \Gamma_{1,m}$.

(MJ2*) ϕ is an eigenfunction of the Casimir operator $\mathcal{C}^{k,\mathcal{M}}$.

(MJ3*) For some $a > 0$,

$$\phi(\tau, z) = O\big(e^{ay}\, e^{2\pi i \mathcal{M}[v]/y}\big) \quad \text{as } y \longrightarrow \infty.$$

Furthermore if $\mathcal{C}^{k,\mathcal{M}}\phi = 0$, it is said to be a *harmonic* Maass-Jacobi form of weight k and index $\mathcal{M}$. We denote by $\mathbb{J}_{k,\mathcal{M}}$ the space of all harmonic Maass-Jacobi forms of weight k and index $\mathcal{M}$.

For the present being we let $\mathcal{M}$ be a positive definite integral even lattice of rank m and k an integer. We identify $\mathcal{M}$ with its Gram matrix with respect to a fixed basis, that is, a positive definite half-integral symmetric matrix of degree m. We write $|\mathcal{M}|$ for the determinant of the Gram matrix of $\mathcal{M}$. Throughout this section n will be an integer and r will be in $\mathbb{Z}^m$. For $r = {}^t(r_1, \cdots, r_m) \in \mathbb{Z}^m$ and $z = {}^t(z_1, \cdots, z_m) \in \mathbb{C}^m$, we put

$$\zeta^r := \prod_{j=1}^{m} e^{2\pi i r_j z_j},$$

where $\zeta = (\zeta_1, \cdots, \zeta_m)$ with $\zeta_j = e^{2\pi i z_j}$ $(1 \leq j \leq m)$. For $a \in \mathbb{C}$, we write $e(a) := e^{2\pi i a}$. For two vectors $\xi = {}^t(\xi_1, \cdots, \xi_m)$ and $\eta = {}^t(\eta_1, \cdots, \eta_m)$ in $\mathbb{C}^m$, we let

$$\langle \xi, \eta \rangle := \sum_{j=1}^{m} \xi_j\, \eta_j$$

be the standard scalar product.

We set

$$D = D_{\mathcal{M}}(n, r) := |\mathcal{M}|\big(4n - \mathcal{M}^{-1}[r]\big) \quad \text{and} \quad h = h_{\mathcal{M}}(r) := |\mathcal{M}|\, \mathcal{M}^{-1}[r]. \tag{6.3}$$

Let $M_{\nu,\mu}(w)$ be the usual M-Whittaker function, which is a solution to the following differential equation

$$\frac{\partial^2}{\partial w^2} f(w) + \left(-\frac{1}{4} + \frac{\nu}{w} + \frac{\frac{1}{4} - \mu^2}{w^2}\right) f(w) = 0. \tag{6.4}$$

For $s \in \mathbb{C}$, $\kappa \in \frac{1}{2}\mathbb{Z}$ and $t \in \mathbb{R}^\times$, we define the function

$$\mathcal{M}_{s,\kappa}(t) := |t|^{-\frac{\kappa}{2}} M_{\mathrm{sgn}(t)\frac{\kappa}{2}, s-\frac{1}{2}}(|t|) \tag{6.5}$$

and

$$\phi^{(n,r)}_{k,\mathcal{M},s}(\tau,z) := \mathcal{M}_{s,k-\frac{m}{2}}\left(\frac{\pi\, Dy}{|\mathcal{M}|}\right)\, e^{2\,\pi\, i\,\left(\langle r,z\rangle\, +\, \frac{i}{4}\,\mathcal{M}^{-1}[r]y\, +\, nx\right)}. \tag{6.6}$$

We define the Poincaré series

$$P^{(n,r)}_{k,\mathcal{M},s}(\tau,z) := \sum_{\gamma\in\Gamma^{\infty}_{1,m}\backslash\Gamma_{1,m}} \left(\phi^{(n,r)}_{s,\mathcal{M},s}\Big|_{k,\mathcal{M}}[r]\right)(\tau,z). \tag{6.7}$$

Obviously $P^{(n,r)}_{k,\mathcal{M},s}$ is holomorphic in $\mathbb{C}^m$. It is easily seen that $P^{(n,r)}_{k,\mathcal{M},s}$ is an eigenfunction of the Casimir operator $\mathcal{C}^{k,\mathcal{M}}$ with eigenvalue

$$-2\,s(1-s) - \frac{1}{2}\left\{k^2 - k(m+2) + \frac{1}{4}\,m(m+4)\right\}.$$

For $s\in\mathbb{C},\ \kappa\in\frac{1}{2}\mathbb{Z}$ and $t\in\mathbb{R}^{\times}$, we set

$$\mathcal{W}_{s,\kappa}(t) := |t|^{-\frac{\kappa}{2}}\, W_{\mathrm{sgn}(t)\frac{\kappa}{2},s-\frac{1}{2}}(|t|), \tag{6.8}$$

where $W_{\nu,\mu}$ denotes the usual W-Whittaker function which is also a solution to the differential equation (6.4).

For $r\in\mathbb{Z}^m$, we define the theta series

$$\theta^{(r)}_{k,\mathcal{M}}(\tau,z) := \sum_{\lambda\in\mathbb{Z}^m} e^{2\,\pi\, i\,\mathcal{M}[\lambda]}\,\zeta^{2\mathcal{M}\lambda}\left\{e^{2\,\pi\, i\,\langle r,\lambda\rangle}\zeta^r \,+\, (-1)^k\, e^{-2\,\pi\, i\,\langle r,\lambda\rangle}\zeta^r\right\}. \tag{6.9}$$

Theorem 6.1(Bringmann-Richter [4] and Conley-Raum [5]). *The Poincaré series $P^{(n,r)}_{s,\mathcal{M},s}(\tau,z)$ has the Fourier expansion*

$$\begin{aligned} P^{(n,r)}_{k,\mathcal{M},s}(\tau,z) \;&=\; \mathcal{M}_{s,k-\frac{m}{2}}\left(\frac{\pi Dy}{|\mathcal{M}|}\right)\, e\left(\frac{-iDy}{4|\mathcal{M}|}\right)\,\theta^{(r)}_{k,\mathcal{M}}(\tau,z)\, e^{2\pi\, i\, n\,\tau} \\ &\quad + \sum_{n'\in\mathbb{Z},\, r'\in\mathbb{Z}^m} c_{y,s}(n',r')\, e^{2\,\pi\, i\, n'\tau}\,\zeta^{r'}. \end{aligned} \tag{6.10}$$

Here the coefficients $c_{y,s}(n',r')$ are

$$c_{y,s}(n',r') := b_{y,s}(n',r') \,+\, (-1)^k\, b_{y,s}(n',-r')$$

with $b_{y,s}$ depending on D and $D' = |\mathcal{M}|\,(4\,n' \,-\, \mathcal{M}^{-1}[r'])$ and $b_{y,s}(n',r')$ is given as follows:

(1) If $D'=0$, there is a constant $a_s(n',r')$ such that

$$b_{y,s}(n',r') = a_s(n',r')\,\frac{y^{1+\frac{m}{4}-\frac{k}{2}-s}}{\Gamma\left(s+\frac{k}{2}-\frac{m}{4}\right)\Gamma\left(s-\frac{k}{2}+\frac{m}{4}\right)}.$$

(2) If $DD' > 0$,

$$b_{y,s}(n',r') = 2^{1-\frac{m}{2}}\,\pi\, i^{-k}\,|\mathcal{M}|^{-\frac{1}{2}}\,\frac{\Gamma(2s)}{\Gamma\left(s-\mathrm{sgn}(D')\left(\frac{k}{2}-\frac{m}{4}\right)\right)} \times \left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{m+2}{4}} e\left(-\frac{i\,D'y}{4|\mathcal{M}|}\right)\mathcal{W}_{s,k-\frac{m}{2}}\left(\frac{\pi D'y}{|\mathcal{M}|}\right) \times \sum_{c\in\mathbb{Z}^+} c^{-\frac{m+2}{2}}\,K_{c,\mathcal{M}}(n,r,n',r')\,J_{2s-1}\left(\frac{\pi\sqrt{DD'}}{c|\mathcal{M}|}\right),$$

where Γ is the usual Gamma function, J_s is the usual J-Bessel function and $K_{c,\mathcal{M}}(n,r,n',r')$ is the Kloosterman sum defined by

$$(6.11)\quad K_{c,\mathcal{M}}(n,r,n',r') := e^{-\pi\, i\, c^{-1}\langle r,\mathcal{M}^{-1}r'\rangle} \times \sum_{\substack{d\in(\mathbb{Z}/c\mathbb{Z})^\times,\\ \lambda\in\mathbb{Z}^m/c\mathbb{Z}^m}} e^{2\,\pi\, i\left(c^{-1}\bar{d}\mathcal{M}[\lambda]+n'd-\langle r',\lambda\rangle+\bar{d}\,n+\bar{d}\,\langle r,\lambda\rangle\right)},$$

where $\bar{d}$ is an integer inverse of d modulo c.

(3) If $DD' < 0$,

$$b_{y,s}(n',r') = 2^{1-\frac{m}{2}}\,\pi\, i^{-k}\,|\mathcal{M}|^{-\frac{1}{2}}\,\frac{\Gamma(2s)}{\Gamma\left(s-\mathrm{sgn}(D')\left(\frac{k}{2}-\frac{m}{4}\right)\right)} \times \left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{m+2}{4}} e\left(-\frac{i\,D'y}{4|\mathcal{M}|}\right)\mathcal{W}_{s,k-\frac{m}{2}}\left(\frac{\pi D'y}{|\mathcal{M}|}\right) \times \sum_{c\in\mathbb{Z}^+} c^{-\frac{m+2}{2}}\,K_{c,\mathcal{M}}(n,r,n',r')\,I_{2s-1}\left(\frac{\pi\sqrt{DD'}}{c|\mathcal{M}|}\right),$$

where I_s is the usual I-Bessel function.

Proof. We refer to [4] for the proof in the case $n = m = 1$ and to [5] in the case $n = 1$, m is arbitrary. □

Remark 6.1. If $s = \frac{k}{2} - \frac{m}{4}$ (resp. $s = 1 + \frac{m}{4} - \frac{k}{2}$) , then the Poincaré series $P^{(n,r)}_{k,\mathcal{M},s}(\tau,z)$ converges for $k > m + 2$ (resp. $k < 0$). In both cases Poincaré series $P^{(n,r)}_{k,\mathcal{M},s}(\tau,z)$ is a *harmonic* Maass-Jacobi form of weight k and index $\mathcal{M}$ which is holomorphic in $\mathbb{C}^m$.

Remark 6.2. The Fourier coefficients $c^{(n,r)}_{y,s} = c^{(n,r)}_{k,\mathcal{M},s}$ of the Poincaré series $P^{(n,r)}_{k,\mathcal{M},s}(\tau,z)$ satisfy the the so-called *Zagier-type duality* with dual weights k and $m+2-k$. More precisely, if $D < 0$ and $D' < 0$, there is a constant $h_{k,s}$ depending only on k and s such that

$$(6.12)\qquad c^{(n,r)}_{k,\mathcal{M},s}(n',r') = h_{k,s}\, c^{(n',r')}_{m+2-k,\mathcal{M},s}(n,r)$$

while if $D<0$ and $D'>0$, there is a constant $\hat{h}_{k,s}$ depending only on k and s such that

$$c_{k,\mathcal{M},s}^{(n,r)}(n',r')=\hat{h}_{k,s}\,c_{m+2-k,\mathcal{M},s}^{(n',r')}(n,r). \tag{6.13}$$

7. Skew-Holomorphic Jacobi Forms

We define the *skew-slash action* of G^J on $C^\infty(\mathbb{H}\times\mathbb{C}^m)$ as follows: If $f\in C^\infty(\mathbb{H}\times\mathbb{C}^m)$, and $(M,(\lambda,\mu;\kappa))\in G^J$ with $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{R})$ and $(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(m)}$,

$$\begin{aligned}&\big(f|_{k,\mathcal{M}}^{sk}[(M,(\lambda,\mu;\kappa))]\big)(\tau,z):\\ =\;&(c\overline{\tau}+d)^{1-k}\,|c\tau+d|^{-1}\,e^{-2\pi i\,\mathcal{M}[z+\lambda\tau+\mu]\,c\,(c\tau+d)^{-1}}\\ &\times e^{2\pi i\,\mathrm{tr}\left(\mathcal{M}(\tau\lambda\,{}^t\lambda+2\,\lambda\,{}^tz+\kappa+\mu\,{}^t\lambda)\right)}\,f\left(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\mu}{c\tau+d}\right),\end{aligned} \tag{7.1}$$

where $\tau\in\mathbb{H}$ and $z\in\mathbb{C}^m$.

Definition 7.1. A smooth $f:\mathbb{H}\times\mathbb{C}^m\longrightarrow\mathbb{C}$ is said to be a *skew-holomorphic* Jacobi form of weight k and index $\mathcal{M}$ if it is real analytic in τ and is holomorphic in $z\in\mathbb{C}^m$ and satisfies the following conditions:

(SK1) $f|_{k,\mathcal{M}}^{sk}[\gamma]=f$ for all $\gamma\in\Gamma^J$.

(SK2) The Fourier expansion of f is of the form

$$f(\tau,z)=\sum_{\substack{n\in\mathbb{Z},\,r\in\mathbb{Z}^m\\ D\gg-\infty}} c(n,r)\,e^{\pi\,Dy/|\mathcal{M}|}\,e^{2\pi i n\tau}\,\zeta^r.$$

We denote by $\mathbb{J}_{k,\mathcal{M}}^{sk}$ the space of all skew-holomorphic Jacobi forms of weight k and index $\mathcal{M}$.

Remark 7.1. The notion of skew-holomorphic Jacobi forms was introduced by N.-P. Skoruppa [18].

Let

$$e_{n,r,\mathcal{M}}(\tau,z):=e^{2\pi i\,(n\tau+\langle r,z\rangle)}\,e^{\pi Dy/|\mathcal{M}|}.$$

We define the Poincaré series

$$P_{k,\mathcal{M}}^{(n,r),sk}(\tau,z):=\sum_{\gamma\in\Gamma_{1,m}^\infty\backslash\Gamma_{1,m}}\big(e_{n,r,\mathcal{M}}|_{k,\mathcal{M}}^{sk}[\gamma]\big)(\tau,z). \tag{7.2}$$

Theorem 7.1. *The Poincaré series $P_{k,\mathcal{M}}^{(n,r),sk}(\tau,z)$ defined in (7.2) is a cuspidal skew-holomorphic Jacobi form of weight k and index $\mathcal{M}$. And it has the Fourier* *expansion*

$$\begin{aligned} P^{(n,r),sk}_{k,\mathcal{M}}(\tau,z) &= e^{\pi D y/|\mathcal{M}|}\,\theta^{(r)}_{k-1,\mathcal{M}}(\tau,z)\,e^{2\pi i n\tau} \\ &\quad + \sum_{\substack{n'\in\mathbb{Z},\, r'\in\mathbb{Z}^m \\ D'>0}} c(n',r')\,e^{\pi D' y/|\mathcal{M}|}\,e^{2\pi i n'\tau}\,\zeta^{r'}, \end{aligned}$$

where $\theta^{(r)}_{k,\mathcal{M}}(\tau,z)$ *is defined in Formula (6.9) and the coefficients* $c(n',r')$ *are*

$$c(n',r') = b(n',r') + (-1)^k b(n',-r').$$

Here

$$\begin{aligned} b(n',r') : &= 2^{1-\frac{m}{2}}\,\pi\,i^{1-k}\left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{m+2}{4}} \\ &\quad \times \sum_{c\in\mathbb{Z}^+} c^{-\frac{m+2}{2}}\,K_{c,\mathcal{M}}(n,r,n',-r')\,J_{k-\frac{m+2}{2}}\left(\frac{\pi\sqrt{DD'}}{c|\mathcal{M}|}\right). \end{aligned}$$

Proof. The proof is analogous to that of Theorem 6.1. □

We define the following lowering operator

$$\begin{aligned} (7.3)\quad D^{(\mathcal{M})}_{-} &= \left(\frac{\tau-\overline{\tau}}{2i}\right)\left\{-(\tau-\overline{\tau})\,\partial_{\overline{\tau}} - {}^t(z-\overline{z})\partial_{\overline{z}} + \frac{\tau-\overline{\tau}}{8\pi i}\,\mathcal{M}^{-1}[\partial_{\overline{z}}]\right\} \\ &= -2iy\left(y\,\partial_{\overline{\tau}} + {}^t v\,\partial_{\overline{z}} - \frac{y}{8\pi i}\,\mathcal{M}^{-1}[\partial_{\overline{z}}]\right). \end{aligned}$$

We note that $D^{(\mathcal{M})}_{-}$ satisfies the following relation

$$(7.4)\qquad \left(D^{(\mathcal{M})}_{-}\phi\right)\Big|_{k-2,\mathcal{M}}[\gamma] = D^{(\mathcal{M})}_{-}\left(\phi|_{k,\mathcal{M}}[\gamma]\right)$$

for all $\phi\in C^\infty(\mathbb{H}\times\mathbb{C}^m)$ and for all $\gamma\in\Gamma_{1,m}$.

Now we define the differential operator

$$(7.5)\qquad \xi_{k,\mathcal{M}} := \left(\frac{\tau-\overline{\tau}}{2i}\right)^{k-\frac{5}{2}} D^{(\mathcal{M})}_{-} = y^{k-\frac{5}{2}}\,D^{(\mathcal{M})}_{-}.$$

It is easily seen that if f is a harmonic Maass-Jacobi form of weight k and index $\mathcal{M}$ which is holomorphic in $\mathbb{C}^m$, then the image $\xi_{k,\mathcal{M}}f$ of f under $\xi_{k,\mathcal{M}}$ is a skew-holomorphic Jacobi form of weight $3-k$ and index $\mathcal{M}$.

Theorem 7.2. *The Poincaré series* $P^{(n,r),sk}_{k,\mathcal{M}}(\tau,z)$ *span the space* $\mathbb{J}^{sk,cusp}_{k,\mathcal{M}}$ *of all cuspidal skew-holomorphic Jacobi forms of weight* k *and index* $\mathcal{M}$.

Proof. The proof can be found in [18]. □

Now we consider the special case $s = \frac{k}{2} - \frac{m}{4}$ and $s = 1 + \frac{m}{4} - \frac{k}{2}$.

Proposition 7.1. *The Poincaré series* $P^{(n,r)}_{k,\mathcal{M},\frac{k}{2}-\frac{m}{4}}$ *with* $k > 2+m$ *is meromorphic. If* $k < 0$,

$$\xi_{k,\mathcal{M}}\left(P^{(n,r)}_{k,\mathcal{M},1+\frac{m}{4}-\frac{k}{2}}\right) = c_{k,\mathcal{M}}\, P^{(n,r),sk}_{3-k,\mathcal{M}},$$

where $c_{k,\mathcal{M}}$ *is a constant depending on* k *and* $\mathcal{M}$.

Proof. We refer to [5], p. 18 for the proof. □

Proposition 7.2. *Let* $\mathbb{J}^{cusp,*}_{k,\mathcal{M}}$ *be the space of all cuspidal harmonic Maass-Jacobi forms of weight* k *and index* $\mathcal{M}$ *which are holomorphic in* $\mathbb{C}^m$. *Then we have the relation*

$$\xi_{k,\mathcal{M}}\left(\mathbb{J}^{cusp,*}_{k,\mathcal{M}}\right) = \mathbb{J}^{sk,cusp}_{k,\mathcal{M}}.$$

Proof. We refer to [5], p. 18 for the proof. □

8. Covariant differential operators on $\mathbb{H} \times \mathbb{C}^m$

Let G be a real Lie group, H a closed subgroup and V a finite dimensional complex vector space. For an element $x \in G$ we denote the coset xH by $\overline{x}$. A 1-cocycle of G on G/H with values in V is a smooth function $\alpha : G \times G/H \longrightarrow GL(V)$ satisfying the following condition

$$\alpha(g_1 g_2, \overline{x}) = \alpha(g_2, \overline{x})\, \alpha(g_1, g_2\overline{x})$$

for all $g_1, g_2, x \in G$. The associated right action of G on $C^\infty\big(G/H\big) \otimes V$ is

$$f|_\alpha[g](\overline{x}) := \alpha(g, \overline{x}) f(g\overline{x}), \qquad g, x \in G$$

and the associated representation of H on V is

$$\pi_\alpha(h) := \alpha(h, \overline{x}),$$

where $h \in H$ and e is the identity element of G.

Definition 8.1. Let V and V' be two finite dimensional complex vector spaces. Let α and α' be two 1-cocycles of G on G/H with values in V and V' respectively. A differential operator $D : C^\infty(G/H) \otimes V \longrightarrow C^\infty(G/H) \otimes V'$ is *covariant* from $|_\alpha$ to $|_{\alpha'}$ if for all $g \in G$ and $f \in C^\infty(G/H) \otimes V$, we have

$$D\big(f|_\alpha[g]\big) = (Df)|_{\alpha'}[g].$$

Let $\mathbb{D}_{\alpha,\alpha'}(G/H)$ be the space of all covariant differential operators from $|_\alpha$ to $|_{\alpha'}$ and $\mathbb{D}^q_{\alpha,\alpha'}(G/H)$ be the space of those of order $\le q$. When $\alpha = \alpha'$, we refer to such operators as $|_\alpha$-invariant, and we write simply $\mathbb{D}_\alpha(G/H)$ and $\mathbb{D}^q_\alpha(G/H)$

We consider our case

$$G^J = SL_2(\mathbb{R}) \ltimes H_{\mathbb{R}}^{(m)} \quad \text{and} \quad K^J = SO(2) \ltimes S(m,\mathbb{R}).$$

We observe that K^J is an abelian closed subgroup of G^J. We define the linear map $\xi : \mathfrak{g}_{\mathbb{C}}^J \longrightarrow \mathfrak{g}_{\mathbb{C}}^J$ by $\xi(X) = \widehat{X}$ with $X \in \mathfrak{g}_{\mathbb{C}}^J$, where

$$\begin{aligned} \widehat{H} : &= i\,(F-E), \qquad \widehat{E} := \frac{1}{2}\,\{H + i\,(E+F)\}\,, \qquad \widehat{F} := \frac{1}{2}\,\{H - i\,(E+F)\}\,, \\ \widehat{R}_{kl} : &= \frac{1}{2}\,R_{kl}, \qquad \widehat{e}_j := \frac{1}{2}\,(e_j - i\,f_j), \qquad \widehat{f}_j := \frac{1}{2}\,(e_j + i\,f_j). \end{aligned}$$

It is easy to see that there is a unique K^J-splitting

$$\mathfrak{g}_{\mathbb{C}}^J = \mathfrak{k}_*^J \oplus \mathfrak{p}_*^J, \tag{8.1}$$

where

$$\mathfrak{k}_*^J = \text{span}\{\widehat{H},\, \widehat{R}_{kl} \mid 1 \leq k \leq l \leq m\}$$

and

$$\mathfrak{p}_*^J = \text{span}\{\widehat{E},\, \widehat{F},\, \widehat{e}_j,\, \widehat{f}_j \mid 1 \leq j \leq m\}.$$

We note that ξ is an automorphism of Lie algebras and so the given basis of $\mathfrak{p}_*^J$ is a K^J-eigenbasis : the $\widehat{H}$=weights of $\widehat{E},\ \widehat{F},\ \widehat{e}_j$ and $\widehat{f}_j$ are 1, -2, -1 and 1 respectively. We take the scalar valued 1-cocycle $\alpha := \gamma_{k,\mathcal{M}}$ defined by (3.7). We set $\mathcal{M} = (\mathcal{M}_{kl})$. We let $\pi_{k,\mathcal{M}} : K^J \longrightarrow GL_1(\mathbb{C})$ be the one-dimensional representation of K^J defined by

$$\pi_{k,\mathcal{M}}(h) := \gamma_{k,\mathcal{M}}(h,\overline{e})^{-1},$$

where $h \in K^J$ and $\overline{e} = (i,0) = eK^J$ with the identity element e in G^J. We remark that ξ maps the Casimir operator Ω_m to $\left(\frac{i}{2}\right)^m \Omega_m$.

Definition 8.2. Let $k \in \mathbb{Z}$ and $\mathcal{M} \in S(m,\mathbb{C})$. We define the raising operators X_+, Y_+ and the lowering operators X_- and Y_-:

$$\begin{aligned} X_+^{k,\mathcal{M}} : &= 2\,i\left(\partial_\tau + y^{-1}\,{}^t v \partial_z + y^{-2}\widetilde{\mathcal{M}}[v]\right), \qquad X_-^{k,\mathcal{M}} := -2\,i\,y\left(y\,\partial_{\overline{\tau}} + {}^t v\,\partial_{\overline{z}}\right), \\ Y_+^{k,\mathcal{M}} : &= i\,\partial_z + 2\,i\,y^{-1}\widetilde{\mathcal{M}}v, \qquad Y_-^{k,\mathcal{M}} := -\,i\,y\,\partial_{\overline{z}}, \qquad \widetilde{\mathcal{M}} := 2\,\pi\,i\,\mathcal{M}. \end{aligned}$$

We write $Y_{\pm,j}^{k,\mathcal{M}}$ for the j-th entry of $Y_{\pm}^{k,\mathcal{M}}$ $(1 \leq j \leq m)$.

For brevity, we write

$$\mathbb{D}(k,\mathcal{M};k',\mathcal{M}') := \mathbb{D}_{\gamma_{k,\mathcal{M}},\gamma_{k',\mathcal{M}'}}\left(G^J/K^J\right)$$

and

$$\mathbb{D}^q(k,\mathcal{M};k',\mathcal{M}') := \mathbb{D}^q_{\gamma_{k,\mathcal{M}},\gamma_{k',\mathcal{M}'}}\left(G^J/K^J\right),$$

where $k,k'\in\mathbb{Z}$, $\mathcal{M},\mathcal{M}'\in S(m,\mathbb{C})$, $q\in\mathbb{Z}\cup\{0\}$ and $G^J/K^J=\mathbb{H}\times\mathbb{C}^m$. We also write

$$\mathbb{D}_{k,\mathcal{M}}:=\mathbb{D}(k,\mathcal{M};k,\mathcal{M})\qquad\text{and}\qquad\text{and}\qquad\mathbb{D}^q_{k,\mathcal{M}}:=\mathbb{D}^q(k,\mathcal{M};k,\mathcal{M}).$$

Conley and Raum [5] obtained the following three results.

Proposition 8.1. *(1) The spaces $\mathbb{D}^1(k,\mathcal{M};k\pm 2,\mathcal{M})$ are one-dimensional. In fact $\mathbb{D}^1(k,\mathcal{M};k\pm 2,\mathcal{M})=\mathbb{C}X_\pm^{k,\mathcal{M}}$.*

(2) $\mathbb{D}^1(k,\mathcal{M};k\pm 1,\mathcal{M})=\mathrm{Span}\{Y^{k,\mathcal{M}}_{\pm,j}\mid 1\leq j\leq m\}$ are m-dimensional.

(3) $\mathbb{D}^0_{k,\mathcal{M}}=\mathbb{D}^1_{k,\mathcal{M}}=\mathbb{C}$.

(4) All other $\mathbb{D}^1(k,\mathcal{M};k',\mathcal{M}')$ are zero.

(5) We have the following commutation relations

$$[X_-,X_+]=-k,\quad [Y_{-,j},Y_{+,j'}]=i\,\widetilde{\mathcal{M}}_{jj'},\quad [X_-,Y_+]=-Y_-,$$
$$[Y_-,X_+]=Y_+,\quad [X_+,Y_+]=[X_-,Y_-]=0.$$

Proposition 8.2. *Any covariant differential operator of order q may be expressed as a linear combination of products up to q raising and lowering operators. There is a unique such expression in which the raising operators are all to the left of the lowering operators. The expression of this form for the Casimir operator $\mathcal{C}^{k,\mathcal{M}}$ is*

$$\begin{aligned}\mathcal{C}^{k,\mathcal{M}} &= -2\,X_+X_-+i\left(X_+\,\widetilde{\mathcal{M}}^{-1}[Y_-]-\widetilde{\mathcal{M}}^{-1}[Y_+]X_-\right)\\ &\quad -\frac{1}{2}\left\{\widetilde{\mathcal{M}}^{-1}[Y_+]\,\widetilde{\mathcal{M}}^{-1}[Y_-]-\,{}^tY_+\big({}^tY_+\,\widetilde{\mathcal{M}}^{-1}Y_-\big)\widetilde{\mathcal{M}}^{-1}Y_-\right\}\\ &\quad -\frac{i}{2}\,(2\,k-m-3)\,{}^tY_+\,\widetilde{\mathcal{M}}^{-1}Y_-.\end{aligned}\tag{8.2}$$

Proposition 8.3. *The algebra $\mathbb{D}_{k,\mathcal{M}}$ is generated by $\mathbb{D}^3_{k,\mathcal{M}}$. Bases for $\mathbb{D}^2_{k,\mathcal{M}}$ and $\mathbb{D}^3_{k,\mathcal{M}}$ are given by*

$$\begin{aligned}\mathbb{D}^2_{k,\mathcal{M}} &= \mathrm{Span}\{1,\ X_+\,X_-,\ Y_{+,i}\,Y_{-,j}\mid 1\leq i,j\leq m\},\\ \mathbb{D}^3_{k,\mathcal{M}} &= \mathrm{Span}\{\,X_+\,Y_{-,i}\,Y_{-,j},\ Y_{+,i}\,Y_{+,j}\,X_-\mid 1\leq i\leq j\leq m\}\oplus\mathbb{D}^2_{k,\mathcal{M}}.\end{aligned}$$

Therefore we have

$$\dim_{\mathbb{C}}\mathbb{D}^2_{k,\mathcal{M}}=m^2+2\qquad\text{and}\qquad\dim_{\mathbb{C}}\mathbb{D}^3_{k,\mathcal{M}}=2\,m^2+m+2.$$

9. Final remarks

In this final section we briefly remark the general case $n > 1$ and $m > 1$.

We let

$$\mathbb{H}_n = \{\,\Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega, \quad \operatorname{Im}\Omega > 0\,\}$$

be the Siegel upper half plane of degree n and let

$$Sp(n,\mathbb{R}) = \{M \in \mathbb{R}^{(2n,2n)} \mid {}^tMJ_nM = J_n\ \}$$

be the symplectic group of degree n, where

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

$Sp(n,\mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{9.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $\Omega \in \mathbb{H}_n$.

For brevity, we write $G_n = Sp(n,\mathbb{R})$. The isotropy subgroup K_n at iI_n for the action (9.1) is a maximal compact subgroup given by

$$K_n = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \,\middle|\, A\,{}^tA + B\,{}^tB = I_n,\ A\,{}^tB = B\,{}^tA,\ A, B \in \mathbb{R}^{(n,n)} \right\}.$$

Let $\mathfrak{k}_n$ be the Lie algebra of K_n. Then the Lie algebra $\mathfrak{g}_n$ of G_n has a Cartan decomposition $\mathfrak{g}_n = \mathfrak{k}_n \oplus \mathfrak{p}_n$, where

$$\mathfrak{g}_n = \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & -\,{}^tX_1 \end{pmatrix} \,\middle|\, X_1, X_2, X_3 \in \mathbb{R}^{(n,n)},\ X_2 = {}^tX_2,\ X_3 = {}^tX_3 \right\},$$

$$\mathfrak{k}_n = \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \,\middle|\, {}^tX + X = 0,\ Y = {}^tY \right\},$$

$$\mathfrak{p}_n = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \,\middle|\, X = {}^tX,\ Y = {}^tY,\ X, Y \in \mathbb{R}^{(n,n)} \right\}.$$

The subspace $\mathfrak{p}_n$ of $\mathfrak{g}_n$ may be regarded as the tangent space of $\mathbb{H}_n$ at iI_n.

We consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \{\, (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\,{}^t\lambda \text{ symmetric } \}$$

endowed with the following multiplication law

$$(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') = (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda\,{}^t\mu' - \mu\,{}^t\lambda')$$

with $(\lambda,\mu;\kappa),(\lambda',\mu';\kappa')\in H_{\mathbb{R}}^{(n,m)}$. We define the semidirect product of $Sp(n,\mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$

$$G^J_{n,m}=Sp(n,\mathbb{R})\ltimes H_{\mathbb{R}}^{(n,m)}$$

endowed with the following multiplication law

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot\big(M',(\lambda',\mu';\kappa')\big)=\big(MM',(\tilde{\lambda}+\lambda',\tilde{\mu}+\mu';\kappa+\kappa'+\tilde{\lambda}\,{}^t\mu'-\tilde{\mu}\,{}^t\lambda')\big)$$

with $M,M'\in Sp(n,\mathbb{R}),(\lambda,\mu;\kappa),\ (\lambda',\mu';\kappa')\in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde{\lambda},\tilde{\mu})=(\lambda,\mu)M'$. Then $G^J_{n,m}$ acts on $\mathbb{H}_n\times\mathbb{C}^{(m,n)}$ transitively by

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot(\Omega,Z)=\Big(M\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}\Big), \tag{9.2}$$

where $M=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega,Z)\in\mathbb{H}_n\times\mathbb{C}^{(m,n)}$.

The stabilizer $K^J_{n,m}$ of $G^J_{n,m}$ at $(iI_n,0)$ for the action (9.2) is given by

$$K^J_{n,m}=\Big\{\big(k,(0,0;\kappa)\big)\ \big|\ k\in K_n,\ \kappa={}^t\kappa\in\mathbb{R}^{(m,m)}\Big\}.$$

Therefore $\mathbb{H}_n\times\mathbb{C}^{(m,n)}\cong G^J_{n,m}/K^J_{n,m}$ is a homogeneous space of *non-reductive type.* The Lie algebra $\mathfrak{g}^J_{n,m}$ of $G^J_{n,m}$ has a decomposition

$$\mathfrak{g}^J_{n,m}=\mathfrak{k}^J_{n,m}+\mathfrak{p}^J_{n,m},$$

where

$$\mathfrak{g}^J_{n,m}=\Big\{\big(Z,(P,Q,R)\big)\ \big|\ Z\in\mathfrak{g}_n,\ P,Q\in\mathbb{R}^{(m,n)},\ R={}^tR\in\mathbb{R}^{(m,m)}\Big\},$$

$$\mathfrak{k}^J_{n,m}=\Big\{\big(X,(0,0,R)\big)\ \big|\ X\in\mathfrak{k}_n,\ R={}^tR\in\mathbb{R}^{(m,m)}\Big\},$$

$$\mathfrak{p}^J_{n,m}=\Big\{\big(Y,(P,Q,0)\big)\ \big|\ Y\in\mathfrak{p}_n,\ P,Q\in\mathbb{R}^{(m,n)}\Big\}.$$

Thus the tangent space of the homogeneous space $\mathbb{H}_{n,m}$ at $(iI_n,0)$ is identified with $\mathfrak{p}^J_{n,m}$. We note that the Jacobi group $G^J_{n,m}$ is *not* a reductive Lie group and that the homogeneous space $\mathbb{H}_n\times\mathbb{C}^{(m,n)}$ is not a symmetric space. From now on, for brevity we write $\mathbb{H}_{n,m}=\mathbb{H}_n\times\mathbb{C}^{(m,n)}$, called the Siegel-Jacobi space of degree n and index m.

For a coordinate $(\Omega,Z)\in\mathbb{H}_{n,m}$ with $\Omega=(\omega_{\mu\nu})\in\mathbb{H}_n$ and $Z=(z_{kl})\in\mathbb{C}^{(m,n)}$, we put

$$\begin{aligned}
\Omega&=X+iY, && X=(x_{\mu\nu}), && Y=(y_{\mu\nu})\ \text{real},\\
Z&=U+iV, && U=(u_{kl}), && V=(v_{kl})\ \text{real},\\
d\Omega&=(d\omega_{\mu\nu}), && d\overline{\Omega}=(d\overline{\omega}_{\mu\nu}), &&\\
dZ&=(dz_{kl}), && d\overline{Z}=(d\overline{z}_{kl}), &&
\end{aligned}$$

$$\frac{\partial}{\partial\Omega} = \left(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial\omega_{\mu\nu}}\right), \quad \frac{\partial}{\partial\overline{\Omega}} = \left(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial\overline{\omega}_{\mu\nu}}\right),$$

$$\frac{\partial}{\partial Z} = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \cdots & \frac{\partial}{\partial z_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_{1n}} & \cdots & \frac{\partial}{\partial z_{mn}} \end{pmatrix}, \quad \frac{\partial}{\partial\overline{Z}} = \begin{pmatrix} \frac{\partial}{\partial \overline{z}_{11}} & \cdots & \frac{\partial}{\partial \overline{z}_{m1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial \overline{z}_{1n}} & \cdots & \frac{\partial}{\partial \overline{z}_{mn}} \end{pmatrix},$$

where δ_{ij} denotes the Kronecker delta symbol.

C. L. Siegel [17] introduced the symplectic metric ds_n^2 on $\mathbb{H}_n$ invariant under the action (9.1) of $Sp(n,\mathbb{R})$ given by

$$ds_n^2 = \sigma\left(Y^{-1}d\Omega\, Y^{-1}d\overline{\Omega}\right) \tag{9.3}$$

and H. Maass [11] proved that the differential operator

$$\Delta_n = 4\,\sigma\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right) \tag{9.4}$$

is the Laplacian of $\mathbb{H}_n$ for the symplectic metric ds_n^2. Here $\sigma(A)$ denotes the trace of a square matrix A. In [23], the author proved that for any two positive real numbers A and B, the following metric

$$\begin{aligned} ds^2_{n,m;A,B} &= A\,\sigma\left(Y^{-1}d\Omega\, Y^{-1}d\overline{\Omega}\right) \\ &\quad + B\left\{\sigma\left(Y^{-1}\,{}^tV\,V\,Y^{-1}d\Omega\,Y^{-1}d\overline{\Omega}\right) + \sigma\left(Y^{-1}\,{}^t(dZ)\,d\overline{Z}\right)\right. \\ &\quad \left. -\sigma\left(V\,Y^{-1}d\Omega\,Y^{-1}\,{}^t(d\overline{Z})\right) - \sigma\left(V\,Y^{-1}d\overline{\Omega}\,Y^{-1}\,{}^t(dZ)\right)\right\} \end{aligned} \tag{9.5}$$

is a Riemannian metric on $\mathbb{H}_{n,m}$ which is invariant under the action (9.2) of the Jacobi group $G^J_{n,m}$.

The author [23] proved that for any two positive real numbers A and B, the Laplacian $\Delta_{n,m;A,B}$ of $(\mathbb{H}_{n,m}, ds^2_{n,m;A,B})$ is given by

$$\begin{aligned} \Delta_{n,m;A,B} &= \frac{4}{A}\left\{\sigma\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right) + \sigma\left(VY^{-1}\,{}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial Z}\right)\right. \\ &\quad \left. +\sigma\left(V\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial Z}\right) + \sigma\left({}^tV\,{}^t\!\left(Y\frac{\partial}{\partial\overline{Z}}\right)\frac{\partial}{\partial\Omega}\right)\right\} \\ &\quad +\frac{4}{B}\,\sigma\left(Y\frac{\partial}{\partial Z}\,{}^t\!\left(\frac{\partial}{\partial\overline{Z}}\right)\right). \end{aligned} \tag{9.6}$$

Using $G^J_{n,m}$-invariant differential operators on the Siegel-Jacobi space $\mathbb{H}_{n,m}$, we introduce a notion of Maass-Jacobi forms.

Definition 9.1. Let

$$\Gamma_{n,m} := Sp(n,\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(n,m)}$$

be the discrete subgroup of G^J, where

$$H_{\mathbb{Z}}^{(n,m)} = \Big\{ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)} \mid \lambda,\mu,\kappa \text{ are integral } \Big\}.$$

A smooth function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is called a Maass-Jacobi form on $\mathbb{H}_{n,m}$ if f satisfies the following conditions (MJ1)-(MJ3):

(MJ1) f is invariant under $\Gamma_{n,m}$.
(MJ2) f is an eigenfunction of the Laplacian $\Delta_{n,m;A,B}$ (cf. (9.6)).
(MJ3) f has a polynomial growth, that is, there exist a constant $C > 0$ and a positive integer N such that

$$|f(X+iY,Z)| \leq C\,|p(Y)|^N \quad \text{as } \det Y \longrightarrow \infty,$$

where $p(Y)$ is a polynomial in $Y = (y_{ij})$.

Remark 9.1. Let $\mathbb{D}_*$ be a commutative subalgebra of $\mathbb{D}(\mathbb{H}_{n,m})$ containing the Laplacian $\Delta_{n,m;A,B}$. We say that a smooth function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is a Maass-Jacobi form with respect to $\mathbb{D}_*$ if f satisfies the conditions $(MJ1)$, $(MJ2)_*$ and $(MJ3)$: the condition $(MJ2)_*$ is given by

$(MJ2)_*$ f is an eigenfunction of any invariant differential operator in $\mathbb{D}_*$.

Let ρ be a rational representation of $GL(n,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbb{R}^{(m,m)}$ be a symmetric half-integral semi-positive definite matrix of degree m. Let $C^\infty(\mathbb{H}_{n,m},V_\rho)$ be the algebra of all C^∞ functions on $\mathbb{H}_{n,m}$ with values in V_ρ. Let $J_{\rho,\mathcal{M}} : G^J_{n,m} \times \mathbb{H}_{n,m} \longrightarrow GL(V_\rho)$ be the canonical automorphic factor for $G^J_{n,m}$ on $\mathbb{H}_{n,m}$ given by

$$(9.7) \qquad J_{\rho,\mathcal{M}}(g,(\Omega,Z)) = e^{2\pi i\,\mathrm{tr}(\mathcal{M}[Z+\lambda\Omega+\mu](C\Omega+D)^{-1}C)} \times e^{-2\pi i\,\mathrm{tr}(\mathcal{M}(\lambda\Omega\,{}^t\lambda + 2\lambda\,{}^tZ + \kappa + \mu\,{}^t\lambda))}\rho(C\Omega+D),$$

where $g = (M,(\lambda,\mu;\kappa)) \in G^J_{n.m}$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $(\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$. We recall the Siegel's notation $\alpha[\beta] = {}^t\beta\alpha\beta$ for suitable matrices α and β.

We define the $|_{\rho,\mathcal{M}}$-slash action of $G^J_{n,m}$ on $C^\infty(\mathbb{H}_{n,m},V_\rho)$ as follows: If $f \in C^\infty(\mathbb{H}_{n,m},V_\rho)$ and $g \in G^J_{n,m}$,

$$(9.8) \qquad \Big(f|_{\rho,\mathcal{M}}[g]\Big)(\Omega,Z) := J_{\rho,\mathcal{M}}(g,(\Omega,Z))^{-1} f(g\cdot(\Omega,Z)).$$

We define $\mathbb{D}_{\rho,\mathcal{M}}$ to be the algebra of all differential operators D on $\mathbb{H}_{n,m}$ satisfying the following condition

$$(9.9) \qquad (Df)|_{\rho,\mathcal{M}}[g] = D(f|_{\rho,\mathcal{M}}[g])$$

for all $f \in C^\infty(\mathbb{H}_{n,m}, V_\rho)$ and for all $g \in G^J_{n,m}$. We denote by $\mathcal{Z}_{\rho,\mathcal{M}}$ the center of $\mathbb{D}_{\rho,\mathcal{M}}$.

We define an another notion of Maass-Jacobi forms as follows.

Definition 9.2. A vector-valued smooth function $\phi : \mathbb{H}_{n,m} \longrightarrow V_\rho$ is called a Maass-Jacobi form on $\mathbb{H}_{n,m}$ of type ρ and index $\mathcal{M}$ if it satisfies the following conditions $(MJ1)_{\rho,\mathcal{M}}$, $(MJ2)_{\rho,\mathcal{M}}$ and $(MJ3)_{\rho,\mathcal{M}}$:

$(MJ1)_{\rho,\mathcal{M}}$ $\phi|_{\rho,\mathcal{M}}[\gamma] = \phi$ for all $\gamma \in \Gamma_{n,m}$.

$(MJ2)_{\rho,\mathcal{M}}$ f is an eigenfunction of all differential operators in the center $\mathcal{Z}_{\rho,\mathcal{M}}$ of $\mathbb{D}_{\rho,\mathcal{M}}$.

$(MJ3)_{\rho,\mathcal{M}}$ f has a growth condition

$$\phi(\Omega, Z) = O\Big(e^{a \det Y} \cdot e^{2\pi \operatorname{tr}(\mathcal{M}[V]Y^{-1})}\Big)$$

as $\det Y \longrightarrow \infty$ for some $a > 0$.

The case $n = 1,\ m = 1$ and $\rho = \det^k (k = 0, 1, 2, \cdots)$ was studied by R. Bendt and R. Schmidt [1], A. Pitale [14] and K. Bringmann and O. Richter [4]. The case $n = 1,\ m =$arbitrary and $\rho = \det^k (k = 1, 2, \cdots)$ was dealt with by C. Conley and M. Raum [5]. In [5] the authors proved that the center $\mathcal{Z}_{\det^k,\mathcal{M}}$ of $\mathbb{D}_{\det^k,\mathcal{M}}$ is the polynomial algebra with one generator $\mathcal{C}^{k,\mathcal{M}}$ (cf. Theorem 3.2), the so-called *Casimir* operator which is a $|_{\det^k,\mathcal{M}}$-slash invariant differential operator of degree three for the case $n = m = 1$ or of degree four for the case $n = 1,\ m \geq 2$. As described in Section 6, Bringmann and Richter [4] considered the Poincaré series $\mathcal{P}^{(n,r)}_{k,\mathcal{M},s}$ (the case $n = m = 1$) (cf. (6.7)) that is a *harmonic* Maass-Jacobi form in the sense of Definition 9.2 and investigated its Fourier expansion and its Fourier coefficients. Here the *harmonicity* of $\mathcal{P}^{(n,r)}_{k,\mathcal{M},s}$ means that $\mathcal{C}^{k,\mathcal{M}}\mathcal{P}^{(n,r)}_{k,\mathcal{M},s} = 0$, i.e., $\mathcal{P}^{(n,r)}_{k,\mathcal{M},s}$ is an eigenfunction of $\mathcal{C}^{k,\mathcal{M}}$ with zero eigenvalue. Conley and Raum [5] generalized the results in [14] and [4] to the case $n = 1$ and m is an arbitrary positive integer.

Remark 9.2. In [3], Bringmann, Conley and Richter proved that the center of the algebra of differential operators invariant under the action of the Jacobi group over a complex quadratic field is generated by two Casimir operators of degree three. They also introduce an analogue of Kohnen's plus space for modular forms of half-integral weight over $K = \mathbb{Q}(i)$, and provide a lift from it to the space of Jacobi forms over K.

Definition 9.3. Let ρ and ρ' be two rational representations of $GL(n, \mathbb{C})$ on finite dimensional complex vector spaces V_ρ and V'_ρ respectively. Let $\mathcal{M}$ and $\mathcal{M}'$ be two symmetric half-integral semi-positive matrices of degree m. A differential operator

$T : C^\infty(\mathbb{H}_{n,m}) \otimes V_\rho \longrightarrow C^\infty(\mathbb{H}_{n,m}) \otimes V_{\rho'}$ is covariant from $|_{\rho,\mathcal{M}}$ to $|_{\rho',\mathcal{M}'}$ if T satisfies the following condition

$$T\Big(f|_{\rho,\mathcal{M}}[g]\Big) = (Tf)|_{\rho',\mathcal{M}'}[g]) \tag{9.10}$$

for all $f \in C^\infty(\mathbb{H}_{n,m}) \otimes V_\rho$ and for all $g \in G^J_{n,m}$.

Let $\mathbb{D}(\rho, \mathcal{M}; \rho', \mathcal{M}')$ be the space of all covariant differential operators on $\mathbb{H}_{n,m}$ from $|_{\rho,\mathcal{M}}$ to $|_{\rho',\mathcal{M}'}$, and let $\mathbb{D}^q(\rho, \mathcal{M}; \rho', \mathcal{M}')$ be the space of all covariant differential operators of order $\leq q$ on $\mathbb{H}_{n,m}$ from $|_{\rho,\mathcal{M}}$ to $|_{\rho',\mathcal{M}'}$. When $\rho = \rho'$ and $\mathcal{M} = \mathcal{M}'$, we refer to such differential operators as $|_{\rho,\mathcal{M}}$-invariant, and we write simply $\mathbb{D}_{\rho,\mathcal{M}}$ and $\mathbb{D}^q_{\rho,\mathcal{M}}$ instead of $\mathbb{D}(\rho, \mathcal{M}; \rho, \mathcal{M})$ and $\mathbb{D}^q(\rho, \mathcal{M}; \rho, \mathcal{M})$ respectively.

We present the natural problems.

Problem 1. Find the generators of the algebra $\mathbb{D}_{\rho,\mathcal{M}}$.

Problem 2. Find all the relations among a complete list of generators of $\mathbb{D}_{\rho,\mathcal{M}}$.

Finally we consider the special case that $\rho = \mathbf{1}$ is a trivial representation of $GL(n, \mathbb{C})$ and $\mathcal{M} = 0$. Let

$$T_{n,m} := S(m, \mathbb{C}) \times \mathbb{C}^{(m,n)}$$

be the complex vector space of dimension $\frac{n(n+1)}{2} + mn$. We obtain the natural action of $U(n)$ on $T_{n,m}$ given by

$$h \cdot (\omega, \zeta) := (h\, \omega\, {}^t h, \zeta\, {}^t h), \quad h \in U(n),\ \omega \in S(m, \mathbb{C}),\ \zeta \in \mathbb{C}^{(m,n)}. \tag{9.11}$$

We refer to [26] for a precise detail. Then the action (9.11) induces the action $\tau_{n,m}$ of $U(n)$ on the polynomial algebra $\mathrm{Pol}(T_{n,m})$ consisting of all polynomial functions on $T_{n,m}$. We denote by $\mathrm{Pol}(T_{n,m})^{U(n)}$ the subalgebra of $\mathrm{Pol}(T_{n,m})$ invariant under the action $\tau_{n,m}$ of $U(n)$. The we have the so-called Helgason map

$$\Theta_{n,m} : \mathrm{Pol}(T_{n,m})^{U(n)} \longrightarrow \mathbb{D}_{\mathbf{1},0} = \mathbb{D}(\mathbf{1}, 0; \mathbf{1}, 0)$$

defined by

$$\Big(\Theta_{n,m}(P)f\Big)(gK^J) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g \exp\left(\sum_{\alpha=1}^{N_\star} t_\alpha \eta_\alpha\right) K^J\right)\right]_{(t_\alpha)=0}, \tag{9.12}$$

where $N_\star = n(n+1) + 2mn$, $\big\{\eta_\alpha \,|\, 1 \leq \alpha \leq N_\star\big\}$ is a basis of $\mathfrak{p}^J_{n,m}$ and $P \in \mathrm{Pol}(T_{n,m})^{U(n)}$. The map $\Theta_{n,m}$ is a linear bijection but is not multiplicative.

The following natural problems arise.

Problem 3. Find a complete list of explicit generators of $\mathrm{Pol}(T_{n,m})^{U(n)}$.

Problem 4. Find all the relations among a complete list of generators of $\mathrm{Pol}(T_{n,m})^{U(n)}$.

Problem 5. Find an easy or effective way to express the images of the above invariant polynomials or generators of $\mathrm{Pol}(T_{n,m})^{U(n)}$ under the Helgason map $\Theta_{n,m}$ explicitly.

Recently Problem 3 was solved completely in [9].

References

[1] R. Berndt and R. Schmidt, *Elements of the Representation Theory of the Jacobi Group*, Progress in Mathematics, **163**, Birkhäuser, Basel, 1998.

[2] W. Borho, *Primitive und vollprimitive Ideale in Einhüllenden von* $\mathfrak{so}(5,\mathbb{C})$, J. Algebra, **43**(1976), 619-654.

[3] K. Bringmann, C. Conley and O. K. Richter, *Jacobi forms over complex quadratic fields via the cubic Casimier operators*, preprint.

[4] K. Bringmann and O. K. Richter, *Zagier-type dualities and lifting maps for harmonic Maass-Jacobi forms*, Advances in Math., **225**(2010), 2298-2315.

[5] C. Conley and M. Raum, *Harmonic Maass-Jacobi forms of degree 1 with higher rank indices*, arXiv:1012.289/v1 [math.NT], 13 Dec 2010.

[6] R. Campoamor-Stursburg and S. G. Low, *Virtual copies of semisimple Lie algebras in enveloping algebras of semidirect products and Casimir operators*, J. Phys. A: Math. Theor., **42**(2009), 065205.

[7] S. Helgason, Groups and geometric analysis, Academic Press, New York (1984).

[8] R. Howe, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond,* The Schur lectures (1992) (Tel Aviv), Israel Math. Conf. Proceedings, **vol. 8** (1995), 1-182.

[9] M. Itoh, H. Ochiai and J.-H. Yang, *Invariant differential operators on Siegel-Jacobi space*, preprint, 2013.

[10] N. N. Lebedev, Special Functions and their Applications, Dover, New York (1972).

[11] H. Maass, *Die Differentialgleichungen in der Theorie der Siegelschen Modulfunktionen*, Math. Ann., **126**(1953), 44-68.

[12] H. Maass, Siegel modular forms and Dirichlet series, Lecture Notes in Math., **vol. 216**, Springer-Verlag, Berlin-Heidelberg-New York (1971).

[13] H. Ochiai, *A remark on the generators of invariant differential operators on Siegel-Jacobi space of the smallest size,* preprint, 2011.

[14] A. Pitale, *Jacobi Maass forms*, Abh. Math. Sem. Univ. Hamburg, **79**(2009), 87-111.

[15] C. Quesne, *Casimir operators of semidirect sum Lie algebras*, J. Phys. A: Math. Gen., **21**(1988), L321-L324.

[16] J.-P. Serre, A Course in Arithmetic, Springer-Verlag, Berlin-Heidelberg-New York (1973).

[17] C. L. Siegel, *Symplectic Geometry,* Amer. J. Math., **65**(1943), 1-86; Academic Press, New York and London (1964); Gesammelte Abhandlungen, no. 41, vol. II, Springer-Verlag (1966), 274-359.

[18] N.-P. Skoruppa, *Explicit formulas for the Fourier coefficients of Jacobi and elliptic modular forms*, Invent. Math., **102**(1990), 501-520.

[19] G. N. Watson, The Treatise on the Theory of Bessel Functions, Cambridge Univ. Press, London (1962).

[20] J.-H. Yang, *The Method of Orbits for Real Lie Groups,* Kyungpook Math. J., **42(2)**(2002), 199-272 *or* arXiv:math.RT/0602056.

[21] J.-H. Yang, *A note on Maass-Jacobi forms,* Kyungpook Math. J. **43(4)**(2003), 547-566 *or* arXiv:math.NT/0612387.

[22] J.-H. Yang, *A note on a fundamental domain for Siegel-Jacobi space,* Houston J. Math., **32(3)**(2006), 701-712 *or* arXiv:math.NT/0507218.

[23] J.-H. Yang, *Invariant metrics and Laplacians on Siegel-Jacobi space,* Journal of Number Theory, **127**(2007), 83-102 *or* arXiv:math.NT/0507215.

[24] J.-H. Yang, *A partial Cayley transform for Siegel-Jacobi disk,* J. Korean Math. Soc., **45**(2008), 781-794 *or* arXiv:math.NT/0507216.

[25] J.-H. Yang, *Invariant metrics and Laplacians on Siegel-Jacobi disk,* Chinese Annals of Mathematics, **31B(1)**(2010), 85-100 *or* arXiv:math.NT/0507217.

[26] J.-H. Yang, *Invariant Differential Operators on the Siegel-Jacobi Space,* arXiv: 1107.0509v1 [math.NT], 4 July 2011.

[E] Theory of Vector Bundles

KIAS (고등과학원)에서 (2005년 4월)

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A NOTE ON HOLOMORPHIC VECTOR BUNDLES OVER COMPLEX TORI

JAE-HYUN YANG

1. Let L be a lattice in $\boldsymbol{C}^n$. A holomorphic automorphic factor of rank r for the lattice L is a holomorphic mapping

$$J : L \times \boldsymbol{C}^n \longrightarrow GL(r;\boldsymbol{C})$$

such that

(1) $J(\alpha, z)$ $(\alpha\in L,\ z\in \boldsymbol{C}^n)$ is holomorphic in z,

(2) $J(\alpha+\beta, z)=J(\alpha, z+\beta)J(\beta, z)$ for all $\alpha, \beta\in L$ and $z\in \boldsymbol{C}^n$.

We have the following free action of L on $\boldsymbol{C}^n\times\boldsymbol{C}^r$ defined by

$$(z, \xi)\alpha=(z+\alpha, J(\alpha, z)\xi),\ \alpha\in L,\ z\in \boldsymbol{C}^n,\ \ \xi\in\boldsymbol{C}^r.$$

The quotient of $\boldsymbol{C}^n\times\boldsymbol{C}^r$ by this group action of L is a holomorphic vector bundle E_J over the complex torus $M=\boldsymbol{C}^n/L$. Holomorphic vector bundles over the complex torus $M=\boldsymbol{C}^n/L$ are always obtained in this way.

In this short paper, we characterize projectively flat vector bundles over a complex torus which is simple.

2. A holomorphic vector bundle over a complex manifold is said to be simple if its endomorphisms are all scalars. It is easy to show that the vector bundle E_J over the complex torus $M=\boldsymbol{C}^n/L$ defined by an automorphic factor J is simple if and only if scalars are holomorphic maps $B : \boldsymbol{C}^n \rightarrow GL(r;\boldsymbol{C})$ such that

$$B(z+\alpha)\ J(\alpha, z)=J(\alpha, z)B(z) \text{ for all } \alpha\in L.$$

In his paper [1], Morikawa shows the following theorem.

THEOREM. *Let J be a holomorphic automorphic factor of rank r for the lattice L in $\boldsymbol{C}^n$ such that*

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(1) *the associated vector bundle E_J is simple,*

(2) $J(\alpha, z+\beta)J(\alpha, z)^{-1}$ $(\alpha, \beta\in L)$ *are constants.*

Then there exists an isogeny $\psi : N\to \boldsymbol{C}^n/L$ *of degree* r *and a line bundle* F *over the complex torus* N *such that* E_J *is the direct image of* F *under the isogeny* ψ.

REMARK. *We can also show that* $T_g{}^*F\not\cong F$ *for all* $g\in \ker\psi$, *where* T_g *is the translation of the complex torus* N *by* $g\in N$.

Let J *be an automorphic factor of rank* r *for the lattice* L *in* $\boldsymbol{C}^n$ *satisfying the hypotheses in the above theorem. Then, by Oda* [2], *the homomorphism* $H^j(M, \mathcal{O})\to H^j(M, \operatorname{End}(E_J))$ *induced by* $\mathcal{O}\to \operatorname{End}(E_J)$ *is an isomorphism for each* j, *where* $M=\boldsymbol{C}^n/L$. *Therefore we obtain, for each* j,

$$\dim_{\boldsymbol{C}} H^j(M, \operatorname{End}(E_J))=\dim_{\boldsymbol{C}}H^j(M, \mathcal{O})=\binom{n}{j}.$$

3. Let E a holomorphic vector bundle of rank r over a complex torus $M=\boldsymbol{C}^n/L$ with the corresponding automorphic factor $J : L\times \boldsymbol{C}^n\to GL(r;\boldsymbol{C})$. Let

$$p : GL(r;\boldsymbol{C})\to PGL(r;\boldsymbol{C})$$

be the natural projection to the projective general linear group and define

$$\hat{J}=p\circ J : L\times\boldsymbol{C}^n\to PGL(r;\boldsymbol{C}).$$

Then $\hat{J}$ is an automorphic factor for the projective bundle $P(E)$ over M.

We assume that E is projectively flat. Then $P(E)$ is defined by a representation of L into $PGL(r;\boldsymbol{C})$. Replacing J by an equivalent automorphic factor, we may assume that $\hat{J}$ is a representation of L into $PGL(r;\boldsymbol{C})$. Since $\hat{J}$ is independent of z, we can write

$$J(\alpha, z)=f(\alpha, z)J(\alpha, 0),\quad \alpha\in L,\quad z\in\boldsymbol{C}^n,$$

where $f : L\times\boldsymbol{C}^n\to\boldsymbol{C}^*$ is a scalar function. Since $\det J$ is an automorphic factor for the line bundle $\det E$, we may write

$$\det J(\alpha, z)=\chi(\alpha)\exp\left\{H(z,\alpha)+\frac{1}{2}H(\alpha,\alpha)\right\},\ \alpha\in L,\quad z\in\boldsymbol{C}^n,$$

where $\chi : L\to\boldsymbol{C}^*$ is a semi-character of L, and H is an Hermitian form on $\boldsymbol{C}^n$. Since $f(\alpha, 0)=f(0, z)=1$,

$$f(\alpha, z)=\exp\left\{\frac{1}{r}H(z,\alpha)\right\},\quad \alpha\in L,\quad z\in\boldsymbol{C}^n.$$

Now we assume that E admits a projectively flat Hermitian structure h. Then we may assume that $\hat{J}$ is a representation of L into $PU(r)$. Let $\tilde{h}$ be the induced Hermitian structure in $\tilde{E}=\pi^*E=\boldsymbol{C}^n\times\boldsymbol{C}^r$, where $\pi: \boldsymbol{C}^n\rightarrow\boldsymbol{C}^n/L$ is the natural projection. Then

(A) $\tilde{h}(z)={}^t\overline{J(\alpha, z)}\tilde{h}(z+\alpha)J(\alpha, z), \quad \alpha\in L, \quad z\in\boldsymbol{C}^n.$

The curvature form Ω of E is of the form

$$\Omega=\delta I_r, \quad \delta \text{ is a 2-form.}$$

Since its trace is the curvature of det E, we have

$$\delta=\frac{1}{r}\sum_{j,k} H_{j\bar{k}}\, dz^j\wedge d\bar{z}^k,$$

with constant coefficient $H_{j\bar{k}}$. Thus

$$\begin{aligned}\tilde{\omega}(z)&=\tilde{h}(z)^{-1}\partial\tilde{h}(z)\\ &=-\frac{1}{r}H(dz, z)I_r+\Xi(z),\end{aligned}$$

where $H(dz, z)=\sum H_{j\bar{k}}\bar{z}^k dz^j$ and Ξ is a holomorphic 1-form with values in the Lie algebra of $CU(r)=\{cU;\ c\in\boldsymbol{C}^*,\ U\in U(r)\}$. Since $\Xi+{}^t\overline{\Xi}=\psi I_r$ (ψ is a 1-form) and Ξ is holomorphic, it follows that

$$\Xi=\theta I_r, \quad \theta \text{ is a holomorphic 1-form.}$$

By a simple calculation, we obtain

$$\Xi(z+\alpha)=\Xi(z), \quad \alpha\in L.$$

That is, θ is a holomorphic 1-form on $M=\boldsymbol{C}^n/L$. Hence

$$\theta=\sum_{j=1}^n C_j dz_j, \quad C_j\text{'s are constants.}$$

If we solve the differential equation

$$\tilde{h}^{-1}\partial\tilde{h}=\tilde{\omega}(z)=-\frac{1}{r}\sum H_{j\bar{k}}\bar{z}^k dz^j+\sum_{j=1}^n C_j dz^j,$$

we obtain

$$\tilde{h}(z)=\tilde{h}(0)\ \exp\left\{-\frac{1}{r}H(z, z)+C(z)+\overline{C(z)}\right\},$$

where $C(z)=\sum_{j=1}^n C_j z^j$.

Using the isomorphism of the bundle $\tilde{E}$ defined by

$$(z, \xi)\in\tilde{E}\longrightarrow(z,\ \exp\ \{C(z)\}\xi)\in\tilde{E},$$

we may assume that $C(z)=0$. By a linear change of coordinates in $\boldsymbol{C}^r$, we may assume that $\tilde{h}(0)=I_r$. Therefore

$$\tilde{h}(z)=\exp\left\{-\frac{1}{r}H(z,z)\right\}I_r,\quad z\in \boldsymbol{C}^n.$$

By the formular (A)

$$J(\alpha,z)=U(\alpha)\ \exp\left\{\frac{1}{r}H(z,\alpha)+\frac{1}{2r}H(\alpha,\alpha)\right\},$$

where $U(\alpha)=J(\alpha,0)\ \exp\left\{-\frac{1}{2r}H(\alpha,\alpha)\right\}$ is a unitary matrix.

In summary, if E admits a projectively flat Hermitian structure h, its associated automorphic factor J can be written as follows:

(B) $J(\alpha,z)=U(\alpha)\ \exp\left\{\frac{1}{r}H(z,\alpha)+\frac{1}{2r}H(\alpha,\alpha)\right\},\quad \alpha\in L,\quad z\in \boldsymbol{C}^n,$

where

(i) H is an Hermitian form on $\boldsymbol{C}^n$ and its imaginary part A satisfies

$$\frac{1}{\pi}A(\alpha,\beta)\in Z \text{ for } \alpha,\beta\in L,$$

(ii) $U:L\rightarrow U(r)$ is a semi-representation in the sense that it satisfies

$$U(\alpha+\beta)=U(\alpha)U(\beta)\ \exp\left\{\frac{i}{r}A(\beta,\alpha)\right\},\quad \alpha,\beta\in L.$$

4. Let E be a simple holomorphic vector bundle of rank r over the compex torus $M=\boldsymbol{C}^n/L$ which admits a projectively flat Hermitian structure. Then its automorphic factor J is given by the formula (B). Then, for all $\alpha,\beta\in L$,

$$\begin{aligned}&J(\alpha,z+\beta)J(\alpha,z)^{-1}\\&=U(\alpha)\ \exp\left\{\frac{1}{r}H(z+\beta,\alpha)+\frac{1}{2r}H(\alpha,\alpha)\right\}\\&\quad U(\alpha)^{-1}\exp\left\{-\frac{1}{r}H(z,\alpha)-\frac{1}{2r}H(\alpha,\alpha)\right\}\\&=\exp\left\{\frac{1}{r}H(\beta,\alpha)\right\}\end{aligned}$$

By Theorem (Morikawa), there exists a sublattice $\tilde{L}$ of L and a line bundle F over $N=\boldsymbol{C}^n/\tilde{L}$ such that $[L:\tilde{L}]=r$ and $\phi_*F\cong E$, where ϕ : $\boldsymbol{C}^n/\tilde{L}\rightarrow \boldsymbol{C}^n/L$ is the natural isogeny. And we have

$$H^j(M,\mathcal{O})\cong H^j(M,\ \mathrm{End}(E)) \text{ for all } j$$

and

$$\dim_{\boldsymbol{C}}H^j(M,\ \mathrm{End}(E))=\binom{n}{j}.$$

We set

$$\omega_\alpha(z)=J(\alpha,z)^{-1}\ dJ(\alpha,z),\quad \alpha\in L.$$

Then we obtain a system of integrable connections satisfying

(1) $d\omega_\alpha(z)+\omega_\alpha(z)\wedge\omega_\alpha(z)=0$

(2) $\omega_{\alpha+\beta}(z)=\omega_\alpha(z)+J(\alpha,z)^{-1}\ \omega_\beta(z+\alpha)J(\alpha,z),\ \alpha,\beta\in L.$

If we write

$$\omega_\alpha(z)=\sum_{l=1}^{n}A_{\alpha l}(z)dz_l\quad(\alpha\in L,\ \ 1\leqq l\leqq n),$$

then all $A_{\alpha l}(z)$ are constants. Indeed,

$$\begin{aligned}&\omega_\alpha(z+\beta)-\omega_\alpha(z)\\&=J(\alpha,z+\beta)^{-1}dJ(\alpha,z+\beta)-J(\alpha,z)^{-1}dJ(\alpha,z)\\&=J(\alpha,z+\beta)^{-1}\ d(J(\alpha,z+\beta)J(\alpha,z)^{-1})\ J(\alpha,z)\\&=0.\end{aligned}$$

Since $M=\boldsymbol{C}^n/L$ is compact and $A_{\alpha l}(z+\beta)=A_{\alpha l}(z)$ $(\beta\in L)$, all $A_{\alpha l}$ are constants.

And we have $\omega_\alpha(z)\wedge\omega_\alpha(z)=0$ and so $[A_{\alpha l},A_{\alpha m}]=0$ for all $\alpha\in L$, $1\leqq l,m\leqq n$. If we define

$$\tilde{J}(\alpha,z)=J(\alpha,0)\ \exp\left(\sum_{l=1}^{n}A_{\alpha l}\ z_l\right),$$

then we have

$$\tilde{J}(\alpha,z)^{-1}d\tilde{J}(\alpha,z)=J(\alpha,z)^{-1}\ dJ(\alpha,z).$$

Since $\tilde{J}(\alpha,0)=J(\alpha,0)$, we have $\tilde{J}(\alpha,z)=J(\alpha,z)$ $(\alpha\in L)$. It is easy to show that the total Chern class of E is given by

$$c(E)=\left(1+\frac{c_1(E)}{r}\right)^r.$$

That is, the k-th Chern class of E is given by

$$c_k(E)=\binom{r}{k}\frac{1}{r^k}c_1(E)^k.$$

It is also easy to show the following identity:

$$c_2(\text{End } E)=-(r-1)c_1^2(E)+2rc_2(E)=0.$$

In summary, we have

THEOREM. *Let E be a simple holomorphic vector bundle of rank r over the complex torus $M=\boldsymbol{C}^n/L$. Assume E admits a projectively flat Hermitian structure. Let J be its associated automorphic factor for L given* *by the formula (B). Let* $\omega_\alpha(z)=J(\alpha,z)^{-1}dJ(\alpha,z)$ $(\alpha\in L)$ *be a system of integrable connections. Then*

(1) *There exists an isogeny* $\psi: N\to M$ *of degree r and a line bundle* F *such that* $\psi_*F\cong E$.

(2) $H^j(M,\mathcal{O})\cong H^j(M, \mathrm{End}(E))$ *for all j.*

(3) *All* $A_{\alpha l}(\alpha\in L,\ 1\leqq l\leqq n)$ *are constants and* $J(\alpha,z)=J(\alpha,0)\exp(\sum_l A_{\alpha l}z_l)$, $\alpha\in L$, $z\in\boldsymbol{C}^n$.

(4) $c(E)=\left(1+\frac{c_1(E)}{r}\right)^r$

and

$$c_2(\mathrm{End}\ E)=0.$$

References

1. H. Morikawa, *A note on holomorphic matric automorphic factors with respect to a lattice in a complex vector space*, Nagoya Math. J. Vol. **63**(1976), 163-171.
2. T. Oda, *Vector bundles on an elliptic curve*, Nagoya Math. J. Vol. **43**(1971), 41-72.

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Holomorphic 벡터속에 관한 연구

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Holomorphic Vector Bundles

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Abstract

In this paper, we survey the recent results on holomorphic vector bundles over complex manifolds.

§0. Introduction

The purpose of this article is to survey the recent results on holomorphic vector bundles which have been obtained during the past decade. It was a formidable task to go through the extensive literature on the subject. Although the author made efforts to make this survey as thorough as possible, I must have overlooked quite a few papers, for the authors of which, and to the reader, I would like to apologize at the outset. But I tried to give the precise bibliography of this survey which would probably be helpful for the reader.

In section 1, we state some vanishing theorems and the results on positive and ample vector bundles. In section 2, we explain the concepts of stability of vector bundles, in the sense of Mumford-Takemoto, Gieseker and Bogomolov. We briefly state the relation between them. In section 3, we write Kobayashi' results on Einstein-Hermitian vector bundles which he introduced as the differential geometric counterpart to the stability. In section 4, we state the recent results on the moduli of simple holomorphic vector bundles. In section 5, we briefly mention the recent results on vector bundles over abelian varieties, homogeneous vector bundles and infinitely extendable vector bundles.

§1. Positive and Ample vector bundles

In this section, we state the notion of positivity and ampleness of vector bundles. First we start with line bundles.

Let L be a holomorphic line bundle over a compact complex manifold M of dimension n. Let $c_1(L) \in H^2(M; R)$ denote the first Chern class of L. We say that $c_1(L)$ is *positive* (resp. semipositive, negative, semi-negative, of rank $\geqq k$) and write $c_1(L) > 0$

(resp. $c_1(L) \geqq 0$, $c_1(L) < 0$, $c_1(L) \leqq 0$, rank $c_1(L) \geqq k$) if the cohomology class $c_1(L)$ can be represented by a closed real (1,1)-form

$$\varphi = -\frac{1}{2\pi i} \sum_{i,j} \varphi_{i\bar{j}} \, dz^i \wedge d\bar{z}^j$$

such that at each point $x \in M$, the hermitian matrix $(\varphi_{ij}(x))$ is positive definite (resp. positive semidefinite, negative, negative semi-definite, of rank $\geqq k$). We say that L is **positive**(resp. negative) if $c_1(L)$ is positive(resp. negative).

We give some of vanishing theorems related to this notion.

Kodaira's vanishing theorem. [30] If L is negative, then

$$H^q(M, L) = (0) \quad \text{for} \quad q \leqq n-1.$$

Nakano's vanishing theorem. [1] If L is negative, then

$$H^q(M, \Omega^p(L)) = (0) \quad \text{for} \quad p+q \leqq n-1$$

Vesentini's vanishing theorem. [62] If M is Kähler and $c_1(L) \leqq 0$ with rank $c_1(L) \geqq k$, then

$$H^q(M, L) = (0) \qquad \text{for} \quad q \leqq k-1,$$
$$H^0(M, \Omega^p(L)) = (0) \quad \text{for} \quad p \leqq k-1.$$

Gigante's vanishing theorem. [13] If M is Kähler and $c_1(L) \geqq 0$ with rank $c_1(L) \geqq k$, then

$$H^q(M, \Omega^p(L)) = (0) \quad \text{for} \quad p+q \leqq k-1.$$

The following vanishing theorem, due to Kawamata [29] and Viewhweg [63] generalizes the vanishing theorem of Kodaira.

Theorem 1.1. Let M be an algebraic manifold of dimension n, and L be a line bundle over M. If

(i) $\int_C c_1(L) \geqq 0$ for every curve C in M,

(ii) $\int_M c_1(L)^n > 0$,

then

$$H^q(M, \Omega^n(L)) = (0) \quad \text{for} \quad q \geqq 1.$$

Nakano's vanishing theorem has been generalized to certain noncompact manifolds. A complex manifold M is said to be **weakly 1-complite** if there is a smooth real function f on M such that

(i) f is plurisubharmonic, i. e., its complex Hessian $(\partial^2 f/\partial z^\alpha \partial \bar{z}^\beta)$ is positive semi-definite,

(ii) $\{x \in M : f(x) < c\}$ is a relatively compact subset of M for every $c \in R$.

Every compact complex manifold is weakly 1-complete since a constant function satisfies the above conditions. Sometimes, the term *"pseudoconvex"* is used for "weakly 1-complete".

A holomorphic line bundle over a (possibly noncompact) complex manifold M is said to be **positive** if there is an hermitian structure h with positive definite curvature. A semi-postive line bundle of rank $\geqq k$ can be defined in a similiar manner.

Theorem 1.2 (Nakano [47]). If L is a positive line buldle over a weakly 1-complete complex manifold M of dimension n, then

$$H^q(M, \Omega^p(L)) = (0) \quad \text{for} \quad p+q \geqq n+1.$$

Theorem 1.3(Takagoshi-Ohsawa [55]). Let M be a weakly 1-complete Kähler manifold of dimension n and L a semi-positive line bundle whose curvature has at least n-k+1 positive eigenvalues outside a proper compact subset K of M. Then

$$H^q(M, \Omega^p(L)) = (0) \quad \text{for} \quad p+q \geqq n+k.$$

We now state a special case of Bott's vanishing theorem which is useful in the vanishing theorems for vector bundles.

Bott's vanishing theorem(Bott [8]). If L is the tautological line bundle over P_n, then

$$H^q(P_n, \Omega^p(L^{-k})) = (0) \quad \text{for} \quad p, q \geqq 0 \text{ and } k \in Z,$$

with the following exceptions:

(1) $p = q$ and $k = 0$,

(2) $q = 0$ and $k > p$,

(3) $q = n$ and $k < p - n$.

Let E be a holomorphic vector bundleof rank r over a compact complex manifold M. Let L(E) be the tautological line bundle over P(E). We say that E is **negative** (resp. **semi-negative of rank** $\geqq$ k) if the first Chern class $c_1(L(E))$ of the line bundle L(E) satisfies $c_1(L(E))<0$ (resp. $c_1(L(E))\leqq 0$ with rank $c_1(L(E))\geqq k+r-1$). We say that E is **positive**(resp. **semi-positive of rank** $\geqq$ k) if its dual E* is negative(resp. semi-negative of rank $\geqq$ k).

Theorem 1.4. Let E be a holomorphic vector bundle of rank r over a compact Kähler manifold M of dimension n. If E is semi-negative of rank $\geqq$ k, then

$$H^q(M, \Omega^p(E))=(0) \quad \text{for} \quad p+q\leqq k-r.$$

Corollary. If E is a positive holomorphic vector bundle of rank r over a compact Kähler manifold of dimension n, then

$$H^q(M, \Omega^p(E))=(0) \quad \text{for} \quad p+q\geqq n+r.$$

We recall Kodaira's embedding theorem which says that if a compact complex manifold M has a positive line bundle, then M admits a projective embedding. We say that a holomorphic line bunele L over a compact complex manifold M is **ample** if there exists some positive integer m such that a basis of sections $(s_0, s_1, \cdots\cdots, s_N)$ of $H^0(M, L^m)$ generates L^m at every point in M and give a projective embedding

$$\varphi_m=(s_0;\cdots\cdots;s_N):M\longrightarrow P^N$$

We say that L is **very ample** if we can take m = 1 in the above condition, so already sections of $H^0(M,L)$ give the projective embedding.

Let K_M be the canonical line bundle over M. We say that M is **canonical** if K_M is ample, and **very canonical** if K_M is very ample.

Definition 1.5. A holomorphic vector bundle E is said to be **negative in the sense of Grauert** if there exists a relatively compact and strongly pseudoconvex neighbourhood of the zero sections of E. A holomorphic vector bundle E is said to be **positive in the sense of Grauert** if its dual E* is negative in the sense of Grauert. We say that a holomorphic vector bundle E over a complex manifold M is **ample** if for any analytic coherent sheaf F over M, there is a positive integer m_0 such that

$$H^q(M, S^m(E)\otimes F)=(0) \quad \text{for} \quad q\geqq 1 \quad \text{and} \quad m\geqq m_0,$$

where $S^m(E)$ is the m-th symmetric product of E over $\mathcal{O}_M$

As an immediate consequence of the definition, we see that every quotient vector bundle of an ample vector bundle is also ample. On the other hand, if E is an ample vector bundle of rank r, then the determinant bundle det E = λ E is also ample.

Remark: R. Hartshorne[17] introduced the notion of ample vector bundles which generalized the notion of ample line bundles.

We recall the well-known facts

Proposition 1.6. A holomorphic line bundle L over a compact complex manifold is positive if and only if it is ample.

Theorem 1.7 (Hartshorne[17] p. 69). A holomorphic vector bundle E is ample if and only if the tautological line bundle L(E) over P(E) is ample.

Theorem 1.8 (Hartshorne[17] and Grauert[14]). A holomorphic vector bundle E is ample if and only if E is positive in the sense of Grauert.

Theorem 1.9 (Hartshorne[18]). Let C be a nonsingular projective curve. Then a vector bundle E over C is ample if and only if all the quotient vector bundles have positive degrees.

Theorem 1.10(Umemura[56]). On a nonsingular projective curve, the following three conditions coincide:

(1) positive, (2) ample and (3) numberically positive.

By a result of W. Fulton,[11] there is a vector bundle on P^2 which is numberically positive but not ample. But the converse seems to be true. In fact,

Theorem 1.11 (Usui and Tango[61]). Let E be a vector bundle on a nonsingular projective variety over a complex number field. Suppose that E is ample and that, in addition, E is generated by global sections. Then E is numerically positive.

Definition 1.12. A holomorphic vector bundle E over a compact complex manifold M is said to be simple if $H^0(M, End(E)) = C$, i.e., every sheaf homomorphism is a scalar multiple of the identity endomorphism.

For a vector bundle of rank 2 over an algebraic surface, the numerical equivalence class of $N(E) = c_1(E)^2 - 4c_2(E)$ can be regarded as an integer. In fact, $N(E) = -c_2(End(E))$ and hence $N(E) = N(E \otimes L)$ for every line bundle L over an algebraic surface.

Theorem 1.13 (Hosoh[21]). If E is a simple vector bundle of rank 2 over P^2 with $c_1(E) \geqq -N(E)/2$, then E is ample.

In the book,[19] Hartshorne proved that if M is an algebraic manifold of dimension 1 or 2, and if the tangent bundle T_M of M is ample, then M is isomorphic to P^1 or P^2. And then, he asked if the following (H-n) is true for all $n \geqq 1$.

(H-n) Let M be an algebraic manifold of dimension n. If the tangent bundle of M is ample, then M is isomorphic to P^n.

This is known as Hartshorne's conjecture and (H-n) is closely related to the famous conjecture of Frankel:

(F-n) A compact Kähler manifold M of dimension n with positive holomorphic bisectional curvature is biholomorphic to the complex projective space P^n.

T. Mabuchi proved that (H-3) holds under the additional assumption that the second Betti number of M is equal to 1.

§2. Stable vector bundles

Let S be a torsion-free coherent sheaf over a compact Kähler manifold (M, g) of dimension n. Let Φ be the Kähler form of (M, g). Let $c_1(S)$ be the first Chern class of S. The degree of S is defined to be

$$\det(S) = \int_M c_1(S) \wedge \Phi^{n-1}$$

The degree/rank ratio $\mu(S)$ is defined to be

$$\mu(S) = \deg(S)/\mathrm{rank}(S).$$

Following Takemoto, we say that S is *Φ-semistable* if for every coherent subsheaf S', $0 \neq S' \subset S$, we have

$$\mu(S') \leqq \mu(S).$$

If moreover the strict inequality

$$\mu(S') < \mu(S)$$

holds for all coherent subsheaf S' with $0 < \mathrm{rank}(S') < \mathrm{rank}(S)$, we say that S is *$\Phi$-stable.* A holomorphic vector bundle E over M is said to be *Φ-semistable* (resp. *Φ-stable)* if the sheaf $\mathcal{O}(E) = \Omega^0(E)$ of germs of holomorphic sections is Φ-semistable (resp. Φ-stable).

If H is an ample line bundle (so that M is projective algebraic) and if Φ is a closed (1,1)-form representing the first Chern class $c_1(H)$, then we say H-*semistable* (resp. H-*stable)* instead of Φ-semistable (resp. Φ-stable). Since $c_1(H)$ and $c_1(S)$ are integral classes, in this case the degree (or the H-degree) of & is an integer.

Proposition 2.1. Let S be a torsion-free coherent sheaf over a compact Kähler manifold (M, g). Then

(1) S is Φ-semistable if and only if $\mu(S) \leqq \mu(S'')$ holds for every quotient sheaf S'';

(ii) S is Φ-stable if and only if $\mu(S) < \mu(S'')$ holds for every quotient sheaf S'' such that $0 < \mathrm{rank}(S'') < \mathrm{rank}(S)$.

Proof. Let

$$0 \to S' \to S \to S'' \to 0$$

be an exact sequence of coherent sheaves over M. Then we see

$$c_1(S) = c_1(S') + c_1(S'')$$

Thus we get

$$(r' + r'')\,\mu(S) = \deg(S) = \deg(S') + \deg(S'') = r'\,\mu(S') + r''\,\mu(S'').$$

$$r'(\mu(S) - \mu(S')) + r''(\mu(S) - \mu(S'')) = 0$$

Hence,
(i) and (ii) follow from the above equality. Q.E.D.

Remark. In Proposition 2.1, we do not have to consider all quotient sheaves. If $\mu(S) \leqq \mu(S'')$ holds for any torsion-free quotient sheaf S'', S is Φ-semistable. The converse is also true.

Proposition 2.2. Let S be a torsion-free coherent sheaf over over a compact Kähler manifold (M, g). Then

(i) If rank(S) = 1, then S is Φ-stable,

(ii) Let L be a line bundle over M. Then $S \otimes L$ is Φ-stable (resp. Φ-semistable) if and only if S is Φ-stable (resp. Φ-semistable),

(iii) S is Φ-stable (resp. Φ-semistable) if and only if S^* is Φ-stable (resp. Φ-semistable).

Proof. Both (i) and (ii) are trivial. For (iii), we first assume that S^* is Φ-stable and consider an exact sequence

$$0 \to S' \to S \to S'' \to 0$$

such that S'' is torsion-free. Dualizing it, we have an exact sequence

$$0 \to S''^* \to S^* \to S'^*$$

Then we have

$$\mu(S) = -\mu(S^*) < -\mu(S''^*) = \mu(S'')$$

Thus S is Φ-stable.

We assume that S is Φ-stable, and consider an exact sequence

$$0 \to F' \to S^* \to F'' \to 0$$

such that F'' is torsion-free. Dualizing it, we have an exact sequence

$$0 \to F''^* \to S^{**} \to F'^*$$

Considering S as a subsheaf of S^{**} under the natural injection i: $S \to S^{**}$, we define S' and S'' by

$$S' = S \cap F''^*, \qquad S'' = S/S'$$

Then define T″ by the exact sequence

$$0 \to F''^*/S' \to S^{**}/S \to T'' \to 0$$

Since S^{**}/S is a torsion-free sheaf, so are F''^*/S' and T″. det (S^{**}) = (det S^*)* = (det S)** = det S. So det (S^{**}/S) is a trivial line bundle. In general, if T is a torsion sheaf such that det T is a trivial line bundle and if

$$0 \to T' \to T \to T'' \to 0$$

is exact, then both det T′ and det T″ are trivial line bundles. Hence det (F''^*) = det S'. In particular, deg (F''^*) = deg (S'). Since rank (F''^*) = rank (S'), we obtain

$$\mu(S') = \mu(F''^*)$$

Hence

$$\mu(F'') = -\mu(F''^*) = -\mu(S') > -\mu(S) = \mu(S^*).$$

The proof for the semistable case is similiar. Q.E.D.

Proposition 2.3. Let S_1 and S_2 ce Φ-semistable sheaves over a compact Kähler manifold (M, g). Let f: $S_1 \to S_2$ be a homomorphism.

(1) If $\mu(S_1) > \mu(S_2)$, then f = 0.

(2) If $\mu(S_1) = \mu(S_2)$ and if S_1 is Φ-stable, then rank (S_1) = rank (f(S_1)) and f is injective unless f = 0.

(3) If $\mu(S_1) = \mu(S_2)$ and if S_2 is Φ-stable, then rank (S_2) = rank (f(S_1)) and f is generically surjective unless f = 0.

Proof. Assume $f \neq 0$. Set $\mathcal{F} = f(S_1)$. Then $\mathcal{F}$ is a torsion-free quotient sheaf of S_1.
For (1), since

$$\mu(\mathcal{F}) \leqq \mu(S_2) < \mu(S_1) \leqq \mu(\mathcal{F}),$$

we have a contradiction.

For (2), if S_1 is Φ-stable and if rank $(S_1) >$ rank $(\mathcal{F})$, then

$$\mu(\mathcal{F}) \leqq \mu(S_2) = \mu(S_1) < \mu(\mathcal{F}),$$

which is impossible. Hence rank (S_1) = rank $(\mathcal{F})$.
For (3), if S_2 is Φ-stable and if rank $(S_2) >$ rank $\mathcal{F}$, then

$$\mu(\mathcal{F}) < \mu(S_2) = \mu(S_1) \leqq \mu(\mathcal{F}),$$

which is impossible. Hence rank (S_1) = rank $(\mathcal{F})$. Q.E.D.

Corollary. Let E_1 and E_2 be Φ-semistable vector bundles over a compact Kähler manifold (M, g) such that rank (E_1) = rank (E_2) and deg (E_1) = deg (E_2). If E_1 or E_2 is Φ-stable, then any nonzero sheaf homomorphism $f: E_1 \rightarrow E_2$ is an isomorphism.

Proof. By the above Proposition, f is an injective sheaf homomorphism. The induced homomorphism det(f): det (E_1) det (E_2) is also nonzero. Consider det(f) as a holomorphic section of the line bundle Hom(det) (E_1), det (E_2)). Then det(f) is an isomorphism. Hence f is an isomorphism. Q.E.D.

Corollary. Every Φ-stable vector bundle E over a compact Kähler manifold (M, g) is simple.

Proof. Given an endomorphism $f: E \rightarrow E$, let c be an eigenvalue of $f: E_x \rightarrow E_x$ at an arbitrarily chosen point $x \in M$. Applying the above Corollary to $f - c I_E$, we see that $f - c I_E = 0$. Q.E.D.

Without proof, we write the Harder-Narasimhan filtration theorem.

Theorem 2.4. Given a torsion-free coherent sheaf S over a compact Kähler manifold (M, g), there is a unique filtration by subsheaves

$$0 = S_0 \subset S_1 \subset S_2 \subset \cdots\cdots \subset S_{s-1} \subset S_s = S$$

such that, for $1 \leqq i \leqq s-1$, S_i / S_{i-1} is the maximal Φ-semistable subsheaf of S / S_{i-1}.

Remark. A good many of algebraic geometers, e.g., D. Mumford, C.S. Seshadri, M. S. Narasimhan, S. Ramanan, P. E. Newstead etc. have been concerned with stable vector bundles on curves and developed, in particular, the theory of moduli of stable vector bundles. In fact, D. Mumford introduced the concept of a stable vector bundle in order to construct good moduli spaces for vector bundles over algebraic curves. F. Takemoto was the first one who generalized this concept to higher dimensional algebraic manifolds.

We give several examples of Φ-stable vector bundles. The tangent and cotangent bundles of a compact irreducible Hermitian symmetric space, the symmetric tensor product $S^p(TP^n)$ and the exterior power $\Lambda^p(TP^n)$ of the tangent bundle of the complex projective space P^n are all Φ-stable. The null correlation bundle over P^{2n+1} is also Φ-stable.

Let M be an algebraic manifold of dimension n. Let S be a torsion-free coherent sheaf over M, and set

$$S(k) = S \otimes (H^k) \quad \text{for} \quad k \in Z,$$
$$\chi(S(k)) = \Sigma (-1)^i \dim H^i(M, S(k)),$$
$$p(S(k)) = \chi(S(k)) / \operatorname{rank}(S),$$

where H is a fixed ample line bundle over M. We say that S is Gieseker H-stable (resp. H-semistable) if, for every coherent subsheaf $\mathcal{F}$ of S with $0 < \operatorname{rank}(\mathcal{F}) < \operatorname{rank}(S)$, the inequality

$$p(\mathcal{F}(k)) < p(S(k)) \quad (\text{resp. } p(\mathcal{F}(k)) \leqq p(S(k)))$$

holds for sufficiently large integers k.

By the Riemann-Roch theorem, we can express $\chi(S(k))$ in terms of Chern classes of M, S and H. Here $\chi(S(k))$ is the Euler-Poincaré characteristic of $S(k)$,

$$\chi(S(k)) = \int_M ch(S)\ ch(H^k)\ td(M),$$

where td(M) denotes the Todd class of M and ch(E) donotes the Chern character of E.

We have the relation between H-stable vector bundles and Gieseker H-stable vector bundles:

Proposition 2.5. Let S be a torsion-free coherent sheaf over a projective algebraic manifold M with an ample line bundle H.

(i) If S is H-stable, then it is Gieseker H-stable,

(ii) If S is Gieseker H-stable, then it is H-semistable.

(iii) A Gieseker H-stable vector bundle over M is simple.

Proposition 2.6. Let E be a Gieseker H-semistable sheaf over an algebraic manifold M, where H is a fixed ample line bundle over M. Then,

(1) there is a filtration $0 = E_0 \subset E_1 \subset \cdots\cdots \subset E_t = E$ by coherent sheaves such that (a) E_i/E_{i-1} is stable $(1 \leqq i \leqq t)$ (b) $p(E_i(m)) = p(E(m))$ $(1 \leqq i \leqq t)$,

(2) if $0 = E_0' \subset E_1' \subset \cdots\cdots \subset E_s' = E$ is another filtration enjoying the properties (a) and (b), then t = s and there is a permutation w of $\{1, 2, \cdots, t\}$ such that $E_i/E_{i-1} \cong E'_{w(i)}/E'_{w(i)-1}$.

Remark. The concept of stability in the sense of Mumford-Takemoto is too strong from the view point of moduli of vector bundles. Moreover when one takes the results on the compactification on curves into consideration, the concept of semistability in the sense of Mumford-Takemoto is too weak. A proper notion is nothing but that of Gieseker semistablity. On an algebraic curve, the notion of stability (resp. semistablity) in the sense of Mumford-Takemoto coincides with that of Gieseker stability (resp. Gieseker semistability).

Now we briefly state the stability of Bogomolov. Let S be a torsion-free coherent sheaf over a compact complex manifold M. A weighted flag of S is a sequence of pairs $\mathfrak{F} = \{(S_i, n_i) : 1 \leqq i \leqq k\}$ consisting of subsheaves

$$S_1 \subset S_2 \subset \cdots\cdots\cdots \subset S_k \subset S$$

with

$$0 < \text{rank } S_1 < \text{rank } S_2 < \cdots\cdots < \text{rank } S_k < \text{rank } S$$

and positive integers $n_1, n_2, \ldots, n_k$. We set

$$r_i = \text{rank } S_i, \quad r = \text{rank}.$$

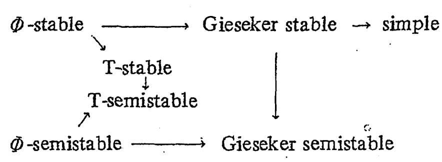
To such a flag $\mathfrak{F}$ we associate a line bundle $T_{\mathfrak{F}}$ by setting

$$T_{\mathfrak{F}} = \prod_{i=1}^{k} ((\det S_i)^r (\det S)^{-r_i})^{n_i}.$$

We say that S is **T-stable** if, for every weighted flag $\mathfrak{F}$ and for a flat line bundle L over M, the line bundle $T_{\mathfrak{F}} \otimes L$ admits no nonzero holomorphic sections. We say that S is **T-semistable** if, for every weighted flag $\mathfrak{F}$ of S and for every flat line bundle L over M, every nonzero holomorphic section of the line bundle $T_{\mathfrak{F}} \otimes L$ (if any) vanishes nowhere on M. We note that if $T_{\mathfrak{F}} \otimes L$ admits a nowhere vanishing holomorphic section, then it is a trivial line bundle and $T_{\mathfrak{F}}$, being isomorphic to L^*, is flat. A T-stable sheave has the properties (i), (ii) and (iii) in Proposition 2.2.

Theorem 2.7. Let S be a torsion-free coherent sheaf over a compact Kähler manifold (M,g) with Kähler form Φ. If S is Φ-stable (resp. Φ-semistable), then it is T-stable, (resp. T-semistable).

The relation between Φ-stable, Gieseker stable and T-stable vector bundles can be described by the following diagram:



Here Φ and H are related by the condition that Φ represents $c_1(H)$.

§ 3. Einstein-Hermitian vector bundles

In § 2, we introduced several notions of stability.

These notions were introduced to construct good moduli for vector bundles. As the differential geometric counterpart to the stability, S. Kobayashi introduced the concept of an Einstein-Hermitian vector bundle, i.e., an hermitian vector bundle satisfying a certain Einstein condition. Einstein-Hermitian vector bundles and stable vector bundles share so many common properties. In a special case, the Einstein condition coincides with the anti-self-duality condtion in Yang-Mills theory. In fact, an Einstein-Hermitian structure can be obtained as the absolute minimum of a Yang-Mills type functional.

Definition 3.1. Let E be a holomorphic hermitian vector bundle of rank r over a compact Kähler manifold (M,g) of dimension n. Let h be a Hermitian metric on E. We choose a holomorphic local coordinate system (z^α) on M and a basis of local holomorphic sections (s_i) of E. We put $h_{i\bar{j}} = h(s_i, s_j)$. The curvature tensor of the hermitian connection corresponding to h has the form

$$R_{ij\alpha\beta} = -\partial^2/\partial z^\alpha \partial \bar{z}^\beta (h_{ij}) + h^{ab}\,\partial/\partial z^\alpha (h_{ib}) \cdot \partial/\partial \bar{z}^\beta (h_{aj}),$$

where $(h^{ab}) = (h_{ij})^{-1}$ Let $\Phi = \sqrt{-1}\, g_{\alpha\beta}\; dz^\alpha \wedge d\bar{z}^\beta$ and $(g^{\gamma\delta}) = (g_{\alpha\beta})^{-1}$ Then (E,h) is called an **Einstein-Hermitian vector bundle** over M if

$$g^{\alpha\beta} R_{ij\alpha\beta} = \lambda h_{ij},$$

where λ is a constant. It can be calculated explicitely as follows:

$$\lambda = 2\pi n \int_M c_1(E) \wedge \Phi^{n-1} \Big/ \left(\operatorname{rank}(E) \int_M \Phi^n\right).$$

On the holomorphic tangent tangent bundle TM over M, there is a hermitian metric $g = 2\, g_{\alpha\beta}\; dz^\alpha\, d\bar{z}^\beta$ Then (M, g) is a Kähler Einstein manifold if (TM, g) is an Einstein-Hermitian vector bundle.

The problem of the existence and uniqueness of an Einstein-Hermitian structure on a holomorphic vector bundle was considered by Kobayashi[24] and Donaldson.[10] Kobayashi proved that the existence of such a structure h implies the semistability. On the other hand, Donaldson showed that on algebraic surfaces (M, g) every stable vector bundle has a unique Einstein-Hermitian structure up to multiplication of h by a constant. The same is true for algebraic curves and, probably, in the general case.

The existence and uniqueness of Kähler-Einstein metrics on Kähler manifolds was studied by Yau[68] and Aubin.[3] From Yau's results it follows that for $c_1(M) = 0$ in every cohomology class of Kähler metrics there is a unique Kähler-Einstein metric. Aubin proved that if there is a form in the class $c_1(M)$ that corresponds to a negative definite metric, then on M there is a unique Kähler-Einstein structure with constant $\lambda = -1$.

As mentioned earlier, Kobayashi obtained the following differential geometric criterion for stability:

Theorem 3.2. Let M be a compact Kähler manifold with a Kähler metric g on M. Let Φ be the Kähler form of g. If (E, h, M, g) is an Einstein-Hermitian vector bundle, then

(a) it is Φ-semistable in the sense of Mumford-Takemoto;

(b) it is a direct sum of Φ-stable Einstein-Hermitian vector bundles $(E_1, h_1, M, g), \ldots, (E_k, h_k, M, g)$ with irreducible holonomy group;

(c) $\mu(E_1) = \cdots = \mu(E_k)$, where $\mu(E)$ denotes the degree-rank ratio of E.

For proof of the above theorem, we refer to Lübke.[33] Lübke[32] obtained the inequality for Einstein-Hermitian vector buneles:

Theorem 3.3. Let (E,h) be an Einstein-Hermitian vector bundle of rank r over a compact Kähler manifold (M, g) with Kähler form Φ. Then

$$\int_M \{(r-1)\, c_1(E, h)^2 - 2r\, c_2(E, h)\} \wedge \Phi^{n-2} \leqq 0$$

and the equality holds if and only if (E,h) is projectively flat.

In the special case where (E,h) is the tangent bundle of a compact Kähler manifold, we know the following theorem of Chen-Ogiue[9] which is sharper than the inequality in Theorem 3.3. Actually the proof of the above theorem is modeled on the proof of the following theorem.

Theorem 3.4. Let (M,g) be a compact Einstein-Kähler manifold of dimension n. Then

$$\int_M \{n c_1(M)^2 - 2(n+1) c_2(M)\} \wedge \Phi^{n-2} \leqq 0,$$

and the equality holds if and only if (M,g) is of constant holomorphic sectional curvature.

Now we consider homogeneous vector bundles over compact homogeneous Kähler manifolds.

Theorem 3.5. Let E be a homogeneous holomorphic vector bundle over a compact irreducible Hermitian symmetric space $M = G/G_0$. Then E admits a G-invariant Hermitian structure unique up to homothety and (E,h) is an irreducible Einstein-Hermitian vector bundle.

Examples 3.6. The following homogeneous vector buncles satisfy the condition of Theorem 3.5.

(a) The tangent and cotangent bundles of a compact Hermitian symmetric space.

(b) The symmetric tensor power $S^m(TP^n)$ of the tangent bundle of the complex projective space P^n.

(c) The exterior power $\overset{m}{\wedge}(TP^n)$ of the tangent bundle of the complex projective space P^n.

Theorem 3.7. Let $M = G/G_0$ be a compact homogeneous Kähler manifold, where G is a connected, compact semi-simple Lie group and $G_0 = C(T)$ is the centralizer of a toral subgroup T of G. Let E be a homogeneous holomorphic vector bundle over M such that G_0 acts irreducibly on E_0. Then with respect to an invariant Hermitian structure h (which exists and is unique up to a homothety), (E, h) is an irreducible Einstein-Hermitian vector bundle.

For proof of the above theorem, we refer to Kobayshi.[26] We mention one example to which the above theorem applies.

Example 3.8. Null correlation bundles. Let $(z^0, z^1, \cdots\cdots, z^{2n+1})$ be a homogeneous coordinate system for the complex projective space P^{2n+1}. Let E be the subbundle of the tangent bundle P^{2n+1} defined by a 1-form α:

$$\alpha = z^0 dz^1 - z^1 dz^0 + \cdots\cdots + z^{2n} dz^{2n+1} - z^{2n+1} dz^{2n} = 0.$$

The form α is defined on $C^{2n+2}-(0)$. Although it is not globally well defined on P^{2n+1}, the equation $\alpha=0$ is well defined on P^{2n+1} if s is a local holomorphic section of the fibering $C^{2n+2}-(0) \rightarrow P^{2n+1}$, then

$$E = \{X \in TP^{2n+1} : s^*\alpha(X) = 0\}$$

defines a subbundle of rank 2n independent of the choice of s. In order to view this bundle E as a homogenous vector bundle, we consider the symplectic form

$$d\alpha = 2(dz^0 \wedge dz^1 + \cdots\cdots + dz^n \wedge dz^{2n+1}).$$

The symplectic group Sp(n+1) is defined as the subgroup of U(2n+2) acting on CP^{2n+2} and leaving $d\alpha$ invariant. Then α itself is invariant by Sp(n+1). We consider P^{2n+1} as a homogeneous space of Sp(n+1) rather than SU(2n+2). Thus,

$$P^{2n+2} = Sp(n+1) / Sp(n) \times T^1$$

Since α is invariant by Sp(n+1), the subbundle $E \subset TP^{2n+1}$ is invariant by Sp(n+1). The isotropy subgroup Sp(n) x T^1 is the centralizer of T^1 in Sp(n+1) and acts irreducibly on the fibre E_0. By Theorem 3.7, with respect to an Sp(n+1)-invariant Hermtian structure h, (E,h) is an irreducible Einstein-Hermitian vector bundle. When n = 1, this rank 2 bundle E over P^3 is known as a **null correlation bundle.** In this particular case, Lübke[31] has shown by explicit calculation that (E,h) is Einstein-Hermitian. The 1-form α defines a complex contact structure on P^{2n+1}. Null correlation bundles and their generalizations are explained in Kobayashi[26] from this view point of complex contact structures.

§ 4. Simple vector bundles over Kähler manifolds

In a recent paper,[44] Mukai has shown that the moduli space of simple sheaves on abelian surface or a K3 surface is smooth and has a holomorphic symplectic structure. In a recent paper,[28] Kobayashi extended Mukai's result to higher dimensional manifolds by a differential geometric method. We now write down his result.

Let M be a compact Kähler manifold of dimension n and E a C^∞ complex vector bundle of rank r over M. Let $A^{p,q}(E)$ be the space of C^∞ (p,q)-forms over M with values in E. A semi-connection in E is a linear map $D'' : A^{0,0}(E) \rightarrow A^{0,1}(E)$ such that

$$D''(as) = d''a\cdot s + aD''s$$

for all functions a on M and all sections s of E. Let $\mathcal{D}''(E)$ denote the space of all semi-connections in E. Every semi-connection D'' extends uniquely to a linear map $D'' : A^{p,q}(E) \rightarrow A^{p,q+1}(E)$ such that

$$D''(\alpha\wedge\sigma) = d''\alpha\wedge\sigma + (-1)^r\alpha\wedge D''\sigma$$

for all r-forms α on M and all $\sigma\in A^{p,q}(E)$, In particular,

$$N(D'') = D''\circ D'' : A^{0,0}(E) \rightarrow A^{0,2}(E),$$

and N(D'') may be considered as an element of $A^{0,2}$ (End(E)). A semi-connection D'' is called a holomorphic structure if N(D'') = 0. Let $H''(E)$ denote the set of holomorphic structures in E. If E is holomorphic, then $d''\in H''(E)$. Conversely, every $D''\in H''(E)$ comes from q unique holomorphic structure in E. The holomorphic vector bundle defined by D'' is denoted by $E^{D''}$ simple if its endomorphisms are all of the form cIE, where $c\in C$ Let

$$\mathrm{End}^0(E^{D''}) = \{u\in \mathrm{End}(E^{D''}) : \mathrm{Tr}(u) = 0\}$$

Then $E^{D''}$ is simple if and only if $H^0(M, \mathrm{End}^0(E^{D''})) = 0$. Let $S''(E)$ denote the set of simple holomorphic structures D'' in E.

Let GL(E) be the group of C^∞ automorphisms of the bundle E. Its Lie algebra gl(E) is nothing but $A^{0,0}$ (End(E)). The group GL(E) acts on $\mathcal{D}''(E)$ by

$$D''f = f^{-1}\circ D''\circ f \text{ for } f\in GL(E),\ D''\in\mathcal{D}''(E).$$

Then GL(E) leaves $H''(E)$ and $S''(E)$ invariant. With the C^∞ topology, the moduli space $S''(E)/GL(E)$ of simple holomorphic structures in E is a (possibly non-Hausdorff) complex analytic space. As was shown by Kim,[22] it is a non-singular complex man ifold in a neighborhood of $[D'']\in S''(E)/GL(E)$ if $H^2(M, \mathrm{End}^0(E^{D''})) = (0)$. This is analogous to Kodaira-Spence-Kuranishi theory of complex structures.

Theorem 4.1. Let M be a compact Kähler manifold with a holomorphic symplectic structure ω_M. Let E be a C^∞ complex vector bundle over M and let $S''(E)/GL(E)$ be the moduli space of simple holomorphic vector bundles in E. Let

$$\mathcal{M}(E) = \{[D'']\in S''(E)/GL(E);\ H^2(M, \mathrm{End}^0(E^{D''})) = 0\}$$

So that $\mathcal{M}(E)$ is a nonsingular (possibly non-Hausdorff) complex manifold. Then ω_M induces in a natural way a holomorphic symplectic structure on

If dim M = 2, then $H^2(M, \mathrm{End}(E^{D''}))$ is dual to $H^0(M, \mathrm{End}(E^{D''}))$ and hence

For the proof of the above theorem, we refer to Kobayashi.[28]

Theorem 4.2. (Mukai[44]). Let M be a K3 surface or a complex torus of dimension 2. Let E be a C^∞ complex vector bundle over M. Then the moduli space of simple holomorphic structures in E is a (possibly non-Hausdorff) complex manifold with a holomorphic symplectic structure. Its dimension is given by

$$2rc_2(E) - (r-1)c_1(E)^2 + r^2h^{0,1} + 2 - 2r^2,$$

where $n^{0,1} = 0$ (resp. = 2) if M is a K3 surface (resp. a torus), provided that the moduli space is nonempty.

§ 5. Vector bundles over special varieties

First we consider vector bundles over abelian varieties. Line bundles over abelian varieties were completely studied by algebraic geometers and differential geometers. We refer to Mumford,[46] Andre Weil,[64] Griffith-Harris[15] and Yang.[67] In fact, the theory of line bundles over abelian varieties is the classical theory of theta functions. But there have been yet unsuccessful attempts at generalizing the classical theory of theta functions to that of vector-valued theta functions with respect to matrix-valued factors of automorphy. This is nothing but the theory of vector bundles over abelian varieties or complex tori. It is desirable to single out a reasonably large class of vector bundles which behave nicely with respect to the group law.

Difinition 5.1. Let M be a complex torus of dimension g. A vector bundle E over M is said to be semi-homogeneous if for every $x \in M$, there is a line bundle L over M such that $T_x^*(E) = E \otimes L$, where T_x denotes the translation of M by x.

Mukai[43] characterized the semi-homogeneous vector bundles as follows:

Theorem 5.2. Let E be a simple vector bundle over an abelian variety of dimension g. Then the following are equivalent to one another:

(1) $\dim H^1(M, End(E)) = g$.

(1') $\dim H^j(M, End(E)) = \binom{g}{j}$ for all $j = 1,\dots, g$.

(2) E is semi-homogeneous.

(3) End(E) is homogeneous.

(4) There exist an isogeny $f: N \to M$ and a line bundle L over N such that $E = f_*(L)$.

Theorem 5.3 (Oda[48]). Let $f: N \to M$ be an isogeny of an abelian variety of dimension g. and L a line bundle over N such that the restriction of φ_L to the kernel of f is an ismorphism. Then $f_*(L)$ is a simple vector bundle over M. Here $\varphi_L : M \to \hat{M}$ is the isogeny defined by $\varphi_L(x) = T_x^*(L) \otimes L^*$ for $x \in M$ and $\hat{M}$ is the dual abelian variety.

Theorem 5.4. (Yang[65]). Let E be a simple vector bundle over a complex torus (M,g). Let Φ be a Kähler form of (M,g). Then if E is semi-homogeneous, then E is Φ stable.

We list here the papers related to abelian varieties: Hano,[16] Matsushima,[38), 39] Morikawa,[40), 41] Mumford,[45), 46] Oda,[48), 49] Takemoto,[53] Umemura,[56), 58), 60] and Yang.[66), 67]

Remark. In his paper,[48] T. Oda first looked into vector bundles constructed by taking direct image of line bundles by isogenies (see Theorem 5.3). This method is superior to Atiyah's[2] succesive extensions of line bundles, at the point, at least, to handle the pull-back by the Frobenius map.

Apart from abelian varieties, there is an interesting result on homogeneous vector bundles (see Theorem 3.5 and Theorem 3.7). Given a representation ρ of H to GL(r;C), we obtain canonically a homogeneous vector bundle E of rank r over G/H E is said to be an irreducible homogenous vector bundle if ρ of H to is irreducible.

Theorem 5.5 (Umemura[59]). Let G be a simply connected, semisimple, linear algebraic group and P a parabolic subgroup which contains no simple factors of G. An irreducible homogeneous vector bundle over M = G/P is stable with respect to any ample line bundle in the sense of Mumford-Takemoto.

This is, in a sense, a generalization of Ramanan's theorem. In fact, when the second Betti number $b_2(M)$ of M is one, Umemura's theorem is subsumed by Ramanan's theorem and its proof.

W. Barth and A. Van de Ven introduced another classes of vector bundles over $\mathbf{P}^n$.

Definition 5.6. A vector bundle E over $\mathbf{P}^n$ is said to be infinitely extendable if for each integer $m > n$, there are a vector bundle E_m over $\mathbf{P}^m$ and a linear embedding $i_m : \mathbf{P}^n \to \mathbf{P}^m$ such that $i_m^*(E_m) \cong E$.

W. Barth, Van de Ven [5] and E. Sato [50], [5] prove the decomposability of an infinitely extendable vector bundle.

Theorem 5.7. An infintely extendable vector bundle over $\mathbf{P}^n$ a projective space is a direct sum of line bundles.

The notion of infinitely extendable vector bundles can be considered on M = G/P for a semi-simple, simply connected, linear algebraic group G of type A, B, C and D and for a maximal parabolic subgroup P of G. E. Sato characterized the infinitely extendable vector bundles over M completely. Infinitely extendable vector bundles are intimately linked with uniform vector bundles. For instance, in the proof of Theorem 5.7, the properties of uniforem vector bundles are used.

Bibliography

1) Y. Akizuki and S. Nakano, Note on Kodaira-Spencer's proof of Lefschetz theorems, *Proc. Japan Acad.*, 30, 1954, 266-272.

2) M. F. Atiyah, Vector bundles over an elliptic curve, *Proc. London Math. Soc.*, 7, 1957, 414-452.

3) T. Aubin, Equations du type Monge-Ampère sur les varietés kähleriennes compactes, *C. R. Acad. Sci.*, Paris 283, 1976, 119-121.

4) T. Aubin, Nolinear Analysis on Manifolds. Monge-Ampère Equations, *Grundlehren Math. Wiss.*, 252, Springer-Verlag, 1982.

5) W. Barth and A. Van de Ven, A decomposibility criterion for algebraic 2-bundles on projective spaces, *inventions Math.*, 25, 1974, 92-106.

6) F. A. Bogomolov, Holomorphic tensors and vector bundles on projective varieties, *Izv. Nauk SSSR*, 42, 1978.

7) ———, Unstable vector bundles and curves on surfaces, Proc. Intern. Congress of Math. Helsinki, 1978, 517-524.

8) R. Bott, Homogeneous vector bundles, *Ann. of Math.*, 66, 1957, 203-248.

9) B. Y. Chen and K. Ogiue, Some characterizations of complex space forms in terms of Chern classes, *Quart. J. Math. Oxford*, 26, 1975, 459-464.

10) S. K. Donaldson, Anti-self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles, *Proc. London Math. Soc.*, 50, 1985, 1-26.

11) W. Fulton Ample vector bundles, Chern classes, and numerical criteria, *Inventiones Math.*, 32, 1976, 171-178.

12) D. Gieseker, On the moduli of vector bundles on an algebraic surface, *Ann. of Math.*, 106, 1977, 45-60.

13) G. Gigantě, Vector bundles with semidefinite curvature and cohomolgy vanishing theorems, *Advance in Math.*, 42, 1981, 40-56.

14) H. Grauert, Über modifikationen und exzeptioelle analytische Mengen, *Math. Ann.*, 146, 1962, 331-368.

15) P. Griffith and J. Harris, *Principle of Algebraic Geometry*, John Wiley & Sons, 1978.

16) J-I. Hano, A geometrical characterization of a class of holomorphic vector bundles over a complex torus, *Nagoya Math. J.*, 61, 1976, 197-202.

17) R. Hartshorne, Ample vector bundles, *Publ. Math. I.H.E.S.*, 29, 1966, 63094.

18) R. Hartshorne, Ample vector bundles on curves, *Nagoya Math. J.*, 43, 1971, 73-89.

19) R. Hartshorne, Ample subvarieties of algebraic varieties, *Lect. Notes in Math.*, 156, Springer, Berlin-Heidelberg-New York, 1970.

20) R. Hartshorne, Stable vector bundles and instantons, *Commn. Math. Phys.*, 59, 1978, 1-15.

21) T. Hosoh, Ample vector bundles on a rational surface, *Nagoya Math. J.*, 59, 1975, 135-148.

22) H. J. Kim, *Curvatures and holomorphic vector bundles*, Ph. D. thesis, Berkeley, 1985.

23) S. Kobayashi, First Chern class and holomorphic tensor fields, *Nagoya Math. J.*, 57, 1975, 153-166.

24) S. Kobayashi, Curvature and stability of vector bundles, *Proc. Japan Acad.*, 58, 1982, 158-162.

25) S. Kobayashi, Differential Geometry of Holomorphic Vector Bundles, *Math. Seminar Notes.*, 41, 1982, Univ. of Tokyo.

26) S. Kobayashi, Homogeneous vector bundles and stability, *Nagoya Math. J.*, 101, 1986.

27) S. Kobayashi, Einstein-Hermitian vector bundles and stability, *Global Riemannian goemetry*, Ellis Horwood Ltd., 1984, 60-64.

28) S. Kobayashi, Simple vector bundles over symplectic Kähler manifolds, *Proc. Japan Acad.*, 62, 1986, 21-24.

29) Y. Kawamata, A generalization of Kodaira-Ramanujam's vanishing theorem, *Math. Ann.*, 261, 1982, 43-46.

30) K. Kodaira, Complex Manifolds and Deformations of Complex Structures, *Grundlehren Math. Wiss.*, 283, Springer-Verlag, 1985.

31) M. Lübke, *Hermite-Einstein-Vektorbundel*, Dissertation, Bayreuth, 1982.

32) M. Lübke, Chernklassen von Hermite-Einstein-Vektorbündeln, *Math. Ann.*, 260, 1982, 133-141.

33) M. Lübke, Stability of Einstein-Hermitian vector bundles, *Manuscripta Math.*, 42, 1983, 245-257.

34) T. Mabuchi, C^3-actions and algebraic threefolds with ample tangent bundle, *Nagoya Math. J.*, 69, 1978, 33-64.

35) M. Maruyama, Openness of a family of torsion free sheaves, *J. Math. Kyoto Univ.*, 16, 1976, 627-637.

36) M. Maruyama, Moduli of stable sheaves, I., *J. Math. Kyoto Univ.*, 17, 1977, 91-126.

37) M. Maruyama, Moduli of stable sheaves, II., *J. Math. Kyoto Univ.*, 18, 1978.

38) Y. Matsushima, Fibrés holomorohes sur un tore complexe, *Nagoya Math. J.*, 14, 1959, 1-24.

39) Y. Matsushima, Heisenberg groups and holomorphic vector bundles over a complex torus, *Nagoya Math. J., Vol.* 61, 1976, 161-195.

40) H. Morikawa, A note on holomorphic vector bundles over complex tori, *Nagoya Math. J.*, 41, 1971, 106-106.

41) H. Morikawa, A note on holomorphic vector vundles over complex tori, *Nagoya Math. J.*, 63, 1976.

42) A. Morimoto, Sur la clssification des espaces fibres vectoriels holomorphes sur un tore complexe admettant des connexions holomorphes, Nagoya *Math. J.*, 1959, 83-154.

43) S. Mukai, Semi-homogeneous vector bundles on an abelian variety, *J. Math. Kyoto Univ.*, 18, 1978, 239-272.

44) S. Mukai, Symplectic structure of the moduli space of sheaves on an abelian or K3 surface, *Invent. Math.*, 77, 1984, 101-106.

45) D. Mumford, On the equations difining abelian varieties, *Invent. Math.*, 1, 1966, 287-354.

46) D. Mumford, *Abelian varieties*, Oxford University Press, 1970.

47) S. Nakano, Vanishing theorems for weakly 1-complete manifolds, Number Theory, Algebraic Geometry & Commutative Algebra, in honor of Y. Akizuki, Kinokuniya, 1973, 169-179; II, *Publ. Res. Inst. Math. Sci., Kyoto Univ.*, 10, 1974, 101-110.

48) T. Oda, Vector bundles on an elliptic curve, *Nagoya Math. J.*, 43, 1971, 41-72.

49) T. Oda, Vector bundles on abelian surfaces, *Inventiones Math.*, 13, 1971, 247-260.

50) E. Sato, On the decomposability of infinitely extendable vector bundles on projective spaces and Grassmann varieties, *J. Math. Kyoto Univ.*, 17, 1977, 127-150.

51) E, Sato, The decomposability of an infinitely extendable vector bundles on the projective space, II, *Proc. Intl. Symp. on Algebraic Geometry, Kyoto,* 1977. Kinokuniya, Tokyo, 1978, 663-672.

52) E. Sato, On infinitely extendable vector bundles on B/P, *J. Math. Kyoto Univ.*

53) F. Takemoto, Stable vector bundles on algebraic surfaces, *Nagoya Math. J.*, 47, 1972, 29-48.

54) F. Takemoto, Stable vector bundles on algebraic surfaces II, *Nagoya Math. J.*, 52, 1973, 173-195.

55) K. Takegoshi and T. Ohsawa, A vanishing theorem for $H^p(X, \Omega^q(B))$ on weakly 1-complete manifolds, *Publ. Res. Inst. Math Sci. Kyoto Univ.*, 17, 1981, 723-733.

56) H. Umemura, Some results in the theory of vector bundles, *Nagoya Math. J.*, 52, 1973, 97-128.

57) H. Umemura, Stable vector bundles with numerically trivial Chern classes over a hyperelliptic surface. *Nagoya Math. J.*, 59, 1975, 107-134.

58) H. Umemura, On a certain type of vector bundles over an abeilan variety, *Nagoya Math. J.*, 64, 1976, 31-45.

59) H. Umemura, On a theorem of Ramanan, *Nagoya Math. J.*, 69, 1978. 131-138.

60) H. Umemura, Moduli spaces of the stable vector bundles over abelian surfaces, *Nagoya Math. J.*, 77, 1980, 47-80.

61) S. Usui and H. Tango, On numerical positivity of ample vector bundles with addtional condition, *J. Math. Kyoto Univ.*, 17, 1977, 151-164.

62) E. Vesentini, Osservazioni sulle strutture fibrate analitiche sopra una varieta kahleriana compatta, *Nota I e II, Atti Accad. Naz. Lincei Rend. Cl. Sci. Nat. Natur.*(8), XXIV, 1958, 505-512; ibid. XXIV, 1957, 232-242.

63) E. Viehweg, Vanishing theorems, *Crelles J.*, 335, 1982, 1-8.

64) A. Weil, *Introduction a letude des varietés kähleriennes*, Hermann, Paris, 1971.

65) J. H. Yang, *Einstein-Hermitian vector bundles*, Ph. D. thesis, Berkeley, 1984.

66) J. H. Yang, Einstein-Hermitian vector bundles over a complex torus, *Bull. Inst. Basic Sci.*, Inha Univ., Vol. 6, 1985.

67) J. H. Yang, Vector bundles over an abelian variety, Preprint, 1986.

68) S. T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equations, *I. Comm. Pure & Appl. Math.*, 31, 1978, 339-411.

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HOLOMORPHIC VECTOR BUNDLES OVER COMPLEX TORI

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1. Introduction

The purpose of this paper is to study the holomorphic vector bundles over a complex torus. The theory of vector bundles of rank r over a g-dimensional complex torus is not sufficiently developed except for $r=1$ and $g=1$. In his paper [1], Atiyah classified the vector bundles over an elliptic curve. Many algebraic geometers, e. g., Weil, Mumford, studied the line bundles over a complex torus. The theory of line bundles is the theory of theta functions. Indeed, the classification of vector bundles over a complex torus $T=V/L$ corresponds to that of automorphic factors for L, where V is a g-dimensional complex vector space and L is a lattice in V. In general, it is a very difficult problem to classify the automorphic factors for a discrete group. Matsushima [1] and Morimoto [1] classified the flat vector bundles over a complex torus. It is equivalent to the problem of the classification of representations of a lattice group L. And Matsushima [2] and Hano [1] characterized the projectively flat vector bundles over a complex torus. Mukai [1] introduced the concept of semi-homogeneous vector bundles over an abelian variety and characterized the semi-homogeneous vector bundles. In fact, semi-homogeneous vector bundles corresponds to projectively flat vector bundles. In this paper, we characterize the projectively flat vector bundles over a complex torus completely and investigate the connection between those vector bundles and the Heisenberg group. I would like to remark that it is so interesting to classify the automorphic factors corresponding to the stable vector bundles.

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In Section 2, we review the basic properties of complex tori. We will omit the proofs and the details. In Section 3, we describe the automorphic factors for the holomorphic vector bundles and in particular, we write the automorphic factors for line bundles explicitly. In Section 4, we review the general results about line bundles over a complex torus we will use in the following sections. For details, we refer to Mumford [1] and Yang [1]. In Section 5, we characterize the projectively flat vector bundles over a complex torus completely and we discuss the stability of those vector bundles. In Section 6, we investigate the connection between the Heisenberg groups and the projectively flat vector bundles.

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2. Complex tori

In this section, we briefly review the theory of complex tori. Proofs will be omitted.

Let $T=V/L$ be a complex torus of dimension g, where V is a g-dimensional comlex vector space and L is a lattice of rank $2g$ in V. Then T is a connected, compact commutative complex Lie group. A complex torus T is called an *abelian variety* if it is a projective variety, i.e., it can be holomorphically embedded in a complex projective space.

DEFINITION 1.1. A morphism $\phi : T_1 \to T_2$ of complex tori T_1, T_2 is said to be an isogeny if it is a surjective homomorphism with finite kernel. The order of the kernel is called the degree of ϕ. We say that two complex tori T_1 and T_2 are isogeneous, denoted by $T_1 \sim T_2$ if there exists an isogeny between T_1 and T_2.

If $\phi_1 : T_1 \sim T_2$ and $\phi_2 : T_2 \to T_3$ are isogenies, then so is $\phi_2 \circ \phi_1$ and the degrees multiply : $\deg(\phi_2 \circ \phi_1) = \deg \phi_2 \deg \phi_1$. The following proposition shows that $\sim$ is an equivalence relation.

PPROPOSITION 1.2. *Let T_1 and T_2 be two complex tori of the same dimension g. If ϕ ; $T_1 \to T_2$ is an isogeny of degree m, then there* *exists a unique isogeny* $\psi : T_2 \rightarrow T_1$ *of degree* m^{2g-1} *such that* $\psi\circ\phi = mI_1$ *and* $\phi\circ\psi = mI_2$, *where* I_1 (*resp.* I_2) *denotes the identity of* T_1 (*resp.* T_2). ψ *is called the dual isogeny to* ϕ.

Let $T=V/L$ be a g-dimensional complex torus. A Hermitian from H on V is called a *Riemann form* for the complex torus $T=V/L$ if

(i) H is nondegenerate,

(ii) $Im\ H=E$ is integral valued on the lattice L.

The following theorem is well known.

THEOREM A (Mumford[1], p. 35). *Let* $T=V/L$ *be a* g*-dimensional complex torus. Then the following are equivalent.*

(1) T *is an abelian variety,*

(2) *there exist* g *algebraically independent meromorphic functions on* T,

(3) *there exists a positive definite Riemann form* H *on* V.

EXAMPLE. Let ω be an elements in the upper-half plane. Now we consider the lattice $L=\{n+m\omega \mid n, m\in Z\}$ in C. Then $T=C/L$ is a one dimensional complex torus. We define a Hermitian form H on C by

$$H(z, w)=\frac{z\cdot \bar{w}}{Im\ \omega} \quad \text{where } z,\ w\in C.$$

Then H is clearly a positive definite Riemann form on C. By Theorem A, there is a projective embedding of T in a complex projective space. In fact, several projective embeddings of T are well-known in the classical theory: for example, the Weierstrass $\wp$-function

$$\wp(z)=\frac{1}{z^2}+\sum_{(n,m)\neq(0,0)}\left[\frac{1}{(z-n-m\omega)^2}-\frac{1}{(n+m\omega)^2}\right]$$

is a meromorphic function, periodid with respect to 1, ω, with poles at the points $n+m\omega\in L$. The map

$$z\longrightarrow(1, \wp(z), \wp'(z))$$

induces an ismorphism of T with a plane cubic curve of the from $X_0X_2^2=4X_1^3+aX_0^2X_1+bX_0^3$ for suitable constants a, b depending on ω. We remark that the Weierstrass $\wp$-function satisfies the following nonlinear differential equation

$$[\varphi'(z)]^2=4\varphi^3(z)-60G_4\varphi(z)-140G_6,$$

where $G_k(k\geq 3)$ is the Eisenstein series of order k. Here, in fact, $a=-60G_4$ and $b=-140G_6$.

Restricting the Riemann from shows that a subtorus of an abelian variety is again an abelian variety. One can show that a quotient of an abelian variety is also an abelian variety. This is a consequence of the following theorem.

THEOREM B (Poincaré Reducibility Theorem). *Suppose A is an abelian variety and $A_1\subset A$ an abelian subvariety. Then there exists an abelian subvariety A_2 such that $A_1\cap A_2$ is finite and A is isogeneous to $A_1\times A_2$.*

Let $T=V/L$ be a complex torus of dimension g. Let $T_x: T\rightarrow T$ be the translation by $x\in T$. We have a Kaehler metric on T which is invariant under the translations. Since T_x preserves a Kaehler metric on T, T_x^* sends harmonic forms into harmonic forms. Since T_x is homotopic to the identity map, the map

$$T_x^*: H^k(T)\rightarrow H^k(T)$$

is the identity map, where $H^k(T)$ denotes the space of all harmonic k-forms on T. If $x\in T$, $T_{x,C}(T)=T_x'(T)+T_x'(T)$. We identify $T_x'(T)$ with V. An element $\theta\in\Lambda^k(T_x'(T))^*\cong\Lambda^k V^*$ extends a holomorphic, translation-invariant k-forms ω_θ on T. Indeed, we define $(\omega_\theta)_y=T_{x-y}^*(\theta)$ for any $y\in T$. Then the map defines a sheaf homomorphism

$$\mathcal{O}_T\otimes_C\Lambda^k V^*\rightarrow\Omega^k$$

which is actually a sheaf homomorphism, where Ω^k is the sheaf of holomorphic k-forms on T and $\mathcal{O}_T$ is the structure sheaf on T, simply denoted by $\mathcal{O}$. Since $T\cong(S^1)^{2g}$ topologically, $\dim_C H^k(T)=\binom{2g}{k}$. Let $I^k(T)$ be the space of all translation-invariant k-forms on T. Then $\dim I^k(T)=\binom{2g}{k}$ and hence $H^k(T)=I^k(T)$. In fact,

$$\begin{aligned}H^k(T)&=I^k(T)=\Lambda^k(V^*\oplus\bar{V}^*)\\&=\oplus_{p+q=k}(\Lambda^p V^*\otimes\Lambda^q\bar{V}^*).\end{aligned}$$

THEOREM C. *$H^q(T,\mathcal{O})\cong\Lambda^q\bar{V}^*$ for all q.*

For the proof of Theorem C, see Mumford ([1], p. 4).

$$\begin{aligned} H^q(T, \Omega^p) &\cong H^q(T, \mathcal{O}\otimes \Lambda^p V^*) \\ &\cong H^q(T, \mathcal{O})\otimes_C \Lambda^p V^* \\ &\cong \Lambda^q \bar{V}^* \otimes_C \Lambda^p V^* \text{ by Theorem } C. \end{aligned}$$

Thus we have

$$\begin{aligned} H^k(T, C) &\cong H^k(T) \text{ by the Hodge theorem} \\ &\cong \oplus_{p+q=k}(\Lambda^p V^* \otimes \Lambda^q \bar{V}^*) \\ &\cong \oplus_{p+q=k} H^q(T, \Omega^p). \end{aligned}$$

This is a so-called famous Hodge theorem.

REMARKS. (1) By Theorem C, the natural map induced by cup product

$$\Lambda^q(H^1(T, \mathcal{O})) \to H^q(T, Q)$$

is an isomorphism.

(2) We consider the three sheaves on T, embedded in one another as follows:

$$Z \subset C \subset \mathcal{O},$$

where Z and C are the constant sheaves on T. Then we have the following:

$$\begin{array}{ccccc} H^1(T, Z) & \xrightarrow{\alpha} & H^1(T, C) & \xrightarrow{\beta} & H^1(T, \mathcal{O}) \\ \wr\| & & \wr\| & & \wr\| \\ \mathrm{Hom}(L, Z) & & V^* \oplus \bar{V}^* & & \bar{V}^* \\ \| & & \| & & \\ L^* & & \mathrm{Hom}_R(V, C) & & \end{array}$$

Let A^k(resp. $A^{k,0}, A^{0,k}$) be the sheaf of C^∞ k-forms(resp. of type $(k, 0)$, of type $(0, k)$) on T. Let $A_{0,1} : A^1 = A^{1,0} \oplus A^{0,1} \to A^{0,1}$ be the projection. Then we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & A^0 & \xrightarrow{d} & A^1 & \longrightarrow & 0 \\ & & & & \| & & \downarrow A_{0,1} & & \\ 0 & \longrightarrow & \mathcal{O} & \longrightarrow & A^{0,0} & \xrightarrow{\partial} & A^{0,1} & \longrightarrow & 0. \end{array}$$

Hence we have

$$\begin{array}{ccccc} H^1(T) = V^* \oplus \bar{V}^* & \longrightarrow & H^0(T, A^1) & \longrightarrow & H^1(T, C) \\ \downarrow \text{proj} & & \downarrow A^{0,1} & & \downarrow \beta \\ \bar{V}^* & \longrightarrow & H^0(T, A^{0,1}) & \longrightarrow & H^1(T, \mathcal{O}). \end{array}$$

Thus β is surjective.

3. Automorphic factors

Let V be a g-dimensional complex vectr space and L a lattice in V. Then V is the universal covering space of the complex torus $T=V/L$. Clearly the projection $\pi: V\to T$ is holomorphic and L is the fudamental group of T.

DEFINITION 3.1. An *automorphic factor* of rank r with respect to the lattice L in V is a holomorphic mapping $J: L\times V\to GL(r\,;C)$ satisfying the condition

$$J(\alpha+\beta, z)=J(\alpha, \beta+z)J(\beta, z) \text{ for all } \alpha, \beta\in L,\ z\in V.$$

Two automorphic factors J and $\tilde{J}$ of the same rank r with respect to a lattice L are said to be holomorphically equivalent (simply equivalent) if there exists a holomorphic mapping $h: V\to GL(r\,;C)$ such that

$$\tilde{J}(\alpha, z)=h(z+\alpha)J(\alpha, z)h(z)^{-1} \text{ for each } \alpha\in L,\ z\in V.$$

An automorphic factor $J: L\times V\to GL(r, C)$ is said to be *flat* if the mapping J is costant on V. Thus a flat automorphic factor consists of an element $J\in\mathrm{Hom}(L, GL(r\,;C))$ of the set of all group homomorphisms from L into $GL(r\,;C)$.

Let E be a holomorphic vector bundle of rank r over a complex torus $T=V/L$. If $\pi: V\to T$ is the projection, then $\pi^*E=\tilde{E}$ is holomorphically trivial since V is a contractible Stein manifold. It is Grauert's result that any topologically trivial holomorphic vector bundle over a Stein manifold is holomorphically trivial (Grauert [1]). Having fixed the isomorphism $\tilde{E}=V\times C^r$, each bundle homomorphism $\tilde{\alpha}: \tilde{E}\to\tilde{E}$ induced by a covering transformation $\alpha: V\to V$ $(\alpha\in L)$ must be of the form

$$\tilde{\alpha}(z, \xi)=(z+\alpha, J(\alpha, z)\xi),\quad \alpha\in L,\quad z\in V,\quad \xi\in C^r,$$

where $J: L\times V\to GL(r\,;C)$ is an automorphic factor of rank r for L. The mapping J is called the automorphic factor for the bundle E. Conversely, given an automorphic factor $J: L\times V\to GL(r:C)$, we may regard L as a group of biholomorphic mappings from $V\times C^r$ to itself by setting $\alpha(z, \xi)=(z+\alpha, J(\alpha, z)\xi)$ for $\alpha\in L$. Then the quotient $E=V\times C^r/L$ is a holomorphic vector bundle over a complex torus $T=V/L$ such that $\pi^*E=V\times C^r$.

Let E and E_1 be holomorphic vector bundles over a complex torus T of rank r and rank s respectively. Any bundle homorphism $\pi : E \to E_1$ induces a bundle homomorphism $\pi : V \times C^r \to V \times C^s$ which commutes with the action of L. Thus π must be of the form

$$\pi(z, \xi) = (z, h(z)\xi), \quad z \in V, \quad \xi \in C^r,$$

where $h : V \to M_{s,r}$ is a holomorphic mapping from V into the space $M_{s,r}$ of all $s \times r$ complex matrices. Moreover, if J and J_1 are the automorphic factors for the bundles E and E_1 respectively,

$$(*) \quad h(z+\alpha)J(\alpha, z) = J_1(\alpha, z)h(z), \quad \alpha \in L, \quad z \in V.$$

Conersely, given a holomorphic mapping $h : V \to M_{s,r}$ satisfying the above condition $(*)$, then h determines a bundle homomorphism $\pi : V \times C^r \to V \times C^s$ which comutes with the action of L and hence determines a bundle homomorphism $\pi : E \to E_1$.

In summary, it has been shown that there exists a one-to-one correspondence between the set of all isomorphic classes of vector bundles of rank r over a complex torus $T = V/L$ and the set of all equivalence classes of automorphic factors of rank r for the lattice L. Therefore the problem of classifying holomorphic vector bundles of rank r over a complex torus $T = V/L$ corresponds to that of classifying automorphic factors of rank r for the lattice L. In general, the determination of all automorphic factors for the lattice L is a difficult problem. However the classification of atomorphic factors of rank one was completely done by Appel. This problem is equivalent to the computation of the group $H^1(T, \mathcal{O}^*) \cong \mathrm{Pic}\, T$ of isomorphic classes of holomorphic line bundles over T. Now we give an explanation in detail.

Let $T = V/L$ be a g-dimensional complex torus and let H be a Riemann form for the complex torus T. A map $\chi : L \to C_1^* = \{z \in C \mid |z| = 1\}$ is said to be a *semi-character* of L with respect to $E = \mathrm{Im}\, H$ if it satisfies the codition

$$\chi(\alpha+\beta) = \chi(\alpha)\chi(\beta)\exp\{i\pi E(\alpha, B)\}, \quad \alpha. \beta \in I.$$

We define the mapping $J_{H,\chi} : L \times V \to C^*$

$$J_{H,\chi}(\alpha, z) = \chi(\alpha)\exp\left\{\pi H(z, \alpha) + \frac{\pi}{2} H(\alpha, \alpha)\right\},$$

where $\alpha \in L$, $z \in V$ and χ denotes a semi-character of L with respect to $E = \mathrm{Im}\, H$. Then it is easily shown that $J_{H,\chi}$ is an automorphic factor of rank one for the lattice L. We denote by $L(H,\chi)$ the holomorphic line bundle over $T=V/L$ defined by the above automorphic factor $J_{H,\chi}$ for L. We can easily show that

$$L(H_1,\chi_1)\otimes L(H_2,\chi_2)\cong L(H_1+L_2,\chi_1\chi_2).$$

Therefore the set of all $L(H,\chi)$ forms a group under tensor product. Now we have the following theorem.

THEOREM (Appell-Humbert). *Any holomorphic line bundle over a complex torus $T=V/L$ is isomorphic to an $L(H,\chi)$ for a uniquely determined Riemann form H for T and a uniquely determined semi-character χ.*

The Chern class of $L(H,\chi)$ is given $E=\operatorname{Im} H\in H^2(T,Z)$. Since $H(x,y)=E(ix,y)+iE(x,y)$ for all $x,y\in V$, according to the above theorem, a holomorphic line bundle over T must be of the form $L(H,\chi)$, where χ is a character of L. Thus group $\operatorname{Pic}^0(T)$ of holomorphic line bundles with Chern class zero is isomorphic to the group $\operatorname{Hom}(L,C_1^*)$ of all characters of L. In fact, $\operatorname{Pic}^0(T)\cong \operatorname{Hom}(L,C_1^*)$ has the structure of a complex torus. By the following exact sequence of sheaves

$$0\longrightarrow Z\longrightarrow\mathcal{O}\longrightarrow\mathcal{O}^*\longrightarrow 0,$$

we have an ismorphism $\operatorname{Pic}^0(T)\cong H^1(T,\mathcal{O})/\operatorname{Im} H^1(T,Z)$. Now we have $H^1(T,\mathcal{O})=H_{0,1}(T)=\bar{V}^*$. We know that the image of $H^1(T,Z)$ in $H^1(T,\mathcal{O})$ is the set of all conjugate linear functionals on V whose real part is half-integral on L. Thus $\operatorname{Pic}^0(T)$ is a complex torus, called the dual complex torus of T and is denoted by $\hat{T}$. If T is an abelian variety, so is $\hat{T}$ and it is called the dual abelian variety of T.

We now compute the curvature of a holomorphic line bundle $F=L(H,\chi)$ over $T=V/L$. We fix a hermitian structure h on F. We pull back h to $\pi^*F=\tilde{F}$ to obtain an hermitian structure on $\tilde{F}=V\times C$. We may consider $\tilde{h}$ as a positive function on V invariant under L. That is, $\tilde{h}$ satisfies

$$\tilde{h}(z)=|J_{H,\chi}(\alpha,z)|^2\tilde{h}(z+\alpha),\quad \alpha\in L,\quad z\in C.$$

Then the connection form $\tilde{\omega}=\partial\log\tilde{h}$ and the curvature form $\tilde{\Omega}=\bar{\partial}\partial\log\tilde{h}$ are given by

$$\tilde{\omega}(z)=\tilde{\omega}(z+\alpha)+\partial\log J_{H,\chi}(\alpha,z),$$

$$\tilde{\Omega}(z)=\tilde{\Omega}(z+\alpha).$$

Thus the curvature form of F is an ordinary 2-form on T. Mutiplying h by a suitable C^∞. positive function on T. we may assume that the curvature form of $F=L(H,\chi)$ is a harmonic (1,1)-form on T. Since $H^2(T,C)=\Lambda^2 V^*\oplus V^*\otimes\bar{V}^*\oplus\Lambda^2\bar{V}^*$, a harmonic form on T has constant coefficients with respect to the natural coordinates $z_1,\cdots,z_g$ in V. In particular, the curvature Ω of $F=L(H,\chi)$ is given by

$$\Omega=\sum c_{kl}dz_k\wedge d\bar{z}_l,$$

where $c_{k\bar{l}}$ are constant.

4. Line Bundles over Complex Tori

In this section, we will review line bundles over complex tori. For details, we refer to Mumford [1] and Yang [1].

LEMMA 4.1. *Let E be a holomorphic vector bundle over a complex torus T. Let $T_x(x\in T)$ be the translation of T by x. Then $T_x^*(E)$ and E have the same Chern classes.*

LEMMA 4.2. *Let E be a holomorphic vector bundle f rank r over a complex torus $T=V/L$. Let J be the automorphic factor for the bundle E. Let $x\in T$. Then an automorphic factor J_x for the bundle $T_x^*(E)$ is given by*

$$J_x(\alpha,z)=J(\alpha,z+a),\quad \alpha\in L,\quad z\in V,$$

where a is an element of V such that $\pi(a)=x$.

DEFINITION 4.3. A holomorphic vector bundle E over a complex torus T is said to be *homogeneous if for all $x\in T$, $T_x^*(E)\cong E$.*

LEMMA 4.4. *For a holomorphic vector bundle E of rank r over a complex torus $T=V/L$, the following conditions are equivalent :*

(1) *E is homogeneous,*

(2) *E is defined by a representation $\rho: L\to GL(r;C)$,*

(3) *E admits a flat connection,*

(4) *E is a flat vector bundle.*

LEMMA 4.5. *If E is a homogeneous vector bundle of rank r over complex torus T, then $c_k(E)=0$ for $k\geq 1$, where $c_k(E)$ denotes the* *k-th Chern class of E.*

REMARK 4.6. If *rank* $E=r\geq 2$, the converse of Lemma 3.5 does not hold. A counter example was given by Oda [2]. However if $r=1$, the converse of Lemma 4.5 is true. For the proof, we refer to Yang [1].

DEFINITION 4.7. Let F be a holomorphic line bundle over a complex torus $T=V/L$. We set $K(F)=\{x\in T|\, T_x^*(F)\cong F\}$. And we define the map $\phi_F: T\to\hat{T}=\mathrm{Pic}^0(T)$ by.

$$\phi_F(x)=T_x^*(\mathrm{F})\otimes F^{-1},\quad x\in T.$$

REMARKS 4.8. (1) IF F is ample, then $K(F)$ is a finite subgroup of T.

(2) ϕ_F is an isogeny and the set $K(F)$ is nothing but the kernel of ϕ_F.

DEFINITION 4.9. Let n be an integer, We define $n_T: T\to T$ by

$$n_T(x)=nx,\quad x\in T.$$

The map n_T is called the *multiplication by n*. The following lemma shows that n_T is an isogeny of degree n^{2g} if T is an abelian variety of dimension g.

LEMMA 4.10. *Let F be an ample line bundle over an abelian variety A. Then*

$$n_T^*(F)=F^{n^2}\otimes F_0 \text{ for some } F_0\in\hat{A}.$$

And the degree of n_T *is* n^{2g}.

For the proof of the above lemma, we refer to Mumford [1], p. 59, p. 63 or Yang [1], p. 14.

THEOREM OF SQUARE. *Let F be an ample line bundle over an abelian variety $A=V/L$. Then for any $x, y\in A$,*

$$T_{x+y}^*(F)\otimes F\cong T_x^*(F)\otimes T_y^*(F).$$

COROLLARY 1. $\phi_F: A\to\hat{A}=Pic^0(A)$ *is a homorphism of A to* $\hat{A}$.

COROLLARY 2. *Let F_1 F_2 be two ample line bundles over an abelian variety A. Then*

$$\phi_{F_1\otimes F_2}=\phi_{F_1}+\phi_{F_2}.$$

COROLLARY 3. $\phi_{T_x^*(F)}=\phi_F$ *for all* $x\in A$.

DEFINITION 4.11. An ample line bundle F over an abelian variety A is said to be *symmetric* if $(-1)_A^*F\cong F$.

LEMMA 4.12. *If F is an ample, symmetric line bundle over an abelian variety A, then $F\otimes(-1)_F^*F$ is also an ample, symmetric line bundle over A.*

PROPOSITION 4.13. *Let A be an abelian variety of dimension g. Then we have*

(1) $F\in\hat{A}$ *if and only if* $\phi_F=0$.

(2) *Let* $f,g:A\rightarrow A$ *be holomorphic maps. If* $F\in\hat{A}$, *then* $(f+g)^*(F)\cong f^*(F)\otimes g^*(F)$.

(3) *If* $F\in\hat{A}$, *then* $n_A^*(F)\cong F^n$.

(4) *If* $F\in H^1(A,\mathcal{O})$ *has finite order, then* $\phi_F\in\hat{A}$.

(5) *If F is an ample line bundle over A, then ϕ_F is surjective.*

(6) *If* $F\in\hat{A}$ *and F is not trivial, then* $H^i(A,F)=0$ *for all* i.

In the previous section, we introduced the line bundle $L(H,\chi)$ over complex torus T. Using Lemma 3.2, we have

PROPOSITION 4.14. Let $a\in V$ *and* $x=\pi(a)$. *Then*

$$T_x^*L(H,\chi)\cong L(H,\chi D_a),$$

where $D_a(\alpha)=exp\{2\pi iA(a,\alpha)\}$.

REMARKS 4.15. If $a\in V$, then $\phi_{L(H,\chi)}(\pi(a))\cong L(0,D_a)$. Thus we have

$$K(L(H,\chi))=L^{\perp}/L,$$

where $L^{\perp}=\{v\in V\,|\,A((\alpha,v)\in Z$ *for all* α *on* $L\}$. Therefore we have

(1) $$L(H,\chi)\in\mathrm{Pic}^0(T)=\hat{T}$$
$$\Longleftrightarrow K(L(H,\chi))=T\Longleftrightarrow L^{\perp}=V$$
$$\Longleftrightarrow A=0\Longleftrightarrow H=0.$$

(2) $K(L(H,\chi))$ is a finite subgroup of T
$\Longleftrightarrow L^{\perp}/L$ is finite $\Longleftrightarrow L^{\perp}$ is a lattice
$\Longleftrightarrow A$ is nondegenerate $\Longleftrightarrow H$ is nondegenerate.

Let $T=V/L$ be an abelian variety of dimension g. From the exact sequence

$$0\longrightarrow Z\longrightarrow\mathcal{O}\longrightarrow\mathcal{O}^*\longrightarrow 0,$$

we obtain the exact sequence

$$H^1(T,Z)\longrightarrow H^1(T,\mathcal{O})\longrightarrow H^1(T,\mathcal{O}^*)\longrightarrow H^2(T,Z).$$

Thus

$$\mathrm{Pic}^0(T)=H^1(T,\mathcal{O})/\mathrm{Im}\,H^1(T,Z).$$

We know that $H^1(T,\mathcal{O})\cong H^{0,1}(T)\cong\bar{V}^*$. On the other hand, $H^1(T,Z)\cong H^1(L,Z)$ is the space of R-linear functionals on V taking integral values on L. The map $i: H^1(T,Z)\rightarrow H^1(\mathrm{T},\mathcal{O})$ is given by

$$w\longrightarrow w^{0,1}.$$

Since

$$\int_\alpha w=2\mathrm{Re}\int_\alpha w^{0,1}\in Z,$$

$\mathrm{Im}\,i$ consists of conjugate linear functionals on V whose real part is half-integral on L. By multiplying a constant $2i$, $\mathrm{Im}\,i$ can be identified with theset $\hat{L}$ given by

$$\hat{L}=\{l\in\bar{V}^*\,|\,\mathrm{Im}\,l(\alpha)\in Z \text{ for all } \alpha\in L\}$$

Thus we obtain

$$\hat{T}=\mathrm{Pic}^0(T)=\bar{V}^*/\hat{L}.$$

LEMMA 4.16. *There exists a unique holomorphic line bundle*

$$P\longrightarrow T\times\hat{T},$$

called the Poincaré line bundle, which is trivial on $\{e\}\times\hat{T}$ *and which satisfies*

$$P|_{T\otimes\{\xi\}}\cong P_\xi \text{ for all } \xi\in\hat{T},$$

where P_ξ is the line bundle over T corresponding to $\xi\in\hat{T}$.

For the proof, we refer Mumford [1; p. 78-80] and also Griffith-Harris [1; p. 328-329].

In fact, we have the following sequence

$$\begin{array}{ccc} H^1(T\times\hat{T},\mathcal{O})\longrightarrow H^1(T\times\hat{T},\mathcal{O}^*) & \xrightarrow{c_1} & H^2(T\times\hat{T},Z) \\ \uparrow & & \uparrow \\ H^1(T,\mathcal{O})\oplus H^1(\hat{T},\mathcal{O}) & & H^1(T,Z)\otimes H^1(\hat{T},Z) \\ & & \wr\| \\ & & H^1(T,Z)\otimes(H^1(T,Z))^* \\ & & \wr\| \\ & & \mathrm{Hom}(H^1(T,Z),H^1(T,Z)) \end{array}$$

The identity $I\in H^2(T\times\hat{T},Z)$ gives a holomorphic line bundle P over $T\times\hat{T}$ such that $c_1(P)=I$ by the Lefschetz theorem on (1, 1) classes. This line bundle P is nothing but the Poincaré line bundle over

$T\times\hat{T}$.

Now we will describe the Poincaré bundle over $T\otimes\hat{T}$ explicitely. We define an Hermitian form H on $V\otimes\bar{V}^*$ by

$$H((z_1, l_1), (z_2, l_2)) = \overline{l_2(z_1)} + l_1(z_2),$$

where $z_1, z_2 \in V$, and $l_1, l_2 \in V^*$. We also define the map $\chi : L\times\hat{L}\to C_1^*$ by

$$\chi(\alpha, l) = \exp\{-\pi i \operatorname{Im} l(\alpha)\}, \quad \alpha\in L, \quad l\in\hat{L}.$$

Then χ is a semicharacter of $L\times\hat{L}$ with respect to H. That is,

$$\chi((\alpha+\beta, l+\hat{l})) = \chi(\alpha, l)\chi(\beta, \hat{l})\exp\{i\pi E((\alpha, l), (\beta, \hat{l}))\},$$

where $\alpha, \beta\in L$, $l, \hat{l}\in\hat{L}$, and $E-\operatorname{Im} H$. Then the line bundle $L(H, \chi)$ over $T\times\hat{T}$ defined by the Hermitian form H and the semi-character χ of L is the Poincaré line bundle over $T\times\hat{T}$. In fact, the corresponding automorphic factor $J : L\times\hat{T}\to C^*$ for $L(H, \chi)$ is given by

$$J((\alpha, \hat{l}), (z, l)) = \chi(\alpha, \hat{l})\exp\{\pi H((z, l), (\alpha, \hat{l})) + \frac{\pi}{2}H((\alpha, \hat{l}), (\alpha, \hat{l}))\},$$

where $\alpha\in L$, $\hat{l}\in\hat{L}$, $z\in V$ and $l\in\bar{V}^*$. The line bundle $L(H, \chi)|_{T\times\pi(l)}$ over T corresponding to a point $\pi(l)\in\hat{T}(l\in\bar{V}^*)$ is defined by a flat automorphic factor $J_l : L\times V\to C^*$ given by

$$J_l(\alpha, z) = \exp\{\pi l(\alpha)\}, \quad \alpha\in L, \quad z\in V.$$

However

$$h(z+\alpha)J_l(\alpha, z) = \exp\{2\pi i \operatorname{Im} l(\alpha)\}h(z),$$

where $\alpha\in L$, $z\in V$ and $h(z) = \exp\{-\pi\overline{l(z)}\}$ is holomorphic in z. Thus we obtain the isomorphism

$$L(H, \chi)|_{T\times\pi(l)} \cong L(0, \chi_l),$$

where $\chi_l(\alpha) = \exp\{2\pi i l(\alpha)\}$ is a semicharacter of $L(\alpha\in L)$. We note that if $l=0$, $L(H, \chi)|_{T\times\hat{e}} \cong L(0, 1) \cong \mathcal{O}_T$. Clearly

$$L(H, \chi)|_{e\times\hat{T}} \cong L(0, 1) \cong \mathcal{O}_{\hat{T}}.$$

By the uniqueness, $L(H, \chi)$ must be the Poincare line bundle over $T\times\hat{T}$.

5. Projectively flat vector bundles

In his paper ([1]), Mukai characterized semihomogeneous vector bundles over an abelian variety and showed that a simple semihomogeneous vector bundle is Gieseker-stable. He dealt with those vector bundles algebraically. In the analytical point of view, those vector bundles corresponds to the projectively flat vector bundles. In this section, we characterize those vector bundles analytically and study the properties of them.

First we give some definitions.

DEFINITION 5.1. Let $\mathscr{E}$ be a torsion-free coherent sheaf over a compact Kaehler manifold (X, g) of dimension n. Let ω be its Kaehler form. It is a real positive closed $(1,1)$-form on X. Let $c_1(\mathscr{E})$ be the first Chern class of $\mathscr{E}$. It is represented by a real closed $(1,1)$-form on X. The degree of $\mathscr{E}$ is defined to be

$$\deg(\mathscr{E}) = \int_X c_1(\mathscr{E}) \wedge \omega^{n-1}.$$

The degree/rank ratio or slope $\mu(\mathscr{E})$ is defined to be

$$\mu(\mathscr{E}) = \deg(\mathscr{E})/\mathrm{rank}(\mathscr{E}).$$

A coherent sheaf $\mathscr{E}$ over a compact Kaehler manifold (X, g) is said to be *stable*(resp. *semistable*) if for every coherent(nontrivial) proper subsheaf $\mathscr{F}$ of lower rank, $\mu(\mathscr{F}) < \mu(\mathscr{E})$ (resp. $\leq$).

DEFINITION 5.2. Let L be an ample line bundle over a compact Kaehler manifold X. For a coherent sheaf $\mathscr{E}$ over X, we denote by $\mathscr{E}(k)$ the sheaf $\mathscr{E} \otimes L^k$ and by $\chi(\mathscr{E}(k))$ the Hilbert polynomial. We define $P_{\mathscr{E}}$ by

$$P_{\mathscr{E}}(k) = \frac{\chi(\mathscr{E}(k))}{\mathrm{rank}(\mathscr{E})}$$

$\mathscr{E}$ is said to be *Gieseker-stable*(resp. *Gieseker-semistable*) if for each proper subsheaf $\mathscr{F}$ of $\mathscr{E}$, we have

$$P_{\mathscr{F}}(k) < P_{\mathscr{E}}(k) \quad (\text{resp. } P_{\mathscr{F}}(k) \leq P_{\mathscr{E}}(k))$$

for all $k \gg 0$.

REMARK 5.3. (1) A holomorphic vector bundle E over X is said to be stable (resp. semistable) if the sheaf $\mathscr{O}(E)$ of germs of holomorphic sections is stable (resp. semistable). Similarly we can say for the Gieseker stability.

(2) It is known that if a coherent sheaf $\mathscr{E}$ is stable (resp. semistable), it is Gieseker-stable (resp. Gieseker-semistable).

The concept of stability is the algebraic geometrical concept. Kobayashi [1] interpreted the concept of stability differential geometrically.

DEFINITION 5.4. Let g be a Kaehler metric on a compact Kaehler manifold X. Then we define $tr_g : A^{1,1}(\mathrm{End}(E)) \to A^0(\mathrm{End}(E))$ as follows: For a section $F=(F_\alpha^\beta) \in A^{1,1}(\mathrm{End}(E))$,

$$tr_g F = (\sum g^{j\bar{k}} F_{\alpha j\bar{k}}^\beta)_{1 \le \alpha, \beta \le n} = \sum_{j,k} g^{j\bar{k}} F_{j\bar{k}},$$

where $F_\alpha^\beta = F_{\alpha j\bar{k}}^\beta dz_j \wedge d\bar{z}_k$ and $F_{j\bar{k}} = (F_{\alpha j\bar{k}}^\beta)_{1 \le \alpha, \beta \le r}$. A holomorphic vector bundle of rank r over a compact Kaehler manifold (X, g) is said to be *Hermitian-Einstein* if there exists an hermitian metric h for which the Hermitian curvature satisfies

$$tr_g F = \mu I,$$

where I is the identity endomorphism of E and μ is a constant.

Kobayashi [1] obtained the following differential geometrical criterion for stability.

THEOREM (Kobayashi). *An indecomposable Hermitian-Einstein vector bundle over a compact Kaehler manifold is stable.*

The fact that the converse of the above theorem is also true was proved by Uhlenbeck and Yau [1].

DEFINITION 5.5. Let E be a holomorphic vector bundle of rank r over a compact Kaehler manifold X and P its associated principal $GL(r\,;C)$-bundle. Then $\hat{P}=P/C^*I_r$ is a principal $PGL(r\,;C)$-bundle. We say that E is *projectively flat* when $\hat{P}$ is provided with a flat sturcture.

Mukai [1] introduced the notion of semihomogeneous vector bundles over a complex torus.

DEFINITION 5.6. A holomorphic vector bundle E over a complex torus T is said to be *semihomogeneous* if for each $x \in T$, there exists a line bundle F over T such that

$$T_x^*(E) \cong E \otimes F,$$

where T_x is the translation of T by x.

LEMMA 5.7. *Let E be a semihomogeneous vector bundle of rank r over an abelian variety A. Then the vector bundle $(r_T)^*(E) \otimes$*

$(\det E)^{-r}$ is homogeneous, where det E denotes thedeterminant line bundle of E.

Proof. $L=(r_T)^*(E)\otimes(\det E)^{-r}$. Then we have

$$\det L=(r_T)^*(\det E)\otimes(\det E)^{-r^2}.$$

By Lemma 4.10 in the preceding section, $c_1(\det L)=0$ and hence det L is homogeneous. It is clear that L is semihomogeneous. Since $T_x^*(L)\cong L\otimes F$ for some $F\in \mathrm{Pic}^0(T)$, we obtain

$$T_{rx}^*(L)\cong H\otimes F^r,$$
$$T_x^*(\det L)\cong \det L\otimes F^r.$$

Therefore

$$T_{rx}^*(L)\cong L\otimes T_x^*(\det L)\otimes(\det L)^{(-1)}$$
$$\cong L(\text{because } \det L \text{ is homogeneous}).$$

Since A is divisible, L is homogeneous.

Proposition 5.8. *Let E be a semihomogeneous vector bundle of rank r over an abelian variety A of dimension g. Then*

$$\chi(E)=r^{1-g}\chi(\det E),$$

where $\chi(E)$ denotes the Euler-Poincaré characteristic of E.

Proof. Since $(r_T)^*(E)\cong(\det E)^r\otimes F$ for some homogeneous vector bundle F, we have

$$\begin{aligned} r^{2g}\chi(E) &= \chi((r_T)^*(E)) \\ &= \chi((\det E)^r\otimes F) \quad (\text{because } c_k(F)=0 \text{ for } k\geq 1) \\ &= r\chi((\det E)^r) \\ &= r^{g+1}\chi(\det E). \end{aligned}$$

Hence $\chi(E)=r^{1-g}\chi(\det E)$.

Mukai [1] and Oda [2] showed the following theorem.

Theorem 5.9 (Mukai, Oda). *Let E be a simple vector bundle over an abelian variety A of dimension g. Then the following are equivalent:*

(1) $\dim_{\mathbb{C}} H^1(A, End(E))=g$,

(2) $\dim_{\mathbb{C}} H^j(A, End(E))=\binom{g}{j}$,

(3) *E is semihomogeneous,*

(4) *There exist an isogeny $f: B\to A$ and a line bundle L on an abelian variety B such that $E=f_*(L)$.*

It is easily seen from the definition that semihomogenous vector bundles over a complex torus are projectively flat. Hano [1] showed that an automorphic factor for a projectively flat vector bundle E over a complex torus $T=V/L$ is given by the following form

$$(*)\quad J(\alpha, z)=G(\alpha)\exp\left\{\frac{\pi}{r}H(z, \alpha)+\frac{\pi}{2r}H(\alpha, \alpha)\right\},\quad \alpha\in L,\quad z\in V,$$

where (i) H is a Riemann form for $T=V/L$,

(ii) $G: L\to GL(r\,;C)$ is a semirepresentation of L in the sense:

$$G(\alpha+\beta)=G(\alpha)G(\beta)\exp\left\{\frac{i\pi}{r}E(\beta, \alpha)\right\},\quad E=\mathrm{Im}\,H.$$

Using Lemma 4.2 in the previous section, we can show that a projectively flat vector bundle over a complex torus is semihomogeneous. Thus the notion of semihomogeneous vector bundles is the same of projectively flat vector bundles.

LEMMA 5.10. *Let $f:\tilde{T}=\tilde{V}/\tilde{L}\to T=V/L$ be an isogeny and let E (resp. F) be a projectively flat vector bundle of rank r over T (resp. $\tilde{T}$). Then $f^*(E)$ (resp. $f_*(F)$) is also projectively flat.*

Proof. f lifts to the linear map $f: V\to V$. Then the automorphic factor J^* for $f^*(E)$ is given by

$$\tilde{J}(\tilde{\alpha}, \tilde{z})=G(f(\tilde{\alpha}))\ \exp\left\{\frac{\pi}{r}H(f(\tilde{z})), f(\tilde{\alpha}))+\frac{\pi}{2r}H(f(\tilde{\alpha}, f(\tilde{\alpha}))\right\},$$

where $\tilde{\alpha}\in\tilde{L}$, $z\in\tilde{V}$. thus $f^*(E)$ is semihomogeneous. We leave to the reader the case of $f_*(E)$.

PROPOSITION 5.11. *Let E be a hermitian vector bundle of rank r over an abelian surface A with $c_2(End(E))=0$. Then the following are equivalent:*

(1) *E is simple,*

(2) *E is simple and semihomogeneous,*

(3) *E is simple and projectively flat,*

(4) *E admits an indecomposable Hermitian-Einstein vector bundle over A,*

(5) *E is stable,*

(6) *E is Gieseker-stable.*

Proof. (1)$\Rightarrow$(2) :

$$\chi(\mathrm{End}(E)) = 2h^0(A, \mathrm{End}(E)) - h^1(A, \mathrm{End}(E))$$
$$= 2 - h^1(A, \mathrm{End}(E)) \text{ (because } E \text{ is simple).}$$

But by the Riemann-Roch theorem,

$$\chi(\mathrm{End}(E)) = -c_2(\mathrm{End}(E)) = 0.$$

Thus we have $h^1(A, \mathrm{End}(E)) = 2$. By Theorem 5.10, E is semihomogeneous. The remaining ones follows from Kobayashi's theorem and the fact that a Gieseker-stable vector bundle is simple.

Since a simple projective flat hermitian vector bundle E over a complex torus T admits an Hermitian-Einstein structure, E is Gieseker-stable by Kobayashi's theorem (or Mukai [1]). Thus E has a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_k = E$$

such that $F_i = E_i/E_{i-1}$ is Gieseker-stable, and $P_{F_i} = P_E$ for all $i = 1, 2, \cdots, k$ (Gieseker [1]).

An automorphic factor J for E is given by the form (*). Now we calculate the curvature form Ω of E. We let $\pi : V \to T = V/L$. We choose an open covering $\{U_i\}$ of T with the following property: U_i are connected and each connected component of $\pi^{-1}(U_i)$ are mapped homeomorphically onto U_i by π. For U_i we choose a connected component $\tilde{U}_i$ of $\pi^{-1}(U_i)$. Then we have $\pi^{-1}(U_i) = \bigcup_{\alpha \in L} T_\alpha \tilde{U}_i$, where $T_\alpha : V \to V$ is the translation of V by $\alpha \in L$. We let

$$\rho_i : U_i \longrightarrow \tilde{U}_i$$

be the inverse of the homeomorphism $\pi : \tilde{U}_i \to U_i$. For each pair (i, j) of indices such that $U_i \cap U_j \neq \phi$, there exists a unique $\sigma_{ji} \in L$ such that

$$\rho_i(x) = \rho_j(x) + \sigma_{ji}$$

for all $x \in U_i \cap U_j$. For all $x \in U_i \cap U_j$, we let

$$g_{ij}(x) = J(\sigma_{ji}, \rho_j(x)).$$

Then $g_{ij} : U_i \cap U_j \to GL(r, C)$ is a holomorphic map and $\{g_{ij}\}$ is a system of transition functions of the vector bundle E over a complex torus T.

We take a basis of V and identify V with C^g and write

$$H(v, w) = \sum_{a,b=1}^{g} h_{a\bar{b}} v_a \bar{\omega}_b.$$

Then we have

$$g_{ij}=G(\sigma_{ji})\exp\left\{\frac{\pi}{r}H(\rho_j,\sigma_{ji})+\frac{\pi}{2r}H(\sigma_{ji},\sigma_{ji})\right\}.$$

We let

$$z_a^{(i)}=v_a\circ\rho_i$$

for each i. Then $\{z_1^{(i)},\cdots,z_g^{(i)}\}$ are local coordinates of T on U_i and we obtain

$$dz_a^{(i)}=dz_a^{(j)} \text{ on } U_i\cap U_j$$

Let ζ_a be the holomorphic 1-form on T such that $\pi^*\zeta_a=dv_a$ Then we have

$$\zeta_a=dz_a^{(i)}$$

on each U_i. We get

$$g_{ij}^{-1}dg_{ij}=\left\{\frac{\pi}{r}\sum_{a,b}h_{a\bar{b}}(\bar{\sigma}_{ji})_b\zeta_a\right\}\cdot I_r,$$

where I_r is the $r\times r$ identity matrix. We let

$$\omega_i=-\left\{\frac{\pi}{r}\sum_{a,b}h_{a\bar{b}}\bar{z}_b^{(i)}\right\}\cdot I_r$$

on each U_i. Then it is easy to show that $\omega=\{\omega_i\}$ is a connection form. The curvature form $\Omega=\{\Omega_i\}$ is the system of 2-forms such that

$$\Omega_i=d\omega_i+\omega_i\wedge\omega_i$$

on U_i. But we have $\omega_i\wedge\omega_i=0$ and hence $\Omega_i=d\omega_i$. Then we have

$$\Omega_i=\left\{\frac{\pi}{r}\sum_{a,b}h_{a\bar{b}}\zeta_a\wedge\bar{\zeta}_b\right\}\}\cdot I_r$$

on U_i and since the left hand side is globally defined, we have globally

$$\Omega=\left\{\frac{\pi}{r}\sum_{a,b}h_{c\bar{b}}\zeta_a\wedge\bar{\zeta}_b\right\}\cdot I_r.$$

The total Chern class $c(E)$ is given by

$$c(E)=\det\left(I_r-\frac{1}{2\pi i}\Omega\right)$$
$$=\left(1+\frac{i}{2r}\sum_{a,b}h_{a\bar{b}}\zeta_a\wedge\bar{\zeta}_b\right)^r.$$

Let E be a holomorphic vector bundle of rank r over a complex torus T. We assume that the total Chern class $c(E)$ of E is

$$c(E)=\left(1+\frac{c_1(E)}{r}\right)^r.$$

We know that $c(E)$ is given by

$$c(E)=\det\left(I_r-\frac{\Omega}{2\pi i}\right),$$

where Ω is the curvature form of E. If we write $\Omega=(\Omega^i_j)$, the k-th Chern class $c_k(E)$ of E is given by

$$c_k(E)=\binom{r}{k}\frac{1}{r^k}c_1(E)^k$$
$$=\frac{(-1)^k}{(2\pi i)^k k!}\sum\delta^{j_1\cdots j_k}_{i_1\cdots i_k}\Omega^{i_1}_{j_1}\wedge\cdots\wedge\Omega^{i_k}_{j_k}.$$

By a tedious calculation, we know that Ω is of the form

$$\Omega=\delta I_r,$$

where δ is a 2-form on T. Hence E is projectively flat.

In summary, we have

THEOREM 5.12. *Let E be a holomorphic vector bundle over rank r over a complex torus $T=V/L$. Then the following conditions are equivalent:*

(1) *E is semihomogeneous,*

(2) *E is projectively flat,*

(3) *The total Chern class $c(E)$ of E is given by*

$$c(E)=\left(1+\frac{c_1(E)}{r}\right)^r,$$

(4) *The Automorphic factor J for E is given by*

$$J(\alpha, z)=G(\alpha)exp\left\{\frac{\pi}{r}H(z,\alpha)+\frac{\pi}{2r}H(\alpha,\alpha)\right\},\quad \alpha\in L,\quad z\in V,$$

where $G:\to GL(r\,;C)$ is a semirepresentation of L and H is a Riemann form for T. Furthermore, if E is simple, (1), (2), (3) *and* (4) *are equivalent to*

(5) *$dim_C H^j(T, End(E))=\binom{g}{j}$ for all $j=1, 2, \cdots, n$.*

(6) *$H^j(T,\mathcal{O})\cong H^j(T, End(E))$ for all j,*

(7) *There exists an isogeny $f:\tilde{T}\to T$ and a line bundle L on $\tilde{T}$ such that $E=f_*(L)$.*

6. Heisenberg Group $\mathcal{G}(E)$

Throughout this section X is assumed to be an abelian variety of dimension g over the field of complex numbers. We recall that

n_X is the multiplication by n for an integer n (see Definition 4. 9).

(6. 1) Let E be a holomorphic vector bundle over X. We define $H(E)$ and $\mathscr{G}(E)$ as follows:

$H(E)=\{x\in X \mid E\cong T_x^*(E)\}$,

$\mathscr{G}(E)=\{(x,\varphi)\mid x\in H(E)$ and φ is an isomorphism of E onto $T_x^*(E)\}$. $\mathscr{G}(E)$ is a group. In fact, let (x,φ), (y,ψ) be elements of $\mathscr{G}(E)$. Then the composition $T_x^*(\psi\circ\varphi)$:

$$E\longrightarrow T_x^*(E)\longrightarrow T_x^*(T_y^*(E))=T_{x+y}^*(E)$$

is an isomorphism of E and T_{x+y}^*. We define the multiplication

$$(y,\psi)\circ(x,\varphi)=(x+y,\ T_x^*(\psi\circ\varphi)).$$

It is easy to check that the set $\mathscr{G}(E)$ forms a group under the above multiplication. And we have the following exact sequence

$$1\longrightarrow \mathrm{Aut}(E)\longrightarrow \mathscr{G}(E)\longrightarrow H(E)\longrightarrow 0.$$

(6. 2) Let L be an ample line bundle over an abelian variety X. We recall the basic results about an ample line bundle (Mumford [1]).

(I) $H(L)$ is finite and $H^0(X,L^n)\neq 0$ for all $n>0$.

(II) If $\dim X=g$, then there exists a positive integer d such that

$$\dim_C H^0(X,L^n)=dn^g \text{ for all } n\geq 1,$$
$$\dim_C H^i(X,L^n)=0 \text{ for all } n\geq 1,\ i\geq 1.$$

The integer d is called the *degree* of L

(III) Let $\hat{X}$ be the dual abelian variety of X. Let $\Lambda(L):\to\hat{X}$ be the homomorphism define by $\Lambda(L)(x)=T_x^*(L)\otimes L^{-1}$ for all $x\in X$. Then we have

$$d^2=|\chi(L)|^2=\text{the degree of } \Lambda(L)=|H(L)|.$$

(IV) For all integers n,

$$(n_X)^*(L)\cong L^{\frac{n^2+n}{2}}\otimes(-1_X)^*(L)^{\frac{n^2-n}{2}}.$$

(6. 3) Let E be a stable ample vector bundle over X. Then E is simple and hence we have the following extension:

$$1\longrightarrow C^*\longrightarrow \mathscr{G}(E)\longrightarrow H(E)\longrightarrow 0.$$

We note that $H(E)$ is a finite abelian group and that C^* is contained in the center of $\mathscr{G}(E)$. The extension defines the following invariant : Given $x,y\in H(E)$, we let $x,y\in\mathscr{G}(E)$ lie over x,y. We set

$$e^E(x,y)=\hat{x}\hat{y}\hat{x}^{-1}\hat{y}^{-1}.$$

It is obvious that this is well-defined, that $e^E(x,y)$ is an element H of C^*, and that e^E is a skew-symmetric bilinear pairing from $H(E)$ to C^*. A subgroup K of $\mathscr{G}(E)$ is said to be a *level subgroup* if $\hat{K}\cap C^*=(0)$, i.e., $\hat{K}$ is isomorphic to its image in $H(E)$. For all subgroup K of $H(E)$, there exists a level subgroup $\hat{K}$ over K if and only if e^E is trivial on K. If e^E is degenerate, then there exists a subgroup K such that e^E is trivial on $\hat{K}$ and such that $|K|^2>|H|$. Hence there exists a level subgroup $\hat{K}$ of order$>|H|^{\frac{1}{2}}$. We now define $U_z : H^0(X,E)\to H^0(X,E)$ by

$$U_z(s)=T^*_{-x}(\varphi(s)) \text{ for all } s\in H^0(X,E),$$

where $z=(x,\varphi)\in\mathscr{G}(E)$. This is an action of the group $\mathscr{G}(E)$ because if $z=(x,\varphi), w=(y,\psi)$, then

$$\begin{aligned} U_w(U_z(s)) &= T^*_y\{\psi(T^*_x(\varphi s))\} \\ &= T^*_{-x-y}\{T^*_x(\psi(T^*_x(\varphi s)))\} \\ &= T^*_{-x-y}\{T^*_x(\psi)(\varphi s)\} \\ &= U_{(x+y,T_x{}^*(\psi)\circ\varphi)}(s). \end{aligned}$$

Also C^* acts on $H^0(X,E)$ by its natural character, that is, $a\in C^*$ acts on $H^0(X,E)$ as multiplication by a. Hence U is the representation of $\mathscr{G}(E)$ on $H^0(X,E)$.

(6.4) Let X be an elliptic curve and let E be an ample indecomposable vector bundle of rank r and of degree d. We assume that r, d are coprime. Then E is stable and hence simple. Thus we have the extension $0\to C^*\to\mathscr{G}(E)\to H(E)\to 0$ and a level subgroup of $\mathscr{G}(E)$ corresponds to a descent data for E. By Oda [1], there exists an isogeny $f : Y\to X$ of degree r and an ample line bundle L of degree d on Y such that E is isomorphic to the direct image $f_*(L)$ and the intersection of $\ker(f)$ and $\ker(\Lambda(L))$ is 0. Moreover, $d=\dim_C H^0(X,E)=\dim_C H^0(Y,L)$. Since $H(L)\cap\ker(f)=0$, a nontrivial translation by an element of $H(L)$ induces a nonzero element of $H(E)$. Hence $H(L)$ is a subgroup of $H(E)$. We have $|H(L)|=d^2$, hence $|H(E)|\geq d^2$. There exists a level subgroup of order $\tilde{d}>d$. If we had $|H(E)|>d^2$ then there would exist an isogeny $\varphi : X\to Z$ of degree $\tilde{d}$ and a vector bundle $\tilde{E}$ over Z such that $\varphi^*(\tilde{E})\cong E$. But we have $d=\chi(X,E)=\tilde{d}\chi(Z,E)$. This is a contradiction. Hence $H(E)=H(L)$ and $\mathscr{G}(E)=\mathscr{G}(L)$. The unique representation of $\mathscr{G}(E)$ is given by $H^0(X, E)$.

Proposiion 6.5. *Let E be a holomorphic vector bundle of rank r over an abelian surface X. Then the following conditions are equivalent:*

(1) *E is a simple homogeneous vector bundle over X,*

(2) *E is an indecomposable projectively flat vector bundle over X,*

(3) *E is stable and satisfies $c_2(End(E))=0$,*

(4) *E is Gieseker stable and satisfies $c_2(End(E))=0$,*

(5) *E is simple and satisfies $c_2(End(E))=0$,*

(6) *$dim_C H^j(X, End(E))=\binom{g}{j}$ for all $j=1, 2, \cdots, g$,*

(7) *$H^j(X, Q)\cong H^j(X, End(E))$ for all j,*

(8) *There exists an isogeny $f: Y\rightarrow X$ of abelian surfaces and an ample line bundle M over Y such that $H(M)\cap ker(f)=0$ and E is isomorphic to the direct image $f_*(M)$,*

(9) *E is simple and, for any ample line bundle L over X and for any sufficiently large integer n, we have the extension of the Heisenberg group*

$$0\longrightarrow C^*\longrightarrow\mathscr{G}(E\otimes L^n)\longrightarrow H(E\otimes L^n)\longrightarrow 0.$$

such that the pairing $e^{E\otimes L^n}(x, y)$ is nondegenerate and $|H(E\otimes L^n)|=\frac{1}{r^2}h^0(X, (det\, E)\otimes L^{rn})^2$,

(10) *E is simple and the same assertion as in (9) holds for an ample line bundle L and for infinitely many $n>0$.*

Proof. The equivalence of (1), (2), $\cdots$, (8) follows from Theorem 5.12 and the equation $c_2(End(E))=-(r-1)c_1(E)^2+2rc_2(E)$. In order to prove (8)$\Rightarrow$(9), we let E be a simple vector bundle of rank r and let L be an ample line bundle over X. By (8), $E\otimes L^n$ is isomorphic to the direct image of $M\otimes f^*(L^n)$. Let $x\in\ker(f)\cap H(M\otimes f^*(L^n))$. Then we have

$$M\otimes f^*(L^n)\cong T_x^*(M\otimes f^*(L^n))\cong T_x^*(M)\otimes T_x^*f^*(L^n)$$
$$\cong T_x^*(M)\otimes f^*(L^n).$$

Hence M is isomorphic to $T_x^*(L)$, i.e., $x\in\ker(f)\cap H(M)$. It follows that $x=0$. As in (6.4), we have the inequality

$$|H(M\otimes f^*(L^n))|\leq|H(E\otimes L^n)|.$$

According to the descent theory, if $M\otimes f^*(L^n)$ is an ample line bundle, then we get $|H(M\otimes f^*(L^n))\cong H(E\otimes L^n)$ and $|H(E\otimes L^n)|=h^0(X, E\otimes L^n)^2$. By the Riemann-Roch theorem,

$$\begin{aligned} h^0(X, E\otimes L^n) &= \frac{1}{2}c_1^2(E\otimes L^n)-c_2(E\otimes L^n) \\ &= \frac{1}{2r}c_1^2(E)+\frac{rn^2}{2}c_1^2(L)+nc_1(E)c_1(L) \\ &= \frac{1}{2r}c_1^2(E\otimes L^n)=\frac{1}{2r}c_1^2((\det E)\otimes L^{rn}) \\ &= \frac{1}{r}h^0(X, (\det E)\otimes L^{rn}). \end{aligned}$$

Thus we have $|H(E\otimes L^n)|=\frac{1}{r^2}h^0(X, (\det E)\otimes L^{rn})$. Obviously (9) implies (10). Thus the proof is complete only if we show that (10) implies (5). We assume that (10) holds. Then there exists an integer n and an ample line bundle L over X such that $H^0(X, E\otimes L^n)\neq 0$, $H^i(X, E\otimes L^n)=0$, $i=1, 2$, and $0\to C^*\to\mathscr{G}(E\otimes L^n)\to H(E\otimes L^n)\to 0$ is a Heisenberg group and $|H(E\otimes L^n)||=\frac{1}{r^2}h^0(X, (\det E)\otimes L^{rn})^2$. Since $E\otimes L^n$ is simple,

$$\begin{aligned} \chi(\mathrm{End}(E\otimes L^n)) &= h^0(X, \mathrm{End}(E\otimes L^n))-h^1(X, \mathrm{End}(E\otimes L^n)) \\ &\quad +h^2(X, \mathrm{End}(E\otimes L^n)) \\ &= 2-h^1(X, \mathrm{End}(E))\leq 0. \end{aligned}$$

By the Riemann-Roch theorem,

$$\begin{aligned} \chi(\mathrm{End}(E\otimes L^n)) &= -c_2(\mathrm{End}(E\otimes L^n)) \\ &= (r-1)c_1^2(E\otimes L^n)-2rc_2(E\otimes L^n), \end{aligned}$$

Thus $(r-1)c_1^2(E\otimes L^n)-2rc_2(E\otimes L^n)\leq 0$. Then we have

$$\begin{aligned} 1\leq h^0(E\otimes L^n) &= \frac{1}{2}c_1^2(E\otimes L^n)-c_2(E\otimes L^n) \\ &\leq \frac{1}{2}c_1^2(E\otimes L^n)-\frac{r-1}{2r}c_1^2(E\otimes L^n) \\ &= \frac{1}{2r}c_1^2(E\otimes L^n)=|H(E\otimes L^n)|^{\frac{1}{2}}. \end{aligned}$$

since $\mathscr{G}(E\otimes L^n)$ is a Heisenberg group and $H^0(X, E\otimes L^n)$ is a representation of $\mathscr{G}(E\otimes L^n)$ in which C^* acts by its natural character, $h^0(E\otimes L^n)$ is divisible by $|H(E\otimes L^n)|^{\frac{1}{2}}$. By the above inequality, we have

$$h^0(X, E\otimes L^n)=|H(E\otimes L^n)|^{\frac{1}{2}}.$$

By the assumtion, we have

$$h^0(X, E\otimes L^n) = \frac{1}{2r} c_1^2(E\otimes L^n).$$

By the way, by the Riemann-Roch theorem, we have

$$h^0(X, E\otimes L^n) = \frac{1}{2} c_1^2(E\otimes L^n) - c_2(E\otimes L^n).$$

Hence we obtain

$$(r-1)c_1^2(E\otimes L^n) - 2rc_2(E\otimes L^n) = 0.$$

So we get the equation

$$c_2(\mathrm{End}(E)) = -(r-1)c_1^2(E) + 2rc_2(E) = 0.$$

REMARK 6.6. Indeed, the classification of projectively flat vector bundles over a complex torus cerresponds to that of representations of the Heisenberg group. Matsushima [2] described the holomorphic vector bundles defined by the representation of the Heisenberg group.

FINAL REMARK 6.7. It is very interesting to characterize the automorphic factors corresponding to stable vector bundles over complex torus. The author believes that a characterization of those automorphic factors will be useful in the study of vector-valued theta functions.

References

1. Atiyah, M. F. [1] *Vector bundles over an elliptic curves*, Proc. London Math. Soc., Vol. **7** (1957), 414-452.
2. Gieseker, D. [1] *On the moduli of vector bundles on an algebraic surface*, Ann. of Math., Vol. **106** (1977), 45-60.
3. Grauert, H. [1] *Analytische Faserung uber holomorphe-vollständigen Räume*, Math. Ann. **135** (1958), 266-273.
4. Griffith, P. and Harris, J. [1] *Principle of Algebraic Geometry*, John Wiely and Sons (1978).
5. Gunning, R. C. [1] *The structure of factors of automorphy*, Amer. of J. Math., **357-382** (1958).
6. _____ [2] *Riemann surfaces and Generalized Theta Functions*, Springer-Verlag (1976).
7. Hano, J. [1] *A geometrical characterization of a class of holomorphic vector bundles over a complex torus*, Nagoya Math. J., Vol. **61** (1976), 196-199.
8. Kobayashi, S. [1] *Curvature and stability of vector bundles*, Proc. Japan

Acad., Vol. **58**(1982), 158-162.
9. _____ [2] *Differential Geometry of Complex Vector Bundles*, Princeton University Press (1987).
10. Maruyama, M. [1] *Moduli of stable sheaves*, II. J. Math. Kyoto Univ., **18**(1978), 557-614.
11. Matsushima, Y. [1] *Fibrés holomorphes sur un tore complexe*, Nagoya Math., J., Vol. **14**(1959), 1-24.
12. _____ [2] *Heisenberg groups and holomorphic vector bundles over a complex torus*, Nagoya Math. J., Vol. **61**(1976), 161-195.
13. Morimoto, A. [1] *Sur la classification des espaces fibrés vectoriels holomorphes sur un tore complexe admettant des connexions holomorphese*, Nagoya Math. J., Vol. **15**(1959), 83-154.
14. Mukai, S. [1] *Semi-homogeneous vector bundles over an abelian variety*, J. Math. Kyoto Univ., Vol. **18**(1978), 239-272.
15. Mumford, D. [1] *Abelian varieties*, Oxford University Press (1970).
16. _____ [2] *On the equations defining abelian varieties*, Invent. Math., Vol. **1**(1966), 287-354.
17. Oda, T. [1] *Vector bundles on an elliptic curve*, Nagoya Math. J., Vol. **43** (1971), 41-72.
18. _____ [2] *Vector bundles on an abelian surface*, Invent. Math., Vol. **13**(1971), 247-260.
19. Takemoto, F. [1] *Stable vector bundles on an algebraic surface*, Nagoya Math. J., Vol. **47**(1972), 29-48.
20. Uhlenbeck, K and Yau, S. T. [1] *On the existence of Hermitian Yang-Mills connections in stable vector bundles*, Comm. on Pure and Applied Math., Vol. XXXXIX, S257-S293(1986).
21. Umemura, H. [1] *Some results in the theory of vector bundles*, Nagoy Math. J., Vol. **52**(1973), 97-128.
22. Yang, J-H. [1] *Vector bundles over an abelian variety*, KOSEF(Preprint), 1986.
23. _____ [2] *A note on holomorphic vector bundles over complex tori*, Bull. Korean Math. Soc., No. 2(1986), 149-154.

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Yang, Jae-Hyun (KR-INHA)
Holomorphic vector bundles over complex tori.
J. Korean Math. Soc. **26** (1989), *no. 1,* 117–142.

Let E be a holomorphic vector bundle of rank r over a complex torus $T = \mathbf{C}^g/L$. The bundle E is said to be semihomogeneous if for each $x \in T$ there exists a line bundle F over T such that $T_x^*(E) \cong E \otimes F$, where T_x is the translation of T by x. The author proves that the following conditions are equivalent: (1) E is semihomogeneous; (2) the associated projective bundle $\mathbf{P}(E)$ is ﬂat (i.e., admits a system of constant transition functions); (3) the total Chern class $c(E)$ is given by $c(E) = (1 + c_1(E)/r)^r$; (4) the automorphy factor J of E is of the form $J(\alpha, z) = G(\alpha)\exp((\pi/r)H(z,\alpha) + (\pi/2r)H(\alpha,\alpha))$, where $\alpha \in L, z \in \mathbf{C}^n$, H is a Riemann form for T, and $G\colon L \to \mathrm{GL}(r, \mathbf{C})$ is a semirepresentation of L, i.e., $G(\alpha,\beta) = G(\alpha)G(\beta)\exp((i\pi/r)E(\beta,\alpha))$, where $E = \operatorname{Im} H$. The assertion $(2) \Rightarrow (4)$ was proved earlier by J. Hano [Nagoya Math. J. **61** (1976), 197–202; MR0419854 (54 #7872)]. Generalizing the result of Mukai and Oda for abelian varieties the author also proves that if E is simple (i.e. $H^0(T, \operatorname{End} E) = \mathbf{C}$), then (1), (2), (3) and (4) are equivalent to any of the following: (5) $\dim H^j(T, \operatorname{End} E) = \binom{g}{j}$ for all $j, j = 1, 2, \cdots, r$; (6) there exists an isogeny $f\colon \tilde{T} \to T$ and a line bundle L on $\tilde{T}$ such that $E = f_*(L)$.

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[F] Representation Theory of a Lie Group

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The Method of Orbits for Real Lie Groups

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In this paper, we outline a development of the theory of orbit method for representations of real Lie groups. In particular, we study the orbit method for representations of the Heisenberg group and the Jacobi group.

Contents

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1. Introduction

Research into representations of Lie groups was motivated on the one hand by physics, and on the other hand by the theory of automorphic forms. The theory of unitary or admissible representations of noncompact reductive Lie groups has been developed systematically and intensively shortly after the end of World War II. In particular, Harish-Chandra, R. Langlands, Gelfand school and some other people made an enormous contribution to the theory of unitary representations of noncompact reductive Lie groups.

Early in the 1960s A.A. Kirillov [47] first initiated the orbit method for a nilpotent real Lie group attaching an irreducible unitary representation to a coadjoint orbit (which is a homogeneous symplectic manifold) in a perfect way. Thereafter Kirillov's work was generalized to solvable groups of type I by L. Auslander and B. Kostant [3] early in the 1970s in a nice way. Their proof was based on the existence of complex polarizations satisfying a positivity condition. Unfortunately Kirillov's work fails to be generalized in some ways to the case of compact Lie groups or semisimple Lie groups. Relatively simple groups like $SL(2,\mathbb{R})$ have irreducible unitary representations that do not correspond to any symplectic homogeneous space. Conversely, P. Torasso [85] found that the double cover of $SL(3,\mathbb{R})$ has a homogeneous symplectic manifold corresponding to no unitary representations. The orbit method for reductive Lie groups is a kind of a philosophy but not a theorem. Many large families of orbits correspond in comprehensible ways to unitary representations, and provide a clear geometric picture of these representations. The coadjoint orbits for a reductive Lie group are classified into three kinds of orbits, namely, hyperbolic, elliptic and nilpotent ones. The hyperbolic orbits are related to the unitary representations obtained by the parabolic induction and on the other hand, the elliptic ones are related to the unitary representations obtained by the cohomological induction. However, we still have no idea of attaching unitary representations to nilpotent orbits. It is known that there are only finitely many nilpotent orbits. In a certain case, some nilpotent orbits are corresponded to the so-called *unipotent representations*. For instance, a minimal nilpotent orbit is attached to a minimal representation. In fact, the notion of unipotent representations is not still well defined. The investigation of unipotent representations is now under way. Recently D. Vogan [93] presented a *new* method for studying the quantization of nilpotent orbits in terms of the restriction to a maximal compact subgroup even though it is not complete and is in a preliminary stage. J.-S. Huang and J.-S. Li [41] attached unitary representations to spherical nilpotent orbits for the real orthogonal and symplectic groups.

In this article, we describe a development of the orbit method for real Lie groups, and then in particular, we study the orbit method for the Heisenberg group and the Jacobi group in detail. This paper is organized as follows. In Section 2, we describe the notion of geometric quantization relating to the theory of unitary representations which led to the orbit method. The study of the geometric quantization was first made intensively by A.A. Kirillov [51]. In Section 3, we outline the beautiful Kirillov's work on the orbit method for a nilpotent real Lie group done early in the 1960s. In Section 4, we describe the work for a solvable Lie group of type I done by L. Auslander and B. Kostant [3] generalizing Kirillov's work. In Section 5, we roughly discuss the cases of compact or semisimple Lie groups where the orbit method does not work nicely. If G is compact or semisimple, the correspondence between G-orbits and irreducible unitary representations of G breaks down. In Section 6, for a real reductive Lie group G with Lie algebra $\mathfrak{g}$, we present some properties of nilpotent orbits for G and describe the Kostant-Sekiguchi correspondence between G-orbits in the cone of all nilpotent elements in $\mathfrak{g}$ and $K_{\mathbb{C}}$-orbits in the cone of nilpotent elements on $\mathfrak{p}_{\mathbb{C}}$, where $K_{\mathbb{C}}$ is the complexification of a fixed maximal compact subgroup K of G and $\mathfrak{g}_{\mathbb{C}}=\mathfrak{k}_{\mathbb{C}}\oplus\mathfrak{p}_{\mathbb{C}}$ is the Cartan decomposition of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. We do not know yet how to quantize a nilpotent orbit in general. But for a maximal compact subgroup K of G, D. Vogan attaches a space with a representation of K to a nilpotent orbit. We explain this correspondence in a rough way. Most of the materials in this section come from the article [93]. In Section 7, we outline the notion of minimal orbits (that are nilpotent orbits), and the relation of the minimal representations to the theory of reductive dual pairs initiated first by R. Howe. We also discuss the recent works for a construction of minimal representations for various groups. For more detail, we refer to [65]. In Section 8, we study the orbit method for the Heisenberg group in some detail. In Section 9, we study the unitary representations of the Jacobi group and their related topics. The Jacobi group appears in the theory of Jacobi forms. That means that Jacobi forms are automorphic forms for the Jacobi group. We study the coadjoint orbits for the Jacobi group.

Notation. We denote by $\mathbb{Z},\mathbb{R}$, and $\mathbb{C}$ the ring of integers, the field of real numbers, and the field of complex numbers respectively. The symbol $\mathbb{C}_1^{\times}$ denotes the multiplicative group consisting of all complex numbers z with $|z|=1$, and the symbol $Sp(n,\mathbb{R})$ the symplectic group of degree n, H_n the Siegel upper half plane of degree n. The symbol ":=" means that the expression on the right hand side is the definition of that on the left. We denote by $\mathbb{Z}^+$ the set of all positive integers, by $F^{(k,l)}$ the set of all $k\times l$ matrices with entries in a commutative ring F. For any $M\in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. For $A\in F^{(k,k)}$, $\sigma(A)$ denotes the trace of A. For $A\in F^{(k,l)}$ and $B\in F^{(k,k)}$, we set $B[A]={}^tABA$. We denote the identity matrix of degree k by E_k. For a positive integer n, $\mathrm{Symm}\,(n,K)$ denotes the vector space consisting of all symmetric $n\times n$ matrices with entries in a field K.

2. Quantization

The problem of quantization in mathematical physics is to attach a quantum mechanical model to a classical physical system. The notion of *geometric quantization* had emerged at the end of the 1960s relating to the theory of unitary group representations which led to the orbit method. The goal of geometric quantization is to construct quantum objects using the geometry of the corresponding classical objects as a point of departure. In this paper we are dealing with the group representations and hence the problem of quantization in representation theory is to attach a unitary group representation to a symplectic homogeneous space.

A classical mechanical system can be modelled by the phase space which is a symplectic manifold. On the other hand, a quantum mechanical system is modelled by a Hilbert space. Each state of the system corresponds to a line in the Hilbert space.

Definition 2.1. A pair (M,ω) is called a *symplectic* manifold with a nondegenerate closed differential 2-form ω. We say that a pair (M,c) is a *Poisson manifold* if M is a smooth manifold with a bivector $c=c^{ij}\partial_i\partial_j$ such that the Poisson brackets

$$\{f_1,f_2\}=c^{ij}\partial_i f_1\partial_j f_2 \tag{2.1}$$

define a Lie algebra structure on $C^\infty(M)$. We define a *Poisson* G-manifold as a pair $(M,\ f^M_{(\cdot)})$ where M is a Poisson manifold with an action of G and $f^M_{(\cdot)}:\mathfrak{g}\to C^\infty(M)\,(X\mapsto f^M_X)$ is a Lie algebra homomorphism such that the following relation holds:

$$\text{s}-\text{grad}(f^M_X)=L_X,\quad X\in\mathfrak{g},\quad X\in\mathfrak{g}. \tag{2.2}$$

Here L_X is the Lie vector field on M associated with $X\in\mathfrak{g}$, and s-grad(f) denotes the *skew gradient* of a function f, that is, the vector field on M such that

$$\text{s-grad}(f)g=\{f,g\}\quad\text{for all}\quad g\in C^\infty(M). \tag{2.3}$$

For a given Lie group G the collection of all Poisson G-manifolds forms the category $\mathcal{P}(G)$ where a morphism $\alpha:(M,f^M_{(\cdot)})\to(N,f^N_{(\cdot)})$ is a smooth map from M to N which preserves the Poisson brackets: $\{\alpha^*(\phi),\ \alpha^*(\psi)\}=\alpha^*(\{\phi,\psi\})$ and makes the following relation holds:

$$\alpha^*(f^N_X)=f^M_X,\quad X\in\mathfrak{g}. \tag{2.4}$$

Observe that the last condition implies that α commutes with the G-action.

First we explain the mathematical model of classical mechanics in the Hamiltonian formalism.

Let (M,ω) be a symplectic manifold of dimension $2n$. According to the Darboux theorem, the sympletic form ω can always be written in the form

$$\omega=\Sigma_{k=1}^n dp_k\wedge dq_k \tag{2.5}$$

in suitable canonical coordinates $p_1, \cdots, p_n, q_1, \cdots, q_n$. However, these canonical coordinates are not uniquely determined.

The symplectic from ω sets up an isomorphism between the tangent and cotangent spaces at each point of M. The inverse isomorphism is given by a bivector c, which has the form

$$c = \Sigma_{k=1}^{n} \frac{\partial}{\partial p_k} \frac{\partial}{\partial q_k} \tag{2.6}$$

in the same system of coordinates in which the equality (2.5) holds. In the general system of coordinates the form ω and the bivecter c are written in the form

$$\omega = \Sigma_{i<j} \omega_{ij} dx_i \wedge dx_j, \quad c = \Sigma_{i<j} c^{ij} \partial_i \partial_j \tag{2.7}$$

with mutually inverse skew-symmetric matrices (ω_{ij}) and (c^{ij}). The set $C^\infty(M)$ of all smooth functions on M forms a commutative associative algebra with respect to the usual multiplication. The Poisson bracket $\{\ ,\ \}$ defined by

$$\{F, G\} := \Sigma_{i,j} c^{ij} \partial_j F \cdot \partial_i G, \quad F, G \in C^\infty(M) \tag{2.8}$$

defines a Lie algebra structure on $C^\infty(M)$. The Jacobi identity for the Poisson bracket is equivalent to the condition $d\omega = 0$ and also to the vanishing of the Schouten bracket

$$[c, c]^{ijk} := \circlearrowleft_{ijk} \Sigma_m c^{im} \partial_m c^{jk} \tag{2.9}$$

where the sign $\circlearrowleft_{ijk}$ denotes the sum over the cyclic permutations of the indices i, j, k.

Physical quantities or *observables* are identified with the smooth functions on M. A state of the system is a linear functional on $C^\infty(M)$ which takes non-negative values on non-negative functions and equals 1 on the function which is identically equal to 1. The general form of such a functional is a probability measure μ on M. By a *pure state* is meant an extremal point of the set of states.

The dynamics of a system is determined by the choice of a Hamiltonian function or energy, whose role can be played by an arbitrary function $H \in C^\infty(M)$. The dynamics of the system is described as follows. The states do *not* depend on time, and the physical quantities are functions of the point of the phase space and of time. If F is any function on $M \times \mathbb{R}$, that is, any observable, the equations of the motion have the form

$$\dot{F} := \frac{\partial F}{\partial t} = \{H, F\}. \tag{2.10}$$

Here the dot denotes the derivative with respect to time. In particular, applying (2.10) to the canonical variables p_k, q_k, we obtain *Hamilton's equations*

$$\dot{q_k} = \frac{\partial H}{\partial p_k}, \quad \dot{p_k} = -\frac{\partial H}{\partial q_k}. \tag{2.11}$$

A set $\{F_1, \cdots, F_m\}$ of physical quantities is called *complete* if the conditions $\{F_i, G\} = 0 \, (1 \leq i \leq m)$ imply that G is a constant.

Definition 2.2. Let (M, ω) be a symplectic manifold and $f \in C^\infty(M)$ a smooth function on M. The Hamiltonian vector field ξ_f of f is defined by

$$\xi_f(g) = \{f, g\}, \quad g \in C^\infty(M) \tag{2.12}$$

where $\{\ ,\ \}$ is the Poisson bracket on $C^\infty(M)$ defined by (2.8). Suppose G is a Lie group with a smooth action of G on M by symplectomorphisms. We say that M is a *Hamiltonian G-space* if there exist a linear map

$$\tilde{\mu} : \mathfrak{g} \longrightarrow C^\infty(M) \tag{2.13}$$

with the following properties (H1)-(H3) :
(H1) $\tilde{\mu}$ intertwines the adjoint action of G on $\mathfrak{g}$ with its action on $C^\infty(M)$;
(H2) For each $Y \in \mathfrak{g}$, the vector field by which Y acts on M is $\xi_{\tilde{\mu}(Y)}$;
(H3) $\tilde{\mu}$ is a Lie algebra homomorphism.

The above definition can be formulated in the category of Poisson manifolds, or even of possibly singular Poisson algebraic varieties. The definition is due to A. A. Kirillov [48] and B. Kostant [57].

The natural quantum analogue of a Hamiltonian G-space is simply a unitary representation of G.

Definition 2.3. Suppose G is a Lie group. A *unitary representation* of G is a pair $(\pi, \mathcal{H})$ with a Hilbert space $\mathcal{H}$, and

$$\pi : G \longrightarrow U(\mathcal{H})$$

a homomorphism from G to the group of unitary operations on $\mathcal{H}$.

We would like to have a notion of quantization passing from Definition 2.2 to Definition 2.3: that is, from Hamiltonian G-space to unitary representations.

Next we explain the mathematical model of quantum mechanics. In quantum mechanics the physical quantities or observables are self-adjoint linear operators on some complex Hilbert space $\mathcal{H}$. They form a linear space on which two bilinear operations are defined:

$$A \circ B := \frac{1}{2}(AB + BA) \quad \text{(Jordan multiplication)} \tag{2.14}$$

$$[A, B]_\hbar := \frac{2\pi i}{\hbar}(AB - BA) \quad \text{(the commutator)} \tag{2.15}$$

where $\hbar$ is the Planck's constant.

With respect to (2.14) the set of observables forms a commutative but not associative algebra. With respect to (2.15) it forms a Lie algebra. These two operations (2.14) and (2.15) are the quantum analogues of the usual multiplication and the Poisson bracket in classical mechanics. The *phase space* in quantum mechanics consists of the non-negative definite operators A with the property that $\operatorname{tr} A = 1$. The *pure states* are the one-dimensional projection operators on $\mathcal{H}$. The *dynamics* of the system is defined by the *energy operator* $\hat{H}$. If the states do not depend on time but the quantities change, then we obtain the *Heisenberg picture*. The equations of motion are given by

$$\dot{\hat{A}} = [\hat{H}, \hat{A}]_\hbar, \quad \text{(Heisenberg's equation).} \tag{2.16}$$

The integrals of the system are all the operators which commute with $\hat{H}$. In particular, the energy operator itself does not change with time.

The other description of the system is the so-called Schrödinger picture. In this case, the operators corresponding to physical quantities do not change, but the states change. A pure state varies according to the law

$$\dot{\Psi} = \frac{2\pi i}{\hbar}\hat{H}\Psi \quad \text{(Schrödinger equation).} \tag{2.17}$$

The eigenfunctions of the Schrödinger operator give the stationary states of the system. We call a set of quantum physical quantities $\hat{A}_1, \cdots, \hat{A_m}$ *complete* if any operator $\hat{B}$ which commutes with $\hat{A}_i$ $(1 \leq i \leq m)$ is a multiple of the identity. One can show that this condition is equivalent to the irreducibility of the set $\hat{A}_1, \cdots, \hat{A_m}$.

Finally we describe the quantization problem relating to the orbit method in group representation theory. As I said earlier, the problem of geometric quantization is to construct a Hilbert space $\mathcal{H}$ and a set of operators on $\mathcal{H}$ which give the quantum analogue of this system from the geometry of a symplectic manifold which gives the model of classical mechanical system. If the initial classical system had a symmetry group G, it is natural to require that the corresponding quantum model should also have this symmetry. That means that on the Hilbert space $\mathcal{H}$ there should be a unitary representation of the group G.

We are interested in homogeneous symplectic manifolds on which a Lie group G acts transitively. If the thesis is true that every quantum system with a symmetry group G can be obtained by quantization of a classical system with the same symmetry group, then the irreducible representations of the group must be connected with homogeneous symplectic G-manifolds. The orbit method in representation theory intiated first by A. A. Kirillov early in the 1960s relates the unitary representations of a Lie group G to the coadjoint orbits of G. Later L. Auslauder, B. Kostaut, M. Duflo, D. Vogan, P. Torasso etc developed the theory of the orbit method to more general cases.

For more details, we refer to [47]- [51], [55], [57]-[58] and [90]-[91], [93].

3. The Kirillov Correspondence

In this section, we review the results of Kirillov on unitary representations of a nilponent real Lie group. We refer to [47]-[51], [55] for more detail.

Let G be a simply connected real Lie group with its Lie algebra $\mathfrak{g}$. Let $Ad_G : G \longrightarrow GL(\mathfrak{g})$ be the adjoint representation of G. That is, for each $g \in G$, $Ad_G(g)$ is the differential map of I_g at the identity e, where $I_g : G \longrightarrow G$ is the conjugation by g given by

$$I_g(x) := gxg^{-1} \quad \text{for} \;\; x \in G.$$

Let $\mathfrak{g}^*$ be the dual space of the vector space $\mathfrak{g}$. Let $Ad_G^* : G \longrightarrow GL(\mathfrak{g}^*)$ be the contragredient of the adjoint representation Ad_G. Ad_G^* is called the coadjoint representation of G. For each $\ell \in \mathfrak{g}^*$, we define the alternating bilinear form B_ℓ on $\mathfrak{g}$ by

$$B_\ell(X,Y) = < [X,Y], \ell >, \quad X, Y \in \mathfrak{g}. \tag{3.1}$$

Definition 3.1. (1) A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be *subordinate* to $\ell \in \mathfrak{g}^*$ if $\mathfrak{h}$ forms a totally isotropic vector space of $\mathfrak{g}$ relative to the alternating bilinear form B_ℓ define by (3.1), i.e., $B_\ell|_{\mathfrak{h}\times\mathfrak{h}} = 0$.

(2) A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ subordinate to $\ell \in \mathfrak{g}^*$ is called a *polarization* of $\mathfrak{g}$ for ℓ if $\mathfrak{h}$ is maximal among the totally isotropic vector subspaces of $\mathfrak{g}$ relative to B_ℓ. In other words, if P is a vector subspace of $\mathfrak{g}$ such that $\mathfrak{h} \subset P$ and $B_\ell|_{P\times P} = 0$, then we have $\mathfrak{h} = P$.

(3) Let $\ell \in \mathfrak{g}^*$ and let $\mathfrak{h}$ be a polarization of $\mathfrak{g}$ for ℓ. We let H the simply connected closed subgroup of G corresponding to the Lie subalgebra $\mathfrak{h}$. We define the unitary character $\chi_{\ell,\mathfrak{h}}$ of H by

$$\chi_{\ell,\mathfrak{h}}(\exp_H(X)) = e^{2\pi i < X, \ell >}, \quad X \in \mathfrak{h}, \tag{3.2}$$

where $\exp_H : \mathfrak{h} \longrightarrow H$ denotes the exponential mapping of $\mathfrak{h}$ to H. It is known that $\exp_H$ is surjective.

Using the Mackey machinary, Dixmier and Kirillov proved the following important theorem.

Theorem 3.1 (Dixmier-Kirillov, [19],[47]). *A simply connected real nilpotent Lie group G is monomial, that is, each irreducible unitary representation of G can be unitarily induced by a unitary character of some closed subgroup of G.*

Remark 3.2. More generally, it can be proved that a simply connected real Lie

group whose exponential mapping is a diffeomorphism is monomial.
Now we may state Theorem 3.1 explicitly.

Theorem 3.3 (Kirillov, [47]). *Let G be a simply connected nilpotent real Lie group with its Lie algebra $\mathfrak{g}$. Assume that there is given an irreducible unitary representation π of G. Then there exist an element $\ell \in \mathfrak{g}^*$ and a polarization $\mathfrak{h}$ of $\mathfrak{g}$ for ℓ such that $\pi \cong Ind_H^G \chi_{\ell,\mathfrak{h}}$, where $\chi_{\ell,\mathfrak{h}}$ is the unitary character of H defined by (3.2).*

Theorem 3.4 (Kirillov, [47]). *Let G be a simply connected nilpotent real Lie group with its Lie algebra $\mathfrak{g}$. If $\ell \in \mathfrak{g}^*$, there exists a polarization $\mathfrak{h}$ of $\mathfrak{g}$ for ℓ such that the monomial representation $Ind_H^G \chi_{\ell,\mathfrak{h}}$ is irreducible and of trace class. If ℓ' is an element of $\mathfrak{g}^*$ which belongs to the coadjoint orbit $Ad_G^*(G)\ell$ and $\mathfrak{h}'$ a polarization of $\mathfrak{g}$ for ℓ', then the monomial representations $Ind_H^G \chi_{\ell,\mathfrak{h}}$ and $Ind_{H'}^G \chi_{\ell',\mathfrak{h}'}$ are unitarily equivalent. Here H and H' are the simply connected closed subgroups corresponding to the Lie subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$ respectively. Conversely, if $\mathfrak{h}$ and $\mathfrak{h}'$ are polarizations of $\mathfrak{g}$ for $\ell \in \mathfrak{g}^*$ and $\ell' \in \mathfrak{g}^*$ respectively such that the monomial representations $Ind_H^G \chi_{\ell,\mathfrak{h}}$ and $Ind_{H'}^G \chi_{\ell',\mathfrak{h}'}$ of G are unitarily equivalent, then ℓ and ℓ' belong to the same coadjoint orbit of G in $\mathfrak{g}^*$. Finally, for each irreducible unitary representation τ of G, there exists a unique coadjoint orbit Ω of G in $\mathfrak{g}^*$ such that for any linear from $\ell \in \Omega$ and each polarization $\mathfrak{h}$ of $\mathfrak{g}$ for ℓ, the representations τ and $Ind_H^G \chi_{\ell,\mathfrak{h}}$ are unitarily equivalent. Any irreducible unitary representation of G is strongly trace class.*

Remark 3.5. (a) The bijection of the space $\mathfrak{g}^*/G$ of coadjoint orbit of G in $\mathfrak{g}^*$ onto the unitary dual $\hat{G}$ of G given by Theorem 3.4 is called the *Kirillov correspondence* of G. It provides a parametrization of $\hat{G}$ by means of the coadjoint orbit space.

(b) The above Kirillov's work was generalized immediately to the class known as exponential solvable groups, which are characterized as those solvable group G whose simply-connected cover $\tilde{G}$ is such that the exponential map $\exp : \tilde{\mathfrak{g}} \longrightarrow \tilde{G}$ is a diffeomorphism. For exponential solvable groups, the bijection between coadjoint orbits and representations holds, and can be realized using induced representations by an explicit construction using a polarization just as in the case of a nilpotent real Lie group. However, two difficulties arise: Firstly not all polarizations yield the same representation, or even an irreducible representation, and secondly not all representations are strongly trace class.

Theorem 3.6 (I.D. Brown). *Let G be a connected simply connected nilpotent Lie group with its Lie algebra $\mathfrak{g}$. The Kirillov correspondence*

$$\hat{G} \longrightarrow \mathcal{O}(G) = \mathfrak{g}^*/G$$

is a homeomorphism.

Theorem 3.7. *Let G be a connected simply connected nilpotent real Lie group with*

Lie algebra $\mathfrak{g}$. *Let* $\mathfrak{h}$ *be a subalgebra of* $\mathfrak{g}$. *Let* $p : \mathfrak{g}^* \longrightarrow \mathfrak{h}^*$ *be the natural projection. Let* H *be the simply connected subgroup of* G *with its Lie algebra* $\mathfrak{h}$. *The following (a),(b) and (c) hold.*

(a) Let π *be an irreducible unitary representation of* G *corresponding to a coadjoint orbit* $\Omega \subset \mathfrak{g}^*$ *of* G *via the Kirillov correspondence. Then* $Res_H^G \pi$ *decomposes into the direct integral of irreducible representations of* H *corresponding to a coadjoint orbit* $\omega (\subset \mathfrak{h}^*)$ *of* H *such that* $\omega \subset p(\Omega)$.

(b) Let τ *be an irreducible unitary representation of* H *corresponding to a coadjoint orbit* $\omega \subset \mathfrak{h}^*$ *of* H. *Then the induced representation* $Ind_H^G \tau$ *decomposes into the direct integral of irreducible representations* π_Ω *of* G *corresponding to coadjoint orbits* $\Omega \subset \mathfrak{g}^*$ *such that* $p(\Omega) \supset \omega$.

(c) Let π_1 *and* π_2 *be the irreducible unitary representations of* G *corresponding to coadjoint orbits* Ω_1 *and* Ω_2 *respectively. Then the tensor product* $\pi_1 \otimes \pi_2$ *decomposes into the direct integral of irreducible representations of* G *corresponding to coadjoint orbits* $\Omega \subset \mathfrak{g}^*$ *such that* $\Omega \subset \Omega_1 + \Omega_2$.

Theorem 3.8. *Let* G *be a connected simply connected nilpotent real Lie group with its Lie algebra* $\mathfrak{g}$. *Let* π *be an irreducible unitary representation of* G *corresponding to an orbit* $\Omega \subset \mathfrak{g}^*$. *Then the character* χ_π *is a distribution on* $\mathcal{S}(G)$ *and its Fourier transform coincides with the canonical measure on* Ω *given by the symplectic structure. Here* $\mathcal{S}(G)$ *denotes the Schwarz space of rapidly decreasing functions on* G.

A.A. Kirillov gave an explicit formula for the Plancherel measure on $\hat{G}$. We observe that for a nilpotent Lie group G, we may choose a subspace Q of $\mathfrak{g}^*$ such that generic coadjoint orbits intersect Q exactly in one point. We choose a basis $x_1, \cdots, x_l, y_1, \cdots, y_{n-l}$ in $\mathfrak{g}$ so that $y_1, \cdots, y_{n-l}$, considered as linear functionals on $\mathfrak{g}^*$, are constant on Q. Then $x_1, \cdots, x_l$ are coordinates on Q and hence on an open dense subset of $\mathcal{O}(G)$. For every $f \in Q$ with coordinates $x_1, \cdots, x_l$, we consider the skew-symmetric matrix $A = (a_{ij})$ with entries

$$a_{ij} =< [y_i, y_j], f >, \quad 1 \leq i, j \leq n - l.$$

We denote by $p(x_1, \cdots, x_l)$ the Pfaffian of the matrix A. We note that $p(x_1, \cdots, x_l)$ is a homogeneous polynomial of degree $\frac{n-l}{2}$.

Theorem 3.9. *Let* G *be a connected simply connected nilpotent Lie group. Then the Plancherel measure on* $\hat{G} \cong \mathcal{O}(G)$ *is concentrated on the set of generic orbits and it has the form*

$$\theta = p(x_1, \cdots, x_l) dx_1 \wedge \cdots \wedge dx_l$$

in the coordinates $x_1, \cdots, x_l$.

4. Auslander-Kostant's Theorem

In this section, we present the results obtained by L. Auslander and B. Kostant in [3] together with some complements suggested by I.M. Shchepochkina [80]-[81]. Early in the 1970s L. Auslander and B. Kostant described the unitary dual of all solvable Lie groups of type I.

Theorem 4.1. *A connected, simply connected solvable Lie group G belongs to type I if and only if the orbit space $\mathcal{O}(G) = \mathfrak{g}^*/G$ is a T_0-space and the canonical symplectic form σ is exact on each orbit.*

Remark 4.2. (a) Let G be a real Lie group with its Lie algebra $\mathfrak{g}$. Let $\Omega_\ell := Ad^*(G)\ell$ be the coadjoint orbit containing ℓ. From now on, we write Ad^* instead of Ad^*_G. Then Ω_ℓ is simply connected if a Lie group G is exponential. We recall that a Lie group G is said to be *exponential* if the exponential mapping $\exp : \mathfrak{g} \longrightarrow G$ is a diffeomorphism. But if G is solvable, Ω_l is not necessarily simply connected.
(b) Let G be a connected, simply connected solvable Lie group and for $\ell \in \mathfrak{g}^*$, we let G_ℓ be the stabilizer at ℓ. Then for any $\ell \in \mathfrak{g}^*$, we have $\pi_1(\Omega_l) \cong G_\ell/G_\ell^0$, where G_ℓ^0 denotes the identity component of G_ℓ in G.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. A pair (ℓ, χ) is called a *rigged momentum* if $\ell \in \mathfrak{g}^*$, and χ is a unitary character of G_ℓ such that $d\chi_e = 2\pi i \ell|_{\mathfrak{g}_\ell}$, where $d\chi_e$ denotes the differential of χ at the identity element e of G_ℓ. We denote by $\mathfrak{g}^*_{\text{rigg}}$ the set of all rigged momenta. Then G acts on $\mathfrak{g}^*_{\text{rigg}}$ by

$$g \cdot (\ell, \chi) := (Ad^*(g)\ell, \chi \circ I_{g^{-1}}) = (\ell \circ Ad(g^{-1}), \chi \circ I_{g^{-1}}) \tag{4.1}$$

for all $g \in G$ and $(\ell, \chi) \in \mathfrak{g}^*_{\text{rigg}}$. Here I_g denotes the inner automorphism of G defined by $I_g(x) = gxg^{-1}\,(x \in G)$. We note that $\chi \circ I_{g^{-1}}$ is a unitary character of $G_{Ad^*(G)\ell} = gG_\ell g^{-1}$. We denote by $\mathcal{O}_{\text{rigg}}(G)$ the set of all orbits in $\mathfrak{g}^*_{\text{rigg}}$ under the action (4.1).

Proposition 4.3. *Let G be a connected, simply connected solvable Lie group. Then the following (a) and (b) hold.*

(a) The G-action commutes with the natural projection

$$\pi : \mathfrak{g}^*_{rigg} \longrightarrow \mathfrak{g}^* \qquad (\ell, \chi) \mapsto \ell. \tag{4.2}$$

(b) For a solvable Lie group G of type I, the projection π is surjective and the fiber over a point $\ell \in \mathfrak{g}^*$ is a torus of dimension equal to the first Betti number $b_1(\Omega_\ell)$ of Ω_ℓ.

Now we mention the main theorem obtained by L. Auslander and B. Kostant in [3].

Theorem 4.4 (Auslander-Kostant). *Let G be a connected, simply connected solvable Lie group of type I. Then there is a natural bijection between the unitary dual $\hat{G}$ and the orbit space $\mathcal{O}_{rigg}(G)$. The correspondence between $\hat{G}$ and $\mathcal{O}_{rigg}(G)$ is given as follows. Let $(\ell, \chi) \in \mathfrak{g}^*_{rigg}$. Then there always exists a complex subalgebra $\mathfrak{p}$ of $\mathfrak{g}_{\mathbb{C}}$ subordinate to ℓ. We let $L(G, \ell, \chi, \mathfrak{p})$ be the space of complex valued functions*

ϕ on G satisfying the following conditions

$$\phi(hg) = \chi(h)\phi(g), \quad h \in G_\ell \tag{4.3}$$

and

$$(L_X + 2\pi i < \ell, X >)\phi = 0, \quad X \in \mathfrak{p}, \tag{4.4}$$

where L_X is the right invariant complex vector field on G defined by $X \in \mathfrak{g}_{\mathbb{C}}$. Then we have the representation T of G defined by

$$(T(g_1)\phi)(g) := \phi(gg_1), \quad g, g_1 \in G. \tag{4.5}$$

We can show that under suitable conditions on $\mathfrak{p}$ including the Pukanszky condition

$$p^{-1}(p(\ell)) = \ell + \mathfrak{p}^{\perp} \subset \Omega_\ell \tag{4.6}$$

and the condition

$$codim_{\mathbb{C}}\, \mathfrak{p} = \frac{1}{2} rank\, B_\ell, \tag{4.7}$$

the representation T is irreducible and its equivalence class depends only on the rigged orbit Ω containing (ℓ, χ). Here $p : \mathfrak{g}_{\mathbb{C}}^* \longrightarrow \mathfrak{p}^*$ denotes the natural projection of $\mathfrak{g}_{\mathbb{C}}^*$ onto $\mathfrak{p}^*$ dual to the inclusion $\mathfrak{p} \hookrightarrow \mathfrak{g}_{\mathbb{C}}$. We denote by T_Ω the representation T of G obtained from $(\ell, \chi) \in \mathfrak{g}_{rigg}^*$ and $\mathfrak{p}$. The correspondence between $\mathcal{O}_{rigg}(G)$ and $\hat{G}$ is given by

$$(\ell, \chi) \in \Omega \mapsto T_\Omega.$$

Definition 4.5. Let H be a closed subgroup of a Lie group G. We say that a rigged orbit $\Omega' \in \mathcal{O}_{\rm rigg}(H)$ lies under a rigged orbit $\Omega \in \mathcal{O}_{\rm rigg}(G)$ (or equivalently, Ω lies over Ω') if there exist rigged momenta $(\ell, \chi) \in \Omega$ and $(\ell', \chi') \in \Omega'$ such that the following conditions are satisfied:

$$p(\ell) = \ell', \quad \chi = \chi' \quad \text{on } H \cap G_\ell. \tag{4.8}$$

We define the *sum of rigged orbits* Ω_1 and Ω_2 as the set of all $(\ell, \chi) \in \mathcal{O}_{\rm rigg}(G)$ for which there exist $(\ell_i, \chi_i) \in \Omega_i,\ i = 1, 2$, such that

$$\ell = \ell_1 + \ell_2, \quad \chi = \chi_1\chi_2 \ \text{ on } G_{\ell_1} \cap G_{\ell_2}. \tag{4.9}$$

I. M. Shchepochkina [80]-[81] proved the following.

Theorem 4.6. *Let G be a connected, simply connected solvable Lie group and H a closed subgroup of G. Then* *(a) The spectrum of* $Ind_H^G S_{\Omega'}$ *consists of those* T_Ω *for which* Ω *lies over* Ω'*, where* $S_{\Omega'}$ *is an irreducible unitary representation of* H *corresponding to a rigged orbit* Ω' *in* $\mathfrak{h}^*_{rigg}$ *by Theorem 4.4.*

(b) The spectrum of $Res_H^G T_\Omega$ *consists of those* $S_{\Omega'}$ *for which* Ω' *lies under* Ω*.*

(c) The spectrum of $T_{\Omega_1} \otimes T_{\Omega_2}$ *consists of those* T_Ω *for which* Ω *lies in* $\Omega_1 + \Omega_2$*.*

5. The Obstacle for the Orbit Method

In this section, we discuss the case where the correspondence between irreducible unitary representations and coadjoint orbits breaks down. If G is a compact Lie group or a semisimple Lie group, the correspondence breaks down.

First we collect some definitions. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\mathcal{S}(G)$ be the Schwarz space of rapidly decreasing functions on G. We define the Fourier transform $\mathcal{F}_f$ for $f \in \mathcal{S}(G)$ by

$$\mathcal{F}_f(\ell) = \int_{\mathfrak{g}} f(\exp X)\, e^{2\pi i \lambda(X)} dX, \quad \lambda \in \mathfrak{g}^*. \tag{5.1}$$

Then (5.1) is a well-defined function on $\mathfrak{g}^*$. As usual, we define the Fourier transform $\mathcal{F}_\chi$ of a distribution $\chi \in \mathcal{S}'(G)$ by

$$< \mathcal{F}_\chi, \mathcal{F}_f > = < \chi, f >, \quad f \in \mathcal{S}(G). \tag{5.2}$$

For $f \in \mathcal{S}(G)$ and an irreducible unitary representation T of G, we put

$$T(f) = \int_{\mathfrak{g}} f(\exp X)\, T(\exp X) dX.$$

Then we can see that for an irreducible unitary representation of a *nilpotent* Lie group G, we obtain the following formula

$$\mathrm{tr}\, T(f) = \int_\Omega \mathcal{F}_f(\lambda) d_\Omega \lambda, \tag{5.3}$$

where Ω is the coadjoint orbit in $\mathfrak{g}^*$ attached to T under the Kirillov correspondence and $d_\Omega\lambda$ is the measure on Ω with dimension $2k$ given by the form $\frac{1}{k!} B_\Omega \wedge \cdots \wedge B_\Omega$ (k factors) with the canonical symplectic form B_Ω on Ω.

Definition 5.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$. A coadjoint orbit Ω in $\mathfrak{g}^*$ is called *integral* if the two dimensional cohomology class defined by the canonical two form B_Ω belongs to $H^2(\Omega, \mathbb{Z})$, namely, the integral of B_Ω over a two dimensional cycle in Ω is an integer.

5.1. Compact Lie Groups

Let G be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Then the G-orbits in $\mathfrak{g}^*$ are simply connected and have Kähler structures(not unique). These Kähler manifolds are called *flag manifolds* because their elements are realized in terms of flags. Let T be a maximal abelian subgroup of G. Then $X = G/T$ is called the *full flag manifold* and other flag manifolds are called *degenerate* ones. From the exact sequence

$$\cdots \longrightarrow \pi_k(G) \longrightarrow \pi_k(X) \longrightarrow \pi_{k-1}(T) \longrightarrow \pi_{k-1}(G) \longrightarrow \cdots$$

and the fact that $\pi_1(G) = \pi_2(G)$ under the assumption that G is simply connected, we obtain

$$H_2(X,\mathbb{Z}) \cong \pi_2(X) \cong \pi_1(T) \cong \mathbb{Z}^{\dim T}. \tag{5.4}$$

Let Ω be a coadjoint orbit in $\mathfrak{g}^*$. We identify $\mathfrak{g}^*$ with $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}^*$ with $\mathfrak{g}_{\mathbb{C}}$ via the Killing form so that $\mathfrak{t}_{\mathbb{C}}^*$ goes to $\mathfrak{t}_{\mathbb{C}}$ and the weight $P \subset \mathfrak{t}_{\mathbb{C}}^*$ corresponds to a lattice in $i\mathfrak{t}^* \subset i\mathfrak{g}^* \cong i\mathfrak{g}$. Then $\Omega \cap \mathfrak{t}^*$ is a finite set which forms a single W-orbit, where W is the Weyl group defined as $W = N_G(T)/Z_G(T)$.

Proposition 5.2. *Let Ω_λ be the orbit passing through the point $i\lambda \in \mathfrak{t}^*$. Then*

(1) the orbit Ω_λ is integral if and only if $\lambda \in P$.

(2) $dim\,\Omega_\lambda$ is equal to the number of roots non-orthogonal to λ.

Let Ω be an integral orbit of maximal dimension in $\mathfrak{g}^*$. Let $\lambda \in \Omega$ and let $\mathfrak{h}$ be a positive admissible polarization for λ. Here the admissibility for λ means that $\mathfrak{h}$ satisfies the following conditions:

(A1) $\mathfrak{h}$ is invariant under the action of G_λ,

(A2) $\mathfrak{h} + \bar{\mathfrak{h}}$ is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

Here G_λ denotes the stabilizer of G at λ.

Let χ_λ be the unitary character of G_λ defined by

$$\chi_\lambda(\exp X) = e^{2\pi i \lambda(X)}, \quad X \in \mathfrak{g}_\lambda, \tag{5.5}$$

where $\mathfrak{g}_\lambda$ is the Lie algebra of G_λ. We note that G_λ is connected. Let L_λ be the hermitian line bundle over $\Omega = G/G_\lambda$ defined by the unitary character χ_λ of G_λ. Then G acts on the space $\Gamma(L_\lambda)$ of holomorphic sections of L_λ as a representation of G. A. Borel and A. Weil proved that $\Gamma(L_\lambda)$ is non-zero and is an irreducible unitary representation of G with highest weight λ. This is the so-called *Borel-Weil Theorem*. Thereafter this theorem was generalized by R. Bott in the late 1950s as follows.

Theorem 5.3 (R. Bott). *Let ρ be the half sum of positive roots of the root system for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Then the cohomology space $H^k(X, L_\lambda)$ is non-zero precisely when*

$$\rho - i\lambda = w(\mu + \rho) \tag{5.6}$$

for some $\mu \in P_+$, $w \in W$ *and* $k = l(w)$, *the length of* w. *In this case the representation of* G *in* $H^k(\Omega, L_\lambda)$ *is equivalent to* $\pi_{-i\mu}$.

We note that the Borel-Weil Theorem strongly suggests relating π_λ to Ω_λ and, on the other hand, the Bott's Theorem suggests the correspondence $\pi_\lambda \leftrightarrow \Omega_{\lambda+\rho}$ which is a bijection between the unitary dual $\hat{G}$ of G and the set of all integral orbits of maximal dimension. It is known that

$$\dim \pi_\lambda = \mathrm{vol}(\Omega_{\lambda+\rho}). \tag{5.7}$$

The character formula (5.3) is valid for a compact Lie group G and provides an integral representation of the character:

$$\chi_\lambda(\exp X) = \frac{1}{p(X)} \int_{\Omega_\lambda} e^{2\pi i \lambda(X)} d_\Omega \lambda. \tag{5.8}$$

In 1990 N.J. Wildberger [96] proved the following.

Theorem 5.4. *Let* $\Phi : C^\infty(\mathfrak{g})' \longrightarrow C^\infty(G)'$ *be the transform defined by*

$$< \Phi(\nu), f > = < \nu, p \cdot (f \circ exp) >, \quad \nu \in C^\infty(\mathfrak{g})', \ f \in C^\infty(G)'. \tag{5.9}$$

Then for $Ad(G)$*-invariant distributions the convolution operators on* G *and* $\mathfrak{g}$ *are related by the transform above:*

$$\Phi(\mu) *_G \Phi(\nu) = \Phi(\mu *_\mathfrak{g} \nu). \tag{5.10}$$

The above theorem says that Φ straightens the group convolution, turning it into the abelian convolution on $\mathfrak{g}$. This implies the following geometric fact.

Corollary 5.5. *For any two coadjoint orbits* $\Omega_1, \Omega_2 \subset \mathfrak{g}$, *we let* $C_1 = exp\,\Omega_1$ *and* $C_2 = exp\,\Omega_2$. *Then the following holds.*

$$C_1 \cdot C_2 \subset \exp(\Omega_1 + \Omega_2). \tag{5.11}$$

5.2. Semisimple Lie Groups

The unitary dual $\hat{G}_u$ of a semisimple Lie group G splits into different series, namely, the principal series, degenerate series, complimentary series, discrete series and so on. These series may be attached to different types of coadjoint orbits. The principal series were defined first for complex semisimple Lie groups and for the real semisimple Lie group G which admit the *split* Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. These series are induced from characters of the Borel subgroup $B \supset H = \exp \mathfrak{h}$. The degenerate series are obtained by replacing B by a parabolic subgroup $P \supset B$. All these series are in a perfect correspondence with the family of coadjoint orbits which have a non-empty intersection with $\mathfrak{h}$. An irreducible unitary representation

π of G is said to be a *discrete series* if it occurs as a direct summand in the regular representation R of G on $L^2(G, dg)$. According to Harish-Chandra, if G is a real semisimple Lie group, $\hat{G}_d \neq 0$ if and only if G has a compact Cartan subgroup. Here $\hat{G}_d$ denotes the set of equivalent classes of discrete series of G. There is an interesting complimentary series of representations which are *not weakly contained* in the regular representation R of G. These can be obtained from the principal series and degenerate series by analytic continuation.

The principal series are related to the semisimple orbits. On the other hand, the nilpotent orbits are related to the so-called *unipotent* representations if they exist. In fact, the hyperbolic orbits are related to the representations obtained by the *parabolic induction* and the elliptic orbits are connected to the representations obtained by the *cohomological parabolic induction*. The notion of unipotent representations are not still well defined and hence not understood well. Recently J.-S. Huang and J.-S. Li [41] attached unitary representations to spherical nilpotent orbits for the real orthogonal and symplectic groups. The study of unipotent representations is under way. For some results and conjectures on unipotent representations, we refer to [1], [16], [41] and [90]-[91].

6. Nilpotent Orbits and the Kostant-Sekiguchi Correspondence

In this section, we present some properties of nilpotent orbits for a reductive Lie group G and describe the Kostant-Sekiguchi correspondence. We also explain the work of D. Vogan that for a maximal compact subgroup K of G, he attaches a space with a K-action to a nilpotent orbit. Most of the materials in this section are based on the article [93].

6.1. Jordan Decomposition

Definition 6.1.1. Let $GL(n)$ be the group of nonsingular real or complex $n \times n$ matrices. The *Cartan involution* of $GL(n)$ is the automorphism conjugate transpose inverse:

$$\theta(g) = {}^t\bar{g}^{-1}, \quad g \in GL(n). \tag{6.1}$$

A *linear reductive group* is a closed subgroup G of some $GL(n)$ preserved by θ and having finitely many connected components. A *reductive* Lie group is a Lie group $\tilde{G}$ endowed with a homomorphism $\pi : \tilde{G} \longrightarrow G$ onto a linear reductive group G so that the kernel of π is finite.

Theorem 6.1.2 (Cartan Decomposition). *Let $\tilde{G}$ be a reductive Lie group with $\pi : \tilde{G} \longrightarrow G$ as in Definition 6.1.1. Let*

$$K = G^\theta = \{g \in G \mid \theta(g) = g\}$$

be a maximal compact subgroup of G. We write $\tilde{K} = \pi^{-1}(K)$, a compact subgroup of $\tilde{G}$, and use $d\pi$ to identify the Lie algebras of $\tilde{G}$ and G. Let $\mathfrak{p}$ be the (-1)-eigenspace *of $d\theta$ on the Lie algebra $\mathfrak{g}$ of G. Then the map*

$$\tilde{K} \times \mathfrak{p} \longrightarrow \tilde{G}, \quad (\tilde{k}, X) \mapsto \tilde{k} \cdot exp\, X, \quad \tilde{k} \in \tilde{K},\ X \in \mathfrak{p} \tag{6.2}$$

is a diffeomorphism from $\tilde{K} \times \mathfrak{p}$ onto $\tilde{G}$. In particular, $\tilde{K}$ is maximal among the compact subgroups of $\tilde{G}$.

Suppose $\tilde{G}$ is a reductive Lie group. We define a map $\theta : \tilde{G} \longrightarrow \tilde{G}$ by

$$\theta(\tilde{k} \cdot \exp X) = \tilde{k} \cdot \exp(-X), \quad \tilde{k} \in \tilde{K},\ X \in \mathfrak{p}. \tag{6.3}$$

Then θ is an involution, that is, the Cartan involution of $\tilde{G}$. The group of fixed points of θ is $\tilde{K}$.

The following proposition makes us identify the Lie algebra of a reductive Lie group with its dual space.

Proposition 6.1.3. *Let G be a reductive Lie group. Identify $\mathfrak{g}$ with a Lie algebra of $n \times n$ matrices (cf. Definition 6.1.1). We define a real valued symmetric bilinear form on $\mathfrak{g}$ by*

$$< X, Y >= Re\, tr(XY), \quad X, Y \in \mathfrak{g}. \tag{6.4}$$

Then the following (a),(b) and (c) hold:

(a) The form $<\,,\,>$ is invariant under $Ad(G)$ and the Cartan involution θ.

(b) The Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is orthogonal with respect to the form $<\,,\,>$, where $\mathfrak{k}$ is the Lie algebra of K (the group of fixed points of θ) which is the $(+1)$-eigenspace of $\theta := d\theta$ on $\mathfrak{g}$ and $\mathfrak{p}$ is the (-1)-eigenspace of θ on $\mathfrak{g}$. The form $<\,,\,>$ is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. And hence the form $<\,,\,>$ is nondegenerate on $\mathfrak{g}$.

(c) There is a G-equivariant linear isomorphism

$$\mathfrak{g}^* \cong \mathfrak{g}, \quad \lambda \mapsto X_\lambda$$

characterized by

$$\lambda(Y) =< X_\lambda, Y >, \quad Y \in \mathfrak{g}. \tag{6.5}$$

Definition 6.1.4. Let G be a reductive Lie group with Lie algebra $\mathfrak{g}$ consisting of $n \times n$ matrices. An element $X \in \mathfrak{g}$ is called *nilpotent* if it is nilpotent as a matrix. An element $X \in \mathfrak{g}$ is called *semisimple* if the corresponding complex matrix is diagonalizable. An element $X \in \mathfrak{g}$ is called *hyperbolic* if it is semisimple and its eigenvalues are real. An element $X \in \mathfrak{g}$ is called *elliptic* if it is semisimple and its eigenvalues are purely imaginary.

Proposition 6.1.5 (Jordan Decomposition). *Let G be a reductive Lie group* *with its Lie algebra* $\mathfrak{g}$ *and let* $G = K \cdot exp\, \mathfrak{p}$ *be the Cartan decomposition of* $\mathfrak{g}$ *(see Theorem 6.1.2). Then the following (1)-(5) hold:*

(1) Any element $X \in \mathfrak{g}$ *has a unique decomposition*

$$X = X_h + X_e + X_n$$

characterized by the conditions that X_h *is hyperbolic,* X_e *is elliptic,* X_n *is nilpotent and* X_h, X_e, X_n *commute with each other.*

(2) After replacing X by a conjugate under $Ad(G)$*, we may assume that* $X_h \in \mathfrak{p}$, $X_e \in \mathfrak{k}$ *and that* $X_n = E$ *belongs to a standard* $\mathfrak{sl}(2)$ *triple. We recall that a triple* $\{H, E, F\} \subset \mathfrak{g}$ *is called a standard* $\mathfrak{sl}(2)$ *triple, if they satisfy the following conditions*

$$\theta(E) = -F, \quad \theta(H) = -H, \quad [H, E] = 2E, \quad [E, F] = H. \tag{6.6}$$

(3) The $Ad(G)$ *orbits of hyperbolic elements in* $\mathfrak{g}$ *are in one-to-one correspondence with the* $Ad(K)$ *orbits in* $\mathfrak{p}$.

(4) The $Ad(G)$ *orbits of elliptic elements in* $\mathfrak{g}$ *are in one-to-one correspondence with the* $Ad(K)$ *orbits in* $\mathfrak{k}$.

(5) The $Ad(G)$ *orbits of nilpotent orbits are in one-to-one correspondence with the* $Ad(K)$ *orbits of standard* $\mathfrak{sl}(2)$ *triples in* $\mathfrak{g}$.

6.2. Nilpotent Orbits

Let G be a real reductive Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. We consider the complex special linear group $SL(2, \mathbb{C})$ which is the complexification of $SL(2, \mathbb{R})$. We define the involution $\theta_0 : SL(2, \mathbb{C}) \longrightarrow SL(2, \mathbb{C})$ by

$$\theta_0(g) = {}^t g^{-1}, \quad g \in SL(2, \mathbb{C}). \tag{6.7}$$

We denote its differential by the same letter

$$\theta_0(Z) = -{}^t Z, \quad Z \in \mathfrak{sl}(2, \mathbb{C}). \tag{6.8}$$

The complex conjugation σ_0 defining the real form $SL(2, \mathbb{R})$ is just the complex conjugation of matrices.

We say that a triple $\{H, X, Y\}$ in a real or complex Lie algebra is a *standard triple* if it satisfies the following conditions:

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H. \tag{6.9}$$

We call the element H (resp. X, Y) a *neutral* (resp. *nilpositive, nilnegative*) element of a standard triple $\{H, X, Y\}$.

We consider the standard basis $\{H_0, E_0, F_0\}$ of $\mathfrak{sl}(2,\mathbb{R})$ given by

$$H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{6.10}$$

Then they satisfy

$$[H_0, E_0] = 2E_0, \quad [H_0, F_0] = -2F_0, \quad [E_0, F_0] = H_0 \tag{6.11}$$

and

$$\theta_0(H_0) = -H_0, \quad \theta_0(E_0) = -F_0, \quad \theta_0(F_0) = -E_0. \tag{6.12}$$

We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and let θ be the corresponding Cartan involution. We say that a standard triple $\{H, X, Y\}$ is a *Cayley triple* in $\mathfrak{g}$ if it satisfies the conditions:

$$\theta(H) = -H, \quad \theta(X) = -Y, \quad \theta(Y) = -X. \tag{6.13}$$

According to (6.11) and (6.12), the triple $\{H_0, E_0, F_0\}$ is a Cayley triple in $\mathfrak{sl}(2,\mathbb{R})$.

For a real Lie algebra $\mathfrak{g}$, we have the following theorems.

Theorem 6.2.1. *Given a Cartan decomposition θ on $\mathfrak{g}$, any triple $\{H, X, Y\}$ in $\mathfrak{g}$ is conjugate under the adjoint group $Ad(G)$ to a Cayley triple $\{H', X', Y'\}$ in $\mathfrak{g}$.*

Theorem 6.2.2 (Jacobson-Morozov). *Let X be a nonzero nilpotent element in $\mathfrak{g}$. Then there exists a standard triple $\{H, X, Y\}$ in $\mathfrak{g}$ such that X is nilpositive.*

Theorem 6.2.3 (Kostant). *Any two standard triples $\{H, X, Y\}$, $\{H', X, Y'\}$ in $\mathfrak{g}$ with the same nilpositive element X are conjugate under G^X, the centralizer of X in the adjoint group of G.*

Let $\{H, X, Y\}$ be a Cayley triple in $\mathfrak{g}$. We are going to look for a semisimple element in $\mathfrak{g}$. For this, we need to introduce an auxiliary standard triple attached to a Cayley triple. We put

$$H' = i(X - Y), \quad X' = \frac{1}{2}(X + Y + iH), \quad Y' = \frac{1}{2}(X + Y - iH). \tag{6.14}$$

Then the triple $\{H', X', Y'\}$ in $\mathfrak{g}_{\mathbb{C}}$ is a standard triple, called the *Cayley transform* of $\{H, X, Y\}$.

Since

$$\theta(H') = H', \quad \theta(X') = -X', \quad \theta(Y') = -Y',$$

we have

$$H' \in \mathfrak{k}_{\mathbb{C}} \quad \text{and} \quad X', Y' \in \mathfrak{p}_{\mathbb{C}}. \tag{6.15}$$

Therefore the subalgebra $\mathbb{C} < H', X', Y' >$ of $\mathfrak{g}c$ spanned by H', X', Y' is stable under the action of θ. A standard triple in $\mathfrak{g}c$ with the property (6.15) is called *normal.*

Theorem 6.2.4. *Any nonzero nilpotent element $X \in \mathfrak{p}_{\mathbb{C}}$ is the nilpositive element of a normal triple (see Theorem 6.2.2).*

Theorem 6.2.5. *Any two normal triples $\{H, X, Y\}$, $\{H', X, Y'\}$ with the same nilpositive element X is $K_{\mathbb{C}}^X$-conjugate, where $K_{\mathbb{C}}^X$ denotes the centralizer of X in the complexification $K_{\mathbb{C}}$ of a maximal compact subgroup K corresponding to the Lie algebra $\mathfrak{k}$.*

Theorem 6.2.6. *Any two normal triples $\{H, X, Y\}$, $\{H, X', Y'\}$ with the same neutral element H are $K_{\mathbb{C}}^H$-conjugate.*

Theorem 6.2.7 (Rao). *Any two standard triples $\{H, X, Y\}$, $\{H', X', Y'\}$ in $\mathfrak{g}$ with $X - Y = X' - Y'$ are conjugate under G^{X-Y}, the centralizer of $X - Y$ in G. In fact, $X - Y$ is a semisimple element which we are looking for.*

Let $\mathcal{A}_{triple}$ be the set of all $Ad(G)$-conjugacy classes of standard triples in $\mathfrak{g}$. Let $\mathcal{O}_{\mathcal{N}}$ be the set of all nilpotent orbits in $\mathfrak{g}$. We define the map

$$\Omega : \mathcal{A}_{triple} \longrightarrow \mathcal{O}_{\mathcal{N}}^{\times} := \mathcal{O}_{\mathcal{N}} - \{0\} \tag{6.16}$$

by

$$\Omega([\{H, X, Y\}]) := \mathcal{O}_X, \quad \mathcal{O}_X := Ad(G) \cdot X, \tag{6.17}$$

where $[\{H, X, Y\}]$ denotes the G-conjugacy class of a standard triple $\{H, X, Y\}$. According to Theorem 6.2.2 (Jacobson-Morozov Theorem) and Theorem 6.2.3 (Kostant's Theorem), the map Ω is bijective.

We put

$$h_0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad x_0 = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad y_0 = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}. \tag{6.18}$$

It is easy to see that the triple $\{h_0, x_0, y_0\}$ in $\mathfrak{sl}(2, \mathbb{C})$ is a normal triple. The complex conjugation σ_0 acts on the triple $\{h_0, x_0, y_0\}$ as follows:

$$\sigma_0(h_0) = -h_0, \quad \sigma_0(x_0) = y_0, \quad \sigma_0(y_0) = x_0. \tag{6.19}$$

We introduce some notations. We denote by $\mathrm{Mor}(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{g}_{\mathbb{C}})$ the set of all nonzero Lie algebra homomorphisms from $\mathfrak{sl}(2, \mathbb{C})$ to $\mathfrak{g}_{\mathbb{C}}$. We define

$$\mathrm{Mor}^{\mathbb{R}}(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{g}_{\mathbb{C}}) = \{\, \phi \in \mathrm{Mor}(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{g}_{\mathbb{C}}) \mid \phi \text{ is defined over } \mathbb{R} \,\},$$

$$\mathrm{Mor}^{\theta}(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{g}_{\mathbb{C}}) = \{\, \phi \in \mathrm{Mor}(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{g}_{\mathbb{C}}) \mid \theta \circ \phi = \phi \circ \theta_0 \,\},$$

$$\mathrm{Mor}^{\mathbb{R},\theta}(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{g}_{\mathbb{C}}) = \mathrm{Mor}^{\mathbb{R}}(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{g}_{\mathbb{C}}) \cap \mathrm{Mor}^{\theta}(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{g}_{\mathbb{C}}), \tag{6.20}$$

$$\mathrm{Mor}^{\sigma}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}}) = \{\, \phi \in \mathrm{Mor}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}}) \mid \sigma\circ\phi = \phi\circ\sigma_0 \,\},$$
$$\mathrm{Mor}^{\sigma,\theta}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}}) = \mathrm{Mor}^{\sigma}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}}) \cap \mathrm{Mor}^{\theta}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}}).$$

We observe that $\mathrm{Mor}^{\mathbb{R}}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$ is naturally isomorphic to $\mathrm{Mor}(\mathfrak{sl}(2,\mathbb{R}),\mathfrak{g})$, the set of all nonzero Lie algebra real homomorphisms from $\mathfrak{sl}(2,\mathbb{R})$ to $\mathfrak{g}$.

Proposition 6.2.8. *Suppose ϕ be a nonzero Lie algebra homomorphism from $\mathfrak{sl}(2,\mathbb{C})$ to a complex reductive Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Write*

$$H = \phi(H_0), \quad E = \phi(E_0), \quad F = \phi(F_0) \quad (\textit{see } (6.2.10)).$$

Then the following hold.
(1)

$$\mathfrak{g}_{\mathbb{C}} = \sum_{k\in\mathbb{Z}} \mathfrak{g}_{\mathbb{C}}(k),$$

where

$$\mathfrak{g}_{\mathbb{C}}(k) = \{X \in \mathfrak{g}_{\mathbb{C}} \mid [H,X] = kX\}, \quad k \in \mathbb{Z}.$$

(2) If we write

$$\mathfrak{l} = \mathfrak{g}_{\mathbb{C}}(0) \quad \textit{and} \quad \mathfrak{u} = \sum_{k>0} \mathfrak{g}_{\mathbb{C}}(k),$$

then $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a Levi decomposition of a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$.
(3) The centralizer of E is graded by the decomposition in (1). More precisely,

$$\mathfrak{g}_{\mathbb{C}}^E = \mathfrak{l}^E + \sum_{k>0} \mathfrak{g}_{\mathbb{C}}(k)^E = \mathfrak{l}^E + \mathfrak{u}^E.$$

(4) The subalgebra $\mathfrak{l}^E = \mathfrak{g}_{\mathbb{C}}^{H,E}$ is equal to $\mathfrak{g}_{\mathbb{C}}^{\phi}$, the centralizer in $\mathfrak{g}_{\mathbb{C}}$ of the image of ϕ. It is a reductive subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Consequently the decomposition in (3) is a Levi decomposition of $\mathfrak{g}_{\mathbb{C}}^E$.

Parallel results hold if $\{H,E,F\}$ are replaced by

$$h = \phi(h_0), \quad x = \phi(x_0), \quad y = \phi(y_0).$$

Proposition 6.2.9. *Suppose G is a real reductive Lie group, and let $\phi_{\mathbb{R}}$ be an element of $Mor^{\sigma}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$. Define $E_{\mathbb{R}}, H_{\mathbb{R}}, F_{\mathbb{R}}$ by*

$$H_{\mathbb{R}} = \phi_{\mathbb{R}}(H_0), \quad E_{\mathbb{R}} = \phi_{\mathbb{R}}(E_0), \quad F_{\mathbb{R}} = \phi_{\mathbb{R}}(F_0).$$

Then the following hold.

(1) $E_{\mathbb{R}}$, $F_{\mathbb{R}}$ are nilpotent, and $H_{\mathbb{R}}$ is hyperbolic.

(2) If we define $L = G^{H_{\mathbb{R}}}$ to be the isotrophy group of the adjoint action at $H_{\mathbb{R}}$ and $U = exp(\mathfrak{u}\cap\mathfrak{g})$, then $Q = LU$ is the parabolic subgroup of G associated to $H_{\mathbb{R}}$.

(3) The isotrophy group $G^{E_{\mathbb{R}}}$ of the adjoint action at $E_{\mathbb{R}}$ is contained in Q, and respects the Levi decomposition :

$$G^{E_{\mathbb{R}}} = \left(L^{E_{\mathbb{R}}}\right)\left(U^{E_{\mathbb{R}}}\right).$$

(4) The subgroup $L^{E_{\mathbb{R}}} = G^{H,E_{\mathbb{R}}}$ is equal to $G^{\phi_{\mathbb{R}}}$, the centralizer in G of the image image of $\phi_{\mathbb{R}}$. It is a reductive subgroup of G. The The subgroup $U^{E_{\mathbb{R}}}$ is simply connected unipotent.

(5) Suppose that $\phi_{\mathbb{R},\theta}$ is an element of $Mor^{\sigma,\theta}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$. Then $G^{\phi_{\mathbb{R},\theta}}$ is stable under the action of θ, and we may take θ as a Cartan involution on this reductive group. In particular, $G^{E_{\mathbb{R}}}$ and $G^{\phi_{\mathbb{R},\theta}}$ have a common maximal compact subgroup

$$K^{\phi_{\mathbb{R},\theta}} = (L\cap K)^{E_{\mathbb{R}}}.$$

The above proposition provides good information about the action of G on the cone $\mathcal{N}_{\mathbb{R}}$ of all nilpotent orbits in $\mathfrak{g}$.

The following proposition gives information about the action of $K_{\mathbb{C}}$ on the cone $\mathcal{N}_\theta$ of all nilpotent elements in $\mathfrak{p}_{\mathbb{C}}$.

Proposition 6.2.10. *Suppose G is a real reductive Lie group with Cartan decomposition $G = K\cdot exp\,\mathfrak{p}$. Let $\phi_\theta \in Mor^{\theta}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$. We define*

$$h_\theta = \phi_\theta(h_0),\quad x_\theta = \phi_\theta(x_0),\quad y_\theta = \phi_\theta(y_0).$$

Then we have the following results.

(1) x_θ and y_θ are nilpotent elements in $\mathfrak{p}c$, $h_\theta\in\mathfrak{k}c$ is hyperbolic and $ih_\theta\in\mathfrak{k}c$ is elliptic.

(2) The parabolic subalgebra $\mathfrak{q}=\mathfrak{l}+\mathfrak{u}$ constructed as in Proposition 6.2.8 using h_θ is stable under θ.

(3) If we define $L_K := (K_{\mathbb{C}})^{h_\theta}$ and $U_K = exp(\mathfrak{u}\cap\mathfrak{k}c)$, then $Q_K = L_K U_K$ is the parabolic subgroup of $K_{\mathbb{C}}$ associated to h_θ.

(4) $K_{\mathbb{C}}^{x_\theta}\subset Q_K$ and $K_{\mathbb{C}}^{x_\theta}$ respects the Levi decomposition

$$K_{\mathbb{C}}^{x_\theta} = (L_K^{x_\theta})\,(U_K^{x_\theta}).$$

(5) $L_K^{x_\theta} = K_{\mathbb{C}}^{h_\theta,x_\theta}$ is equal to $K_{\mathbb{C}}^{\phi_\theta}$, the centralizer in $K_{\mathbb{C}}$ of the image of ϕ_θ. It is a reductive algebraic subgroup of $K_{\mathbb{C}}$. The subgroup U_K^{x} is simply connected unipotent. In particular, the decomposition of (4) is a Levi decomposition of $K_{\mathbb{C}}^{x}$.

(6) Let $\phi_{\mathbb{R},\theta}$ be an element of $Mor^{\sigma,\theta}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$. Then $K_{\mathbb{C}}^{\phi_{\mathbb{R},\theta}}$ is stable under σ, and we may take σ as complex conjugation for a compact real form of this reductive algebraic group. In particular, $K_{\mathbb{C}}^{x}$ and $K_{\mathbb{C}}^{\phi_{\mathbb{R},\theta}}$ have a common maximal compact subgroup

$$K_{\mathbb{C}}^{\phi_{\mathbb{R},\theta}} = L_K^{x_\theta}\cap K.$$

6.3. The Kostant-Sekiguchi Correspondence

J. Sekiguchi [79] and B. Kostant (unpublished) established a bijection between the set of all nilpotent G-orbits in $\mathfrak{g}$ on the one hand and, on the other hand, the set of all nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathbb{C}}$. The detail is as follows.

Theorem 6.3.1. *Let G be a real reductive Lie group with Cartan involution θ and its corresponding maximal compact subgroup K. Let σ be the complex conjugation on the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. Then the following sets are in one-to-one correspondence.*

(a) G-orbits on the cone $\mathcal{N}_{\mathbb{R}}$ of nilpotent elements in $\mathfrak{g}$.

(b) G-conjugacy classes of Lie algebra homomorphisms $\phi_{\mathbb{R}}$ in $Mor^{\sigma}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$.

(c) K-conjugacy classes of Lie algebra homomorphisms $\phi_{\mathbb{R},\theta}$ in $Mor^{\sigma,\theta}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$.

(d) $K_{\mathbb{C}}$-conjugacy classes of Lie algebra homomorphisms ϕ_{θ} in $Mor^{\theta}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$.

(e) $K_{\mathbb{C}}$-orbits on the cone $\mathcal{N}_{\theta}$ of nilpotent elements in $\mathfrak{p}_{\mathbb{C}}$.

Here G acts on $\mathrm{Mor}^{\sigma}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$ via the adjoint action of G in $\mathfrak{g}$:

$$(6.21)\qquad (g\cdot\phi_{\mathbb{R}})(\zeta)=Ad(g)(\phi_{\mathbb{R}}(\zeta)),\quad g\in G \text{ and } \quad \phi_{\mathbb{R}}\in \mathrm{Mor}^{\sigma}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}}).$$

Similarly K and $K_{\mathbb{C}}$ act on $\mathrm{Mor}^{\sigma,\theta}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$ and $\mathrm{Mor}^{\theta}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$ like (6.21) respectively. The correspondence between (a) and (e) is called the *Kostant-Sekiguchi correspondence* between the G-orbits in $\mathcal{N}_{\mathbb{R}}$ and the $K_{\mathbb{C}}$-orbits in $\mathcal{N}_{\theta}$. If $\phi_{\mathbb{R},\theta}$ is an element in $\mathrm{Mor}^{\sigma,\theta}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$ as in (c) , then the correspondence is given by

$$(6.22)\qquad E=\phi_{\mathbb{R},\theta}(E_0)\leftrightsquigarrow x=\phi_{\mathbb{R},\theta}(x_0).\quad \text{(see (6.18))}$$

The proof of the above theorem can be found in [90], pp. 348-350.

M. Vergne [87] showed that the orbits $G\cdot E=Ad(G)E$ and $K_{\mathbb{C}}\cdot x=Ad(K_{\mathbb{C}})x$ are diffeomorphic as manifolds with K-action under the assumption that they are in the Kostant-Sekiguchi correspondence. Here E and x are given by (6.22).

Theorem 6.3.2 (M. Vergne). *Suppose $G=K\cdot exp(\mathfrak{p})$ is a Cartan decomposition of a real reductive Lie group G and $E\in\mathfrak{g}$, $x\in\mathfrak{p}c$ are nilpotent elements. Assume that the orbits $G\cdot E$ and $K_{\mathbb{C}}\cdot x$ correspond under the Kostant-Sekiguchi correspondence. Then there is a K-equivariant diffeomorphism from $G\cdot E$ onto $K_{\mathbb{C}}\cdot x$.*

Remark 6.3.3. The Kostant-Sekiguchi correspondence sends the zero orbit to the zero orbit, and the nilpotent orbit through the nilpositive element of a Cayley triple in $\mathfrak{g}$ to the orbit through the nilpositive element of its Cayley transform.

Remark 6.3.4. Let $G\cdot E$ and $K_{\mathbb{C}}\cdot x$ be in the Kostant-Sekiguchi correspondence, where $E\in\mathcal{N}_{\mathbb{R}}\subset\mathfrak{g}$ and $x\in\mathcal{N}_{\theta}\subset\mathfrak{p}_{\mathbb{C}}$. Then the following hold.

(1) $G_{\mathbb{C}}\cdot E=G_{\mathbb{C}}\cdot x$, where $G_{\mathbb{C}}$ denotes the complexification of G.

(2) $\dim_{\mathbb{C}}(K_{\mathbb{C}}\cdot x)=\frac{1}{2}\dim_{\mathbb{R}}(G\cdot E)=\frac{1}{2}\dim_{\mathbb{C}}(G_{\mathbb{C}}\cdot x)$.

(3) The centralizers G^E, $K_{\mathbb{C}}^x$ have a common maximal compact subgroup $K^{E,x}$ which is the centralizer of the span of E and x in K.

Remark 6.3.5. Let π be an irreducible, admissible representation of a reductive Lie group G. Recently Schmid and Vilonen gave a new geometric description of the Kostant-Sekiguchi correspondence (cf. [76], Theorem 7.22) and then using this fact proved that the associated cycle $\text{Ass}(\pi)$ of π coincides with the wave front cycle $\text{WF}(\pi)$ via the Kostant-Sekiguchi correspondence (cf. [77], Theorem 1.4).

6.4. The Ouantization of the K-action

It is known that a hyperbolic orbit could be quantized by the method of a parabolic induction, and on the other hand an elliptic orbit may also be quantized by the method of cohomological induction. However, we do not know yet how to quantize a nilpotent orbit. But D. Vogan attached a space with a representation of K to a nilpotent orbit.

We first fix a nonzero nilpotent element $\lambda_n \in \mathfrak{g}^*$. Let E be the unique element in $\mathfrak{g}$ given from λ_n via (6.5). According to Jacobson-Morosov Theorem or Theorem 6.3.1, there is a non-zero Lie algebra homomorphism $\phi_{\mathbb{R}}$ from $\mathfrak{sl}(2,\mathbb{R})$ to $\mathfrak{g}$ with $\phi_{\mathbb{R}}(E_0) = E$. We recall that E_0 is given by (6.10). $\phi_{\mathbb{R}}$ extends to an element $\phi_{\mathbb{R}} \in \text{Mor}^{\sigma}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$. After replacing λ_n by a conjugate under G, we may assume that $\phi_{\mathbb{R}} = \phi_{\mathbb{R},\theta}$ intertwines θ_0 and θ, that is, $\phi_{\mathbb{R},\theta} \in \text{Mor}^{\sigma,\theta}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$.

We define

$$x = \phi_{\mathbb{R},\theta}(x_0) \in \mathfrak{p}_{\mathbb{C}}, \quad x_0 = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}. \tag{6.23}$$

The isomorphism in (c), Proposition 6.1.3 associates to x a linear functional

$$\lambda_\theta \in \mathfrak{p}_{\mathbb{C}}^*, \quad \lambda_\theta(Y) = < x, Y >, \quad Y \in \mathfrak{g}. \tag{6.24}$$

We note that the element λ_θ is not uniquely determined by λ_n, but the orbit $K_{\mathbb{C}} \cdot \lambda_\theta$ is determined by $G \cdot \lambda_n$. According to Theorem 6.3.2, we get a K-equivariant diffeomorphism

$$G \cdot \lambda_n \cong K_{\mathbb{C}} \cdot \lambda_\theta. \tag{6.25}$$

Definition 6.4.1. (1) Let $\mathfrak{g}_{im}^*$ be the space of purely imaginary-valued linear functionals on $\mathfrak{g}$. We fix an element $\lambda_{im} \in \mathfrak{g}_{im}^*$. We denote the G-orbit of λ_{im} by

$$\mathcal{O}_{im} := G \cdot \lambda_{im} = Ad^*(G) \cdot \lambda_{im}. \tag{6.26}$$

We may define an imaginary-valued symplectic form ω_{im} on the tangent space

$$T_{\lambda_{im}}(\mathcal{O}_{im}) \cong \mathfrak{g}/\mathfrak{g}^{\lambda_{im}}, \tag{6.27}$$

where $\mathfrak{g}^{\lambda_{im}}$ is the Lie algebra of the isotropy subgroup $G^{\lambda_{im}}$ of G at λ_{im}. We denote by $Sp(\omega_{im})$ the group of symplectic real linear transformations of the tangent space (6.27). Then the isotropy action gives a natural homomorphism

$$j : G^{\lambda_{im}} \longrightarrow Sp(\omega_{im}). \tag{6.28}$$

On the other hand, we let $Mp(\omega_{im})$ be the metaplectic group of $Sp(\omega_{im})$. That is, we have the following exact sequence

$$1 \longrightarrow \{1, \epsilon\} \longrightarrow Mp(\omega_{im}) \longrightarrow Sp(\omega_{im}) \longrightarrow 0. \tag{6.29}$$

Pulling back (6.29) via (6.28), we have the so-called *metaplectic double cover* of the isotropy group $G^{\lambda_{im}}$:

$$1 \longrightarrow \{1, \epsilon\} \longrightarrow \tilde{G}^{\lambda_{im}} \longrightarrow G^{\lambda_{im}}. \tag{6.30}$$

That is, $\tilde{G}^{\lambda_{im}}$ is defined by

$$\tilde{G}^{\lambda_{im}} = \{(g, m) \in G^{\lambda_{im}} \times Mp(\omega_{im}) |\; j(g) = p(m)\}. \tag{6.31}$$

A representation χ of $\tilde{G}^{\lambda_{im}}$ is called *genuine* if $\chi(\epsilon) = -I$. We say that χ is *admissible* if it is genuine, and the differential of χ is a multiple of λ_{im}: namely, if

$$\chi(\exp x) = \exp(\lambda_{im}(x)) \cdot I, \quad x \in \mathfrak{g}^{\lambda_{im}}. \tag{6.32}$$

If admissible representations exist, we say that λ_{im} (or the orbit $\mathcal{O}_{im}$) is *admissible.* A pair (λ_{im}, χ) consisting of an element $\lambda_{im} \in \mathfrak{g}^*_{im}$ and an irreducible admissible representation χ of $\tilde{G}^{\lambda_{im}}$ is called an *admissible G-orbit datum.* Two such are called *equivalent* if they are conjugate by G.

We observe that if $G^{\lambda_{im}}$ has a finite number of connected components, an irreducible admissible representation of $\tilde{G}^{\lambda_{im}}$ is unitarizable. The notion of admissible G-orbit data was introduced by M. Duflo [25].

(2) Suppose $\lambda_\theta \in \mathfrak{p}^*_{\mathbb{C}}$ is a non-zero nilpotent element. Let $K_{\mathbb{C}}^{\lambda_\theta}$ be the isotropy subgroup of $K_{\mathbb{C}}$ at λ_θ. Define 2ρ to be the algebraic character of $K_{\mathbb{C}}^{\lambda_\theta}$ by which it acts on the top exterior power of the cotangent space at λ_θ to the orbit:

$$2\rho(k) := \det\left(Ad^*(k)|_{(\mathfrak{k}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}}^{\lambda_\theta})^*}\right), \quad k \in K_{\mathbb{C}}^{\lambda_\theta}. \tag{6.33}$$

The differential of 2ρ is a one-dimensional representation of $\mathfrak{k}_{\mathbb{C}}^{\lambda_\theta}$, which we denote also by 2ρ. We define $\rho \in (\mathfrak{k}_{\mathbb{C}}^{\lambda_\theta})^*$ to be the half of 2ρ. More precisely,

$$\rho(Z) = \frac{1}{2}\operatorname{tr}\left(ad^*(Z)|_{(\mathfrak{k}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}}^{\lambda_\theta})^*}\right), \quad Z \in \mathfrak{k}_{\mathbb{C}}^{\lambda_\theta}. \tag{6.34}$$

A *nilpotent admissible $K_{\mathbb{C}}$-orbit datum* at λ_θ is an irreducible algebraic representation (τ, V_τ) of $K_{\mathbb{C}}^{\lambda_\theta}$ whose differential is equal to $\rho \cdot I_\tau$, where I_τ denotes the identity map on V_τ. The nilpotent element $\lambda_\theta \in \mathfrak{p}_{\mathbb{C}}^*$ is called *admissible* if a nilpotent admissible $K_{\mathbb{C}}$-orbit datum at λ_θ exists. Two such data are called *equivalent* if they are conjugate.

Theorem 6.4.2 (J. Schwarz). *Suppose G is a real reductive Lie group, K is a maximal compact subgroup, and $K_{\mathbb{C}}$ is its complexification. Then there is a natural bijection between equivalent classes of nilpotent admissible G-orbit data and equivalent classes of nilpotent admissible $K_{\mathbb{C}}$-orbit data.*

Suppose $\lambda_n \in \mathfrak{g}^*$ is a non-zero nilpotent element. Let $\lambda_\theta \in \mathfrak{p}_{\mathbb{C}}^*$ be a nilpotent element which corresponds under the Kostant-Sekiguchi correspondence. We fix a nilpotent admissible $K_{\mathbb{C}}$-orbit datum (τ, V_τ) at λ_θ. We let

$$\mathcal{V}_\tau := K_{\mathbb{C}} \times_{K_{\mathbb{C}}^{\lambda_\theta}} V_\tau \tag{6.35}$$

be the corresponding algebraic vector bundle over the nilpotent orbit $K_{\mathbb{C}} \cdot \lambda_\theta \cong K_{\mathbb{C}}/K_{\mathbb{C}}^{\lambda_\theta}$. We note that a complex structure on $\mathcal{V}_\tau$ is preserved by K but not preserved by G. We now *assume* that the boundary of $\overline{K_{\mathbb{C}} \cdot \lambda_\theta}$ (that is, $\overline{K_{\mathbb{C}} \cdot \lambda_\theta} - K_{\mathbb{C}} \cdot \lambda_\theta$) has a complex codimension at least two. We denote by denote by $X_K(\lambda_n, \tau)$ the space of algebraic sections of $\mathcal{V}_\tau$. Then $X_K(\lambda_n, \tau)$ is an algebraic representation of $K_{\mathbb{C}}$. That is, if $k_1 \in K_{\mathbb{C}}$ and $s \in X_K(\lambda_n, \tau)$, then $(k_1 \cdot s)(k\lambda_\theta) = s((k_1^{-1}k)\lambda_\theta)$. We call the representation $(K_{\mathbb{C}},\, X_K(\lambda_n, \tau))$ of $K_{\mathbb{C}}$ the *quantization of the K-action* on $G \cdot \lambda_n$ for the admissible orbit datum (τ, V_τ).

What this definition amounts to is a desideratum for the quantization of the G-action on $G \cdot \lambda_\theta$. That is, whatever a unitary representation $\pi_G(\lambda_n, \tau)$ we associate to these data, we hope that we have

$$K\text{-finite part of } \pi_G(\lambda_n, \tau) \cong X_K(\lambda_n, \tau). \tag{6.36}$$

When G is a complex Lie group, the coadjoint orbit is a complex symplectic manifold and hence of real dimension $4m$. Consequently the codimension condition is automatically satisfied in this case.

Remark 6.4.3. Nilpotent admissible orbit data may or may not exist. When they exist, there is a one-dimensional admissible datum (τ_0, V_{τ_0}). In this case, all admissible data are in one-to-one correspondence with irreducible representations of the group of connected components of $K_{\mathbb{C}}^{\lambda_\theta}$; the correspondence is obtained by tensoring with τ_0. If G is connected and simply connected, then this component group is just the fundamental group of the nilpotent orbit $K_{\mathbb{C}} \cdot \lambda_\theta$.

7. Minimal Representations

Let G be a real reductive Lie group. Let π be an admissible representation of G. Let $\mathfrak{g}$ be the Lie algebra of G. Three closely related invariants $WF(\pi)$, $AS(\pi)$ and $Ass(\pi)$ in $\mathfrak{g}^*$ which are called the *wave front set* of π, the *asymptotic support of the character* of π and the *associated variety* of π respectively, are attached to a given admissible representation π. The subsets $WF(\pi)$, $AS(\pi)$, and $Ass(\pi)$ are contained in the cone $\mathcal{N}^*$ consisting of nilpotent elements in $\mathfrak{g}^*$. They are all invariant under the coadjoint action of G. Each of them is a closed subvariety of $\mathfrak{g}^*$, and is the union of finitely many nilpotent orbits. It is known that $WF(\pi) = AS(\pi)$. W. Schmid and K. Vilonen [77] proved that $Ass(\pi)$ coincides with $WF(\pi)$ via the Kostant-Sekuguchi correspondence. The dimensions of all three invariants are the same, and is always even. We define the *Gelfand-Kirillov dimension* of π by

$$\dim_{G-K} \pi := \frac{1}{2} \dim WF(\pi) \tag{7.1}$$

If $\mathfrak{g}_{\mathbb{C}}$ is simple, there exist a unique nonzero nilpotent $G_{\mathbb{C}}$-orbit $\mathcal{O}_{min} \subset \mathfrak{g}_{\mathbb{C}}^*$ of minimal dimension, which is contained in the closure of any nonzero nilpotent $G_{\mathbb{C}}$-orbit. In this case, we have $\mathcal{O}_{min} = \mathcal{O}_{X_\alpha}$, where $\mathcal{O}_{X_\alpha}$ is the $G_{\mathbb{C}}$-orbit of a nonzero highest root vector X_α.

A nilpotent G-orbit $\mathcal{O} \subset \mathfrak{g}^*$ is said to be *minimal* if

$$\dim_{\mathbb{R}} \mathcal{O} = \dim_{\mathbb{C}} \mathcal{O}_{min}, \tag{7.2}$$

equivalently, $\mathcal{O}$ is nonzero and contained in $\mathcal{O}_{min} \cap \mathfrak{g}^*$. An irreducible unitary representation π of G is called *minimal* if

$$\dim_{G-K} \pi = \frac{1}{2} \dim_{\mathbb{C}} \mathcal{O}_{min}. \tag{7.3}$$

Remark 7.1. (1) If G is not of type A_n, there are at most finity many minimal representations. These are the unipotent representations attached to the minimal orbit $\mathcal{O}_{min}$.

(2) In many cases, the minimal representations are isolated in the unitary dual $\hat{G}_u$ of G.

(3) A minimal representation π is almost always *automorphic*, namely, π occurs in $L^2(\Gamma\backslash G)$ for some lattice Γ in G. The theory of minimal representation is the basis for the construction of large families of other interesting automorphic representations. For example, it is known that the end of complementary series of $Sp(n,1)$ and $F_{4,1}$ are both automorphic.

Remark 7.2. Let (π, V) be an irreducible admissible representation of π. We denote by V^K the space of K-finite vectors for π, where K is a maximal compact subgroup of G. Then V^K is a $U(\mathfrak{g}_{\mathbb{C}})$-module. We fix any vector $0 \neq x_\pi \in V^K$. Let $U_n(\mathfrak{g}_{\mathbb{C}})$ be the subspace of $U(\mathfrak{g}_{\mathbb{C}})$ spanned by products of at most n elements of $\mathfrak{g}_{\mathbb{C}}$. Put

$$X_n(\pi) := U_n(\mathfrak{g}_{\mathbb{C}}) x_\pi.$$

D. Vogan [88] proved that $\dim X_n(\pi)$ is asymptotic to $\frac{c(\pi)}{d!} \cdot n^d$ as $n \to \infty$. Here $c(\pi)$ and d are positive integers independent of the choice of x_π. In fact, d is the Gelfand-Kirillov dimension of π. We may say that $d = \dim_{G-K} \pi$ is a good measurement of the size of π.

Suppose $F = \mathbb{R}$ or $\mathbb{C}$. Let G be a connected simple Lie group over F and K a maximal compact subgroup of G. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition of $\mathfrak{g}$. If G/K is hermitian symmetric, all the minimal representations are known to be either holomorphic or antiholmorphic. They can be found in the list of unitary highest weight modules given in [27]. D. Vogan [89] proved the existence and unitarity of the minimal representations for a family of split simple group including E_8, F_4 and all classical groups except for the B_n-case ($n \geq 4$), which no minimal representation seems to exist. The construction of the minimal representations for G_2 was given by M. Duflo [24] in the complex case and by D. Vogan [93] in the real case. D. Kazhdan and G. Savin [46] constructed the spherical minimal representation for every simple, split, simply laced group. B. Gross and N. Wallach [32] constructed minimal representations of all exceptional groups of real rank 4. R. Brylinski and B. Kostant [13]-[14] gave a construction of minimal representations for any simple real Lie group G under the assumption that G/K is not hermitian symmetric and minimal representations exist.

For a complex group G not of type A_n, the Harish-Chandra module of the spherical minimal representation can be realized on $U(\mathfrak{g})/J$, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ and $J \subset U(\mathfrak{g})$ is the Joseph ideal of $\mathfrak{g}$ [43]. It is known that $Sp(2n, \mathbb{C})$ has a non-spherical minimal representation, that is, the odd piece of the Weil representation. The following natural question arises.

Question. Are there non-spherical minimal representations for complex Lie groups other than $Sp(2n, \mathbb{C})$?

Recently P. Torasso [86] gave a uniform construction of minimal representations for a simple group over any local field of characteristic 0 with split rank ≥ 3. He constructs a minimal representation for each set of admissible datum associated to the minimal orbit defined over F.

Let π_{min} be a minimal representation of G. It is known that the annihilator of the Harish-Chandra module of π_{min} in $U(\mathfrak{g})$ is the Joseph ideal. D. Vogan [89] proved that the restriction of π_{min} to K is given by

$$\pi_{min}|_K = \oplus_{n=0}^{\infty} V(\mu_0 + n\beta),$$

where β is a highest weight for the action of K on $\mathfrak{p}$, μ_0 is a fixed highest weight depending on π_{min} and $V(\mu_0 + n\beta)$ denotes the highest weight module with highest weight $\mu_0 + n\beta$. Indeed, there are two or one possibilities for β depending on whether G/K is hermitian symmetric or not.

Definition 7.3. (1) A *reductive dual pair* in a reductive Lie group G is a pair (A, B) of closed subgroups of G, which are both reductive and are centralizers of each other.

(2) A reductive dual pair (A, B) in G is said to be *compact* if at least one of A and B is compact.

Duality Conjecture. Let (A, B) be a reductive dual pair in a reductive Lie group G. Let π_{min} be a minimal representation of G. Can you find a Howe type correspondence between suitable subsets of the admissible duals of A and B by restricting π_{min} to $A \times B$?

In the 1970s R. Howe [39] first formulated the duality conjecture for the Weil representation (which is a minimal representation) of the symplectic group $Sp(n, F)$ over any local field F. He [40] proved the duality conjecture for $Sp(n, F)$ when F is archimedean and J. L. Waldspurger [94] proved the conjecture when F is non-archimedean with odd residue characteristic.

Example 7.4. Let G be the simply connected split real group E_8. Then $K = Spin(16)$ is a maximal compact subgroup of G. We take $A = B = Spin(8)$. It is easy to see that the pair (A, B) is not only a reductive dual pair in $Spin(16)$, but also a reductive dual pair in G. Let π_{min} be a minimal representation of G. J.-S. Li [64] showed that the restriction of π_{min} to $A \times B \subset G$ is decomposed as follows:

$$\pi_{min}|_{A\times B} = \oplus_\pi m(\pi) \cdot (\pi \otimes \pi),$$

where π runs over all irreducible representations of $Spin(8)$ and $m(\pi)$ is the multiplicity with which $\pi \otimes \pi$ occurs. It turns out that $m(\pi) = +\infty$ for all π.

Example 7.5. Let G be the simply connected quaternionic E_8 with split rank 4. We let $A = Spin(8)$ and $B = Spin(4, 4)$. Then the pair (A, B) is a reductive dual pair in G. Let π_{min} be the minimal representation of G. H. Y. Loke [62] proved that the restriction of π_{min} to $A \times B$ is decomposed as follows.

$$\pi_{min}|_{A\times B} = \oplus_\pi m(\pi) \cdot (\pi \otimes \pi'),$$

where π runs over all irreducible representations of A and π' is the discrete series representation of B which is uniquely determined by π. All the multiplicities are *finite*. But they are *unbounded.*

The following interesting problem is proposed by Li.

Problem 7.6. Let (A, B) be a *compact* reductive dual pair in G. Describe the explicit decomposition of the restriction $\pi_{min}|_{A\times B}$ of a minimal representation π_{min} of G to $A \times B$.

Remark 7.7. In [42], J. Huang, P. Paudzic and G. Savin dealt with the family of dual pairs (A, B), where A is the split exceptional group of type G_2 and B is compact.

8. The Heisenberg Group $H_{\mathbb{R}}^{(g,h)}$

For any positive integers g and h, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(g,h)} = \left\{ (\lambda, \mu, \kappa) \mid \lambda, \mu \in \mathbb{R}^{(h,g)},\ \kappa \in \mathbb{R}^{(h,h)},\ \kappa + \mu\, {}^t\lambda \text{ symmetric} \right\}$$

with the multiplication law

$$(\lambda,\mu,\kappa)\circ(\lambda',\mu',\kappa')=(\lambda+\lambda',\mu+\mu',\kappa+\kappa'+\lambda\,{}^t\mu'-\mu\,{}^t\lambda').$$

Here $\mathbb{R}^{(h,g)}$ (resp. $\mathbb{R}^{(h,h)}$) denotes the all $h\times g$ (resp. $h\times h$) real matrices.

The Heisenberg group $H_{\mathbb{R}}^{(g,h)}$ is embedded to the symplectic group $Sp(g+h,\mathbb{R})$ via the mapping

$$H_{\mathbb{R}}^{(g,h)}\ni(\lambda,\mu,\kappa)\longmapsto\begin{pmatrix}E_g&0&0&{}^t\mu\\ \lambda&E_h&\mu&\kappa\\ 0&0&E_g&-{}^t\lambda\\ 0&0&0&E_h\end{pmatrix}\in Sp(g+h,\mathbb{R}).$$

This Heisenberg group is a 2-step nilpotent Lie group and is important in the study of toroidal compactifications of Siegel moduli spaces. In fact, $H_{\mathbb{R}}^{(g,h)}$ is obtained as the unipotent radical of the parabolic subgroup of $Sp(g+h,\mathbb{R})$ associated with the rational boundary component F_g (cf. [28] p. 123 or [69] p. 21). For the motivation of the study of this Heisenberg group we refer to [103]-[107] and [110]. We refer to [98]-[102] for more results on $H_{\mathbb{R}}^{(g,h)}$.

In this section, we describe the Schrödinger representations of $H_{\mathbb{R}}^{(g,h)}$ and the coadjoint orbits of $H_{\mathbb{R}}^{(g,h)}$. The results in this section are based on the article [108] with some corrections.

8.1. Schrödinger Representations

First of all, we observe that $H_{\mathbb{R}}^{(g,h)}$ is a 2-step nilpotent Lie group. It is easy to see that the inverse of an element $(\lambda,\mu,\kappa)\in H_{\mathbb{R}}^{(g,h)}$ is given by

$$(\lambda,\mu,\kappa)^{-1}=(-\lambda,-\mu,-\kappa+\lambda\,{}^t\mu-\mu\,{}^t\lambda).$$

Now we set

$$[\lambda,\mu,\kappa]:=(0,\mu,\kappa)\circ(\lambda,0,0)=(\lambda,\mu,\kappa-\mu\,{}^t\lambda). \tag{8.1}$$

Then $H_{\mathbb{R}}^{(g,h)}$ may be regarded as a group equipped with the following multiplication

$$[\lambda,\mu,\kappa]\diamond[\lambda_0,\mu_0,\kappa_0]=[\lambda+\lambda_0,\mu+\mu_0,\kappa+\kappa_0+\lambda\,{}^t\mu_0+\mu_0\,{}^t\lambda]. \tag{8.2}$$

The inverse of $[\lambda,\mu,\kappa]\in H_{\mathbb{R}}^{(g,h)}$ is given by

$$[\lambda,\mu,\kappa]^{-1}=[-\lambda,-\mu,-\kappa+\lambda\,{}^t\mu+\mu\,{}^t\lambda].$$

We set

$$K=\left\{\,[0,\mu,\kappa]\in H_{\mathbb{R}}^{(g,h)}\;\middle|\;\mu\in\mathbb{R}^{(h,g)},\ \kappa={}^t\kappa\in\mathbb{R}^{(h,h)}\,\right\}. \tag{8.3}$$

Then K is a commutative normal subgroup of $H_{\mathbb{R}}^{(g,h)}$. Let $\hat{K}$ be the Pontrajagin dual of K, i.e., the commutative group consisting of all unitary characters of K. Then $\hat{K}$ is isomorphic to the additive group $\mathbb{R}^{(h,g)} \times \text{Symm}\,(h,\mathbb{R})$ via

$$(8.4) \qquad < a, \hat{a} >= e^{2\pi i \sigma(\hat{\mu}\,{}^t\mu + \hat{\kappa}\kappa)}, \quad a = [0,\mu,\kappa] \in K,\ \hat{a} = (\hat{\mu},\hat{\kappa}) \in \hat{K}.$$

We put

$$(8.5) \qquad S = \left\{ [\lambda, 0, 0] \in H_{\mathbb{R}}^{(g,h)} \,\middle|\, \lambda \in \mathbb{R}^{(h,g)} \right\} \cong \mathbb{R}^{(h,g)}.$$

Then S acts on K as follows:

$$(8.6) \qquad \alpha_\lambda([0,\mu,\kappa]) = [0,\mu,\kappa + \lambda\,{}^t\mu + \mu\,{}^t\lambda], \quad [\lambda,0,0] \in S.$$

It is easy to see that the Heisenberg group $\left(H_{\mathbb{R}}^{(g,h)}, \diamond\right)$ is isomorphic to the semi-direct product $S \ltimes K$ of S and K whose multiplication is given by

$$(\lambda, a) \cdot (\lambda_0, a_0) = (\lambda + \lambda_0, a + \alpha_\lambda(a_0)), \quad \lambda, \lambda_0 \in S,\ a, a_0 \in K.$$

On the other hand, S acts on $\hat{K}$ by

$$(8.7) \qquad \alpha_\lambda^*(\hat{a}) = (\hat{\mu} + 2\hat{\kappa}\lambda, \hat{\kappa}), \quad [\lambda,0,0] \in S, \quad a = (\hat{\mu},\hat{\kappa}) \in \hat{K}.$$

Then, we have the relation $< \alpha_\lambda(a), \hat{a} >=< a, \alpha_\lambda^*(\hat{a}) >$ for all $a \in K$ and $\hat{a} \in \hat{K}$.

We have three types of S-orbits in $\hat{K}$.

Type I. Let $\hat{\kappa} \in \text{Sym}\,(h,\mathbb{R})$ be nondegenerate. The S-orbit of $\hat{a}(\hat{\kappa}) := (0,\hat{\kappa}) \in \hat{K}$ is given by

$$(8.8) \qquad \hat{\mathcal{O}}_{\hat{\kappa}} = \left\{ (2\hat{\kappa}\lambda, \hat{\kappa}) \in \hat{K} \,\middle|\, \lambda \in \mathbb{R}^{(h,g)} \right\} \cong \mathbb{R}^{(h,g)}.$$

Type II. Let $(\hat{\mu},\hat{\kappa}) \in \mathbb{R}^{(h,g)} \times \text{Sym}\,(h,\mathbb{R})$ with degenerate $\hat{\kappa} \neq 0$. Then

$$(8.9) \qquad \hat{\mathcal{O}}_{(\hat{\mu},\hat{\kappa})} = \left\{ \hat{\mu} + 2\hat{\kappa}\lambda, \hat{\kappa}) \,\middle|\, \lambda \in \mathbb{R}^{(h,g)} \right\} \subsetneqq \mathbb{R}^{(h,g)} \times \{\hat{\kappa}\}.$$

Type III. Let $\hat{y} \in \mathbb{R}^{(h,g)}$. The S-orbit $\hat{\mathcal{O}}_{\hat{y}}$ of $\hat{a}(\hat{y}) := (\hat{y}, 0)$ is given by

$$(8.10) \qquad \hat{\mathcal{O}}_{\hat{y}} = \{ (\hat{y}, 0) \} = \hat{a}(\hat{y}).$$

We have

$$\hat{K} = \left(\bigcup_{\substack{\hat{\kappa} \in \text{Sym}(h,\mathbb{R}) \\ \hat{\kappa} \text{ nondegenerate}}} \hat{\mathcal{O}}_{\hat{\kappa}} \right) \bigcup \left(\bigcup_{\hat{y} \in \mathbb{R}^{(h,g)}} \hat{\mathcal{O}}_{\hat{y}} \right) \bigcup \left(\bigcup_{\substack{(\hat{\mu},\hat{\kappa}) \in \mathbb{R}^{(h,g)} \times \text{Sym}(h,\mathbb{R}) \\ \hat{\kappa} \neq 0 \text{ degenerate}}} \hat{\mathcal{O}}_{(\hat{\mu},\,\hat{\kappa})} \right)$$

as a set. The stabilizer $S_{\hat{\kappa}}$ of S at $\hat{a}(\hat{\kappa}) = (0, \hat{\kappa})$ is given by

$$S_{\hat{\kappa}} = \{0\}. \tag{8.11}$$

And the stabilizer $S_{\hat{y}}$ of S at $\hat{a}(\hat{y}) = (\hat{y}, 0)$ is given by

$$S_{\hat{y}} = \left\{ [\lambda, 0, 0] \,\middle|\, \lambda \in \mathbb{R}^{(h,g)} \right\} = S \cong \mathbb{R}^{(h,g)}. \tag{8.12}$$

From now on, we set $G = H_{\mathbb{R}}^{(g,h)}$ for brevity. It is known that K is a closed, commutative normal subgroup of G. Since $(\lambda, \mu, \kappa) = (0, \mu, \kappa + \mu\, {}^t\lambda) \circ (\lambda, 0, 0)$ for $(\lambda, \mu, \kappa) \in G$, the homogeneous space $X = K \backslash G$ can be identified with $\mathbb{R}^{(h,g)}$ via

$$Kg = K \circ (\lambda, 0, 0) \longmapsto \lambda, \quad g = (\lambda, \mu, \kappa) \in G.$$

We observe that G acts on X by

$$(Kg) \cdot g_0 = K\,(\lambda + \lambda_0, 0, 0) = \lambda + \lambda_0, \tag{8.13}$$

where $g = (\lambda, \mu, \kappa) \in G$ and $g_0 = (\lambda_0, \mu_0, \kappa_0) \in G$.

If $g = (\lambda, \mu, \kappa) \in G$, we have

$$k_g = (0, \mu, \kappa + \mu\, {}^t\lambda), \quad s_g = (\lambda, 0, 0) \tag{8.14}$$

in the Mackey decomposition of $g = k_g \circ s_g$ (cf.[67]). Thus if $g_0 = (\lambda_0, \mu_0, \kappa_0) \in G$, then we have

$$s_g \circ g_0 = (\lambda, 0, 0) \circ (\lambda_0, \mu_0, \kappa_0) = (\lambda + \lambda_0, \mu_0, \kappa_0 + \lambda\, {}^t\mu_0) \tag{8.15}$$

and so

$$k_{s_g \circ g_0} = (0, \mu_0, \kappa_0 + \mu_0\, {}^t\lambda_0 + \lambda\, {}^t\mu_0 + \mu_0\, {}^t\lambda). \tag{8.16}$$

For a real symmetric matrix $c = {}^tc \in \mathbb{R}^{(h,h)}$ with $c \neq 0$, we consider the one-dimensional unitary representation σ_c of K defined by

$$\sigma_c\left((0, \mu, \kappa)\right) = e^{2\pi i \sigma(c\kappa)}\, I, \quad (0, \mu, \kappa) \in K, \tag{8.17}$$

where I denotes the identity mapping. Then the induced representation $U(\sigma_c) := \mathrm{Ind}_K^G\, \sigma_c$ of G induced from σ_c is realized in the Hilbert space $\mathcal{H}_{\sigma_c} = L^2(X, d\dot{g}, \mathbb{C}) \cong L^2\left(\mathbb{R}^{(h,g)}, d\xi\right)$ as follows. If $g_0 = (\lambda_0, \mu_0, \kappa_0) \in G$ and $x = Kg \in X$ with $g = (\lambda, \mu, \kappa) \in G$, we have

$$\left(U_{g_0}(\sigma_c) f\right)(x) = \sigma_c\left(k_{s_g \circ g_0}\right)\left(f(x g_0)\right), \quad f \in \mathcal{H}_{\sigma_c}. \tag{8.18}$$

It follows from (8.16) that

$$\left(U_{g_0}(\sigma_c) f\right)(\lambda) = e^{2\pi i \sigma\{c(\kappa_0 + \mu_0\, {}^t\lambda_0 + 2\lambda\, {}^t\mu_0)\}}\, f(\lambda + \lambda_0). \tag{8.19}$$

Here, we identified $x = Kg$ (resp. $xg_0 = Kgg_0$) with λ (resp. $\lambda + \lambda_0$). The induced representation $U(\sigma_c)$ is called the *Schrödinger representation* of G associated with σ_c. Thus $U(\sigma_c)$ is a monomial representation.

Now, we denote by $mathcalH^{\sigma_c}$ the Hilbert space consisting of all functions $\phi : G \longrightarrow \mathbb{C}$ which satisfy the following conditions:

(1) $\phi(g)$ is measurable measurable with respect to dg,
(2) $\phi\left((0,\mu,\kappa)\circ g)\right) = e^{2\pi i \sigma(c\kappa)}\phi(g)$ for all $g \in G$,
(3) $\| \phi \|^2 := \int_X |\phi(g)|^2 \, d\dot{g} < \infty, \quad \dot{g} = Kg$,

where dg (resp. $d\dot{g}$) is a G-invariant measure on G (resp. $X = K\backslash G$). The inner product $(\ ,\ )$ on $\mathcal{H}^{\sigma_c}$ is given by

$$(\phi_1, \phi_2) = \int_G \phi_1(g)\, \overline{\phi_2(g)}\, dg \qquad \text{for } \phi_1,\ \phi_2 \in \mathcal{H}^{\sigma_c}.$$

We observe that the mapping $\Phi_c : \mathcal{H}_{\sigma_c} \longrightarrow \mathcal{H}^{\sigma_c}$ defined by

$$(8.20) \qquad \left(\Phi_c(f)\right)(g) = e^{2\pi i \sigma\{c(\kappa + \mu\, {}^t\lambda)\}}\, f(\lambda), \quad f \in \mathcal{H}_{\sigma_c},\ g = (\lambda,\mu,\kappa) \in G$$

is an isomorphism of Hilbert spaces. The inverse $\Psi_c : \mathcal{H}^{\sigma_c} \longrightarrow \mathcal{H}_{\sigma_c}$ of Φ_c is given by

$$(8.21) \qquad \left(\Psi_c(\phi)\right)(\lambda) = \phi((\lambda, 0, 0)), \quad \phi \in \mathcal{H}^{\sigma_c},\ \lambda \in \mathbb{R}^{(h,g)}.$$

The Schrödinger representation $U(\sigma_c)$ of G on $\mathcal{H}^{\sigma_c}$ is given by

$$(8.22) \qquad \left(U_{g_0}(\sigma_c)\phi\right)(g) = e^{2\pi i \sigma\{c(\kappa_0 + \mu_0\, {}^t\lambda_0 + \lambda\, {}^t\mu_0 - \lambda_0\, {}^t\mu)\}}\, \phi\left((\lambda_0, 0, 0)\circ g\right),$$

where $g_0 = (\lambda_0, \mu_0, \kappa_0),\ g = (\lambda, \mu, \kappa) \in G$ and $\phi \in \mathcal{H}^{\sigma_c}$. (8.22) can be expressed as follows.

$$(8.23) \qquad \left(U_{g_0}(\sigma_c)\phi \right)(g) = e^{2\pi i \sigma\{c(\kappa_0 + \kappa + \mu_0\, {}^t\lambda_0 + \mu\, {}^t\lambda + 2\lambda\, {}^t\mu_0)\}}\, \phi((\lambda_0 + \lambda, 0, 0)).$$

Theorem 8.1. *Let c be a positive symmetric half-integral matrix of degree h. Then the Schrödinger representation $U(\sigma_c)$ of G is irreducible.*

Proof. The proof can be found in [99], Theorem 3.

8.2. The Coadjoint Orbits of Picture

In this subsection, we find the coadjoint orbits of the Heisenberg group $H_{\mathbb{R}}^{(g,h)}$ and describe the connection between the coadjoint orbits and the unitary dual of $H_{\mathbb{R}}^{(g,h)}$ explicitly.

For brevity, we let $G := H_{\mathbb{R}}^{(g,h)}$ as before. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$. We observe that $\mathfrak{g}$ can be regarded as the subalgebra consisting of all $(g+h)\times(g+h)$ real matrices of the form

$$X(\alpha,\beta,\gamma)=\begin{pmatrix} 0 & 0 & 0 & {}^t\beta \\ \alpha & 0 & \beta & \gamma \\ 0 & 0 & 0 & -{}^t\alpha \\ 0 & 0 & 0 & 0 \end{pmatrix},\ \alpha,\beta\in\mathbb{R}^{(h,g)},\ \gamma={}^t\gamma\in\mathbb{R}^{(h,h)}$$

of the lie algebra $\mathfrak{sp}(g+h,\mathbb{R})$ of the symplectic group $Sp(g+h,\mathbb{R})$. An easy computation yields

$$[X(\alpha,\beta,\gamma),\ X(\delta,\epsilon,\xi)]=X(0,0,\alpha^t\epsilon+\epsilon^t\alpha-\beta^t\delta-\delta^t\beta).$$

The dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ can be identified with the vector space consisting of all $(g+h)\times(g+h)$ real matrices of the form

$$F(a,b,c)=\begin{pmatrix} 0 & {}^ta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & {}^tb & 0 & 0 \\ b & c & -a & 0 \end{pmatrix},\ a,b\in\mathbb{R}^{(h,g)},\ c={}^tc\in\mathbb{R}^{(h,h)}$$

so that

(8.24) $< F(a,b,c),X(\alpha,\beta,\gamma) >= \sigma(F(a,b,c)\,X(\alpha,\beta,\gamma))=2\sigma({}^t\alpha a+{}^tb\beta)+\sigma(c\gamma).$

The adjoint representation Ad of G is given by $Ad_G(g)X=gXg^{-1}$ for $g\in G$ and $X\in\mathfrak{g}$. For $g\in G$ and $F\in\mathfrak{g}^*$, gFg^{-1} is not of the form $F(a,b,c)$. We denote by $(gFg^{-1})_*$ the

$$\begin{pmatrix} 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} - \text{part}$$

of the matrix gFg^{-1}. Then it is easy to see that the coadjoint representation $Ad_G^*: G\longrightarrow GL(\mathfrak{g}^*)$ is given by $Ad_G^*(g)F=(gFg^{-1})_*$, where $g\in G$ and $F\in\mathfrak{g}^*$. More precisely,

$$Ad_G^*(g)F(a,b,c)=F(a+c\mu,b-c\lambda,c), \tag{8.25}$$

where $g=(\lambda,\mu,\kappa)\in G$. Thus the coadjoint orbit $\Omega_{a,b}$ of G at $F(a,b,0)\in\mathfrak{g}^*$ is given by

$$\Omega_{a,b}=Ad_G^*(G)\,F(a,b,0)=\{F(a,b,0)\},\ \text{a single point} \tag{8.26}$$

and the coadjoint orbit Ω_c of G at $F(0,0,c)\in\mathfrak{g}^*$ with $c\neq 0$ is given by

$$\Omega_c=Ad_G^*(G)\,F(0,0,c)=\{F(a,b,c)|a,b\in\mathbb{R}^{(h,g)}\}\cong\mathbb{R}^{(h,g)}\times\mathbb{R}^{(h,g)}. \tag{8.27}$$

Therefore the coadjoint orbits of G in $\mathfrak{g}^*$ fall into two classes :

(I) The single point $\{\Omega_{a,b} | a, b \in \mathbb{R}^{(h,g)}\}$ located in the plane $c = 0$.

(II) The affine planes $\{\Omega_c | c = {}^t c \in \mathbb{R}^{(h,h)},\ c \neq o\}$ parallel to the homogeneous plane $c = 0$.

In other words, the orbit space $\mathcal{O}(G)$ of coadjoint orbits is parametrized by

$$\begin{cases} c-\text{axis},\ c \neq 0,\ c = {}^t c \in \mathbb{R}^{(h,h)}; \\ (a,b) - \text{plane} \approx \mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)}. \end{cases}$$

The single point coadjoint orbits of the type $\Omega_{a,b}$ are said to be the *degenerate* orbits of G in $\mathfrak{g}^*$. On the other hand, the flat coadjoint orbits of the type Ω_c are said to be the *non-degenerate* orbits of G in $\mathfrak{g}^*$. Since G is connected and simply connected 2-step nilpotent Lie group, according to A. Kirillov (cf. [47] or [48] p.249, Theorem 1), the unitary dual $\widehat{G}$ of G is given by

$$\widehat{G} = \left(\mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)}\right) \coprod \left\{ z \in \mathbb{R}^{(h,h)} \mid z = {}^t z,\ z \neq 0 \right\}, \tag{8.28}$$

where $\coprod$ denotes the disjoint union. The topology of $\widehat{G}$ may be described as follows. The topology on $\{c - \text{axis} - (0)\}$ is the usual topology of the Euclidean space and the topology on $\{F(a,b,0) | a, b \in \mathbb{R}^{(h,g)}\}$ is the usual Euclidean topology. But a sequence on the c-axis which converges to 0 in the usual topology converges to the whole Euclidean space $\mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)}$ in the topology of $\hat{G}$. This is just the quotient topology on $\mathfrak{g}^*/G$ so that algebraically and topologically $\widehat{G} = \mathfrak{g}^*/G$.

It is well known that each coadjoint orbit is a symplectic manifold. We will state this fact in detail. For the present time being, we fix an element F of $\mathfrak{g}^*$ once and for all. We consider the alternating $\mathbb{R}$-bilinear form $\mathbf{B}_F$ on $\mathfrak{g}$ defined by

$$\mathbf{B}_F(X,Y) = < F, [X,Y] > = < ad_{\mathfrak{g}}^*(Y)F, X >,\ X, Y \in \mathfrak{g}, \tag{8.29}$$

where $ad_{\mathfrak{g}}^* : \mathfrak{g} \longrightarrow \text{End}(\mathfrak{g}^*)$ denotes the differential of the coadjoint representation $Ad_G^* : G \longrightarrow GL(\mathfrak{g}^*)$. More precisely, if $F = F(a,b,c)$, $X = X(\alpha, \beta, \gamma)$, and $Y = X(\delta, \epsilon, \xi)$, then

$$\mathbf{B}_F(X,Y) = \sigma\{c(\alpha^t\epsilon + \epsilon^t\alpha - \beta^t\delta - \delta^t\beta)\}. \tag{8.30}$$

For $F \in \mathfrak{g}^*$, we let

$$G_F = \{g \subset G | Ad_G^*(g)F = F\}$$

be the stabilizer of the coadjoint action Ad^* of G on $\mathfrak{g}^*$ at F. Since G_F is a closed subgroup of G, G_F is a Lie subgroup of G. We denote by $\mathfrak{g}_F$ the Lie subalgebra of $\mathfrak{g}$ corresponding to G_F. Then it is easy to show that

$$\mathfrak{g}_F = rad\,\mathbf{B}_F = \{X \in \mathfrak{g} | ad_{\mathfrak{g}}^*(X)F = 0\}. \tag{8.31}$$

Here $rad\,\mathbf{B}_F$ denotes the radical of $\mathbf{B}_F$ in $\mathfrak{g}$. We let $\dot{\mathbf{B}_F}$ be the non-degenerate alternating $\mathbb{R}$-bilinear form on the quotient vector space $\mathfrak{g}/rad\,\mathbf{B}_F$ induced from $\mathbf{B}_F$.

Since we may identify the tangent space of the coadjoint orbit $\Omega_F \cong G/G_F$ with $\mathfrak{g}/\mathfrak{g}_F = \mathfrak{g}/rad\mathbf{B}_F$, we see that the tangent space of Ω_F at F is a symplectic vector space with respect to the symplectic form $\dot{\mathbf{B}}_F$.

Now we are ready to prove that the coadjoint orbit $\Omega_F = Ad_G^*(G)F$ is a symplectic manifold. We denote by $\widetilde{X}$ the smooth vector field on $\mathfrak{g}^*$ associated to $X \in \mathfrak{g}$. That means that for each $\ell \in \mathfrak{g}^*$, we have

$$\widetilde{X}(\ell) = ad_{\mathfrak{g}}^*(X)\,\ell. \tag{8.32}$$

We define the differential 2-form B_{Ω_F} on Ω_F by

$$B_{\Omega_F}(\widetilde{X}, \widetilde{Y}) = B_{\Omega_F}(ad_{\mathfrak{g}}^*(X)F, ad_{\mathfrak{g}}^*(Y)F) := \mathbf{B}_F(X, Y), \tag{8.33}$$

where $X, Y \in \mathfrak{g}$.

Lemma 8.2. *B_{Ω_F} is non-degenerate.*

Proof. Let $\widetilde{X}$ be the smooth vector field on $\mathfrak{g}^*$ associated to $X \in \mathfrak{g}$ such that $B_{\Omega_F}(\widetilde{X}, \widetilde{Y}) = 0$ for all $\widetilde{Y}$ with $Y \in \mathfrak{g}$. Since $B_{\Omega_F}(\widetilde{X}, \widetilde{Y}) = \mathbf{B}_F(X, Y) = 0$ for all $Y \in \mathfrak{g}$, $X \in \mathfrak{g}_F$. Thus $\widetilde{X} = 0$. Hence B_{Ω_F} is non-degenerate.

Lemma 8.3. *B_{Ω_F} is closed.*

Proof. If $\widetilde{X_1}$, $\widetilde{X_2}, \text{and} \widetilde{X_3}$ are three smooth vector fields on $\mathfrak{g}^*$ associated to X_1, X_2, $X_3 \in \mathfrak{g}$, then

$$\begin{aligned} dB_{\Omega_F}(\widetilde{X_1}, \widetilde{X_2}, \widetilde{X_3}) &= \widetilde{X_1}(B_{\Omega_F}(\widetilde{X_2}, \widetilde{X_3})) - \widetilde{X_2}(B_{\Omega_F}(\widetilde{X_1}, \widetilde{X_3})) + \widetilde{X_3}(B_{\Omega_F}(\widetilde{X_1}, \widetilde{X_2})) \\ &\quad - B_{\Omega_F}([\widetilde{X_1}, \widetilde{X_2}], \widetilde{X_3}) + B_{\Omega_F}([\widetilde{X_1}, \widetilde{X_3}], \widetilde{X_2}) - B_{\Omega_F}([\widetilde{X_2}, \widetilde{X_3}], \widetilde{X_1}) \\ &= - < F, [[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] > \\ &= 0 \qquad \text{(by the Jacobi identity).} \end{aligned}$$

Therefore B_{Ω_F} is closed. □

In summary, (Ω_F, B_{Ω_F}) is a symplectic manifold of dimension $2hg$ or 0.

In order to describe the irreducible unitary representations of G corresponding to the coadjoint orbits under the Kirillov correspondence, we have to determine the polarizations of $\mathfrak{g}$ for the linear forms $F \in \mathfrak{g}^*$.

Case **I.** $F = F(a, b, 0)$; the degenerate case.

According to (8.26), $\Omega_F = \Omega_{a,b} = \{F(a, b, 0)\}$ is a single point. It follows from (8.30) that $\mathbf{B}_F(X, Y) = 0$ for all $X, Y \in \mathfrak{g}$. Thus $\mathfrak{g}$ is the unique polarization of $\mathfrak{g}$ for F. The Kirillov correspondence says that the irreducible unitary representation $\pi_{a,b}$ of G corresponding to the coadjoint orbit $\Omega_{a,b}$ is given by

$$\pi_{a,b}(\exp X(\alpha, \beta, \gamma)) = e^{2\pi i < F, X(\alpha,\beta,\gamma) >} = e^{4\pi i \sigma({}^t a\alpha + {}^t b\beta)}. \tag{8.34}$$

That is, $\pi_{a,b}$ is a one-dimensional degenerate representation of G.

Case **II.** $F = F(0,0,c),\ 0 \neq c = {}^tc \in \mathbb{R}^{(h,h)}$: the non-degenerate case.

According to (8.27), $\Omega_F = \Omega_c = \{F(a,b,c)|a,b \subset \mathbb{R}^{(h,g)}\}$. By (8.30), we see that

$$\mathfrak{k} = \{\ X(0,\beta,\gamma)|\beta \in \mathbb{R}^{(h,g)},\ \gamma = {}^t\gamma \in \mathbb{R}^{(h,h)}\} \tag{8.35}$$

is a polarization of $\mathfrak{g}$ for F, i.e.,$\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ subordinate to $F \in \mathfrak{g}^*$ which is maximal among the totally isotropic vector subspaces of $\mathfrak{g}$ relative to the alternating $\mathbb{R}$-bilinear form $\mathbf{B}_F$. Let K be the simply connected Lie subgroup of G corresponding to the Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$. We let

$$\chi_{c,\mathfrak{k}} : K \longrightarrow \mathbb{C}_1^\times$$

be the unitary character of K defined by

$$\chi_{c,\mathfrak{k}}(\exp X(0,\beta,\gamma)) = e^{2\pi i<F,X(0,\beta,\gamma)>} = e^{2\pi i\sigma(c\gamma)}. \tag{8.36}$$

The Kirillov correspondence says that the irreducible unitary representation $\pi_{c,\mathfrak{k}}$ of G corresponding to the coadjoint orbit $\Omega_F = \Omega_c$ is given by

$$\pi_{c,\mathfrak{k}} = \mathrm{Ind}_K^G\, \chi_{c,\mathfrak{k}}. \tag{8.37}$$

According to Kirillov's Theorem (cf. [47]), we know that the induced representation $\pi_{c,\mathfrak{k}}$ is, up to equivalence, independent of the choice of a polarization of $\mathfrak{g}$ for F. Thus we denote the equivalence class of $\pi_{c,\mathfrak{k}}$ by π_c. π_c is realized on the representation space $L^2(\mathbb{R}^{(h,g)}, d\xi)$ as follows:

$$(\pi_c(g)f)(\xi) = e^{2\pi i\sigma\{c(\kappa+\mu^t\lambda+2\xi^t\mu)\}} f(\xi+\lambda), \tag{8.38}$$

where $g = (\lambda,\mu,\kappa) \in G$ and $\xi \in \mathbb{R}^{(h,g)}$. Using the fact that

$$\exp X(\alpha,\beta,\gamma) = (\alpha,\beta,\gamma + \frac{1}{2}(\alpha^t\beta - \beta^t\alpha)),$$

we see that π_c is nothing but the Schrödinger representation $U(\sigma_c)$ of G induced from the one-dimensional unitary representation σ_c of K given by $\sigma_c((0,\mu,\kappa)) = e^{2\pi i\sigma(c\kappa)}I$. We note that π_c is the non-degenerate representation of G with central character $\chi_c : Z \longrightarrow \mathbb{C}_1^\times$ given by $\chi_c((0,0,\kappa)) = e^{2\pi i\sigma(c\kappa)}$. Here $Z = \{(0,0,\kappa)|\kappa = {}^t\kappa \in \mathbb{R}^{(h,h)}\}$ denotes the center of G.

It is well known that the monomial representation $(\pi_c, L^2(\mathbb{R}^{(h,g)}, d\xi))$ of G extends to an operator of operator of trace class

$$\pi_c(\phi) : L^2(\mathbb{R}^{(h,g)}, d\xi) \longrightarrow L^2(\mathbb{R}^{(h,g)}, d\xi) \tag{8.39}$$

for all $\phi \in C_c^\infty(G)$. Here $C_c^\infty(G)$ is the vector space of all smooth functions on G with compact support. We let $C_c^\infty(\mathfrak{g})$ and $C(\mathfrak{g}^*)$ the vector space of all smooth functions on $\mathfrak{g}$ with compact support and the vector space of all continuous functions on $\mathfrak{g}^*$ respectively. If $f \in \mathcal{C}_c^\infty(\mathfrak{g})$, we define the Fourier cotransform

$$\mathcal{C}F_{\mathfrak{g}} : C_c^\infty(\mathfrak{g}) \longrightarrow C(\mathfrak{g}^*)$$

by

$$(\mathcal{C}F_{\mathfrak{g}}(f))\,(F') := \int_{\mathfrak{g}} f(X)\, e^{2\pi i <F',X>} dX, \tag{8.40}$$

where $F' \in \mathfrak{g}^*$ and dX denotes the usual Lebesgue measure on $\mathfrak{g}$. According to A. Kirillov (cf. [47]), there exists a measure β on the coadjoint orbit $\Omega_c \approx \mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)}$ which is invariant under the coadjoint action og G such that

$$\operatorname{tr} \pi_c^1(\phi) = \int_{\Omega_c} \mathcal{C}F_{\mathfrak{g}}(\phi \circ \exp)(F') d\beta(F') \tag{8.41}$$

holds for all test functions $\phi \in C_c^\infty(G)$, where exp denotes the exponentional mapping of $\mathfrak{g}$ onto G. We recall that

$$\pi_c^1(\phi)(f) = \int_G \phi(x)\,(\pi_c(x)f)\,dx,$$

where $\phi \in C_c^\infty(G)$ and $f \in L^2(\mathbb{R}^{(h,g)}, d\xi)$. By the Plancherel theorem, the mapping

$$S(G/Z) \ni \varphi \longmapsto \pi_c^1(\varphi) \in TC(L^2(\mathbb{R}^{(h,g)}, d\xi))$$

extends to a unitary isometry

$$\pi_c^2 : L^2(G/Z, \chi_c) \longrightarrow HS(L^2(\mathbb{R}^{(h,g)}, d\xi)) \tag{8.42}$$

of the representation space $L^2(G/Z), \chi_c)$ of $\mathrm{Ind}_Z^G\, \chi_c$ onto the complex Hilbert space $HS(L^2(\mathbb{R}^{(h,g)}, d\xi))$ consisting of all Hilbert-Schmidt operators on $L^2(\mathbb{R}^{(h,g)}, d\xi)$, where $S(G/z)$ is the Schwartz space of all infinitely differentiable complex-valued functions on $G/Z \cong \mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)}$ that are rapidly decreasing at infinity and $TC(L^2(\mathbb{R}^{(h,g)}, d\xi))$ denotes the complex vector space of all continuous $\mathbb{C}$-linear mappings of $L^2(\mathbb{R}^{(h,g)}, d\xi)$ into itself which are of trace class.

In summary, we have the following result.

Theorem 8.4. *For $F = F(a, b, 0) \in \mathfrak{g}^*$, the irreducible unitary representation $\pi_{a,b}$ of G corresponding to the coadjoint orbit $\Omega_F = \Omega_c$ under the Kirillov correspondence is degenerate representation of G given by*

$$\pi_{a,b}(exp\, X(\alpha, \beta, \gamma)) = e^{4\pi i \sigma({}^t a\alpha - {}^t b\beta)}.$$

On the other hand, for $F = F(0, 0, c) \in \mathfrak{g}^*$ with $0 \neq c = {}^t c \in \mathbb{R}^{(h,h)}$, the irreducible unitary representation $(\pi_c, L^2(\mathbb{R}^{(h,g)}, d\xi))$ of G corresponding to the coadjoint orbit

Ω_c under the Kirillov correspondence is unitary equivalent to the Schrödinger representation $U(\sigma_c), L^2(\mathbb{R}^{(h,g)}, d\xi))$ and this non-degenerate representation π_c is square integrable module its center Z. For all test functions $\phi \in C_c^\infty(G)$, the character formula

$$tr\pi_c^2(\phi) = \mathcal{C}(\phi, c) \int_{\mathbb{R}^{(h,g)}} \phi(0,0,\kappa)\, e^{2\pi i\sigma(c\kappa)} d\kappa$$

holds for some constant $\mathcal{C}(\phi, c)$ depending on ϕ and c, where $d\kappa$ is the Lebesgue measure on the Euclidean space $\mathbb{R}^{(h,h)}$.

Now we consider the subgroup K of G given by

$$K = \{(0,0,\kappa) \in G \,|\, \mu \in \mathbb{R}^{(h,g)},\ \kappa = {}^t\kappa \in \mathbb{R}^{(h,h)}\}.$$

The Lie algebra $\mathfrak{k}$ of K is given by (8.35). The dual space $\mathfrak{k}^*$ of $\mathfrak{k}$ may be identified with the space

$$\{F(0,b,c) \,|\, b \in \mathbb{R}^{(h,g)},\ c = {}^tc \in \mathbb{R}^{(h,h)}\}.$$

We let $\mathrm{Ad}_K^* : K \longrightarrow GL(\mathfrak{k}^*)$ be the coadjoint representation of K on $\mathfrak{k}^*$. The coadjoint orbit $\omega_{b,c}$ of K at $F(0,b,c) \in \mathfrak{k}^*$ is given by

(8.43) $$\omega_{b,c} = Ad_K^*(K)\, F(0,b,c) = \{F(0,b,c)\}, \text{ a single point.}$$

Since K is a commutative group, $[\mathfrak{k},\mathfrak{k}] = 0$ and so the alternating $\mathbb{R}$-bilinear form $\mathbf{B}_f$ on $\mathfrak{k}$ associated to $F := F(0,b,c)$ identically vanishes on $\mathfrak{k} \times \mathfrak{k}$(cf. (8.29)). $\mathfrak{k}$ is the unique polarization of $\mathfrak{k}$ for $F = F(0,b,c)$. The Kirillov correspondence says that the irreducible unitary representation $\chi_{b,c}$ of K corresponding to the coadjoint orbit $\omega_{b,c}$ is given by

(8.44) $$\chi_{b,c}(\exp X(0,\beta,\gamma)) = e^{2\pi i < F(0,b,c), X(0,\beta,\gamma)>} = e^{2\pi i\sigma(2\,{}^tb\beta + c\gamma)}$$

or

(8.45) $$\chi_{b,c}((0,\mu,\kappa)) = e^{2\pi i\sigma(2\,{}^tb\mu + c\kappa)}, \quad (0,\mu,\kappa) \in K.$$

For $0 \neq c = {}^tc \in \mathbb{R}^{(h,h)}$, we let π_c be the Schrödinger representation of G given by (8.38). We know that the irreducible unitary representation of G corresponding to the coadjoint orbit

$$\Omega_c = \mathrm{Ad}_G^*(G)\, F(0,0,c) = \{F(a,b,c) \,|\, a,b \in \mathbb{R}^{(h,g)}\}.$$

Let $p : \mathfrak{g}^* \longrightarrow \mathfrak{k}^*$ be the natural projection defined by $p(F(a,b,c)) = F(o,b,c)$. Obviously we have

$$p(\Omega_c) = \left\{F(0,b,c) \,|\, b \in \mathbb{R}^{(h,g)}\right\} = \cup_{b\in\mathbb{R}^{(h,g)}}\, \omega_{b,c}.$$

According to Kirillov Theorem (cf. [48] p.249, Theorem1), The restriction $\pi_c|_K$ of π_c to K is the direct integral of all one-dimensional representations $\chi_{b,c}$ of K ($b \in$

$\mathbb{R}^{(h,g)}$). Conversely, we let $\chi_{b,c}$ be the element of $\hat{K}$ corresponding to the coadjoint orbit $\omega_{b,c}$ of K. The induced representation $\mathrm{Ind}_K^G\, \chi_{b,c}$ is nothing but the Schrödinger representation π_c. The coadjoint orbit Ω_c of G is the only coadjoint orbit such that $\Omega_c \cap p^{-1}(\omega_{b,c})$ is nonempty.

9. The Jacobi Group

In this section, we study the unitary representations of the Jacobi group which is a semi-product of a a symplectic group and a Heisenberg group, and their related topics. In the subsection 9.1, we present basic ingredients of the Jacobi group and the Iwasawa decomposition of the Jacobi group. In the subsection 9.2, we find the Lie algebra of the Jacobi group in some detail. In the subsection 9.3, we give a definition of Jacobi forms. In the subsection 9.4, we characterize Jacobi forms as functions on the Jacobi group satisfying certain conditions. In the subsection 9.5, we review some results on the unitary representations, in particular, the Weil representation of the Jacobi group. Most of the materials here are contained in [82]-[84]. In the subsection 9.6, we describe the duality theorem for the Jacobi group. In the final subsection, we study the coadjoint orbits for the Jacobi group and relate these orbits to the unitary representations of the Jacobi group.

9.1 The Jacobi Group G^J

In this section, we give the standard coordinates of the Jacobi group G^J and an Iwasawa decomposition of G^J.

9.1.1. The Standard Coordinates of the Jacobi Group G^J

Let m and n be two fixed positive integers. Let

$$Sp(n,\mathbb{R}) = \{M \in \mathbb{R}^{(2n,2n)} \mid {}^tMJ_nM = J_n\}$$

be the symplectic group of degree n, where

$$J_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$$

is the symplectic matrix of degree n. We let

$$H_n = \{\, Z \in \mathbb{C}^{(n,n)} \mid Z = {}^tZ, \;\; \mathrm{Im}\, Z > 0 \,\}$$

be the Siegel upper half plane of degree n. Then it is easy to see that $Sp(n,\mathbb{R})$ acts on H_n transitively by

$$M < Z >= (AZ+B)(CZ+D)^{-1}, \tag{9.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $Z \in H_n$.

We consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \left\{ (\lambda, \mu, \kappa) \Big| \lambda, \mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\, {}^t\lambda \text{ symmetric } \right\}$$

endowed with the following multiplication law

$$(9.2) \qquad (\lambda, \mu, \kappa) \circ (\lambda', \mu', \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda').$$

We already studied this Heisenberg group in the previous section.

Now we let

$$G^J_{n,m} := Sp(n, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

the semidirect product of the symplectic group $Sp(n, \mathbb{R})$ and the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ endowed with the following multiplication law

$$(9.3) \quad (M, (\lambda, \mu, \kappa)) \cdot (M', (\lambda', \mu', \kappa')) = (MM', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu', \kappa + \kappa' + \tilde{\lambda}\, {}^t\mu' - \tilde{\mu}\, {}^t\lambda'))$$

with $M, M' \in Sp(n, \mathbb{R}), (\lambda, \mu, \kappa),\ (\lambda', \mu', \kappa') \in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde{\lambda}, \tilde{\mu}) := (\lambda, \mu)M'$. We call $G^J_{n,m}$ the *Jacobi group* of *degree* (n, m). If there is no confusion about the degree (n, m), we write G^J briefly instead of $G^J_{n,m}$. It is easy to see that G^J acts on $H_{n,m} := H_n \times \mathbb{C}^{(m,n)}$ transitively by

$$(9.4) \qquad (M, (\lambda, \mu, \kappa)) \cdot (Z, W) = (M < Z >, (W + \lambda Z + \mu)(CZ + D)^{-1}),$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}),\ (\lambda, \mu, \kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(Z, W) \in H_{n,m}$.

Now we define the linear mapping

$$Q\ :\ (\mathbb{R}^{(m,n)} \times \mathbb{R}^{(m,n)}) \times (\mathbb{R}^{(m,n)} \times \mathbb{R}^{(m,n)}) \longrightarrow \mathbb{R}^{(m,m)}$$

by

$$Q((\lambda, \mu), (\lambda', \mu')) = \lambda^t\mu' - \mu^t\lambda', \quad \lambda, \mu, \lambda', \mu' \in \mathbb{R}^{(m,n)}.$$

Clearly we have

$$(9.5) \qquad {}^tQ(\xi, \eta) = -Q(\eta, \xi) \ \text{ for all} \xi, \eta \in \mathbb{R}^{(m,n)} \times \mathbb{R}^{(m,n)},$$

$$(9.6) \qquad Q(\xi M, \eta M) = Q(\xi, \eta) \ \text{ for all} M \in Sp(n, \mathbb{R}).$$

For a reason of the convenience, we write an element of G^J as

$$g = [M, (\lambda, \mu, \kappa)] := (E_{2n}, (\lambda, \mu, \kappa)) \cdot (M, (0, 0, 0)).$$

Then the multiplication becomes

$$[M,(\xi,\kappa)]\circ[M',(\xi',\kappa')]=[MM',(\xi+\xi'M^{-1},\kappa+\kappa'+Q(\xi,\xi'M^{-1}))].$$

For brevity, we set $G=Sp(n,\mathbb{R})$. We note that the stabilizer K of G at iE_n under the symplectic action is given by

$$K=\left\{\begin{pmatrix} A & -B\\ B & A\end{pmatrix}\in GL(2n,\mathbb{R})\ \middle|\ {}^tAB={}^tBA,\ \ {}^tAA+{}^tBB=E_n\right\}$$

and is a maximal compact subgroup of G. We also recall that the Jacobi group G^J acts on $H_{n,m}$ transitively via (9.4). Then it is easy to see that the stabilizer K^J of G^J at $(iE_n,0)$ under this action is given by

$$\begin{aligned} K^J&=\left\{[k,(0,0,\kappa)]\ \middle|\ k\in K,\ \ k={}^tk\in\mathbb{R}^{(m,m)}\right\}\\ &\cong K\times\{\ (0,0,\kappa)\mid\ \kappa={}^t\kappa\in\mathbb{R}^{(m,m)}\ \}\cong K\times \mathrm{Symm}^2(\mathbb{R}^m).\end{aligned}$$

Thus on $G^J/K^J\cong H_{n,m}$, we have the coordinate

$$g\cdot(iE_n,0):=(Z,W):=(X+iY,\lambda Z+\mu),\quad g\in G^J.$$

In fact, if $g=[M,(\lambda,\mu,\kappa)]\in G^J$ with $M=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in G$,

$$\begin{aligned} Z&=M<iE_n>=(iA+B)(iC+D)^{-1}=X+iY,\\ W&=\{i(\lambda A+\mu C)+\lambda B+\lambda D\}(iC+D)^{-1}\\ &=\{\lambda(iA+B)+\mu(iC+D)\}(iC+D)^{-1}\\ &=\lambda Z+\mu.\end{aligned}$$

We set

$$dX=\begin{pmatrix} dX_{11} & \dots & dX_{1n}\\ \vdots & \ddots & \vdots\\ dX_{n1} & \dots & dX_{nn}\end{pmatrix},\quad dW=\begin{pmatrix} dW_{11} & \dots & dW_{1n}\\ \vdots & \ddots & \vdots\\ dW_{m1} & \dots & dW_{mn}\end{pmatrix}$$

and

$$\frac{\partial}{\partial X}=\begin{pmatrix} \frac{\partial}{\partial X_{11}} & \cdots & \frac{\partial}{\partial X_{1n}}\\ \vdots & \ddots & \vdots\\ \frac{\partial}{\partial X_{n1}} & \cdots & \frac{\partial}{\partial X_{nn}}\end{pmatrix},\quad \frac{\partial}{\partial W}=\begin{pmatrix} \frac{\partial}{\partial W_{11}} & \cdots & \frac{\partial}{\partial W_{m1}}\\ \vdots & \ddots & \vdots\\ \frac{\partial}{\partial W_{1n}} & \cdots & \frac{\partial}{\partial W_{mn}}\end{pmatrix}.$$

Similarly we set $dY=(dY_{ij}),\ d\lambda=(d\lambda_{pq}),\ d\mu=(d\mu_{pq}),\cdots$etc. By an easy calculation, we have

$$\begin{aligned}
\frac{\partial}{\partial W} &= \frac{1}{2i} Y^{-1}\left(\frac{\partial}{\partial \lambda} - \bar{Z}\frac{\partial}{\partial \mu}\right),\\
\frac{\partial}{\partial \overline{W}} &= \frac{i}{2} Y^{-1}\left(\frac{\partial}{\partial \lambda} - Z\frac{\partial}{\partial \mu}\right),\\
\frac{\partial}{\partial X} &= \frac{\partial}{\partial Z} + \frac{\partial}{\partial \bar{Z}} + \frac{\partial}{\partial W}\lambda + \frac{\partial}{\partial \overline{W}}\lambda,\\
\frac{\partial}{\partial Y} &= i\frac{\partial}{\partial Z} - i\frac{\partial}{\partial \bar{Z}} + i\frac{\partial}{\partial W}\lambda - i\frac{\partial}{\partial \overline{W}}\lambda.
\end{aligned}$$

We set

$$P_+ = \frac{1}{2}\left(\frac{\partial}{\partial X} - i\frac{\partial}{\partial Y}\right) = \frac{\partial}{\partial Z} + \frac{\partial}{\partial W}\,(\mathrm{Im}\,W)\,Y^{-1}$$

and

$$P_- = \frac{1}{2}\left(\frac{\partial}{\partial X} + i\frac{\partial}{\partial Y}\right) = \frac{\partial}{\partial \bar{Z}} + \frac{\partial}{\partial \overline{W}}\,(\mathrm{Im}\,W)\,Y^{-1}.$$

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Then

$$\mathfrak{g}_{\mathbb{C}} = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \;\middle|\; B = {}^tB, \quad C = {}^tC \right\}.$$

We let $\hat{J} := iJ_n$ with $J_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$. We define an involution σ of G by

$$\sigma(g) := \hat{J} g \hat{J}^{-1}, \quad g \in G. \tag{9.7}$$

The differential map $d\sigma = \mathrm{Ad}\,(\hat{J})$ of σ extends complex linearly to the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. $\mathrm{Ad}\,(\hat{J})$ has 1 and -1 as eigenvalues. The (+1)-eigenspace of $\mathrm{Ad}\,(\hat{J})$ is given by

$$\mathfrak{k}_{\mathbb{C}} = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \;\middle|\; {}^tA + A = 0,\ B = {}^tB \right\}. \tag{9.8}$$

We note that $\mathfrak{k}_{\mathbb{C}}$ is the complexification of the Lie algebra $\mathfrak{k}$ of a maximal compact subgroup $K = G \cap SO(2n, \mathbb{R}) \cong U(n)$ of G. The (-1)-eigenspace of $\mathrm{Ad}\,(\hat{J})$ is given by

$$\mathfrak{p}_{\mathbb{C}} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \;\middle|\; A = {}^tA,\ B = {}^tB \right\}. \tag{9.9}$$

We observe that $\mathfrak{p}_{\mathbb{C}}$ is not a Lie algebra. But $\mathfrak{p}_{\mathbb{C}}$ has the following decomposition

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-,$$

where

$$\mathfrak{p}_+ = \left\{ \begin{pmatrix} X & iX \\ iX & -X \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \;\middle|\; X = {}^tX \right\} \tag{9.10}$$

and

$$\mathfrak{p}_- = \left\{ \begin{pmatrix} Y & -iY \\ -iY & -Y \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \;\middle|\; Y = {}^tY \right\}. \tag{9.11}$$

We observe that $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are abelian subalgebras of $\mathfrak{g}_{\mathbb{C}}$. Since $\mathrm{Ad}\,(\hat{J})[X,Y] = [\mathrm{Ad}\,(\hat{J})X,\ \mathrm{Ad}\,(\hat{J})Y]$ for all $X, Y \in \mathfrak{g}_{\mathbb{C}}$, we have

$$[\mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}] \subset \mathfrak{k}_{\mathbb{C}}, \quad [\mathfrak{k}_{\mathbb{C}}, \mathfrak{p}_{\mathbb{C}}] \subset \mathfrak{p}_{\mathbb{C}}, \quad [\mathfrak{p}_{\mathbb{C}}, \mathfrak{p}_{\mathbb{C}}] \subset \mathfrak{k}_{\mathbb{C}}. \tag{9.12}$$

Since $\mathrm{Ad}(k)X = kXk^{-1}\,(\,k \in K,\ X \in \mathfrak{g}_{\mathbb{C}}\,)$, we obtain

$$\mathrm{Ad}(k)\mathfrak{p}_+ \subset \mathfrak{p}_+, \quad \mathrm{Ad}(k)\mathfrak{p}_- \subset \mathfrak{p}_-. \tag{9.13}$$

For instance, if $k = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K$, then

$$\mathrm{Ad}(k) \begin{pmatrix} X & \pm iX \\ \pm iX & -X \end{pmatrix} = \begin{pmatrix} X' & \pm iX' \\ \pm iX' & -X' \end{pmatrix}, \quad X = {}^tX, \tag{9.14}$$

where

$$X' = (A+iB)X\,{}^t(A+iB).$$

If we identify $\mathfrak{p}_-$ with $\mathrm{Symm}^2(\mathbb{C}^n)$ and K with $U(n)$ as a subgroup of $GL(n,\mathbb{C})$ via the mapping $K \ni \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \longrightarrow A + iB \in U(n)$, then the action of K on $\mathfrak{p}_-$ is compatible with the natural representation $\rho^{[1]}$ of $GL(n,\mathbb{C})$ on $\mathrm{Symm}^2(\mathbb{C}^n)$ given by

$$\rho^{[1]}(g)X = gX\,{}^tg, \quad g \in GL(n,\mathbb{C}), \quad X \in \mathrm{Symm}^2(\mathbb{C}^n).$$

The Lie algebra $\mathfrak{g}$ of G has a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{9.15}$$

where

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \;\middle|\; A + {}^tA = 0, \quad B = {}^tB \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \;\middle|\; A = {}^tA, \quad B = {}^tB \right\}.$$

Then $\theta := \mathrm{Ad}\,(\hat{J})$ is a Cartan involution because

$$-B(W, \theta(W)) = -B(X,X) + B(Y,Y) > 0$$

for all $W = X + Y,\ X \in \mathfrak{k},\ Y \in \mathfrak{p}$. Here B denotes the Cartan-Killing form for $\mathfrak{g}$. Indeed,

$$B(X,Y) = 2(n+1)\,\sigma(XY), \quad X,Y \in \mathfrak{g}. \tag{9.16}$$

The vector space $\mathfrak{p}$ is identified with the tangent space of H_n at iE_n. The correspondence

$$\frac{1}{2}\begin{pmatrix} B & A \\ A & -B \end{pmatrix} \longmapsto A + iB \tag{9.17}$$

yields an isomorphism of $\mathfrak{p}$ onto $\mathrm{Symm}^2(\mathbb{C}^n)$. The Lie algebra $\mathfrak{g}^J$ of the Jacobi group G^J has a decomposition

$$\mathfrak{g}^J = \mathfrak{k}^J + \mathfrak{p}^J, \tag{9.18}$$

where

$$\mathfrak{k}^J = \left\{ (X, (0,0,\kappa) \mid\ X \in \mathfrak{k},\ \ \kappa = {}^t\kappa \in \mathbb{R}^{(m,m)} \right\},$$
$$\mathfrak{p}^J = \left\{ (Y, (P,Q,0) \ \mid\ Y \in \mathfrak{p},\ \ P,Q \in \mathbb{R}^{(m,n)} \right\}.$$

Thus the tangent space of the homogeneous space $H_{n,m} \cong G^J/K^J$ at $(iE_n, 0)$ is given by

$$\mathfrak{p}^J \cong \mathfrak{p} \oplus (\mathbb{R}^{(n,m)} \times \mathbb{R}^{(n,m)}) \cong \mathfrak{p} \oplus \mathbb{C}^{(n,m)}.$$

We define a complex structure I^J on the tangent space $\mathfrak{p}^J$ of $H_{n,m}$ at iE_n by

$$I^J\left(\begin{pmatrix} Y & X \\ X & -Y \end{pmatrix}, (P,Q) \right) := \left(\begin{pmatrix} X & -Y \\ -Y & -X \end{pmatrix}, (Q,-P) \right). \tag{9.19}$$

Identifying $\mathbb{R}^{(m,n)} \times \mathbb{R}^{(m,n)}$ with $\mathbb{C}^{(m,n)}$ via

$$(P,Q) \longmapsto iP + Q, \quad P,Q \in \mathbb{R}^{(m,n)}, \tag{9.20}$$

we may regard the complex structure I^J as a real linear map

$$I^J(X+iY, Q+iP) = (-Y+iX, -P+iQ), \tag{9.21}$$

where $X + iY \in \mathrm{Symm}^2(\mathbb{C}^n),\ Q + iP \in \mathbb{C}^{(m,n)}$. I^J extends complex linearly on the complexification $\mathfrak{p}^J_{\mathbb{C}} = \mathfrak{p} \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathfrak{p}$. $\mathfrak{p}_{\mathbb{C}}$ has a decomposition

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^J_+ \oplus \mathfrak{p}^J_-, \tag{9.22}$$

where $\mathfrak{p}^J_+$ (resp. $\mathfrak{p}^J_-$) denotes the $(+i)$-eigenspace (resp. $(-i)$-eigenspace) of I^J. Precisely, both $\mathfrak{p}^J_+$ and $\mathfrak{p}^J_-$ are given by

$$\mathfrak{p}^J_+ = \left\{ \left(\begin{pmatrix} X & iX \\ iX & -X \end{pmatrix}, (P, iP) \right) \ \middle|\ X \in \mathrm{Symm}^2(\mathbb{C}^n),\ \ P \in \mathbb{C}^{(m,n)} \right\}$$

and

$$\mathfrak{p}_-^J = \left\{ \left(\begin{pmatrix} X & -iX \\ -iX & -X \end{pmatrix}, (P, -iP) \right) \;\middle|\; X \in \mathrm{Symm}^2(\mathbb{C}^n), \quad P \in \mathbb{C}^{(m,n)} \right\}.$$

With respect to this complex structure I^J, we may say that f is *holomorphic* if and only if $\xi f = 0$ for all $\xi \in \mathfrak{p}_-^J$.

We fix $g = [M, (l, \mu; \kappa)] \in G^J$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$. Let $T_g : H_n \longrightarrow H_n$ be the mapping defined by (9.4). We consider the behavior of the differential map dT_g of T_g at $(iE_n, 0)$

$$dT_g : T_{(iE_n,0)}(H_{n,m}) \longrightarrow T_{(Z,W)}(H_{n,m}), \quad (Z, W) := g \cdot (iE_n, 0).$$

Now we let $\alpha(t) = (Z(t), \xi(t))$ be a smooth curve in $H_{n,m}$ passing through $(iE_n, 0)$ with $\alpha'(0) = (V, iP + Q) \in T_{(iE_n,0)}(H_{n,m})$. Then

$$\begin{aligned} \gamma(t) :&= g \cdot \alpha(t) = (Z(g;t), \xi(g;t)) \\ &= (M < Z(t) >, (\xi(t) + \tilde{\lambda} Z(t) + \tilde{\mu})(CZ(t) + D)^{-1}) \end{aligned}$$

is a curve in $H_{n,m}$ passing through $\gamma(0) = (Z, W)$ with $(\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)M$. Using the relation

$$\left.\frac{\partial}{\partial t}\right|_{t=0} (CZ(t) + D)^{-1} = -(iC + D)^{-1} C Z'(0)(iC + D)^{-1},$$

we have

$$\begin{aligned} \left.\frac{\partial}{\partial t}\right|_{t=0} Z(g;t) &= AZ'(0)(iC + D)^{-1} + (iA + B) \left.\frac{\partial}{\partial t}\right|_{t=0} (CZ(t) + D)^{-1} \\ &= AZ'(0)(iC + D)^{-1} - (iA + B)(iC + D)^{-1} CZ'(0)(iC + D)^{-1} \\ &= \{A\,{}^t(iC + D) - (iA + B)\,{}^tC\}\,{}^t(iC + D)^{-1} Z'(0)(iC + D)^{-1} \\ &= \,{}^t(iC + D)^{-1} Z'(0)(iC + D)^{-1}. \end{aligned}$$

and

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \xi(g;t) = (\xi'(0) + \tilde{\lambda} Z'(0))(iC+D)^{-1}$$
$$+ (\xi(0) + i\tilde{\lambda} + \tilde{\mu}) \left.\frac{\partial}{\partial t}\right|_{t=0} (CZ(t)+D)^{-1}$$
$$= (iP + Q + \tilde{\lambda} Z'(0))(iC+D)^{-1}$$
$$- (i\tilde{\lambda} + \tilde{\mu})(iC+D)^{-1} C Z'(0)(iC+D)^{-1}$$
$$= (iP+Q)(iC+D)^{-1}$$
$$+ \left\{ \tilde{\lambda}\, {}^t(iC+D) - (i\tilde{\lambda} + \tilde{\mu})\, {}^tC \right\} {}^t(iC+D)^{-1} Z'(0)(iC+D)^{-1}$$
$$= (iP+Q)(iC+D)^{-1} + \lambda\, {}^t(iC+D)^{-1} Z'(0)(iC+D)^{-1}.$$

Here we used the fact that $(iC+D)^{-1}C$ is symmetric and the relation

$$\tilde{\lambda} = \lambda A + \mu C, \quad \tilde{\mu} = \lambda B + \mu D.$$

Therefore we obtain

$$Z'(g;0) = {}^t(iC+D)^{-1} Z'(0)(iC+D)^{-1},$$
$$\xi'(g;0) = \xi'(0)(iC+D)^{-1} + \lambda\, {}^t(iC+D)^{-1} Z'(0)(iC+D)^{-1}.$$

In summary, we have

Proposition 9.1. *Let* $g = [M, (\lambda, \mu; \kappa)] \in G^J$ *with* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ *and let* $(Z, W) = g \cdot (iE_n, 0)$. *Then the differential map* $dT_g : T_{(iE_n,0)}(H_{n,m}) \longrightarrow T_{(Z,W)}(H_{n,m})$ *is given by*

$$(v, w) \longmapsto (v(g), w(g)), \quad v \in \mathrm{Symm}^2(\mathbb{C}^n), \quad w \in \mathbb{C}^{(m,n)} \tag{9.23}$$

with

$$v(g) = {}^t(iC+D)^{-1} v (iC+D)^{-1},$$
$$w(g) = w(iC+D)^{-1} + \lambda\, {}^t(iC+D)^{-1} v (iC+D)^{-1}.$$

9.1.2. An Iwasawa Decomposition of the Jacobi Group G^J

First of all, we give the Iwasawa decomposition of $G = Sp(n, \mathbb{R})$. For a positive diagonal matrix H of degree n, we put

$$t(H) = \begin{pmatrix} H & 0 \\ 0 & H^{-1} \end{pmatrix}$$

and for an upper triangular matrix A with 1 in every diagonal entry and $B \in \mathbb{R}^{(m,n)}$, we write

$$n(A,B) = \begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix}.$$

We let A be the set of such all $t(H)$ and let N be the set of such all $n(A,B)$ such that $n(A,B) \in G$, namely, $A\,{}^tB = B\,{}^tA$. It is clear that A is an abelian subgroup of G and N is a nilpotent subgroup of G. Then we have the so-called *Iwasawa decomposition*

$$G = NAK = KAN. \tag{9.24}$$

Now we define the subgroups A^J, N^J and $\tilde{N}^J$ of G^J by

$$A^J = \left\{ t(H,\lambda) := [t(H), (\lambda, 0, 0)] \,\middle|\, t(H) \in A,\ \lambda \in \mathbb{R}^{(m,n)} \right\},$$

$$N^J = \left\{ n(A,B;\mu) := [n(A,B), (0,\mu,0)] \,\middle|\, n(A,B) \in N,\ \mu \in \mathbb{R}^{(m,n)} \right\}$$

and

$$\tilde{N}^J := \left\{ \tilde{n}(A,B;\mu,\kappa) = [n(A,B),(0,\mu,\kappa)] \,\middle|\, n(A,B) \in N,\ \mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)} \right\}.$$

For $t(H,\lambda),\ t(H',\lambda') \in A^J$, we have

$$t(H,\lambda) \circ t(H',\lambda') = t(HH', \lambda + \lambda' H^{-1}).$$

Thus A^J is the semidirect product of $\mathbb{R}^{(m,n)}$ and $\mathbb{D}^+$, where $\mathbb{D}^+$ denotes the subgroup of $GL(n,\mathbb{R})$ consisting of positive diagonal matrices of degree n. Furthermore we have for $t(H,\lambda) \in A^J$ and $\tilde{n}(A,B;\mu,\kappa) \in \tilde{N}^J$

$$\tilde{n}(A,B;\mu,\kappa) \circ t(H,\lambda) = [n(A,B)t(H), (\lambda A^{-1},\ \mu - \lambda A^{-1} B\,{}^tA,\ -\mu\,{}^tA^{-1}\,{}^t\lambda)]$$

and

$$\begin{aligned} t(H,\lambda) \circ \tilde{n}(A,B;\mu,\kappa) &= [t(H)n(A,B), (\lambda, \mu H,\ \kappa + \lambda H\,{}^t\mu)] \\ &= \left[\begin{pmatrix} HA & HB \\ 0 & H^{-1}\,{}^tA^{-1} \end{pmatrix}, (\lambda, \mu H, \kappa + \lambda H\,{}^t\mu) \right]. \end{aligned}$$

Therefore we have

$$\begin{aligned} & t(H,\lambda) \circ \tilde{n}(A,B;\mu,\kappa) \circ t(H,\lambda)^{-1} \\ =& [t(H)n(X)t(H^{-1}), (\lambda - \lambda H A^{-1} H^{-1},\ \mu H + \lambda H A^{-1} B\,{}^tA H, \\ & \ \kappa + \lambda H A\,{}^tB\,{}^tA^{-1} H\,{}^t\lambda - \mu\,{}^tA^{-1} H\,{}^t\lambda)] \\ =& [n(HAH^{-1}, HBH),\ (\lambda - \lambda H A^{-1} H^{-1}, \mu H + \lambda H A^{-1} B\,{}^tA H, \\ & \ \kappa + \lambda H A\,{}^tB\,{}^tA^{-1} H\,{}^t\lambda - \mu\,{}^tA^{-1} H\,{}^t\lambda)]. \end{aligned}$$

Thus there is a decomposition

$$G^J = \tilde{N}^J A^J K. \tag{9.25}$$

For $g \in G^J$, one has

$$\begin{aligned} g &= [n(A,B)t(H)\kappa, (\lambda,\mu,\kappa)], \quad \kappa \in K \\ &= \tilde{n}(A,B;\mu^*,\kappa^*) \circ t(H,\lambda^*) \circ k \end{aligned}$$

with

$$\lambda^* = \lambda H, \quad \mu^* = \mu + \lambda B^t A \ \text{ and } \ \kappa^* = \kappa + \mu^t\lambda + \lambda B\,{}^t(\lambda A).$$

Recalling the subgroup K^J of G^J defined by

$$K^J = \{\, [k,(0,0,\kappa)] \mid k \in K,\ \kappa = {}^t\kappa \in \mathbb{R}^{(m,m)} \,\},$$

we also have a decomposition

$$G^J = N^J A^J K^J. \tag{9.26}$$

For $g \in G^J$, one has

$$\begin{aligned} g &= [n(A,B)t(H)k, (\lambda,\mu,\kappa)] \\ &= n(A,B;\tilde{\mu}) \circ t(H,\tilde{\lambda}) \circ [k,(0,0,\tilde{\kappa})] \end{aligned}$$

with

$$\tilde{\lambda} = \lambda A, \quad \tilde{\mu} = \mu + \lambda A^{-1} B\,{}^t A, \quad \tilde{\kappa} = \kappa + (\mu + \lambda A^{-1}B\,{}^tA)\,{}^t\lambda.$$

We call the decomposition (9.25) or (9.26) an *Iwasawa decomposition* of G^J. Finally we note that the decomposition (9.25) or (9.26) may be understood as the product of the usual Iwasawa decomposition (9.24) of G with a decomposition

$$H_{\mathbb{R}}^{(n,m)} = \tilde{N}_0 A_0 \tag{9.27}$$

of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ into the group $A_0 = \{\, (\lambda,0,0) \mid \lambda \in \mathbb{R}^{(m,n)} \,\}$ which normalizes the maximal abelian subgroup $\tilde{N}_0 = \{\ (0,\mu,\kappa)\ \mid\ \mu \in \mathbb{R}^{(m,n)}, \kappa = {}^t\kappa \in \mathbb{R}^{(m,n)} \,\}$.

9.2. The Lie Algebra of the Jacobi Group G^J

In this section, we describe the Lie algebra $\mathfrak{g}^J$ of the Jacobi group G^J explicitly.

First of all, we observe that $\mathfrak{g}$ of G may be regarded as a subalgebra of $\mathfrak{g}^J$ by identifying $\mathfrak{g}$ with $\mathfrak{g} \times \{0\}$ and the Lie algebra $\mathfrak{h}$ of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ may be regarded as an ideal of $\mathfrak{g}^J$ by identifying $\mathfrak{h}$ with $\{0\} \times \mathfrak{h}$. We denote by E_{ij} the matrix with entry 1 where the i-th row and the j-th column meet, all other entries 0.

For $1 \leq a, b, p \leq m,\ 1 \leq i, j, q \leq n$, we set

$$A_{ij} := \begin{pmatrix} E_{ij}+E_{ji} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(E_{ij}+E_{ji}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_{ij} := \begin{pmatrix} 0 & 0 & E_{ij}+E_{ji} & 0 \\ 0 & 0 & 0 & 0 \\ E_{ij}+E_{ji} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_{ij} := \begin{pmatrix} E_{ij}-E_{ji} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E_{ij}-E_{ji} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$T_{ij} := \begin{pmatrix} 0 & 0 & E_{ij}+E_{ji} & 0 \\ 0 & 0 & 0 & 0 \\ -(E_{ij}+E_{ji}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D^0_{ab} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(E_{ab}+E_{ba}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D_{pq} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ E_{pq} & 0 & 0 & 0 \\ 0 & 0 & 0 & -E_{qp} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{D}_{pq} := \begin{pmatrix} 0 & 0 & 0 & E_{qp} \\ 0 & 0 & E_{pq} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We observe that the set

$$\left\{ S_{ij},\, T_{kl},\, D^0_{ab} \,\middle|\, 1 \leq i < j \leq n,\ 1 \leq k \leq l \leq n,\ 1 \leq a \leq b \leq m \right\}$$

form a basis of $\mathfrak{k}^J$ and the set

$$\left\{ A_{ij},\, B_{ij},\, D_{pq},\, \hat{D}_{rs} \,\middle|\, 1 \leq i \leq j \leq n,\ 1 \leq p, r \leq m,\ 1 \leq q, s \leq n \right\}$$

form a basis of $\mathfrak{p}^J$ (cf. (9.15)). We note that

$$A_{ij} = A_{ji}, \quad B_{ij} = B_{ji}, \quad S_{ij} = -S_{ji}, \quad T_{ij} = T_{ji},$$
$$D^0_{ab} = D^0_{ba}, \quad D^2_{pq} = \hat{D}^2_{pq} = 0.$$

Lemma 9.2. *We have the following commutation relation :*

$$
\begin{aligned}
[A_{ij}, A_{kl}] &= \delta_{ik}S_{jl} + \delta_{il}S_{jk} + \delta_{jk}S_{il} + \delta_{jl}S_{ik},\\
[A_{ij}, B_{kl}] &= \delta_{ik}T_{jl} + \delta_{il}T_{jk} + \delta_{jk}T_{il} + \delta_{jl}T_{ik},\\
[A_{ij}, S_{kl}] &= \delta_{ik}A_{jl} - \delta_{il}A_{jk} + \delta_{jk}A_{il} - \delta_{jl}A_{ik},\\
[A_{ij}, T_{kl}] &= \delta_{ik}B_{jl} + \delta_{il}B_{jk} + \delta_{jk}B_{il} + \delta_{jl}B_{ik},\\
[B_{ij}, B_{kl}] &= \delta_{ik}S_{jl} + \delta_{il}S_{jk} + \delta_{jk}S_{il} + \delta_{jl}S_{ik},\\
[B_{ij}, S_{kl}] &= \delta_{ik}B_{jl} - \delta_{il}B_{jk} + \delta_{jk}B_{il} - \delta_{jl}B_{ik},\\
[B_{ij}, T_{kl}] &= -\delta_{ik}A_{jl} - \delta_{il}A_{jk} - \delta_{jk}A_{il} - \delta_{jl}A_{ik},\\
[S_{ij}, S_{kl}] &= -\delta_{ik}S_{jl} + \delta_{il}S_{jk} + \delta_{jk}S_{il} - \delta_{jl}S_{ik},\\
[S_{ij}, T_{kl}] &= -\delta_{ik}T_{jl} - \delta_{il}T_{jk} + \delta_{jk}T_{il} + \delta_{jl}T_{ik},\\
[T_{ij}, T_{kl}] &= -\delta_{ik}S_{jl} - \delta_{il}S_{jk} - \delta_{jk}S_{il} - \delta_{jl}S_{ik},\\
[D^0_{ab}, A_{ij}] &= [D^0_{ab}, B_{ij}] = [D^0_{ab}, S_{ij}] = [D^0_{ab}, T_{ij}] = 0,\\
[D^0_{ab}, D^0_{cd}] &= [D^0_{ab}, D_{pq}] = [D^0_{ab}, \hat{D}_{pq}] = 0,\\
[D_{pq}, A_{ij}] &= \delta_{qi}D_{pj} + \delta_{qj}D_{pi},\\
[D_{pq}, B_{ij}] &= [D_{pq}, T_{ij}] = \delta_{qi}\hat{D}_{pj} + \delta_{qj}\hat{D}_{pi},\\
[D_{pq}, S_{ij}] &= \delta_{qi}D_{pj} - \delta_{qj}D_{pi},\\
[D_{pq}, D_{rs}] &= 0, \quad [D_{pq}, \hat{D}_{rs}] = 2\delta_{qs}D^0_{pr},\\
[\hat{D}_{pq}, A_{ij}] &= -\delta_{qi}\hat{D}_{pj} - \delta_{qj}\hat{D}_{pi},\\
[\hat{D}_{pq}, B_{ij}] &= \delta_{qi}D_{pj} + \delta_{qj}D_{pi},\\
[\hat{D}_{pq}, S_{ij}] &= \delta_{qi}D_{pj} - \delta_{qj}\hat{D}_{pi},\\
[\hat{D}_{pq}, T_{ij}] &= -\delta_{qi}D_{pj} - \delta_{qj}D_{pi},\\
[\hat{D}_{pq}, \hat{D}_{rs}] &= 0.
\end{aligned}
$$

Here $1 \le a, b, c, d, p, r \le m,\ 1 \le i, j, k, l, q, s \le n$ and δ_{ij} denotes the Kronecker delta symbol.

Proof. The proof follows from a straightforward calculation. □

Corollary 9.3. *We have the following relation :*

$$
[\mathfrak{k}^J, \mathfrak{k}^J] \subset \mathfrak{k}^J, \quad [\mathfrak{k}^J, \mathfrak{p}^J] \subset \mathfrak{p}^J,
$$
$$
[\mathfrak{p}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}.
$$

Proof. It follows immediately from Lemma 9.2. □

Remark 9.4. We remark that the relation

$$[\mathfrak{p}^J, \mathfrak{p}^J] \subset \mathfrak{k}^J$$

does not hold.

Now we set

$$\begin{aligned}
Z_{ab}^0 &:= -\sqrt{-1}D_{ab}^0,\\
Y_{pq}^{\pm} &:= \frac{1}{2}(D_{pq} \pm \sqrt{-1}\hat{D}_{pq}),\\
Z_{ij}^{+} &:= -S_{ij},\\
Z_{ij}^{-} &:= -\sqrt{-1}T_{ij},\\
X_{ij}^{\pm} &:= \frac{1}{2}(A_{ij} \pm \sqrt{-1}B_{ij}).
\end{aligned}$$

Lemma 9.5. *We have the following commutation relation :*

$$\begin{aligned}
&[Z_{ab}^0, Z_{cd}^0] = [Z_{ab}^0, Y_{pq}^{\pm}] = [Z_{ab}^0, Z_{ij}^{\pm}] = [Z_{ab}^0, X_{ij}^{\pm}] = 0,\\
&[Y_{pq}^{+}, Y_{rs}^{+}] = 0, \quad [Y_{pq}^{+}, Y_{rs}^{-}] = \delta_{qs} Z_{pr}^0,\\
&[Y_{pq}^{+}, Z_{ij}^{+}] = -\delta_{qi} Y_{pj}^{+} + \delta_{qj} Y_{pi}^{+},\\
&[Y_{pq}^{+}, Z_{ij}^{-}] = -\delta_{qi} Y_{pj}^{+} - \delta_{qj} Y_{pi}^{+},\\
&[Y_{pq}^{+}, X_{ij}^{+}] = 0,\\
&[Y_{pq}^{+}, X_{ij}^{-}] = \delta_{qi} Y_{pj}^{-} + \delta_{qj} Y_{pi}^{-},\\
&[Y_{pq}^{-}, Y_{rs}^{-}] = 0,\\
&[Y_{pq}^{-}, Z_{ij}^{+}] = -\delta_{qi} Y_{pj}^{-} + \delta_{qj} Y_{pi}^{-},\\
&[Y_{pq}^{-}, Z_{ij}^{-}] = \delta_{qi} Y_{pj}^{-} + \delta_{qj} Y_{pi}^{-},\\
&[Y_{pq}^{-}, X_{ij}^{+}] = \delta_{qi} Y_{pj}^{+} + \delta_{qj} Y_{pi}^{+},\\
&[Y_{pq}^{-}, X_{ij}^{-}] = 0,
\end{aligned}$$

$$
\begin{aligned}
[Z_{ij}^+, Z_{kl}^+] &= \delta_{ik} Z_{jl}^+ - \delta_{il} Z_{jk}^+ - \delta_{jk} Z_{il}^+ + \delta_{jl} Z_{ik}^+, \\
[Z_{ij}^+, Z_{kl}^-] &= \delta_{ik} Z_{jl}^- - \delta_{il} Z_{jk}^- + \delta_{jk} Z_{il}^- - \delta_{jl} Z_{ik}^-, \\
[Z_{ij}^+, X_{kl}^\pm] &= \delta_{ik} X_{jl}^\pm - \delta_{jk} X_{il}^\pm + \delta_{il} X_{jk}^\pm - \delta_{jl} X_{ik}^\pm, \\
[Z_{ij}^-, Z_{kl}^-] &= -\delta_{ik} Z_{jl}^+ - \delta_{il} Z_{jk}^+ - \delta_{jk} Z_{il}^+ - \delta_{jl} Z_{ik}^+, \\
[Z_{ij}^-, X_{ij}^+] &= \delta_{ik} X_{jl}^+ + \delta_{il} X_{jk}^+ + \delta_{jk} X_{il}^+ + \delta_{jl} X_{ik}^+, \\
[Z_{ij}^-, X_{ij}^-] &= -\delta_{ik} X_{jl}^- - \delta_{il} X_{jk}^- - \delta_{jk} X_{il}^- - \delta_{jl} X_{ik}^-, \\
[X_{ij}^+, X_{kl}^+] &= [X_{ij}^-, X_{kl}^-] = 0, \\
[X_{ij}^+, X_{kl}^-] &= -\frac{1}{2}(\delta_{ik} Z_{jl}^+ + \delta_{il} Z_{jk}^+ + \delta_{jk} Z_{il}^+ + \delta_{jl} Z_{ik}^+) \\
&\quad + \frac{\sqrt{-1}}{2}(\delta_{ik} Z_{jl}^- + \delta_{il} Z_{jk}^- + \delta_{jk} Z_{il}^- + \delta_{jl} Z_{ik}^-).
\end{aligned}
$$

Proof. It follows from Lemma 9.2. □

Corollary 9.6. *The set*

$$
\left\{ Z_{ab}^0,\ Z_{ij}^+,\ Z_{kl}^- \,\middle|\, 1 \le a \le b \le m,\ 1 \le i < j \le n,\ 1 \le k < l \le n \right\}
$$

form a basis of the complexification $\mathfrak{k}_{\mathbb{C}}^J$ *of* $\mathfrak{k}^J$ *and the set*

$$
\left\{ X_{ij}^\pm,\ Y_{pq}^\pm \,\middle|\, 1 \le i \le j \le n,\ 1 \le p \le m,\ 1 \le q \le n \right\}
$$

form a basis of $\mathfrak{p}_{\mathbb{C}}^J$. *And* $\left\{ X_{ij}^+,\ Y_{pq}^+ \,\middle|\, 1 \le i \le j \le n,\ 1 \le p \le m,\ 1 \le q \le n \right\}$ *form a basis of* $\mathfrak{p}_+^J$ *and* $\left\{ X_{ij}^-,\ Y_{pq}^- \,\middle|\, 1 \le i \le j \le n,\ 1 \le p \le m,\ 1 \le q \le n \right\}$ *form a basis of* $\mathfrak{p}_-^J$. *Both* $\mathfrak{p}_+^J$ *and* $\mathfrak{p}_-^J$ *are all abelian subalgebras of* $\mathfrak{g}_{\mathbb{C}}^J$. *We have the relation*

$$
[\mathfrak{k}_{\mathbb{C}}^J,\ \mathfrak{p}_+^J] \subset \mathfrak{p}_+^J, \qquad [\mathfrak{k}_{\mathbb{C}}^J,\ \mathfrak{p}_-^J] \subset \mathfrak{p}_-^J.
$$

$\mathfrak{g}_{\mathbb{C}}$ *is a subalgebra of* $\mathfrak{g}_{\mathbb{C}}^J$ *and* $\mathfrak{h}_{\mathbb{C}}$, *the complexification of* $\mathfrak{h}$, *is an ideal of* $\mathfrak{g}_{\mathbb{C}}^J$.

Proof. It follows immediately from Lemma 9.5. □

9.3. Jacobi Forms

Let ρ be a rational representation of $GL(n, \mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbb{R}^{(m,m)}$ be a symmetric half-integral semi-positive definite matrix of degree m. Let $C^\infty(H_{n,m}, V_\rho)$ be the algebra of all C^∞ functions on $H_{n,m}$ with values in V_ρ. For $f \in C^\infty(H_{n,m}, V_\rho)$, we define

$$\begin{aligned} &(f|_{\rho,\mathcal{M}}[(M,(\lambda,\mu,\kappa))])(Z,W) \\ (9.28) \qquad &:= e^{-2\pi i\sigma(\mathcal{M}[W+\lambda Z+\mu](CZ+D)^{-1}C)} \times e^{2\pi i\sigma(\mathcal{M}(\lambda Z^t\lambda+2\lambda^tW+(\kappa+\mu^t\lambda)))} \\ &\times \rho(CZ+D)^{-1} f(M<Z>,(W+\lambda Z+\mu)(CZ+D)^{-1}), \end{aligned}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R}),\ (\lambda,\mu,\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(Z,W) \in H_{n,m}$.

Definition 9.7. Let ρ and $\mathcal{M}$ be as above. Let

$$H_{\mathbb{Z}}^{(n,m)} := \{(\lambda,\mu,\kappa) \in H_{\mathbb{R}}^{(n,m)} \,|\, \lambda,\mu \in \mathbb{Z}^{(m,n)},\ \kappa \in \mathbb{Z}^{(m,m)}\ \}.$$

A *Jacobi form* of index $\mathcal{M}$ with respect to ρ on Γ_n is a holomorphic function $f \in C^\infty(H_{n,m}, V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho,\mathcal{M}}[\tilde\gamma] = f$ for all $\tilde\gamma \in \Gamma_n^J := \Gamma_n \ltimes H_{\mathbb{Z}}^{(n,m)}$.

(B) f has a Fourier expansion of the following form :

$$f(Z,W) = \sum_{\substack{T\geq 0 \\ \text{half-integral}}} \sum_{R\in\mathbb{Z}^{(n,m)}} c(T,R)\cdot e^{2\pi i\,\sigma(TZ)}\cdot e^{2\pi i\sigma(RW)}$$

with $c(T,R) \neq 0$ only if $\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}\,{}^tR & \mathcal{M} \end{pmatrix} \geq 0$.

If $n \geq 2$, the condition (B) is superfluous by Köcher principle (cf. [110] Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma_n)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ_n. Ziegler (cf. [110] Theorem 1.8 or [26] Theorem 1.1) proves that the vector space $J_{\rho,\mathcal{M}}(\Gamma_n)$ is finite dimensional. For more results on Jacobi forms with $n > 1$ and $m > 1$, we refer to [61], [103]-[107] and [110].

Definition 9.8. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma)$ is said to be a *cusp* (or *cuspidal*) form if $\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}\,{}^tR & \mathcal{M} \end{pmatrix} > 0$ for any $T,\ R$ with $c(T,R) \neq 0$. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma)$ is said to be *singular* if it admits a Fourier expansion such that a Fourier coefficient $c(T,R)$ vanishes unless $\det \begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}\,{}^tR & \mathcal{M} \end{pmatrix} = 0$.

Example 9.9. Let $S \in \mathbb{Z}^{(2k,2k)}$ be a symmetric, positive definite unimodular even integral matrix and $c \in \mathbb{Z}^{(2k,m)}$. We define the theta series

$$\vartheta_{S,c}^{(g)}(Z,W) := \sum_{\lambda\in\mathbb{Z}^{(2k,n)}} e^{\pi i\{\sigma(S\lambda Z\,{}^t\lambda)+2\sigma({}^tcS\lambda,{}^tW)\}}, \quad Z \in H_n,\ W \in \mathbb{C}^{(m,n)}.$$

We put $\mathcal{M} := \frac{1}{2}\,{}^tcSc$. We assume that $2k < g + \operatorname{rank}(\mathcal{M})$. Then it is easy to see that $\vartheta_{S,c}^{(g)}$ is a singular form in $J_{k,\mathcal{M}}(\Gamma_g)$ (cf. [110] p. 212).

9.4. Characterization of Jacobi Forms as Functions on the Jacobi Group G^J

In this section, we lift a Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ to a smooth function Φ_f on the Jacobi group G^J and characterize the lifted function Φ_f on G^J.

We recall that for given ρ and $\mathcal{M}$, the canonical automorphic factor $J_{\mathcal{M},\rho} : G^J \times H_{n,m} \longrightarrow GL(V_\rho)$ is given by

$$J_{\mathcal{M},\rho}(g,(Z,W)) = e^{-2\pi i\sigma(\mathcal{M}[W+\lambda Z+\mu](CZ+D)^{-1}C)} \times e^{2\pi i\sigma(\mathcal{M}(\lambda Z\,{}^t\lambda+2\lambda\,{}^tW+\kappa+\mu\,{}^t\lambda))}\rho(CZ+D)^{-1},$$

where $g = (M,(\lambda,\mu;\kappa)) \in G^J$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$. It is easy to see that the automorphic factor $J_{\mathcal{M},\rho}$ satisfies the cocycle condition:

$$J_{\mathcal{M},\rho}(g_1g_2,(Z,W)) = J_{\mathcal{M},\rho}(g_2,(Z,W))\, J_{\mathcal{M},\rho}(g_1, g_2\cdot(Z,W)) \tag{9.29}$$

for all $g_1,\ g_2 \in G^J$ and $(Z,W) \in H_{n,m}$.

Since the space $H_{n,m}$ is diffeomorphic to the homogeneous space G^J/K^J, we may lift a function f on $H_{n,m}$ with values in V_ρ to a function Φ_f on G^J with values in V_ρ in the following way. We define the lifting

$$\varphi_{\rho,\mathcal{M}} : \mathcal{F}(H_{n,m},V_\rho) \longrightarrow \mathcal{F}(G^J,V_\rho), \quad \varphi_{\rho,\mathcal{M}}(f) := \Phi_f \tag{9.30}$$

by

$$\begin{aligned}\Phi_f(g) :&= (f|_{\rho,\mathcal{M}}[g])(iE_n,0)\\ &= J_{\mathcal{M},\rho}(g,(iE_n,0))\, f(g\cdot(iE_n,0)),\end{aligned}$$

where $g \in G^J$ and $\mathcal{F}(H_{n,m},V_\rho)$ (resp. $\mathcal{F}(G^J,V_\rho)$) denotes the vector space consisting of functions on $H_{n,m}$ (resp. G^J) with values in V_ρ.

For brevity, we set $\Gamma := \Gamma_n = Sp(n,\mathbb{Z})$ and $\Gamma^J = \Gamma \ltimes H_{\mathbb{Z}}^{(n,m)}$. We let $\mathcal{F}^{\Gamma}_{\rho,\mathcal{M}}$ be the space of all functions f on $H_{n,m}$ with values in V_ρ satisfying the transformation formula

$$f|_{\rho,\mathcal{M}}[\gamma] = f \quad \text{for all } \gamma \in \Gamma^J. \tag{9.31}$$

And we let $\mathcal{F}^{\Gamma}_{\rho,\mathcal{M}}(G^J)$ be the space of functions $\Phi : G^J \longrightarrow V_\rho$ on G^J with values in V_ρ satisfying the following conditions (9.32) and (9.33):

$$\Phi(\gamma g) = \Phi(g) \qquad \text{for all} \quad \gamma \in \Gamma^J \quad \text{and} \quad g \in G^J. \tag{9.32}$$

$$\Phi(g\ r(k,\kappa)) = e^{2\pi i\sigma(\mathcal{M}\kappa)}\rho(k)^{-1}\Phi(g), \qquad \forall\ \ r(k,\kappa) := [k,(0,0;\kappa)] \in K^J. \tag{9.33}$$

Lemma 9.10. *The space* $\mathcal{F}^{\Gamma}_{\rho,\mathcal{M}}$ *is isomorphic to the space* $\mathcal{F}^{\Gamma}_{\rho,\mathcal{M}}(G^J)$ *via the lifting* $\varphi_{\rho,\mathcal{M}}$.

Proof. Let $f \in \mathcal{F}^{\Gamma}_{\rho,\mathcal{M}}$. If $\gamma \in \Gamma^J,\ g \in G^J$ and $r(k,\kappa) = [k,(0,0;\kappa)] \in K^J$, then we have

$$\begin{aligned}\Phi_f(\gamma g) &= (f|_{\rho,\mathcal{M}}[\gamma g])(iE_n,0)\\ &= ((f|_{\rho,\mathcal{M}}[\gamma])|_{\rho,\mathcal{M}}[g])(iE_n,0)\\ &= (f|_{\rho,\mathcal{M}}[g])(iE_n,0) \qquad\qquad (\text{since } f \in \mathcal{F}^{\Gamma}_{\rho,\mathcal{M}})\\ &= \Phi_f(g)\end{aligned}$$

and

$$\begin{aligned}\Phi_f(g\,r(k,\kappa)) &= J_{\mathcal{M},\rho}(g\,r(k,\kappa),(iE_n,0))\,f(g\,r(k,\kappa)\cdot(iE_n,0))\\ &= J_{\mathcal{M},\rho}(r(k,\kappa),(iE_n,0))\,J_{\mathcal{M},\rho}(g,(iE_n,0))\,f(g\cdot(iE_n,0))\\ &= e^{2\pi i\sigma(\mathcal{M}\kappa)}\rho(k)^{-1}\Phi_f(g).\end{aligned}$$

Here we identified $k = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K$ with $A+iB \in U(n)$.

Conversely, if $\Phi \in \mathcal{F}^{\Gamma}_{\rho,\mathcal{M}}(G^J),\ G^J$ acting on $H_{n,m}$ transitively, we may define a function f_Φ on $H_{n,m}$ by

$$f_\Phi(g\cdot(iE_n,0)) := J_{\mathcal{M},\rho}(g,(iE_n,0))^{-1}\Phi(g). \tag{9.34}$$

Let $\gamma \in \Gamma^J$ and $(Z,W) = g\cdot(iE_n,0)$ for some $g \in G^J$. Then using the cocycle condition (9.29), we have

$$\begin{aligned}(f_\Phi|_{\rho,\mathcal{M}}[\gamma])(Z,W) &= J_{\mathcal{M},\rho}(\gamma,(Z,W))f_\Phi(\gamma\cdot(Z,W))\\ &= J_{\mathcal{M},\rho}(\gamma,g\cdot(iE_n,0))\,f_\Phi(\gamma g\cdot(iE_n,0))\\ &= J_{\mathcal{M},\rho}(\gamma,g\cdot(iE_n,0))J_{\mathcal{M},\rho}(\gamma g,(iE_n,0))^{-1}\Phi(\gamma g)\\ &= J_{\mathcal{M},\rho}(\gamma,g\cdot(iE_n,0))\,J_{\mathcal{M},\rho}(\gamma,g\cdot(iE_n,0))^{-1}\\ &\quad J_{\mathcal{M},\rho}(g,(iE_n,0))^{-1}\Phi(g)\\ &= J_{\mathcal{M},\rho}(g,(iE_n,0))^{-1}\Phi(g)\\ &= f_\Phi(g\cdot(iE_n,0)) = f_\Phi(Z,W).\end{aligned}$$

This completes the proof. □

Now we have the following two algebraic representations $T_{\rho,\mathcal{M}}$ and $\dot{T}_{\rho,\mathcal{M}}$ of G^J defined by

$$T_{\rho,\mathcal{M}}(g)f := f|_{\rho,\mathcal{M}}[g^{-1}],\quad g \in G^J,\ f \in \mathcal{F}^{\Gamma}_{\rho,\mathcal{M}} \tag{9.35}$$

and

$$\dot{T}_{\rho,\mathcal{M}}(g)\Phi(g') := \Phi(g^{-1}g'), \quad g,\, g' \in G^J, \quad \Phi \in \mathcal{F}^{\Gamma}_{\rho,\mathcal{M}}(G^J). \tag{9.36}$$

Then it is easy to see that these two models $T_{\rho,\mathcal{M}}$ and $\dot{T}_{\rho,\mathcal{M}}$ are intertwined by the lifting $\varphi_{\rho,\mathcal{M}}$.

Proposition 9.11. The vector space $J_{\rho,\mathcal{M}}(\Gamma_n)$ is isomorphic to the space $A_{\rho,\mathcal{M}}(\Gamma^J)$ of smooth functions Φ on G^J with values in V_ρ satisfying the following conditions:

(1a) $\Phi(\gamma g) = \Phi(g)$ for all $\gamma \in \Gamma^J$.
(1b) $\Phi(g r(k,\kappa)) = e^{2\pi i \sigma(\mathcal{M}\kappa)} \rho(k)^{-1} \Phi(g)$ for all $g \in G^J,\ r(k,\kappa) \in K^J$.
(2) $X^-_{ij}\Phi = Y^-_{ij}\Phi = 0, \quad 1 \le i, j \le n.$
(3) For all $M \in Sp(n,\mathbb{R})$, the function $\psi : G^J \longrightarrow V_\rho$ defined by

$$\psi(g) := \rho(Y^{-\frac{1}{2}})\, \Phi(Mg), \quad g \in G^J$$

is bounded in the domain $Y \ge Y_0$. Here $g\cdot(iE_n, 0) = (Z, W)$ with $Z = X + iY,\ Y > 0$.

Corollary 9.12. $J^{\mathrm{cusp}}_{\rho,\mathcal{M}}(\Gamma_n)$ is isomorphic to the subspace $A^0_{\rho,\mathcal{M}}(\Gamma^J)$ of $A_{\rho,\mathcal{M}}(\Gamma^J)$ with the condition (3') the function $g \longmapsto \Phi(g)$ is bounded.

9.5. Unitary Representations of the Jacobi Group G^J

In this section, we review some results of Takase (cf. [82]-[84]) on the unitary representations of the Jacobi group G^J. We follow the notations in the previous sections.

First we observe that G^J is not reductive because the center of G^J is given by

$$\mathcal{Z} = \left\{ [E_{2n}, (0,0;\kappa)] \in G^J \,\middle|\, \kappa = {}^t\kappa \in \mathbb{R}^{(m,m)} \right\} \cong \mathrm{Sym}^2(\mathbb{R}^m).$$

Let $d_K(k)$ be a normalized Haar measure on K so that $\int_K d_K(k) = 1$ and $d_{\mathcal{Z}}(\kappa) = \prod_{i\le j} d\kappa_{ij}$ a Haar measure on $\mathcal{Z}$. We let $d_{K^J} = d_K \times d_{\mathcal{Z}}$ be the product measure on $K^J = K \times \mathcal{Z}$. The Haar measure d_{G^J} on G^J is normalized so that

$$\int_{G^J} f(g) d_{G^J}(g) = \int_{G^J/K^J} \left(\int_{K^J} f(gh) d_{K^J}(h) \right) d_{G^J/K^J}(\dot{g})$$

for all $f \in C_c(G^J)$.

From now on, we will fix a real positive definite symmetric matrix $S \in \mathrm{Sym}^2(\mathbb{R}^m)$ of degree m. For any fixed $Z = X + iY \in H_n$, we define a measure $\nu_{S,Z}$ on $\mathbb{C}^{(m,n)}$ by

$$d\nu_{S,Z}(W) = (\det 2S)^n\, (\det Y)^{-m}\, \kappa_S(Z,W)\, dU dV, \tag{9.37}$$

where $W = U + iV \in \mathbb{C}^{(m,n)}$ with $U, V \in \mathbb{R}^{(m,n)}$ and

$$\kappa_S(Z, W) = e^{-4\pi\sigma({}^tVSVY^{-1})}. \tag{9.38}$$

Let $H_{S,Z}$ be the complex Hilbert space consisting of all $\mathbb{C}$-valued holomorphic functions φ on $\mathbb{C}^{(m,n)}$ such that $\int_{\mathbb{C}^{(m,n)}} |\varphi(W)|^2 d\nu_{S,Z} < +\infty$. The inner product on $H_{S,Z}$ is given by

$$(\varphi, \psi) = \int_{\mathbb{C}^{(m,n)}} \varphi(W)\overline{\psi(W)} d\nu_{S,Z}(W), \qquad \varphi,\, \psi \in H_{S,Z}.$$

We put

$$\eta_S = J_{S,\delta}^{-1} \quad \text{(see subsection 9.4)},$$

where δ denotes the trivial representation of $GL(n, \mathbb{C})$. Now we define a unitary representation $\Xi_{S,Z}$ of $H_{\mathbb{R}}^{(n,m)}$ by

$$\left(\Xi_{S,Z}(h)\varphi\right)(W) = \eta_S(h^{-1}, (Z, W))^{-1} \cdot \varphi(W - \lambda Z - \mu), \tag{9.39}$$

where $h = (\lambda, \mu, \kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $\varphi \in H_{S,Z}$. It is easy to see that $(\Xi_{S,Z}, H_{S,Z})$ is irreducible and $\Xi_{S,Z}(0, 0, \kappa) = e^{-2\pi i\sigma(S\kappa)}$.

Let

$$\mathcal{X} = \left\{ T \in \mathbb{C}^{(n,n)} \,\middle|\, T = {}^tT, \quad \mathrm{Re}\, T > 0 \right\}$$

be a connected simply connected open subset of $\mathbb{C}^{(n,n)}$. Then there exists uniquely a holomorphic function $\det^{\frac{1}{2}}$ on $\mathcal{X}$ such that

$$\begin{aligned} &(1) \qquad \left(\det^{\frac{1}{2}} T\right)^2 = \det T \qquad \text{for all } T \in T \in \mathcal{X}, \\ &(2) \qquad \det^{\frac{1}{2}} T = (\det T)^{\frac{1}{2}} \qquad \text{for all } T \in \mathcal{X} \cap \mathbb{R}^{(n,n)}. \end{aligned}$$

For any integer $k \in \mathbb{Z}$, we set

$$\det^{\frac{k}{2}} T = \left(\det^{\frac{1}{2}} T\right)^k, \quad T \in \mathcal{X}. \tag{9.40}$$

For any $g = (\sigma, h) \in G^J$ with $\sigma \in G$, we define an integral operator $T_{S,Z}(g)$ from $H_{S,Z}$ to $H_{S,\sigma<Z>}$ by

$$\left(T_{S,Z}(g)\varphi\right)(W) = \eta_S(g^{-1}, (Z, W))^{-1}\varphi(W'), \tag{9.41}$$

where $(Z', W') = g^{-1} \cdot (Z, W)$ and $\varphi \in H_{S,Z}$. And for any fixed Z and Z' in H_n, we define a unitary mapping

$$U_{Z',Z}^S : H_{S,Z} \longrightarrow H_{S,Z'} \tag{9.42}$$

by

$$\left(U^S_{Z',Z}\varphi\right)(W') = \gamma(Z',Z)^m \cdot \int_{\mathbb{C}^{(m,n)}} \kappa_S((Z',W'),(Z,W))^{-1}\varphi(W)\,d\nu_{S,Z}(W),$$

where

$$\gamma(Z',Z) = \det{}^{-\frac{1}{2}}\left(\frac{Z'-\bar{Z}}{2i}\right)\cdot \det(\operatorname{Im} Z')^{\frac{1}{4}} \cdot \det(\operatorname{Im} Z)^{\frac{1}{4}}$$

and

$$\kappa_S((Z',W'),(Z,W)) = e^{2\pi i \sigma(S[W'-\overline{W}]\cdot(Z'-\overline{Z})^{-1})}.$$

For any $g=(\sigma,h)\in G^J$, we define a unitary operator $T_S(g)$ of H_{S,iE_n} by

$$T_S(g) = T_{S,\sigma^{-1}<iE_n>}(g)\circ U^S_{\sigma^{-1}<iE_n>,iE_n}. \tag{9.43}$$

We put, for any $\sigma_1,\sigma_2\in G$,

$$\beta(\sigma_1,\sigma_2) = \frac{\gamma(\sigma_1^{-1}<iE_n>,iE_n)}{\gamma(\sigma_2^{-1}\sigma_1^{-1}<iE_n>,\sigma_1^{-1}<iE_n>)}\quad . \tag{9.44}$$

Then the function $\beta(\sigma_1,\sigma_2)$ satisfies the cocycle condition

$$\beta(\sigma_2,\sigma_3)\beta(\sigma_1\sigma_2,\sigma_3)^{-1}\beta(\sigma_1,\sigma_2\sigma_3)\beta(\sigma_1,\sigma_2)^{-1} = 1$$

for all $\sigma_1,\sigma_2,\sigma_3\in G$. Thus $\beta(\sigma_1,\sigma_2)$ defines a group extension $G\ltimes\mathbb{C}_1$ by $\mathbb{C}_1 = \{z\in\mathbb{C}\,|\,|z|=1\}$. Precisely, $G\ltimes\mathbb{C}_1$ is a topological group with multiplication

$$(\sigma_1,\epsilon_1)\cdot(\sigma_2,\epsilon_2) = (\sigma_1\sigma_2,\beta(\sigma_1,\sigma_2)\epsilon_1\epsilon_2)$$

for all $(\sigma_i,\epsilon_i)\in G\times\mathbb{C}_1\ (i=1,2)$. If we put

$$\epsilon(\sigma) = \frac{\det J(\sigma^{-1},iE_n)}{|\det J(\sigma^{-1},iE_n)|}\quad ,\qquad \sigma\in G, \tag{9.45}$$

then we have the relation

$$\beta(\sigma_1,\sigma_2)^2 = \epsilon(\sigma_1)\cdot\epsilon(\sigma_1\sigma_2)^{-1}\cdot\epsilon(\sigma_2),\quad \sigma_1,\,\sigma_2\in G.$$

Therefore we have a closed normal subgroup

$$G_2 = \left\{(\sigma,\epsilon)\in G\ltimes\mathbb{C}_1\;\middle|\;\epsilon^2 = \epsilon(\sigma)^{-1}\right\} \tag{9.46}$$

of $G_2\ltimes\mathbb{C}_1$ which is a connected two-fold covering group of G. Since G_2 acts on the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ via the projection $p: G_2\longrightarrow G$, we may put

$$G_2^J = G_2\ltimes H_{\mathbb{R}}^{(n,m)}.$$

Now we define the unitary representation ω_S of G_2^J by

$$\omega_S(g) = \epsilon^m \cdot T_S(\sigma, h), \quad g = ((\sigma, \epsilon), h) \in G_2^J. \tag{9.47}$$

It is easy to see that (ω_S, H_{S,iE_n}) is irreducible and the restriction of ω_S to G_2 is the m-fold tensor product of the Weil representation. ω_S is called the *Weil representation* of the Jacobi group G^J.

We set

$$\begin{aligned} &p : G_2 \longrightarrow G, \quad p(\sigma, \epsilon) = \sigma, \\ &p^J : G_2^J \longrightarrow G^J, \quad p^J((\sigma, \epsilon), h) = (\sigma, h), \\ &q : G^J \longrightarrow G, \quad q(\sigma, h) = \sigma, \\ &q^J : G_2^J \longrightarrow G_2, \quad q^J((\sigma, \epsilon), h) = (\sigma, \epsilon). \end{aligned}$$

Proposition 9.13. *Let χ_S be the character of $\mathcal{Z} \cong Sym^2(\mathbb{R}^m)$ defined by $\chi_S(\kappa) = e^{2\pi i \sigma(S\kappa)}$, $\kappa \in \mathcal{Z}$. We denote by $\hat{G}_2^J(\bar{\chi}_S)$ the set of all equivalence classes of irreducible unitary representations τ of G_2^J such that $\tau(\kappa) = \chi_S(\kappa)^{-1}$ for all $\kappa \in \mathcal{Z}$. We put $\tilde{\pi} = \pi \circ q^J \in \hat{G}_2^J$ for any $\pi \in \hat{G}_2$. The correspondence*

$$\pi \longmapsto \tilde{\pi} \otimes \omega_S$$

is a bijection from $\hat{G}_2$ to $\hat{G}_2^J(\bar{\chi}_S)$. And $\tilde{\pi} \otimes \omega_S$ is square-integrable modulo $\mathcal{Z}$ if and only if π is square integrable.

Proof. See [82], Proposition 11.8. □

Proposition 9.14. *Let m be even. We put $\breve{\pi} = \pi \circ q \in \hat{G}^J$ for any $\pi \in \hat{G}$. Then the correspondence*

$$\pi \longmapsto \breve{\pi} \otimes \omega_S$$

is a bijection of $\hat{G}$ to $\hat{G}^J$. And $\breve{\pi} \otimes \omega_S$ is square integrable modulo A if and only if π is square integrable.

Proof. See [82]. □

The above proposition was proved by Satake [75] or by Berndt [6] in the case $m = 1$.

Let (ρ, V_ρ) be an irreducible representation of $K = U(n)$ with highest weight $l = (l_1, l_2, \cdots, l_n) \in \mathbb{Z}^n$, $l_1 \geq \cdots \geq l_n \geq 0$. Then ρ is extended to a rational representation of $GL(n, \mathbb{C})$ which is also denoted by ρ. The representation space V_ρ of ρ has an hermitian inner product $(\ ,\ )$ such that $(\rho(g)u, v) = (u, \rho(g^*)v)$ for all $g \in GL(n, \mathbb{C})$, $u, v \in V_\rho$, where $g^* = {}^t\bar{g}$. We let the mapping $J : G \times H_n \longrightarrow GL(n, \mathbb{C})$ be the automorphic factor defined by

$$J(\sigma, Z) = CZ + D, \quad \sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G.$$

We define a unitary representation τ_l of K by

$$\tau_l(k) = \rho(J(k, iE_n)), \quad k \in K. \tag{9.48}$$

We set $J_{\rho,S} = J_{S,\rho}^{-1}$ (cf. subsection 9.4). According to the definition, we have

$$J_{\rho,S}(g,(Z,W)) = \eta_S(g,(Z,W))\,\rho(J(\sigma,Z))$$

for all $g = (\sigma, h) \in G^J$ and $(Z,W) \in H_{n,m}$. For any $g = (\sigma,h) \in G^J$ and $(Z,W) \in H_{n,m}$, we set

$$\begin{aligned} \overline{J_{\rho,S}(g,(Z,W))} &= \overline{\eta_S(g,(Z,W))}\,\rho(\overline{J(\sigma,Z)}),\\ {}^tJ_{\rho,S}(g,(Z,W)) &= \eta_S(g,(Z,W))\,\rho({}^tJ(\sigma,Z)),\\ J_{\rho,S}(g,(Z,W))^* &= {}^t\overline{J_{\rho,S}(g,(Z,W))}. \end{aligned}$$

Then for all $g \in G^J$, $(Z,W) \in H_{n,m}$ and $u, v \in V_\rho$, we have

$$(J_{\rho,S}(g,(Z,W))u, v) = (u, J_{\rho,S}(g,(Z,W))^* v)$$

We denote by $E(\rho, S)$ the Hilbert space consisting of V_ρ-valued measurable functions φ on $H_{n,m}$ such that

$$|\varphi|^2 = \int_{H_{n,m}} (\rho(Im\,Z)\,\varphi(Z,W), \varphi(Z,W))\ \kappa_S(Z,W)\,d(Z,W),$$

where

$$d(Z,W) = (\det Y)^{-(m+n+1)}dX\,dY\,dU\,dV, \quad Z = X+iY,\ W = U+iV$$

denotes a G^J-invariant volume element on $H_{n,m}$. The induced representation $\mathrm{Ind}_{K^J}^{G^J}(\rho\otimes\bar{\chi}_S)$ is realized on $E(\rho,S)$ as follows: For any $g \in G^J$ and $\varphi \in E(\rho,S)$, we have

$$\left(\mathrm{Ind}_{K^J}^{G^J}(\rho\otimes\bar{\chi}_S)(g)\varphi\right)(Z,W) = J_{\rho,S}(g^{-1},(Z,W))^{-1}\varphi(g^{-1}\cdot(Z,W)).$$

We recall that χ_S is the unitary character of A defined by $\chi_S(\kappa) = e^{2\pi i\sigma(S\kappa)}$, $\kappa \in \mathcal{Z}$. Let $H(\rho,S)$ be the subspace of $E(\rho,S)$ consisting of $\varphi \in E(\rho,S)$ which is holomorphic on $H_{n,m}$. Then $H(\rho,S)$ is a closed G^J-invariant subspace of $E(\rho,S)$. Let $\pi^{\rho,S}$ be the restriction of the induced representation $\mathrm{Ind}_{K^J}^{G^J}(\rho\otimes\bar{\chi}_S)$ to $H(\rho,S)$.

Takase (cf. [83], Theorem 1.1) proved the following

Theorem 9.15. *Suppose $l_n > n + \frac{m}{2}$. Then $H(\rho,S) \neq 0$ and $\pi^{\rho,S}$ is an irreducible unitary representation of G^J which is square integrable modulo $\mathcal{Z}$. The multiplicity of ρ_l in $\pi^{\rho,S}|_K$ is equal to one.*

We put

$$K_2 = p^{-1}(K) = \left\{ (k,\epsilon) \in K \times \mathbb{C}_1 \,\middle|\, \epsilon^2 = \det J(k, iE_n) \right\}.$$

The Lie algebra $\mathfrak{k}$ of K_2 and its Cartan algebra are given by

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \,\middle|\, A + {}^tA = 0,\ B = {}^tB \right\}$$

and

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \,\middle|\, C = \operatorname{diag}(c_1, c_2, \cdots, c_n) \right\}.$$

Here $\operatorname{diag}(c_1, c_2, \cdots, c_n)$ denotes the diagonal matrix of degree n. We define $\lambda_j \in \mathfrak{h}_{\mathbb{C}}^*$ by $\lambda_j \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix} := \sqrt{-1}c_j$. We put

$$M^+ = \left\{ \sum_{j=1}^n m_j\lambda_j \,\middle|\, m_j \in \frac{1}{2}\mathbb{Z},\ m_1 \geq \cdots \geq m_n,\ m_i - m_j \in \mathbb{Z} \text{ for all } i,j \right\}.$$

We take an element $\lambda = \sum_{j=1}^n m_j\lambda_j \in M^+$. Let ρ be an irreducible representation of K with highest weight $l = (l_1, \cdots, l_n) \in \mathbb{Z}^n$, where $l_j = m_j - m_n \ (1 \leq j \leq n-1)$. Let $\rho_{[\lambda]}$ be the irreducible representation of K_2 defined by

$$\rho_{[\lambda]}(k,\epsilon) = \epsilon^{2m_n} \cdot \rho(J(k, iE_n)), \quad (k,\epsilon) \in K_2. \tag{9.49}$$

Then $\rho_{[\lambda]}$ is the irreducible representation of K_2 with highest weight $\lambda = (m_1, \cdots, m_n)$ and $\lambda \longmapsto \rho_{[\lambda]}$ is a bijection from M^+ to $\hat{K}_2$, the unitary dual of K_2.

The following proposition is a special case of [44], Theorem 7.2.

Proposition 9.16. *We have an irreducible decomposition*

$$\omega_S \Big|_{K_2} = \oplus_\lambda\, m(\lambda)\rho_{[\lambda]},$$

where λ runs over

$$\lambda = \sum_{j=1}^{\nu} l_j\lambda_j + \frac{m}{2}\sum_{j=1}^{n}\lambda_j \in M^+ \quad (\nu = min\{m,n\}),$$

$$\lambda_j \in \mathbb{Z} \ \textit{such that}\ l_1 \geq l_2 \cdots \geq l_\nu \geq 0$$

and the multiplicity $m(\lambda)$ is given by

$$m(\lambda) = \prod_{1 \leq i < j \leq m} \left(1 + \frac{l_i - l_j}{j - i} \right),$$

where $l_j = 0$ if $j > \nu$. Let $\hat{G}_{2,d}$ be the set of all the unitary equivalence classes of square integrable irreducible unitary representations of G_2. The correspondence

$$\pi \longmapsto \textit{Harish-Chandra parameter of } \pi$$

is a bijection from $\hat{G}_{2,d}$ to Λ^+, where

$$\Lambda^+ = \left\{ \sum_{j=1}^{n} m_j \lambda_j \in M^+ \;\middle|\; m_1 > \cdots > m_n,\; m_i - m_j \neq 0 \textit{ for all } i, j,\; i \neq j \right\}.$$

See [95], Theorem 10.2.4.1 for the details.

We take an element $\lambda = \sum_{j=1}^{n} m_j \lambda_j \in M^+$. Let $\pi^\lambda \in \hat{G}_{2,d}$ be the representation corresponding to the Harish-Chandra parameter

$$\sum_{j=1}^{n} (m_j - j)\lambda_j \in \Lambda^+.$$

The representation π^λ is realized as follows (see [56], Theorem 6.6): Let (ρ, V_ρ) be the irreducible representation of K with highest weight $l = (l_1, \cdots, l_n)$, $l_i = m_i - m_n$ ($1 \leq j \leq n$). Let H^λ be a complex Hilbert space consisting of the V_ρ-valued holomorphic functions φ on H_n such that

$$|\varphi|^2 = \int_{H_n} (\rho(\operatorname{Im} Z)\,\varphi(Z), \varphi(Z)) \cdot (\det \operatorname{Im} Z)^{m_n}\, dZ < +\infty,$$

where dZ is the usual G_2-invariant measure on H_n. Then π^λ is defined by

$$\left(\pi^\lambda(g)\varphi\right)(Z) = J_\lambda(g^{-1}, Z)^{-1} \varphi(g^{-1} < Z >)$$

for all $g = (\sigma, \epsilon) \in G_2$ and $\varphi \in H^\lambda$. Here

$$J_\lambda(g, Z) = \rho(J(\sigma, Z)) \cdot J_{\frac{1}{2}}(g, Z)^{m_n},$$

where

$$J_{\frac{1}{2}}(g, Z) = \frac{\gamma(\sigma < Z >, \sigma < iE_n >)}{\gamma(Z, iE_n)} \cdot \beta(\sigma, \sigma^{-1}) \cdot \epsilon \cdot |det\, J(\sigma, Z)|^{\frac{1}{2}}\ .$$

Proposition 9.17. *Suppose $l_n > n + \frac{m}{2}$. We put $\lambda = \sum_{j=1}^{n} (l_j - \frac{m}{2})\lambda_j \in M^+$. Then $\pi^{\rho,S}$ is an irreducible unitary representation of G^J and we have a unitary equivalence*

$$(\pi^\lambda \circ q^J) \otimes \omega_S \longrightarrow \pi^{\rho,S} \circ p^J$$

via the intertwining operator $\Lambda_{\rho,S} : H^\lambda \otimes H_{S,iE_n} \longrightarrow H(\rho, S)$ defined by

$$(\Lambda_{\rho,S}(\varphi \otimes \psi))\,(Z, W) = (\,det 2S\,)^n (\,det\, Im\, Z\,)^{-\frac{m}{4}} \varphi(Z)(U^S_{Z,iE_n}\psi)(W)$$

for all $\varphi \in H^\lambda$ *and* $\psi \in H_{S,iE_n}$.

9.6. Duality Theorem for G^J

In this subsection, we state the duality theorem for the Jacobi group G^J.

Let E_{ij} denote a square matrix of degree $2n$ with entry 1 where the i-th row and the j-th column meet, all other entries being 0. We put

$$H_i = E_{ii} - E_{n+i,n+i}\,(1 \le i \le n), \quad \mathfrak{h} = \sum_{i=1}^{n} \mathbb{C} H_i.$$

Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $e_j : \mathfrak{h} \longrightarrow \mathbb{C}\,(1 \le j \le n)$ be the linear form on $\mathfrak{h}$ defined by

$$e_j(H_i) = \delta_{ij},$$

where δ_{ij} denotes the Kronecker delta symbol. The roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ are given by

$$\pm 2e_i \ (1 \le i \le n), \ \pm e_k \pm e_l \,(1 \le k < l \le n).$$

The set Φ^+ of positive roots is given by

$$\Phi^+ = \{2e_i\,(1 \le i \le n),\ e_k + e_l\,(\,1 \le k < l \le n\,)\}.$$

Let

$$\mathfrak{g}_\alpha = \{\, X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\,\}$$

be the root space corresponding to a root α of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We put $\mathfrak{n} = \sum_{\Phi^+} \mathfrak{g}_\alpha$. We define

$$N^J = \left\{ [\exp X, (0, \mu, 0)] \in G^J \,\Big|\, X \in \mathfrak{n} \right\},$$

where $\exp : \mathfrak{g} \longrightarrow G$ denotes the exponential mapping from $\mathfrak{g}$ to G. A subgroup N^g of G^J is said to be *horosherical* if it is conjugate to N^J, that is, $N^g = gN^Jg^{-1}$ for some $g \in G$. A horospherical subgroup N^g is said to be *cuspidal* for $\Gamma^J = \Gamma_n \ltimes H_{\mathbb{Z}}^{(n,m)}$ in G^J if $(N^g \cap \Gamma^J)\backslash N^g$ is compact. Let $L^2(\Gamma^J\backslash G^J, \rho)$ be the complex Hilbert space consisting of all Γ^J-invariant V_ρ-valued measurable functions Φ on G^J such that $||\Phi|| < \infty$, where $||\ \ ||$ is the norm induced from the norm $|\ \ |$ on $E(\rho, \mathcal{M})$ by the lifting from $H_{n,m}$ to G^J. We denote by $L_0^2(\Gamma^J\backslash G^J, \rho)$ the subspace of $L^2(\Gamma^J\backslash G^J, \rho)$ consisting of functions φ on G^J such that $\varphi \in L^2(\Gamma^J\backslash G^J, \rho)$ and

$$\int_{N^g \cap \Gamma^J \backslash N^g} \varphi(ng_0)dn = 0$$

for any cuspidal subgroup N^g of G^J and almost all $g_0 \in G^J$. Let R be the right regular representation of G^J on $L_0^2(\Gamma^J\backslash G^J, \rho)$.

Now we state the duality theorem for the Jacobi group G^J.

Duality Theorem. *Let ρ be an irreducible representation of K with highest weight $l = (l_1, \cdots, l_n) \in \mathbb{Z}^n$, $l_1 \geq l_2 \geq \cdots \geq l_n$. Suppose $l_n > n + \frac{1}{2}$ and let $\mathcal{M}$ be a half integrable positive definite symmetric matrix of degree m. Then the multiplicity $m_{\rho,\mathcal{M}}$ of $\pi^{\rho,\mathcal{M}}$ in the right regular representation R of G^J in $L_0^2(\Gamma^J \backslash G^J, \rho)$ is equal to the dimension of $J_{\rho,\mathcal{M}}^{cusp}(\Gamma_n)$, that is,*

$$m_{\rho,\mathcal{M}} = \dim_{\mathbb{C}} J_{\rho,\mathcal{M}}^{\mathrm{cusp}}(\Gamma_n).$$

We may prove the above theorem following the argument of [10] in the case $m = n = 1$. So we omit the detail of the proof.

9.7. Coadjoint Orbits for the Jacobi Group G^J

We observe that the Jacobi group G^J is embedded in $Sp(n+m, \mathbb{R})$ via

$$(M, (\lambda, \mu, \kappa)) \mapsto \begin{pmatrix} A & 0 & B & A^t\mu - B^t\lambda \\ \lambda & E_m & \mu & \kappa \\ C & 0 & D & C^t\mu - D^t\lambda \\ 0 & 0 & 0 & E_m \end{pmatrix}, \tag{9.50}$$

where $(M, (\lambda, \mu, \kappa)) \in G^J$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$. The Lie algebra $\mathfrak{g}^J$ of G^J is given by

$$\mathfrak{g}^J = \left\{ (X, (P, Q, R)) \mid X \in \mathfrak{g},\ P, Q \in \mathbb{R}^{(m,n)},\ R = {}^tR \in \mathbb{R}^{(m,m)} \right\} \tag{9.51}$$

with the bracket

$$[(X_1, (P_1, Q_1, R_1)), (X_2, (P_2, Q_2, R_2))] = (\tilde{X}, (\tilde{P}, \tilde{Q}, \tilde{R})), \tag{9.52}$$

where

$$X_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & -{}^ta_1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & -{}^ta_2 \end{pmatrix} \in \mathfrak{g}$$

and

$$\begin{aligned} \tilde{X} &= X_1X_2 - X_2X_1, \\ \tilde{P} &= P_1a_2 + Q_1c_2 - P_2a_1 - Q_2c_1, \\ \tilde{Q} &= P_1b_2 - Q_1\,{}^ta_2 - P_2b_1 + Q_2\,{}^ta_1, \\ \tilde{R} &= P_1\,{}^tQ_2 - Q_1\,{}^tP_2 - P_2\,{}^tQ_1 + Q_2\,{}^tP_1. \end{aligned}$$

Indeed, an element $(X, (P, Q, R))$ in $\mathfrak{g}^J$ with $X = \begin{pmatrix} a & b \\ c & -{}^ta \end{pmatrix} \in \mathfrak{g}$ may be identified with the matrix

$$\begin{pmatrix} a & 0 & b & {}^tQ \\ P & 0 & Q & R \\ c & 0 & -{}^ta & -{}^tP \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad b = {}^tb,\ c = {}^tc,\ R = {}^tR \tag{9.53}$$

in $\mathfrak{sp}(n+m,\mathbb{R})$.

Let us identify $\mathfrak{g}_{n+m} := \mathfrak{sp}(n+m,\mathbb{R})$ with its dual $\mathfrak{g}^*_{n+m}$ (see Proposition 6.1.3. (6.5)). In fact, there exists a G-equivariant linear isomorphism

$$\mathfrak{g}^*_{n+m} \longrightarrow \mathfrak{g}_{n+m}, \quad \lambda \mapsto X_\lambda$$

characterized by

$$\lambda(Y) = \mathrm{tr}(X_\lambda Y), \quad Y \in \mathfrak{g}_{n+m}. \tag{9.54}$$

Then the dual $(\mathfrak{g}^J)^*$ of $\mathfrak{g}^J$ consists of matrices of the form

$$\begin{pmatrix} x & p & y & 0 \\ 0 & 0 & 0 & 0 \\ z & q & -\,{}^t x & 0 \\ {}^t q & r & -{}^t p & 0 \end{pmatrix}, \quad y = {}^t y,\ z = {}^t z,\ r = {}^t r. \tag{9.55}$$

There is a family of coadjoint orbits Ω_δ which have the minimal dimension $2n$, depending on a nonsingular $m \times m$ real symmetric matrix parameter δ and are defined by the equation

$$\delta = r, \quad XJ_n = \begin{pmatrix} p \\ q \end{pmatrix} \delta^{-1}\, {}^t\!\begin{pmatrix} p \\ q \end{pmatrix}, \tag{9.56}$$

where $X = \begin{pmatrix} x & y \\ z & -\,{}^t x \end{pmatrix}$ with $y = {}^t y$ and $z = {}^t z$ in (9.55). Let us denote by $\mathfrak{h}_{n,m}$ the Lie algebra of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$. Then the family Ω_δ ($\delta = {}^t\delta,\ \delta \in GL(m,\mathbb{R})$) have the following properties ($\Omega1$)-($\Omega2$):

($\Omega1$) Under the natural projection on $\mathfrak{h}^*_{n,m}$, the orbit Ω_δ goes to the orbit which corresponds to the irreducible unitary representation $U(\delta)$ of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$, namely, the Schrödinger representation of $H_{\mathbb{R}}^{(n,m)}$ (cf. (8.19)).

($\Omega2$) Under the projection on $\mathfrak{g}^* = \mathfrak{sp}(n,\mathbb{R})^*$, the orbit Ω_δ goes to $\Omega_{\mathrm{sign}(\det(\delta))}$.

In fact, there is an irreducible unitary representation π_δ ($\delta = {}^t\delta,\ \delta \in GL(m,\mathbb{R})$) of G^J (or its universal cover) with properties

$$\mathrm{Res}^{G^J}_{H_{\mathbb{R}}^{(n,m)}} \pi_\delta \cong U(\delta), \quad \mathrm{Res}^{G^J}_{G}\, \pi_\delta \cong \pi_{\mathrm{sign}(\det(\delta))}, \tag{9.57}$$

where $\pi_\pm$ are some representations of G (or its universal cover) corresponding to the minimal orbits $\Omega_\pm \subset \mathfrak{g}^*$. Indeed, $\pi_\pm$ are two irreducible components of the Weil representation of G and π_δ is one of the irreducible components of the Weil representation of G^J (cf. (9.47)). These are special cases of the so-called *unipotent* representations of G^J. We refer to [90]-[91], [93] for a more detail on unipotent representations of a reductive Lie group.

Now we consider the case $m = n = 1$. If

$$g^{-1} = \begin{pmatrix} a & 0 & b & a\mu - b\lambda \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & c\mu - d\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is an element of the Jacobi group G^J, then its inverse is given by

$$g = \begin{pmatrix} d & 0 & -b & -\mu \\ c\mu - d\lambda & 1 & \lambda b - \mu a & -\kappa \\ -c & 0 & a & \lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We put

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then according to (9.55), X, Y, Z, P, Q, R form a basis for $(\mathfrak{g}^J)^*$. By an easy computation, we see that the coadjoint orbits Ω_X, Ω_Y, Ω_Z, Ω_P, Ω_Q, Ω_R of X, Y, Z, P, Q, R respectively are given by

$$\Omega_X = \left\{ \begin{pmatrix} ad+bc & 0 & -2ab & 0 \\ 0 & 0 & 0 & 0 \\ 2cd & 0 & -(ad+bc) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; ad - bc = 1,\ a, b, c, d \in \mathbb{R} \right\},$$

$$\Omega_Y = \left\{ \begin{pmatrix} bd-ac & 0 & a^2-b^2 & 0 \\ 0 & 0 & 0 & 0 \\ d^2-c^2 & 0 & ac-bd & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; ad - bc = 1,\ a, b, c, d \in \mathbb{R} \right\},$$

$$\Omega_Z = \left\{ \begin{pmatrix} -(ac+bd) & 0 & a^2+b^2 & 0 \\ 0 & 0 & 0 & 0 \\ -(c^2+d^2) & 0 & ac+bd & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; ad - bc = 1,\ a, b, c, d \in \mathbb{R} \right\},$$

$$\Omega_P = \left\{ \begin{pmatrix} (2ad-1)\lambda - 2ac\mu & a & 2ab\lambda - 2a^2\mu & 0 \\ 0 & 0 & 0 & 0 \\ 2c^2\mu - 2cd\lambda & c & (1-2ad)\lambda + 2ac\mu & 0 \\ c & 0 & -a & 0 \end{pmatrix} \;\middle|\; \begin{matrix} ad - bc = 1, \\ a, b, c, d, \lambda, \mu \in \mathbb{R} \end{matrix} \right\},$$

$$\Omega_Q = \left\{ \begin{pmatrix} (2ad-1)\mu - 2bd\lambda & b & 2b^2\lambda - 2ab\mu & 0 \\ 0 & 0 & 0 & 0 \\ 2cd\mu - 2d^2\lambda & d & (1-2ad)\mu + 2bd\lambda & 0 \\ d & 0 & -b & 0 \end{pmatrix} \;\middle|\; \begin{matrix} ad-bc=1, \\ a,b,c,d,\lambda,\mu \in \mathbb{R} \end{matrix} \right\}$$

and

$$\Omega_R = \left\{ \begin{pmatrix} (a\mu - b\lambda)(c\mu - d\lambda) & a\mu - b\lambda & -(a\mu - b\lambda)^2 & 0 \\ 0 & 0 & 0 & 0 \\ (c\mu - d\lambda)^2 & c\mu - d\lambda & -(a\mu - b\lambda)(c\mu - d\lambda) & 0 \\ c\mu - d\lambda & 1 & b\lambda - a\mu & 0 \end{pmatrix} \;\middle|\; \begin{matrix} ad-bc=1, \\ a,b,c,d,\lambda,\mu \in \mathbb{R} \end{matrix} \right\}.$$

Moreover we put

$$S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the coadjoint orbits Ω_S and Ω_T of S and T are given by

$$\Omega_S = \left\{ \begin{pmatrix} -ab & 0 & a^2 & 0 \\ 0 & 0 & 0 & 0 \\ -b^2 & 0 & ab & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; a,b \in \mathbb{R} \right\}$$

and

$$\Omega_T = \left\{ \begin{pmatrix} ab & 0 & -a^2 & 0 \\ 0 & 0 & 0 & 0 \\ b^2 & 0 & -ab & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; a,b \in \mathbb{R} \right\}.$$

For an element of $(\mathfrak{g}^J)^*$, we write

$$\begin{pmatrix} x & p & y+z & 0 \\ 0 & 0 & 0 & 0 \\ y-z & q & -x & 0 \\ q & r & -p & 0 \end{pmatrix} = xX + yY + zZ + pP + qQ + rR. \tag{9.58}$$

The coadjoint orbit Ω_X is represented by the one-sheeted hyperboloid

$$x^2 + y^2 - z^2 = 1 > 0, \quad p = q = r = 0. \tag{9.59}$$

The coadjoint orbit Ω_Y is also represented by the one-sheeted hyperboloid (9.59). The coadjoint orbit Ω_Z is represented by the two-sheeted hyperboloids

$$x^2 + y^2 = z^2 - 1 > 0, \quad p = q = r = 0. \tag{9.60}$$

The coadjoint G^J-orbit Ω_S of S is represented by the the cone

$$(9.61) \qquad x^2+y^2=z^2>0, \quad z>0, \quad p=q=r=0.$$

On the other hand, the coadjoint G^J-orbit Ω_T of T is represented by the cone

$$(9.62) \qquad x^2+y^2=z^2>0, \quad z<0, \quad p=q=r=0.$$

The coadjoint orbit Ω_P is represented by the variety

$$(9.63) \qquad 2pqx+(q^2-p^2)y+(p^2+q^2)z=0, \quad (p,q)\in\mathbb{R}^2-\{(0,0)\}, \quad r=0$$

in $\mathbb{R}^6$. The coadjoint orbit orbit Ω_Q is represented by the variety (9.63) in $\mathbb{R}^6$. particular, we are interested in the coadjoint orbits orbits $\Omega_{hR}\,(h\in\mathbb{R},\ h\neq 0)$ of hR which are represented by

$$(9.64) \qquad x^2+y^2=z^2,\ x=h^{-1}pq,\ y+z=-h^{-1}p^2,\ y-z=h^{-1}q^2 \text{ and } r=h.$$

For a fixed $h\neq 0$, we note that Ω_{hR} is two dimensional and satisfies the equation (9.56). Indeed, from the above expression of Ω_{hR} and (9.58), we have $X=\begin{pmatrix} x & y+z \\ y-z & -x\end{pmatrix}$ and

$$\begin{aligned} x&=h(a\mu-b\lambda)(c\mu-d\lambda),\\ y+z&=-h(a\mu-b\lambda)^2,\\ y-z&=h(c\mu-d\lambda)^2,\\ p&=h(a\mu-b\lambda), \quad q=h(c\mu-d\lambda), \quad r=h. \end{aligned}$$

Hence these satisfy the equation (9.56). An irreducible unitary representation π_h that corresponds to a coadjoint orbit Ω_{hR} satisfies the properties (9.57). In fact, π_h is one of the irreducible components of the so-called (Schrödinger-)Weil representation of G^J (cf. (9.47)). A coadjoint orbit $\Omega_{mR+\alpha X}$ or $\Omega_{mR+\alpha Y}\,(m\in\mathbb{R}^\times,\ \alpha\in\mathbb{R})$ is corresponded to a principal series $\pi_{m,\alpha,\frac{1}{2}}$, the coadjoint orbit $\Omega_{mR+kZ}\,(m\in\mathbb{R}^\times,\ k\in\mathbb{Z}^+)$ of $mR+kZ$ is attached to the discrete series $\pi^{\pm}_{m,k}$ of G^J. There are no coadjoint G^J-orbits which correspond to the complimentary series $\pi_{m,\alpha,\nu}\,(m\in\mathbb{R}^\times,\ \alpha\in\mathbb{R},\ \alpha^2<\frac{1}{2},\ \nu=\pm\frac{1}{2})$. See [11], pp. 47-48. There are no unitary representations of G^J corresponding to the G^J-orbits of $\alpha P_*+\beta Q_*$ with $(\alpha,\beta)\neq(0,0)$.

Finally we mention that the coadjoint orbit $\Omega_{mR+\alpha X}$ or $\Omega_{mR+\alpha Y}\,(m\in\mathbb{R}^\times,\ \alpha\in\mathbb{R})$ is characterized by the variety

$$(9.65) \qquad x^2+y^2-(z^2+\alpha^2)=\frac{2}{m}pqx+\frac{1}{m}(q^2-p^2)y+\frac{1}{m}(p^2+q^2)z, \quad r=m.$$

and the coadjoint orbit $\Omega_{mR+kZ}(m\in\mathbb{R}^\times,k\in\mathbb{Z}^+)$ of $mR+kZ$ is represented by the variety

$$(9.66) \quad x^2+y^2-(z^2-k^2)=\frac{2}{m}pqx+\frac{1}{m}(q^2-p^2)y+\frac{1}{m}(p^2+q^2)z,\ z>0,\ r=m.$$

or

$$(9.67)\quad x^2 + y^2 - (z^2 - k^2) = \frac{2}{m}pqx + \frac{1}{m}(q^2 - p^2)y + \frac{1}{m}(p^2 + q^2)z,\ z < 0,\ r = m$$

depending on the sign $\pm$.

References

[1] Arthur J., *Unipotent Automorphic Representation,: Conjectures*, Société France, Astérique, **171-172**(1989), 13-71.

[2] Arnold V.I. and Givental A.B., Symplectic geometry, Encyclopaedia of Mathematical Sciences, Dynamical systems IV, Springer,**4**(1990), 1-136.

[3] Auslander L. and Kostant B.,*Polarization and unitary representations of solvable Lie groups* , Invent. Math., **14**(1971), 255-354.

[4] Bargmann V., *Irreducible unitary representations of the Lorentz group*, Ann. of Math., **48**(1947), 568-640.

[5] Bernat P., Conze N., Duflo M. et al., Représentations des groupes de Lie résolubles, Dunod Paris, (1972), 1-272.

[6] Berndt R., *Die Jacobigruppe und die Wärmeleitungsgleichung*, Math. Z., **191**(1986), 351-361.

[7] Berndt R., *Some remarks on automorphic forms for the Jacobi group*, IHES/M, (1989).

[8] Berndt R., *The Continuous Part of $L^2(\Gamma^J\backslash G^J)$ for the Jacobi Group*, Abh. Math. Sem. Univ. Hamburg., **60**(1990), 225-248.

[9] Berndt R., *On Automorphic Forms for the Jacobi Group*, Jb. d. Dt. Math.-Verein., **97**(1995), 1-18.

[10] Berndt R. and Böcherer S., *Jacobi Forms and Discrete Series Representations of the Jacobi Group*, Math. Z., **204**(1990), 13-44.

[11] Berndt R. and Schmidt R., Elements of the Representation Theory of the Jacobi Group, Birkhäuser, **163**(1998).

[12] Borel A. and Jacquet H., *Automorphic forms and automorphic representations*, Proc. Symposia in Pure Math., **XXXIII, Part 1**(1979), 189-202.

[13] Brylinski, R. and Kostant, B., *Minimal representations of E_6, and E_8 and the generalized Capelli identity*, Proc. Nat. Acad. Sci. U.S.A., **91, No.7**(1994), 2469-2472.

[14] Brylinski, R. and Kostant, B., *Minimal representations, geometric quantization, and unitarity*, Proc. Nat. Acad. Sci. U.S.A., **91, No.13**(1994), 6026-6029.

[15] MathbbRown I.D., *Dual topology of a nilpotent Lie group*, Ann. Sci. École Sup., (1973), 407-411.

[16] Bump D., Automorphic Forms and Representations, Cambridge University Press, (1997).

[17] Busyatskaya I.K., *Representations of exponential Lie groups*, Funct. Anal. and Appl., **7, No.2**(1973), 151-152.

[18] Brylinski R.K. and Kostant B., *Minimal representations, geometric quantization and unitarity*, Proc. Natl. Acad. Sci. USA, **91**(1994), 6026-6029.

[19] Dixmier J., *Sur les représentation unitaires des groupes de Lie nilpotents, III* , Canad. J. Math., **10**(1958), 321-348.

[20] Dixmier J., Enveloping algebras, North Holland, Amsterdam, (1977).

[21] Dooley A.H. and Wildberger N.J., *Harmonic analysis and the global exponential map for compact Lie groups*, Funct. Anal. and Appl., **27**(1993), 25-32.

[22] Duflo M., *Fundamental-series representations of a semisimple Lie group*, Funct. Anal. and Appl., **4**(1970), 122-126.

[23] Duflo M., *Opérateurs différentiels bi-invariants sur un groupe de Lie*, Ann. Sci. École Norm. Sup., **IV 10 Ser. 3**(1977), 265-283.

[24] Duflo M., *Représentations unitaires irréducibles des groupes semi-simples complexes de rang deux*, Bull. Soc. Math. Fr., **107**(1979), 55-96.

[25] Duflo M., *Théorie de Mackey pour les groupes de Lie algèbriques*, Acta Math., **149**(1982), 153-213.

[26] M. Eichler and D. Zagier, The Theory of Jacobi Forms, Progress in Math., Birkhäuser, Boston-Basel- Stuttgart, **55**(1985).

[27] Enright T., Howe R. and Wallach N., *A Classification of Unitary Highest weight Modules*, in: Representation Theory of Reductive Groups, Birkhäuser, Boston (1983), 97-144.

[28] Faltings G. and Chai C.-L., Degeneration of Abelian Varieties, EMG, Band, Springer-Verlag, New York/Berlin, **22**(1990).

[29] Fell J.M.G., *Weak containment and induced representations of groups*, Canad. J. Math., **14**(1962), 237-268.

[30] Ginzburg V.A., *The method of orbits in the representation theory of complex Lie groups*, Funct. Anal. and Appl., **15**(1981), 18-28.

[31] Ginzburg V.A., $\mathfrak{g}$*-modules, Springer's representations and bivariant Chern classes*, Adv. in Math., **61**(1986) , 1-48.

[32] Gross B. and Wallach N., *A distinguished family of unitary representations for the exceptional groups of real rank=4*, in: Lie theory and geometry, in honor of B. Kostant, Birkhäuser, (1994), 289-304.

[33] Guillemin V. and Sternberg S., *Quantization and multiplicities of group representations*, Invent. Math., **67**(1982), 515-538.

[34] Guillemin V. and Sternberg S., Symplectic techniques in physics, Cambridge Univ. Press, (1994), MR 86f: 58054

[35] Harish-Chandra, *Discrete series for semisimple Lie groups II*, Acta Math., **116**(1966), 1-111.

[36] Heckman G.J., *Projections of orbits and asymptotic behavior of multiplicities for compact connected Lie groups*,Invent. Math., **67**(1983), 333-356.

[37] Howe R., *On Frobenius reciprocity for unipotent algebraic groups over* $\mathbb{Q}$, American Jour. Math., **93**(1971), 163-172.

[38] Howe R., *θ series and invariant theory* , Automorphic forms, representations, and L-functions, Proc. Symp. Pure Math., **33**(1979), 275-285.

[39] Howe R., *Reciprocity laws in the theory of dual pairs*, Representation Theory of Reductive Groups (P. Trombi, ed.), Progress in Math., Birkhäuser, Boston, **40**(1983).

[40] Howe R., *Transcending Classical Invariant Theory* , Journal American Math. Soc., **2**(1989), 535-552.

[41] Huang J.-S. and Li J.-S., *Unipotent representations attached to spherical nilpotent orbits* , Amer. Jour. Math., **121**(1999), 497-517.

[42] Huang J.-S., Pandzic P. and Savin G., *New dual pair correspondences*, Duke Math. J., **82**(1996), 447-471.

[43] Joseph A., *The minimal orbit in a simple simple Lie algemathbbRa and its associated matrix ideal* , Ann. Scient. Ec. Norm. Sup., **4**(1976), 1-30.

[44] Kashiwara M. and Vergne M., *On the Segal-Shale-Weil representations and harmonic polynomials*, Invent. Math., **44**(1978), 1-47.

[45] Kashiwara M. and Vergne M., *The Campbell-Hausdorff formula and invariant hyperfunctions*, Invent. Math., **47**(1978), 249-272.

[46] Kazhdan D. and Savin G., *The smallest representation of simply laced groups*, Israel Math. Conf. Proceedings, **2**(1990), 209-223.

[47] Kirillov A.A., *Unitary representations of nilpotent Lie groups*, Russian Math. Surveys, **17**(1962), 57-110.

[48] Kirillov A.A., Elements of the theory of representations, Springer, (1976), (Russian editions 1972, 1978).

[49] Kirillov A.A., Introduction to the theory of representations and noncommutative harmonic analysis, Encyclopaedia of Mathematical Sciences, Representation theory and noncommutative harmonic analysis I, Springer, **22**(1994), 1-156, CMP95: 06.

[50] Kirillov A.A., (Adams J., Kleppner A., Li J.-S., Lipsman R., Rosenberg J. , eds), Introduction to the orbit method. I, II., Progress in Math., **145**(1992).

[51] Kirillov A.A., *Geometric quantization*, Encyclopaedia of Mathematical Sciences, Dynamical systems, Springer, **4**(1990) (1997, the second edition), 137-172.

[52] Kirillov A.A., *Variations on the triangular theme*, AMS Translations, AMS, **169**(1995), 43-74.

[53] Kirillov A.A., *The characters of unitary representations of Lie groups*, Funct. Anal. and Appl., **2, No.2**(1968), 133-146.

[54] Kirillov A.A., *Geometric approach to Discrete Series of Unirreps for Vir*, Comm. Pures Appl. Math., **77**(1998), 735-746.

[55] Kirillov A.A., *Merits and Demerits of the Orbit Method*, Bull. AMS, **36, No.4**(1999), 433-488.

[56] Knapp, A.W., *Representation Theory of Semisimple Groups*, Princeton Univ. Press, Princeton, (1986).

[57] Kostant B., *Quantization and unitary representations*, Lecture Notes in Math., Springer, **170**(1970), 87-208.

[58] Kostant B., *Graded manifolds, graded Lie theory and prequantization*, Lecture Notes in Math., Springer, **570**(1970), 177-306.

[59] Kostant B., *On the existence and irreducibility of certain series of representations*, Lie groups and their representations, Proceedings of the Summer School of the Bolyai Janos Math. Soc., I.M. Gelfand, Editor, John Wiley and Sons, (1975), 231-331.

[60] Kostant B., *The solution to a generalized Toda lattice and representation theory*, Adv. in Math., **34**(1979), 195-338.

[61] Kramer J., *An arithmetic theory of Jacobi forms in higher dimensions*, J. reine angew. Math., **458**(1995), 157-182.

[62] Lang S., $SL_2(R)$, Springer-Verlag, (1985).

[63] Leptin H. and Ludwig J., *Unitary representation theory of exponential Lie groups*, Walter de Gruyter & Co., (1994), X + 200.

[64] Li J.-S., *The correspondence of infinitesimal characters for reductive dual pairs in simple Lie groups*, Duke Math. J., **97**(1999), 347-377.

[65] Li J.-S., Minimal Representations and Reductive Dual Pairs in Representation Theory of Lie Groups, IAS/PARK CITY, Math. Series, AMS and IAS, **8**(2000), 179-238.

[66] Loke H.Y., *Dual pairs correspondences of* $E_{8,4}$ *and* $E_{7,4}$, Thesis, Harvard University, (1997).

[67] Mackey G.W., *Induced Representations of Locally Compact Groups I*, Ann. of Math., **55**(1952), 101-139.

[68] Moore C.C.paper, *Decomposition of unitary representations defined by a discrete subgroup of nilpotent groups*, Ann. of Math., **82**(1965), 146-182.

[69] Namikawa Y., *Toroidal Compactification of Siegel Spaces*, Lect. Notes in Math., Springer-Verlag, New York /Berlin, **812**(1980).

[70] Pukanszky L., *On the theory of exponential groups*, Trans. Amer. Math. Soc., **126** (1967), 487-507.

[71] Pukanszky L., *Unitary representations of solvable Lie groups*, Ann. Sci. Ec. Norm. Sup., **IV, Ser. 4**(1971), 457-608.

[72] Rossmann W., *Kirillov's character formula for reductive Lie groups*, Invent. Math., **48**(1978), 207-220.

[73] Runge B., *Theta functions and Siegel-Jacobi functions*, Acta Math., **175**(1995), 165-196.

[74] I. Satake, *Fock Representations and Theta Functions*, Ann. Math. Studies, **66**(1971), 393-405.

[75] I. Satake, *Unitary representations of a semi-direct products of Lie groups on* $\bar{\partial}$*-cohomology spaces*, Math. Ann., **190**(1971), 177-202.

[76] Schmid W. and Vilonen K., *On the geometry of nilpotent orbits*, Asian J. Math., **3, No.1**(1999), 233-274.

[77] Schmid W. and Vilonen K., *Characteristic cycles and wave front cycles of representations of reductive Lie groups*, Ann. of Math.(2), **3, No.3**(2000), 1071-1118.

[78] Schwartz J., *The determination of the admissible orbits in the real classical groups*, Ph.D. dissertation, MIT.

[79] Sekiguchi J., *Remarks on nilpotent orbits of a symmetric pair*, Jour. Math. Soc. Japan, **39**(1987), 127-138.

[80] Shchepochkina I.M., *On representations of solvable Lie groups*, Funct. Anal. and Appl., **11**(1977), 93-94.

[81] Shchepochkina I.M, *Orbit method for the restriction-induction problems for normal subgroup of a solvable Lie group*, C.R. de l'Acad. bulgare des Sci., **33**(1980), 1039-1042.

[82] Takase K., *A note on automorphic forms*, J. reine angew. Math., **409**(1990), 138-171.

[83] Takase K., *On unitary representations of Jacobi groups*, J. reine angew. Math., **430**(1992), 130-149.

[84] Takase K., *Correction to the paper "On unitary representations of Jacobi groups"*, J. reine angew. Math., **441**(1993), 215-216.

[85] Torasso P., *Quantification géométrique, opérateurs d'entrelacement et représentations unitaires de* $\tilde{SL}_3(\mathbb{R})$, Acta Math., **150**(1983), 153-242.

[86] Torasso P., *Méthod des orbites de Kirillov-Duflo et représentations minimales des groupes simples sur un corps local de caractéristique nulle*, Duke Math. J., **90**(1997), 261-377.

[87] Vergne M., *Instantons et correspondence de Kostant-Sekiguchi*, C.R. Acad. Sci. Paris, **320**(1995), 901-906.

[88] Vogan D., *Gelfand-Kirillov dimension for Harish-Chandra modules*, Invent. Math., **48**(1978), 75-98.

[89] Vogan D., *Singular unitary representations*, Non-commutative Harmonic Analysis and Lie Groups, Springer Lecture Notes, **880**(1980), 506-534.

[90] Vogan D., *Associated Varieties and Unipotent Representations*, Harmonic Analysis on Reductive Groups, Progr. Math., Birkhäuser, Boston **101**(1991), 315-388.

[91] Vogan D., Unitary representations of reductive Lie groups, Princeton Univ. Press, (1987).

[92] Vogan D., *The unitary dual of* G_2, Invent. Math., **116**(1994), 677-791.

[93] Vogan D., The Methods of Coadjoint Orbits for Real Reductive Groups in Representation Theory of Lie Groups, IAS/PARK CITY, Math. Series, AMS and IAS, **8**(2000), 179-238.

[94] Waldspurger, J.L., *Démonstration d'une conjecture de dualité de Howe dans le cas p-adique,* $p \neq 2$, Festschrift in honor of I.I. Piatetski-Shapiro on the occasion of his sixties birthday, Part 1, Israel Math. Conf. Proc., **2**(1989), 267-324.

[95] G. Warner, Harmonic Analysis on Semisimple Lie Groups, I, II, Springer-Verlag, Berlin-Heidelberg-New York, (1972).

[96] Wildberger N.J., *On a relationship between adjoint orbits and conjugacy classes of a Lie group*, Canad. Math. Bull., **33, No.3**(1990), 297-304.

[97] Yang, J.-H., Represntation Theory of Lie Groups (Korean), Minumsa, (1998).

[98] Yang, J.-H., *Theta Functions and Heisenberg groups*, Pyungsan Institute for Math. Sci., PIMS, **99, No.5**(2000), 1-68.

[99] Yang, J.-H., *Harmonic Analysis on the Quotient Spaces of Heisenberg Groups*, Nagoya Math. J., **123**(1991), 103-117.

[100] Yang, J.-H., *Harmonic Analysis on the Quotient Spaces of Heisenberg Groups, II*, J. Number Theory, **49**(1994), 63-72.

[101] Yang, J.-H., *A decomposition theorem on differential polynomials of theta functions of high level*, Japanese J. Math., Math. Soc. Japan, New Series, **22**(1996), 37-49.

[102] Yang, J.-H., *Lattice representations of* $H_{\mathbb{R}}^{(g,h)}$, Math. Ann., **317**(2000), 309-323.

[103] Yang, J.-H., *The Siegel-Jacobi Operator*, Abh. Math. Sem. Univ. Hamburg, **63**(1993), 135-146.

[104] Yang, J.-H., *Remarks on Jacobi forms of higher degree*, Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, PIMS, (1993), 33-58.

[105] Yang, J.-H., *Singular Jacobi Forms*, Trans. Amer. Math. Soc., **347**(1995), 2041-2049.

[106] Yang, J.-H., *Construction of Vector-Valued Modular Forms from Jacobi Forms*, Canadian J. Math., **47**(1995), 1329-1339.

[107] Yang, J.-H., *A geometrical theory of Jacobi forms of higher degree*, Kyungpook Math. J., **40**(2000), 209-237.

[108] Yang J.-H. and Son J.-W., *Unitary representations of Heisenberg groups*, Proc. of Japan-Korea Joint Seminar, (1997), 143-157.

[109] Zelevinsky A.V., *Representations of finite classical groups*, In: Lecture Notes in Math., Springer, **869**(1981), 1-184.

[110] Ziegler C., *Jacobi Forms of Higher Degree*, Abh. Math. Sem. Univ. Hamburg, **59**(1989), 191-224.

The Weil representations of the Jacobi group

Jae-Hyun Yang

1 Introduction

The Weil representation of a symplectic group was first introduced by A. Weil in his remarkable paper [51] to reformulate Siegel's analytic theory of quadratic forms [43] in group theoretical terms. The Weil representation plays a central role in the study of the transformation behaviors of theta series and has many applications to the theory of automorphic forms (cf. [18, 27, 28, 29, 30, 34, 41, 42]). A Jacobi group is defined to be a semi-direct product of a symplectic group and a Heisenberg group. A Jacobi group is an important object in the framework of quantum mechanics, geometric quantization and optics (cf. [1, 3, 4, 5, 6, 7, 19, 20, 21, 31, 44, 52, 73]). The squeezed states in quantum optics represent a physical realization of the coherent states associated with a Jacobi group (cf. [21, 31, 44, 73]). In this paper, we show that we can construct several types of the Weil representations of a Jacobi group and present their applications to the theory of automorphic forms on a Jacobi group and representation theory of a Jacobi group.

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For a given fixed positive integer n, we let

$$\mathbb{H}_n = \{\, \Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega, \quad \operatorname{Im}\Omega > 0 \}$$

be the Siegel upper half plane of degree n and let

$$Sp(n,\mathbb{R}) = \{ g \in \mathbb{R}^{(2n,2n)} \mid {}^t g J_n g = J_n \,\}$$

be the symplectic group of degree n, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l, ${}^tM$ denotes the transpose of a matrix M, $\operatorname{Im}\Omega$ denotes the imaginary part of Ω and

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Here I_n denotes the identity matrix of degree n. We see that $Sp(n,\mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

$$g \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{1.1}$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $\Omega \in \mathbb{H}_n$.

For two positive integers n and m, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \{\, (\lambda,\mu;\kappa) \mid \lambda,\mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \ \kappa + \mu\,{}^t\lambda \text{ symmetric } \}$$

endowed with the following multiplication law

$$(\lambda,\mu;\kappa)\circ(\lambda',\mu';\kappa') = (\lambda+\lambda',\mu+\mu';\kappa+\kappa'+\lambda\,{}^t\mu' - \mu\,{}^t\lambda').$$

We let

$$G^J = Sp(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)} \qquad \text{(semi-direct product)}$$

be the Jacobi group endowed with the following multiplication law

$$\Big(g,(\lambda,\mu;\kappa)\Big)\cdot\Big(g',(\lambda',\mu';\kappa')\Big) = \Big(gg',(\widetilde{\lambda}+\lambda',\widetilde{\mu}+\mu';\kappa+\kappa'+\widetilde{\lambda}\,{}^t\mu' - \widetilde{\mu}\,{}^t\lambda')\Big)$$

with $g,g' \in Sp(n,\mathbb{R}), (\lambda,\mu;\kappa),\ (\lambda',\mu';\kappa') \in H_{\mathbb{R}}^{(n,m)}$ and $(\widetilde{\lambda},\widetilde{\mu}) = (\lambda,\mu)g'$.

Then we have the *natural action* of G^J on the Siegel-Jacobi space $\mathbb{H}_{n,m} := \mathbb{H}_n \times \mathbb{C}^{(m,n)}$ defined by

$$\Big(g,(\lambda,\mu;\kappa)\Big)\cdot(\Omega,Z) = \Big(g\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}\Big), \tag{1.2}$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$. We refer to [2], [9], [17], [58]-[62], [65]-[68], [71], [72] for more details on materials related to the Siegel-Jacobi space.

The aim of this article is to introduce three types of the Weil representations of the Jacobi group G^J and to study their applications to the theory of automorphic forms and representation theory. They are slightly different each other. They are essentially isomorphic. However each has its own advantage in applications to the theory of automorphic forms and representation theory.

This article is organized as follows. In Section 2, we review the Weil representation of a symplectic group and the Maslov index briefly following G. Lion and M. Vergne [30]. In Section 3, we define the Weil representation of the Jacobi group G^J using a cocycle class of G^J in $H^2(G^J, T)$ with a circle $T = \{z \in \mathbb{C} \mid |z| = 1\}$. In Section 4, we define the Schrödinger-Weil representation of the Jacobi group G^J that is used to study the transformation behaviors of certain theta series with toroidal variables. The Schrödinger-Weil representation plays an important role in the construction of Jacobi forms, the theory of Maass-Jacobi forms and the study of Jacobi's theta sums. We deal with these applications in detail in Section 7. In Section 5, we recall the Weil-Satake representation of the Jacobi group G^J formulated by Satake [40] on the Fock model of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$. In Section 6, we recall the concept of Jacobi forms of half integral weight to be used in a subsequent section. We review Siegel modular forms of half integral weight. In Section 7, we present the applications of the Schrödinger-Weil representation to constructing of Jacobi forms via covariant maps for the Schrödinger-Weil representation, the study of Maass-Jacobi forms and Jacobi's theta sums. We describe the works of the author [69], A. Piale [37] and J. Marklof [32]. In Section 8, we provides some applications of the Weil-Takase representation of G^J to the study of representations of G^J which were obtained by Takase [45, 46, 47]. Takase [45] showed that there is a bijective correspondence between the unitary equivalence classes of unitary representations of a two-fold covering group of the symplectic group and the unitary equivalence classes of unitary representations of the Jacobi group. Using this representation theoretical fact, Takase [48] established a bijective correspondence between the space of cuspidal Jacobi forms and the space of Siegel cusp forms of half integral weight which is compatible with the action of Hecke operators.

Notations : We denote by $\mathbb{Z}$ and $\mathbb{C}$ the ring of integers, and the field of complex numbers respectively. We denote by $\mathbb{R}^*_+$ the multiplicative group of positive real numbers. $\mathbb{C}^*$ (resp. $\mathbb{R}^*$) denotes the multiplicative group of nonzero complex (resp. real) numbers. We denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers respectively. $T = \{z \in \mathbb{C} \mid |z| = 1\}$ denotes the multiplicative group of complex numbers of modulus one. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose of a matrix M. I_n denotes the identity matrix of degree n. We put $i = \sqrt{-1}$. For $z \in \mathbb{C}$, we define $z^{1/2} = \sqrt{z}$ so that $-\pi/2 < \arg(z^{1/2}) \leqq \pi/2$. Furthermore we put $z^{\kappa/2} = \big(z^{1/2}\big)^{\kappa}$ for every $\kappa \in \mathbb{Z}$. For a rational number field $\mathbb{Q}$, we denote by $\mathbb{A}$ and $\mathbb{A}^*$ the ring of adeles of $\mathbb{Q}$ and the multiplicative group of ideles of $\mathbb{Q}$ respectively. For a positive integer m we denote by $S(m)$ the set of all $m \times m$ symmetric real matrices.

2 The Weil Representation of a Symplectic Group

In this section we review the Weil representation of a symplectic group and the Maslov index following G. Lion and M. Vergne [30].

Let (V, B) be a symplectic real vector space of dimension $2n$ with a non-degenerate alternating bilinear form B. We consider the Lie algebra $\mathfrak{h} = V + \mathbb{R}E$ with the Lie bracket satisfying the following properties (2.1) and (2.2):

$$[X, Y] = B(X, Y)E \quad \text{for all } X, Y \in V; \tag{2.1}$$

$$[Z, E] = 0 \quad \text{for all } Z \in \mathfrak{h}. \tag{2.2}$$

Let H be the Heisenberg group with its Lie algebra $\mathfrak{h}$. Via the exponential map $\exp : \mathfrak{h} \longrightarrow H$, H is identified with the $(2n+1)$-dimensional vector space with following multiplication law:

$$\exp(v_1 + t_1E) \cdot \exp(v_2 + t_2E) = \exp\left(v_1 + v_2 + \left(t_1 + t_2 + \frac{B(v_1, v_2)}{2}\right)E\right),$$

where $v_1, v_2 \in V$ and $t_1, t_2 \in \mathbb{R}$. Let

$$Sp(B) = \big\{ g \in GL(V) \mid B(gx, gy) = B(x, y) \quad \text{for all } x, y \in V \big\}$$

be the symplectic group of (V, B). Then $Sp(B)$ acts on H by

$$g \cdot \exp(v + tE) = \exp(gv + tE), \quad g \in Sp(B),\ v \in V,\ t \in \mathbb{R}.$$

For a fixed nonzero real number m, we let $\chi_m : H \longrightarrow T$ be the function defined by

$$\chi_m\big(\exp(v + tE)\big) = e^{2\pi i m t}, \quad v \in V,\ t \in \mathbb{R}.$$

Let $\mathfrak{l}$ be a Lagrangian subspace in (V, B). We put $L = \exp(\mathfrak{l} + \mathbb{R}E)$. Obviously the restriction of χ_m to L is a character of L. The induced representation

$$W_{\mathfrak{l},m} = \mathrm{Ind}_L^H \chi_m$$

is the so-called Schrödinger representation of the Heisenberg group H. The representation $H_{\mathfrak{l},m}$ of $W_{\mathfrak{l},m}$ is the completion of the space of continuous functions φ on H satisfying the following properties (2.3) and (2.4):

$$\varphi(hl) = \chi_m(l)^{-1}\varphi(h), \quad h \in H,\ l \in L \tag{2.3}$$

and

$$h \mapsto |\varphi(h)| \text{ is square integrable w.r.t an invariant measure on } H/L. \tag{2.4}$$

We observe that

$$W_{\mathfrak{l},m}\big(\exp(tE)\big) = e^{2\pi i m t} I_{H_{\mathfrak{l},m}},$$

where $I_{H_{\mathfrak{l},m}}$ denotes the identity operator on $H_{\mathfrak{l},m}$.

For brevity, we put $G = Sp(B)$. For a fixed element $g \in G$, we consider the representation $W^g_{\mathfrak{l},m}$ of H on $H_{\mathfrak{l},m}$ defined by

$$W^g_{\mathfrak{l},m}(h) = W_{\mathfrak{l},m}(g \cdot h), \quad h \in H. \tag{2.5}$$

Since $W_{\mathfrak{l},m}(\exp tE) = W^g_{\mathfrak{l},m}(\exp tE)$ for all $t \in \mathbb{R}$, according to Stone-von Neumann theorem, there exists a unitary operator $R_{\mathfrak{l},m}(g) : H_{\mathfrak{l},m} \longrightarrow H_{\mathfrak{l},m}$ such that

$$W^g_{\mathfrak{l},m}(h)R_{\mathfrak{l},m}(g) = R_{\mathfrak{l},m}(g)W_{\mathfrak{l},m}(h)$$

for all $h \in H$. For convenience, we choose $R_{\mathfrak{l},m}(\mathbf{1}) = I_{H_{\mathfrak{l},m}}$, where $\mathbf{1}$ denotes the identity element of G. We note that $R_{\mathfrak{l},m}(g)$ is determined uniquely up to a scalar of modulus one. Since $R_{\mathfrak{l},m}(g_2)^{-1}R_{\mathfrak{l},m}(g_1)^{-1}R_{\mathfrak{l},m}(g_1g_2)$ is the unitary operator on $H_{\mathfrak{l},m}$ commuting with $W_{\mathfrak{l},m}$, according to Schur's lemma, we have a map $c_{\mathfrak{l},m} : G \times G \longrightarrow T$ satisfying the condition

$$R_{\mathfrak{l},m}(g_1g_2) = c_{\mathfrak{l},m}(g_1, g_2)R_{\mathfrak{l},m}(g_1)R_{\mathfrak{l},m}(g_2) \tag{2.6}$$

for all $g_1, g_2 \in G$. Therefore $R_{\mathfrak{l},m}$ is a projective representation of G with multiplier $c_{\mathfrak{l},m}$. It is easy to see that the map $c_{\mathfrak{l},m}$ satisfies the cocycle condition

$$c_{\mathfrak{l},m}(g_1g_2, g_3)\, c_{\mathfrak{l},m}(g_1, g_2) = c_{\mathfrak{l},m}(g_1, g_2g_3)\, c_{\mathfrak{l},m}(g_2, g_3) \tag{2.7}$$

for all $g_1, g_2, g_3 \in G$. The cocycle $c_{\mathfrak{l},m}$ produces the central extension $G_{\mathfrak{l},m}$ of G by T. The group $G_{\mathfrak{l},m}$ is the set $G \times T$ with the following group multiplication law:

$$(g_1, t_1) \cdot (g_2, t_2) := \big(g_1g_2,\ t_1t_2\, c_{\mathfrak{l},m}(g_1, g_2)^{-1}\big) \tag{2.8}$$

for all $g_1, g_2 \in G,\ t_1, t_2 \in T$. We see that the map $\tilde{R}_{\mathfrak{l},m} : G_{\mathfrak{l},m} \longrightarrow GL(H_{\mathfrak{l},m})$ defined by

$$\tilde{R}_{\mathfrak{l},m}(g, t) := t\, R_{\mathfrak{l},m}(g), \quad g \in G,\ t \in \mathbb{R} \tag{2.9}$$

is a true representation of $G_{\mathfrak{l},m}$.

We now express the cocycle $c_{\mathfrak{l},m}$ in terms of the Maslov index. Let $\mathfrak{l}_1$, $\mathfrak{l}_2$, $\mathfrak{l}_3$ be three Lagrangian subspaces of (V,B). The Maslov index $\tau(\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_3)$ of $\mathfrak{l}_1$, $\mathfrak{l}_2$ and $\mathfrak{l}_3$ is defined to be the signature of the quadratic form Q on the $3n$ dimensional vector space $\mathfrak{l}_1\oplus\mathfrak{l}_2\oplus\mathfrak{l}_3$ given by

$$Q(x_1+x_2+x_3)=B(x_1,x_2)+B(x_2,x_3)+B(x_3,x_1)$$

for all $x_i\in\mathfrak{l}_i,\ i=1,2,3.$

For a sequence $\{\mathfrak{l}_1,\mathfrak{l}_2,\cdots,\mathfrak{l}_k\}$ of Lagrangian subspaces $\mathfrak{l}_1,\mathfrak{l}_2,\cdots,\mathfrak{l}_k$ $(k\geq 4)$ in (V,B), we define the Maslov index $\tau(\mathfrak{l}_1,\mathfrak{l}_2,\cdots,\mathfrak{l}_k)$ by

$$\tau(\mathfrak{l}_1,\mathfrak{l}_2,\cdots,\mathfrak{l}_k)=\tau(\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_3)+\tau(\mathfrak{l}_1,\mathfrak{l}_3,\mathfrak{l}_4)+\cdots+\tau(\mathfrak{l}_1,\mathfrak{l}_{k-1},\mathfrak{l}_k).$$

For a Lagrangian subspace $\mathfrak{l}$ in (V,B), we put $\tau_{\mathfrak{l}}(g_1,g_2)=\tau(\mathfrak{l},g_1\mathfrak{l},g_1g_2\mathfrak{l})$ for $g_1,g_2\in G$.

Lemma 2.1. *Let $\mathfrak{l}_1,\mathfrak{l}_2,\cdots,\mathfrak{l}_k$ be Lagrangian subspaces in (V,B) with $k\geq 4$. Then we have*

(a) $\tau(\mathfrak{l}_1,\mathfrak{l}_2,\cdots,\mathfrak{l}_k)$ *is invariant under the action of G and its value is unchanged under circular permutations.*
(b) $\tau(\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_3)=-\tau(\mathfrak{l}_2,\mathfrak{l}_1,\mathfrak{l}_3)=-\tau(\mathfrak{l}_1,\mathfrak{l}_3,\mathfrak{l}_2).$
(c) *For any four Lagrangian subspaces $\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_3,\mathfrak{l}_4$ in (V,B),*

$$\tau(\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_3)=\tau(\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_4)+\tau(\mathfrak{l}_2,\mathfrak{l}_3,\mathfrak{l}_4)+\tau(\mathfrak{l}_3,\mathfrak{l}_1,\mathfrak{l}_4).$$

(d) $\tau(\mathfrak{l}_1,\mathfrak{l}_2,\cdots,\mathfrak{l}_d)=\tau(\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l})+\tau(\mathfrak{l}_2,\mathfrak{l}_3,\mathfrak{l})+\cdots+\tau(\mathfrak{l}_{d-1},\mathfrak{l}_d,\mathfrak{l})+\tau(\mathfrak{l}_d,\mathfrak{l}_1,\mathfrak{l})$ *for any Lagrangian subspace $\mathfrak{l}$ in (V,B) and $d\geq 3$.*
(e) $\tau(\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_3,\mathfrak{l}_4)=-\tau(\mathfrak{l}_2,\mathfrak{l}_1,\mathfrak{l}_4,\mathfrak{l}_3).$
(f) *For any Lagrangian subspaces $\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_3,\mathfrak{l}_1',\mathfrak{l}_2',\mathfrak{l}_3'$ in (V,B), we have*

$$\tau(\mathfrak{l}_1',\mathfrak{l}_2',\mathfrak{l}_3')=\tau(\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_3)+\tau(\mathfrak{l}_1',\mathfrak{l}_2',\mathfrak{l}_2,\mathfrak{l}_1)+\tau(\mathfrak{l}_2',\mathfrak{l}_3',\mathfrak{l}_3,\mathfrak{l}_2)+\tau(\mathfrak{l}_3',\mathfrak{l}_1',\mathfrak{l}_1,\mathfrak{l}_3).$$

(g) $\tau_{\mathfrak{l}}(g_1g_2,g_3)+\tau_{\mathfrak{l}}(g_1,g_2)=\tau_{\mathfrak{l}}(g_1,g_2g_3)+\tau_{\mathfrak{l}}(g_2,g_3)$ *for all $g_1,g_2,g_3\in G$.*

Proof. The proof can be found in [30]. □

Theorem 2.1. *For a Lagrangian subspace $\mathfrak{l}$ in (V,B) and a real number m, we have*

$$c_{\mathfrak{l},m}(g_1,g_2)=e^{-\frac{i\pi m}{4}\,\tau(\mathfrak{l},g_1\mathfrak{l},g_1g_2\mathfrak{l})}\quad \textit{for all } g_1,g_2\in G.$$

Proof. The proof can be found in [30]. □

An *oriented vector space* of dimension n is defined to be a pair (U,e), where U is a real vector space of dimension n and e is an orientation of U, i.e., a connected component of $\bigwedge^n U-\{0\}$. For two oriented vector space $(\mathfrak{l}_1,e_1)$ and $(\mathfrak{l}_2,e_2)$ in a symplectic vector space (V,B), we define

$$s\big((\mathfrak{l}_1,e_1),(\mathfrak{l}_2,e_2)\big) := i^{n-\dim(\mathfrak{l}_1\cap\mathfrak{l}_2)}\,\varepsilon\big((\mathfrak{l}_1,e_1),(\mathfrak{l}_2,e_2)\big). \qquad (2.10)$$

We refer to [30, pp. 64–66] for the precise definition of $\varepsilon\big((\mathfrak{l}_1,e_1),(\mathfrak{l}_2,e_2)\big)$. Let M be the space of all Lagrangian subspaces in (V,B) and $\widetilde{M}$ the manifold of all oriented Lagrangian subspaces in (V,B). Let $p:\widetilde{M}\longrightarrow M$ be the natural projection from $\widetilde{M}$ onto M. Now we will write $\tilde{\mathfrak{l}}$ for a Lagrangian oriented subspace $(\mathfrak{l},e)$.

Theorem 2.2. *Let* $\tilde{\mathfrak{l}}_1,\,\tilde{\mathfrak{l}}_2,\,\tilde{\mathfrak{l}}_3\in\widetilde{M}$. *Then*

$$e^{-\frac{i\pi}{2}\,\tau(p(\tilde{\mathfrak{l}}_1),p(\tilde{\mathfrak{l}}_1),p(\tilde{\mathfrak{l}}_1))} = s\big(\tilde{\mathfrak{l}}_1,\,\tilde{\mathfrak{l}}_2\big)\,s\big(\tilde{\mathfrak{l}}_2,\,\tilde{\mathfrak{l}}_3\big)\,s\big(\tilde{\mathfrak{l}}_3,\,\tilde{\mathfrak{l}}_1\big).$$

Proof. The proof can be found in [30, pp. 67–70]. □

Let $\mathfrak{l}$ be a Lagrangian subspace in (V,B). We choose an orientation $\mathfrak{l}^+$ on $\mathfrak{l}$. Then G acts on oriented Lagrangian subspace in (V,B). We define

$$s_{\mathfrak{l},m}(g) := s\big(\mathfrak{l}^+,g\mathfrak{l}^+\big)^m,\quad g\in G. \qquad (2.11)$$

The above definition is well defined, i.e., does not depend on the choice of orientation on $\mathfrak{l}$. Since $s_{\mathfrak{l},m}(g^{-1}) = s_{\mathfrak{l},m}(g)^{-1}$, according to Theorem 2.1 and Theorem 2.2, we get

$$c_{\mathfrak{l},m}(g_1,g_2)^2 = s_{\mathfrak{l},m}(g_1)^{-1}\,s_{\mathfrak{l},m}(g_2)^{-1}\,s_{\mathfrak{l},m}(g_1g_2) \qquad (2.12)$$

for all $g_1,g_2\in G$. Hence we can see that

$$G_{2,\mathfrak{l},m} := \big\{\,(g,t)\in G_{\mathfrak{l},m}\mid t^2 = s_{\mathfrak{l},m}(g)^{-1}\,\big\} \qquad (2.13)$$

is the subgroup of $G_{\mathfrak{l},m}$ (cf. Formula (2.8)) that is called the metaplectic group associated with a pair $(\mathfrak{l},m)$. We know that $G_{2,\mathfrak{l},m}$ is a two-fold covering group of G. The restriction $R_{2,\mathfrak{l},m}$ of $\tilde{R}_{\mathfrak{l},m}$ to $G_{2,\mathfrak{l},m}$ is a true representation of $G_{2,\mathfrak{l},m}$ that is called the Weil representation of G associated with a pair $(\mathfrak{l},m)$. We note that

$$R_{2,\mathfrak{l},m}(g,t) = t\,R_{\mathfrak{l},m}(g) = s_{\mathfrak{l},m}(g)^{-1/2}R_{\mathfrak{l},m}(g) \qquad (2.14)$$

for all $(g,t)\in G_{2,\mathfrak{l},m}$. We refer to [18, 24, 30] for more detail on the Weil representation.

3 The Weil Representation of the Jacobi Group G^J

Let $V=\mathbb{R}^{(m,n)}\times\mathbb{R}^{(m,n)}$ be the symplectic real veactor space with a nondegenerate alternating bilinear form on V given by

$$B((\lambda,\mu),(\lambda',\mu')) := \sigma(\lambda\,{}^t\mu-\mu\,{}^t\lambda),\quad (\lambda,\mu),(\lambda',\mu')\in\mathbb{R}^{(m,n)}.$$

We assume that $\mathcal{M}$ is a positive definite symmetric real matrix of degree m. We denote by $S(m)$ the set of all $m\times m$ symmetric real matrices. We let

$$\mathscr{W}_{\mathcal{M}} : H_{\mathbb{R}}^{(n,m)} \longrightarrow U(H_{\mathcal{M}}) \tag{3.1}$$

be the Schrödinger representation with central character $\mathscr{W}_{\mathcal{M}}((0,0;\kappa)) = e^{2\pi i\,\sigma(\mathcal{M}\kappa)} I_{H_{\mathcal{M}}}$,
$\kappa\in S(m)$. Here $H_{\mathcal{M}}$ denotes the representation space of $\mathscr{W}_{\mathcal{M}}$. We note that $\mathscr{W}_{\mathcal{M}}$ is realized on $L^2\big(\mathbb{R}^{(m,n)}\big)\cong H_{\mathcal{M}}$ by

$$\left(\mathscr{W}_{\mathcal{M}}(h)f\right)(x) = e^{2\pi i\,\sigma\left(\mathcal{M}(\kappa+\mu\,{}^t\lambda+2\,x\,{}^t\mu)\right)} f(x+\lambda), \tag{3.2}$$

where $x\in\mathbb{R}^{(m,n)},\ h=(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$ and $f\in L^2\big(\mathbb{R}^{(m,n)}\big)$. We refer to [53, 54, 55, 56, 57] for more detail about $\mathscr{W}_{\mathcal{M}}$. The Jacobi group G^J acts on $H_{\mathbb{R}}^{(n,m)}$ by conjugation inside G^J. Fix an element $\widetilde{g}\in G^J$. The irreducible unitary representation $\mathscr{W}_{\mathcal{M}}^{\widetilde{g}}$ of $H_{\mathbb{R}}^{(n,m)}$ defined by

$$\mathscr{W}_{\mathcal{M}}^{\widetilde{g}}(h) := \mathscr{W}_{\mathcal{M}}(\widetilde{g}\,h\,\widetilde{g}^{-1}),\quad h\in H_{\mathbb{R}}^{(n,m)} \tag{3.3}$$

has the property that $\mathscr{W}_{\mathcal{M}}^{\widetilde{g}}((0,0;\kappa)) = \mathscr{W}_{\mathcal{M}}((0,0;\kappa)) = e^{2\pi i\sigma(\mathcal{M}\kappa)}\cdot I_{H_{\mathcal{M}}}$ for all $\kappa\in S(m)$. According to Stone-von Neumann theorem, there exists a unitary operator $T_{\mathcal{M}}(\widetilde{g})$ on $H_{\mathcal{M}}$ such that $T_{\mathcal{M}}(\widetilde{g})\,\mathscr{W}_{\mathcal{M}}(h) = \mathscr{W}_{\mathcal{M}}^{\widetilde{g}}(h)\,T_{\mathcal{M}}(\widetilde{g})$ for all $h\in H_{\mathbb{R}}^{(n,m)}$. We observe that $T_{\mathcal{M}}(\widetilde{g})$ is determined uniquely up to a scalar of modulus one. According to Schur's lemma, we have a map $\widetilde{c}_{\mathcal{M}} : G^J\times G^J \longrightarrow T$ satisfying the relation

$$T_{\mathcal{M}}(\widetilde{g}_1\widetilde{g}_2) = \widetilde{c}_{\mathcal{M}}(\widetilde{g}_1,\widetilde{g}_2)\,T_{\mathcal{M}}(\widetilde{g}_1)T_{\mathcal{M}}(\widetilde{g}_2) \tag{3.4}$$

for all $\widetilde{g}_1,\widetilde{g}_2\in G^J$. Therefore $T_{\mathcal{M}}$ is a projective representation of G^J and $\widetilde{c}_{\mathcal{M}}$ defines the cocycle class in $H^2(G^J,T)$. The cocycle $\widetilde{c}_{\mathcal{M}}$ satisfies the following properties

$$\widetilde{c}_{\mathcal{M}}(h_1,h_2)=1\quad\text{for all } h_1,h_2\in H_{\mathbb{R}}^{(n,m)}; \tag{3.5}$$

$$\widetilde{c}_{\mathcal{M}}(\tilde{g},e)=\widetilde{c}_{\mathcal{M}}(e,\tilde{g})=\widetilde{c}_{\mathcal{M}}(e,e)=1\quad\text{for all } \tilde{g}\in G^J; \tag{3.6}$$

$$\widetilde{c}_{\mathcal{M}}(\tilde{g},\tilde{g}^{-1})=\widetilde{c}_{\mathcal{M}}(\tilde{g}^2,\tilde{g}^{-1})\,\widetilde{c}_{\mathcal{M}}(\tilde{g},\tilde{g})\quad\text{for all } \tilde{g}\in G^J; \tag{3.7}$$

$$T_{\mathcal{M}}(\tilde{g}^{-1}) = \widetilde{c}_{\mathcal{M}}(\tilde{g},\tilde{g}^{-1})^{-1}\,T_{\mathcal{M}}(\tilde{g})^{-1}\quad\text{for all } \tilde{g}\in G^J, \tag{3.8}$$

where e is the identity element of G^J. The cocycle $\widetilde{c}_{\mathcal{M}}$ yields the central extension $G^J_{\mathcal{M}}$ of G^J by T. The extension group $G^J_{\mathcal{M}}$ is the set $G^J\times T$ with the following group multiplication law :

$$(\widetilde{g}_1,t_1)\cdot(\widetilde{g}_2,t_2) = \big(\widetilde{g}_1\widetilde{g}_2, t_1t_2\,\widetilde{c}_{\mathcal{M}}(\widetilde{g}_1,\widetilde{g}_2)^{-1}\big) \tag{3.9}$$

for all $\widetilde{g}_1,\widetilde{g}_2\in G^J,\ t_1,t_2\in T$. It is easily checked that $(I_{2n},1)$ is the identity element of $G^J_{\mathcal{M}}$ and

$$(\tilde{g},t)^{-1} = \big(\tilde{g}^{-1}, t^{-1}\,\widetilde{c}_{\mathcal{M}}(\tilde{g},\tilde{g}^{-1})\big)$$

if $(\tilde{g},t) \in G^J_{\mathcal{M}}$. We see easily that the map $\widetilde{T}_{\mathcal{M}} : G^J_{\mathcal{M}} \longrightarrow U(H_{\mathcal{M}})$ defined by

$$\widetilde{T}_{\mathcal{M}}(\widetilde{g},t) := t\,T_{\mathcal{M}}(\widetilde{g}), \quad (\widetilde{g},t) \in G^J_{\mathcal{M}} \tag{3.10}$$

is a true representation of $G^J_{\mathcal{M}}$. Here $U(H_{\mathcal{M}})$ denotes the group of unitary operators of $H_{\mathcal{M}}$. For the Lagrangian subspace $\mathfrak{l} = \big\{(0,\mu) \in V \mid \mu \in \mathbb{R}^{(m,n)}\big\}$, as (2.11) and (2.12) in Section 2, we can define the function $\widetilde{s}_{\mathcal{M}} : G^J \longrightarrow T$ satisfying the relation

$$\widetilde{c}_{\mathcal{M}}(\widetilde{g}_1,\widetilde{g}_2)^2 = \widetilde{s}_{\mathcal{M}}(\widetilde{g}_1)^{-1}\,\widetilde{s}_{\mathcal{M}}(\widetilde{g}_2)^{-1}\,\widetilde{s}_{\mathcal{M}}(\widetilde{g}_1\widetilde{g}_2) \tag{3.11}$$

for all $\widetilde{g}_1, \widetilde{g}_2 \in G^J$. Then it is easily seen that

$$G^J_{\mathcal{M},2} := \big\{\,(\widetilde{g},t) \in G^J_{\mathcal{M}} \mid t^2 = \widetilde{s}_{\mathcal{M}}(\widetilde{g})^{-1}\,\big\} \tag{3.12}$$

is a two-fold covering group of G^J. The restriction $\widetilde{\omega}_{\mathcal{M}}$ of $\widetilde{T}_{\mathcal{M}}$ to $G^J_{\mathcal{M},2}$ is called the Weil representation of G^J associated with $\mathcal{M}$.

4 The Schrödinger-Weil Representation

Let $\mathscr{W}_{\mathcal{M}}$ be the Schrödinger representation of $H^{(n,m)}_{\mathbb{R}}$ defined by (3.1) in Section 3. The symplectic group $G = Sp(n,\mathbb{R})$ acts on $H^{(n,m)}_{\mathbb{R}}$ by conjugation inside G^J. We fix an element $g \in G$. We consider the unitary representation $\mathscr{W}^g_{\mathcal{M}}$ of $H^{(n,m)}_{\mathbb{R}}$ defined by

$$\mathscr{W}^g_{\mathcal{M}}(h) = \mathscr{W}_{\mathcal{M}}(ghg^{-1}), \qquad h \in H^{(n,m)}_{\mathbb{R}}. \tag{4.1}$$

Since $\mathscr{W}^g_{\mathcal{M}}\big((0,0;\kappa)\big) = \mathscr{W}_{\mathcal{M}}\big((0,0;\kappa)\big) = e^{2\,\pi\,i\,\sigma(\mathcal{M}\kappa)} I_{H_{\mathcal{M}}}$ for all $\kappa \in S(m)$, according to Stone-von Neumann theorem, $\mathscr{W}^g_{\mathcal{M}}$ is unitarily equivalent to $\mathscr{W}_{\mathcal{M}}$. Thus there exists a unitary operator $R_{\mathcal{M}}(g)$ of $H_{\mathcal{M}}$ satisfying the commutation relation $R_{\mathcal{M}}(g)\,\mathscr{W}_{\mathcal{M}}(h) = \mathscr{W}^g_{\mathcal{M}}(h)\,R_{\mathcal{M}}(g)$ for all $h \in H^{(n,m)}_{\mathbb{R}}$. We observe that $R_{\mathcal{M}}$ is determined uniquely up to a scalar of modulus one. According to Schur's lemma, we have a map $c_{\mathcal{M}} : G \times G \longrightarrow T$ satisfying the relation

$$R_{\mathcal{M}}(g_1g_2) = c_{\mathcal{M}}(g_1,g_2)\,R_{\mathcal{M}}(g_1)R_{\mathcal{M}}(g_2), \quad g_1,g_2 \in G. \tag{4.2}$$

Therefore $R_{\mathcal{M}}$ is a projective representation of G and $c_{\mathcal{M}}$ defines the cocycle class in $H^2(G,T)$. The cocycle $c_{\mathcal{M}}$ gives rise to the central extension $G_{\mathcal{M}}$ of G by T. The extension group $G_{\mathcal{M}}$ is the set $G \times T$ with the following group multiplication law:

$$(g_1,t_1)\cdot(g_2,t_2) = \big(g_1g_2, t_1t_2\, c_{\mathcal{M}}(g_1,g_2)^{-1}\big) \tag{4.3}$$

for all $g_1,g_2 \in G,\ t_1,t_2 \in T$. We see that the map $\widetilde{R}_{\mathcal{M}} : G_{\mathcal{M}} \longrightarrow U(H_{\mathcal{M}})$ defined by

$$\widetilde{R}_{\mathcal{M}}(g,t) = t\, R_{\mathcal{M}}(g), \quad (g,t)\in G_{\mathcal{M}} \tag{4.4}$$

is a true representation of $G_{\mathcal{M}}$. For the Lagrangian subspace

$$\mathfrak{l} = \Big\{ (0,\mu)\in V \mid \mu \in \mathbb{R}^{(m,n)} \Big\},$$

as (2.11) and (2.12) in Section 2, we can define the function $s_{\mathcal{M}} : G \longrightarrow T$ satisfying the relation

$$c_{\mathcal{M}}(g_1,g_2)^2 = s_{\mathcal{M}}(g_1)^{-1} s_{\mathcal{M}}(g_2)^{-1} s_{\mathcal{M}}(g_1g_2) \tag{4.5}$$

for all $g_1,g_2\in G$. Hence we see that

$$G_{2,\mathcal{M}} = \big\{ (g,t)\in G_{\mathcal{M}} \mid t^2 = s_{\mathcal{M}}(g)^{-1} \big\} \tag{4.6}$$

is the metaplectic group associated with $\mathcal{M}\in S(m)$ that is a two-fold covering group of G. The restriction $R_{2,\mathcal{M}}$ of $\widetilde{R}_{\mathcal{M}}$ to $G_{2,\mathcal{M}}$ is the Weil representation of G associated with $\mathcal{M}\in S(m)$. Now we define the projective representation $\pi_{\mathcal{M}}$ of G^J by

$$\pi_{\mathcal{M}}(hg) := \mathscr{W}_{\mathcal{M}}(h)\, R_{\mathcal{M}}(g), \quad h \in H_{\mathbb{R}}^{(n,m)},\ g\in G. \tag{4.7}$$

We observe that any element $\widetilde{g}$ of G^J can be expressed in the form $\widetilde{g} = hg$ with $h\in H_{\mathbb{R}}^{(n,m)}$ and $g\in G$. Indeed, if $g,g_1\in G$ and $h,h_1\in H_{\mathbb{R}}^{(n,m)}$, then we have

$$\begin{aligned}
\pi_{\mathcal{M}}(hgh_1g_1) &= \pi_{\mathcal{M}}(hgh_1g^{-1}gg_1)\\
&= \mathscr{W}_{\mathcal{M}}(hgh_1g^{-1})\, R_{\mathcal{M}}(gg_1)\\
&= c_{\mathcal{M}}(g,g_1)\, \mathscr{W}_{\mathcal{M}}(h)\, \mathscr{W}_{\mathcal{M}}(gh_1g^{-1})\, R_{\mathcal{M}}(g) R_{\mathcal{M}}(g_1)\\
&= c_{\mathcal{M}}(g,g_1)\, \mathscr{W}_{\mathcal{M}}(h)\, \mathscr{W}_{\mathcal{M}}^{g}(h_1)\, R_{\mathcal{M}}(g) R_{\mathcal{M}}(g_1)\\
&= c_{\mathcal{M}}(g,g_1)\, \mathscr{W}_{\mathcal{M}}(h)\, R_{\mathcal{M}}(g)\, \mathscr{W}_{\mathcal{M}}(h_1)\, R_{\mathcal{M}}(g_1)\\
&= c_{\mathcal{M}}(g,g_1)\, \pi_{\mathcal{M}}(hg)\, \pi_{\mathcal{M}}(h_1g_1).
\end{aligned}$$

In the second equality, we used the fact that $H_{\mathbb{R}}^{(n,m)}$ is a normal subgroup of G^J. Therefore we get the relation

$$\pi_{\mathcal{M}}(hgh_1g_1) = c_{\mathcal{M}}(g,g_1)\, \pi_{\mathcal{M}}(hg)\, \pi_{\mathcal{M}}(h_1g_1) \tag{4.8}$$

for all $g,g_1\in G$ and $h,h_1\in H_{\mathbb{R}}^{(n,m)}$. From (4.8) we obtain the relation

$$T_{\mathcal{M}}(g) = R_{\mathcal{M}}(g), \quad \tilde{c}_{\mathcal{M}}(g,g') = c_{\mathcal{M}}(g,g') \tag{4.9}$$

for all $g, g' \in G$. $T_{\mathcal{M}}$ and $\tilde{c}_{\mathcal{M}}$ were defined in (3.4). Thus the representation $\pi_{\mathcal{M}}$ of G^J is naturally extended to the true representation $\omega_{\rm SW}^{\mathcal{M}}$ of $G^J_{2,\mathcal{M}} := G_{2,\mathcal{M}} \ltimes H_{\mathbb{R}}^{(n,m)}$. The representation $\omega_{\rm SW}^{\mathcal{M}}$ is called *Schrödinger-Weil representation* of the Jacobi group G^J associated with $\mathcal{M} \in S(m)$. Indeed we have

$$\omega_{\rm SW}^{\mathcal{M}}(h\cdot(g,t)) = t\,\pi_{\mathcal{M}}(hg), \quad h \in H_{\mathbb{R}}^{(n,m)},\ (g,t) \in G_{2,\mathcal{M}}. \tag{4.10}$$

We recall that the following matrices

$$t(b) := \begin{pmatrix} I_n & b \\ 0 & I_n \end{pmatrix} \text{ with any } b = {}^t b \in \mathbb{R}^{(n,n)},$$
$$g(\alpha) := \begin{pmatrix} {}^t\alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \text{ with any } \alpha \in GL(n,\mathbb{R}),$$
$$\sigma_n := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

generate the symplectic group $G = Sp(n,\mathbb{R})$ (cf. [16, p. 326], [33, p. 210]).

The Weil representation $R_{2,\mathcal{M}}$ is realized on the Hilbert space $L^2\big(\mathbb{R}^{(m,n)}\big)$ as follows:

$$(R_{2,\mathcal{M}}(t(b)f)\,(x) = e^{2\,\pi\,i\,\sigma(\mathcal{M}\,x\,b\,{}^t x)} f(x), \quad b = {}^t b \in \mathbb{R}^{(n,n)}\,; \tag{4.11}$$

$$(R_{2,\mathcal{M}}(g(\alpha)f)\,(x) = (\det\alpha)^{\frac{m}{2}} f(x\,{}^t\alpha), \quad \alpha \in GL(n,\mathbb{R}), \tag{4.12}$$

$$(R_{2,\mathcal{M}}(\sigma_n)f)\,(x) = \left(\frac{1}{i}\right)^{\frac{mn}{2}} (\det\mathcal{M})^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} e^{-4\,\pi\,i\,\sigma(\mathcal{M}\,y\,{}^t x)} f(y)dy. \tag{4.13}$$

We refer to [51] and [24] for more detail.

According to Formulas (4.11)-(4.13), $R_{2,\mathcal{M}}$ is decomposed into two irreducible representations $R^{\pm}_{2,\mathcal{M}}$

$$R_{2,\mathcal{M}} = R^{+}_{2,\mathcal{M}} \oplus R^{-}_{2,\mathcal{M}}, \tag{4.14}$$

where $R^{+}_{2,\mathcal{M}}$ and $R^{-}_{2,\mathcal{M}}$ are the even Weil representation and the odd Weil representation respectively. Obviously the center $\mathcal{Z}^J_{2,\mathcal{M}}$ of $G^J_{2,\mathcal{M}}$ is given by

$$\mathcal{Z}^J_{2,\mathcal{M}} = \big\{\big((I_{2n},1),(0,0;\kappa)\big) \in G^J_{2,\mathcal{M}}\big\} \cong S(m).$$

We note that $\omega_{\rm SW}^{\mathcal{M}}|_{G_{2,\mathcal{M}}} = R_{2,\mathcal{M}}$ and $\omega_{\rm SW}^{\mathcal{M}}(h) = \mathscr{W}_{\mathcal{M}}(h)$ for all $h \in H_{\mathbb{R}}^{(n,m)}$.

5 The Weil-Satake Representation

In this section we discuss the realization of the Weil representation on the Fock model and the Weil-Satake representation due to Satake (cf. [40]). We follow the notations in Section 3 and Section 4.

For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$, we set

$$J(g,\Omega) = C\Omega + D, \quad \Omega \in \mathbb{H}_n. \tag{5.1}$$

Let $\mathcal{M}$ be an $m\times m$ symmetric real matrix. We define the map $J_{\mathcal{M}} : G^J \times \mathbb{H}_{n,m} \longrightarrow \mathbb{C}^\times$ by

$$\begin{aligned} J_{\mathcal{M}}\big(\widetilde{g},(\Omega,Z)\big) = \;& e^{2\pi i\,\sigma\left(\mathcal{M}[Z+\lambda\Omega+\mu](C\Omega+D)^{-1}C\right)} \\ & \times\, e^{-2\pi i\,\sigma\left(\mathcal{M}(\lambda\Omega\,{}^t\lambda+2\lambda\,{}^tZ+\kappa+\mu\,{}^t\lambda)\right)}, \end{aligned} \tag{5.2}$$

where $\widetilde{g} = (g,(\lambda,\mu;\kappa)) \in G^J$ with $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ and $(\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$. Here $M[N] := {}^tNMN$ is a Siegel's notation for two matrices M and N. The $J_{\mathcal{M}}$ satisfies the cocycle condition

$$J_{\mathcal{M}}(\widetilde{g}_1\,\widetilde{g}_2,(\Omega,Z)) = J_{\mathcal{M}}(\widetilde{g}_1,\widetilde{g}_2\cdot(\Omega,Z))\,J_{\mathcal{M}}(\widetilde{g}_2,(\Omega,Z))$$

for all $\widetilde{g}_1,\widetilde{g}_2 \in G^J$ and $(\Omega,Z) \in \mathbb{H}_{n,m}$. We refer to [40] and [61] for a construction of $J_{\mathcal{M}}$.

We introduce the coordinates (Ω,Z) on $\mathbb{H}_{n,m}$ and some notations.

$$\begin{aligned} \Omega &= X + iY, \quad & X &= (x_{\mu\nu}), \quad & Y &= (y_{\mu\nu}) \ \text{ real}, \\ Z &= U + iV, \quad & U &= (u_{kl}), \quad & V &= (v_{kl}) \ \text{ real}, \end{aligned}$$

$$[dX] = \bigwedge_{\mu\le\nu} dx_{\mu\nu}, \quad [dY] = \bigwedge_{\mu\le\nu} dy_{\mu\nu},$$

$$[dU] = \bigwedge_{k,l} du_{kl}, \quad [dV] = \bigwedge_{k,l} dv_{kl}.$$

Now we assume that $\mathcal{M}$ is *positive definite*. We define the function $\kappa_{\mathcal{M}} : \mathbb{H}_{n,m} \longrightarrow \mathbb{R}$ by

$$\kappa_{\mathcal{M}}(\Omega,Z) := e^{-4\pi\,\sigma({}^tV\mathcal{M}VY^{-1})}. \tag{5.3}$$

We fix an element Ω in $\mathbb{H}_n$. We let $H_{\mathcal{M},\Omega}$ be the complex Hilbert space consisting of all complex valued holomorphic functions f on $\mathbb{C}^{(m,n)}$ such that

$$\int_{\mathbb{C}^{(m,n)}} |f(Z)|^2\, d\nu_{\mathcal{M},\Omega}(Z) < \infty,$$

where

$$d\nu_{\mathcal{M},\Omega}(Z) = (\det 2\mathcal{M})^n (\det \operatorname{Im} \Omega)^{-m}\, \kappa_{\mathcal{M}}(\Omega, Z)\, [dU] \wedge [dV].$$

We define an irreducible unitary representation $\mathscr{U}_{\mathcal{M},\Omega}$ of $H_{\mathbb{R}}^{(n,m)}$ on $H_{\mathcal{M},\Omega}$ by

$$\big(\mathscr{U}_{\mathcal{M},\Omega}(h)f\big)(Z) := J_{\mathcal{M}}\big(h^{-1},(\Omega,Z)\big)^{-1} f(Z-\lambda\Omega-\mu), \tag{5.4}$$

where $h=(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$, $f\in H_{\mathcal{M},\Omega}$ and $Z\in\mathbb{C}^{(m,n)}$. It is known that for any two elements Ω_1 and Ω_2 of $\mathbb{H}_n$, $\mathscr{U}_{\mathcal{M},\Omega_1}$ is equivalent to $\mathscr{U}_{\mathcal{M},\Omega_2}$ (cf. [40]). Therefore $\mathscr{U}_{\mathcal{M},\Omega}$ is called the Fock representation of $H_{\mathbb{R}}^{(n,m)}$ associated to $\mathcal{M}$. Clearly $\mathscr{U}_{\mathcal{M},\Omega}((0,0;\kappa)) = e^{-2\pi i\,\sigma(\mathcal{M}\kappa)}$. According to Stone-von Neumann theorem, $\mathscr{U}_{\mathcal{M},\Omega}$ is equivalent to $\mathscr{W}_{-\mathcal{M}}$
(cf. Formula (3.1)). Since the representation $\mathscr{U}^g_{\mathcal{M},\Omega}$ $(g\in G)$ of $H_{\mathbb{R}}^{(n,m)}$ defined by $\mathscr{U}^g_{\mathcal{M},\Omega}(h)=\mathscr{U}_{\mathcal{M},\Omega}(ghg^{-1})$ is equivalent to $\mathscr{U}_{\mathcal{M},\Omega}$, there exists a unitary operator $U_{\mathcal{M},\Omega}(g)$ of $H_{\mathcal{M},\Omega}$ such that $U_{\mathcal{M},\Omega}(g)\mathscr{U}_{\mathcal{M},\Omega}(h)=\mathscr{U}^g_{\mathcal{M},\Omega}(h)U_{\mathcal{M},\Omega}(g)$ for all $h\in H_{\mathbb{R}}^{(n,m)}$. Thus we obtain a projective representation $U_{\mathcal{M},\Omega}$ of G on $H_{\mathcal{M},\Omega}$ and a cocycle $\widehat{c}_{\mathcal{M},\Omega}: G\times G\longrightarrow T$ satisfying the condition

$$U_{\mathcal{M},\Omega}(g_1g_2)=\widehat{c}_{\mathcal{M},\Omega}(g_1,g_2)\,U_{\mathcal{M},\Omega}(g_1)\,U_{\mathcal{M},\Omega}(g_2),\quad g_1,g_2\in G.$$

Now $\widehat{c}_{\mathcal{M},\Omega}$ and $U_{\mathcal{M},\Omega}(g)$ will be determined explicitly (cf. [40], [45]). In fact,

$$\widehat{c}_{\mathcal{M},\Omega}(g_1,g_2)=\left(\frac{\gamma(g_2^{-1}g_1^{-1}\Omega,\, g_2^{-1}\Omega)}{\gamma(g_1^{-1}\Omega,\Omega)}\right)^m, \tag{5.5}$$

where for $\Omega_1,\Omega_2\in\mathbb{H}_n$,

$$\gamma(\Omega_1,\Omega_2):=\left(\det\left(\frac{\Omega_1-\overline{\Omega}_2}{2i}\right)\right)^{-\frac12}(\det\operatorname{Im}\Omega_1)^{\frac14}\,(\det\operatorname{Im}\Omega_2)^{\frac14}.$$

We define the projective representation $\tau_{\mathcal{M},\Omega}$ of G^J by

$$\tau_{\mathcal{M},\Omega}(hg):=\mathscr{U}_{\mathcal{M},\Omega}(h)\,U_{\mathcal{M},\Omega}(g)\quad\text{for all } h\in H_{\mathbb{R}}^{(n,m)},\ g\in G. \tag{5.6}$$

Then $\tau_{\mathcal{M},\Omega}$ satisfies the following relation

$$\tau_{\mathcal{M},\Omega}(\widetilde{g}_1\,\widetilde{g}_2)=\widehat{c}_{\mathcal{M},\Omega}(g_1,g_2)\,\tau_{\mathcal{M},\Omega}(\widetilde{g}_1)\,\tau_{\mathcal{M},\Omega}(\widetilde{g}_2) \tag{5.7}$$

for all $\widetilde{g}_1=(g_1,h_1)$, $\widetilde{g}_2=(g_2,h_2)\in G^J$ with $g_1,g_2\in G$ and $h_1,h_2\in H_{\mathbb{R}}^{(n,m)}$.

We put

$$\beta_\Omega(g_1,g_2) := \widehat{c}_{\mathcal{M},\Omega}(g_1,g_2)^{-\frac{1}{m}}, \quad g_1,g_2\in G. \tag{5.8}$$

Then β_Ω satisfies the cocycle condition and the following relation

$$\beta_\Omega(g_1,g_2)^2 = \widehat{s}_\Omega(g_1)^{-1}\,\widehat{s}_\Omega(g_2)^{-1}\,\widehat{s}_\Omega(g_1g_2), \quad g_1,g_2\in G,$$

where

$$\widehat{s}_\Omega(g) = \,|\det J(g^{-1},\Omega)|\,\left(\det J(g^{-1},\Omega)\right)^{-1}, \quad g\in G.$$

The cocycle class $[\beta_\Omega]$ in $H^2(G,T)$ defines the central extension $G_\Omega = G\times T$ of G by T with the following multiplication law

$$(g_1,t_1)\cdot(g_2,t_2) = \big(g_1g_2, t_1t_2\,\beta_\Omega(g_1,g_2)^{-1}\big).$$

We obtain a normal closed subgroup $G_{2,\Omega}$ of G_Ω given by

$$G_{2,\Omega} = \left\{\,(g,t)\in G_\Omega\,|\,t^2 = \widehat{s}_\Omega(g)^{-1}\,\right\}. \tag{5.9}$$

We can show that $G_{2,\Omega}$ is a two-fold covering group of G. We set for any $g\in G$ and $\Omega_1,\Omega_2\in\mathbb{H}_n$,

$$\varepsilon(g;\Omega_1,\Omega_2) := \frac{\gamma(g\cdot\Omega_1, g\cdot\Omega_2)}{\gamma(\Omega_1,\Omega_2)}. \tag{5.10}$$

We can see that for any element $g\in G$ and $\Omega\in\mathbb{H}_n$, the topological group $G_{2,\Omega}$ is isomorphic to $G_{2,g\cdot\Omega}$ via the correspondence

$$(g_0,t_0)\mapsto\big(g_0, t_0\,\varepsilon(g_0^{-1}; g\cdot\Omega,\Omega)\big), \quad (g_0,t_0)\in G_{2,\Omega}.$$

Therefore it is enough to consider the case $\Omega = iI_n$. We set $G_2 := G_{2,\,iI_n}$. We let

$$G_2^J := G_2\ltimes H_{\mathbb{R}}^{(n,m)}$$

be the two-fold covering group of G^J endowed with the multiplication law

$$\begin{aligned}&\Big((g,t),(\lambda,\mu;\kappa)\Big)\cdot\Big((g',t'),(\lambda',\mu';\kappa')\Big)\\ =\ &\Big((g,t)\cdot(g',t'),(\tilde{\lambda}+\lambda',\tilde{\mu}+\mu';\kappa+\kappa'+\tilde{\lambda}\,{}^t\mu'-\tilde{\mu}\,{}^t\lambda')\Big)\end{aligned}$$

with $(g,t),(g',t')\in G_2$, $(\lambda,\mu;\kappa),(\lambda',\mu';\kappa')\in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde{\lambda},\tilde{\mu}) = (\lambda,\mu)g'$.

We observe that any element $\widetilde{\sigma}$ of G_2^J can be written in the form $\widetilde{\sigma} = h(g,t)$ with $h\in H_{\mathbb{R}}^{(n,m)}$ and $(g,t)\in G_2$. We define a unitary representation $\widehat{\omega} := \widehat{\Omega}_{\mathcal{M},iI_n}$ of G_2^J by

$$\widehat{\omega}_{\mathcal{M}}(h(g,t)) := t^m\tau_{\mathcal{M},iI_n}(hg), \quad h\in H_{\mathbb{R}}^{(n,m)},\ (g,t)\in G_2. \tag{5.11}$$

In fact, if $h, h_1 \in H_{\mathbb{R}}^{(n,m)}$ and $(g,t), (g_1, t_2) \in G_2$, then we obtain

$$\begin{aligned}
&\widehat{\omega}_{\mathcal{M}}\left(h(g,t)h_1(g_1,t_1)\right)\\
&= \widehat{\omega}_{\mathcal{M}}\left((h(g,t)h_1(g,t)^{-1}(g,t)(g_1,t_1)\right)\\
&= \widehat{\omega}_{\mathcal{M}}\left((h(g,t)h_1(g,t)^{-1}\big(gg_1, t\,t_1\beta_{iI_n}(g,g_1)^{-1}\big)\right)\\
&= (t\,t_1)^m \beta_{iI_n}(g,g_1)^{-m}\, \tau_{\mathcal{M},\,iI_n}\left(h(g,t)h_1(g,t)^{-1}gg_1\right)\\
&= (t\,t_1)^m \beta_{iI_n}(g,g_1)^{-m}\, \mathscr{U}_{\mathcal{M},\,iI_n}\left(h(g,t)h_1(g,t)^{-1}\right)\, U_{\mathcal{M},\,iI_n}(gg_1)\\
&= (t\,t_1)^m\, \mathscr{U}_{\mathcal{M},\,iI_n}(h)\, \mathscr{U}^g_{\mathcal{M},\,iI_n}(h_1)\, U_{\mathcal{M},\,iI_n}(g)\, U_{\mathcal{M},\,iI_n}(g_1)\\
&= (t\,t_1)^m\, \mathscr{U}_{\mathcal{M},\,iI_n}(h)\, U_{\mathcal{M},\,iI_n}(g)\, \mathscr{U}_{\mathcal{M},\,iI_n}(h_1)\, U_{\mathcal{M},\,iI_n}(g_1)\\
&= t^m\, t_1^m\, \tau_{\mathcal{M},iI_n}(hg)\, \tau_{\mathcal{M},iI_n}(h_1g_1)\\
&= \widehat{\omega}_{\mathcal{M}}\left(h(g,t)\right)\, \widehat{\omega}_{\mathcal{M}}\left(h_1(g_1,t_1)\right).
\end{aligned}$$

$\widehat{\omega}_{\mathcal{M}}$ is called the Weil-Satake representation of G^J associated with $\mathcal{M}$. In Section 8, we discuss some applications of the Weil-Satake representation $\widehat{\omega}_{\mathcal{M}}$ to the study of unitary representation of G^J.

6 Jacobi Forms

Let ρ be a rational representation of $GL(n,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbb{R}^{(m,m)}$ be a symmetric half-integral semi-positive definite matrix of degree m. Let $C^\infty(\mathbb{H}_{n,m}, V_\rho)$ be the algebra of all C^∞ functions on $\mathbb{H}_{n,m}$ with values in V_ρ. For $f \in C^\infty(\mathbb{H}_{n,m}, V_\rho)$, we define

$$\begin{aligned}
&(f|_{\rho,\mathcal{M}}[(g,(\lambda,\mu;\kappa))])(\Omega,Z)\\
&:= e^{-2\pi i\,\sigma\left(\mathcal{M}(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}C\,{}^t(Z+\lambda\,\Omega+\mu)\right)}\\
&\quad\times e^{2\pi i\,\sigma\left(\mathcal{M}(\lambda\,\Omega\,{}^t\lambda+2\,\lambda\,{}^tZ+\kappa+\mu\,{}^t\lambda)\right)}\\
&\quad\times \rho(C\Omega+D)^{-1} f(g\cdot\Omega, (Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}),
\end{aligned} \tag{6.1}$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$.

Definition 1. Let ρ and $\mathcal{M}$ be as above. Let

$$H_{\mathbb{Z}}^{(n,m)} := \{(\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)} \,|\, \lambda,\mu \in \mathbb{Z}^{(m,n)},\ \kappa \in \mathbb{Z}^{(m,m)}\ \}.$$

A *Jacobi form* of index $\mathcal{M}$ with respect to ρ on a subgroup Γ of Γ_n of finite index is a holomorphic function $f \in C^\infty(\mathbb{H}_{n,m}, V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho,\mathcal{M}}[\tilde{\gamma}] = f$ for all $\tilde{\gamma} \in \widetilde{\Gamma} := \Gamma \ltimes H_{\mathbb{Z}}^{(n,m)}$.

(B) For each $M \in \Gamma_n$, $f|_{\rho,\mathcal{M}}$ has a Fourier expansion of the following form :

$$f(\Omega, Z) = \sum_{\substack{T = {}^tT \geq 0 \\ \text{half-integral}}} \sum_{R \in \mathbb{Z}^{(n,m)}} c(T,R) \cdot e^{\frac{2\pi i}{\lambda_\Gamma} \sigma(T\Omega)} \cdot e^{2\pi i \sigma(RZ)}$$

with a suitable $\lambda_\Gamma \in \mathbb{Z}$ and $c(T,R) \neq 0$ only if $\begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2} {}^tR & \mathcal{M} \end{pmatrix} \geqq 0$.

If $n \geq 2$, the condition (B) is superfluous by Köcher principle (cf. [74] Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ. Ziegler (cf. [74] Theorem 1.8 or [15] Theorem 1.1) proves that the vector space $J_{\rho,\mathcal{M}}(\Gamma)$ is finite dimensional. In the special case $\rho(A) = (\det(A))^k$ with $A \in GL(n,\mathbb{C})$ and a fixed $k \in \mathbb{Z}$, we write $J_{k,\mathcal{M}}(\Gamma)$ instead of $J_{\rho,\mathcal{M}}(\Gamma)$ and call k the *weight* of the corresponding Jacobi forms. For more results on Jacobi forms with $n > 1$ and $m > 1$, we refer to [58]-[62] and [74]. Jacobi forms play an important role in lifting elliptic cusp forms to Siegel cusp forms of degree $2n$ (cf. [23]).

Definition 2. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma)$ is said to be *cuspidal* if

$$\begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2} {}^tR & \mathcal{M} \end{pmatrix} > 0$$

for any T, R with $c(T,R) \neq 0$. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma)$ is said to be *singular* if it admits a Fourier expansion such that a Fourier coefficient $c(T,R)$ vanishes unless $\det \begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2} {}^tR & \mathcal{M} \end{pmatrix} = 0$.

Remark 1. Singular Jacobi forms were characterized by a certain differential operator and the weight by the author [60].

Without loss of generality we may assume that ρ is irreducible. Then we choose a hermitian inner product $\langle\ ,\ \rangle$ on V_ρ that is preserved under the unitary group $U(n) \subset GL(n,\mathbb{C})$. For two Jacobi forms f_1 and f_2 in $J_{\rho,\mathcal{M}}(\Gamma)$, we define the Petersson inner product formally by

$$(f_1, f_2) = \int_{\Gamma_{n,m} \backslash \mathbb{H}_{n,m}} \langle\, \rho(Y^{\frac{1}{2}}) f_1(\Omega, Z), \rho(Y^{\frac{1}{2}}) f_2(\Omega, Z) \rangle\, \kappa_{\mathcal{M}}(\Omega, Z)\, dv, \quad (6.2)$$

where

$$dv = (\det Y)^{-(n+m+1)} [dX] \wedge [dY] \wedge [dU] \wedge [dV] \quad (6.3)$$

is a G^J-invariant volume element on $\mathbb{H}_{n,m}$. A Jacobi form f in $J_{\rho,\mathcal{M}}(\Gamma)$ is said to be square integrable if $\langle f, f \rangle < \infty$. We note that cusp Jacobi forms are square integrable and that $\langle f_1, f_2\rangle$ is finite if one of f_1 and f_2 is a cusp Jacobi form (cf. [74], p. 203).

We define the map $J_{\rho,\mathcal{M}} : G^J \times \mathbb{H}_{n,m} \longrightarrow GL(V_\rho)$ by

$$J_{\rho,\mathcal{M}}(\widetilde{g},(\Omega,Z)) = J_{\mathcal{M}}(\widetilde{g},(\Omega,Z))\,\rho(J(g,\Omega)) \qquad \text{(cf. (5.1) and (5.2))},$$

where $\widetilde{g}=(g,h)\in G^J$ with $g\in G$ and $h\in H_{\mathbb{R}}^{(n,m)}$. For a function f on $\mathbb{H}_n$ with values in V_ρ, we can lift f to a function Φ_f on G^J :

$$\begin{aligned}\Phi_f(\sigma) : &= (f|_{\rho,\mathcal{M}}[\sigma])(iI_n,0)\\ &= J_{\rho,\mathcal{M}}(\sigma,(iI_n,0))^{-1} f(\sigma\cdot(iI_n,0)), \quad \sigma\in G^J.\end{aligned}$$

A characterization of Φ_f for a cusp Jacobi form f in $J_{\rho,\mathcal{M}}(\Gamma)$ was given by Takase [45, pp. 162–164] and the author [63, pp. 252–254].

We allow a weight k to be half-integral. For brevity, we set $G=Sp(n,\mathbb{R})$. For any $g\in G$ and $\Omega,\Omega'\in\mathbb{H}_n$, we note that

$$\begin{aligned}\varepsilon(g;\Omega',\Omega) = \det{}^{-\frac{1}{2}}&\left(\frac{g\cdot\Omega' - \overline{g\cdot\Omega}}{2\,i}\right)\det{}^{\frac{1}{2}}\left(\frac{\Omega'-\overline{\Omega}}{2\,i}\right)\\ &\times|\det J(g,\Omega')|^{-1/2}\,|\det J(g,\Omega)|^{-1/2}.\end{aligned} \tag{6.4}$$

Here $J(g,\Omega)=C\Omega+D$ for $g=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in G$ (cf. (5.1)).

Let

$$\mathfrak{S}=\left\{\,S\in\mathbb{C}^{(n,n)}\mid S={}^tS,\ \mathrm{Re}\,(S)>0\,\right\}$$

be a connected simply connected complex manifold. Then there is a uniquely determined holomorphic function $\det^{-\frac{1}{2}}$ on $\mathfrak{S}$ such that

$$\left(\det{}^{\frac{1}{2}} S\right)^2 = \det S \qquad \text{for all } S\in\mathfrak{S}, \tag{6.5}$$

$$\det{}^{\frac{1}{2}} S = (\det S)^{\frac{1}{2}} \qquad \text{for all } S\in\mathfrak{S}\cap\mathbb{R}^{(n,n)}. \tag{6.6}$$

For each integer $k\in\mathbb{Z}$ and $S\in\mathfrak{S}$, we put

$$\det{}^{\frac{k}{2}} S = \left(\det{}^{\frac{1}{2}} S\right)^k.$$

For each $\Omega\in\mathbb{H}_n$, we define the function $\beta_\Omega : G\times G\longrightarrow T$ by

$$\beta_\Omega(g_1,g_2)=\epsilon(g_1;\Omega,g_2(\Omega)),\quad g_1,g_2\in G. \tag{6.7}$$

Then β_Ω satisfies the cocycle condition and the cohomology class of β_Ω of order two :

$$\beta_\Omega(g_1,g_2)^2=\alpha_\Omega(g_2)\,\alpha_\Omega(g_1g_2)^{-1}\,\alpha_\Omega(g_1), \tag{6.8}$$

where

$$\alpha_\Omega(g) = \frac{\det J(g,\Omega)}{|\det J(g,\Omega)|}, \quad g \in G,\ \Omega \in \mathbb{H}_n. \tag{6.9}$$

For any $\Omega \in \mathbb{H}_n$, we let

$$G_\Omega = \left\{ (g,\epsilon) \in G \times T \mid \epsilon^2 = \alpha_\Omega(g)^{-1} \right\}$$

be the two-fold covering group with the multiplication law

$$(g_1,\epsilon_1)(g_2,\epsilon_2) = \big(g_1 g_2,\ \epsilon_1\epsilon_2\beta_\Omega(g_1,g_2)\big).$$

The covering group G_Ω depends on the choice of $\Omega \in \mathbb{H}_n$, i.e., the choice of a maximal compact subgroup of G. However for any two element $\Omega_1, \Omega_2 \in \mathbb{H}_n$, G_{Ω_1} is isomorphic to G_{Ω_2} (cf. [47]). We put

$$G_* := G_{iI_n}.$$

We define the automorphic factor $J_{1/2} : G_* \times \mathbb{H}_n \longrightarrow \mathbb{C}^*$ by

$$J_{1/2}(g_\epsilon,\Omega) := \epsilon^{-1}\varepsilon(g;\Omega,iI_n)|\det J(g,\Omega)|^{1/2}, \tag{6.10}$$

where $g_\epsilon = (g,\epsilon) \in G_\Omega$ with $g \in G$ and $\Omega \in \mathbb{H}_n$. It is easily checked that

$$J_{1/2}(g_* h_*,\Omega) = J_{1/2}(g_*, h\cdot\Omega)J_{1/2}(h_*,\Omega) \tag{6.11}$$

for all $g_* = (g,\epsilon),\ h_* = (h,\eta) \in G_*$ and $\Omega \in \mathbb{H}_n$. Moreover

$$J_{1/2}(g_*,\Omega)^2 = \det(C\Omega + D) \tag{6.12}$$

for all $g_* = (g,\epsilon) \in G_*$ with $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$.

Let $\pi_* : G_* \longrightarrow G$ be the projection defined by $\pi_*(g,\epsilon) = g$. Let Γ be a subgroup of the Siegel modular group Γ_n of finite index. Let $\Gamma_* = \pi_*^{-1}(\Gamma) \subset G_*$. Let χ be a finite order unitary character of Γ_*. Let $k \in \mathbb{Z}^+$ be a positive integer. We say that a holomorphic function $\phi : \mathbb{H}_n \longrightarrow \mathbb{C}$ is a *Siegel modular form* of a *half-integral weight* $k/2$ with level Γ if it satisfies the condition

$$\phi(\gamma_* \cdot \Omega) = \chi(\gamma_*)J_{1/2}(\gamma_*,\Omega)^k\phi(\Omega) \tag{6.13}$$

for all $\gamma_* \in \Gamma_*$ and $\Omega \in \mathbb{H}_n$. We denote by $M_{k/2}(\Gamma,\chi)$ be the vector space of all Siegel modular forms of weight $k/2$ with level Γ. Let $S_{k/2}(\Gamma,\chi)$ be the subspace of $M_{k/2}(\Gamma,\chi)$ consisting of $\phi \in M_{k/2}(\Gamma,\chi)$ such that

$$|\phi(\Omega)|\det(\operatorname{Im}\Omega)^{k/4} \text{ is bounded on } \mathbb{H}_n.$$

An element of $S_{k/2}(\Gamma,\chi)$ is called a *Siegel cusp form* of weight $k/2$.

Definition 3. Let $\Gamma \subset \Gamma_n$ be a subgroup of finite index. We put $\Gamma_* = \pi_*^{-1}(\Gamma)$ and

$$\widetilde{\Gamma}_* = \Gamma_* \ltimes H_{\mathbb{Z}}^{(n,m)}.$$

A holomorphic function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is said to be a Jacobi form of a weight $k \in \frac{1}{2}\mathbb{Z}$ (k : odd) with level Γ and index $\mathcal{M}$ for the character χ of Γ_* of if it satisfies the following transformation formula

$$f(\widetilde{\gamma}_* \cdot (\Omega, Z)) = \chi(\gamma_*)\, J_{k,\mathcal{M}}(\widetilde{\gamma}_*, (\Omega, Z)) f(\Omega, Z) \quad \text{for all } \widetilde{\gamma}_* \in \widetilde{\Gamma}_* \tag{6.14}$$

where $J_{k,\mathcal{M}} : \widetilde{\Gamma}_* \times \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is an automorphic factor defined by

$$\begin{aligned} J_{k,\mathcal{M}}\big(\widetilde{\gamma}_*, (\Omega, Z)\big) :&= e^{2\pi i\, \sigma\left(\mathcal{M}(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}C\,{}^t(Z+\lambda\Omega+\mu)\right)} \\ &\quad \times e^{-2\pi i \sigma\left(\mathcal{M}(\lambda\,\Omega\,{}^t\lambda + 2\lambda\,{}^tZ + \kappa + \mu\,{}^t\lambda)\right)} J_{1/2}(\gamma_*, \Omega)^k, \end{aligned} \tag{6.15}$$

where $\widetilde{\gamma}_* = (\gamma_*, (\lambda, \mu; \kappa)) \in \widetilde{\Gamma}_*$ with $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma,\ \gamma_* = (\gamma, \epsilon),\ (\lambda, \mu, \kappa) \in H_{\mathbb{Z}}^{(n,m)}$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$.

7 Applications of the Schrödinger-Weil Representation

7.1. Construction of Jacobi Forms

We assume that $\mathcal{M}$ is a positive definite symmetric integral matrix of degree m. Let $\omega_{\mathcal{M}}$ be the Schrödinger-Weil representation of G^J constructed in Section 4. We recall that $\omega_{\mathcal{M}}$ is realized on the Hilbert space $L^2\big(\mathbb{R}^{(m,n)}\big)$ by Formulas (4.11)–(4.13). We define the mapping $\mathscr{F}^{(\mathcal{M})} : \mathbb{H}_{n,m} \longrightarrow L^2\big(\mathbb{R}^{(m,n)}\big)$ by

$$\mathscr{F}^{(\mathcal{M})}(\Omega, Z)(x) = e^{\pi i\, \sigma\{\mathcal{M}(x\,\Omega\,{}^tx + 2\,x\,{}^tZ)\}}, \tag{7.1}$$

where $(\Omega, Z) \in \mathbb{H}_{n,m},\ x \in \mathbb{R}^{(m,n)}$.

For brevity we put $\mathscr{F}_{\Omega,Z}^{(\mathcal{M})} := \mathscr{F}^{(\mathcal{M})}(\Omega, Z)$ for $(\Omega, Z) \in \mathbb{H}_{n,m}$. Let $J : G^J \times \mathbb{H}_{n,m} \longrightarrow \mathbb{C}^\times$ be an automorphic factor for G^J on $\mathbb{H}_{n,m}$ defined by

$$\begin{aligned} J_{\mathcal{M}}^*(\widetilde{g}, (\Omega, Z)) &= e^{\pi i\, \sigma\left(\mathcal{M}(Z+\lambda\,\Omega+\mu)(C\Omega+D)^{-1}C\,{}^t(Z+\lambda\,\Omega+\mu)\right)} \\ &\quad \times e^{-\pi i\, \sigma\left(\mathcal{M}(\lambda\,\Omega\,{}^t\lambda + 2\,\lambda\,{}^tZ + \kappa + \mu\,{}^t\lambda)\right)} \det(C\Omega + D)^{\frac{m}{2}}, \end{aligned} \tag{7.2}$$

where $\widetilde{g} = (g, (\lambda, \mu; \kappa)) \in G^J$ with $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$, $(\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$.

Theorem 7.3. *The map $\mathscr{F}^{(\mathcal{M})} : \mathbb{H}_{n,m} \longrightarrow L^2\big(\mathbb{R}^{(m,n)}\big)$ defined by (7.1) is a covariant map for the Schrödinger-Weil representation $\omega_{\mathcal{M}}$ of G^J and the automorphic factor $J^*_{\mathcal{M}}$ for G^J on $\mathbb{H}_{n,m}$ defined by Formula (7.2). In other words, $\mathscr{F}^{(\mathcal{M})}$ satisfies the following covariance relation*

$$\omega_{\mathcal{M}}(\widetilde{g})\mathscr{F}^{(\mathcal{M})}_{\Omega,Z} = J^*_{\mathcal{M}}\big(\widetilde{g}, (\Omega, Z)\big)^{-1} \mathscr{F}^{(\mathcal{M})}_{\widetilde{g}\cdot(\Omega,Z)} \tag{7.3}$$

for all $\widetilde{g} \in G^J$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$.

Proof. The proof can be found in [69]. □

For a positive definite integral matrix $\mathcal{M}$ of degree m, we define the holomorphic function $\Theta_{\mathcal{M}} : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ by

$$\Theta_{\mathcal{M}}(\Omega, Z) = \sum_{\xi \in \mathbb{Z}^{(m,n)}} e^{\pi\, i\, \sigma\left(\mathcal{M}(\xi\, \Omega\, {}^t\xi + 2\, \xi\, {}^tZ)\right)}, \quad (\Omega, Z) \in \mathbb{H}_{n,m}. \tag{7.4}$$

We can prove the following theorem.

Theorem 7.4. *The function $\Theta_{\mathcal{M}}$ is a Jacobi form of weight $\frac{m}{2}$ and index $\frac{1}{2}\mathcal{M}$ with respect to a discrete subgroup $\Gamma^J_{\mathcal{M}} := \Gamma_{\mathcal{M}} \ltimes H_{\mathbb{Z}}^{(n,m)}$ of Γ^J with a suitable arithmetic subgroup $\Gamma_{\mathcal{M}}$ of Γ_n. That is, $\Theta_{\mathcal{M}}$ satisfies the functional equation*

$$\Theta_{\mathcal{M}}\big(\widetilde{\gamma} \cdot (\Omega, Z)\big) = \rho_{\mathcal{M}}(\widetilde{\gamma})\, J^*_{\mathcal{M}}\big(\widetilde{\gamma}, (\Omega, Z)\big)\Theta_{\mathcal{M}}(\Omega, Z), \tag{7.5}$$

where $(\Omega, Z) \in \mathbb{H}_{n,m}$ and $\rho_{\mathcal{M}}(\widetilde{\gamma})$ is a suitable character of $\Gamma^J_{\mathcal{M}}$.
Proof. The proof can be found in [69] when $\mathcal{M}$ is unimodular and even integral. In a similar way we can prove the above theorem. □

According to Theorem 7.3 and Theorem 7.4, we see that the theta series $\Theta_{\mathcal{M}}$ is closely related to the Schrödinger-Weil representation of the Jacobi group G^J. We note that the theta series

$$\Theta(\Omega) = \sum_{A \in \mathbb{Z}^n} e^{\pi\, i\, \sigma(A\, \Omega\, {}^tA)}, \quad \Omega \in \mathbb{H}_n \tag{7.6}$$

is a Siegel modular form of weight $\frac{1}{2}$ with respect to the theta subgroup Γ_Θ of Γ_n, that is, Θ satisfies the following functional equation

$$\Theta(\gamma \cdot \Omega) = \zeta(\gamma)\ (\det(C\Omega + D))^{\frac{1}{2}}\ \Theta(\Omega), \quad \Omega \in \mathbb{H}_n, \tag{7.7}$$

where $\zeta(\gamma)$ is a character of Γ_Θ with $|\zeta(\gamma)|^8 = 1$ and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_\Theta$. We refer to [33, pp. 189-201] for more detail. Indeed the function $\mathscr{F} : \mathbb{H}_n \longrightarrow L^2(\mathbb{R}^n)$ defined by

$$\mathscr{F}(\Omega)(x) = e^{\pi\, i\, \sigma(x\, \Omega\, {}^t x)}, \quad \Omega \in \mathbb{H}_n \text{ and } x \in \mathbb{R}^n. \tag{7.8}$$

is a covariant map for the Weil representation ω of $Sp(n,\mathbb{R})$ and the automorphic form $J_{\frac{1}{2}} : Sp(n,\mathbb{R}) \times \mathbb{H}_n \longrightarrow \mathbb{C}^\times$ defined by

$$J_{\frac{1}{2}}(g,\Omega) = \left(\det(C\Omega + D)\right)^{\frac{1}{2}}, \quad \Omega \in \mathbb{H}_n \tag{7.9}$$

with $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$. More precisely, if we put $\mathscr{F}_\Omega := \mathscr{F}(\Omega)$ for brevity, the vector valued map $\mathscr{F}$ satisfies the following covariance relation

$$\omega(g)\mathscr{F}_\Omega = \left(\det(C\Omega + D)\right)^{-\frac{1}{2}} \mathscr{F}_{g\cdot\Omega} \tag{7.10}$$

for all $g \in Sp(n,\mathbb{R})$ and $\Omega \in \mathbb{H}_n$. We refer to [30] for more detail. This is a special case of Theorem 7.3 and Theorem 7.4.

7.2. Maass-Jacobi Forms

Recently in the case $n = m = 1$ A. Pitale [37] gave a new definition of nonholomorphic Maass-Jacobi forms of weight k and $m \in \mathbb{Z}^+$ as eigenfunctions of a certain differential operator $\mathcal{C}^{k,m}$, and constructed new examples of cuspidal Maass-Jacobi forms F_f of even weight k and index 1 from Maass forms f of weight half integral weight $k - 1/2$ with respect to $\Gamma_0(4)$. Moreover he also showed that the map $f \mapsto F_f$ is Hecke equivariant and compatible with the representation theory of the Jacobi group G^J. We will describe his results in some detail.

For a positive integer N, we let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod{N} \right\}$$

be the congruence subgroup of $SL(2,\mathbb{Z})$ called the *Hecke subgroup* of level N. Let $\mathfrak{G}$ be the group which consists of all pairs $(\gamma, \phi(\tau))$, where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R})^+$ and $\phi(\tau)$ is a function on $\mathbb{H}$ such that

$$\phi(\tau) = t\, \det(\gamma) \left(\frac{(c\tau + d)}{|c\tau + d|} \right)^{1/2} \quad \text{with } t \in \mathbb{C},\ |t| = 1.$$

The group law is given by

$$(\gamma_1, \phi_1(\tau)) \cdot (\gamma_2, \phi_2(\tau)) = \big(\gamma_1\gamma_2, \phi_1(\gamma_2 \cdot \tau)\phi_2(\tau)\big), \tag{7.11}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R})^+$. Then there is an injective homomorphism $\Gamma_0(4) \mapsto \mathfrak{G}$ given by

$$\gamma \mapsto \gamma^* := \big(\gamma, j(\gamma,\tau)\big), \tag{7.12}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ and

$$j(\gamma,\tau) := \left(\frac{c}{d}\right) \epsilon_d^{-1} \left(\frac{(c\tau+d)}{|c\tau+d|}\right)^{1/2} = \frac{\theta(\gamma\cdot\tau)}{\theta(\tau)}$$

with

$$\theta(\tau) := y^{1/4} \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 \tau}$$

and

$$\epsilon_d = \begin{cases} 1, & \text{if } d \equiv 1 \pmod 4, \\ i, & \text{if } d \equiv 3 \pmod 4 \end{cases}.$$

And $\left(\frac{c}{d}\right)$ is defined as in [41, p. 442].

For an integer $k \in \mathbb{Z}$, we define the slash operator $||_{k-1/2}$ on functions on $\mathbb{H}$ as follows:

$$\big(f||_{k-1/2}(\gamma,\phi)\big)(\tau) := f(\gamma\cdot\tau)\,\phi(\tau)^{-(2k-1)}. \tag{7.13}$$

Definition 4. A smooth function $f : \mathbb{H} \longrightarrow \mathbb{C}$ is called a *Maass form* of weight $k - 1/2$ with respect to $\Gamma_0(4)$ if it satisfies the following properties (M1)-(M3):

(M1) $f||_{k-1/2}\gamma^* = f$ for all $\gamma \in \Gamma_0(4)$.

(M2) $\Delta_{k-1/2} f = \Lambda f$ for some $\Lambda \in \mathbb{C}$, where $\Delta_{k-1/2}$ is the Laplace-Beltrami operator given by

$$\Delta_{k-1/2} = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - i\,(k-1/2)\,y\frac{\partial}{\partial x}. \tag{7.14}$$

(M3) $f(\tau) = O(y^N)$ as $y \longrightarrow \infty$ for some $N > 0$.

If, in addition, f vanishes at all the cusps of $\Gamma_0(4)$, then we say that f is a Maass cusp form.

We denote by $M_{k-1/2}(4)$ (resp. $S_{k-1/2}(4)$) be the vector space of all Maass forms (resp. Maass cusp forms) of weight $k-1/2$ with respect to Γ_0. As shown in [25, 37], if $f \in M_{k-1/2}(4)$, then f has the following Fourier expansion

$$f(\tau) = \sum_{n \in \mathbb{Z}} c(n) W_{\mathrm{sgn}\,\frac{k-1/2}{2}, \frac{il}{2}}(2\pi|n|y)\, e^{2\pi i n x}, \tag{7.15}$$

where $\Lambda = -\left\{1/4 + (l/2)^2\right\}$ and $W_{\mu,\nu}(y)$ is the classical Whittaker function which is normalized so that $W_{\mu,\nu}(y) \sim e^{-y/2} y^{\mu}$ as $y \longrightarrow \infty$. If $f \in S_{k-1/2}(4)$, then we have $c(0) = 0$ in (7.15). We define the plus space by

$$M^{+}_{k-1/2}(4) = \left\{ f \in M_{k-\frac{1}{2}}(4) \mid c(n) = 0 \text{ if } (-1)^{k-1} n \equiv 2, 3 \bmod 4) \right\}. \tag{7.16}$$

We set

$$S^{+}_{k-1/2}(4) := M^{+}_{k-1/2}(4) \cap S_{k-1/2}(4).$$

For a given integer $k \in \mathbb{Z}$ and $m \in \mathbb{Z}^{+}$, we let

$$j^{\mathrm{nh}}_{k,m}(\tilde{g}, (\tau, z)) := e^{2\pi i\, m \left\{\kappa - c(z+\lambda\tau+\mu)(c\tau+d)^{-1} + \lambda^2\tau + 2\lambda z + \lambda\mu\right\}} \times \left(\frac{c\tau+d}{|c\tau+d|}\right)^{-k} \tag{7.17}$$

be the nonholomorphic automorphic factor for G^J on $\mathbb{H} \times \mathbb{C}$, where $\tilde{g} = (g, (\lambda, \mu; \kappa))$ with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}),\ \lambda, \mu, \kappa \in \mathbb{R}$ and $(\tau, z) \in \mathbb{H} \times \mathbb{C}$. For $\tilde{g} \in G^J(\mathbb{R}),\ (\tau, z) \in \mathbb{H} \times \mathbb{C}$ and a smooth function $F : \mathbb{H} \times \mathbb{C} \longrightarrow \mathbb{C}$, we set

$$(F|_{k,m}\tilde{g})(\tau, z) := j^{\mathrm{nh}}_{k,m}(\tilde{g}, (\tau, z)) F(\tilde{g} \cdot (\tau, z)). \tag{7.18}$$

Let $\Gamma^J := SL(2, \mathbb{Z}) \ltimes H^{(1,1)}_{\mathbb{Z}}$ be the discrete subgroup of $G^J(\mathbb{R}) := SL(2, \mathbb{R}) \ltimes H^{(1,1)}_{\mathbb{R}}$.

Definition 5. A smooth function $F : \mathbb{H} \times \mathbb{C} \longrightarrow \mathbb{C}$ is called a *Maass-Jacobi form* of weight k and index m with respect to Γ^J) if it satisfies the following properties (MJ1)–(MJ3) :

(M1) $F(\tilde{\gamma} \cdot (\tau, z)) = j^{\mathrm{nh}}_{k,m}(\tilde{\gamma}, (\tau, z))^{-1} F(\tau, z)$ for all $\tilde{\gamma} \in \Gamma^J$ and $(\tau, z) \in \mathbb{H} \times \mathbb{C}$.

(M2) $\mathcal{C}^{k,m} F = \lambda_{k,m}(f) F$ for some $\lambda_{k,m}(f) \in \mathbb{C}$.

(M3) $F(\tau, z) = O(y^N)$ as $y \longrightarrow \infty$for some $N > 0$.

If, in addition, f satisfies the following cuspidal condition

$$\int_0^1 \int_0^1 F\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} (0, \mu; 0)(\tau, z)\right) e^{-2\pi i(nx + r\mu)} dx d\mu = 0 \tag{7.19}$$

for all $n, r \in \mathbb{Z}$ such that $4mn - r^2 = 0$, then we say that f is a Maass-Jacobi cusp form.

We denote by $J^{\rm nh}_{k,m}$ (resp. $J^{{\rm nh},cusp}_{k,m}$) the vector space of all Maass-Jacobi forms (resp. Maass-Jacobi cusp forms) of weight k and index m with respect to Γ^J.

For a Maass form $f \in M^+_{k-1/2}(4)$ with $k \in 2\mathbb{Z}$, he defined the function F_f on $\mathbb{H} \times \mathbb{C}$ by

$$F_f(\tau, z) := f^{(0)}(\tau)\, \widetilde{\Theta}^{(0)}(\tau, z) + f^{(1)}(\tau)\, \widetilde{\Theta}^{(1)}(\tau, z) \tag{7.20}$$

for all $(\tau, z) \in \mathbb{H} \times \mathbb{C}$. We refer to [37, pp. 96-97] for the precise definition of $f^{(0)}$, $f^{(1)}$, $\widetilde{\Theta}^{(0)}$ and $\widetilde{\Theta}^{(1)}$. Pitale [37] showed that if $f \in M^+_{k-1/2}(4)$ with $k \in 2\mathbb{Z}$, then $F_f \in J^{\rm nh}_{k,1}$, and $F_f \in J^{{\rm nh},cusp}_{k,1}$ if and only if $f \in S^+_{k-1/2}(4)$. Furthermore he showed that if $\Delta_{k-1/2} f = \Lambda f$, then $\mathcal{C}^{k,1} F_f = 2\,\Lambda\, F_f$ under the assumption $f \in M^+_{k-1/2}(4)$ with $k \in 2\mathbb{Z}$.

For an odd prime p, the Jacobi Hecke operator T_p on $J^{\rm nh}_{k,1}$ (cf. [10, p. 168] or [15, p. 41] is defined by

$$T_p F = \sum_{\substack{M \in SL(2,\mathbb{Z})/\mathbb{Z}^{(2,2)} \\ \det(M)=p^2 \\ \gcd(M)=1}} \ \sum_{(\lambda,\mu)\in (\mathbb{Z}/p\mathbb{Z})^2} F|_{k,1}\Big(\det(M)^{-1/2} M(\lambda, \mu; 0)\Big). \tag{7.21}$$

Theorem 7.5. *Let $f \in S^+_{k-1/2}(4)\,(k \in 2\mathbb{Z})$ be a Hecke eigenform with eigenvalue λ_p for every odd prime p. Then $T_p = p^{k-3/2}\lambda_p\, F_f$ for all odd prime p. Namely F_f is also an eigenfunction of all T_p for every odd prime p.*

Proof. The proof can be found in [37, pp. 104-106]. □

Let f be a Hecke eigenform in $S^+_{k-1/2}(4)\,(k \in 2\mathbb{Z})$ such that for every odd prime p we have $T_p f = \lambda_p f$ and $\Delta_{k-1/2} f = \Lambda f$ with $\Lambda = \frac{1}{4}(s^2 - 1)$. Let $\widetilde{\pi}_f = \otimes \widetilde{\pi}_{f,p}$ be the irreducible cuspidal genuine automorphic representation of a two-fold covering group $\widetilde{SL(2,\mathbb{A})}$ of $SL(2,\mathbb{A})$ corresponding to f (cf. [49, p. 386]). Now we let F_f be the Maass-Jacobi cusp form in $J^{{\rm nh},cusp}_{k,1}$ constructed from an eigenform $f \in S^+_{k-1/2}(4)\,(k \in 2\mathbb{Z})$ by Formula (7.20). Then F_f is an eigenform of all T_p for every odd prime p and is an eigenfunction of the differential operator $\mathcal{C}^{k,1}$. We lift F_f to the function Φ_{F_f} on $G^J(\mathbb{A})$ as follows. By the strong approximation theorem for $G^J(\mathbb{A})$, we have the decomposition

$$G^J(\mathbb{A}) = G^J(\mathbb{Z})\, G^J(\mathbb{R})\, \Pi_{p<\infty} G^J(\mathbb{Z}_p). \tag{7.22}$$

If $\tilde{g} = \gamma \tilde{g}_\infty k_0 \in G^J(\mathbb{A})$ with $\gamma \in G^J(\mathbb{Z})$, $\tilde{g}_\infty \in G^J(\mathbb{R})$, $k_0 \in \Pi_{p<\infty} G^J(\mathbb{Z}_p)$, we define

$$\Phi_{F_f}(\tilde{g}) := (F_f|_{k,m}\tilde{g}_\infty)(i, 0). \tag{7.23}$$

Let Π_{F_f} be the space of all right translates of Φ_{F_f} on which $G^J(\mathbb{A})$ acts by right translation. Pitale [37] proved that

$$\Pi_{F_f} = \widetilde{\pi}_f \otimes \omega_{\rm SW}^1, \tag{7.24}$$

where $\omega_{\rm SW}^1$ is the Schrödinger-Weil representation of $G^J(\mathbb{A})$ (cf. [10]).

Remark 2. For a Siegel cusp form of half integral weight, we have a result similar to Formula (7.24). See [49] for the case $n = 1$ and [47, 48] for the case $n \geq 1$.

Remark 3. Berndt and Schmidt [10] gave a definition of Maass-Jacobi forms different from Definition 7.2. Yang [64, 66, 70] gave a definition of Maass-Jacobi forms using the Laplacian of an invariant metric on the Siegel-Jacobi space $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ in the aspect of the spectral theory on $L^2\big(\Gamma_n^J\backslash\mathbb{H}_n \times \mathbb{C}^{(m,n)}\big)$. We refer to [11, 12, 13, 14] for another notion of Maass-Jacobi forms.

7.3. Theta Sums

We embed $SL(2,\mathbb{R})$ into $Sp(n,\mathbb{R})$ by

$$SL(2,\mathbb{R}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} aI_n & bI_n \\ cI_n & dI_n \end{pmatrix} \in Sp(n,\mathbb{R}). \tag{7.25}$$

Every map $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ admits the unique Iwasawa decomposition

$$M = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = (\tau,\theta),$$

where $\tau = x + i\,y \in \mathbb{H}_1$ and $0 \leq \theta < 2\pi$. Then $SL(2,\mathbb{R})$ acts on $\mathbb{H}_1 \times [0, 2\pi)$ by

$$M \cdot (\tau,\theta) := \big(M\cdot\tau,\ \theta + \arg(c\tau + d) \bmod 2\pi\big), \tag{7.26}$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}),\ \tau \in \mathbb{H}_1$ and $\theta \in [0, 2\pi)$.

We put

$$G_{n,1}^J := Sp(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,1)}.$$

We take $\mathcal{M} = 1$ in section 4. Then we let $\mathscr{W} = \mathscr{W}_{\mathcal{M}},\ R = R_{\mathcal{M}}$ and $c = c_{\mathcal{M}}$ (see section 4). If $M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in SL(2,\mathbb{R})$ for $i = 1, 2, 3$ with $M_3 = M_1 M_2$, then the cocycle c is given by

$$c(M_1, M_2) = e^{-i\pi n\,{\rm sign}(c_1 c_2 c_3)/4},$$

where

$$\mathrm{sign}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x < 0. \end{cases}$$

For $(\tau, \theta) \in SL(2, \mathbb{R})$, we define

$$\widetilde{R}(\tau, \theta) := e^{-i\pi n\, \sigma_\theta/4}\, R(\tau, \theta), \tag{7.27}$$

where

$$\sigma_\theta = \begin{cases} 2\nu & \text{if } \theta = \nu\pi,\ \nu \in \mathbb{Z}, \\ 2\nu + 1 & \text{if } \nu\pi < \theta < (\nu+1)\pi,\ \nu \in \mathbb{Z}. \end{cases}$$

Then $\widetilde{R}$ is a unitary representation of the double covering group of $SL(2, \mathbb{R})$ (cf. [30]). Obviously $\widetilde{R}(i, \theta)\widetilde{R}(i, \theta') = \widetilde{R}(i, \theta + \theta')$.

We see that

$$\omega^1_{\rm SW}((\xi; t)(\tau, \theta)) = \mathscr{W}((\xi; t))\, \widetilde{R}(\tau, \theta), \tag{7.28}$$

where $\omega^1_{\rm SW}$ denotes the Schrödinger-Weil representation of $G^J_{n,1}$ (see Formula (4.10)). Here $(\xi; t) \in H^{(n,1)}_{\mathbb{R}}$ and (τ, θ) is considered as an element of $Sp(n, \mathbb{R})$ by the embedding (7.25).

We denote by $\mathcal{S}(\mathbb{R}^n)$ the vector space of C^∞-functions on $\mathbb{R}^n$ that, as well as their derivatives, decrease rapidly at ∞. For any $f \in \mathcal{S}(\mathbb{R}^n)$, *Jacobi's theta sum* for f is defined to be the function

$$\Theta_f(\tau, \theta; \xi, t) := \sum_{\alpha \in \mathbb{Z}^n} \left[\omega^1_{\rm SW}\big((\xi; t)(\tau, \theta)\big) f\right](\alpha), \tag{7.29}$$

where $(\tau, \theta) \in SL(2, \mathbb{R}) \hookrightarrow Sp(n, \mathbb{R})$ and $(\xi; t) \in H^{(n,1)}_{\mathbb{R}}$ with $\xi = (\lambda, \mu),\ \lambda, \mu \in \mathbb{R}^n$ and $t \in \mathbb{R}$. For $f, g \in \mathcal{S}(\mathbb{R}^n)$, the product of theta sums of the form

$$\Theta_f(\tau, \theta; \xi, t)\, \overline{\Theta_g(\tau, \theta; \xi, t)}$$

is independent of the t-variable.

Let us therefore define the semi-direct product group

$$G[n] := SL(2, \mathbb{R}) \ltimes \mathbb{R}^{2n}$$

with multiplication law

$$(M, \xi)(M', \xi') = (MM',\, \xi + M\xi'), \quad M, M' \in SL(2, \mathbb{R}),\ \xi, \xi' \in \mathbb{R}^{2n}.$$

The set

$$\Gamma[n] =: \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a\, b\, \mathfrak{s} \\ c\, d\, \mathfrak{s} \end{pmatrix} + \alpha \right) \ \Big|\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}),\ \alpha \in \mathbb{Z}^{2n} \right\}$$

with $\mathfrak{s} = {}^t(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}) \in \mathbb{R}^n$ is a subgroup of $G[n]$. We can show that $\Gamma[n]$ is generated by

$$\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0\right), \quad \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \mathfrak{s} \\ 0 \end{pmatrix}\right), \quad \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \alpha\right), \quad \alpha \in \mathbb{Z}^{2n}.$$

We put, for brevity,

$$\Theta_f(\tau, \theta; \xi) := \Theta_f(\tau, \theta; \xi, 0).$$

J. Marklof [32] proved the following properties of Jacobi's theta sums.

Theorem 7.6. *Let f and g be two elements in $\mathcal{S}(\mathbb{R}^n)$. Then*

(1) *$\Theta_f(\tau, \theta; \xi)\,\overline{\Theta_g(\tau, \theta; \xi)}$ is invariant under the action of the left action of $\Gamma[n]$.*

(2) *For any real number $R > 1$, we have*

$$\begin{aligned} &\Theta_f(\tau, \theta; \xi)\,\overline{\Theta_g(\tau, \theta; \xi)} \\ &= y^{n/2} \sum_{\alpha \in \mathbb{Z}^n} f_\theta\big((\alpha - \mu)\, y^{1/2}\big)\, \overline{g_\theta\big((\alpha - \mu)\, y^{1/2}\big)} + O_R\big(y^{-R}\big), \end{aligned}$$

where $\tau = x + i\,y \in \mathbb{H}_1$, $\xi = (\lambda, \mu)$ with $\lambda, \mu \in \mathbb{R}^n$ and

$$f_\theta = \widetilde{R}(i, \theta) f.$$

Proof. The proof can be found in [32, pp. 432-433]. □

The above properties of Jacobi's theta sums together with Ratner's classification of measures invariant under unipotent flows (cf. [38, 39]) are used to prove the important fact that under explicit diophantine conditions on $(\alpha, \beta) \in \mathbb{R}^2$, the local two-point correlations of the sequence given by the values $(m - \alpha)^2 + (n - \beta)^2$ with $(m, n) \in \mathbb{Z}^2$, are those of a Poisson process (see [32] for more detail).

8 Applications of the Weil-Satake Representation

In this section we provide some applications of the Weil-Satake Representation $\widehat{\omega}_{\mathcal{M}, iI_n}$ to the theory of representations of the Jacobi group G^J. Throughout this section, for brevity, we put $G := Sp(n, \mathbb{R})$ and $\widehat{\omega}_{\mathcal{M}} := \widehat{\omega}_{\mathcal{M}, iI_n}$. We will keep the notations and the conventions in Section 5. We recall the notations $G_2 = G_{2, iI_n}$ and $G_2^J = G_2 \ltimes H_{\mathbb{R}}^{(n,m)}$ in Section 5. For a real Lie group $\mathfrak{G}$, we denote by $\widehat{\mathfrak{G}}$ the unitary dual of $\mathfrak{G}$. We define the following projections

$$\begin{aligned} p_2 : G_2 &\longrightarrow G, && (g,t) \longmapsto g, \\ p^J : G^J &\longrightarrow G, && (g,h) \longmapsto g, \\ p_2^J : G_2^J &\longrightarrow G^J, && ((g,t),h) \longmapsto (g,h), \\ p_{2,J} : G_2^J &\longrightarrow G_2, && \big((g,t),h\big) \longmapsto (g,t). \end{aligned}$$

Let $\mathscr{Z}$ be the center of G^J. Obviously $\mathscr{Z} \cong S(m)$.

Proposition 1. *Let $\chi_{\mathcal{M}}$ be the character of $\mathscr{Z}$ defined by $\chi_{\mathcal{M}}(\kappa) = e^{2\pi i\,\sigma(\mathcal{M}\kappa)}$ with $\kappa \in \mathscr{Z}$. We denote by $\widehat{G_2^J}\big(\overline{\chi}_{\mathcal{M}}\big)$ the set of all equivalence classes of irreducible representations η of G_2^J such that $\eta(\kappa) = \chi_{\mathcal{M}}(\kappa)^{-1}$ for all $\kappa \in \mathscr{Z}$. We put $\widetilde{\pi} = \pi \circ p_{2,J}$ for any $\pi \in \widehat{G_2^J}$. Then the correspondence*

$$\widehat{G_2} \longrightarrow \widehat{G_2^J}\big(\overline{\chi}_{\mathcal{M}}\big), \qquad \pi \longmapsto \widetilde{\pi} \otimes \widehat{\omega}_{\mathcal{M}}$$

is a bijection from $\widehat{G_2}$ to $\widehat{G_2^J}\big(\overline{\chi}_{\mathcal{M}}\big)$. Furthermore π is square integrable if and only if $\widetilde{\pi} \otimes \widehat{\omega}_{\mathcal{M}}$ is square square integrable modulo $\mathscr{Z}$.

Proof. The proof can be found in [45]. □

We now consider a holomorphic discrete series representation of G^J. Let K be the stabilizer of the action (1.1) at iI_n. Then

$$K = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in G \;\middle|\; A + i\,B \in U(n) \right\}.$$

Thus K can be identified with the unitary group $U(n)$. Let (ρ, V_ρ) be an irreducible representation of K with highest weight $\rho = (\rho_1, \cdots, \rho_n) \in \mathbb{Z}^n$ such that $\rho_1 \geq \cdots \geq \rho_n \geq 0$. Then ρ can be extended to a rational representation of $GL(n,\mathbb{C})$ that is also denoted by ρ. The representation space V_ρ of ρ has a Hermitian inner product $\langle\ ,\ \rangle$ on V_ρ such that $\langle \rho(g)u, v\rangle = \langle v, \rho(g^*)v\rangle$ for all $g \in GL(n,\mathbb{C}),\ u, v \in V_\rho$, where $g^* = {}^t\overline{g}$. We define the unitary representation τ_ρ of K by

$$\tau_\rho(k) := \rho\big(J(k, iI_n)\big), \quad k \in K. \tag{8.1}$$

For all $\widetilde{g} = (g,h) \in G^J$ with $g \in G$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$, we define

$$J_{\rho,\mathcal{M}}\big(\widetilde{g}, (\Omega, Z)\big) := J_{\mathcal{M}}(\widetilde{g}, (\Omega, Z))\, \rho(J(g,\Omega)). \quad \text{(see (5.1) and (5.2))} \tag{8.2}$$

We note that for all $\widetilde{g} \in G^J$, $(\Omega, Z) \in \mathbb{H}_{n,m}$ and $u, v \in V_\rho$, we have the relation

$$\langle\, J_{\rho,\mathcal{M}}\big(\widetilde{g}, (\Omega, Z)\big)u, v\,\rangle = \langle\, u, J_{\rho,\mathcal{M}}\big(\widetilde{g}, (\Omega, Z)\big)^* v\,\rangle,$$

where

$$J_{\rho,\mathcal{M}}\big(\widetilde{g}, (\Omega, Z)\big)^* = \overline{J_{\mathcal{M}}\big(\widetilde{g}, (\Omega, Z)\big)}\, \rho\big({}^t\overline{J(g,\Omega)}\big).$$

We let $\mathbb{E}_{\rho,\mathcal{M}}$ be the Hilbert space consisting of V_ρ-valued measurable functions f on $\mathbb{H}_{n,m}$ satisfying the condtion

$$(f,f) = ||f||^2 = \int_{\mathbb{H}_{n,m}} \langle\, \rho(Y) f(\Omega,Z), f(\Omega,Z)\,\rangle\, \kappa_{\mathcal{M}}(\Omega,Z)\, dv,$$

where $\kappa_{\mathcal{M}}(\Omega,Z)$ and dv are defined in (5.3) and (6.3) respectively. We let $K^J := K \times \mathscr{Z}$ be a subgroup of G^J. The representation $\Pi_{\rho,\mathcal{M}} := \mathrm{Ind}_{K^J}^{G^J}(\rho \otimes \overline{\chi}_{\mathcal{M}})$ induced from a representation $\rho \otimes \overline{\chi}_{\mathcal{M}}$ is realized on $\mathbb{E}_{\rho,\mathcal{M}}$ as follows: for any $\widetilde{g} \in G^J$ and $f \in \mathbb{E}_{\rho,\mathcal{M}}$, $\Pi_{\rho,\mathcal{M}}$ is given by

$$\big(\Pi_{\rho,\mathcal{M}}(\widetilde{g})f\big)(\Omega,Z) = J_{\rho,\mathcal{M}}(\widetilde{g},(\Omega,Z))^{-1} f(\widetilde{g}^{-1}\cdot(\Omega,Z)). \qquad (8.3)$$

Let $\mathbb{H}_{\rho,\mathcal{M}}$ be the subspace of $\mathbb{E}_{\rho,\mathcal{M}}$ consisting of holomorphic functions in $\mathbb{E}_{\rho,\mathcal{M}}$. It is easily seen that $\mathbb{H}_{\rho,\mathcal{M}}$ is a closed subspace of $\mathbb{E}_{\rho,\mathcal{M}}$ invariant under the action of $\Pi_{\rho,\mathcal{M}}$. We let $\pi_{\rho,\mathcal{M}}$ be the restriction of $\Pi_{\rho,\mathcal{M}}$ to $\mathbb{H}_{\rho,\mathcal{M}}$.

Takase [46] proved the following result.

Theorem 8.7. *Suppose $\rho_n > n + \frac{m}{2}$. Then $\mathbb{H}_{\rho,\mathcal{M}} \neq 0$ and $\pi_{\rho,\mathcal{M}}$ is an irreducible representation of G^J which is square integrable modulo $\mathscr{Z}$. Moreover the multiplicity of ρ in the restriction $\pi_{\rho,\mathcal{M}}|_K$ of $\pi_{\rho,\mathcal{M}}$ to K is equal to one.*

We let

$$K_2 = p_2^{-1}(K) = \left\{\, (k,t) \in K \times T \mid\ t^2 = \det J(k, iI_n)\,\right\}.$$

The Lie algebra $\mathfrak{k}$ of K_2 and its Cartan subalgebra $\mathfrak{h}$ are given by

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \,\middle|\, A + {}^tA = 0,\ B = {}^tB \right\}$$

and

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \,\middle|\, C = \mathrm{diag}\,(c_1, c_2, \cdots, c_n) \right\}.$$

Here $\mathrm{diag}\,(c_1, c_2, \cdots, c_n)$ denotes the diagonal matrix of degree n. We define $\lambda_j \in \mathfrak{h}_{\mathbb{C}}^*$ by $\lambda_j \left(\begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix} \right) := \sqrt{-1}\, c_j$. We put

$$\mathbb{M}^+ = \left\{ \sum_{j=1}^n m_j \lambda_j \,\middle|\, m_j \in \frac{1}{2}\mathbb{Z},\ m_1 \geq \cdots \geq m_n,\ m_i - m_j \in \mathbb{Z} \text{ for all } i,j \right\}.$$

We take an element $\lambda = \sum_{j=1}^n m_j \lambda_j \in \mathbb{M}^+$. Let τ be an irreducible representation of K with highest weight $\tau = (\tau_1, \cdots, \tau_n) \in \mathbb{Z}^n$, where $\tau_j = m_j - m_n\,(1 \leq j \leq n-1)$. Let $\tau_{[\lambda]}$ be the irreducible representation of K_2 defined by

$$\tau_{[\lambda]}(k,t) := t^{2m_n} \cdot \tau(J(k, iI_n)), \quad (k,t) \in K_2. \qquad (8.4)$$

Then $\tau_{[\lambda]}$ is the irreducible representation of K_2 with highest weight $\lambda = (m_1, \cdots, m_n)$ and $\lambda \longmapsto \tau_{[\lambda]}$ is a bijection from $\mathbb{M}^+$ to $\widehat{K}_2$, the unitary dual of K_2. According to [24, Theorem 7.2], we have a decomposition of the restriction $\widehat{\omega}_{\mathcal{M}}|_{K_2}$ into irreducible components:

$$\widehat{\omega}_{\mathcal{M}}|_{K_2} = \bigoplus_{\lambda} m_\lambda\, \tau_{[\lambda]},$$

where λ runs over

$$\lambda = \sum_{j=1}^{s} \tau_j \lambda_j + \frac{m}{2} \sum_{j=1}^{n} \lambda_j \in \mathbb{M}^+ \quad (s = \mathrm{Min}\,\{m, n\}),$$
$$\tau_j \in \mathbb{Z} \text{ such that } \tau_1 \geq \tau_2 \geq \cdots \geq \tau_s \geq 0$$

and the multiplicity m_λ is given by

$$m_\lambda = \prod_{1 \leq i < j \leq m} \left(1 + \frac{\tau_i - \tau_j}{j - i} \right),$$

where $\tau_j = 0$ if $j > s$. Let $\widehat{G}_{2,d}$ be the set of all equivalence classes of square integrable irreducible unitary representations of G_2. The correspondence

$$\pi \longmapsto \text{Harish-Chandra parameter of } \pi$$

is a bijection from $\widehat{G}_{2,d}$ to Λ^+, where

$$\Lambda^+ = \left\{ \sum_{j=1}^{n} m_j \lambda_j \in \mathbb{M}^+ \;\middle|\; m_1 > \cdots > m_n,\ m_i - m_j \neq 0 \text{ for all } i, j,\ i \neq j \right\}.$$

See [50], Theorem 10.2.4.1 for the details.

We choose an element $\lambda = \sum_{j=1}^{n} m_j\, \lambda_j \in \mathbb{M}^+$. Let $\pi^\lambda \in \widehat{G}_{2,d}$ be the representation corresponding to the Harish-Chandra parameter

$$\sum_{j=1}^{n} (m_j - j)\, \lambda_j \in \Lambda^+.$$

The representation π^λ is realized as follows (see [26], Theorem 6.6): Let (τ, V_τ) be the irreducible representation of K with highest weight $\tau = (\tau_1, \cdots, \tau_n)$, $\tau_i = m_i - m_n\ (1 \leq j \leq n-1)$. Let $\mathscr{H}^\lambda$ be a Hilbert space consisting of V_τ-valued holomorphic functions φ on $\mathbb{H}_n$ such that

$$|\varphi|^2 = \int_{\mathbb{H}_n} (\tau(Y)\, \varphi(\Omega), \varphi(\Omega))\, (\det Y)^{m_n}\, dv_\Omega < \infty,$$

where $dv_\Omega = (\det Y)^{-(n+1)}[dX] \wedge [dY]$ is a G-invariant volume element on $\mathbb{H}_n$. Then π^λ is defined realized on $\mathscr{H}^\lambda$ as follows: for any $\sigma = (g,t) \in G_2$ and $f \in \mathscr{H}^\lambda$,

$$\left(\pi^\lambda(\sigma)f\right)(\Omega) = J_{[\lambda]}(\sigma^{-1},\Omega)^{-1} f(\sigma^{-1}\Omega) \tag{8.5}$$

for all $\sigma = (g,t) \in G_2$ and $f \in \mathscr{H}^\lambda$. Here

$$J_{[\lambda]}(\sigma,\Omega) = \left\{ t\,\beta_{iI_n}(g,g^{-1})\,|\det J(g,\Omega)|^{\frac{1}{2}}\,\frac{\gamma(g\Omega, g(iI_n))}{\gamma(\Omega, iI_n)} \right\}^{m_n} \tau(J(g,\Omega)).$$

Proposition 2. *Suppose $\tau_n > n + \frac{m}{2}$. We put $\lambda = \sum_{j=1}^n (\tau_j - \frac{m}{2})\lambda_j \in \mathbb{M}^+$. Then the unitary representation $\pi_{\tau,\mathcal{M}} \circ p_2^J$ of G_2^J is unitarily equivalent to the representation $(\pi^\lambda \circ p_{2,J}) \otimes \widehat{\omega}_{\mathcal{M}}$.*

Proof. The proof can be found in [46]. □

Using Theorem 8.7, Takase [48] established a bijective correspondence between the space of cuspidal Jacobi forms and the space of Siegel cusp forms of half integral weight which is compatible with the action of Hecke operators. For example, the classical result (cf. [15] and [22])

$$J_{k,1}^{\text{cusp}}(\Gamma_n) \cong S_{k-1/2}(\Gamma_0(4)) \tag{8.6}$$

can be obtained by the method of the representation theory. Here Γ_n denotes the Siegel modular group of degree n and $\Gamma_0(4)$ denotes the Hecke subgroup of Γ_n.

References

1. S. T. Ali, J. P. Antoine and J.-P. Gazeau, *Coherent states, wavelets, and their generalizations*, Springer-Verlag, New York (2000).
2. E. Balslev, Spectral theory of the Laplacian on the modular Jacobi group manifold, preprint, Aarhus University (2012).
3. S. Berceau, *Coherent states associated to the Jacobi group – a variation on a theme by Erich Kähler*, Jour. of Geometry and Symmetry in Physics, Vol. **9** (2007), 1–8.
4. S. Berceau, *Coherent states associated to the Jacobi group*, Romanian Reports in Physics, Vol. **59**, No. 4 (2007), 1089-1101.
5. S. Berceau, *Generalized squeezed states for the Jacobi group*, arXiv:0812.071v1 [math.DG] 3 Dec 2008.
6. S. Berceau, *Consequences of the fundamental conjecture for the motion on the Siegel-Jacobi disk*, arXiv:1110.5469v1 [math.DG] 25 Oct 2011.
7. S. Berceau, *A convenient coordinatization of on the Siegel-Jacobi domains*, arXiv:1204. 5610v1 [math.DG] 25 Apr 2012.
8. S. Berceau and A. Gheorghe, *Holomorphic discrete series representations on Siegel-Jacobi domains*, 22-26 November, 2010, Lorentz Center, Leiden.
9. S. Berceau and A. Gheorghe, *On the geometry of Siegel-Jacobi domains*, arXiv:1011. 3317v1 [math.DG] 15 Nov 2010.

10. R. Berndt and R. Schmidt, *Elements of the Representation Theory of the Jacobi Group*, Birkhäuser, 1998.
11. K. Bringmann, C. Conley and O. K. Richter, *Maass-Jacobi forms over complex quadratic fields*, Math. Res. Lett. **14**, No. 1 (2007), 137–156.
12. K. Bringmann, C. Conley and O. K. Richter, *Jacobi forms over complex quadratic fields via the cubic Casimier operators*, preprint.
13. K. Bringmann and O. K. Richter, *Zagier-type dualities and lifting maps for harmonic Maass-Jacobi forms*, Advances in Math. **225** (2010), 2298–2315.
14. C. Conley and M. Raum, *Harmonic Maass-Jacobi forms of degree 1 with higher rank indices*, arXiv:1012.289/v1 [math.NT] 13 Dec 2010.
15. M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progress in Math., **55**, Birkhäuser, Boston, Basel and Stuttgart, 1985.
16. E. Freitag, *Siegelsche Modulfunktionen*, Grundlehren de mathematischen Wissenschaften **55**, Springer-Verlag, Berlin-Heidelberg-New York (1983).
17. F. Gay-Balmaz and C. Tronci, *Vlasov moment flows and geodesics on the Jacobi group*, arXiv:1105.1734v1 [nlin.SI] 9 May 2011.
18. S. Gelbart, *Weil's Representation and the Spectrum of the Metaplectic Group*, Lecture Notes in Math. **530**, Springer-Verlag, Berlin and New York, 1976.
19. V. Guillemin and S. Sternberg, *Geometric Asymptotics*, Amer. Math. Soc., Providence, R. I. (1977).
20. V. Guillemin and S. Sternberg, *Symplectic Technique in Physics,* Cambridge University Press, Cambridge (1984).
21. J. N. Hollenhors, Phy. Rev. D, **19** (1979), 1669-1679.
22. T. Ibukiyama, *On Jacobi forms and Siegel modular forms of half integral weights*, Comment. Math. Univ. Sancti Pauli **41** (1992), 109-124.
23. T. Ikeda, *On the lifting of elliptic cusp forms to Siegel cusp forms of degree* $2n$, Ann. Math. **154** (2001), 641–681.
24. M. Kashiwara and M. Vergne, *On the Segal-Shale-Weil Representations and Harmonic Polynomials*, Invent. Math. **44** (1978), 1–47.
25. S. Katok and P. Sarnak, *Heegner points, cycles and Maass forms*, Israel Math. J. **84** (1993), 237–268.
26. A. W. Knapp, *Representation Theory of Semisimple Groups*, Princeton Univ. Press, Princeton, (1986).
27. S. Kudla and S. Rallis, *On the Weil-Siegel formula*, J. reine angew. Math. **387** (1988), 1–68.
28. S. Kudla and S. Rallis, *On the Weil-Siegel formula II*, J. reine angew. Math. **391** (1988), 65-84.
29. S. Kudla and S. Rallis, *A regularized Weil-Siegel formula : the first term identity*, Ann. of Math.(2) **140** (1994), no. 1, 1-80.
30. G. Lion and M. Vergne, *The Weil representation, maslov index and Theta seires*, Progress in Math., **6**, Birkhäuser, Boston, Basel and Stuttgart, 1980.
31. E. Y. C. Lu, Lett. Nuovo. Cimento **2** (1971), 1241–1244.
32. J. Marklof, *Pair correlation densities of inhomogeneous quadratic forms*, Ann. Math. **158** (2003), 419-471.
33. D. Mumford, *Tata Lectures on Theta I,* Progress in Math. **28**, Boston-Basel-Stuttgart (1983).
34. S. Niwa, *Modular forms of half-integral weight and the integral of certain theta-functions*, Nagoya Math. J., **56** (1975), 147–161.
35. A. M. Perelomov, *Generalized Coherent States and their Applications*, Springer-Verlag, Berlin (1986).
36. A. Pitale, *Lifting from* $\widetilde{SL_2}$ *to* $GSpin(1,4)$, Int. math. Res. Notices, Vol. **63** (2005), 3919–3966.
37. A. Pitale, *Jacobi Maass forms*, Abh. Math. Sem. Hamburg **79** (2009), 87–111.

38. M. Ratner, *On Raghunathan's measure conjecture* , Ann. of Math., **134** (1991), 545–607.
39. M. Ratner, *Raghunathan's topological conjecture and distributions of unipotent flows,*, Duke Math. J. **63** (1991), 235–280.
40. I. Satake, *Fock representations and theta functions*, Ann. Math. Study **66** (1969), 393–405.
41. G. Shimura, *On modular forms of half integral weight* , Ann. of Math., **97**(1973), 440–481.
42. T. Shintani, *On construction of holomorphic cusp forms of half integral weight*, Nagoya Math. J., **58** (1975), 83–126.
43. C. L. Siegel, *Indefinite quadratische Formen und Funnktionentheorie I and II*, Math. Ann. **124** (1951), 17–54 and Math. Ann. **124** (1952), 364–387; Gesammelte Abhandlungen, Band III, Springer-Verlag (1966), 105–142 and 154–177.
44. P. Stoler, Phys. rev. D. **1** (1970), 3217–3219.
45. K. Takase, *A note on automorphic forms*, J. reine angew. Math., **409**(1990), 138-171.
46. K. Takase, *On unitary representations of Jacobi groups*, J. reine angew. Math., **430**(1992), 130-149.
47. K. Takase, *On Two-fold Covering Group of* $Sp(n,\mathbb{R})$ *and Automorphic Factor of Weight 1/2*, Comment. Math. Univ. Sancti Pauli **45** (1996), 117-145.
48. K. Takase, *On Siegel Modular Forms of Half-integral Weights and Jacobi Forms*, Trans. of American Math. Soc. **351**, No. 2 (1999), 735–780.
49. J.-L. Waldspurger, *Sur les coefficients de Fourier de formes modulaires de poids demi-entier*, J. Math. Pures Appl. **60** (1981), 375–484.
50. G. Warner, Harmonic Analysis on Semisimple Lie Groups, I, II, Springer-Verlag, Berlin-Heidelberg-New York, (1972).
51. A. Weil, *Sur certains groupes d'operateurs unitares*, Acta Math., **111** (1964), 143–211.
52. K. B. Wolf, *Geometric Optics on Phase Space*, Springer (2004).
53. J.-H. Yang, *Harmonic Analysis on the Quotient Spaces of Heisenberg Groups*, Nagoya Math. J., **123**(1991), 103–117.
54. J.-H. Yang, *Harmonic Analysis on the Quotient Spaces of Heisenberg Groups II* , J. Number Theory, **49**(1)(1994), 63–72.
55. J.-H. Yang, *A decomposition theorem on differential polynomials of theta functions of high level*, Japanese J. of Mathematics, The Mathematical Society of Japan, New Series, **22**(1)(1996), 37–49.
56. J.-H. Yang, *Fock Representations of the Heisenberg Group* $H_{\mathbb{R}}^{(g,h)}$, J. Korean Math. Soc., **34**, no. 2 (1997), 345–370.
57. J.-H. Yang, *Lattice Representations of the Heisenberg Group* $H_{\mathbb{R}}^{(g,h)}$, Math. Annalen, **317**(2000), 309–323.
58. J.-H. Yang, *The Siegel-Jacobi Operator*, Abh. Math. Sem. Univ. Hamburg **63** (1993), 135–146.
59. J.-H. Yang, *Remarks on Jacobi forms of higher degree*, Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, the Pyungsan Institute for Mathematical Sciences, Seoul (1993), 33–58.
60. J.-H. Yang, *Singular Jacobi Forms*, Trans. of American Math. Soc. **347**, No. 6 (1995), 2041–2049.
61. J.-H. Yang, *Construction of Modular Forms from Jacobi Forms*, Canadian J. of Math. **47** (1995), 1329–1339.
62. J.-H. Yang, *A geometrical theory of Jacobi forms of higher degree*, Proceedings of Symposium on Hodge Theory and Algebraic Geometry (edited by Tadao Oda), Sendai, Japan (1996), 125-147 *or* Kyungpook Math. J. **40 (2)**, 209–237 (2000) *or* arXiv:math.NT/0602267.
63. J.-H. Yang, *The method of orbits for real Lie groups*, Kyungpook Math. J. **42** (2002), 199–272.

64. J.-H. Yang, *A Note on Maass-Jacobi Forms*, Kyungpook Math. J. **43** (2003), 547–566.
65. J.-H. Yang, *A note on a fundamental domain for Siegel-Jacobi space,* Houston Journal of Mathematics, Vol. **32**, No. 3 (2006), 701–712.
66. J.-H. Yang, *Invariant metrics and Laplacians on Siegel-Jacobi space,* arXiv: math.NT/0507215 v1 or Journal of Number Theory **127** (2007), 83-102.
67. J.-H. Yang, *A partial Cayley transform for Siegel-Jacobi disk,* J. Korean Math. Soc. **45**, No. 3 (2008), 781-794 *or* arXiv:math.NT/0507216.
68. J.-H. Yang, *Invariant metrics and Laplacians on Siegel-Jacobi disk,* Chinese Annals of Mathematics, Vol. 31B(1), 2010, 85-100 *or* arXiv:math.NT/0507217.
69. J.-H. Yang, *The Schrödinger-Weil Representation and Jacobi Forms of Half Integral Weight*, arXiv:0908.0252v1 [math.NT] 3 Aug 2009.
70. J.-H. Yang, *A Note on Maass-Jacobi Forms II,* to appear in Kyungpook Math. J.(2013).
71. J.-H. Yang, *Invariant Differential Operators on Siegel-Jacobi Space,* arXiv:1107.0509 v1 [math.NT] 4 July 2011.
72. J. Yang and L. Yin, Derivation of Jacobi forms from connections, arXiv:1301.1156v1 [math.NT] 7 Jan 2013.
73. H. P. Yuen, Phys. Rev. A, **13** (1976), 2226–2243.
74. C. Ziegler, *Jacobi Forms of Higher Degree*, Abh. Math. Sem. Hamburg **59** (1989), 191–224.

[G] Theory of Elliptic Curves

전북대학교에서 개최된 국제학술회의에서 (2002년 7월)

PROCEEDINGS OF THE 1994 CONFERENCES
OCTOBER 11 AND DECEMBER 20-21

NUMBER THEORY AND RELATED TOPICS

Editors
Jin-Woo Son
Jae-Hyun Yang

The Pyungsan Institute for Mathematical Sciences
Seoul, Republic of Korea

NUMBER THEORY

AND

RELATED TOPICS

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CONTENTS

PREFACE

The present volume contains six papers based on the talks given at two conferences which were held in Masan on October 11, 1994 and in Pusan on December 20-21, 1994. One of these conferences was organized by Professor Jin-Woo Son and supported financially by Kyungnam University. The other conference was held at Pusan National University and supported financially by the Pyungsan Institute for Mathematical Sciences. These two conferences were origanized to stimulate Korean young mathematicians and graduate students to make a research on more advanced areas in mathematics and to upgrade Korean Mathematics.

Pusan is the largest seaport in Korea and is famous for its beatuiful scenery and a political activity. For instance, Pusan has beautiful beaches, mountains, parks and so on. One of the editors was born and raised in this beautiful city. Masan is also a seaport and is quite famous for a political reason. The present president, Young-Sam Kim was born and grew up in the small island close to Masan. At the critical moment politically, many people in the city, Masan had taken a rally to the democracy in Korea. In this sense the above two cities are meaningful places in Korea. In fact, two editors were affected politically while they grew up in these cities. Most people in the fifties and forties had spent a hard time sacrificing their precious youth for the democracy past three decades. We would like to dedicate this proceedings to the people in Korea who have fought against the dictatorship, injustice and corruption at a hard time.

Finally, we would like to give our sincere gratitude to all the invited speakers for their enthusiastic talks and their contributed articles. Also we would like to give our hearty and deep thanks to Kyungnam University, Pusan National University and the colleagues who helped us preparing these conferences for their effort.

Jin-Woo Son

Jae-Hyun Yang

December, 1995

Masan, Korea

LIST OF TALKS

[1] The conference at Kyungnam University, Masan, Korea

(October 11, 1994)

- Hi-Joon Chae : KAIST, Taejeon, Korea
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- Winfried Kohnen : MPI, Bonn, Germany
  Elliptic Curves and Automorphic Forms

- Jae-Hyun Yang : Inha University, Incheon, Korea
  Note on Taniyama-Shimura-Weil Conjecture

[2] The conference at Pusan University, Pusan, Korea

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- Yukihiko Namikawa : Nagoya University, Nagoya, Japan
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- Jae-Hyun Yang : Inha University, Incheon, Korea
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  (2) Supplementary Note on the Freund-Witten Adelic Formula for Veneziano Amplitudes

- Young Ho Im : Pusan National University, Pusan, Korea
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NOTE ON TANIYAMA–SHIMURA–WEIL CONJECTURE

JAE-HYUN YANG

ABSTRACT. In this article, we discuss Taniyama-Shimura-Weil conjecture relating to Wiles' work on Fermat's Last Theorem.

Introduction

On June 21-23 in 1993, Professor Andrew Wiles had given a series of lectures under the title, "Elliptic curves, modular forms and Galois representations" at the Issac Newton Institute for Mathematical Sciences in Cambridge, England. In this last lecture Wiles commented that he had proved a part of Taniyama-Shimura-Weil conjecture to the effect that every semistable elliptic curve over $\mathbb{Q}$ is modular. With the aid of the works of Frey [4], Serre [17] and Ribet [12], his proof solves Fermat's Last Theorem which had been unsolved for more than 350 years. The news spread out all over the world through the well-known newspapers and magazines because Fermat's Last Theorm holds great fascination for amateurs and professionals alike. But in the fall that year it turned out that the proof of Wiles was incomplete and flawed. Precisely, his construction of the Euler system used to extend Flach's method was not complete. He was very concerned with filling the gaps in his flawed proof. At that time he was completely isolated from outside. In January 1994, he proposed Dr. Richard Taylor in Cambridge University, UK to join him in the attempt to repair the Euler system argument. Still Wiles was convinced that his method was correct. Dr. Taylor accepted his proposal and joined in that project. On September 19th in 1994, he was quite convinced that his method was correct and his gap could be filled up. After he invited Dr. Taylor to Princeton again, he completed his proof with the aid of Dr. Taylor on October, 1994 and submitted his paper to Annals of Mathematics on October 14, 1994. Finally his paper appeared in that journal this May [22].

The Taniyama-Shimura-Weil conjecture, briefly TSW conjecture, is of great significance for modern mathematics. So in the fall of 1993, the author had the need to introduce TSW conjecture in particular to Korean young mathematicians and graduate students. In May and October in 1994, the author gave a survey talk on TSW conjecture relating to Wiles' work on Fermat's Last Theorm in the 2nd Workshop of the Pyungsan Institute for Mathematical Sciences and the conference on Algebra and Number Theory which were held in Kyungju, the one-thousand years old capital of the Silla Dynasty and in Masan respectively. I remember that the audience was quite interested in this conjecture and was pleased to know the reality of TSW conjecture and its connection to Fermat's Last Theorem.

This work was financially supported by Inha University 1995 .

Returning to the history of TSW conjecture, this conjecture was first proposed by Yutaka Taniyama at the Tokyo-Nikko conference in the mid 1950s. Its statement was refined through the efforts of Goro Shimura and Andre Weil. In fact, TSW conjecture only became widely known through its publication in the paper of Weil [21] in 1967, in which Weil gave conceptual evidence for the conjecture. TSW conjecture associates objects of representation theory to those of algebraic geometry. It states the L-series of an elliptic curve over $\mathbb{Q}$, which measures the behaviour of the curve mod p for all primes p, can be identified with an integral transform of the Fourier series derived from a modular form. In 1985 Frey made the remarkable observation that TSW conjecture should imply Fermat's Last Theorem. The precise mechanism relating the two was formulated by Serre [17] as the ϵ-conjecture and this was proved by Ribet [12] in the summer of 1986. Ribet's result only require one to prove TSW conjecture for semistable elliptic curves over $\mathbb{Q}$ in order to deduce Fermat's Last Theorem. As soon as Wiles learned of Ribet's result, he began working on TSW conjecture for semistable elliptic curves over $\mathbb{Q}$ in the late summer of 1986. Finally Wiles proved TSW conjecture for semistable ones and so proved Fermat's Last Theorem in October, 1994. TSW conjecture is a particular case of the "Langlands Philosophy" made by R. P. Langlands and his colleagues. Although the Langlands conjectures require a substantial background in automorphic forms, TSW conjecture has been rephrased in such a way that only complex-analytic maps appear (cf. see section 1 and [8]). TSW conjecture is quite interesting in the sense that it brings us closer to tying together automorphic representations and algebraic varieties.

This paper is organized as follows. In section one, we state equivalent statements of TSW conjecture without proof. We mention that Wiles proved a part of TSW conjecture, that is, every semistable elliptic curve over $\mathbb{Q}$ is modular. We observe the fact that his proof solves Fermat's Last Theorem. We present the connections of TSW conjecture to other conjectures, in particular, the Hasse-Weil conjecture. In section two, we sketch the Wiles' work on a part of TSW conjecture following the content of the survey paper of K. Rubin and A. Silverberg [16] and the paper of Wiles [22]. In the appendices, we discuss the Eichler-Shimura Theorem and state the relation of an elliptic curve to an elliptic integral briefly. Finally I would like to recommend to the reader a good survey paper of Ribet [15], in which he discuss Wiles' proof of Fermat's Last Theorem.

ACKNOWLEDGEMENT. I would like to give my hearty thanks to Professor W. Kohnen and my Korean colleagues for their interest in this work.

1 Taniyama-Shimura-Weil Conjecture

First of all, we state the Taniyama-Shimura-Weil Conecture.

[Taniyama – Shimura – Weil Conjecture]

The following statements are equivalent :

(a) If E is an arithmetic elliptic curve, there exists a positive integer $N \in \mathbb{Z}^+$ for which there is a nonconstant morphism $X_0(N) \longrightarrow E$, where $X_0(N)$ is the modular curve of level N.

(b) An arithmetic elliptic curve has a modular parametrization.

(c) An arithmetic elliptic curve is modular, i.e., there is an eigenform

$$f(z) = \sum_{n=1}^{\infty} a_n \, e^{2\pi i n z}$$

such that for almost all primes q,

$$a_q = q + 1 - \sharp(E(\mathbf{F}_q)) .$$

(d) An arithmetic elliptic curve is a Weil curve.

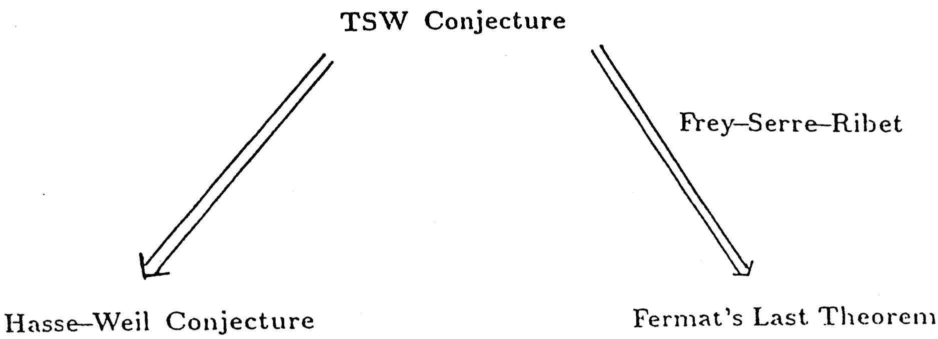
(e) Given an elliptic curve $y^2 = ax^3 + bx^2 + cx + d$ over $\mathbb{Q}$, there exist nonconstant modular functions $f(z)$, $g(z) \in A_0(\Gamma_0(N))$ of the same level N such that

$$f(z)^2 = ag(z)^3 + bg(z)^2 + cg(z) + d,$$

that is, an arithmetic elliptic curve can be parametrized by modular functions of the same level.

(f) If E is an arithmetic elliptic curve, then there exists a surjective holomorphic map $X_0(N) \longrightarrow E$.

(g) Any arithmetic elliptic curve admits a hyperbolic uniformization of arithmetic type.



From now on, we write *TSW Conjecture* briefly instead of Taniyama-Shimura-Weil Conjecture.

Definition. An elliptic curve over $\mathbb{Q}$ is said to be *semistable at the prime* q if it is isomorphic to an elliptic curve over $\mathbb{Q}$ which modulo q either is nonsingular or has a node. An elliptic curve over $\mathbb{Q}$ is called *semistable* if it is semistable at every prime.

Theorem 1.1 (Wiles [22]). Every semistable elliptic curve over $\mathbb{Q}$ is modular. This is a part of TSW Conjecture.

(*) TSW Conjecture implies FLT

We assume that we have a solution $(a, b, c) \in \mathbb{Z}^3$ such that

$$a^p + b^p = c^p, \qquad abc \neq 0.$$

We may assume that p is a prime greater than 3 and $g.c.d.(a, b, c) = 1$ with b even and $a \equiv -1\,(mod\ 4)$.

We consider the socalled *Frey curve*

$$F(a, b : p) \ : \ y^2 = x(x - a^p)(x + b^p).$$

Then the discriminant D of $F(a, b : p)$ is given by

$$D = a^{2p} b^{2p} c^{2p} = (abc)^{2p}. \quad \textit{unusual}\,!!!$$

This is our first hint that the Frey curve $F(a, b : p)$ is very *special.*

Remark. Y. Helloguarch [5] considered the Frey curve for a solution to FLT of exponent $2p^h$. It seems that Frey was the first one to suspect that the Frey curve couldn't exist because of the TSW Conjecture.

1*st Proof of* $(*)$:

Lemma 1. The Frey curve $F(a, b : p)$ is semistable.

Proof. See [3], [4] and [16]. □

Lemma 2. For any odd prime $l|N$, the j-invariant of the Frey curve $F(a, b : p)$ can be written as $j = l^{-mp} \cdot q$, where $m \in \mathbb{Z}^+$ and q is a fraction not involving l. In this case, we say that the j-invariant is exactly divisible by l^{-mp}. Here N is the conductor of the Frey curve.

Proof.

$$j = \frac{2^8\,(a^{2p} + b^{2p} + a^p b^p)^3}{(abc)^{2p}} = \frac{2^8(c^{2p} - b^p c^p)}{(abc)^{2p}}.$$

The power of l dividing the denominator $(abc)^{2p}$ is a multiple of p. Since $g.c.d.(a, b, c) = 1$, $g.c.d.((c^{2p} - b^p c^p), (abc)^{2p}) = 1$. Since N is the product of the primes dividing abc, the lemma follows. □

Since the Frey curve $F(a, b : p)$ is modular by Theorem 1.1, there is an eigenform $f \in S_2(\Gamma_0(N))$, where N is the conductor of $F(a, b : p)$. The Frey curve also has a Galois representation ρ on the points of order p on the curve. According to Serre's level reduction conjecture, there is a cusp form $f_l \in S_2(\Gamma_0(N/l))$ with

$$f_l \equiv f \ (\text{mod } p)$$

and f_l is also an eigenform for a suitable Hecke algebra. If l' is another odd prime dividing N, we apply the level reduction conjecture to the pair (ρ, f_l) and get a cusp form $f_{ll'} \in S_2(\Gamma_0(N/ll'))$. Continuing this process, we get a cusp form $f \in S_2(\Gamma_0(2))$ because b is even. But it is known that $S_2(\Gamma_0(2)) = 0$. This is a contradiction. Consequently FLT is proved ! □

2nd Proof of $(*)$:

Definition. A *hyperbolic uniformization* of an elliptic curve over E is a mapping

$$\alpha : H_1 - \{\text{finite union of } \Gamma - \text{orbits}\} \longrightarrow E - \{\text{finite set of points}\}$$

which is a covering mapping of the domain onto the range preserving the orientation, is a local isomorphism of conformal geometries and is periodic with respect to a subgroup of finite index in the elliptic modular group $\Gamma = SL(2,\mathbb{Z})$. The last condition means that there exists a subgroup $\Gamma' \subset \Gamma$ of finite index such that

$$\alpha(\gamma(z)) = \alpha(z) \quad \text{for all } \gamma \in \Gamma'.$$

In other words, α factors through the orbit space of H_1 under the action of Γ'.

Remark. (1) For any $\Gamma' \subset \Gamma$ with $[\Gamma : \Gamma'] < \infty$, there are only a finite number of distinct elliptic curves admitting a hyperbolic uniformization periodic with respect to Γ'.

(2) Weierstrass proved that an elliptic curve E admits a Euclidean uniformization

$$\mathbb{C} \longrightarrow \mathbb{C}/\Lambda$$

with respect to a lattice $\Lambda \subset \mathbb{C}$ up to complex scalar change. On the other hand, Bely proved that an elliptic curve admits a hyperbolic uniformization if and only if it has a Weierstrass equation

$$y^2 = 4x^3 + Ax + B, \quad A, B \in \overline{\mathbb{Q}}.$$

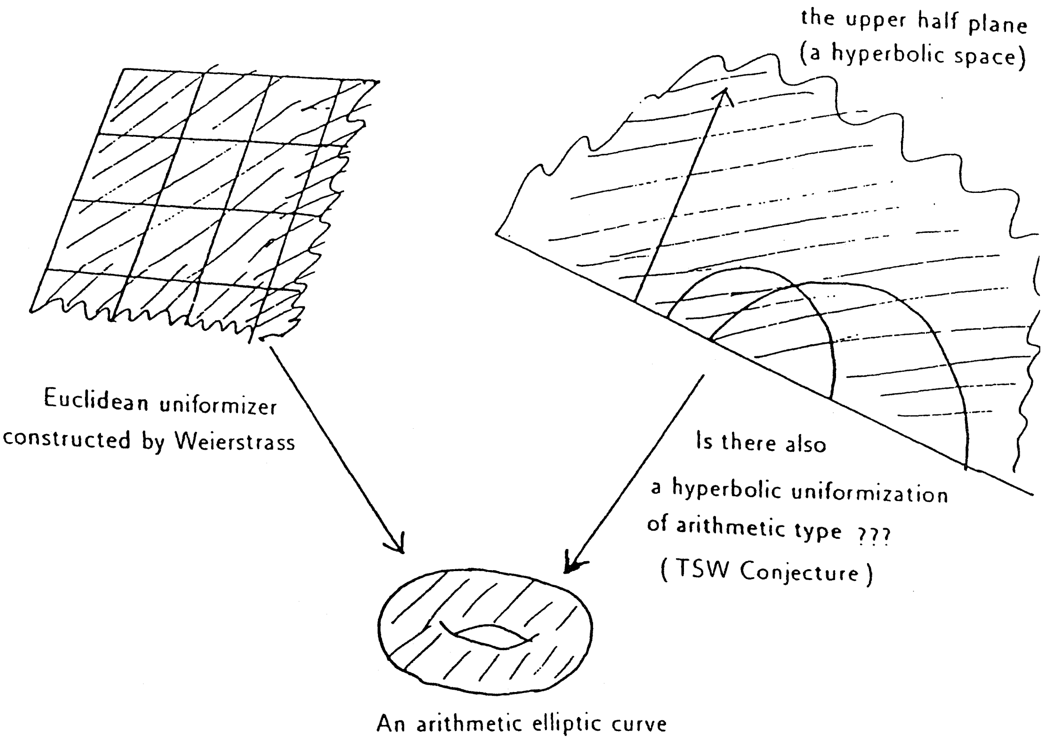
Definition. Let E be an elliptic curve. A *hyperbolic uniformization of arithmetic type* is a hyperbolic uniformization of E which is periodic with respect to a congruence subgroup $\Gamma' \subset \Gamma$.

Theorem 1.2 (Ribet [12]). Let $p\,(\geq 5)$ be a prime. Then the Frey curve $F(a,b:p)$ does not admit a hyperbolic uniformization of arithmetic type, that is, $F(a,b:p)$ is not modular.

Proof. We refer a sketchy proof of the above theorem to [11]. Suppose that the Frey curve admits a hyperbolic uniformization α of arithmetic type. If ω is a regular differential 1-form on $F(a,b:p)$, then $\alpha^*\omega$ is a differential 1-form on H_1 which after suitable normalization can be written as $f(z)dz$, where $f(z) \in S_2(\Gamma_0(N))$ is an eigenform of the Hecke operator with integral Fourier coefficients:

$$f(z) = \sum_{n=1}^{\infty} c_n e^{2\pi i n z}, \quad c_1 = 1, \quad c_n \in \mathbb{Z}.$$

Using an approach suggested by Serre's level reduction conjecture and using the special form of the discriminant and the conductor of $F(a,b:p)$, Ribet has shown that $c_1 = 1$, c_2, $c_3, \cdots$ are congruent modulo p to the Fourier coefficients of a modular form $\varphi \in A_2(\Gamma_0(2))$. He makes use of the "Drinfeld switch" which occurs in the description of the "bad fibres" of Shimura curves. Theorem of Eichler-Shimura links the Fourier coefficients c_j modulo p to the questions of $\mathbb{Q}$-rational p-torsion on $F(a,b:p)$. From our explicit knowledge of α_j modulo p, we deduce that there exists a rational point of order p on $F(a,b:p)$. But any arithmetic elliptic curve, which like the Frey curve $F(a,b:p)$ has all four points of order 2 rational over $\mathbb{Q}$, cannot have any rational p-torsion for $p \geq 5$ (cf. [9]). This is a contradiction !!! $\square$



Folk Theorem 1.3 (cf. Mazur [8]). If an elliptic curve E over $\mathbb{Q}$ admits a nonconstant mapping from $X_0(N)/\mathbb{C} \longrightarrow E/\mathbb{C}$ for some N, then E admits a nonconstant mapping $X_0(N)/\mathbb{Q} \longrightarrow E/\mathbb{Q}$ (but possibly for a different value of N).

Let E be an elliptic curve over $\mathbb{Q}$. We define

$$a_p := p + 1 - \sharp(E(\mathbf{F}_p)), \quad p \ \ a\ prime. \tag{1.2}$$

The L function $L(s,E)$ of E is defined as the product of the local L factors:

$$L(s,E) := \prod_{p|D} \left(\frac{1}{1 - a_p p^{-s}} \right) \cdot \prod_{p \nmid D} \left(\frac{1}{1 - a_p p^{-s} + p^{1-2s}} \right). \tag{1.3}$$

Then $L(s,E)$ converges absolutely for $\mathrm{Re}\, s > \frac{3}{2}$ and is given by an absolutely convergent Dirichlet series.

Expectations: (1) Deep arithmetic information is encoded in the behaviour of $L(s,E)$ beyond the region of convergence. For example, the Birch and Swinnerton-Dyer Conjecture predicts that the rank of $E(\mathbb{Q})$ is the order of vanishing of $L(s,E)$ at $s = 1$.
(2) $L(s,E)$ has an analytic continuation. In view of the behaviour of a Dirichlet series, $L(s,E)$ satisfies the functional equation.

Let $f(z) = \sum_{n=1}^{\infty} c_n e^{2\pi i n z}$ be a newform in $S_k^{\pm}(\Gamma_0(n))$ with

$$S_k(\Gamma_0(N)) = S_k^{+}(\Gamma_0(N)) \oplus S_k^{-}(\Gamma_0(n)), \quad \omega_N^2 = 1. \tag{1.4}$$

We define the L-series $L(s,f)$ of f by

$$L(s,f) := \sum_{n=1}^{\infty} \frac{c_n}{n^s}. \quad (formally) \tag{1.5}$$

Let m be an integer relatively prime to N and χ be a primitive Dirichlet character modulo m. We put

$$L(s,f,\chi) := \sum_{n=1}^{\infty} \frac{c_n \chi(n)}{n^s}. \tag{1.6}$$

We define

$$\Lambda(s,f,\chi) := (m^2 N)^{\frac{s}{2}} (2\pi)^{-s} \Gamma(s) L(s,f,\chi). \tag{1.7}$$

We recall the *Gauss sum*

$$c(m,\chi) := \sum_{l=0}^{m-1} e^{\frac{2\pi i l}{m}} \chi(l). \tag{1.8}$$

Relationships of TSW Conjecture to other Conjectures

First we state some important theorems.

Theorem 1.4. Let $f \in S_k^{\epsilon}(\Gamma_0(N))$ with $\epsilon = \pm$, $g.c.d.(m,N) = 1$ and χ be a primitive Dirichlet character modulo m. Let $L(s,f,\chi)$ and $\Lambda(s,f,\chi)$ be the L-series defined by (1.6) and (1.7) respectively. Then $L(s,f,\chi)$ converges for $\operatorname{Re} s > \frac{k}{2}+1$ and extends to be entire in s. Moreover, $\Lambda(s,f,\chi)$ is entire and bounded in the vertical strips. And $\Lambda(s,f,\chi)$ satisfies the functional equation

$$\Lambda(s,f,\chi) = \epsilon\,(-1)^{\frac{k}{2}} \frac{c(m,\chi)\chi(-N)}{c(m,\overline{\chi})} \Lambda(k-s,f,\overline{\chi}), \tag{1.9}$$

where $c(m,\chi)$ is the Gauss sum (1.8).

If $L(s,E) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is the L function of an elliptic curve E and χ is a primitive Dirichlet character, we define

$$L(s,E,\chi) := \sum_{n=1}^{\infty} \frac{a_n \chi(n)}{n^s}. \tag{1.10}$$

Conjecture 1.5 (Hasse-Weil). Let E be an arithmetic elliptic curve. Then the function $L(s,E)$ extends to be entire and for some $N \in \mathbb{Z}^+$, so does $L(s,E,\chi)$ for every primitive Dirichlet character whose conductor m is prime to N. Moreover, the modified functions

$$\Lambda(s,E) := N^{\frac{s}{2}} (2\pi)^{-s} \Gamma(s) L(s,E) \tag{1.11}$$

and

$$\Lambda(s,E,\chi) := (m^2N)^{\frac{s}{2}}(2\pi)^{-s}\Gamma(s)L(s,E,\chi) \tag{1.12}$$

extends to be entire and, for a suitable sign ϵ, satisfies the functional equation

$$\Lambda(s,E) = -\epsilon\,\Lambda(2-s,E) \tag{1.13}$$

and

$$\Lambda(s,E,\chi) = -\frac{\epsilon\, c(m,\chi)\chi(-N)}{c(m,\overline{\chi})}\,\Lambda(2-s,E,\overline{\chi}). \tag{1.14}$$

Question (Weil): How close is $L(s,E)$ to coming from a modular form if it satisfies functional equations (1.13) and (1.14)?

Theorem 1.6 (Weil [21]). Let $L(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}$ be a Dirichlet series with $c_n = O(n^c)$ for some $c>0$. Fix a positive integer $N\in\mathbb{Z}^+$, an even positive integer k and a sign ϵ. Suppose that
(a) the function

$$\Lambda(s) := N^{\frac{s}{2}}(2\pi)^{-s}\Gamma(s)L(s)$$

is entire, is bounded in every vertical strip and satisfies

$$\Lambda(s) = \epsilon\,(-1)^{\frac{k}{2}}\Lambda(k-s),$$

(b) for every integer $m\in\mathbb{Z}^+$ with $g.c.d.(m,N)=1$ and every primitive Dirichlet character χ modulo m, the modified function

$$L(s,\chi) := \sum_{n=1}^{\infty}\frac{a_n\chi(n)}{n^s}$$

is such that the function

$$\Lambda(s,\chi) := (m^2N)^{\frac{s}{2}}(2\pi)^{-s}\Gamma(s)L(s,\chi)$$

is entire, is bounded in every vertical strip and satisfies the functional equation

$$\Lambda(s,\chi) = \epsilon\,(-1)^{\frac{k}{2}}\frac{c(m,\chi)\chi(-N)}{c(m,\overline{\chi})}\,\Lambda(k-s,\overline{\chi}),$$

(c) the series $L(s)$ converges absolutely at $s=k-\delta$ for some $\delta>0$. Then the function

$$f(z) = \sum_{n=1}^{\infty}a_n e^{2\pi i n z}$$

is a cusp form in $S_k(\Gamma_0(N))$.

According to Theorem 1.4 and 1.6, the Hasse-Weil Conjecture is completely equivalent to the following fundamental conjecture.

Conjecture 1.7 (Hasse-Weil). If an arithmetic elliptic curve E is given, then there exist a positive integer N and a sign ϵ such that the function $L(s,E)$ equals $L(s,f)$ for some newform $f \in S_2^{\epsilon}(\Gamma_0(N))$.

Conjecture 1.7 provides us with the following implication and questions.

(A) Conjecture 1.7 implies the identification of geometric L functions as automorphic L functions.
(B) What are N and ϵ?
(C) Is there a deeper relationship between E and f?

The Eichler-Shimura theory gives part of the answer to the question (C). Roughly speaking, the E-S theory says that starting from any normalized newform $f \in S_2(\Gamma_0(N))$ whose Fourier coefficients are integers, the E-S theory provides us with a mapping from $X_0(N)$ to a *canonical elliptic curve* over $\mathbb{Q}$. Weil made a similiar suggestion, i.e., TSW Conjecture as the E-S theory unfolded.

Theorem 1.8. If E has a modular parametrization of level N but of no level M for $M < N$, then E comes via the E-S theory from the map defined by a normalized newform in $S_2(\Gamma_0(N))$ followed by an isogeny defined over $\mathbb{Q}$.

Theorem 1.9 (Carayol). Let $f \in S_2(\Gamma_0(N))$ be a normalized newform whose Fourier coefficients are integers and let E be the elliptic curve over $\mathbb{Q}$ associated to f by the E-S theory. Then $L(s,E) = L(s,f)$ and N is the conductor of E.

Corollary 1.10. TSW Conjecture implies the Hasse-Weil Conjecture.

Proof. Let E be an arithmetic elliptic curve. We choose $N \in \mathbb{Z}^+$ to be the smallest positive integer such that E has a modular parametrization of level N. Theorem 1.8 says that E comes from the map defined by a normalized newform $f \in S_2(\Gamma_0(N))$, possibly followed by an isogeny over $\mathbb{Q}$. Since an isogeny defined over $\mathbb{Q}$ does not affect the L function, Theorem 1.9 implies that $L(s,E) = L(s,f)$. Thus we prove Conjecture 1.7. □

Corollary 1.11. The Hasse-Weil Conjecture almost implies TSW Conjecture.

Proof. Let E be an arithmetic elliptic curve with conductor N. By the Hasse-Weil Conjecture 1.7, there is a normalized newform $f \in S_2(\Gamma_0(N'))$ such that $L(s,E) = L(s,f)$. By the E-S theory and Theorem 1.9, there is an arithmetic elliptic curve E' of conductor N' with $L(s,E') = L(s,f)$. Since $L(s,E) = L(s,E')$, a theorem of Serre allows us to conclude that E and E' are isogeneous over $\mathbb{Q}$ if $j(E)$ is not an integer. In this case, the modular parametrization for E gives for E' by composition. □

Theorem 1.12. Let E and E' be twists of each other which are related as

$$(s^2c_4, s^3c_6, s^6D) = (r^2c_4', r^3c_6', r^6D'), \quad r, s \in \mathbb{Z}, \ g.c.d.(r,s) = 1.$$

Let the L-function $L(s,E)$ and $L(s,E')$ have the coefficients a_p and a_p' respectively. If p is a prime with $p|6DD'$, then

$$a_p' = \left(\frac{rs}{p}\right) a_p.$$

Theorem 1.12 almost proves that the question whether E is a Weil curve depends only on $j(E)$.

We have natural questions.

(D) Determine whether a particular elliptic curve is a Weil curve.
(E) How can we write down equations for the Weil curves that arise from $X_0(N)$?

The following theorem bounds the denominator of $j(E)$ in the case of a prime conductor.

Theorem 1.13 (Mestre-Oesterlé). If a Weil curve E is in global minimal form and its conductor N is a prime, then the discriminant D of E divides N^5.

Theorem 1.14 (Ribet). Suppose that E is a Weil curve with a modular parametrization of level N via a normalized newform

$$f(z) = 1 \cdot e^{2\pi i z} + a_2 \cdot e^{4\pi i z} + a_3 \cdot e^{6\pi i z} + \cdots, \quad a_1 = 1$$

in $S_2(\Gamma_0(N))$. Suppose E is given in global minimal form and let

$$D = \prod_{p|D} p^{\delta_p}, \qquad N = \prod_{p|D} p^{f_p}$$

be the discriminant and the conductor of E respectively. Fix a prime l and let

$$N_1 := \frac{N}{\prod_p' p},$$

where p runs through primes with $f_p = 1$ and $l|\delta_p$. Then there is $f_1 \in S_2(\Gamma_0(N_1))$ such that $f_1(z) = \sum_{n=1}^{\infty} b_n e^{2\pi i n z}$ satisfying the conditions
(a) $b_n \in \mathbb{Z}$ for all $n \in \mathbb{Z}^+$,
(b) $a_n \equiv b_n \mod l$ for all $n \in \mathbb{Z}^+$.

2 Wiles' Work on TSW Conjecture

Wiles's work on TSW Conjecture is based on the *Modular Lifting Conjecture* and *Mazur's deformation theory* (cf. Deforming Galois representations, Galois groups over **Q**, MSRI (1989), 385-437). The following flowchart shows how FLT would follow from the Modular Lifting Conjecture, simply MLC.

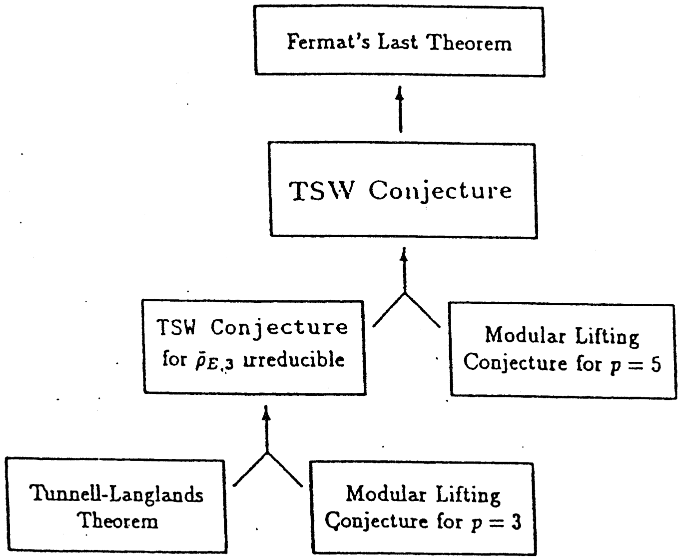


FIGURE 1. Modular Lifting Conjecture ⇒ Fermat's Last Theorem

MLC is still an open problem, even for the primes 3 and 5. However, Wiles proves enough of MLC so that, with some additional work, he can still obtain enough of TSW Conjecture to prove FLT.

If p is a prime, we write

$$\omega_p : G_{\mathbb{Q}} \longrightarrow \mathbf{F}_p^\times, \quad G_{\mathbb{Q}} = Gal\,(\overline{\mathbb{Q}}/\mathbb{Q})$$

for the character giving the action of $G_{\mathbb{Q}}$ on the p-th roots of unity. If p is a prime, we denote by $E[p]$ the subgroup of $E(\overline{\mathbb{Q}})$ of order dividing p. Then $E[p] \cong \mathbf{F}_p^2$. The action of $G_{\mathbb{Q}}$ on $E[p]$ gives a continuous representation

$$\bar{\rho}_{E,p} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbf{F}_p) \tag{2.1}$$

(defined up to isomorphism) such that

$$det\,(\bar{\rho}_{E,p}) = \omega_p \tag{2.2}$$

and for all but finitely many primes q

$$trace\,(\bar{\rho}_{E,p}(Frob_q)) \equiv q + 1 - \sharp\,(E(\mathbf{F}_q)) \quad (mod\,p). \tag{2.3}$$

Definition. Let $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ be an eigenform in $S_2(\Gamma_0(N))\,(a_1 = 1)$. We denote by $\mathcal{O}_f$ the ring of integers of the number field $\mathbb{Q}(a_2, a_3, \cdots)$.

Modular Lifting Conjecture. Suppose p is a prime and E is an arithmetic elliptic curve satisfying
(a) $\overline{\rho}_{E,p}$ is irreducible,
(b) there are an eigenform $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z} \in S_2(\Gamma_0(N))$ and a prime ideal λ of $\mathcal{O}_f$ such that $p \in \lambda$ and for all but finitely many primes q,

$$a_q \equiv q + 1 - \sharp(E(\mathbf{F}_q)) \quad (mod\, \lambda).$$

Then E is modular.

Remarks. When an arithmetic elliptic curve E is fixed, it is known that $\overline{\rho}_{E,p}$ can be reducible only for a finite number of p (cf. J.-P. Serre, Invent. Math. **15** (1972), 259-331). In fact, B. Mazur proved that $\overline{\rho}_{E,p}$ is irreducible for all primes not in the set $\{2, 3, 5, 7, 11, 13, 17, 19, 37, 43, 67, 163\}$ (cf. *Rational isogenies of prime degree, Invent. Math.* **44** *(1978), 129-162).*

Important! Wiles does not prove the full MLC, but proves it subject to some additional hypotheses on $\overline{\rho}_{E,p}$. MLC is *a prióri* weaker than TSW Conjecture because of the extra hypotheses (a) and (b) in MLC. The more serious condition is (b); there is no known way to produce such a form in general. But when $p = 3$, the existence of such a form follows from Langlands-Tunnell Theorem. Wiles then gets around condition (a) by a clever argument which, when $\overline{\rho}_{E,3}$ is *not* irreducible, allows him to use $p = 5$ instead.

Langlands-Tunnell Theorem. Suppose $\rho : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{C})$ is a continuous irreducible representation whose image in $PGL_2(\mathbb{C})$ is a subgroup of S_4, τ is a complex conjugation and $det\,(\rho(\tau)) = 1$. Then there is a cusp form $f(z) = \sum_{n=1}^{\infty} b_n e^{2\pi i n z} \in S_1(\Gamma(N))$ for some $N \in \mathbb{Z}^+$, which is an eigenfunction for all the corresponding Hecke operators such that for all but finitely many primes q,

$$b_q = trace\,(\rho(Frob_q)). \tag{2.4}$$

The theorem as stated by Langlands [7] and by Tunnell [20] produces an automorphic representation, rather than a cusp form. Using the fact that $det\,(\rho(\tau)) = -1$, standard technique shows that this automorphic representation corresponds to a wight one cusp form as in Langlands-Tunnell Theorem.

Proposition 2.1. Suppose that MLC is true for $p = 3$, E is an arithmetic elliptic curve and $\overline{\rho}_{E,3}$ is irreducible. Then E is modular.

Proof. It suffices to show that the condition (b) of MLC is satisfied with the given curve E for $p = 3$. Using Langlands-Tunnell Theorem, we obtain the desired result. □

Proposition 2.2 (Wiles [22]). Suppose that MLC is true for $p = 3$ and $p = 5$, E is an elliptic curve over $\mathbb{Q}$ and $\overline{\rho}_{E,3}$ is reducible. Then E is modular.

Proof. The elliptic curves over $\mathbb{Q}$ for which both $\overline{\rho}_{E,3}$ and $\overline{\rho}_{E,5}$ are reducible are all known to be modular. So we can suppose that $\overline{\rho}_{E,5}$ is irreducible. It suffices to produce an eigenform as in (b) of MLC, but this time there is no analogue of Langlands-Tunnel Theorem to help. Wiles uses the *Hilbert Irreduciblity Theorem,*

applied to a parameter space of elliptic curves, to produce another elliptic curve E' over $\mathbb{Q}$ satisfying
(i) $\overline{\rho}_{E',5}$ is isomorphic to $\overline{\rho}_{E,5}$ and
(ii) $\overline{\rho}_{E',3}$ is irreducible.
Now by Proposition 2.1, E' is modular. Let $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ be the corresponding eigenform. Then for all but finitely many primes q

$$\begin{aligned} a_q = q + 1 - \sharp\,(E'(\mathbf{F}_q)) &\equiv trace\,(\overline{\rho}_{E',5}(Frob_q)) \\ &\equiv trace\,(\overline{\rho}_{E,5}(Frob_q)) \\ &\equiv q + 1 - \sharp\,(E(\mathbf{F}_q)) \quad (mod\ 5) \end{aligned}$$

Therefore f satisfies the hypothesis (b) of MLC and we conclude that E is modular. □

Theorem 2.3 (Wiles [22]). Suppose p, $\overline{\rho}$, ω_p, D_q, R, $\mathbf{T}$ are as those given in [16]. Assume $\overline{\rho}$ satisfies the following conditions:

(i) $det\,(\overline{\rho}) = \omega_p$,
(ii) $Sym^2(\overline{\rho})$ is absolutely irreducible,
(iii) $\overline{\rho}$ is ramified at q and $q \neq p$, then the restriction of $\overline{\rho}$ to D_q is reducible.
(iv) if p is 3 or 5, then for some prome q, $p | \sharp\,(\overline{\rho}(I_q))$.

Then $\varphi : R \longrightarrow \mathbf{T}$ is an isomorphism.

Theorem 2.4 (Wiles [22]). Every semistable elliptic curve over $\mathbb{Q}$ is modular.

Proof. See [22], pp. 542-543. □

Using a variant of the Wiles argument in [22], F. Diamond deduces the following generalization of Theorem 2.4.

Theorem 2.5. Suppose that E is an arithmetic elliptic curve which is semistable both at 3 and at 5. Then E is modular.

Finally we make a rough sketch of Wiles' approach to TSW conjecture for semistable arithmetic elliptic curves. One first fixes a prime p and consider the family of group $E[p^\nu]$ for $\nu = 1, 2, \cdots$. The resulting sequence of representations

$$\overline{\rho}_{E,p^\nu} : G_{\mathbf{Q}} \longrightarrow GL_2(\mathbf{Z}/p^\nu\mathbf{Z})$$

may be packaged as a *single* representation

$$\overline{\rho}_{E,p^\infty} : G_{\mathbf{Q}} \longrightarrow GL_2(\mathbf{Z}_p).$$

To prove that E is modular, it suffices to show that $\overline{\rho}_{E,p^\infty}$ is modular in an appropriate sense. Indeed, $trace(\overline{\rho}_{E,p^\infty}(Frob_q))$ coincides with the rational integer b_q for all $q \nmid pN$, where N is the conductor of E. As soon as one finds an eigenform f in $S_2(\Gamma_0(N))$ whose eigenvalues are related to the traces of $\overline{\rho}_{E,p^\infty}(Frob_q))$, one essentially proves that E is modular. We note that if $\overline{\rho}_{E,p^\infty}$ is modular, then so is $\overline{\rho}_{E,p}$. To prove that E is modular by p-adic method, we need only work with a *single* prime p. Observing that both $GL_2(\mathbf{F}_2)$ and $GL_2(\mathbf{F}_3)$ are solvable and using Langlands-Tunnell Theorem, Wiles proves that $\overline{\rho}_{E,p}$ is modular for $p \leq 3$. Wiles' basic idea is to prove that if p is a prime for which $\overline{\rho}_{E,p}$ is modular, then $\overline{\rho}_{E,p^\infty}$ is automatically modular and hence E is modular.

Appendix A. Definitions and Notations

An elliptic curve over $\mathbb{Q}$ is a singular curve defined by an equation of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \quad a_i \in \mathbb{Z}. \tag{A.1}$$

The solution (∞, ∞) will be viewed as a point on the elliptic curve. Two elliptic curves over $\mathbb{Q}$ are said to be *isomorphic* if one can be obtained from the other by changing coordinates

$$x = A^2x' + B, \quad y = A^3y' + Cx' + D, \quad A, B, C, D \in \mathbb{Q} \tag{A.2}$$

and dividing through by A^6. It is shown that every elliptic curve over $\mathbb{Q}$ is isomorphic to one of the form

$$y^2 = x^3 + a_2x^2 + a_4x + a_6, \quad a_2, a_4, a_6 \in \mathbb{Z}. \tag{A.3}$$

For a positive integer $N \in \mathbb{Z}^+$, we define

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$

and

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,\middle|\, N|c, i.e., c \equiv 0 (\mathrm{mod}\ N) \right\}.$$

$\Gamma(N)$ (resp.$\Gamma_0(N)$) is called the principal congruence subgroup of level N (resp. the Hecke subgroup of level N).

$$\Gamma(N) \subset \Gamma_0(N) \subset \Gamma = SL(2, \mathbb{Z}).$$

$\Gamma(N)$ is a normal subgroup of Γ because it is the kernel of the reduction-modulo-N homomorphism $SL(2, \mathbb{Z}) \longrightarrow SL(2, \mathbb{Z}_N)$. We set

$$H_1^* := H_1 \cup \mathbb{Q} \cup \{i\infty\}$$

and

$$X_0(N) := \Gamma_0(N)\backslash H_1^* \quad \text{(a compact Riemann surface)}.$$

Definition. An elliptic curve E is called *modular* if there exists a holomorphic map from $X_0(N)$ onto E for some integer $N \in \mathbb{Z}^+$.

Example. The arithmetic elliptic curve $E : y^2 = x(x + 3^2)(x - 4^2)$ is modular (indeed, there is a holomorphic isomorphism from $X_0(15)$ onto E). E is semistable because it is isomorphic to $y^2 + xy + y = x^3 + x^2 - 10x - 10$.

For a fixed $N \in \mathbb{Z}^+$, we have commuting linear operators (called Hecke operators) T_m $(m \geq 1)$ on $S_2(\Gamma_0(N))$. If $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$, then

$$T_m f(z) := \sum_{n=1}^{\infty} \left(\sum_{\substack{(d,N)=1 \\ d|(n,m)}} d a_{\frac{nm}{d^2}} \right) e^{2\pi i n z}. \tag{A.4}$$

We denote by $T(N)$ the Hecke algebra generated by these operators T_m.

An *eigenform* means a normalized cusp form in $S_2(\Gamma_0(N))$ for some $N \in \mathbb{Z}^+$ which is an eigenfunction for all the Hecke operators. If $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$: $(a_1 = 1)$ is an eigenform, according to $(A.4)$, $T_m f = a_m f$ for all $m \in \mathbb{Z}^+$.

Appendix B. The Eichler-Shimura Theorem

For a positive integer $N \in \mathbb{Z}^+$. Let $\alpha_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\alpha_N \, \gamma \, \alpha_N^{-1} = \begin{pmatrix} d & -\frac{c}{N} \\ -Nb & a \end{pmatrix}.$$

Therefore

$$\text{(B.1)} \qquad \alpha_N \Gamma_0(N) \alpha_N^{-1} \subset \Gamma_0(N).$$

For $f \in A_k(\Gamma_0(N))$, we consider the map

$$\text{(B.2)} \qquad w_N f := f \circ [\alpha_N]_k.$$

We recall that $(f \circ [\alpha]_k)(z) := J(\alpha, z)^{-k} f(\alpha(z))$ for $\alpha \in SL(2,\mathbb{R})$, where $J(\alpha, z) = cz + d$ for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$. For a matrix α of positive determinant, we define

$$\text{(B.3)} \qquad f \circ [\alpha]_k := f \circ [\alpha^{\#}]_k, \quad \alpha^{\#} := (\det\alpha)^{-\frac{1}{2}} \alpha.$$

It is easy to check that

$$\text{(B.4)} \qquad f \circ [\alpha_1 \alpha_2]_k = (f \circ [\alpha_1]_k) \circ [\alpha_2]_k, \quad \alpha_1, \alpha_2 \in GL^+(2,\mathbb{R}).$$

For $\gamma \in \Gamma_0(N)$ and $f \in A_k(\Gamma_0(N))$,

$$(f \circ [\alpha_N]_k) \circ [\gamma]_k = (f \circ [\alpha_N \gamma \alpha_N^{-1}]_k) \circ [\alpha_N]_k = f \circ [\alpha_N]_k.$$

It is easy to see that

$$\text{(B.5)} \qquad w_N(A_k(\Gamma_0(N)) \subset A_k(\Gamma_0(N)), \quad w_N(S_k(\Gamma_0(N)) \subset S_k(\Gamma_0(N)).$$

Since $(\alpha_N^{\#})^2 = -I_2$, we see that w_N is an involution on $S_k(\Gamma_0(N))$. We denote by $S_k^{\pm}(\Gamma_0(N))$ the eigenspaces corresponding to the eigenvalues ± 1 respectively.

Definition. An eigenform f for $\Gamma_0(N/(r_1 r_2))$ with $r_1 r_2 | N$ is called an *oldform* if $f(r_2 z)$ is an eigenform for $\Gamma_0(N)$ with the same eigenvalues. The linear span of the oldforms is denoted by $S_k^{old}(\Gamma_0(N))$ and its orthogonal complement is denoted by $S_k^{new}(\Gamma_0(N))$. The eigenforms in $S_k^{new}(\Gamma_0(N))$ are called *newforms* for $\Gamma_0(N)$.

Theorem (Atkin-Lehner). If $f \in S_k(\Gamma_0(N))$ is a new form, then its equivalence class is one-dimensional, i.e., consists of multiple of f.

Let $\pi : H_1^* \longrightarrow X_0(N)$ be the projection and $\Phi : X_0(N) \longrightarrow J(X_0(N))$. We write $\tilde{\Phi} = \Phi \circ \pi$. We define the mapping $\mu : S_2(\Gamma_0(N)) \longrightarrow \Omega^1(J)$, (where $J := J(X_0(N))$) by the relation

$$\text{(B.6)} \qquad \tilde{\Phi}^*(\mu(f)) = f(z)dz, \quad f \in S_2(\Gamma_0(N)).$$

Let

$$M(n,N):=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in M(2,\mathbb{Z})\,\middle|\, ad-bc=n,\ \ (a,N)=1,\ \ N|c\right\}.$$

Then we can write

$$M(n,N)=\cup_{i=0}^{K}\Gamma_0(N)\alpha_i \quad \text{(disjoint union)}.$$

We define the Hecke operator $T(n):X_0(N)\longrightarrow Div(X_0(N))$ by

$$T(n)(\pi(z)):=\sum_{i=1}^{K}(\pi(\alpha_i z)). \tag{B.7}$$

The mapping $\Phi:X_0(N)\longrightarrow J$ extends additively to a map

$$\Phi^{\#}:Div(X_0(N))\longrightarrow J. \tag{B.8}$$

We put

$$T^{\#}(n):=\Phi^{\#}\circ T(n):X_0(N)\longrightarrow J. \tag{B.9}$$

$T^{\#}$ is a morphism of varieties over $\mathbb{C}$. Applying the universal mapping property to $T^{\#}(n):X_0(N)\longrightarrow J$, we obtain an element $t(n)\in End(J)$ such that

$$T^{\#}(n)(\pi(z))=t(n)(\tilde{\Phi}(z))+T^{\#}(n)(\pi(z_0)),\quad z\in H_1. \tag{B.10}$$

We can show that $t(n)$ is defined over $\mathbb{Q}$ for every $n\in\mathbb{Z}^+$.

Theorem (Eichler-Shimura). Let $f(z)=\sum_{n=1}^{\infty}a_n e^{2\pi i n z}$ be a normalized newform in $S_2(\Gamma_0(N))$ such that all a_n are in $\mathbb{Z}$. Then there exists a pair (E,ν) such that

(a) E is an elliptic curve defined over $\mathbb{Q}$ and (E,ν) is a quotient of J by an abelian subvariety A of J defined over $\mathbb{Q}$,
(b) the members $t(n)$ of $End(J)$ leave A stable and act on the quotient E as multiplication by the integers a_n,
(c) $\mu(f)$ is a nonzero multiple of $\nu^*(\omega)$, where ω is the invariant differential of E,
(d) if

$$\Lambda_f:=\left\{\Phi_f(\gamma)=\int_{z_0}^{\gamma(z_0)}f(\tau)d\tau\,\middle|\,\gamma\in\Gamma(N)\right\},$$

then Λ_f is a lattice in $\mathbb{C}$, and E is isomorphic to $\mathbb{C}/\Lambda_f$ over $\mathbb{C}$,
(e) the L functoins of E and f coincide as Euler products except possibly at finitely many primes.

Moreover, the properties (a) and (b) characterize A uniquely and therefore determine (E,ν) up to isomorphism defined over $\mathbb{Q}$.

Appendix C. Relation of an elliptic curve to an elliptic integral

Let

$$E : y^2 = Ax^3 + Bx^2 + Cx + D$$

be a modular elliptic curve. Then an elliptic integral of the first kind is given by

$$\int \frac{dx}{y} = \int \frac{dx}{\sqrt{Ax^3 + Bx^2 + Cx + D}}.$$

Then

$$\frac{dx}{y} = \frac{df}{g} = \frac{f'(z)}{g(z)}dz = F(z)dz, \quad F(z) = \frac{f'(z)}{g(z)}$$

for suitable modular functions $f(z)$ and $g(z)$.

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, we have

$$F(\gamma(z)) = \frac{f'(\gamma(z))}{g(\gamma(z))} = (cz+d)^2 \frac{f'(z)}{g(z)} = (cz+d)^2 F(z).$$

Remark.

(1) It is easy to see that $F \in S_2(\Gamma_0(N))$.
(2) $F(z)$ is an eigenform for the action of a certain Hecke algebra on $S_2(\Gamma_0(N))$.

Miracle. $F(z)$ is intimately connected to the elliptic curve to E. Roughly speaking, we can construct $F(z)$ simply by knowing the number of solutions of the congruence $E_p : y^2 \equiv Ax^3 + Bx^2 + Cx + d \pmod{p}$ for all primes p. The fact $F \in S_2(\Gamma_0(N))$ tells us profound things about E. This is one reason why TSW Conjecture is wonderful!!

Appendix D. Frobenius Elements

If q is a prime and λ is a prime ideal dividing q in the ring of integers of $\overline{\mathbb{Q}}$, there is a filtration

$$G_{\mathbb{Q}} \supset D_\lambda \supset I_\lambda,$$

where the decomposition group D_λ and the inertia group I_λ are defined by

$$D_\lambda := \{\sigma \in G_{\mathbb{Q}} | \sigma(\lambda) = \lambda\}$$

and

$$I_\lambda := \{\sigma \in D_\lambda | \sigma(x) \equiv x \pmod{\lambda} \text{ for all algebraic integers } x\}.$$

There are natural identifications

$$D_\lambda \cong Gal(\overline{\mathbb{Q}}_q/\mathbb{Q}_q), \quad D_\lambda/I_\lambda \cong Gal(\overline{\mathbf{F}}_q/\mathbf{F}_q),$$

and $Frob_\lambda \in D_\lambda/I_\lambda$ denotes the inverse image of the canonical generator $x \mapsto x^q$ of $Gal(\overline{\mathbf{F}}_q/\mathbf{F}_q)$. If λ' is another prime ideal above q, then $\lambda' = \sigma(\lambda)$ for some $\sigma \in G_{\mathbb{Q}}$ and

$$D_{\lambda'} = \sigma D_\lambda \sigma^{-1}, \quad I_{\lambda'} = \sigma I_\lambda \sigma^{-1}, \quad Frob_{\lambda'} = \sigma Frob_\lambda \sigma^{-1}.$$

Since we will care about these objects only up to conjugation, we will write D_λ and I_λ. We will write $Frob_q \in G_{\mathbb{Q}}$ for any representative of a $Frob_\lambda$. If ρ is a representation of $G_{\mathbb{Q}}$ which is unramified at q, then $trace(\rho(Frob_q))$ and $det(\rho(Frob_q))$ are well defined, i.e., are independent of the choice of λ.

References

1. H. Caroyol, On the l-adic representations associated to Hilbert modular forms, Ann. Scient. Ecole Norm. Sup. **19** (1986), 409-468.
2. D. Cox, Introduction to FLT, Amer. Math. Monthly (1994), 1-14.
3. G. Frey, Rationale Punkte auf fermatkurven und getwisten Modulkurven, Crelle, **331** (1982), 185-191.
4. G. Frey, Links between stable elliptic curves and certain diophantine equations, Annales Universitatis Saraviensis **1** (1986), 1-40.
5. Y. Helloguarch, Points d'order $2p^h$ sur les courbes elliptique, Acta Arith. **26** (1975), 253-263.
6. A. W. Knapp, Elliptic Curves, Math. Notes **40**, Princeton Univ. Press (1992).
7. R. Langlands, Base change for $GL(2)$, Ann. of Math. Studies **96**, Princeton (1980).
8. B. Mazur, Number Theory as Gadfly, Amer. Math. Monthly **98** (1991), 593-610.
9. B. Mazur, Modular curves and the Eisenstein ideal, Publ. IHES. **47** (1977), 33-186.
10. J.-F. Mestre and J. Oesterle, Semistable Weil curves with the discriminant of the m-th power, Crelle **400** (1989), 173-184.
11. J. Oesterle, New Approach to FLT, Sem. Bourbaki **694** (1987-88), Asterisque **161-162** (1988), 165-186.
12. K. Ribet, On modular representations of $Gal\,(\overline{\mathbb{Q}}/\mathbb{Q})$ arising from modular forms, Invent. Math. **100**(1990), 431-476.
13. K. Ribet, Wiles proves Taniyama's Conjecture; FLT follows, Notices AMS **40** (1993), 575-576.
14. K. Ribet, Update and Overview on FLT, AMS-MAA Invited Hour Talk, Jan. 13 (1994).
15. K. Ribet, Galois representations and modular forms, Bull. Amer. Math. Soc. **32**, No. 4 (1995), 375-402.
16. K. Rubin and A. Silverberg, Wiles' Proof of FLT, Bull. Amer. Math. Soc. **31** (1994), 15-38.
17. J.-P. Serre, Sur les representations modulaires de degre 2 de $Gal\,(\overline{\mathbb{Q}}/\mathbb{Q})$, Duke Math. J. **54** (1987), 179-230.
18. G. Shimura, Introduction to the Arithmetic Theory of Automorphic Functions, Princeton Univ. Press (1971).
19. J. H. Silverman, The Arithmetic of Elliptic Curves, Springer-Verlag (1986).
20. J. Tunnell, Artin's conjecture for representations of octahedral type, Bull. AMS **5** (1981), 173-175.
21. A. Weil, Über die Bestimmung Dirichletscher Reihnen durch Funktionalgleichungen, Math. Ann. **168** (1967), 149-156.
22. A. Wiles, Modular elliptic curves and Fermat's Last Theorem, Ann. Math. **141** (1995), 443-551.
23. R. Taylor and A. Wiles, Ring-theoretic properties of certain Hecke algebras, Ann. Math. **141** (1995), 553-572.

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초청논문

타원곡선에 관한 지난 20년 간의 연구 동향

양 재 현

요약문. 지난 20여 년 동안의 타원곡선의 이론에 관한 연구 동향을 알기 쉽게 설명하는 것이 이 논문의 목적이다. Birch-Swinnerton-Dyer 가설과 Shimura-Taniyama 가설을 소개하고 최근의 A. Wiles의 업적과 Shimura-Taniyama 가설과의 관계를 설명하며 Fermat 마지막 정리의 증명 (Wiles의 증명)을 스케치한다. 그리고 Heegner 점과 야코비 형식의 이론 사이의 관계를 간략하게 기술한다.

머리말

타원곡선 (elliptic curve) 이란 3차 방정식

(1) $$E : y^2 = x^3 + ax + b \quad (\text{단}, a, b \text{ 는 상수}, 4a^3 + 27b^2 \neq 0)$$

의 해에 의하여 주어질 수 있는 곡선이다. 이 곡선은 종수 (genus)가 1이며 특이점이 없는 사영곡선 (nonsingular projective curve)이다. 이 곡선에 무한 점 ∞를 첨가하면 이 집합 상에 자연스런 기하학적인 덧셈 연산을 얻을 수 있으며 이 연산에 대하여 이 집합은 군 구조 (group structure)를 지니고 무한 점 ∞는 이 군의 단위원소의 역할을 한다. 그래서 타원곡선은 간단한 공간인데다가 기하학적인 군 구조를 지니고 있기 때문에 다른 일반적인 다양체에서 찾아

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볼 수 없는 매우 아름다운 성질뿐만 아니라 풍부한 정보를 타원곡선의 이론에서 발견할 수 있다. 타원곡선은 1 차원 아벨 다양체 (abelian variety)임을 지적하고 싶다. 현대 대수기하학뿐만 아니라 수론 등의 분야에서 중요한 역할을 하고 있는 아벨 다양체에 관한 참고문헌으로 [1], [22]와 [42]를 추천한다.

타원곡선의 이론은 19세기 경에 C. F. Gauss (1777 ~ 1855), N. H. Abel (1802 ~ 1829), C. G. J. Jacobi (1804 ~ 1851) 등의 위대한 수학자들에 의해 연구되었던 타원함수 이론에서 나타난다. 이제는 타원곡선의 이론은 현대 수학의 여러 분야 (가령, 수론, 복소 함수론, 대수기하학 등)와 밀접하게 연관되어 있어서 지난 30여 년 동안 세계 수학계에서 각광을 받으면서 심도 있게 연구되어 왔던 분야이다. 1994년 9월에 프린스턴 대학의 교수인 Andrew Wiles (1953 ~) 가 350여 년 동안 풀리지 않았던 Fermat 마지막 정리를 타원곡선의 한 심오한 결과를 이용하여 해결함으로써 타원곡선 이론의 심오함과 중요성이 세계 수학계뿐만 아니라 일반 대중들에게도 널리 보급되면서 인식되었다.

저자는 이 논문에서 타원곡선의 수론적인 측면을 Birch-Swinnerton-Dyer 가설, Shimura-Taniyama 가설, Wiles의 최근 업적과 관련지어 가면서 설명하고자 한다. 그리고 Heegner 점의 이론과 야코비 형식의 이론과의 관계를 간략하게 설명한다.

기호: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$는 각각 정수환, 유리수체, 실수체, 복소수체 등을 나타낸다. $\mathbb{H}$는 Poincaré 상반평면을 나타내고 $q := e^{2\pi i\tau}$ (단, $\tau \in \mathbb{H}$)이다. $G_{\mathbb{Q}} :=$Gal $(\bar{\mathbb{Q}}/\mathbb{Q})$ 는 $\mathbb{Q}$의 절대적 Galois 군이다. 여기서 $\bar{\mathbb{Q}}$는 $\mathbb{Q}$의 대수적 폐포 (algebraic closure)이다.

1. Mordell-Weil 군

$a, b \in \mathbb{Q}$일 때 (1)에 의하여 주어지는 타원곡선 E상의 점들 중에서 이들의 좌표가 모두 유리수인 점들과 무한 점 ∞으로 이루어진 집합을 $E(\mathbb{Q})$로 표기한다. 이제부터는 $E(\mathbb{Q})$의 원소를 유리점이라 부르기로 한다. 그러면 $E(\mathbb{Q})$는 타원곡선 E로부터 이어받은 덧셈 연산에 대하여 가환군이 된다. 1900년에 H. Poincaré (1854 ~ 1912)는 $E(\mathbb{Q})$는 유한하게 생성된다는 사실을 증명 없이 예

상하였다. 이 예상은 1922년에 영국 수학자 L. J. Mordell (1888～1972)에 의하여 증명되었다.

MORDELL 정리. 아벨 가환군 $E(\mathbb{Q})$는 유한하게 생성된다.

이 정리의 증명 과정에서 Mordell는 높이 함수 (height function)의 개념을 소개하였다. 이 정리는 1940년에 A. Weil (1906～1998)에 의하여 임의의 수체 상에 정의되는 아벨 다양체에 일반화되어 증명되었다.

$E(\mathbb{Q})$ 는 Mordell 정리에 의하여

$$(2) \qquad E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus E_{tor}(\mathbb{Q}) \quad (\text{단, } r \text{ 은 } 0 \text{ 또는 자연수})$$

의 형태로 주어진다. 여기서, $E_{tor}(\mathbb{Q})$는 유한 가환군이며 E의 비틀림 군 (torsion group)이라 한다. 정수 r을 E의 대수적 계수 (algebraic rank), 또는 간단히 계수 (rank)라고 한다. E의 비틀림 군 $E_{tor}(\mathbb{Q})$의 실체는 1977년에 B. Mazur (1938～)에 의하여 완전히 밝혀졌다. 그의 논문 [18]에 의하면 $E_{tor}(\mathbb{Q})$는 아래의 15개의 가환군들 중의 하나이다.

$$\mathbb{Z}/n\mathbb{Z} \ (\text{단, } 1 \le n \le 10,\ n = 12), \quad \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z} \quad (\text{단, } 1 \le n \le 4).$$

$E_{tor}(\mathbb{Q})$의 형태는 간단하지만 이 결과는 모듈러 곡선 (modular curve)의 심오한 이론으로부터 얻어진다.

$E_{tor}(\mathbb{Q})$에 관한 아래의 결과는 E. Lutz (1914～)와 T. Nagell (1895～?)에 의하여 각각 1937년과 1935년에 독립적으로 얻어졌다.

LUTZ-NAGELL 정리. 타원곡선 E는 (1)의 형태로 주어지며 a, b는 정수라고 하자. 만일, $P = (x_0, y_0) \in E$가 $E_{tor}(\mathbb{Q})$의 원소이면

(a) $x_0, y_0 \in \mathbb{Z}$ 이고

(b) $2P = 0$ 이든가 $y_0^2 \mid (4a^3 + 27b^2)$.

Lutz-Nagell 정리는 타원곡선의 비틀림 점들을 구하는 방법을 제공한다는 점에서 의미가 있다. 1994년에 L. Merel (참고문헌 [21])은 K가 수체일 때 $E_{tor}(\mathbb{Q})$의 개수는 K의 차수 $[K : \mathbb{Q}]$에만 의존하는 어떤 상수에 의하여 균일하게 유계 (uniformly bounded) 되어 있음을 증명하였다. 그러나 E의 계수에

관해서는 거의 알려져 있지 않다. $r = 14$ 인 경우가 지금까지 알려진 가장 큰 타원곡선의 계수이다. 이 사실은

$$a = -35971713708112, \qquad b = 8508621384829839400 0$$

인 타원곡선 (1)의 계수가 14 임이 1984년에 J.-F. Mestre에 의하여 입증되었다. 당연히, 임의의 큰 자연수를 계수로 갖는 타원곡선이 있을까 하는 의문이 생긴다.

가설 A. 임의의 자연수 n이 주어져 있을 때, 계수가 n인 타원곡선이 존재한다.

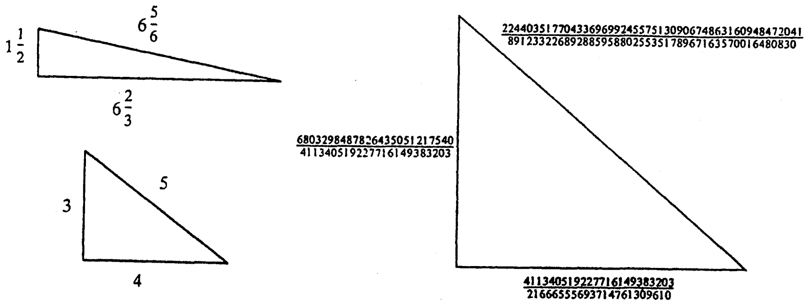
이것으로 미루어보아 타원곡선의 계수는 신비롭고 흥미로운 불변량 (invariant) 임을 짐작할 수 있다. 여기서 불변량이란 동종사상 (isogeny)의 작용에 변하지 않는 어떤 양을 의미하고 있다. 지난 약 20여 년 동안 타원곡선의 계수에 대한 가설과 관련하여 여러 유명 수학자, 예를 들면, J. Coates, A. Wiles, K. Rubin, B. Gross, D. Zagier, V. A. Kolyvagin 등의 수학자들에 의하여 특별한 타원곡선인 경우 (가령, CM-타원곡선, Weil 곡선)에 연구되었다.

왜 타원곡선 E의 유리점들의 집합 $E(\mathbb{Q})$를 구체적으로 구하는 것이 수학적으로 의미가 있고 중요한가? 실제로 $E(\mathbb{Q})$의 구조를 정확히 밝히면 타원곡선 E의 여러 기하학적 및 수적인 성질을 알 수 있다.

한 예로서 소위 합동수 문제와 아주 특별한 타원곡선

$$\text{(3)} \qquad E_n : y^2 = x^3 - n^2x, \ \text{(단, } n\text{은 자연수)}$$

과의 밀접한 연관성을 간략하게 설명하면서 상기의 질문에 부분적으로 답할까 한다. 자연수 n이 세 변이 모두 유리수인 직각삼각형



(computed by D. Zagier)

의 넓이일 때 n을 합동수 (congruence number)라고 한다. 예를 들면, 5, 6, 7, 157 은 합동수이지만 1, 2, 3, 4 는 합동수가 아니다. L. Euler는 7 이 합동수임을 보였고 앞쪽에 있는 직각삼각형의 넓이는 각각 5, 6, 157 이다. 또 1 이 합동수가 아니라는 사실은 Fermat 방정식

$$x^4 + y^4 = z^4 \qquad (\text{단, } xyz \neq 0)$$

이 정수해를 갖지 않는다는 사실과 동치임을 쉽게 알 수 있다.

(CNP) 과연 어떤 자연수가 합동수인가?

상기의 질문 (CNP)가 소위 합동수 문제 (congruence number problem)이다. 1983년에 J. Tunnell은 그의 논문 [39]에서 타원곡선 E_n과 반정수 무게의 모듈러 형식의 이론을 이용하여 (CNP)의 질문을 부분적으로 해결하였다. 이것에 관하여 아주 간략하게 설명하겠다.

합동수 n이 세 변의 길이가 유리수 X, Y, Z (단, Z는 빗변의 길이) 인 직각삼각형의 넓이라고 하면

$$X^2 + Y^2 = Z^2, \ XY = 2n$$

이므로

$$\left(\frac{Z}{2}\right)^2, \left(\frac{Z}{2}\right)^2 - n = \left(\frac{X-Y}{2}\right)^2, \left(\frac{Z}{2}\right)^2 + n = \left(\frac{X+Y}{2}\right)^2$$

은 유리수의 제곱이다.

역으로 $x, x-n, x+n$ (단, n은 자연수)이 유리수의 제곱이 되게 하는 유리수 x가 존재한다면

$$X = \sqrt{x+n} - \sqrt{x-n}, \ Y = \sqrt{x+n} + \sqrt{x-n}, \ Z = 2\sqrt{x}$$

는 유리수이고 직각삼각형의 세 변의 길이가 되어 자연수 n은 합동수가 된다. 그리고

$$x = \left(\frac{Z}{2}\right)^2, \ x - n = \left(\frac{X-Y}{2}\right)^2, \ x + n = \left(\frac{X+Y}{2}\right)^2$$

임을 관찰하라. 그러므로 자연수 n이 합동수이기 위한 필요충분조건은 $x, x-n, x+n$ 이 유리수의 제곱이 되게 하는 유리수 x가 존재한다는 것임을 알 수 있다.

만일, 어떤 유리수 $x_0 \in \mathbb{Q}$가 있어 x_0, $x_0 - n$, $x_0 + n$ (단, n은 자연수)이 유리수의 제곱이면 $y_0 = \pm\{x_0(x_0-n)(x_0+n)\}^{\frac{1}{2}} \neq 0$ 도 역시 유리수가 되어 (x_0, y_0)은 $E_n(\mathbb{Q})$의 원소이다. 역으로, $(x_0, y_0) \in E_n(\mathbb{Q})$이고 $y_0 \neq 0$이면 $x_1, x_1 - n, x_1 + n$ (단, $(x_1, y_1) := 2(x_0, y_0)$) 은 간단한 계산에 의하여 유리수의 제곱임을 보일 수 있다. 따라서 자연수 n이 합동수인가를 알기 위해서는 타원곡선 E_n의 유리점들의 집합 $E_n(\mathbb{Q})$안에 $y_0 \neq 0$인 유리점 (x_0, y_0)의 존재여부를 밝히면 된다.

Tunnell은 그의 논문 [39]에서 합동수를 아래와 같이 특징지었다.

TUNNELL 정리. 자연수 n을 합동수이라 가정하고 n의 소인수분해가 $n = p_1 \cdots p_k$ (단, $p_1, \cdots, p_k$는 서로 다른 소수)의 형태로 주어진다고 가정하자. 그러면,

(ㄱ) n이 홀수라고 하면

$$\begin{aligned}&\#\left\{(x,y,z) \in \mathbb{Z}^3 \middle| n = 2x^2 + y^2 + 32z^2\right\}\\ &= \frac{1}{2}\#\left\{(x,y,z) \in \mathbb{Z}^3 \middle| n = 2x^2 + y^2 + 8z^2\right\}\end{aligned}$$

의 관계가 성립한다.

(ㄴ) n이 짝수라고 하면

$$\begin{aligned}&\#\left\{(x,y,z) \in \mathbb{Z}^3 \middle| n = 8x^2 + y^2 + 64z^2\right\}\\ &= \frac{1}{2}\#\left\{(x,y,z) \in \mathbb{Z}^3 \middle| n = 8x^2 + 2y^2 + 16z^2\right\}\end{aligned}$$

이다.

만일, 타원곡선 E_n에 대하여 Birch-Swinnerton-Dyer 가설이 옳다고 하면 상기의 역도 성립한다. 이 가설에 관하여 제 2 장에서 자세히 서술하겠다.

상기의 정리의 증명은 참고문헌 [13]에 아주 쉽고 자세하게 서술되어 있다.

2. Birch-Swinnerton-Dyer 가설

이제부터는 Birch-Swinnerton-Dyer 가설을 간단히 BSD 가설이라 표기한다.

2.1. 타원곡선의 L-급수

$\mathbb{Q}$상에서 정의되는 타원곡선 E의 L-급수 $L(E,s)$는

$$L(E,s) = \prod_{p|\Delta_E}(1-a_p p^{-s})^{-1} \cdot \prod_{p\nmid\Delta_E}(1-a_p p^{-s}+p^{1-2s})^{-1}, \ s \in \mathbb{C} \tag{4}$$

으로 정의된다. 여기서, Δ_E는 E의 판별식 (determinant, [34, p. 46])이고 p는 소수를 나타낸다. 그리고 소수 p가 $p|\Delta_E$인 경우에, 축소된 곡선 (reduced curve) $\bar{E}$가 p에서 첨점 (cusp)을 가지면 $a_p = 0$, $\bar{E}$가 p에서 분리 노드 (split node)를 가지면 $a_p = 1$, $\bar{E}$가 p에서 비분리 노드 (nonsplit node)를 가지면 $a_p = -1$이라 정의하고 소수 p가 $p \nmid \Delta_E$인 경우에는

$$a_p := p + 1 - \#\bar{E}(\mathbb{F}_p) \tag{5}$$

이라 정의한다. 그러면 $L(E,s)$는 $\mathrm{Re}s > \frac{3}{2}$ 인 영역에서 절대 수렴한다. 유사한 방법으로 수체 K상에 정의된 타원곡선 E/K의 L-급수를 정의할 수 있다 (참고문헌 [34, pp. 360-361]). E의 L-급수 $L(E,s)$는 E의 국소적인 성질들을 측정하는 함수라고 대충 말할 수 있다 불행하게도 아직까지 $L(E,s)$가 전 복소평면 $\mathbb{C}$상으로 해석적으로 접속이 가능하다는 사실이 아직 밝혀지지 않았다.

가설 B. E를 $\mathbb{Q}$상에 정의된 타원곡선이라 하고 N_E를 E의 전도체 (conductor, 참조 [34, p. 361])라고 하자. 그러면, 아래의 함수

$$\Lambda(E,s) := N^{s/2}(2\pi)^{-s}\Gamma(s)L(E,s) \tag{6}$$

는 전 복소 평면 상으로의 해석적 접속을 지니며 다음의 함수 방정식

$$\Lambda(E,s) := \epsilon\,\Lambda(E,2-s), \quad \epsilon = \pm 1 \tag{7}$$

을 만족한다.

2.2. 모듈러 곡선

자연수 N에 대하여

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \middle| \ N|c \right\}$$

이라 하자. $\Gamma_0(N)$은 수준(level) N인 Hecke 부분군이라 불린다.

$$Y_0(N) = \mathbb{H}/\Gamma_0(N) \tag{8}$$

은 비긴밀 곡면이며

$$X_0(N) = \mathbb{H}^*/\Gamma_0(N), \qquad \mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\} \tag{9}$$

는 $Y_0(N)$의 긴밀화 (compactification)인 긴밀 리만 곡면이다. $X_0(N)$은 $\mathbb{Q}$상에서 정의되는 비특이 사영곡선이다. $X_0(N)$을 수준 (level) N인 모듈러 곡선 (modular curve)이라 한다. 모듈러 곡선 $X_0(N)$의 종수는 쉽게 구할 수 있다 (참고문헌 [32, pp. 23-26]). 다음의 집합

$$X_0(N) - Y_0(N) := \mathbb{P}^1(\mathbb{Q})/\Gamma_0(N), \ \mathbb{P}^1(\mathbb{Q}) := \mathbb{Q} \cup \{\infty\} \tag{10}$$

은 유한집합이며 이의 원소를 첨점 (cusp)이라고 한다 (참고문헌 [32, p. 13]).

Hecke 부분군 $\Gamma_0(N)$에 대하여 무게 (weight)가 2인 첨점 형식들로 이루어진 복소 공간을 $S_2(\Gamma_0(N))$으로 표기하기로 한다. 즉, $S_2(\Gamma_0(N))$의 원소 $f : \mathbb{H} \to \mathbb{C}$는 아래의 (C1), (C2)의 성질을 만족한다.

(C1) $f((a\tau+b)(c\tau+d)^{-1}) = (c\tau+d)^{-2}f(\tau), \ \tau \in \mathbb{H}, \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

(C2) ∞ 에서의 f의 푸리에 전개에서 상수항이 0이다.

여기서는 설명하지 않겠지만 임의의 자연수 m에 대하여 벡터공간 $S_2(\Gamma_0(N))$에 작용하는 Hecke 작용소 T_m이 정의되며, 이러한 T_m들은 Hecke 대수 $T(N)$을 생성한다. $S_2(\Gamma_0(N))$의 원소인 첨점 형식 $f(\tau) = \sum_{n\geq 1} c_n q^n$가 있어 $c_1 = 1$ 이고 모든 Hecke 작용소에 대하여 동시에 고유함수가 될 때 첨점 형식 $f(\tau)$를 고

유형식 (eigenform)이라고 한다. $\Omega^1(X_0(N))$을 모듈러 곡선 $X_0(N)$상의 차수가 1 인 해석적 미분형식들의 벡터공간이라 하면 아래의 대응

$$(11) \qquad S_2(\Gamma_0(N)) \to \Omega^1(X_0(N)), \quad f \mapsto f d\tau$$

은 동형사상이다.

예를 들면, $X_0(11)$은 종수가 1 이므로

$$\dim_{\mathbb{C}} S_2(\Gamma_0(11)) = \dim_{\mathbb{C}} \Omega^1(X_0(11)) = 1$$

이다. 실제로 함수

$$f(\tau) := \eta(\tau)^{12}\,\eta(11\tau)^2$$

가 $S_2(\Gamma_0(11))$의 생성원 (generator)이며 모든 Hecke 작용소에 대하여 고유함수이다. 여기서, $\eta(\tau)$는

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n\geq 1}(1-q^n), \ \tau \in \mathbb{H}$$

으로 주어지는 Dedekind eta 함수이다.

정의 1. E를 $\mathbb{Q}$상에서 정의되는 타원곡선이라 하자. 만약에 적당한 자연수 N이 있어 고유형식 $f(\tau) = \sum_{n\geq 1} c_n q^n \in S_2(\Gamma_0(N))$가 존재하여 거의 모든 소수 p에 대하여

$$(12) \qquad c_p = p + 1 - \#\bar{E}(\mathbb{F}_p)$$

인 관계식을 만족하는 경우에 E를 모듈러 타원곡선이라 한다. 모듈러 타원곡선을 보통 *Weil* 곡선이라고도 한다.

도움말 1. E가 모듈러 곡선이면 전사 해석적 함수 $f : X_0(N) \to E$가 존재한다는 사실을 보일 수 있다. 여기서 N은 적당한 자연수이다.

2.3. BSD 가설

이 절에서는 E를 $\mathbb{Q}$상에 정의되는 타원곡선이라 가정한다. 이미 언급하였지만 가환군 $E(\mathbb{Q})$의 계수는 신비스러운 불변량이고 지금까지 거의 알려져 있는 것이 없다. 1960년대 초반 경에 영국의 수학자 B. Birch (1931∼)와 H. P. F. Swinnerton-Dyer (1927∼)는 E의 계수에 관한 아래의 가설을 제시하였다.

BSD 가설.
[a] rank $E(\mathbb{Q}) = \mathrm{ord}_{s=1} L(E,s)$
[b] $r :=$ rank $E(\mathbb{Q})$ 이라 두면

$$\lim_{s\to 1}(s-1)^{-r}L(E,s) = \Omega \cdot \#\text{III}(E)\cdot 2^r \cdot R(E)\cdot \left\{\#E_{tor}(\mathbb{Q})\right\}^{-2}\cdot \prod_p c_p.$$

여기서 Ω는 E의 실 주기 (real period), III(E)는 Tate-Shafarevich 군 ([34, p. 297]), $R(E)$는 E의 타원 조정자 (elliptic regulator, [34, p. 233])이고 $c_p := \#E(\mathbb{Q}_p)/E_0(\mathbb{Q}_p)$ ([34, p. 183])이다. E의 계수가 0 이면 $R(E)=1$ 이고 E의 계수가 1 이면 $R(E)=\hat{h}(P_0)$이다. (단, $\hat{h}$는 Néron-Tate 높이 함수이고 P_0는 $E(\mathbb{Q})/E_{tor}(\mathbb{Q})$의 생성원 (generator)이다.)

도움말 2. 보통 rank $E(\mathbb{Q})$를 E의 대수적 계수 (algebraic rank), $\mathrm{ord}_{s=1}$ $L(E,s)$를 E의 해석적 계수 (analytic rank)라고 한다.

Tate-Shafarevich 군 III(E)도 E의 계수처럼 역시 흥미로운 연구의 대상이다. 실제로 III(E)가 유한 집합인지 무한 집합인지 아직까지도 모르고 있다. 그러나 III(E)가 유한 군이 되는 여러 예가 발견되어, 아래와 같은 가설이 이 분야의 수학자들 사이에서 제시되고 있다.

가설 C. Tate-Shaferevich 군 III(E)는 유한 군이다.

BSD 가설을 입증할 만한 여러 예가 발견되었지만 이의 완전한 증명을 얻으려면 아마도 몇 십 년을 기다려야 할 것 같다. 이 가설이 제시된 후로 지난 약 40년 동안 이 가설의 해결에 거의 진전이 없다. 완벽한 증명을 요구하는 수학이란 학문의 특성 때문에 수학의 역사적인 흐름이 다른 분야보다 느린 것 같다.

지난 20여 년 동안 BSD 가설의 옳음을 입증할 수 있는 부분적인 연구 결과들을 연대별로 간략하게 소개하겠다.

[**BSD1**] 1977년에 영국 수학자 J. Coates (1945~)와 A. Wiles (1953~) (참고문헌 [3])는 E가 CM-곡선이고 $E(\mathbb{Q})$가 무한 군이라 가정하면 $L(E,1) = 0$ 이 된다는 사실을 증명하였다. CM-곡선은 모듈러 타원곡선이다. E가 End$E \not\subset \mathbb{Z}$의 성질을 만족할 때 E를 CM-곡선이라 한다.

[**BSD2**] 1983년에 R. Greenberg (참고문헌 [9])는 E가 CM-곡선이고 $\mathrm{ord}_{s=1} L(E,s)$가 홀수라고 가정하면 rank $E(\mathbb{Q}) \geq 1$이든가 어떤 소수들에 대하여 $\text{III}(E)[p^\infty]$는 무한 집합이라는 사실을 증명하였다.

[**BSD3**] 1986년에 B. Gross 와 D. Zagier (1950~)는 E가 모듈러 타원곡선이면서 $\mathrm{ord}_{s=1}L(E,s) = 1$이라고 가정하면 rank $E(\mathbb{Q}) \geq 1$ 이란 사실을 증명하였다. 이 증명 과정에서 Heegner 점들의 이론을 이용하고 있다 (참고문헌 [10]).

[**BSD4**] B. Gross와 D. Zagier는 E가 모듈러 타원곡선이며 $L(E,1) = 0$, rank $E(\mathbb{Q}) = 1$ 이라고 가정하면

$$L'(E,1) = c \cdot \Omega \cdot R(E), \quad \text{단 },\ 0 \neq c \in \mathbb{Q}$$

의 관계식이 성립함을 상기의 논문에서 증명하였다.

[**BSD5**] 상기의 논문에서 역시 B. Gross와 D. Zagier는 $\mathrm{ord}_{s=1}L(E,s) =$ rank $E(\mathbb{Q}) = 3$ 인 타원곡선 E가 존재한다는 사실을 보였다. 가령, 아래의 타원곡선

$$E_1 : -139\,y^2 = x^3 + 10\,x^2 - 20\,x + 8$$

의 계수는 3 이고 $\mathrm{ord}_{s=1}L(E_1,s) = 3$ 이다.

[**BSD6**] K. Rubin (참고문헌 [28])은 E가 CM-타원곡선이며 $\mathrm{ord}_{s=1}L(E,s) = 0$ 이라고 가정하면 Tate-Shafarevich 군 $\text{III}(E)$는 유한 군이라는 사실을 증명하였다.

[**BSD7**] V. A. Kolyvagin (참고문헌 [15])은 E가 모듈러 타원곡선이며 $\mathrm{ord}_{s=1} L(E,s) = 0$ 이라고 가정하면 $E(\mathbb{Q})$와 $\text{III}(E)$가 모두 유한 군임을 증명하였다.

도움말 3. Galois 코호모로지 이론을 이용하여 Cassels는 BSD 가설이 동종사상 (isogeny)과 양립한다는 사실을 증명하였다. 보다 자세히 설명하면 두 타원곡선 E와 E'이 $\mathbb{Q}$상에서 서로 동종적 (isogeneous)이다라고 가정하고 E에 대하여 BSD 가설이 성립한다고 가정한다면 E'에 대하여서도 BSD 가설이 성립한다.

도움말 4. Birch는 1980년대 초반에 곡선 E와 허수 이차체 (imaginary quadratic field) K에 대하여 Heegner 점의 개념을 소개하면서 이 점의 order가 무한일 것이라는 사실을 추측하였다. B. Gross와 D. Zagier는 Birch의 추측을 검토하며

$$\hat{h}(P_K) = c \cdot L'(E/K, 1), \quad \text{단, } c \text{ 는 } c \neq 0 \text{ 인 상수} \tag{13}$$

의 관계식을 발견하였다. 여기서 $\hat{h}$는 E/K상의 Néron-Tate 높이 함수 ([34, pp. 228-229])이며 $P_K \in E(K)$는 Heegner 점이다. 공식 (13)으로부터 ord P_K $= \infty$ 이기 위한 필요충분조건은 $L'(E/K, 1) \neq 0$ 이다 (참고문헌 [34, Theorem 9.3 (d)]).

3. Shimura-Taniyama 가설

이 장에서는 타원곡선 이론에서 정수 (精髓, essense)라고 할 수 있는 (필자의 사견이지만) Shimura-Taniyama 가설에 관하여 설명하겠다. 이제부터는 이 가설을 편의상 S-T 가설이라 쓰기로 한다. 또 이 가설과 Fermat 마지막 정리와의 연관성을 설명하면서 약 5년 전에 얻어졌던 Wiles의 업적을 간략하게 소개한다. 그리고 Fontaine-Mazur 가설, Serre 가설, Artin 가설 등을 소개하면서 S-T 가설과의 연관성을 설명한다.

Shimura-Taniyama 가설. $\mathbb{Q}$상에서 정의되는 타원곡선 E는 모둘러 타원곡선이다.

이 가설의 주장은 간단히 한 줄로 쓰여져 있지만 이의 밑바닥에는 매우 심오하고 아름답기 짝이 없는 내용이 깊숙이 깔려 있다.

이제 S-T 가설의 깊은 의미를 이해하기 위해서 먼저 Eichler-Shimura 이론을 대충 상기할 필요가 있다. $f(\tau) = \sum_{n\geq 1} c_n q^n$ (단, $c_1 = 1$)이 수준 N인 고유형식인 $S_2(\Gamma_0(N))$의 원소이며 모든 푸리에 계수 c_n $(n \geq 1)$이 정수라고 하자.

기호 편의상

$$X := X_0(N), \quad \Gamma := \Gamma_0(N)$$

이라 두자. 그리고 $\Omega^1(X)^\vee$을 $\Omega^1(X)$의 쌍대 공간이라 하자. 즉

$$\Omega^1(X)^\vee := \mathrm{Hom}(\Omega^1(X), \mathbb{C}).$$

g를 리만 곡면 X의 종수라고 하면 $\Omega^1(X)^\vee$는 g차원의 복소 공간이다.

(14)
$$\begin{aligned} \phi : H_1(X,\mathbb{Z}) &\rightarrow \Omega^1(X)^\vee, \\ \phi(\gamma)(\omega) &:= \int_\gamma \omega \quad (\text{단}, \gamma \in H_1(X,\mathbb{Z}),\ \omega \in \Omega^1(X)) \end{aligned}$$

와 같이 정의되는 사상 ϕ는 단사 사상이며 $\Lambda := \mathrm{Im}\ \phi$ 는 복소 공간 $\Omega^1(X)^\vee$ 안에서 계수가 $2g$ 인 격자 (lattice)가 된다.

(15)
$$J(X) := \Omega^1(X)^\vee / \Lambda$$

이라 정의하면 $J(X)$는 g차원의 아벨 다양체이다. $J(X)$를 X의 야코비 다양체 (Jacobian variety)라고 한다. 그리고 아벨-야코비 사상 $\Phi_{\mathrm{AJ}} : X \rightarrow J(X)$는

(16)
$$\Phi_{\mathrm{AJ}}(P)(\alpha) := \int_{P_0}^{P} \alpha, \quad P \in X,\ \alpha \in \Omega^1(X)$$

와 같이 정의된다는 사실을 상기하라. 여기서 P_0는 어떤 고정된 X의 한 점이다. 그러면 Φ_{AJ}는 선형적으로 인자군 (因子群, the divisor group) Div(X) 상으로 자연스럽게 확장될 수 있으며 차수가 0 인 인자들로 이루어진 $\mathrm{Div}^0(X)$에 제한한 사상

(17)
$$\Phi^0_{\mathrm{AJ}} : \mathrm{Div}^0(X) \rightarrow J(X)$$

는 전사 사상임을 쉽게 알 수 있다. 그래서

$$\mathrm{Pic}^0(X) = \mathrm{Div}^0(X)/\mathrm{Ker}\ \Phi^0_{AJ} \ \cong\ J(X).$$

(11)에서 $S_2(\Gamma)$는 $\Omega^1(X)$와 동형이라는 사실을 언급하였다. Hecke 대수 $T(N)$은 $S_2(\Gamma)$상에서 작용하기 때문에 $\Omega^1(X)^\vee$ 상에도 작용한다. 뿐만 아니라 $T(N)$은 Λ상에도 작용함을 보일 수 있다. 즉, $T(N)(\Lambda) \subset \Lambda$.

사상 $\Phi_f : \Omega^1(X)^\vee \ \to\ \mathbb{C}$를

(18) $$\Phi_f(\alpha) := \alpha(f d\tau), \quad \alpha \in \Omega^1(X)^\vee$$

와 같이 정의한다. 그리고 $\Lambda_f := \Phi_f(\Lambda)$이라 두면 Λ_f는 $\mathbb{C}$안에서 격자가 된다.

(19) $$E_f := \mathbb{C}/\Lambda_f$$

는 1 차원 복소 토러스가 되며 $\mathbb{Q}$상에서 정의되는 타원곡선이 된다는 사실을 Hecke 이론을 이용하여 증명할 수 있다. 사상 Φ_f는 자연스럽게 전사사상 $\Phi_f{}^* : J(X) \ \to\ E_f$를 유도한다. 따라서 합성사상 $\Theta_f := \Phi_f{}^* \circ \Phi_{AJ} : X \ \to\ E_f$를 얻는다. 실제로 Θ_f는 $\mathbb{Q}$상에서 정의되는 사상 (morphism)이다.

E가 앞에서 언급한 고유형식 f와 관련된 타원곡선이라 가정하자 (제 2 장의 정리 1을 보라). 그러면 E와 E_f는 서로 동종적임을 증명할 수 있다. 이 사실로부터

$$L(E,s) = L(E_f,s) = L(f,s)$$

의 관계식이 성립하여 E의 L-급수 $L(E,s)$는 $\mathbb{C}$상으로 해석적 접속이 가능하며 적당한 함수 방정식을 만족한다.

타원곡선 E로부터 유도되는 Galois 표현 $\rho_{E,\ell}$을 간략하게 설명하겠다. 소수 ℓ에 대하여 동종사상 $[\ell] : E \ \to\ E$를

$$[\ell]P := P + P + \cdots + P \ (\ell\text{번 합}),\ P \in E$$

와 같이 정의한다. 그러면 임의의 자연수 ℓ에 대하여

$$\mathrm{Ker}\ [\ell^k] \cong \mathbb{Z}/\ell^k\mathbb{Z}\ \times\ \mathbb{Z}/\ell^k\mathbb{Z}$$

임을 보일 수 있다. 이의 증명은 [34, p. 89]를 보라. 그리고

$$T_\ell(E) := \varprojlim \mathrm{Ker}\,[\ell^k] \;\cong\; \mathbb{Z}_\ell \times \mathbb{Z}_\ell$$

임을 쉽게 알 수 있다. $T_\ell(E)$를 소수 ℓ에 대하여 E의 Tate 모듈이라고 한다. $G_{\mathbb{Q}}$는 Ker $[\ell^k]$상에 작용하므로 Tate 모듈 $T_\ell(E) \cong (\mathbb{Z}_\ell)^2$ 상에 작용한다. 그러므로 자연스럽게 ℓ-adic 표현

$$\rho_{E,\ell} : G_{\mathbb{Q}} \;\rightarrow\; \mathrm{Aut}(T_\ell(E)) \;\cong\; GL_2(\mathbb{Z}_\ell) \tag{20}$$

을 얻는다.

S-T 가설은 아래의 주장 (ST1)–(ST8)과 각각 동치관계가 있음을 알 수 있다. 이의 증명은 독자에게 남겨 두겠다. 이제 당분간 E는 $\mathbb{Q}$상에서 정의되는 타원곡선이라 하자.

(ST1) 적당한 자연수 N에 대하여 상수 함수가 아닌 사상 $X_0(N) \;\rightarrow\; E$가 존재한다.

(ST2) 적당한 자연수 N에 대하여 해석적 전사사상 $X_0(N)(\mathbb{C}) \;\rightarrow\; E(\mathbb{C})$가 존재한다.

(ST3) E는 $X_0(N)$상에 정의되는 동일한 수준을 갖는 모듈러 함수 $f(\tau)$, $g(\tau)$에 의하여 매개화될 수 있다.

(ST4) E는 수적 타입 (arithmetic type)의 쌍곡적 균일화 (hyperbolic uniformization)를 허용한다.

(ST5) 적당한 고유형식 f가 존재하여 E는 E_f와 동종적이다. 여기서 E_f는 (19)에서 정의된 타원곡선이다.

(ST6) 임의의 소수 ℓ에 대하여 Galois 표현 (ℓ-adic 표현) $\rho_{E,\ell} : G_{\mathbb{Q}} \;\rightarrow\; GL_2(\mathbb{Z}_\ell)$은 모듈러이다.

(ST7) 어떤 소수 ℓ에 대하여 Galois 표현 $\rho_{E,\ell} : G_{\mathbb{Q}} \;\rightarrow\; GL_2(\mathbb{Z}_\ell)$은 모듈러이다.

(ST8) $S_2(\Gamma_0(N))$의 원소로서 푸리에 계수가 모두 유리수인 고유형식들의 집합과 전도체가 N인 타원곡선들의 동종류 (isogeny class)들의 집합사이에는 일대일 대응관계가 존재한다.

(ST4)에 관한 보다 상세한 것은 [19]와 [47]을 참고하길 바란다. 여기서 $\rho_{E,\ell}$이 모듈러이다라는 것은 어떤 고유형식 $f(\tau) = \sum_{n\geq 1} c_n q^n \in S_2(\Gamma_0(N))$가 존재하여 임의의 소수 p에 대하여

$$\mathrm{Tr}\ (\mathbb{F}_p) = c_p \in \mathbb{Z}$$

의 조건을 만족한다는 의미이다. 단, $\mathbb{F}_p$는 $G_{\mathbb{Q}}$의 p에서의 Frobenius 원소이다.

그러면, 왜 S-T 가설이 수론 분야에서 중요한가? 이 질문에 답하기는 쉽지 않지만 이 가설의 중요성을 입증할 몇 가지의 예를 소개하겠다. 만일, S-T 가설이 옳다고 가정하자. 첫째는 Hasse-Weil 가설이 성립하고, 둘째는 모듈러 타원곡선에 관한 BSD 가설의 연구가 어느 정도 되어 있기 때문에 BSD 가설의 해결을 앞당길 수가 있고, 셋째는 Fermat 마지막 정리가 S-T 가설의 일부가 해결됨으로써 마침내 증명되었다는 것이다. 참고로 CM-타원곡선이 모듈러이다라는 사실은 1971년에 G. Shimura (1930～)에 의하여 증명되었다 (참고문헌 [33]). 그 이 후 CM-타원곡선에 대한 BSD 가설이 여러 유명 수학자들에 의하여 연구되어 왔다는 사실은 이미 앞에서 언급하였다.

S-T 가설과 관련하여 Fontaine-Mazur 가설, Serre 가설, Artin 가설, Deligne-Serre 정리, Langlands-Tunnell 정리 등을 간략하게 열거하겠다.

FONTAINE-MAZUR 가설. K를 ℓ-adic 수체 $\mathbb{Q}_\ell$의 유한 확장체라고 하자. Galois 표현 $\rho:\ G_{\mathbb{Q}} \ \rightarrow\ GL_2(K)$가 절대적으로 기약인 ℓ-adic 표현이고 분해군 (decomposition group) G_ℓ에 제한한 표현 $\rho|_{G_\ell}$이 반안정적 (semistable)이라고 가정하자. 그러면 ρ는 모듈러이다. 즉, $\rho \sim \rho_f$이다. 여기서, ρ_f는 무게가 2 인 신형식 $f \in S_2(\Gamma_0(N_f))$로부터 구성되는 Galois 표현이다.

SERRE 가설. K를 유한체라고 하자. Galois 표현 $\bar{\rho}:\ G_{\mathbb{Q}} \ \rightarrow\ GL_2(k)$가 절대적으로 기약이고 홀수 표현이라고 가정하자. 여기서 $\mathrm{Aut}(\bar{\mathbb{Q}})$의 원소인 복소공액사상 c에 대하여 $\det \rho(c) = -1$ 인 조건을 만족하면 표현 ρ를 홀수(odd) 표현이라 한다. 그러면 $\bar{\rho}$는 모듈러이다. 즉, $\bar{\rho} \sim \bar{\rho}_f$이다. $\bar{\rho}_f$는 상기의 ρ_f의 축소 (reduction)의 반단순화 (semi-simplication)를 나타내고 있다.

도움말 5. $k = \mathbb{F}_3$이거나 $\bar{\rho}$의 사영적 상 (projective image)이 정이면체 군 (dihedral group)인 경우에는 Serre 가설이 성립한다는 사실이 증명되었다.

Artin 가설을 기술하기 전에 먼저 Deligne-Serre 정리를 소개하겠다.

DELIGNE-SERRE 정리. $f = \sum_{n\geq 1} c_n q^n$이 무게 1, 수준이 N_f, 지표가 ψ_f인 신형식이라고 하자. 그러면 아래의 성질 (A1)-(A3)의 조건을 만족하는 기약인 Artin 표현 $\rho_f : G_{\mathbb{Q}} \to GL_2(\mathbb{C})$가 존재한다.

(A1) ρ_f의 전도체(conductor)는 N_f이다.

(A2) $p \nmid N_f$이면 $\rho_f(\mathrm{Frob}_p)$의 고유다항식은

$$X^2 - c_p X + \psi_f(p)$$

이다.

(A3) $\rho_f(c) \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

ARTIN 가설. $\rho : G_{\mathbb{Q}} \to GL_2(\mathbb{C})$를 기약인 홀수 표현이라고 가정하자. 그러면 무게가 1 인 고유형식 f가 존재하여 $\rho \sim \rho_f$이다. 여기서 ρ_f는 Deligne-Serre에 의하여 구성된 Galois 표현이다.

Artin 가설에 관한 부분적인 결과가 R. Langlands (1936~)와 J. Tunnell에 의하여 얻어졌다.

LANGLANDS-TUNNELL 정리. Galois 표현 $\rho : G_{\mathbb{Q}} \to GL_2(\mathbb{C})$가 기약인 홀수 표현이라 하고 $\mathrm{Im}\,\rho = \rho(G_{\mathbb{Q}})$ 가 가해군 (solvable group)이라고 가정하자. 그러면 $\rho \sim \rho_f$이다. 여기서 ρ_f는 Deligne-Serre에 의하여 구성된 Galois 표현이다.

질문. $\mathrm{Im}\,\rho = A_5$ 인 경우에 Artin 가설은 성립하는가 ?

도움말 6. (1) Fontaine-Mazur 가설이 어떤 소수 ℓ에 대하여 성립한다면 S-T 가설도 성립한다.

(2) 무한히 많은 소수 ℓ에 대하여 Serre 가설이 성립한다면 S-T 가설이 성립한다.

K. A. Ribet은 S-T 가설을 아벨 다양체 상으로 일반화하였지만 필자의 사견으로는 자연스럽지 못한 것 같다. 하여튼 그가 S-T 가설을 일반화한 것을 간략하게 소개하겠다.

정의 2. $\mathbb{Q}$상에 정의된 아벨 다양체 A가 어떤 신형식 $f \in S_2(\Gamma_0(N))$가 존재하여 A와 A_f가 서로 동종적일 때 A를 모듈러이다라고 한다. 여기서 A_f는 Shimura 구성법에 의하여 f로부터 얻어지는 아벨 다양체이다.

정의 3 **(K. Ribet).** $\mathbb{Q}$상에 정의된 g차원 아벨 다양체 A의 $\mathrm{End}_{\mathbb{Q}} A$가 $[K:\mathbb{Q}] = g$인 어떤 수체 K의 order를 포함할 때 A를 GL_2-타입이라고 한다.

가설. $\mathbb{Q}$상에 정의된 단순 (simple) 아벨 다양체 A가 GL_2-타입이라고 가정하자. 그러면 A는 모듈러이다.

예를 들면 $g = 1$인 경우 타원곡선 A는 GL_2-타입이다.

이제 S-T 가설과 Fermat 마지막 정리를 서로 연관시키면서 Wiles의 연구 결과를 설명하겠다. $\mathbb{Q}$상에 정의된 타원곡선 E가 있어 임의의 소수 p에서의 축소 곡선 (reduced curve) $\bar{E}_p$가 특이점이 없는 곡선이든가 아니면 나빠야 특이점이 노드 (node) 인 경우에 E를 반안정 (semistable) 타원곡선이라고 한다.

정리 A **(Wiles-Taylor).** E가 반안정 타원곡선이라고 하자. 그리고 어떤 소수 ℓ에 대하여 Galois 표현

$$\bar{\rho}_{E,\ell} : G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{F}_\ell)$$

이 모듈러인 기약 표현이라고 가정하자. 그러면 E는 모듈러 타원곡선이다.

Hecke 이론과 B. Mazur가 창안한 Galois 표현의 변형이론 (deformation theory)을 이용하여 이 정리를 증명하고 있다.

정리 B **(Wiles).** 반안정인 타원곡선은 항상 모듈러이다.

이 정리는 Fermat 마지막 정리를 증명하는데 결정적인 역할을 한다. Wiles는 $\bar{\rho}_{E,3}$과 $\bar{\rho}_{E,5}$를 정리 A와 Langlands-Tunnell 정리에 적용함과 동시에 보조 역할을 하는 또 다른 타원곡선을 발견하여 연구함으로써 정리 B를 증명하였다.

Fred Diamond는 상기의 정리를 보다 일반화하였다.

정리 C (**Diamond**). E가 소수 2 와 3 에서 모두 반안정이라 가정하자. 그러면 E는 모둘러 타원곡선이다.

끝으로 Fermat 마지막 정리의 증명을 스케치하겠다. 우선 정수의 세 짝 $(a,b,c) \in \mathbb{Z}^3$, $abc \neq 0$가 Fermat 방정식

$$x^p + y^p = z^p \tag{21}$$

의 해라고 가정하자. 여기서 p는 $p > 3$인 소수, $\text{g.c.d}\,(a,b,c) = 1$, b는 짝수, $a \equiv -1 (\text{mod } 4)$라고 가정해도 무방하다. 독일 수학자 G. Frey는 1980년대 초에

$$F \ : \ y^2 = x(x - a^p)(x + b^p) \tag{22}$$

으로 주어지는 타원곡선을 연구 조사하였다. 이 곡선은 그의 이름을 따서 Frey 곡선이라고 한다. F의 판별식은

$$\Delta_F = (abc)^{2p} \tag{23}$$

로 주어져서 타원곡선의 판별식으로서는 아주 특이한 것이다. 그래서 Frey는 이러한 곡선이 있을 수 없다는 생각을 하게 되었다. 그는 Fermat 마지막 정리와 Frey 곡선과 연관시켜 가며 조사한 끝에 이러한 곡선이 존재하지 않는다는 사실만 보일 수만 있다면 350여 년 동안 풀리지 않았던 Fermat 마지막 정리를 완벽하게 증명할 수 있다는 놀랄만한 사실을 발견하였다. 이 사실을 전해들은 J.-P. Serre (1927~)는 그 동안 연구하여 왔던 그의 가설들을 다시 쓰기 시작하여 소위 ϵ-가설을 제시하였다 (참고문헌 [31]). 1986년 여름에 K. Ribet는 Serre의 가설의 일부를 해결하면서 『반안정인 타원곡선이 모둘러이다』 라는 사실만 보일 수만 있다면 Fermat 마지막 정리를 증명하는 것이라고 설명하였다. 그 당시에 Ribet의 강연을 듣고 난 후 A. Wiles는 정리 B를 해결하기 위해 약 7여 년 동안 조용히 연구하였다. 그는 자신이 정리 B를 증명하였다는 확신을 갖고 1993년 6월에 영국의 캠브리지에 있는 아이삭 뉴턴 수리연구소에서 열린 정수론 학술회의에서 「Elliptic curves, modular forms and Galois representations」의 제목으로 6월 21-23일의 사흘 동안 세 차례의 강연을 하였다. 그의 마지막 날의 강연에서 그는 정리 B의 증명을 간략하게 스케치하면

서 Fermat 마지막 정리를 해결하였다고 선언하였다. 그 당시에 거기에 있었던 모든 수학자들은 매우 놀랐으며 기쁨에 휩싸여 있었다. 이 기쁜 소식은 뉴욕 타임즈, 뉴스위크, 타임즈 등의 여러 매스 미디어를 통하여 세계 방방곡곡에 퍼져 나갔다. 이러는 동안 정리 B의 증명의 검증이 이 분야의 전문 수학자들에 시작되었다. 그러나 불행히도 검증 과정에서 오일러 시스템의 구성에 결함이 발견되어 정리 B의 증명에 대한 암운이 드리워지기 시작하였다. 아주 낙담한 Wiles는 이 결함을 메우기 위하여 악전 고투하였지만 반년이 지나도 진전이 없었다. 그는 그의 증명 방법이 올바르다는 데에 확신을 갖고 있었지만 결함을 고칠 수가 없었다. 결국에는 1994년 1월에 그의 제자였던 R. Taylor에게 도움을 청하였다. 함께 1994년 2월부터 연구한 지 9개월 후에 그의 결함을 메울 수 있었다. 그의 논문 ([44])에 의하면 1994년 9월 19일에 그의 증명의 결함을 고쳤으며 1994년 10월 14일에 「Annals of Mathematics」에 투고하였다고 한다. 마침내 그의 논문은 1995년 5월에 상기의 저널에 발표되었다. 그 후 그는 유명 인사가 되어 지난 수년 동안 울프 상, Fields 특별상 등 여러 상을 수상하였다. 보다 상세한 것은 필자의 저서 [48]을 읽어 보길 바란다.

다시 Frey 곡선으로 돌아가자. Frey는 Frey 곡선이 반안정인 타원곡선이라는 사실을 보였고 Ribet는 Frey 곡선이 모듈러이지 않다는 사실을 1986년에 증명하였다. 그런데 Wiles의 정리 B에 의하면 이 타원곡선은 모듈러이어야 하므로 Ribet의 주장에 어긋난다. 따라서 Fermat 방정식 (21)을 만족하는 정수해가 존재할 수 없다. 이로부터 마침내 350여 년 동안 풀리지 않았던 Fermat 마지막 정리가 해결된 것이다.

4. Heegner 점과 야코비 형식

E가 모듈러 타원곡선이면 이의 L-함수 $L(E,s)$는 제 2 장 (7)로 주어지는 함수 방정식을 만족한다. 정의 1에 의하여 적당한 자연수 N이 있어 공식 (12)를 만족하는 고유형식 $f \in S_2(\Gamma_0(N))$가 존재한다. 편의상

$$m := \mathrm{ord}_{s=1} L(E,s) \quad (E\text{의 해석적 계수})$$

이라 둔다. 그러면

$$m = \text{짝수} \iff \epsilon = 1 \iff f(-1/N\tau) = -N\tau^2 f(\tau),\ \tau \in \mathbb{H}$$

이고

$$m = \text{홀수} \iff \epsilon = -1 \iff f(-1/N\tau) = +N\tau^2 f(\tau),\ \tau \in \mathbb{H}$$

임을 알 수 있다. E가 모듈러이기 때문에 전사 해석적 함수

(24) $$\phi_E : X_0(N) \rightarrow E$$

가 존재한다. K를 판별식이 D인 허수 이차체라고 하고 N의 임의의 소수 인자 (prime divisor)가 K안에서 분할 (split)된다고 가정하자. 그러면 g.c.d $(D, N) = 1$ 이고 $D \equiv r^2 \pmod{4N}$ 이다. 단, r은 적당한 정수이다.

이차 방정식

$$a\tau^2 + b\tau + c = 0, \qquad a, b, c \in \mathbb{Z},\ N|a,$$
$$b \equiv r (\text{mod } 2N), \qquad D = b^2 - 4ac$$

의 조건을 만족하는 모든 $\tau \in \mathbb{H}$의 점들의 집합을 Θ라고 표기하자. 그러면 임의의 $r \in \Gamma_0(N)$에 대하여 $\gamma(\Theta) \subset \Theta$이고 Θ는 오직 h_K개의 $\Gamma_0(N)$-궤적을 갖는다. 여기서 h_K는 K의 유수 (class number)이다. $z_1, \cdots, z_{h_K}$가 이 궤적들의 대표원소 (representative)라고 하면 $\phi_E(z_1), \cdots, \phi_E(z_{h_K})$는 일반적으로 K상에서는 정의되지 않고 K의 Hilbert 유체상에서 정의된다. (이는 CM-이론으로부터 이 사실을 알 수 있다.)

Heegner 점

(25) $$P_{D,r} := \sum_{i=1}^{h_K} \phi_E(z_i)$$

은 K상에서 정의되고 $c(P_{D,r}) = -\epsilon P_{D,r}$이란 사실을 보일 수 있다. 여기서, c는 복소 공액사상 (complex conjugation)이다.

먼저 $\epsilon = -1$ 인 경우를 검토하여 보자. B. Gross와 D. Zagier는 논문 [10]에서 $L(E,1) = 0$ 이고 $2P_{D,r} \in E(\mathbb{Q})$ 인 경우에는

$$(26) \qquad L(E^{(D)},1)L'(E,1) = c \cdot \Omega_{E^{(D)}} \cdot \Omega_E \cdot \hat{h}(2P_{D,r})$$

의 공식을 발견하였다. 여기서 $E^{(D)}$는

$$(27) \qquad E^{(D)} \ : \ Dy^2 = x^3 + ax + b$$

으로 주어지는 뒤틀린 곡선 (twisted curve)이고, $\Omega_E, \Omega_{E^{(D)}}$는 타원곡선 E와 $E^{(D)}$의 실 주기, $\hat{h}$는 E의 Néron-Tate 높이 함수이다. 게다가 c는 0 이 아닌 유리수이다. 다음 $\epsilon = 1$ 인 경우에도 상기의 두 수학자는

$$(28) \qquad L(E,1)L'(E^{(D)},1) = d \cdot \Omega_{E^{(D)}} \cdot \Omega_E \cdot \hat{h}_{E^{(D)}}(2P_{D,r})$$

의 공식을 발견하였다. 게다가 d는 0 이 아닌 유리수이다. 여기서 $\hat{h}_{E^{(D)}}$는 $E^{(D)}(\mathbb{Q})$상의 Néron-Tate 높이 함수이다.

관계식 (26)을 이용하여 제 2 장에서 언급하였던 [BSD3]과 [BSD4]를 증명할 수 있다. 이 증명을 간략하게 스케치하겠다. 만일 E가 해석적 계수 $m = 1$ 인 모둘러 타원곡선이라 하면 $\epsilon = -1$ 이고 $L'(E,1) \neq 0$ 이다. Waldspurger 정리에 의하여 적당한 정수 D가 존재하여 $L'(E^{(D)},1) \neq 0$ 이다. (26)에 의하여 $\hat{h}(2P_{D,r}) \neq 0$ 이므로 $\mathrm{ord}(P_{D,r}) = \infty$ 이다. 따라서 [BSD3]을 얻는다. 그리고 만약에 E의 대수적 계수가 1 이라고 하자. 그러면 $L(E^{(D)},1)\,\Omega_{E^{(D)}}^{-1}$이 유리수이기 때문에 (28)에 의하여 $L'(E,1)\,\Omega_E^{-1}\,\hat{h}(2P_{D,r})^{-1}$ 는 유리수이다. 따라서 [BSD4]가 증명된다.

도움말 7. 실제로 공식 (26)과 (28)의 우변은 r 의 선택에 무관하다. 그리고 $\hat{h}(P_{D,r})$도 역시 r의 선택에 무관하다.

H를 K의 Hilbert 유체라고 하고 $\mathrm{Gal}(H/K)$의 원소 σ를 고정시키자. 그러면 Artin 동형사상에 의하여 $\mathrm{Gal}(H/K)$는 K의 클라스 (class) 군 Cl_K와 동형이다. σ에 대응되는 클라스 Ξ에 대하여 세타함수

$$(29) \qquad \theta_\Xi(\tau) := \frac{1}{2u} + \sum_{\mathfrak{a} \in \Xi} e^{2\pi N\mathfrak{a}\tau} = \sum_{n \geq 0} r_\Xi(n)\, e^{2\pi i n\tau}$$

를 정의한다. 여기서, u는 유한군 $\mathcal{O}/\{\pm 1\}$ (단, $\mathcal{O}$는 K의 정수환이다)의 개수이고, $D = -3$이면 $u = 3$, $D = -4$ 이면 $u = 2$, 이외의 경우는 $u = 1$이다.

또, $r_\Xi(0) = \frac{1}{2u}$ 이고 $r_\Xi(n)$ $(n \geq 1)$은 $N\mathfrak{a} = n$ 인 Ξ 안에 있는 정수 이데알들의 개수이다. 그러면 $\theta_\Xi(\tau)$는 $\Gamma_1(D)$와 K와 관련되어 있는 지표 $\epsilon : (\mathbb{Z}/D\mathbb{Z})^\times \to \{\pm 1\}$에 대하여 무게가 1 인 모둘러 형식이다.

고유형식 $f \in S_2(\Gamma_0(N))$와 아이디얼 클라스 Ξ에 대하여 L-함수

$$L_\Xi(f, s) := \sum_{n\geq 1} \epsilon(n), n^{1-2s} \cdot \sum_{n \geq 1} a_n, r_\Xi(n), n^s \tag{30}$$

를 정의한다. 게다가 χ를 K의 클라스 군 Cl_K의 복소 지표라고 하고 이에 대하여 L-함수

$$L(f, \chi, s) := \sum_{\Xi} \chi(\Xi) L_\Xi(f, s) \tag{31}$$

을 정의한다. 그러면 $L_\Xi(f, s)$와 $L(f, \chi, s)$는 $\mathrm{Re}s > \frac{3}{2}$ 인 영역에서 절대 수렴하며 Rankin 방법에 의하여 이들은 전 복소 평면에 해석적 접속을 가지며 적당한 함수 방정식을 만족한다.

J를 모둘러 곡선 $X_0(N)$의 아벨 다양체라고 하자. 그리고 $x := P_{D,r}$를 Heegner 점이라 하고 c를 인자 $(x) - (\infty)$ 의 클라스라고 하자. $c_\chi := \sum_\sigma \chi^{-1}(\sigma) c^\sigma$ 이라 두면 $c_\chi \in J(H) \otimes \mathbb{C}$ 이고 $c_{\chi, f}$ 를 Hecke 대수 $\mathbf{T}$의 작용에 대하여 $J(H) \otimes \mathbb{C}$ 의 f-등방 성분이라 하자. B. Gross와 D. Zagier는

$$L'(f, \chi, 1) = \frac{8\pi^2 (f, f)}{h_K u^2 |D|^{\frac{1}{2}}} \hat{h}(c_{\chi, f}) \tag{32}$$

의 공식을 발견하였다. 여기서

$$(f, f) = \int_{\Gamma_0(N)\backslash \mathbb{H}} |f(\tau)|^2 dx dy, \;\; \tau = x + iy$$

이고 $\hat{h}$는 J상의 Néron-Tate 높이 함수이다.

$$P_K = \sum_{\sigma \in \mathrm{Gal}(H/K)} \phi_E(x^\sigma)$$

이라 두면 P_K는 $E(K)$의 원소이고

$$L'(E/K,1) = \frac{||\omega||^2 \cdot \hat{h}(P_K)}{c^2u^2|D|^{\frac{1}{2}}} \tag{33}$$

인 관계식이 성립한다. 여기서 ω는 E상의 Néron 미분형식이고

$$||\omega||^2 = \int_{E(\mathbb{C})} |\omega \wedge \bar{\omega}|^2$$

이다. (33)의 증명은 참고문헌 [10]을 참고하길 바란다.

야코비 형식의 이론과 무게가 반정수인 모듈러 형식의 이론으로부터

$$[\Gamma_2, k]^M \cong J_{k,1}(\Gamma_1) \cong M^+_{k-\frac{1}{2}}(\Gamma_0(4)) \cong [\Gamma_1, 2k-2] \tag{34}$$

의 관계식을 얻는다. 여기서 Γ_n은 Siegel 모듈러 군이고, $[\Gamma_2, k]^M$은 Maass 공간을 나타내고, $J_{k,1}(\Gamma_1)$은 무게가 k이고 지수 (index)가 1 인 야코비 형식들로 이루어진 벡터공간이고, $M^+_{k-\frac{1}{2}}(\Gamma_0(4))$는 Kohnen 공간이고, $[\Gamma_1, 2k-2]$는 무게가 $2k-2$인 모듈러 형식으로 이루어진 벡터공간을 나타낸다. (34)에서 맨 오른쪽에 있는 동형사상이 다름 아닌 Shimura 대응이다. 상기의 동형사상들은 Hecke 작용과 양립한다. (34)에 대한 보다 상세한 설명은 [5]와 [45]를 참조하길 바란다. 야코비 형식에 관한 참고문헌으로는 [5]를, 보다 고차원 공간에서의 야코비 형식의 이론에 관한 참고문헌으로는 [45],[46]을 추천한다.

모듈러 타원곡선 E와 이와 관련된 고유형식 $f \in S_2(\Gamma_0(N))$로 돌아가자. $g_f(\tau) = \sum_{n\geq 1} c(n)q^n$을 Shimura 대응에 의하여 f로부터 주어지는 무게가 $\frac{3}{2}$인 첨점 형식이라 하자. 그러면 Waldspurger 정리에 의하여 특수한 값 $L(E^{(D)}, 1)$은 푸리에 계수 $c(|D|)$의 제곱에 비례한다. $\langle\ ,\ \rangle$ 를 Néron-Tate 높이 함수 $\hat{h}$ 로부터 유도되는 높이 짝 (height pairing)이라 하자. 그러면 서로 다른 Heegner 점 P_{D_1,r_1}과 P_{D_2,r_2}에 대하여 값 $\langle P_{D_1,r_1}, P_{D_2,r_2}\rangle$와 $c(|D_1|)c(|D_2|)$와의 관계가 존재하지 않을까하는 의문이 자연스럽게 생긴다. $P_{D,r}$의 높이 $\hat{h}(P_{D,r})$은 r의 선택에 무관하지만 $\langle P_{D_1,r_1}, P_{D_2,r_2}\rangle$의 값은 r_1, r_2의 선택에 의존한다. 그러므로 무게가 $\frac{3}{2}$ 인 모듈러 형식의 이론은 상기의 의문을 해결하는데 도움이 되지 않는다. 그래서 D. Zagier는 무게가 $\frac{3}{2}$ 인 모듈러 형식의 이론대신에 무게가

2 인 야코비 형식의 이론을 연구 · 검토하여야 한다는 생각을 하게 되었다. 무게가 k이고 지수가 N인 야코비 형식 ϕ는

$$\phi(\tau, z) = \sum_{n,r\in\mathbb{Z}} c(n,r)e^{2\pi i(n\tau+rz)}, \ \tau \in \mathbb{H}, \ z \in \mathbb{C} \tag{35}$$

와 같은 푸리에 급수를 갖는다. 푸리에 계수 $c(n,r)$은 판별식 $D = r^2 - 4Nn$ 과 잉여류 $r (\mathrm{mod}\ 2N)$ 에만 의존한다. N.-P. Skoruppa와 D. Zagier는 논문 [37]에서 $J_{k,N}(\Gamma_1)$과 $[\Gamma_0(N), 2k-2]$의 어떤 부분공간 사이에 Hecke 대수의 작용에 양립하는 대응관계가 있음을 보였다. 여기서, $[\Gamma_0(N), 2k-2]$는 $\Gamma_0(N)$위에서의 무게가 $2k-2$ 인 모듈러 형식들로 이루어진 벡터공간을 나타낸다.

GROSS-KOHNEN-ZAGIER 정리. E를 전도체가 N인 모듈러 타원곡선이라 하고 $\epsilon = -1$, 해석적 계수 $m = 1$ 이라고 가정하자. 또, D_1과 D_2가 서로 소인 음의 정수이고 $D_i \equiv r_i^2 \ (\mathrm{mod}\ 4N)$ (단, $i = 1, 2$) 이라고 가정하자. 그러면

$$L'(E,1)\, c(n_1, r_1)\, c(n_2, r_2) = c_E \left\langle P_{D_1,r_1}, P_{D_2,r_2} \right\rangle \tag{36}$$
$$(\text{단}, \ D_i = r_i^2 - 4Nn_i, \ i = 0, 1)$$

의 관계가 성립한다. 상수 c_E,는 양수로서 D_1, r_1, D_2, r_2 에 의존하지 않는다. 여기서 $c(n,r)$은 Skoruppa-Zagier에 의하여 f에 의하여 대응되는 야코비 형식 ϕ_f의 푸리에 계수이다. 실제로 상수 c_E는 E의 실 주기 Ω_E로서 나타낼 수 있다.

따름정리. 조건 $D \equiv r^2 \ (\mathrm{mod}\ 4N)$ 과 조건 g.c.d $(D, 2N)$=1 을 만족하는 임의의 D와 r,에 대하여

$$P_{D,r} = c\left(\frac{r^2 - D}{4N}, r\right) P_0$$

의 관계식을 만족하는 점 $P_0 \in E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{R}$ 가 존재한다.

도움말 8. R. E. Borcherds (참고문헌 [2])는 Gross-Kohnen-Zagier 정리를 어떤 고차원의 에르미트 등질공간상으로 일반화하였다.

5. 끝맺음 말

지난 30여 년 동안 $\mathbb{Q}$ 또는 수체 상에서 정의되는 타원곡선의 이론이 수론, 대수기하학, 보형형식의 이론 등의 여러 분야와 밀접하게 관련되어 현대 수학에서 중요한 분야 중의 하나가 되었다. 이러한 타원곡선의 기하학적인 측면과 수론적인 측면을 더욱더 이해하기 위해서는 이 논문에서는 다루지 않았지만, 오일러 시스템 (Euler system)이론, Heegner 점 이론, Galois 표현과 이의 변형 이론, Tate-Shafarevich 군 이론 등의 여러 토픽을 그전보다 더욱더 심도있게 연구・검토하여야 할 필요가 있다고 생각한다. 이러한 토픽에 관한 참고문헌으로 [6], [10], [11], [15], [16], [20], [27], [29]를 독자에게 추천한다.

타원곡선에 대한 실체가 상당 부분 드러났지만 이에 대한 본질적인 부분은 아직까지도 해결되지 않았다. 이 논문에서 소개되었던 BSD 가설과 S-T 가설 등의 여러 가설의 해결을 위해서는 아주 새로운 아이디어와 보다 고차원적인 이론이 도입되어야 한다고 생각한다. 가까운 장래에 대한민국의 수학자에 의하여 이 가설이 해결되어 세계 수학계에서 대한민국 수학의 위상이 세워지기를 바란다.

참고문헌

[1] C. Birkenhake and H. Lange, *Complex abelian varieties*, Springer-Verlag, 1992.

[2] R. E. Borcherds, *The Gross-Kohnen-Zagier theorem in higher dimensions*, preprint.

[3] J. Coates and A. Wiles, *On the Birch-Swinnerton-Dyer conjecture*, Invent. Math. **39** (1977), 223-251.

[4] H. Darmon, F. Diamond and R. Taylor, *Fermat's last theorem*, Current Development in Math. 1995, International Press, 1995, pp. 1-154.

[5] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Prog. Math. **55** (1995), Birkhäuser.

[6] J.-M. Fontaine and B. Mazur, *Geometric Galois representations*, Proceedings of the conference on Elliptic Curves, Modular Forms and Fermat's Last Theorem, edited by J. Coates and S.-T. Yau, International Press, Cambridge, 1995, pp. 41-78.

[7] G. Frey, *Rationale Punkte auf Fermatkurven und getwisten Modulkurven*, Crelle **331** (1982), 185-19.

[8] ______, *Links between stable elliptic curves and certain diophantine equations*, Annales Universitatis Saraviensis **1** (1986), 1-40.

[9} R. Greenberg, *On the Birch-Swinnerton-Dyer conjecture*, Invent. Math. **72** (1983), 241-265.

[10] B. Gross and D. Zagier, *Heegner points and derivatives of L-series*, Invent. Math. **72** (1983), 241-265.

[11] B. Gross, W. Kohnen and D. Zagier, *Heegner points and derivatives of L-series II*, Math. Ann. **278** (1987), 497-562.

[12] A. W. Knapp, *Elliptic Curves*, Math. Notes **40**, Princeton Univ. Press, 1992.

[13] N. Koblitz, *Introduction to elliptic curves and modular forms*, Springer-Verlag, 1984.

[14] W. Kohnen, *Elliptic curves and automorphic forms*, Proceedings of the 1994 Conference on Number Theory and Related Topics, Pyungsan Institute for Mathematical Sciences (1995), 19-28.

[15] V. A. Kolyvagin, *Finiteness of* $E(\mathbb{Q})$ *and* $\text{III}(E, \mathbb{Q})$ *for a subclass of Weil curves*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **52** (1988), 522-540; English translation in Math. USSR-Izv. **32** (1989), 523-541.

[16] ______, *Euler systems, the Grothendieck Festschrift* (vol. II), edited by P. Cartier et al. Prog. in Math. **87**, Birkhäuser (1990), 435-483.

[17] R. Langlands, *Base change for* $GL(2)$, Ann. of Math. Studies **96**, Princeton, 1980.

[18] B. Mazur, *Modular curves and the Eisenstein ideals*, Publ. IHES. **47** (1977), 33-186.

[19] ______, *Number Theory as Gadfly*, Amer. Math. Monthly **98** (1991), 593-610.

[20] ______, *Deforming Galois representations in Galois Groups over* $\mathbb{Q}$, MSRI Publ., vol. 16, Springer-Verlag, Berlin, 1989.

[21] L. Merel, *Bornes pour la torsion des courbes elliptiques sur les corps de nombres*, Invent. Math. **124** (1996), 437-449.

[22] D. Mumford, *Abelian varieties*, Oxford University Press, 1970.

[23] K. Ribet, *On modular representations of* $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ *arising from modular forms*, Invent. Math. **100** (1990), 431-476.

[24] ______, *Wiles proves Taniyama's conjecture; FLT follows*, Notices of AMS **40** (1993), 575-576.

[25] ———, *Update and Overview on FLT*, AMS-MAA Invited Hour Talk, 1994.

[26] ———, *Galois representations and modular forms*, Bull. AMS. **32** (1995), no. 4, 375-402.

[27] ———, *Irreducible Galois representations*, Proceedings of the conference on Elliptic Curves, Modular Forms and Fermat's Last Theorem, edited by J. Coates and S.-T. Yau, International Press, Cambridge (1995), pp. 131-145.

[28] K. Rubin, *Elliptic curves with complex multiplication and the BSD conjecture*, Invent. Math. **64** (1981), 455-470.

[29] ———, *Euler systems*, preprint, Stanford University, 1999.

[30] K. Rubin and A. Silverberg, *Wiles' Proof of FLT*, Bull. AMS. **31** (1994), 15-38.

[31] J.-P. Serre, *Sur les representations modularies de degree* 2 *de* $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, Duke Math. J. **54** (1987), 179-230.

[32] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Iwanami Shoten, Publishers and Princeton University Press, 1971.

[33] ———, *On elliptic curves with complex multiplication as factors of the Jacobians of modular function fields*, Nagoya Math. J. **43** (1971), 199-208.

[34] J. H. Silverman, *The Arithmetic of Elliptic Curves*, Springer-Verlag, 1992.

[35] ———, *Advanced Topics in the Arithmetic of Elliptic Curves*, Springer-Verlag, 1994.

[36] J. H. Silverman and J. Tate, *Rational points on elliptic curves*, Springer-Verlag, 1992.

[37] N.-P. Skoruppa and D. Zagier, *Jacobi forms and a certain space of modular forms*, Invent. Math. **94** (1988), 113-146.

[38] J. Tunnell, *Artin's conjecture for representations of octahedral type*, Bull. AMS. **5** (1981), 173-175.

[39] ———, *A classical Diophantine problem and modular forms of weight* 3/2, Invent. Math. **72** (1983), 323-334.

[40] R. Taylor and A. Wiles, *Ring theoretic properties of certain Hecke algebras*, Ann. Math. **141** (1995), 553-572.

[41] J.-L. Waldspurger, *Sur les coefficients de Fourier des formes modulaires de poids demi entier*, J. Math. Pures Appl. **60** (1981), 375-484.

[42] A. Weil, *Variétés abéliennes et courbes algébriques*, Hermann, Paris, 1948.

[43] ———, *Über die bestimmung Drichlescher Reihnen durch Funktionalgleichungen*, Math. Ann. **168** (1967), 149-156.

[44] A. Wiles, *Modular elliptic curves and Fermat's last theorem*, Ann. Math. **141** (1995), 443-551.

[45] 양재현, *Jacobi* 형식에 관하여, 평산 연구회 소식 제 3호 (1993), 3-29.

[46] ______, *Remarks on Jacobi forms of higher degree*, Proceedings of the 1993 Conference on Automorphic Forms and Related Topics, edited by J.-W. Son and J.-H. Yang, Pyungsan Institute for Mathematical Sciences (1993), 33-58.

[47] ______, *Note on Taniyama-Shimura-Weil conjecture*, Proceedings of the 1994 Conference on Number Theory and Related Topics, edited by J.-W. Son and J.-H. Yang, Pyungsan Institute for Mathematical Sciences (1995), 29-46.

[48] ______, 『20세기 수학자들과의 만남』 (1998년 [초판], 1999년 [중판]), 도서출판 애플기획.

[49] D. Zagier, *L-series of Elliptic Curves, the BSD Conjecture, and the Class Number Problem of Gauss*, Notices of AMS **31** (1984), 739-743.

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THE BIRCH-SWINNERTON-DYER CONJECTURE

JAE-HYUN YANG

ABSTRACT. We give a brief description of the Birch-Swinnerton-Dyer conjecture which is one of the seven Clay problems.

1 INTRODUCTION

On May 24, 2000, the Clay Mathematics Institute (CMI for short) announced that it would award prizes of 1 million dollars each for solutions to seven mathematics problems. These seven problems are

Problem 1. The "P versus NP" Problem:

Problem 2. The Riemann Hypothesis:

Problem 3. The Poincaré Conjecture:

Problem 4. The Hodge Conjecture:

Problem 5. The Birch-Swinnerton-Dyer Conjecture:

Problem 6. The Navier-Stokes Equations: Prove or disprove the existence and smoothness of solutions to the three dimensional Navier-Stokes equations.

Problem 7. Yang-Mills Theory: Prove that quantum Yang-Mills fields exist

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

and have a mass gap.

Problem 1 is arisen from theoretical computer science, Problem 2 and Problem 5 from number theory, Problem 3 from topology, Problem 4 from algebraic geometry and topology, and finally problem 6 and 7 are related to physics. For more details on some stories about these problems, we refer to Notices of AMS, vol. 47, no. 8, pp. 877-879 (September 2000) and the homepage of CMI.

In this paper, I will explain Problem 5, that is, the Birch-Swinnerton-Dyer conjecture which was proposed by the English mathematicians, B. Birch and H. P. F. Swinnerton-Dyer around 1960 in some detail. This conjecture says that if E is an elliptic curve defined over $\mathbb{Q}$, then the algebraic rank of E equals the analytic rank of E. Recently the Taniyama-Shimura conjecture stating that any elliptic curve defined over $\mathbb{Q}$ is modular was shown to be true by Breuil, Conrad, Diamond and Taylor [BCDT]. This fact shed some lights on the solution of the BSD conjecture. In the final section, we describe the connection between the heights of Heegner points on modular curves $X_0(N)$ and Fourier coefficients of modular forms of half integral weight or of the Jacobi forms corresponding to them by the Skoruppa-Zagier correspondence.

Notations : We denote by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the fields of rational numbers, real numbers and complex numbers respectively. $\mathbb{Z}$ and $\mathbb{Z}^+$ denotes the ring of integers and the set of positive integers respectively.

2 The Mordell-Weil Group

A curve E is said to be an *elliptic curve* over $\mathbb{Q}$ if it is a nonsingular projective curve of genus 1 with its affine model

$$y^2 = f(x), \tag{2.1}$$

where $f(x)$ is a polynomial of degree 3 with integer coefficients and with 3 distinct roots over $\mathbb{C}$. An elliptic curve over $\mathbb{Q}$ has an abelian group structure with distinguished element ∞ as an identity element. The set $E(\mathbb{Q})$ of rational points given by

$$E(\mathbb{Q}) = \{\, (x,y) \in \mathbb{Q}^2 \mid y^2 = f(x) \,\} \cup \{\infty\} \tag{2.2}$$

also has an abelian group structure.

L. J. Mordell (1888-1972) proved the following theorem in 1922.

Theorem A (Mordell, 1922). $E(\mathbb{Q})$ is finitely generated, that is,

$$E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus E_{\mathrm{tor}}(\mathbb{Q}),$$

where r is a nonnegative integer and $E_{\mathrm{tor}}(\mathbb{Q})$ is the torsion subgroup of $E(\mathbb{Q})$.

Definition 1. Around 1930, A. Weil (1906-1998) proved the set $A(\mathbb{Q})$ of rational points on an abelian variety A defined over $\mathbb{Q}$ is finitely generated. An elliptic curve is an abelian variety of dimension one. Therefore $E(\mathbb{Q})$ is called the *Mordell-Weil group* and the integer r is said to be the *algebraic rank* of E.

In 1977, B. Mazur (1937-)[Ma1] discovered the structure of the torsion subgroup $E_{\text{tor}}(\mathbb{Q})$ completely using a deep theory of elliptic modular curves.

Theorem B (Mazur, 1977). Let E be an elliptic curve defined over $\mathbb{Q}$. Then the torsion subgroup $E_{\text{tor}}(\mathbb{Q})$ is isomorphic to the following 15 groups

$$\mathbb{Z}/n\mathbb{Z} \quad (1 \leq n \leq 10,\ n = 12),$$

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z} \quad (1 \leq n \leq 4).$$

E. Lutz (1914-?) and T. Nagell (1895-?) obtained the following result independently.

Theorem C (Lutz, 1937; Nagell, 1935). Let E be an elliptic curve defined over $\mathbb{Q}$ given by

$$E: \quad y^2 = x^2 + ax + b, \quad a, b \in \mathbb{Z},\ 4a^3 + 27b^2 \neq 0.$$

Suppose that $P = (x_0, y_0)$ is an element of the torsion subgroup $E_{\text{tor}}(\mathbb{Q})$. Then

(a) $x_0, y_0 \in \mathbb{Z}$, and

(b) $2P = 0$ or $y_0^2 | (4a^3 + 27b^2)$.

We observe that the above theorem gives an effective method for bounding $E_{\text{tor}}(\mathbb{Q})$. According to Theorem B and C, we know the torsion part of $E(\mathbb{Q})$ satisfactorily. But we have no idea of the free part of $E(\mathbb{Q})$ so far. As for the algebraic rank r of an elliptic curve E over $\mathbb{Q}$, it is known by J.-F. Mestre in 1984 that values as large as 14 occur. Indeed, the elliptic curve defined by

$$y^2 = x^3 - 35971713708112\,x + 85086213848298394000$$

has its algebraic rank 14.

Conjecture D. Given a nonnegative integer n, there is an elliptic curve E over $\mathbb{Q}$ with its algebraic rank n.

The algebraic rank of an elliptic curve is an invariant under the isogeny. Here an isogeny of an elliptic curve E means a holomorphic map $\varphi : E(\mathbb{C}) \longrightarrow E(\mathbb{C})$ satisfying the condition $\varphi(0) = 0$.

3 Modular Elliptic Curves

For a positive integer $N \in \mathbb{Z}^+$, we let

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \mid N|c \right\}$$

be the Hecke subgroup of $SL(2,\mathbb{Z})$ of level N. Let $\mathbb{H}$ be the upper half plane. Then

$$Y_0(N) = \mathbb{H}/\Gamma_0(N)$$

is a noncompact surface, and

$$X_0(N) = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}/\Gamma_0(N) \tag{3.1}$$

is a compactification of $Y_0(N)$. We recall that a *cusp form* of weight $k \geq 1$ and level $N \geq 1$ is a holomorphic function f on $\mathbb{H}$ such that for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and for all $z \in \mathbb{H}$, we have

$$f((az+b)/(cz+d)) = (cz+d)^k f(z)$$

and $|f(z)|^2(\operatorname{Im} z)^k$ is bounded on $\mathbb{H}$. We denote the space of all cusp forms of weight k and level N by $S_k(N)$. If $f \in S_k(N)$, then it has a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} c_n(f) q^n, \quad q := e^{2\pi i z}$$

convergent for all $z \in \mathbb{H}$. We note that there is no constant term due to the boundedness condition on f. Now we define the L-series $L(f,s)$ of f to be

$$L(f,s) = \sum_{n=1}^{\infty} c_n(f)\, n^{-s}. \tag{3.2}$$

For each prime $p \nmid N$, there is a linear operator T_p on $S_k(N)$, called the Hecke operator, defined by

$$(f|T_p)(z) = p^{-1} \sum_{i=0}^{p-1} f((z+i)/p) + p^{k-1}(cpz+d)^k \cdot f((apz+d)/(cpz+d))$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ with $c \equiv 0\,(N)$ and $d \equiv p\,(N)$. The Hecke operators T_p for $p \nmid N$ can be diagonalized on the space $S_k(N)$ and a simultaneous eigenvector is called an *eigenform.* If $f \in S_k(N)$ is an eigenform, then the corresponding eigenvalues, $a_p(f)$, are algebraic integers and we have $c_p(f) = a_p(f)\, c_1(f)$.

Let λ be a place of the algebraic closure $\bar{\mathbb{Q}}$ in $\mathbb{C}$ above a rational prime ℓ and $\bar{\mathbb{Q}}_\lambda$ denote the algebraic closure of $\mathbb{Q}_\ell$ considered as a $\bar{\mathbb{Q}}$-algebra via λ. It is known that if $f \in S_k(N)$, there is a unique continuous irreducible representation

$$\rho_{f,\lambda} : \operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\bar{\mathbb{Q}}_\lambda) \tag{3.3}$$

such that for any prime $p \nmid N\ell$, $\rho_{f,\lambda}$ is unramified at p and $\operatorname{tr} \rho_{f,\lambda}(\mathrm{Frob}_p) = a_p(f)$. The existence of $\rho_{f,\lambda}$ is due to G. Shimura (1930-) if $k = 2$ [Sh], to P. Deligne (1944-

) if $k > 2$ [D] and to P. Deligne and J.-P. Serre (1926-) if $k = 1$ [DS]. Its irreducibility is due to Ribet if $k > 1$ [R], and to Deligne and Serre if $k = 1$ [DS]. Moreover $\rho_{f,\lambda}$ is odd and potentially semi-stable at ℓ in the sense of Fontaine. We may choose a conjugate of $\rho_{f,\lambda}$ which is valued in $GL_2(\mathcal{O}_{\bar{\mathbb{Q}}_\lambda})$, and reducing modulo the maximal ideal and semi-simplifying yields a continuous representation

$$\bar{\rho}_{f,\lambda} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\bar{\mathbb{F}}_\ell), \tag{3.4}$$

which, up to isomorphism, does not depend on the choice of conjugate of $\rho_{f,\lambda}$.

Definition 2. Let $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\bar{\mathbb{Q}}_\ell)$ be a continuous representation which is unramified outside finitely many primes and for which the restriction of ρ to a decomposition group at ℓ is potentially semi-stable in the sense of Fontaine. We call ρ *modular* if ρ is isomorphic to $\rho_{f,\lambda}$ for some eigenform f and some $\lambda|\ell$.

Definition 3. An elliptic curve E defined over $\mathbb{Q}$ is said to be *modular* if there exists a surjective holomorphic map $\varphi : X_0(N) \longrightarrow E(\mathbb{C})$ for some positive integer N.

Recently C. Breuil, B. Conrad, F. Diamond and R. Taylor [BCDT] proved that the Taniyama-Shimura conjecture is true.

Theorem E ([BCDT], 2001). An elliptic curve defined over $\mathbb{Q}$ is modular.

Let E be an elliptic curve defined over $\mathbb{Q}$. For a positive integer $n \in \mathbb{Z}^+$, we define the isogeny $[n] : E(\mathbb{C}) \longrightarrow E(\mathbb{C})$ by

$$[n]P := nP = P + \cdots + P \ (n \text{ times}), \quad P \in E(\mathbb{C}). \tag{3.5}$$

For a negative integer n, we define the isogeny $[n] : E(\mathbb{C}) \longrightarrow E(\mathbb{C})$ by $[n]P := -[-n]P$, $P \in E(\mathbb{C})$, where $-[-n]P$ denotes the inverse of the element $[-n]P$. And $[0] : E(\mathbb{C}) \longrightarrow E(\mathbb{C})$ denotes the zero map. For an integer $n \in \mathbb{Z}$, $[n]$ is called the multiplication-by-n homomorphism. The kernel $E[n]$ of the isogeny $[n]$ is isomorphic to $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$. Let

$$\mathrm{End}(E) = \{\varphi : E(\mathbb{C}) \longrightarrow E(\mathbb{C}), \text{ an isogeny}\}$$

be the endomorphism group of E. An elliptic curve E over $\mathbb{Q}$ is said to have *complex multiplication* (or CM for short) if

$$\mathrm{End}(E) \not\subseteq \mathbb{Z} \cong \{[n] \mid n \in \mathbb{Z}\},$$

that is, there is a nontrivial isogeny $\varphi : E(\mathbb{C}) \longrightarrow E(\mathbb{C})$ such that $\varphi \neq [n]$ for all integers $n \in \mathbb{Z}$. Such an elliptic curve is called a CM *curve*. For most of elliptic curves E over $\mathbb{Q}$, we have $\mathrm{End}(E) \cong \mathbb{Z}$.

4 The L-Series of an Elliptic Curve

Let E be an elliptic curve over $\mathbb{Q}$. The L-series $L(E, s)$ of E is defined as the product of the local L-factors:

$$(4.1) \qquad L(E,s) = \prod_{p \mid \Delta_E} (1 - a_p p^{-s})^{-1} \cdot \prod_{p \nmid \Delta_E} (1 - a_p p^{-s} + p^{1-2s})^{-1},$$

where Δ_E is the discriminant of E, p is a prime, and if $p \nmid \Delta_E$,

$$a_p := p + 1 - |\bar{E}(\mathbb{F}_p)|,$$

and if $p|\Delta_E$, we set $a_p := 0,\ 1,\ -1$ if the reduced curve $\bar{E}/\mathbb{F}_p$ has a cusp at p, a split node at p, and a nonsplit node at p respectively. Then $L(E,s)$ converges absolutely for $\operatorname{Re} s > \frac{3}{2}$ from the classical result that $|a_p| < 2\sqrt{p}$ for each prime p due to H. Hasse (1898-1971) and is given by an absolutely convergent Dirichlet series. We remark that $x^2 - a_p x + p$ is the characteristic polynomial of the Frobenius map acting on $\bar{E}(\mathbb{F}_p)$ by $(x,y) \mapsto (x^p, y^p)$.

Conjecture F. Let $N(E)$ be the conductor of an elliptic curve E over $\mathbb{Q}$ ([S], p. 361). We set

$$\Lambda(E,s) := N(E)^{s/2}\,(2\pi)^{-s}\,\Gamma(s)\,L(E,s), \quad \operatorname{Re} s > \frac{3}{2}.$$

Then $\Lambda(E,s)$ has an analytic continuation to the whole complex plane and satisfies the functional equation

$$\Lambda(E,s) = \epsilon\,\Lambda(E, 2-s), \quad \epsilon = \pm 1.$$

The above conjecture is now true because the Taniyama-Shimura conjecture is true (cf. Theorem E). We have some knowledge about analytic properties of E by investigating the L-series $L(E,s)$. The order of $L(E,s)$ at $s=1$ is called the *analytic rank* of E.

Now we explain the connection between the modularity of an elliptic curve E, the modularity of the Galois representation and the L-series of E. For a prime ℓ, we let $\rho_{E,\ell}$ (resp. $\bar{\rho}_{E,\ell}$) denote the representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the ℓ-adic Tate module (resp. the ℓ-torsion) of $E(\bar{\mathbb{Q}})$. Let $N(E)$ be the conductor of E. Then it is known that the following conditions are equivalent:

(1) The L-function $L(E,s)$ of E equals the L-function $L(f,s)$ for some eigenform f.

(2) The L-function $L(E,s)$ of E equals the L-function $L(f,s)$ for some eigenform f of weight 2 and level $N(E)$.

(3) For some prime ℓ, the representation $\rho_{E,\ell}$ is modular.

(4) For all primes ℓ, the representation $\rho_{E,\ell}$ is modular.

(5) There is a non-constant holomorphic map $X_0(N) \longrightarrow E(\mathbb{C})$ for some positive integer N.

(6) There is a non-constant morphism $X_0(N(E)) \longrightarrow E$ which is defined over $\mathbb{Q}$.

(7) E admits a hyperbolic uniformization of arithmetic type (cf. [Ma2] and [Y1]).

5 The Birch-Swinnerton-Dyer conjecture

Now we state the BSD conjecture.

The BSD Conjecture. Let E be an elliptic curve over $\mathbb{Q}$. Then the algebraic rank of E equals the analytic rank of E.

I will describe some historical backgrounds about the BSD conjecture. Around 1960, Birch (1931-) and Swinnerton-Dyer (1927-) formulated a conjecture which determines the algebraic rank r of an elliptic curve E over $\mathbb{Q}$. The idea is that an elliptic curve with a large value of r has a large number of rational points and should therefore have a relatively large number of solutions modulo a prime p on the average as p varies. For a prime p, we let $N(p)$ be the number of pairs of integers $x, y \pmod p$ satisfying (2.1) as a congruence (mod p). Then the BSD conjecture in its crudest form says that we should have an asymptotic formula

$$\prod_{p<x} \frac{N(p)+1}{p} \sim C\ (\log p)^r \quad \text{as } x \longrightarrow \infty \tag{5.1}$$

for some constant $C > 0$. If the L-series $L(E,s)$ has an analytic continuation to the whole complex plane (this fact is conjectured; cf. Conjecture F), then $L(E,s)$ has a Taylor expansion

$$L(E,s) = c_0(s-1)^m + c_1(s-1)^{m+1} + \cdots$$

at $s = 1$ for some non-negative integer $m \geq 0$ and constant $c_0 \neq 0$. The BSD conjecture says that the integer m, in other words, the analytic rank of E, should equal the algebraic rank r of E and furthermore the constant c_0 should be given by

$$c_0 = \lim_{s \to 1} \frac{L(E,s)}{(s-1)^m} = \alpha \cdot R \cdot |E_{\mathrm{tor}}(\mathbb{Q})|^{-1} \cdot \Omega \cdot S, \tag{5.2}$$

where $\alpha > 0$ is a certain constant, R is the elliptic regulator of E, $|E_{\mathrm{tor}}(\mathbb{Q})|$ denotes the order of the torsion subgroup $E_{\mathrm{tor}}(\mathbb{Q})$ of $E(\mathbb{Q})$, Ω is a simple rational multiple (depending on the bad primes) of the elliptic integral

$$\int_\gamma^\infty \frac{dx}{\sqrt{f(x)}} \qquad (\gamma = \text{the largest root of } f(x) = 0)$$

and S is an integer square which is supposed to be the order of the Tate-Shafarevich group Ш(E) of E.

The Tate-Shafarevich group $\text{III}(E)$ of E is a very intersting subject to be investigated in the future. Unfortunately $\text{III}(E)$ is still not known to be finite. So far an elliptic curve whose Tate-Shafarevich group is infinite has not been discovered. So many mathematicians propose the following.

Conjecture G. The Tate-Shafarevich group $\text{III}(E)$ of E is finite.

There are some evidences supporting the BSD conjecture. I will list these evidences chronologically.

Result 1 (Coates-Wiles [CW], 1977). Let E be a CM curve over $\mathbb{Q}$. Suppose that the analytic rank of E is zero. Then the algebraic rank of E is zero.

Result 2 (Rubin [R], 1981). Let E be a CM curve over $\mathbb{Q}$. Assume that the analytic rank of E is zero. Then the Tate-Shafarevich group $\text{III}(E)$ of E is finite.

Result 3 (Gross-Zagier [GZ], 1986 ; [BCDT], 2001). Let E be an elliptic curve over $\mathbb{Q}$. Assume that the analytic rank of E is equal to one and $\epsilon = -1$ (cf. Conjecture F). Then the algebraic rank of E is equal to or bigger than one.

Result 4 (Gross-Zagier [GZ], 1986). There exists an elliptic curve E over $\mathbb{Q}$ such that $\text{rank}\, E(\mathbb{Q}) = \text{ord}_{s=1} L(E, s) = 3$. For instance, the elliptic curve $\tilde{E}$ given by

$$\tilde{E} \; : \quad -139\, y^2 = x^3 + 10\, x^2 - 20\, x + 8$$

satisfies the above property.

Result 5 (Kolyvagin [K], 1990 : Gross-Zagier [GZ], 1986 : Bump-Friedberg-Hoffstein [BFH], 1990 : Murty-Murty [MM], 1990 : [BCDT], 2001). Let E be an elliptic curve over $\mathbb{Q}$. Assume that the analytic rank of E is 1 and $\epsilon = -1$. Then algebraic rank of E is equal to 1.

Result 6 (Kolyvagin [K], 1990 : Gross-Zagier [GZ], 1986 : Bump-Friedberg-Hoffstein [BFH], 1990 : Murty-Murty [MM], 1990 : [BCDT], 2001). Let E be an elliptic curve over $\mathbb{Q}$. Assume that the analytic rank of E is zero and $\epsilon = 1$. Then algebraic rank of E is equal to zero.

Cassels proved the fact that if an elliptic curve over $\mathbb{Q}$ is isogeneous to another elliptic curve E' over $\mathbb{Q}$, then the BSD conjecture holds for E if and only if th e BSD conjecture holds for E'.

6 Jacobi Forms and Heegner Points

In this section, I shall describe the result of Gross-Kohnen-Zagier [GKZ] roughly.

First we begin with giving the definition of Jacobi forms. By definition a Jacobi form of weight k and index m is a holomorphic complex valued function $\phi(z,w)$ ($z \in \mathbb{H}$, $z \in \mathbb{C}$) satisfying the transformation formula

$$\phi\left(\frac{az+b}{cz+d}, \frac{w+\lambda z+\mu}{cz+d}\right) = e^{-2\pi i\left\{cm(w+\lambda z+\mu)^2(cz+d)^{-1} - m(\lambda^2 z + 2\lambda w)\right\}} \times (cz+d)^k\, \phi(z,w) \tag{6.1}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ and $(\lambda,\mu) \in \mathbb{Z}^2$, and having a Fourier expansion of the form

$$\phi(z,w) = \sum_{\substack{n,r \in \mathbb{Z}^2 \\ r^2 \le 4mn}} c(n,r)\, e^{2\pi i(nz+rw)}. \tag{6.2}$$

We remark that the Fourier coefficients $c(n,r)$ depend only on the discrimnant $D = r^2 - 4mn$ and the residue $r \pmod{2m}$. From now on, we put $\Gamma_1 := SL(2,\mathbb{Z})$. We denote by $J_{k,m}(\Gamma_1)$ the space of all Jacobi forms of weight k and index m. It is known that one has the following isomorphisms

$$(6.3) \qquad [\Gamma_2, k]^M \cong J_{k,1}(\Gamma_1) \cong M^+_{k-\frac{1}{2}}(\Gamma_0(4)) \cong [\Gamma_1, 2k-2],$$

where Γ_2 denotes the Siegel modular group of degree 2, $[\Gamma_2, k]^M$ denotes the Maass space introduced by H. Maass (1911-1993) (cf. [M1-3]), $M^+_{k-\frac{1}{2}}(\Gamma_0(4))$ denotes the Kohnen space introduced by W. Kohnen [Koh] and $[\Gamma_1, 2k-2]$ denotes the space of modular forms of weight $2k-2$ with respect to Γ_1. We refer to [Y1] and [Y3], pp. 65-70 for a brief detail. The above isomorphisms are compatible with the action of the Hecke operators. Moreover, according to the work of Skoruppa and Zagier [SZ], there is a Hecke-equivariant correspondence between Jacobi forms of weight k and index m, and certain usual modular forms of weight $2k-2$ on $\Gamma_0(N)$.

Now we give the definition of Heegner points of an elliptic curve E over $\mathbb{Q}$. By [BCDT], E is modular and hence one has a surjective holomorphic map $\phi_E : X_0(N) \longrightarrow E(\mathbb{C})$. Let K be an imaginary quadratic field of discriminant D such that every prime divisor p of N is split in K. Then it is easy to see that $(D,N) = 1$ and D is congruent to a square r^2 modulo $4N$. Let Θ be the set of all $z \in \mathbb{H}$ satisfying the following conditions

$$az^2 + bz + c = 0, \quad a, b, c \in \mathbb{Z},\ N|a,$$

$$b \equiv r \pmod{2N}, \qquad D = b^2 - 4ac.$$

Then Θ is invariant under the action of $\Gamma_0(N)$ and Θ has only a h_K $\Gamma_0(N)$-orbits, where h_K is the class number of K. Let $z_1, \cdots, z_{h_K}$ be the representatives for these $\Gamma_0(N)$-orbits. Then $\phi_E(z_1), \cdots, \phi_E(z_{h_K})$ are defined over the Hilbert class field $H(K)$ of K, i.e., the maximal everywhere unramified extension of K. We define the Heegner point $P_{D,r}$ of E by

$$P_{D,r} = \sum_{i=1}^{h_K} \phi_E(z_i). \tag{6.4}$$

We observe that $\epsilon = 1$, then $P_{D,r} \in E(\mathbb{Q})$.

Let $E^{(D)}$ be the elliptic curve (twisted from E) given by

$$E^{(D)} \; : \; Dy^2 = f(x). \tag{6.5}$$

Then one knows that the L-series of E over K is equal to $L(E,s)\,L(E^{(D)},s)$ and that $L(E^{(D)},s)$ is the twist of $L(E,s)$ by the quadratic character of $K/\mathbb{Q}$.

Theorem H (Gross-Zagier [GZ], 1986; [BCDT], 2001). Let E be an elliptic curve of conductor N such that $\epsilon = -1$. Assume that D is odd. Then

$$L'(E,1)\,L(E^{(D)},1) = c_E\, u^{-2}\, |D|^{-\frac{1}{2}}\, \hat{h}(P_{D,r}), \tag{6.6}$$

where c_E is a positive constant not depending on D and r, u is a half of the number of units of K and $\hat{h}$ denotes the canonical height of E.

Since E is modular by [BCDT], there is a cusp form of weight 2 with respect to $\Gamma_0(N)$ such that $L(f,s) = L(E,s)$. Let $\phi(z,w)$ be the Jacobi form of weight 2 and index N which corresponds to f via the Skoruppa-Zagier correspondence. Then $\phi(z,w)$ has a Fourier series of the form (6.2).

B. Gross, W. Kohnen and D. Zagier obtained the following result.

Theorem I (Gross-Kohnen-Zagier, [GKZ] ; BCDT], 2001). Let E be a modular elliptic curve with conductor N and suppose that $\epsilon = -1$, $r = 1$. Suppose that $(D_1, D_2) = 1$ and $D_i \equiv r_i^2 \pmod{4N}$ $(i = 1, 2)$. Then

$$L'(E,1)\, c((r_1^2 - D_1)/(4N), r_1)\, c((r_2^2 - D_2)/(4N), r_2) = c_E' < P_{D_1,r_1}, P_{D_2,r_2} >,$$

where c_E' is a positive constant not depending on D_1, r_1 and D_2, r_2 and $< , >$ is the height pairing induced from the Néron-Tate height function $\hat{h}$, that is, $\hat{h}(P_{D,r}) = < P_{D,r}, P_{D,r} >$.

We see from the above theorem that the value of $< P_{D_1,r_1}, P_{D_2,r_2} >$ of two distinct Heegner points is related to the product of the Fourier coefficients $c((r_1^2 - D_1)/(4N), r_1)\, c((r_2^2 - D_2)/(4N), r_2)$ of the Jacobi forms of weight 2 and index N corresponded to the eigenform f of weight 2 associated to an elliptic curve E. We refer to [Y4] and [Z] for more details.

Corollary. There is a point $P_0 \in E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{R}$ such that

$$P_{D,r} = c((r^2 - D)/(4N), r) P_0$$

for all D and r $(D \equiv r^2 \pmod{4N})$ with $(D, 2N) = 1$.

The corollary is obtained by combining Theorem H and Theorem I with the Cauchy-Schwarz inequality in the case of equality.

Remark 4. R. Borcherds [B] generalized the Gross-Kohnen-Zagier theorem to certain more general quotients of Hermitian symmetric spaces of high dimension, namely to quotients of the space associated to an orthogonal group of signature $(2, b)$ by the unit group of a lattice. Indeed he relates the Heegner divisors on the given quotient space to the Fourier coefficients of vector-valued holomorphic modular forms of weight $1 + \frac{b}{2}$.

REFERENCES

[BSD1] B. Birch and H.P.F. Swinnerton-Dyer, *Notes on elliptic curves (I)*, J. Reine Angew. Math. **212** (1963), 7-25.

[BSD2] B. Birch and H.P.F. Swinnerton-Dyer, *Notes on elliptic curves (II)*, J. Reine Angew. Math. **218** (1965), 79-108.

[B] R. Borcherds, *The Gross-Kohnen-Zagier theorem in higher dimensions*, Duke Math. J. **97, no. 2** (1999), 219-233.

[BCDT] C. Breuil, B. Conrad, F. Diamond and R. Taylor, *On the modularity of elliptic curves over* $\mathbb{Q}$, Journal of AMS **109** (2001), 843-939.

[BFH] B. Bump, S. Friedberg and J. Hoffstein, *Nonvanishing theorems for L-functions of modular forms and their derivatives*, Invent. Math. **102** (1990), 543-618.

[CW] J. Coates and A. Wiles, *On the Birch-Swinnerton-Dyer conjecture*, Invent. Math. **39** (1977), 223-252.

[EZ] M. Eichler and D. Zagier, *The theory of Jacobi forms*, vol. 55, Birkhäuser, 1985.

[GZ] B. Gross and D. Zagier, *Heegner points and derivatives of L-series*, Invent. Math. **84** (1986), 225-320.

[GKZ] B. Gross, W. Kohnen and D. Zagier, *Heegner points and derivatives of L-series. II*, Math. Ann. **278** (1987), 497-562.

[Koh] W. Kohnen, *Modular forms of half integral weight on* $\Gamma_0(4)$, Math. Ann. **248** (1980), 249-266.

[K1] V. A. Kolyvagin, *Finiteness of* $E(\mathbb{Q})$ *and* $III(E, \mathbb{Q})$ *for a subclass of Weil curves (Russian)*, Izv. Akad. Nauk SSSR Ser. Mat. **52** (1988), 522-54; English translation in Math. USSR-IZv. **32** (1980), 523-541.

[K2] ______, *Euler systems, the Grothendieck Festschrift (vol. II), edited by P. Cartier and et al*, Birkhäuser **87** (1990), 435-483.

[M1] H. Maass, *Über eine Spezialschar von Modulformen zweiten Grades I*, Invent. Math. **52** (1979), 95-104.

[M2] H. Maass, *Über eine Spezialschar von Modulformen zweiten Grades II*, Invent. Math. **53** (1979), 249-253.

[M3] H. Maass, *Über eine Spezialschar von Modulformen zweiten Grades III*, Invent. Math. **53** (1979), 255-265.

[Ma1] B. Mazur, *Modular curves and the Eisenstein series*, Publ. IHES **47** (1977), 33-186.

[Ma2] ______, *Number Theory as Gadfly*, Amer. Math. Monthly **98** (1991), 593-610.

[MM] M.R. Murty and V.K. Murty, *Mean values of derivatives of modular L-series*, Ann. Math. **133** (1991), 447-475.

[R] K. Rubin, *Elliptic curves with complex multiplication and the BSD conjecture*, Invent. Math. **64** (1981), 455-470.

[S] J.H. Silvermann, *The Arithmetic of Elliptic Curves*, vol. Graduate Text in Math. 106, Springer-Verlag, 1986.

[SZ] N.-P. Skoruppa and D. Zagier, *Jacobi forms and a certain space of modular forms*, Invent. Math. **94** (1988), 113-146.

[Y1] J.-H. Yang, *Remarks on Jacobi forms of higher degree*, Proceedings of the 1993 Conference on Automorphic Forms and Related Topics, edited by J.-W. Son and J.-H. Yang, Pyungsan Institute for Mathematical Sciences **1** (1993), 33-58.

[Y2] ———, *Note on Taniyama-Shimura-Weil conjecture*, Proceedings of the 1994 Conference on Number Theory and Related Topics, edited by J.-W. Son and J.-H. Yang, Pyungsan Institute for Mathematical Sciences **2** (1995), 29-46.

[Y3] ———, *Kac-Moody algebras, the Monstrous Moonshine, Jacobi Forms and Infinite Products*, Proceedings of the 1995 Symposium on Number Theory, Geometry and Related Topics, edited by J.-W. Son and J.-H. Yang, Pyungsan Institute for Mathematical Sciences **3** (1996), 13-82.

[Y4] ———, *Past twenty years of the theory of elliptic curves (Korean)*, Comm. Korean Math. Soc. **14** (1999), 449-477.

[Z] D. Zagier, *L-series of Elliptic Curves, the BSD Conjecture, and the Class Number Problem of Gauss*, Notices of AMS **31** (1984), 739-743.

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[H] Theory of Siegel Modular Forms

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Geometric Theory of Siegel Modular Forms

JAE-HYUN YANG

Abstract

The purpose of this short article is to survey the geometry of the Siegel modular variety.

1 Singular Modular Forms

Let H_n be the Siegel upper half space of degree n and Γ_n the Siegel modular group of degree n. Let ρ be a rational representation of $GL(n,\mathbb{C})$ on a finite dimensional vector space V_ρ. A V_ρ–vector valued holomorphic function $f : H_n \longrightarrow V_\rho$ is called a *modular form of type* ρ if for any $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n = Sp(n,\mathbb{Z})$,

$$f(M\langle Z\rangle) = \rho(CZ+D)\,f(Z), \quad Z \in H_n.$$

We denote by $[\Gamma_n, \rho]$ the vector space of modular forms on H_n of type ρ.

A vector valued modular form $f(Z)$ of type ρ has a Fourier series

$$f(Z) = \sum_{H\geq 0} a(H)\, e^{\pi i \sigma(HZ)}, \quad Z \in H_n,$$

where H runs over the set of all nonnegative integral matrices of degree n. If its coefficients satisfies the condition (*),

$$a(H) \neq 0 \Longrightarrow det(H) = 0 \quad (*)$$

then f is said to be *singular*.

In this section, we let $\rho : GL(n, \mathbb{C}) \longrightarrow GL(V_\rho)$ be a rational representation of $GL(n, C)$ on a finite dimensional vector space V_ρ.

Definition 1.1 A harmonic form P on $M_{m,n}(\mathbb{C})$ with respect to ρ is a polynomial

$$P : M_{m,n}(\mathbb{C}) \longrightarrow V_\rho$$

satisfying the following conditions (1), (2) :

(1) $P(XA) = \rho({}^tA)P(X)$ for all $A \in GL(n, \mathbb{C})$, $X \in M_{m,n}(\mathbb{C})$,

(2) $\triangle P = 0$, where $\triangle$ denotes the Laplacian on $M_{m,n}(\mathbb{C})$.

We denote by $H(m, n, \rho)$ the set of all harmonic forms on $M_{m,n}(\mathbb{C})$ with respect to a representation ρ.

Let $S = {}^tS > 0$ be a positive definite integral even unimodular matrix of degree m. To $P \in H(m, n, \rho)$ and S we associate the theta series

$$\vartheta_{S,P}(Z) = \sum_G P(S^{\frac{1}{2}}G)\, e^{\pi i\sigma(S[G]Z)}, \quad Z \in H_n,$$

where G runs through all $m \times n$ integral matrices. Then it is known that $\vartheta_{S,P}(Z)$ is a modular form of type $\rho \otimes det^{\frac{m}{2}}$. If $n > m$, then it is easy to see that $\vartheta_{S,P}(Z)$ is singular.

In his paper([Fr2]), Freitag proved that every singular modular form can be written in a finite linear combination of the above mentioned theta series and suggested the problem to characterize singular modular forms. His student, Weissauer, gave the following criterion.

Theorem 1.2([**W1**]) Let ρ be the irreducible finite dimensional representation of $GL(n, \mathbb{C})$ with highest weight $(\lambda_1, \cdots, \lambda_n)$. Let $f \in [\Gamma_n, \rho]$. Then the following are equivalent.

(1) f is singular.

(2) $2\lambda_n < n$.

Now we describe how the concept of singular modular forms is related to the geometry of the Siegel modular variety. Let X be the Satake compactification of the quotient space $\Gamma_n \backslash H_n$. Then $\Gamma_n \backslash H_n$ is embedded in X as a quasiprojective algebraic subvariety of codimension n. Let X_s be the smooth part of $\Gamma_n \backslash H_n$ and $\tilde{X}$ the desingularization of X. Without loss of generality, we assume $X_s \subset \tilde{X}$. Let $\Omega^p(\tilde{X})$ (resp. $\Omega^p(X_s)$) be the space of holomorphic p-form on $\tilde{X}$(resp. X_s). In [Fr–Po], they showed that if $n > 1$, then the restriction map

$$\Omega^p(\tilde{X}) \longrightarrow \Omega^p(X_s)$$

is an isomorphism for $p < dim(\tilde{X}) = \frac{n(n+1)}{2}$. Since the singular part of $\Gamma_n \backslash H_n$ is of at least codimension 2 for $n > 1$, we have an isomorphism

$$\Omega^p(\tilde{X}) \xrightarrow{\sim} \Omega^p(H_n)^{\Gamma_n}.$$

Here $\Omega^p(H_n)^{\Gamma_n}$ denotes the space of Γ_n– invariant holomorphic p-forms on H_n. Let $\rho^{[1]} = Symm^2(\mathbb{C}^n)$ be the symmetric power of the canonical representation of $GL(n, \mathbb{C})$ on $\mathbb{C}^n$. Then we obtain an isomorphism

$$\Omega^p(H_n)^{\Gamma_n} \xrightarrow{\sim} [\Gamma_n, \wedge^p \rho^{[1]}].$$

Definition 1.3 Let ρ be an irreducible representation of $GL(n,\mathbb{C})$ with highest weight $(\lambda_1,\cdots,\lambda_n)$. We call the number of $j\,(1\le j\le n\,)$ such that $\lambda_j=\lambda_n$ the *corank* of ρ, denoted by $corank(\rho)$.

Theorem 1.4([W1]) Let ρ_α be the irreducible representation of $GL(n,C)$ with highest weight

$$(n+1,\cdots,n+1,n-\alpha,\cdots,n-\alpha)$$

such that $corank(\rho_\alpha)=\alpha$ for $1\le\alpha\le n$. If $\alpha=-1$, we let $\rho_\alpha=(n+1,\cdots,n+1)$. Then

$$\Omega^p(H_n)^{\Gamma_n}=\begin{cases}[\Gamma_n,\rho_\alpha] & \text{if } p=\frac{(n(n+1)}{2}-\frac{\alpha(\alpha+1)}{2}\\ 0 & \text{otherwise}\end{cases}$$

Remark 1.5 If $2\alpha>n$, any $f\in[\Gamma_n,\rho_\alpha]$ is singular. Thus if $p<\frac{n(3n+2)}{8}$, then any Γ_n–invariant holomorphic p–form on H_n can be expressed in terms of vector valued theta series with harmonic coefficients. It can be shown with a suitable modification that the just mentioned statement holds for a sufficiently small congruence subgroup of Γ_n.

Thus the natural question is how to determine the Γ_n– invariant holomorphic p–forms on H_n for the nonsingular range $\frac{n(3n+2)}{8}\le p\le\frac{n(n+1)}{2}$. Weissauer([W3]) answered the above question for $n=2$. For $n>2$, the question is still open. By Harish–Chandra, the vector space of vector valued modular forms of type ρ is finite dimensional. The computation or estimate of the dimension of $\Omega^p(H_n)^{\Gamma_n}$ is also interesting because its dimension is finite even though the quotient space is noncompat. In my opinion, there will be possibly the connection the dimension of the above modular forms and the arithmetic or geometric invariants of the Siegel modular variety.

2 Differential Forms on $\Gamma\backslash H_n$

Let Γ be a congruence subgroup of Γ_n, the Siegel modular group of degree n. Since $\Gamma\backslash H_n$ is noncompact and highly singular, it is difficult to compute the cohomology groups $H^*(\Gamma\backslash H_n)$. So in order to study their structure, it is natural to ask if they have pure Hodge structures or mixed Hodge structures. As mentioned in section 1, holomorphic forms on $\Gamma\backslash H_n$ are corresponded to modular forms on H_n associated to some representation of the general group $GL(n,\mathbb{C})$. By Freitag([Fr2]), holomorphic forms on $\Gamma\backslash H_n$ in the singular range was characterized in terms of theta series with harmonic coefficients. The natural question is to characterize or determine the holomorphic forms on $\Gamma\backslash H_n$ in the nonsingular range. In his paper([W2]), Weissauer gives an answer for the question in case $n = 2$.

Now let Γ be a congruence subgroup of Γ_2, the Siegel modular group of degree 2. By Theorem 1.4, holomorphic forms in $\Omega^2(H_2)^\Gamma$ are corresponded to modular forms of type $(3,1)$ We note that these holomorphic 2–form are contained in the nonsingular range. And if these modular forms are not cusp forms, they are mapped under the Siegel Φ–operator to cusp forms of weight 3 with respect to some congruence subgroup (dependent on Γ) of the elliptic modular group. Since there are finitely many cusps, it is easy to deal with these modular forms in the adelic version. Observing these facts, he showed that any 2–holomorphic forms on $\Gamma\backslash H_2$ can be expressed in terms of theta series with harmonic coefficients associated to binary positive definite quadratic forms. Moreover he proves that $H^2(\Gamma\backslash H_2,\mathbb{C})$ has a pure Hodge structure and that the Tate conjecture holds for a suitable smooth compactification of $\Gamma\backslash H_2$. For the proofs of the above statements, we refer to [W3].

Remark For $n > 3$, the characterization of holomorphic forms on $\Gamma\backslash H_n$ in the nonsingular range was not done so far.

3 Subvarieties of the Siegel modular variety

Let $A_g = \Gamma_g\backslash H_g$ be the Siegel modular variety of degree g, i.e., the moduli space of principally polarized abelian varieties of dimension g. It was proved that A_g is of general type for $g \geq 7$. At first Freitag([Fr1]) proved this if $24 \mid g$. Tai([T]) proved this for $g \geq 9$. Mumford([M]) proved this for $g \geq 7$. On the other hand, A_g is known to be unirational for $g \leq 5$: Donagi([D]) for $g = 5$, Clemens([C]) for $g = 4$, classical for $g \leq 3$. For $g = 3$, using the moduli theory of curves, Riemann([R]), Weber([We]), Frobenius([Fro]) showed that $A_3(2) = \Gamma_3(2)\backslash H_3$ is a rational variety and moreover gave 6 generators of the modular function field $K(\Gamma_3(2))$ written explicitly in terms of derivatives of odd theta functions at the origin. So A_3 is a unirational variety with a Galois covering of a rational variety of degree $[\Gamma_3 : \Gamma_3(2)] = 1,451,520$. Here $\Gamma_3(2)$ denotes the principal congruence subgroup Γ_3 of level 2. Furthermore it was shown that A_3 is stably rational(see [Ko-Sch] and [Bo-Ka]). The remaining unsolved problems are summarized as follows.

Problem 1 Is A_3 rational?

Problem 2 Are A_4, A_5 stably rational or rational?

Problem 3 Discuss the (uni)rationality of A_6.

We already mentioned that A_6 is of general type if $g \geq 7$. It is natural to ask if the subvarieties of $A_g (g \geq 7)$ are of general type, in particular the subvarieties of A_g of codimension one. Freitag([Fr4]) showed that there exists a certain bound g_0 such that for $g \geq g_0$, each irreducible subvariety of A_g of codimension one is of general type. Weissauer([W2]) proved that every irreducible divisor of A_g is of general type for $g \geq 10$. Moreover he proved that every subvariety of codimension $\leq g-13$ in A_g is of general type for $g \geq 13$. We observe that the smallest known codimension for which there exist subvarieties of A_g for large g which are not of general type is $g-1$. $A_1 \times A_{g-1}$ is a subvariety of A_g of codimension $g-1$ which is not of general type.

Remark Let $\mathcal{M}_g$ be the coarse moduli space of curves of genus g over $\mathbb{C}$. Then $\mathcal{M}_g$ is an analytic subvariety of A_g of dimension $3g-3$. It is known that $\mathcal{M}_g$ is unirational for $g \leq 10$. So the Kodaira dimension of $\mathcal{M}_g$ is ∞ for $g \leq 10$. In [H-M], they prove that $\mathcal{M}_g$ is of general type for odd g with $g \geq 25 \,(\kappa(\mathcal{M}_g) = dim\, \mathcal{M}_g)$, and $\kappa(\mathcal{M}_{23}) \geq 0$. Here $\kappa(\mathcal{M}_g)$ denotes the Kodaira dimension of $\mathcal{M}_g$.

4 Hecke Algebras

Let $\mathbf{G} = Sp(2g)$ be the algebraic group defined over $\mathbb{Q}$, $\mathbf{P} \subset \mathbf{G}$ the parabolic subgroup of upper triangular matrices, $\mathbf{U} \subset \mathbf{P}$ its unipotent radical, $\mathbf{M} \subset \mathbf{P}$ the standard Levi factor and $\mathbf{T} \subset \mathbf{M} \subset \mathbf{G}$ the maximal torus consisting of diagonal matrices. We denote by $\mathbf{G}_p$, $\mathbf{P}_p$, $\mathbf{M}_p$, $\mathbf{U}_p$, $\mathbf{T}_p$ $\mathbb{Q}_p$-valued points of the above groups respectively. Let $\mathbf{K} \subset \mathbf{G}_p$ be the maximal compact subgroup of $\mathbb{Z}_p$-valued points of $\mathbf{G}$. We normalize Haar measures on all these groups by requiring that their intersection with $\mathbf{K}$ has measure one.

Let $\mathcal{H}_p(\mathbf{G})$(resp. $\mathcal{H}_p(\mathbf{M})$) be the Hecke algebra associated with $\mathbf{G}$(resp. $\mathbf{M}$). Then we can define an injection $S : \mathcal{H}_p(\mathbf{G}) \hookrightarrow \mathcal{H}_p(\mathbf{M})$ by

$$S(\phi)(m) = \int_{\mathbf{U}_p} \phi(mu)\, du, \quad \phi \in \mathcal{H}_p(\mathbf{G}), \quad m \in \mathbf{M}.$$

$\mathcal{H}_p(\mathbf{G})$ acts on the l–adic etale cohomology of the Siegel modular variety A_g. Faltings and Chai proved that this action can be extented to the Hecke algebra $\mathcal{H}_p(\mathbf{G})$ and that there exists a well–defined element $\mathsf{F}_p \in \mathcal{H}_p(\mathbf{M})$ which is mapped to the Frobenius. In [F], Faltings states the following questions: Associated to a family of characters of the various $\mathcal{H}_p(\mathbf{G})$ for primes p, there is an eigenspace in the singular cohomology. He claims without proof that this eigenspace has 2^g dimension and the Galois representation on it is a spin–representation of $Sp(2g+1)$. Furthermore he says without proof that the polynomial equation for the Frobenius describing the relation between the Hecke operator and the Frobenius should be the characteristic equation on this space. He expects to prove these results by using the trace formula. I was recently told from Prof. Kazhdan that Zahrin and Drinfeld have some results related to the above statements.

5 Conclusion

I omitted the following important topics:

- *Compactification of A_g (see [F])*
- *the Ramanujan Conjecture (see [S], [H – P] and [F – K])*
- *Automorphic L – functions*

In the future I will discuss these topics. Finally I refer the reader to Oda's good survey paper([O]).

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References

[A] Andrianov, A.N., Quadratic Forms and Hecke Operators, Springer–Verlag(1987).

[Bo-Ka] Bogomolov and Katsylo, P.I., Rationality of some quotient varieties, Math. USSR Sbornik **54** (1986), 571-576.

[C] Clemens, H., Double Solids, Adv. Math. **47** (1983), 107-230.

[D] Donagi, R., The unirationality of A_5, Ann. Math. **119**(1984), 269-307.

[F] Faltings, G., Arithmetic Theory of Siegel Modular Forms, Springer Lecture Notes **1240**(1984), 101-108.

[F-K] Flicker, Y. and Kazhdan, D., Geometric Ramanujan Conjecture and Drinfeld Reciprocity Law, Academic Press(Number Theory, Trace Formula and Discrete Groups), 201-218(1989).

[Fr1] Freitag, E., Die Kodairadimension von Körpen automorpher Funktionen, J. reine angew. Math., **296**(1977).

[Fr2] Freitag, E., Thetareihnen mit harmonischen Koeffizienten zur Siegelschen Modulgruppe, Math. Ann. **254**(1980), 27-51.

[Fr3] Freitag, E., Die Wirkung von Heckeoperatoren auf Thetareihnen mit harmonischen Koeffizienten, Math. Ann. **258** (1982), 419-440.

[Fr4] Freitag, E., Holomorphic tensors on subvarieties of the Siegel modular variety, Birkhäuser, Prog. Math. Boston **46**(1984), 93-113.

[Fr-Po] Freitag, E. and Pommerening, K., J. Reine Angew. Math. **331**(1982), 207-220.

[Fro] Frobenius, G., Über die Beziehungen zwischenden 28 Doppeltangenten einer eben Curve vierte Ordnung, J. reine angew. Math. **99**(1886), 275-314.

[H-M] Harris, J. and Mumford, D., On the Kodaira dimension of the moduli space of curves, Inven. Math. **67** (1982), 23-88.

[H-P] Howe, R. and Piatetski-Shapiro, I.I., A counterexample to the generalized Ramanujan conjecture for (quasi-) split groups, Proc. Sympo. Pure Math. **33**(1979), Part1, 315-322.

[Ko-Sch] Kollar, J. and Schreyer, F.O., The moduli of curves is stably rational for $g \leq 6$, Duke Math. J. **51** (1984), 239-242.

[M] Mumford, D., On the Kodaira dimension of the Siegel Modular Variety, Springer Lecture Notes **997**(1982), 425-439.

[O] Oda, T., Hodge structures attached to geometric automorphic forms, Adv. Studies in Pure Math. **7**, Automorphic and Number Theory(1985), 223-276.

[R] Riemann, B., Zur Theorie der Abel'schen Funktionen für den $p = 3$, Math. Werke, Teubener, Leipzig (1876), 456-476.

[S] Satake, I., Spherical functions and Ramanujan conjecture, Proc. Sympo. Pure Math. **9**(1966), 258-264.

[T] Tai, Y., On the Kodaira dimension of the moduli space of abelian varieties, Invent. Math. **68**(1982), 425-439.

[We] Weber, H., Theorie der Abel'schen Funktionen von Geschlecht 3, Berlin(1876).

[W1] Weissauer, R., Vektorwertige Siegelsche Modulformen kleine Gewichtes, J. reine angew. Math. **343**(1983), 184-202.

[W2] Weissauer, R., Untervarietäten der Siegelschen Modulmannigfatigkeiten von allgemeinen Typ, Math. Ann. **275**(1986), 209-220.

[W3] Weissauer, R., Differentialformen zu Untergruppen der Siegelschen Modulgruppe zweiten Grades, J. reine angew. Math. **391**(1988), 100-156.

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THEORY OF THE SIEGEL MODULAR VARIETY

JAE-HYUN YANG

ABSTRACT. In this paper, we discuss the theory of the Siegel modular variety in the aspects of arithmetic and geometry. This article covers the theory of Siegel modular forms, the Hecke theory, a lifting of elliptic cusp forms, geometric properties of the Siegel modular variety, (hypothetical) motives attached to Siegel modular forms.

To the memory of my mother

Table of Contents

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1. Introduction

For a given fixed positive integer g, we let

$$\mathbb{H}_g = \{\, \Omega \in \mathbb{C}^{(g,g)} \mid \Omega = {}^t\Omega, \quad \operatorname{Im}\Omega > 0\,\}$$

be the Siegel upper half plane of degree g and let

$$Sp(g,\mathbb{R}) = \{M \in \mathbb{R}^{(2g,2g)} \mid {}^tMJ_gM = J_g\,\}$$

be the symplectic group of degree g, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l, ${}^tM$ denotes the transposed matrix of a matrix M and

$$J_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

Then $Sp(g,\mathbb{R})$ acts on $\mathbb{H}_g$ transitively by

$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{1.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R})$ and $\Omega \in \mathbb{H}_g$. Let

$$\Gamma_g = Sp(g,\mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R}) \;\Big|\; A, B, C, D \text{ integral} \right\}$$

be the Siegel modular group of degree g. This group acts on $\mathbb{H}_g$ properly discontinuously. C. L. Siegel investigated the geometry of $\mathbb{H}_g$ and automorphic forms on $\mathbb{H}_g$ systematically. Siegel [131] found a fundamental domain $\mathcal{F}_g$ for $\Gamma_g\backslash\mathbb{H}_g$ and described it explicitly. Moreover he calculated the volume of $\mathcal{F}_g$. We also refer to [65], [92], [131] for some details on $\mathcal{F}_g$. Siegel's fundamental domain is now called the Siegel modular variety and is usually denoted by $\mathcal{A}_g$. In fact, $\mathcal{A}_g$ is one of the important arithmetic varieties in the sense that it is regarded as the moduli of principally polarized abelian varieties of dimension g. Suggested by Siegel, I. Satake [117] found a canonical compactification, now called the Satake compactification of $\mathcal{A}_g$. Thereafter W. Baily [6] proved that the Satake compactification of $\mathcal{A}_g$ is a normal projective variety. This work was generalized to bounded symmetric domains by W. Baily and A. Borel [7] around the 1960s. Some years later a theory of smooth compactification of bounded symmetric domains was developed by Mumford school [5]. G. Faltings and C.-L. Chai [30] investigated the moduli of abelian varieties over the integers and could give the analogue of the Eichler-Shimura theorem that expresses Siegel modular forms in terms of the cohomology of local systems on $\mathcal{A}_g$. I want to emphasize that Siegel modular forms play an important role in the theory of the arithmetic and the geometry of the Siegel modular variety $\mathcal{A}_g$.

The aim of this paper is to discuss a theory of the Siegel modular variety in the aspects of arithmetic and geometry. Unfortunately two important subjects, which are the theory of harmonic analysis on the Siegel modular variety, and the Galois representations associated to Siegel modular forms are not covered in this article. These two topics shall be discussed in the near future in the separate papers. This article is organized as follows. In Section 2, we review the results of Siegel and Maass on invariant metrics and their Laplacians on $\mathbb{H}_g$. In Section 3, we investigate differential operators on $\mathbb{H}_g$ invariant under the action (1.1). In Section 4, we review Siegel's fundamental domain $\mathcal{F}_g$ and expound the spectral theory of the abelian variety A_Ω associated to an element Ω of $\mathcal{F}_g$. In Section 5, we review some properties of vector valued Siegel modular forms, and also discuss construction of Siegel modular forms and singular modular forms. In Section 6, we review the structure of the Hecke algebra of the group $GSp(g,\mathbb{Q})$ of symplectic similitudes and investigate the action of the Hecke algebra on Siegel modular forms. In Section 7, we briefly illustrate the basic notion of Jacobi forms which are needed in the next section. We also give a short historical survey on the theory of Jacobi forms. In Section 8, we deal with a lifting of elliptic cusp forms to Siegel modular forms and give some recent results on the lifts obtained by some people. A lifting of modular forms plays an important role arithmetically and geometrically. One of the interesting lifts is the so-called Duke-Imamoğlu-Ikeda lift. We discuss this lift in some detail. In Section 9, we give a short survey of toroidal compactifications of the Siegel modular variety $\mathcal{A}_g$ and illustrate a relationship between Siegel modular forms and holomorphic differential forms on $\mathcal{A}_g$. Siegel modular forms related to holomorphic differential forms on $\mathcal{A}_g$ play an important role in studying the geometry of $\mathcal{A}_g$. In Section 10, we investigate the geometry of subvarieties of the Siegel modular variety. Recently Grushevsky and Lehavi [45] announced that they proved that the Siegel modular variety $\mathcal{A}_6$ of genus 6 is of general type after constructing a series of new effective geometric divisors on $\mathcal{A}_g$. Before 2005 it had been known that $\mathcal{A}_g$ is of general type for $g \geq 7$. In fact, in 1983 Mumford [102] proved that $\mathcal{A}_g$ is of general type for $g \geq 7$. Nearly past twenty years nobody had known whether $\mathcal{A}_6$ is of general type or not. In Section 11, we formulate the proportionality theorem for an automorphic vector bundle on the Siegel modular variety following the work of Mumford (cf. [101]). In Section 12, we explain roughly Yoshida's interesting results about the fundamental periods of a motive attached to a Siegel modular form. These results are closely related to Deligne's conjecture about critical values of an L-function of a motive and the (pure or mixed) Hodge theory.

Finally I would like to give my hearty thanks to Hiroyuki Yoshida for explaining his important work kindly and sending two references [162, 163] to me.

Notations: We denote by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers respectively. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transposed matrix of M. I_n denotes the identity matrix of degree n. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. For a complex matrix A, $\overline{A}$ denotes the complex *conjugate* of A. For $A \in \mathbb{C}^{(k,l)}$ and $B \in \mathbb{C}^{(k,k)}$, we use the abbreviation $B\{A\} = {}^t\overline{A}BA$. For a number field F, we denote by $\mathbb{A}_F$ the ring of adeles of F. If $F = \mathbb{Q}$, the subscript will be omitted. We denote by $\mathbb{A}_{F,f}$ and $\mathbb{A}_f$ the finite part of $\mathbb{A}_F$ and $\mathbb{A}$ respectively. By $\overline{\mathbb{Q}}$ we mean the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

2. Invariant Metrics and Laplacians on Siegel Space

For $\Omega = (\omega_{ij}) \in \mathbb{H}_g$, we write $\Omega = X + iY$ with $X = (x_{ij}),\ Y = (y_{ij})$ real and $d\Omega = (d\omega_{ij})$. We also put

$$\frac{\partial}{\partial \Omega} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial \omega_{ij}} \right) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{\Omega}} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial \overline{\omega}_{ij}} \right).$$

C. L. Siegel [131] introduced the symplectic metric ds^2 on $\mathbb{H}_g$ invariant under the action (1.1) of $Sp(g,\mathbb{R})$ given by

$$ds^2 = \sigma(Y^{-1} d\Omega\, Y^{-1} d\overline{\Omega}) \tag{2.1}$$

and H. Maass [91] proved that its Laplacian is given by

$$\Delta = 4\,\sigma\left(Y\ {}^t\!\left(Y \frac{\partial}{\partial \overline{\Omega}} \right) \frac{\partial}{\partial \Omega} \right). \tag{2.2}$$

And

$$dv_g(\Omega) = (\det Y)^{-(g+1)} \prod_{1 \le i \le j \le g} dx_{ij} \prod_{1 \le i \le j \le g} dy_{ij} \tag{2.3}$$

is a $Sp(g,\mathbb{R})$-invariant volume element on $\mathbb{H}_g$ (cf. [133], p. 130).

Theorem 2.1. (**Siegel** [131]). *(1) There exists exactly one geodesic joining two arbitrary points Ω_0, Ω_1 in $\mathbb{H}_g$. Let $R(\Omega_0, \Omega_1)$ be the cross-ratio defined by*

$$R(\Omega_0, \Omega_1) = (\Omega_0 - \Omega_1)(\Omega_0 - \overline{\Omega}_1)^{-1}(\overline{\Omega}_0 - \overline{\Omega}_1)(\overline{\Omega}_0 - \Omega_1)^{-1}. \tag{2.4}$$

For brevity, we put $R_* = R(\Omega_0, \Omega_1)$. *Then the symplectic length* $\rho(\Omega_0, \Omega_1)$ *of the geodesic joining* Ω_0 *and* Ω_1 *is given by*

$$\rho(\Omega_0, \Omega_1)^2 = \sigma\left(\left(\log \frac{1+R_*^{\frac{1}{2}}}{1-R_*^{\frac{1}{2}}}\right)^2\right), \tag{2.5}$$

where

$$\left(\log \frac{1+R_*^{\frac{1}{2}}}{1-R_*^{\frac{1}{2}}}\right)^2 = 4\,R_* \left(\sum_{k=0}^{\infty} \frac{R_*^k}{2k+1}\right)^2 .$$

(2) For $M \in Sp(g,\mathbb{R})$, *we set*

$$\tilde{\Omega}_0 = M \cdot \Omega_0 \quad \textit{and} \quad \tilde{\Omega}_1 = M \cdot \Omega_1.$$

Then $R(\Omega_1, \Omega_0)$ *and* $R(\tilde{\Omega}_1, \tilde{\Omega}_0)$ *have the same eigenvalues.*
(3) All geodesics are symplectic images of the special geodesics

$$\alpha(t) = i\; diag(a_1^t, a_2^t, \cdots, a_g^t), \tag{2.6}$$

where $a_1, a_2, \cdots, a_g$ *are arbitrary positive real numbers satisfying the condition*

$$\sum_{k=1}^{g} (\log a_k)^2 = 1.$$

The proof of the above theorem can be found in [131], pp. 289-293.

Let

$$\mathbb{D}_g = \left\{ W \in \mathbb{C}^{(g,g)} \mid W = {}^tW,\ I_g - W\overline{W} > 0 \right\}$$

be the generalized unit disk of degree g. The Cayley transform $\Psi : \mathbb{D}_g \longrightarrow \mathbb{H}_g$ defined by

$$\Psi(W) = i\,(I_g + W)(I_g - W)^{-1}, \quad W \in \mathbb{D}_g \tag{2.7}$$

is a biholomorphic mapping of $\mathbb{D}_g$ onto $\mathbb{H}_g$ which gives the bounded realization of $\mathbb{H}_g$ by $\mathbb{D}_g$ (cf. [131]). A. Korányi and J. Wolf [81] gave a realization of a bounded symmetric domain as a Siegel domain of the third kind investigating a generalized Cayley transform of a bounded symmetric domain that generalizes the Cayley transform Ψ of $\mathbb{D}_g$.

Let

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} I_g & I_g \\ iI_g & -iI_g \end{pmatrix} \tag{2.8}$$

be the $2g \times 2g$ matrix represented by Ψ. Then

(2.9)

$$T^{-1} Sp(g,\mathbb{R})\, T = \left\{ \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \,\Big|\, {}^tP\overline{P} - {}^t\overline{Q}Q = I_g,\ {}^tP\overline{Q} = {}^t\overline{Q}P \right\}.$$

Indeed, if $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R})$, then

$$T^{-1}MT = \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \tag{2.10}$$

where

$$P = \frac{1}{2}\Big\{(A+D) + i\,(B-C)\Big\} \tag{2.11}$$

and

$$Q = \frac{1}{2}\Big\{(A-D) - i\,(B+C)\Big\}. \tag{2.12}$$

For brevity, we set

$$G_* = T^{-1}Sp(g,\mathbb{R})T.$$

Then G_* is a subgroup of $SU(g,g)$, where

$$SU(g,g) = \Big\{\, h \in \mathbb{C}^{(g,g)} \;\big|\; {}^t h I_{g,g} \overline{h} = I_{g,g} \,\Big\}, \quad I_{g,g} = \begin{pmatrix} I_g & 0 \\ 0 & -I_g \end{pmatrix}.$$

In the case $g = 1$, we observe that

$$T^{-1}Sp(1,\mathbb{R})T = T^{-1}SL_2(\mathbb{R})T = SU(1,1).$$

If $g > 1$, then G_* is a *proper* subgroup of $SU(g,g)$. In fact, since ${}^tT J_g T = -\,i\,J_g$, we get

$$G_* = \Big\{\, h \in SU(g,g) \;\big|\; {}^t h J_g h = J_g \,\Big\} = SU(g,g) \cap Sp(g,\mathbb{C}), \tag{2.13}$$

where

$$Sp(g,\mathbb{C}) = \Big\{\alpha \in \mathbb{C}^{(2g,2g)} \;\big|\; {}^t\alpha\, J_g\, \alpha = J_g \,\Big\}.$$

Let

$$P^+ = \left\{ \begin{pmatrix} I_g & Z \\ 0 & I_g \end{pmatrix} \;\Big|\; Z = {}^tZ \in \mathbb{C}^{(g,g)} \right\}$$

be the P^+-part of the complexification of $G_* \subset SU(g,g)$. We note that the Harish-Chandra decomposition of an element $\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}$ in G_* is

$$\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} = \begin{pmatrix} I_g & Q\overline{P}^{-1} \\ 0 & I_g \end{pmatrix} \begin{pmatrix} P - Q\overline{P}^{-1}\overline{Q} & 0 \\ 0 & \overline{P} \end{pmatrix} \begin{pmatrix} I_g & 0 \\ \overline{P}^{-1}\overline{Q} & I_g \end{pmatrix}.$$

For more detail, we refer to [75, p. 155]. Thus the P^+-component of the following element

$$\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \cdot \begin{pmatrix} I_g & W \\ 0 & I_g \end{pmatrix}, \quad W \in \mathbb{D}_g$$

of the complexification of G_*^J is given by

$$\begin{pmatrix} I_g & (PW+Q)(\overline{Q}W+\overline{P})^{-1} \\ 0 & I_g \end{pmatrix}. \tag{2.14}$$

We note that $Q\overline{P}^{-1} \in \mathbb{D}_g$. We get the Harish-Chandra embedding of $\mathbb{D}_g$ into P^+ (cf. [75, p. 155] or [120, pp. 58-59]). Therefore we see that G_* acts on $\mathbb{D}_g$ transitively by

(2.15)

$$\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \cdot W = (PW+Q)(\overline{Q}W+\overline{P})^{-1}, \quad \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*, \ W \in \mathbb{D}_g.$$

The isotropy subgroup K_* of G_* at the origin o is given by

$$K_* = \left\{ \begin{pmatrix} P & 0 \\ 0 & \overline{P} \end{pmatrix} \,\Big|\, P \in U(g) \right\}.$$

Thus G_*/K_* is biholomorphic to $\mathbb{D}_g$. It is known that the action (1.1) is compatible with the action (2.15) via the Cayley transform Ψ (cf. (2.7)). In other words, if $M \in Sp(g,\mathbb{R})$ and $W \in \mathbb{D}_g$, then

$$M \cdot \Psi(W) = \Psi(M_* \cdot W), \tag{2.16}$$

where $M_* = T^{-1}MT \in G_*$.

For $W = (w_{ij}) \in \mathbb{D}_g$, we write $dW = (dw_{ij})$ and $d\overline{W} = (d\overline{w}_{ij})$. We put

$$\frac{\partial}{\partial W} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial w_{ij}} \right) \quad \text{and} \quad \frac{\partial}{\partial \overline{W}} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial \overline{w}_{ij}} \right).$$

Using the Cayley transform $\Psi : \mathbb{D}_g \longrightarrow \mathbb{H}_g$, Siegel showed (cf. [131]) that

$$ds_*^2 = 4\sigma\Big((I_g - W\overline{W})^{-1} dW\, (I_g - \overline{W}W)^{-1} d\overline{W}\Big) \tag{2.17}$$

is a G_*-invariant Riemannian metric on $\mathbb{D}_g$ and Maass [91] showed that its Laplacian is given by

$$\Delta_* = \sigma\left((I_g - W\overline{W})\,{}^t\!\left((I_g - W\overline{W}) \frac{\partial}{\partial \overline{W}} \right) \frac{\partial}{\partial W} \right). \tag{2.18}$$

3. Invariant Differential Operators on Siegel Space

For brevity, we write $G = Sp(g,\mathbb{R})$. The isotropy subgroup K at iI_g for the action (1.1) is a maximal compact subgroup given by

$$K = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \,\Big|\, A\,{}^tA + B\,{}^tB = I_g,\ A\,{}^tB = B\,{}^tA,\ A,B \in \mathbb{R}^{(g,g)} \right\}.$$

Let $\mathfrak{k}$ be the Lie algebra of K. Then the Lie algebra $\mathfrak{g}$ of G has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{p} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \,\Big|\, X = {}^tX,\ Y = {}^tY,\ X,Y \in \mathbb{R}^{(g,g)} \right\}.$$

The subspace $\mathfrak{p}$ of $\mathfrak{g}$ may be regarded as the tangent space of $\mathbb{H}_g$ at iI_g. The adjoint representation of G on $\mathfrak{g}$ induces the action of K on $\mathfrak{p}$ given by

$$(3.1) \qquad k\cdot Z = kZ\,{}^tk, \quad k\in K,\ Z\in\mathfrak{p}.$$

Let T_g be the vector space of $g\times g$ symmetric complex matrices. We let $\psi:\mathfrak{p}\longrightarrow T_g$ be the map defined by

$$(3.2) \qquad \psi\left(\begin{pmatrix} X & Y\\ Y & -X\end{pmatrix}\right) = X+i\,Y, \quad \begin{pmatrix} X & Y\\ Y & -X\end{pmatrix}\in\mathfrak{p}.$$

We let $\delta: K\longrightarrow U(g)$ be the isomorphism defined by

$$(3.3) \qquad \delta\left(\begin{pmatrix} A & -B\\ B & A\end{pmatrix}\right) = A+i\,B, \quad \begin{pmatrix} A & -B\\ B & A\end{pmatrix}\in K,$$

where $U(g)$ denotes the unitary group of degree g. We identify $\mathfrak{p}$ (resp. K) with T_g (resp. $U(g)$) through the map Ψ (resp. δ). We consider the action of $U(g)$ on T_g defined by

$$(3.4) \qquad h\cdot Z = hZ\,{}^th, \quad h\in U(g),\ Z\in T_g.$$

Then the adjoint action (3.1) of K on $\mathfrak{p}$ is compatible with the action (3.4) of $U(g)$ on T_g through the map ψ. Precisely for any $k\in K$ and $\omega\in\mathfrak{p}$, we get

$$(3.5) \qquad \psi(k\,\omega\,{}^tk) = \delta(k)\,\psi(\omega)\,{}^t\delta(k).$$

The action (3.4) induces the action of $U(g)$ on the polynomial algebra $\mathrm{Pol}(T_g)$ and the symmetric algebra $S(T_g)$ respectively. We denote by $\mathrm{Pol}(T_g)^{U(g)}$ $\Big($resp. $S(T_g)^{U(g)}\Big)$ the subalgebra of $\mathrm{Pol}(T_g)$ $\Big($resp. $S(T_g)\Big)$ consisting of $U(g)$-invariants. The following inner product $(\ ,\ )$ on T_g defined by

$$(Z,W) = \mathrm{tr}\big(Z\overline{W}\big), \quad Z,W\in T_g$$

gives an isomorphism as vector spaces

$$(3.6) \qquad T_g\cong T_g^*, \quad Z\mapsto f_Z, \quad Z\in T_g,$$

where T_g^* denotes the dual space of T_g and f_Z is the linear functional on T_g defined by

$$f_Z(W) = (W,Z), \quad W\in T_g.$$

It is known that there is a canonical linear bijection of $S(T_g)^{U(g)}$ onto the algebra $\mathbb{D}(\mathbb{H}_g)$ of differential operators on $\mathbb{H}_g$ invariant under the action (1.1) of G. Identifying T_g with T_g^* by the above isomorphism (3.6), we get a canonical linear bijection

$$(3.7) \qquad \Phi:\mathrm{Pol}(T_g)^{U(g)}\longrightarrow\mathbb{D}(\mathbb{H}_g)$$

of $\mathrm{Pol}(T_g)^{U(g)}$ onto $\mathbb{D}(\mathbb{H}_g)$. The map Φ is described explicitly as follows. Similarly the action (3.1) induces the action of K on the polynomial algebra $\mathrm{Pol}(\mathfrak{p})$ and $S(\mathfrak{p})$ respectively. Through the map ψ, the subalgebra $\mathrm{Pol}(\mathfrak{p})^K$ of $\mathrm{Pol}(\mathfrak{p})$ consisting of K-invariants is isomorphic to $\mathrm{Pol}(T_g)^{U(g)}$. We put $N = g(g+1)$. Let $\{\xi_\alpha \mid 1 \leq \alpha \leq N\}$ be a basis of $\mathfrak{p}$. If $P \in \mathrm{Pol}(\mathfrak{p})^K$, then

$$(3.8)\qquad \Big(\Phi(P)f\Big)(gK) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g\exp\left(\sum_{\alpha=1}^{N} t_\alpha \xi_\alpha\right)K\right)\right]_{(t_\alpha)=0},$$

where $f \in C^\infty(\mathbb{H}_g)$. We refer to [53, 54] for more detail. In general, it is hard to express $\Phi(P)$ explicitly for a polynomial $P \in \mathrm{Pol}(\mathfrak{p})^K$.

According to the work of Harish-Chandra [46, 47], the algebra $\mathbb{D}(\mathbb{H}_g)$ is generated by g algebraically independent generators and is isomorphic to the commutative ring $\mathbb{C}[x_1, \cdots, x_g]$ with g indeterminates. We note that g is the real rank of G. Let $\mathfrak{g}_\mathbb{C}$ be the complexification of $\mathfrak{g}$. It is known that $\mathbb{D}(\mathbb{H}_g)$ is isomorphic to the center of the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$ (cf. [129]).

Using a classical invariant theory (cf. [58, 147]), we can show that $\mathrm{Pol}(T_g)^{U(g)}$ is generated by the following algebraically independent polynomials

$$(3.9)\qquad q_j(Z) = \mathrm{tr}\Big((Z\overline{Z})^j\Big), \quad j = 1, 2, \cdots, g.$$

For each j with $1 \leq j \leq g$, the image $\Phi(q_j)$ of q_j is an invariant differential operator on $\mathbb{H}_g$ of degree $2j$. The algebra $\mathbb{D}(\mathbb{H}_g)$ is generated by g algebraically independent generators $\Phi(q_1), \Phi(q_2), \cdots, \Phi(q_g)$. In particular,

$$(3.10)\qquad \Phi(q_1) = c_1\,\mathrm{tr}\left(Y\,{}^t\!\left(Y\frac{\partial}{\partial\overline{\Omega}}\right)\frac{\partial}{\partial\Omega}\right) \quad \text{for some constant } c_1.$$

We observe that if we take $Z = X + i\,Y$ with real X, Y, then $q_1(Z) = q_1(X,Y) = \mathrm{tr}\big(X^2 + Y^2\big)$ and

$$q_2(Z) = q_2(X,Y) = \mathrm{tr}\Big(\big(X^2+Y^2\big)^2 + 2\,X\big(XY - YX\big)Y\Big).$$

We propose the following problem.

Problem. Express the images $\Phi(q_j)$ explicitly for $j = 2, 3, \cdots, g$.

We hope that the images $\Phi(q_j)$ for $j = 2, 3, \cdots, g$ are expressed in the form of the *trace* as $\Phi(q_1)$.

Example 3.1. We consider the case $g = 1$. The algebra $\mathrm{Pol}(T_1)^{U(1)}$ is generated by the polynomial

$$q(z) = z\,\overline{z}, \quad z \in \mathbb{C}.$$

Using Formula (3.8), we get

$$\Phi(q) = 4\,y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Therefore $\mathbb{D}(\mathbb{H}_1) = \mathbb{C}\big[\Phi(q)\big]$.

Example 3.2. We consider the case $g = 2$. The algebra $\mathrm{Pol}(T_2)^{U(2)}$ is generated by the polynomial

$$q_1(Z) = \sigma\big(Z\,\overline{Z}\big), \quad q_2(Z) = \sigma\Big(\big(Z\,\overline{Z}\big)^2\Big), \quad Z \in T_2.$$

Using Formula (3.8), we may express $\Phi(q_1)$ and $\Phi(q_2)$ explicitly. $\Phi(q_1)$ is expressed by Formula (3.10). The computation of $\Phi(q_2)$ might be quite tedious. We leave the detail to the reader. In this case, $\Phi(q_2)$ was essentially computed in [19], Proposition 6. Therefore $\mathbb{D}(\mathbb{H}_2) = \mathbb{C}\big[\Phi(q_1), \Phi(q_2)\big]$. The authors of [19] computed the center of $U(\mathfrak{g}_{\mathbb{C}})$.

4. Siegel's Fundamental Domain

We let

$$\mathcal{P}_g = \Big\{\, Y \in \mathbb{R}^{(g,g)} \mid Y = {}^tY > 0 \,\Big\}$$

be an open cone in $\mathbb{R}^N$ with $N = g(g+1)/2$. The general linear group $GL(g,\mathbb{R})$ acts on $\mathcal{P}_g$ transitively by

$$g \circ Y := gY\,{}^tg, \qquad g \in GL(g,\mathbb{R}),\ Y \in \mathcal{P}_g. \tag{4.1}$$

Thus $\mathcal{P}_g$ is a symmetric space diffeomorphic to $GL(g,\mathbb{R})/O(g)$.

The fundamental domain $\mathcal{R}_g$ for $GL(g,\mathbb{Z})\backslash\mathcal{P}_g$ which was found by H. Minkowski [97] is defined as a subset of $\mathcal{P}_g$ consisting of $Y = (y_{ij}) \in \mathcal{P}_g$ satisfying the following conditions (M.1)–(M.2) (cf. [65] p. 191 or [92] p. 123):

(M.1) $aY\,{}^ta \geq y_{kk}$ for every $a = (a_i) \in \mathbb{Z}^g$ in which $a_k, \cdots, a_g$ are relatively prime for $k = 1, 2, \cdots, g$.

(M.2) $y_{k,k+1} \geq 0$ for $k = 1, \cdots, g-1$.

We say that a point of $\mathcal{R}_g$ is *Minkowski reduced* or simply *M-reduced.* $\mathcal{R}_g$ has the following properties (R1)–(R4):

(R1) For any $Y \in \mathcal{P}_g$, there exist a matrix $A \in GL(g,\mathbb{Z})$ and $R \in \mathcal{R}_g$ such that $Y = R[A]$ (cf. [65] p. 191 or [92] p. 139). That is,

$$GL(g,\mathbb{Z}) \circ \mathcal{R}_g = \mathcal{P}_g.$$

(R2) $\mathcal{R}_g$ is a convex cone through the origin bounded by a finite number of hyperplanes. $\mathcal{R}_g$ is closed in $\mathcal{P}_g$ (cf. [92] p. 139).

(R3) If Y and $Y[A]$ lie in $\mathcal{R}_g$ for $A \in GL(g,\mathbb{Z})$ with $A \neq \pm I_g$, then Y lies on the boundary $\partial\mathcal{R}_g$ of $\mathcal{R}_g$. Moreover $\mathcal{R}_g \cap (\mathcal{R}_g[A]) \neq \emptyset$ for only finitely many $A \in GL(g,\mathbb{Z})$ (cf. [92] p. 139).

(R4) If $Y = (y_{ij})$ is an element of $\mathcal{R}_g$, then

$$y_{11} \le y_{22} \le \cdots \le y_{gg} \quad \text{and} \quad |y_{ij}| < \frac{1}{2} y_{ii} \quad \text{for } 1 \le i < j \le g.$$

We refer to [65] p. 192 or [92] pp. 123-124.

Remark. Grenier [43] found another fundamental domain for $GL(g,\mathbb{Z})\backslash\mathcal{P}_g$.

For $Y = (y_{ij}) \in \mathcal{P}_g$, we put

$$dY = (dy_{ij}) \qquad \text{and} \qquad \frac{\partial}{\partial Y} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial y_{ij}} \right).$$

Then we can see easily that

$$ds^2 = \sigma((Y^{-1}dY)^2) \tag{4.2}$$

is a $GL(g,\mathbb{R})$-invariant Riemannian metric on $\mathcal{P}_g$ and its Laplacian is given by

$$\Delta = \sigma\left(\left(Y \frac{\partial}{\partial Y} \right)^2 \right).$$

We also can see that

$$d\mu_g(Y) = (\det Y)^{-\frac{g+1}{2}} \prod_{i \le j} dy_{ij}$$

is a $GL(g,\mathbb{R})$-invariant volume element on $\mathcal{P}_g$. The metric ds^2 on $\mathcal{P}_g$ induces the metric $ds^2_{\mathcal{R}}$ on $\mathcal{R}_g$. Minkowski [97] calculated the volume of $\mathcal{R}_g$ for the volume element $[dY] := \prod_{i \le j} dy_{ij}$ explicitly. Later Siegel computed the volume of $\mathcal{R}_g$ for the volume element $[dY]$ by a simple analytic method and generalized this case to the case of any algebraic number field.

Siegel [131] determined a fundamental domain $\mathcal{F}_g$ for $\Gamma_g\backslash\mathbb{H}_g$. We say that $\Omega = X + iY \in \mathbb{H}_g$ with X, Y real is *Siegel reduced* or *S-reduced* if it has the following three properties:

(S.1) $\det(\operatorname{Im}(\gamma \cdot \Omega)) \le \det(\operatorname{Im}(\Omega))$ for all $\gamma \in \Gamma_g$;

(S.2) $Y = \operatorname{Im}\Omega$ is M-reduced, that is, $Y \in \mathcal{R}_g$;

(S.3) $|x_{ij}| \le \frac{1}{2}$ for $1 \le i, j \le g$, where $X = (x_{ij})$.

$\mathcal{F}_g$ is defined as the set of all Siegel reduced points in $\mathbb{H}_g$. Using the highest point method, Siegel proved the following (F1)–(F3) (cf. [65] pp. 194-197 or [92] p. 169):

(F1) $\Gamma_g \cdot \mathcal{F}_g = \mathbb{H}_g$, i.e., $\mathbb{H}_g = \cup_{\gamma \in \Gamma_g} \gamma \cdot \mathcal{F}_g$.

(F2) $\mathcal{F}_g$ is closed in $\mathbb{H}_g$.

(F3) $\mathcal{F}_g$ is connected and the boundary of $\mathcal{F}_g$ consists of a finite number of hyperplanes.

The metric ds^2 given by (2.1) induces a metric $ds^2_{\mathcal{F}}$ on $\mathcal{F}_g$.

Siegel [131] computed the volume of $\mathcal{F}_g$

$$\text{vol}\,(\mathcal{F}_g) = 2\prod_{k=1}^{g} \pi^{-k}\,\Gamma(k)\,\zeta(2k), \tag{4.3}$$

where $\Gamma(s)$ denotes the Gamma function and $\zeta(s)$ denotes the Riemann zeta function. For instance,

$$\text{vol}\,(\mathcal{F}_1) = \frac{\pi}{3},\ \text{vol}\,(\mathcal{F}_2) = \frac{\pi^3}{270},\ \text{vol}\,(\mathcal{F}_3) = \frac{\pi^6}{127575},\ \text{vol}\,(\mathcal{F}_4) = \frac{\pi^{10}}{200930625}.$$

For a fixed element $\Omega \in \mathbb{H}_g$, we set

$$L_\Omega := \mathbb{Z}^g + \mathbb{Z}^g\Omega, \qquad \mathbb{Z}^g = \mathbb{Z}^{(1,g)}.$$

It follows from the positivity of $\text{Im}\,\Omega$ that L_Ω is a lattice in $\mathbb{C}^g$. We see easily that if Ω is an element of $\mathbb{H}_g$, the period matrix $\Omega_* := (I_g, \Omega)$ satisfies the Riemann conditions (RC.1) and (RC.2) :

(RC.1) $\quad \Omega_* J_g\, {}^t\Omega_* = 0.$

(RC.2) $\quad -\frac{1}{i}\Omega_* J_g\, {}^t\overline{\Omega}_* > 0.$

Thus the complex torus $A_\Omega := \mathbb{C}^g/L_\Omega$ is an abelian variety.

We fix an element $\Omega = X+iY$ of $\mathbb{H}_g$ with $X = \text{Re}\,\Omega$ and $Y = \text{Im}\,\Omega$. For a pair (A,B) with $A, B \in \mathbb{Z}^g$, we define the function $E_{\Omega;A,B} : \mathbb{C}^g \longrightarrow \mathbb{C}$ by

$$E_{\Omega;A,B}(Z) = e^{2\pi i\left(\sigma({}^tAU)+\sigma((B-AX)Y^{-1}\,{}^tV)\right)},$$

where $Z = U + iV$ is a variable in $\mathbb{C}^g$ with real U, V.

Lemma 4.1. *For any $A, B \in \mathbb{Z}^g$, the function $E_{\Omega;A,B}$ satisfies the following functional equation*

$$E_{\Omega;A,B}(Z + \lambda\Omega + \mu) = E_{\Omega;A,B}(Z), \quad Z \in \mathbb{C}^g$$

for all $\lambda, \mu \in \mathbb{Z}^g$. Thus $E_{\Omega;A,B}$ can be regarded as a function on A_Ω.

Proof. The proof can be found in [157]. □

We let $L^2(A_\Omega)$ be the space of all functions $f : A_\Omega \longrightarrow \mathbb{C}$ such that

$$||f||_\Omega := \int_{A_\Omega} |f(Z)|^2 dv_\Omega,$$

where dv_Ω is the volume element on A_Ω normalized so that $\int_{A_\Omega} dv_\Omega = 1$. The inner product $(\ ,\ )_\Omega$ on the Hilbert space $L^2(A_\Omega)$ is given by

$$(f, g)_\Omega := \int_{A_\Omega} f(Z)\,\overline{g(Z)}\,dv_\Omega, \quad f, g \in L^2(A_\Omega).$$

Theorem 4.1. *The set* $\{ E_{\Omega;A,B} \mid A, B \in \mathbb{Z}^g \}$ *is a complete orthonormal basis for* $L^2(A_\Omega)$. *Moreover we have the following spectral decomposition of* Δ_Ω*:*

$$L^2(A_\Omega) = \bigoplus_{A,B\in\mathbb{Z}^g} \mathbb{C} \cdot E_{\Omega;A,B}.$$

Proof. The complete proof can be found in [157]. □

5. Siegel Modular Forms

5.1. Basic Properties of Siegel Modular Forms

Let ρ be a rational representation of $GL(g,\mathbb{C})$ on a finite dimensional complex vector space V_ρ.

Definition. A holomorphic function $f : \mathbb{H}_g \longrightarrow V_\rho$ is called a *Siegel modular form* with respect to ρ if

$$f(\gamma \cdot \Omega) = f\big((A\Omega + B)(C\Omega + D)^{-1}\big) = \rho(C\Omega + D)f(\Omega) \tag{5.1}$$

for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$ and all $\Omega \in \mathbb{H}_g$. Moreover if $g = 1$, we require that f is holomorphic at the cusp ∞.

We denote by $M_\rho(\Gamma_g)$ the vector space of all Siegel modular forms with respect to Γ_g. If $\rho = \det^k$ for $k \in \mathbb{Z}$, a Siegel modular form f with respect to ρ satisfies the condition

$$f(\gamma \cdot \Omega) = \det(C\Omega + D)^k f(\Omega), \tag{5.2}$$

where γ and Ω are as above. In this case f is called a (classical) Siegel modular form on $\mathbb{H}_g$ of weight k. We denote by $M_k(\Gamma_g)$ the space of all Siegel modular forms on $\mathbb{H}_g$ of weight k.

Remark. (1) If $\rho = \rho_1 \oplus \rho_2$ is a direct sum of two finite dimensional rational representations of $GL(g,\mathbb{C})$, then it is easy to see that $M_\rho(\Gamma_g)$ is isomorphic to $M_{\rho_1}(\Gamma_g) \oplus M_{\rho_1}(\Gamma_g)$. Therefore it suffices to study $M_\rho(\Gamma_g)$ for an irreducible representation ρ of $GL(g,\mathbb{C})$.
(2) We may equip V_ρ with a hermitian inner product $(\ ,\ )$ satisfying the following condition

$$\big(\rho(x)v_1, v_2\big) = \big(v_1, \overline{\rho({}^t x)}v_2\big), \quad x \in GL(g,\mathbb{C}),\ v_1, v_2 \in V_\rho. \tag{5.3}$$

For an irreducible finite dimensional representation (ρ, V_ρ) of $GL(g,\mathbb{C})$, there exist a highest weight $k(\rho) = (k_1, \cdots, k_g) \in \mathbb{Z}^g$ with $k_1 \geq \cdots \geq k_g$ and a highest weight vector $v_\rho (\neq 0) \in V_\rho$ such that

$$\rho\big(\mathrm{diag}(a_1, \cdots, a_g)\big)v_\rho = \prod_{i=1}^{g} a_i^{k_i}\, v_\rho, \quad a_1, \cdots, a_g \in \mathbb{C}^\times.$$

Such a vector v_ρ is uniquely determined up to scalars. The number $k(\rho) := k_g$ is called the *weight* of ρ. For example, if $\rho = \det^k$, its highest weight is $(k, k, \cdots, k)$ and hence its weight is k.

Assume that (ρ, V_ρ) is an irreducible finite dimensional rational representation of $GL(g, \mathbb{C})$. Then it is known [65, 92] that a Siegel modular form f in $M_\rho(\Gamma_g)$ admits a Fourier expansion

$$f(\Omega) = \sum_{T \geq 0} a(T)\, e^{2\pi i\, \sigma(T\Omega)}, \tag{5.4}$$

where T runs over the set of all half-integral semi-positive symmetric matrices of degree g. We recall that T is said to be *half-integral* if $2T$ is an integral matrix whose diagonal entries are even.

Theorem 5.1. *(1) If kg is odd, then $M_k(\Gamma_g) = 0$.*
(2) If $k < 0$, then $M_k(\Gamma_g) = 0$.
(3) Let ρ be a non-trivial irreducible finite dimensional representation of $GL(g, \mathbb{C})$ with
highest weight $(k_1, \cdots, k_g)$. If $M_\rho(\Gamma_g) \neq \{0\}$, then $k_g \geq 1$.
(4) If $f \in M_\rho(\Gamma_g)$, then f is bounded in any subset $\mathcal{H}(c)$ of $\mathbb{H}_g$ given by the form

$$\mathcal{H}(c) := \{\Omega \in \mathbb{H}_g \mid Im\Omega > c I_g\}$$

with any positive real number $c > 0$.

5.2. The Siegel Operator

Let (ρ, V_ρ) be an irreducible finite dimensional representation of $GL(g, \mathbb{C})$. For any positive integer r with $0 \leq r < g$, we define the operator $\Phi_{\rho,r}$ on $M_\rho(\Gamma_g)$ by
(5.5)

$$(\Phi_{\rho,r} f)(\Omega_1) := \lim_{t \longrightarrow \infty} f\left(\begin{pmatrix} \Omega_1 & 0 \\ 0 & itI_{g-r} \end{pmatrix}\right), \quad f \in M_\rho(\Gamma_g),\ \Omega_1 \in \mathbb{H}_r.$$

We see that $\Phi_{\rho,r}$ is well-defined because the limit of the right hand side of (5.5) exists (cf. Theorem 5.1. (4)). The operator $\Phi_{\rho,r}$ is called the *Siegel operator*. A Siegel modular form $f \in M_\rho(\Gamma_g)$ is said to be a *cusp form* if $\Phi_{\rho,g-1} f = 0$. We denote by $S_\rho(\Gamma_g)$ the vector space of all cusp forms on $\mathbb{H}_g$ with respect to ρ. Let $V_\rho^{(r)}$ be the subspace of V_ρ spanned by the values

$$\{(\Phi_{\rho,r} f)(\Omega_1) \mid \Omega_1 \in \mathbb{H}_r,\ f \in M_\rho(\Gamma_g)\}.$$

According to [143], $V_\rho^{(r)}$ is invariant under the action of the subgroup

$$\left\{\begin{pmatrix} a & 0 \\ 0 & I_{g-r} \end{pmatrix} \;\middle|\; a \in GL(r, \mathbb{C})\right\}.$$

Then we have an irreducible rational representation $\rho^{(r)}$ of $GL(r,\mathbb{C})$ on $V_\rho^{(r)}$ defined by

$$\rho^{(r)}(a)v := \rho\left(\begin{pmatrix} a & 0 \\ 0 & I_{g-r}\end{pmatrix}\right)v, \quad a \in GL(r,\mathbb{C}),\ v \in V_\rho^{(r)}.$$

We observe that if $(k_1, \cdots, k_g)$ is the highest weight of ρ, then $(k_1, \cdots, k_r)$ is the highest weight of $\rho^{(r)}$.

Theorem 5.2. *The Siegel operator* $\Phi_{\det^k, r} : M_k(\Gamma_g) \longrightarrow M_k(\Gamma_r)$ *is surjective for* k *even with* $k > \frac{g+r+3}{2}$.

The proof of Theorem 5.2 can be found in [144].

We define the Petersson inner product $\langle\ ,\ \rangle_P$ on $M_\rho(\Gamma_g)$ by

(5.6)

$$\langle f_1, f_2\rangle_P := \int_{\mathcal{F}_g} \big(\rho(\operatorname{Im}\Omega) f_1(\Omega), f_2(\Omega)\big)\, dv_g(\Omega), \quad f_1, f_2 \in M_\rho(\Gamma_g),$$

where $\mathcal{F}_g$ is the Siegel's fundamental domain, $(\ ,\ )$ is the hermitian inner product defined in (5.3) and $dv_g(\Omega)$ is the volume element defined by (2.3). We can check that the integral of (5.6) converges absolutely if one of f_1 and f_2 is a cusp form. It is easily seen that one has the orthogonal decomposition

$$M_\rho(\Gamma_g) = S_\rho(\Gamma_g) \oplus S_\rho(\Gamma_g)^\perp,$$

where

$$S_\rho(\Gamma_g)^\perp = \big\{ f \in M_\rho(\Gamma_g) \mid \langle f, h\rangle_P = 0 \text{ for all } h \in S_\rho(\Gamma_g) \big\}$$

is the orthogonal complement of $S_\rho(\Gamma_g)$ in $M_\rho(\Gamma_g)$.

5.3. Construction of Siegel Modular Forms

In this subsection, we provide several well-known methods to construct Siegel modular forms.

(A) KLINGEN'S EISENSTEIN SERIES

Let r be an integer with $0 \le r < g$. We assume that k is a positive *even* integer. For $\Omega \in \mathbb{H}_g$, we write

$$\Omega = \begin{pmatrix} \Omega_1 & * \\ * & \Omega_2 \end{pmatrix}, \quad \Omega_1 \in \mathbb{H}_r,\ \Omega_2 \in \mathbb{H}_{g-r}.$$

For a fixed *cusp* form $f \in S_k(\Gamma_r)$ of weight k, H. Klingen [73] introduced the Eisenstein series $E_{g,r,k}(f)$ formally defined by

(5.7)

$$E_{g,r,k}(f)(\Omega) := \sum_{\gamma \in P_r \backslash \Gamma_g} f\big((\gamma\cdot\Omega)_1\big)\cdot \det(C\Omega + D)^{-k}, \quad \gamma = \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in \Gamma_g,$$

where

$$P_r = \left\{ \begin{pmatrix} A_1 & 0 & B_1 & * \\ * & U & * & * \\ C_1 & 0 & D_1 & * \\ 0 & 0 & 0 & {}^tU^{-1} \end{pmatrix} \in \Gamma_g \;\middle|\; \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \Gamma_r, U \in GL(g-r,\mathbb{Z}) \right\}$$

is a parabolic subgroup of Γ_g. We note that if $r=0$, and if $f=1$ is a constant, then

$$E_{g,0,k}(\Omega) = \sum_{C,D} \det(C\Omega + D)^{-k},$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ runs over the set of all representatives for the cosets $GL(g,\mathbb{Z})\backslash\Gamma_g$.

Klingen [73] proved the following:

Theorem 5.3. *Let $g \geq 1$ and let r be an integer with $0 \leq r < g$. We assume that k is a positive even integer with $k > g+r+1$. Then for any cusp form $f \in S_k(\Gamma_r)$ of weight k, the Eisenstein series $E_{g,r,k}(f)$ converges to a Siegel modular form on $\mathbb{H}_g$ of the same weight k and one has the following property*

$$\Phi_{\det^k, r} E_{g,r,k}(f) = f. \tag{5.8}$$

The proof of the above theorem can be found in [73, 74, 92].

(B) Theta Series

Let (ρ, V_ρ) be a finite dimensional rational representation of $GL(g,\mathbb{C})$. We let $H_\rho(r,g)$ be the space of pluriharmonic polynomials $P : \mathbb{C}^{(r,g)} \longrightarrow V_\rho$ with respect to (ρ, V_ρ). That is, $P \in H_\rho(r,g)$ if and only if $P : \mathbb{C}^{(r,g)} \longrightarrow V_\rho$ is a V_ρ-valued polynomial on $\mathbb{C}^{(r,g)}$ satisfying the following conditions (5.9) and (5.10): if $z = (z_{kj})$ is a coordinate in $\mathbb{C}^{(r,g)}$,

$$\sum_{k=1}^{r} \frac{\partial^2 P}{\partial z_{ki} \partial z_{kj}} = 0 \quad \text{for all } i,j \text{ with } 1 \leq i,j \leq g \tag{5.9}$$

and

(5.10)

$$P(zh) = \rho({}^t h) \det(h)^{-\frac{r}{2}} P(z) \quad \text{for all } z \in \mathbb{C}^{(r,g)} \text{ and } h \in GL(g,\mathbb{C}).$$

Now we let S be a positive definite even unimodular matrix of degree r. To a pair (S,P) with $P \in H_\rho(r,g)$, we attach the theta series

$$\Theta_{S,P}(\Omega) := \sum_{A \in \mathbb{Z}^{(r,g)}} P(S^{\frac{1}{2}} A)\, e^{\pi i\, \sigma(S[A]\Omega)} \tag{5.11}$$

which converges for all $\Omega \in \mathbb{H}_g$. E. Freitag [34] proved that $\Theta_{S,P}$ is a Siegel modular form on $\mathbb{H}_g$ with respect to ρ, i.e., $\Theta_{S,P} \in M_\rho(\Gamma_g)$.

Next we describe a method of constructing Siegel modular forms using the so-called *theta constants*.

We consider a theta characteristic

$$\epsilon = \begin{pmatrix} \epsilon' \\ \epsilon'' \end{pmatrix} \in \{0,1\}^{2g} \quad \text{with} \quad \epsilon', \epsilon'' \in \{0,1\}^g.$$

A theta characteristic $\epsilon = \begin{pmatrix} \epsilon' \\ \epsilon'' \end{pmatrix}$ is said to be *odd* (resp. *even*) if ${}^t\epsilon'\epsilon''$ is odd (resp. even). Now to each theta characteristic $\epsilon = \begin{pmatrix} \epsilon' \\ \epsilon'' \end{pmatrix}$, we attach the theta series

$$(5.12) \qquad \theta[\epsilon](\Omega) := \sum_{m \in \mathbb{Z}^g} e^{\pi i \left\{ \Omega\left[m + \frac{1}{2}\epsilon'\right] + {}^t\left(m + \frac{1}{2}\epsilon'\right)\epsilon'' \right\}}, \quad \Omega \in \mathbb{H}_g.$$

If ϵ is odd, we see that $\theta[\epsilon]$ vanishes identically. If ϵ is even, $\theta[\epsilon]$ is a Siegel modular form on $\mathbb{H}_g$ of weight $\frac{1}{2}$ with respect to the principal congreuence subgroup $\Gamma_g(2)$ (cf. [65, 103]). Here

$$\Gamma_g(2) = \left\{ \sigma \in \Gamma_g \mid \sigma \equiv I_{2g} \ (\mathrm{mod}\ 2) \right\}$$

is a congruence subgroup of Γ_g of level 2. These theta series $\theta[\epsilon]$ are called *theta constants*. It is easily checked that there are $2^{g-1}(2^g+1)$ even theta characteristics. These theta constants $\theta[\epsilon]$ can be used to construct Siegel modular forms with respect to Γ_g. We provide several examples. For $g = 1$, we have

$$(\theta[\epsilon_{00}]\, \theta[\epsilon_{01}]\, \theta[\epsilon_{11}])^8 \in S_{12}(\Gamma_1),$$

where

$$\epsilon_{00} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \epsilon_{01} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \epsilon_{11} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

For $g = 2$, we get

$$\chi_{10} := -2^{-14} \prod_{\epsilon \in \mathbb{E}} \theta[\epsilon]^2 \in S_{10}(\Gamma_2)$$

and

$$\left(\prod_{\epsilon \in \mathbb{E}} \theta[\epsilon] \right) \cdot \sum_{\epsilon_1, \epsilon_2, \epsilon_3} (\theta[\epsilon_1]\, \theta[\epsilon_2]\, \theta[\epsilon_3])^{20} \in S_{35}(\Gamma_2),$$

where $\mathbb{E}$ denotes the set of all even theta characteristics and $(\epsilon_1, \epsilon_2, \epsilon_3)$ runs over the set of triples of theta characteristics such that $\epsilon_1 + \epsilon_2 + \epsilon_3$ is odd. For $g = 3$, we have

$$\prod_{\epsilon \in \mathbb{E}} \theta[\epsilon] \in S_{18}(\Gamma_3).$$

We refer to [65] for more details.

5.4. Singular Modular Forms

We know that a Siegel modular form $f \in M_\rho(\Gamma_g)$ has a Fourier expansion

$$f(\Omega) = \sum_{T \geq 0} a(T)\, e^{2\pi i\, \sigma(T\Omega)},$$

where T runs over the set of all half-integral semi-positive symmetric matrices of degree g. A Siegel modular form $f \in M_\rho(\Gamma_g)$ is said to be *singular* if $a(T) \neq 0$ implies $\det(T) = 0$. We observe that the notion of singular modular forms is opposite to that of cusp forms. Obviously if $g = 1$, singular modular forms are constants.

We now characterize singular modular forms in terms of the weight of ρ and a certain differential operator. For a coordinate $\Omega = X + iY$ in $\mathbb{H}_g$ with X real and $Y = (y_{ij}) \in \mathcal{P}_g$ (cf. Section 4), we define the differential

$$M_g := \det(Y) \cdot \det\left(\frac{\partial}{\partial Y}\right) \tag{5.13}$$

which is invariant under the action (4.1) of $GL(g, \mathbb{R})$. Here

$$\frac{\partial}{\partial Y} = \left(\frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial y_{ij}}\right).$$

Using the differential operator M_g, Maass [92, pp. 202-204] proved that if a nonzero singular modular form on $\mathbb{H}_g$ of weight k exists, then $nk \equiv 0 \pmod 2$ and $0 < 2k \leq g - 1$. The converse was proved by Weissauer (cf. [143, Satz 4]).

Theorem 5.4. *Let ρ be an irreducible rational finite dimensional representation of $GL(g, \mathbb{C})$ with highest weight $(k_1, \cdots, k_g)$. Then a non-zero Siegel modular form $f \in M_\rho(\Gamma_g)$ is singular if and only if $2k(\rho) = 2k_g < g$.*

The above theorem was proved by Freitag [33], Weissauer [143] et al. By Theorem 5.4, we see that the weight of a singular modular form is small. For instance, W. Duke and Ö. Imamoḡlu [27] proved that $S_6(\Gamma_g) = 0$ for all g. In a sense we say that there are *no* cusp forms of *small weight.*

Theorem 5.5. *Let $f \in M_\rho(\Gamma_g)$ be a Siegel modular form with respect to a rational representation ρ of $GL(g, \mathbb{C})$. Then the following are equivalent:*

(1) f is a singular modular form.
(2) f satisfies the differential equation $M_g f = 0$.

We refer to [92] and [152] for the proof.

Let $f \in M_k(\Gamma_g)$ be a nonzero singular modular form of weight k. According to Theorem 5.4, $2k < g$. We can show that k is divisible by 4. Let $S_1, \cdots, S_h$ be a complete system of representatives of positive definite even unimodular integral matrices of degree $2k$. Freitag [33, 34] proved that $f(\Omega)$ can be written as a linear combination of theta series $\theta_{S_1}, \cdots, \theta_{S_h}$, where θ_{S_ν} $(1 \le \nu \le h)$ is defined by

$$\theta_{S_\nu}(\Omega) := \sum_{A \in \mathbb{Z}^{(2k,g)}} e^{\pi i\, \sigma(S_\nu[A]\Omega)}, \quad 1 \le \nu \le h. \tag{5.14}$$

According to Theorem 5.5, we need to investigate some properties of the weight of ρ in order to understand singular modular forms. Let $(k_1, \cdots, k_g)$ be the highest weight of ρ. We define the *corank* of ρ by

$$\operatorname{corank}(\rho) := \left| \left\{ j \mid 1 \le j \le g,\ k_j = k_g \right\} \right|.$$

Let

$$f(\Omega) = \sum_{T \ge 0} a(T)\, e^{2\pi i\, \sigma(T\Omega)}$$

be a Siegel modular form in $M_\rho(\Gamma_g)$. The notion of the *rank* of f and that of the *corank* of f were introduced by Weissauer [143] as follows:

$$\operatorname{rank}(f) := \max\left\{ \operatorname{rank}(T) \mid a(T) \ne 0 \right\}$$

and

$$\operatorname{corank}(f) := g - \min\left\{ \operatorname{rank}(T) \mid a(T) \ne 0 \right\}.$$

Weissauer [143] proved the following.

Theorem 5.6. *Let ρ be an irreducible rational representation of $GL(g, \mathbb{C})$ with highest weight $(k_1, \cdots, k_g)$ such that corank$(\rho) < g - k_g$. Assume that*

$$\left| \left\{ j \mid 1 \le j \le g,\ k_j = k_g + 1 \right\} \right| < 2\big(g - k_g - \mathit{corank}(\rho)\big).$$

Then $M_\rho(\Gamma_g) = 0$.

6. The Hecke Algebra

6.1. The Structure of the Hecke Algebra

For a positive integer g, we let $\Gamma_g = Sp(g, \mathbb{Z})$ and let

$$\Delta_g := GSp(g, \mathbb{Q}) = \left\{ M \in GL(2g, \mathbb{Q}) \mid {}^tMJ_gM = l(M)J_g,\ l(M) \in \mathbb{Q}^\times \right\}$$

be the group of symplectic similitudes of the rational symplectic vector space $(\mathbb{Q}^{2g}, \langle\ ,\ \rangle)$. We put

$$\Delta_g^+ := GSp(g, \mathbb{Q})^+ = \left\{ M \in \Delta_g \mid l(M) > 0 \right\}.$$

Following the notations in [34], we let $\mathscr{H}(\Gamma_g, \Delta_g)$ be the complex vector space of all formal finite sums of double cosets $\Gamma_g M \Gamma_g$ with $M \in \Delta_g^+$. A double coset $\Gamma_g M \Gamma_g \, (M \in \Delta_g^+)$ can be written as a finite disjoint union of right cosets $\Gamma_g M_\nu \, (1 \leq \nu \leq h)$:

$$\Gamma_g M \Gamma_g = \cup_{\nu=1}^h \Gamma_g M_\nu \quad \text{(disjoint)}.$$

Let $\mathscr{L}(\Gamma_g, \Delta_g)$ be the complex vector space consisting of formal finite sums of right cosets $\Gamma_g M$ with $M \in \Delta^+$. For each double coset $\Gamma_g M \Gamma_g = \cup_{\nu=1}^h \Gamma_g M_\nu$ we associate an element $j(\Gamma_g M \Gamma_g)$ in $\mathscr{L}(\Gamma_g, \Delta_g)$ defined by

$$j(\Gamma_g M \Gamma_g) := \sum_{\nu=1}^h \Gamma_g M_\nu.$$

Then j induces a linear map

$$j_* : \mathscr{H}(\Gamma_g, \Delta_g) \longrightarrow \mathscr{L}(\Gamma_g, \Delta_g). \tag{6.1}$$

We observe that Δ_g acts on $\mathscr{L}(\Gamma_g, \Delta_g)$ as follows:

$$\Big(\sum_{j=1}^h c_j \, \Gamma_g M_j\Big) \cdot M = \sum_{j=1}^h c_j \, \Gamma_g M_j M, \quad M \in \Delta_g.$$

We denote

$$\mathscr{L}(\Gamma_g, \Delta_g)^{\Gamma_g} := \big\{ \, T \in \mathscr{L}(\Gamma_g, \Delta_g) \mid T \cdot \gamma = T \text{ for all } \gamma \in \Gamma_g \, \big\}$$

be the subspace of Γ_g-invariants in $\mathscr{L}(\Gamma_g, \Delta_g)$. Then we can show that $\mathscr{L}(\Gamma_g, \Delta_g)^{\Gamma_g}$ coincides with the image of j_* and the map

$$j_* : \mathscr{H}(\Gamma_g, \Delta_g) \longrightarrow \mathscr{L}(\Gamma_g, \Delta_g)^{\Gamma_g} \tag{6.2}$$

is an isomorphism of complex vector spaces (cf. [34, p. 228]). From now on we identify $\mathscr{H}(\Gamma_g, \Delta_g)$ with $\mathscr{L}(\Gamma_g, \Delta_g)^{\Gamma_g}$.

We define the multiplication of the double coset $\Gamma_g M \Gamma_g$ and $\Gamma_g N$ by

$$(\Gamma_g M \Gamma_g) \cdot (\Gamma_g N) = \sum_{j=1}^h \Gamma_g M_j N, \quad M, N \in \Delta_g, \tag{6.3}$$

where $\Gamma_g M \Gamma_g = \cup_{j=1}^h \Gamma_g M_j$ (disjoint). The definition (6.3) is well defined, i.e., independent of the choice of M_j and N. We extend this multiplication to $\mathscr{H}(\Gamma_g, \Delta_g)$ and $\mathscr{L}(\Gamma_g, \Delta_g)$. Since

$$\mathscr{H}(\Gamma_g, \Delta_g) \cdot \mathscr{H}(\Gamma_g, \Delta_g) \subset \mathscr{H}(\Gamma_g, \Delta_g),$$

$\mathscr{H}(\Gamma_g, \Delta_g)$ is an associative algebra with the identity element $\Gamma_g I_{2g} \Gamma_g = \Gamma_g$. The algebra $\mathscr{H}(\Gamma_g, \Delta_g)$ is called the *Hecke algebra* with respect to Γ_g and Δ_g.

We now describe the structure of the Hecke algebra $\mathscr{H}(\Gamma_g, \Delta_g)$. For a prime p, we let $\mathbb{Z}[1/p]$ be the ring of all rational numbers of the form $a \cdot p^\nu$ with $a, \nu \in \mathbb{Z}$. For a prime p, we denote

$$\Delta_{g,p} := \Delta_g \cap GL(2g, \mathbb{Z}[1/p]).$$

Then we have a decomposition of $\mathscr{H}(\Gamma_g, \Delta_g)$

$$\mathscr{H}(\Gamma_g, \Delta_g) = \bigotimes_{p:\,\text{prime}} \mathscr{H}(\Gamma_g, \Delta_{g,p})$$

as a tensor product of local Hecke algebras $\mathscr{H}(\Gamma_g, \Delta_{g,p})$. We denote by $\check{\mathscr{H}}(\Gamma_g, \Delta_g)$ (resp. $\check{\mathscr{H}}(\Gamma_g, \Delta_{g,p})$) the subring of $\mathscr{H}(\Gamma_g, \Delta_g)$ (resp. $\mathscr{H}(\Gamma_g, \Delta_{g,p})$) by integral matrices.

In order to describe the structure of local Hecke operators $\mathscr{H}(\Gamma_g, \Delta_{g,p})$, we need the following lemmas.

Lemma 6.1. *Let $M \in \Delta_g^+$ with ${}^tMJ_gM = lJ_g$. Then the double coset $\Gamma_g M \Gamma_g$ has a unique representative of the form*

$$M_0 = diag(a_1, \cdots, a_g, d_1, \cdots, d_g),$$

where $a_g | d_g$, $a_j > 0$, $a_j d_j = l$ for $1 \le j \le g$ and $a_k | a_{k+1}$ for $1 \le k \le g-1$.

For a positive integer l, we let

$$O_g(l) := \{\, M \in GL(2g, \mathbb{Z}) \mid {}^tMJ_gM = lJ_g \,\}.$$

Then we see that $O_g(l)$ can be written as a finite disjoint union of double cosets and hence as a finite union of right cosets. We define $T(l)$ as the element of $\mathscr{H}(\Gamma_g, \Delta_g)$ defined by $O_g(l)$.

Lemma 6.2. *(a) Let l be a positive integer. Let*

$$O_g(l) = \cup_{\nu=1}^h \Gamma_g M_\nu \quad (\textit{disjoint})$$

be a disjoint union of right cosets $\Gamma_g M_\nu$ $(1 \le \nu \le h)$. Then each right coset $\Gamma_g M_\nu$ has a representative of the form

$$M_\nu = \begin{pmatrix} A_\nu & B_\nu \\ 0 & D_\nu \end{pmatrix}, \quad {}^tA_\nu D_\nu = lI_g, \quad A_\nu \textit{ is upper triangular.}$$

(b) Let p be a prime. Then

$$T(p) = O_g(p) = \Gamma_g \begin{pmatrix} I_g & 0 \\ 0 & pI_g \end{pmatrix} \Gamma_g$$

and

$$T(p^2) = \sum_{i=0}^g T_i(p^2),$$

where

$$T_k(p^2) := \begin{pmatrix} I_{g-k} & 0 & 0 & 0 \\ 0 & pI_k & 0 & 0 \\ 0 & 0 & p^2 I_{g-k} & 0 \\ 0 & 0 & 0 & pI_k \end{pmatrix} \Gamma_g, \quad 0 \le k \le g.$$

Proof. The proof can be found in [34, p. 225 and p. 250]. □

For example, $T_g(p^2) = \Gamma_g(pI_{2g})\Gamma_g$ and

$$T_0(p^2) = \Gamma_g \begin{pmatrix} I_g & 0 \\ 0 & p^2 I_g \end{pmatrix} \Gamma_g = T(p)^2.$$

We have the following

Theorem 6.1. *The local Hecke algebra* $\check{\mathscr{H}}(\Gamma_g, \Delta_{g,p})$ *is generated by algebraically independent generators* $T(p), T_1(p^2), \cdots, T_g(p^2)$.

Proof. The proof can be found in [34, p. 250 and p. 261]. □

On Δ_g we have the anti-automorphism $M \mapsto M^* := l(M)M^{-1}$ $(M \in \Delta_g)$. Obviously $\Gamma_g^* = \Gamma_g$. By Lemma 6.1, $(\Gamma_g M \Gamma_g)^* = \Gamma_g M^* \Gamma_g = \Gamma_g M \Gamma_g$. According to [125], Proposition 3.8, $\mathscr{H}(\Gamma_g, \Delta_g)$ is commutative.

Let $X_0, X_1, \cdots, X_g$ be the $g+1$ variables. We define the automorphisms

$$w_j : \mathbb{C}\big[X_0^{\pm 1}, X_1^{\pm 1}, \cdots, X_g^{\pm 1}\big] \longrightarrow \mathbb{C}\big[X_0^{\pm 1}, X_1^{\pm 1}, \cdots, X_g^{\pm 1}\big], \quad 1 \le j \le g$$

by

$$w_j(X_0) = X_0 X_j^{-1}, \quad w_j(X_j) = X_j^{-1}, \quad w_j(X_k) = X_k \ \text{ for } k \ne 0, j.$$

Let W_g be the finite group generated by $w_1, \cdots, w_g$ and the permutations of variables $X_1, \cdots, X_g$. Obviously w_j^2 is the identity map and $|W_g| = 2^g g!$.

Theorem 6.2. *There exists an isomorphism*

$$Q : \mathscr{H}(\Gamma_g, \Delta_{g,p}) \longrightarrow \mathbb{C}\big[X_0^{\pm 1}, X_1^{\pm 1}, \cdots, X_g^{\pm 1}\big]^{W_g}.$$

In fact, Q *is defined by*

$$Q\Big(\sum_{j=1}^h \Gamma_g M_j\Big) = \sum_{j=1}^h Q(\Gamma_g M_j) = \sum_{j=1}^h X_0^{-k_0(j)} \prod_{\nu=1}^g \big(p^{-\nu} X^\nu\big)^{k_\nu(j)} |\det A_j|^{g+1},$$

where we choose the representative M_j *of* $\Gamma_g M_j$ *of the form*

$$M_j = \begin{pmatrix} A_j & B_j \\ 0 & D_j \end{pmatrix}, \quad A_j = \begin{pmatrix} p^{k_1(j)} & \dots & * \\ 0 & \ddots & \vdots \\ 0 & 0 & p^{k_g(j)} \end{pmatrix}.$$

We note that the integers $k_1(j), \cdots, k_g(j)$ *are uniquely determined.*

Proof. The proof can be found in [34]. □

For a prime p, we let

$$\mathscr{H}(\Gamma_g, \Delta_{g,p})_{\mathbb{Q}} := \left\{ \sum c_j \, \Gamma_g M_j \Gamma_g \in \mathscr{H}(\Gamma_g, \Delta_{g,p}) \mid c_j \in \mathbb{Q} \right\}$$

be the $\mathbb{Q}$-algebra contained in $\mathscr{H}(\Gamma_g, \Delta_{g,p})$. We put

$$G_p := GSp(g, \mathbb{Q}_p) \quad \text{and} \quad K_p = GSp(g, \mathbb{Z}_p).$$

We can identify $\mathscr{H}(\Gamma_g, \Delta_{g,p})_{\mathbb{Q}}$ with the $\mathbb{Q}$-algebra $\mathscr{H}_{g,p}^{\mathbb{Q}}$ of $\mathbb{Q}$-valued locally constant, K_p-biinvariant functions on G_p with compact support. The multiplication on $\mathscr{H}_{g,p}^{\mathbb{Q}}$ is given by

$$(f_1 * f_2)(h) = \int_{G_p} f_1(g)\, f_2(g^{-1}h) dg, \quad f_1, f_2 \in \mathscr{H}_{g,p}^{\mathbb{Q}},$$

where dg is the unique Haar measure on G_p such that the volume of K_p is 1. The correspondence is obtained by sending the double coset $\Gamma_g M \Gamma_g$ to the characteristic function of $K_p M K_p$.

In order to describe the structure of $\mathscr{H}_{g,p}^{\mathbb{Q}}$, we need to understand the p-adic Hecke algebras of the diagonal torus $\mathbb{T}$ and the Levi subgroup $\mathbb{M}$ of the standard parabolic group. Indeed, $\mathbb{T}$ is defined to be the subgroup consisting of diagonal matrices in Δ_g and

$$\mathbb{M} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \Delta_g \right\}$$

is the Levi subgroup of the parabolic subgroup

$$\left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Delta_g \right\}.$$

Let Y be the co-character group of $\mathbb{T}$, i.e., $Y = \mathrm{Hom}(\mathbb{G}_m, \mathbb{T})$. We define the local Hecke algebra $\mathscr{H}_p(\mathbb{T})$ for $\mathbb{T}$ to be the $\mathbb{Q}$-algebra of $\mathbb{Q}$-valued, $\mathbb{T}(\mathbb{Z}_p)$-biinvariant functions on $\mathbb{T}(\mathbb{Q}_p)$ with compact support. Then $\mathscr{H}_p(\mathbb{T}) \cong \mathbb{Q}[Y]$, where $\mathbb{Q}[Y]$ is the group algebra over $\mathbb{Q}$ of Y. An element $\lambda \in Y$ corresponds the characteristic function of the double coset $D_\lambda = K_p \lambda(p) K_p$. It is known that $\mathscr{H}_p(\mathbb{T})$ is isomorphic to the ring $\mathbb{Q}\big[(u_1/v_1)^{\pm 1}, \cdots, (u_g/v_g)^{\pm 1}, (v_1 \cdots v_g)^{\pm 1}\big]$ under the map

$$(a_1, \cdots, a_g, c) \mapsto (u_1/v_1)^{a_1} \cdots (u_g/v_g)^{a_g} (v_1 \cdots v_g)^c.$$

Similarly we have a p-adic Hecke algebra $\mathscr{H}_p(\mathbb{M})$. Let $W_{\Delta_g} = N(\mathbb{T})/\mathbb{T}$ be the Weyl group with respect to $(\mathbb{T}, \Delta_g)$, where $N(\mathbb{T})$ is the normalizer of $\mathbb{T}$ in Δ_g. Then $W_{\Delta_g} \cong S_g \ltimes (\mathbb{Z}/2\mathbb{Z})^g$, where the generator of the i-th factor $\mathbb{Z}/2\mathbb{Z}$ acts on a matrix of the form $\mathrm{diag}(a_1, \cdots, a_g, d_1, \cdots, d_g)$ by interchanging a_i and d_i, and the symmetry group S_g acts by permuting the a_i's and d_i's. We note that W_{Δ_g} is isomorphic to W_g. The Weyl group $W_{\mathbb{M}}$ with respect to $(\mathbb{T}, \mathbb{M})$ is isomorphic to S_g. We can prove that the algebra $\mathscr{H}_p(\mathbb{T})^{W_{\Delta_g}}$ of W_{Δ_g}-invariants in $\mathscr{H}_p(\mathbb{T})$ is isomorphic to $\mathbb{Q}\big[Y_0^{\pm 1}, Y_1, \cdots, Y_g\big]$ (cf. [34]). We let

$$B = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Delta_g \,\middle|\, A \text{ is upper triangular, } D \text{ is lower triangular} \right\}$$

be the Borel subgroup of Δ_g. A set Φ^+ of positive roots in the root system Φ determined by B. We set $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$.

Now we have the map $\alpha_{\mathbb{M}} : \mathbb{M} \longrightarrow \mathbb{G}_m$ defined by

$$\alpha_{\mathbb{M}}(M) := l(M)^{-\frac{g(g+1)}{2}}\big(\det A\big)^{g+1}, \quad M = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathbb{M}$$

and the map $\beta_{\mathbb{T}} : \mathbb{T} \longrightarrow \mathbb{G}_m$ defined by

$$\beta_{\mathbb{T}}(\mathrm{diag}(a_1, \cdots, a_g, d_1, \cdots, d_g)) := \prod_{i=1}^{g} a_1^{g+1-2i}, \ \mathrm{diag}(a_1, \cdots, a_g, d_1, \cdots, d_g) \in \mathbb{T}.$$

Let $\theta_{\mathbb{T}} := \alpha_{\mathbb{M}}\,\beta_{\mathbb{T}}$ be the character of $\mathbb{T}$. The *Satake's spherical map* $S_{p,\mathbb{M}} : \mathscr{H}_{g,p}^{\mathbb{Q}} \longrightarrow \mathscr{H}_p(\mathbb{M})$ is defined by

$$S_{p,\mathbb{M}}(\phi)(m) := |\alpha_{\mathbb{M}}(m)|_p \int_{U(\mathbb{Q}_p)} \phi(mu)du, \quad \phi \in \mathscr{H}_{g,p}^{\mathbb{Q}},\ m \in \mathbb{M}, \tag{6.4}$$

where $|\ \ |_p$ is the p-adic norm and $U(\mathbb{Q}_p)$ denotes the unipotent radical of Δ_g. Also another *Satake's spherical map* $S_{\mathbb{M},\mathbb{T}} : \mathscr{H}_p(\mathbb{M}) \longrightarrow \mathscr{H}_p(\mathbb{T})$ is defined by

$$S_{\mathbb{M},\mathbb{T}}(f)(t) := |\beta_{\mathbb{T}}(t)|_p \int_{\mathbb{M}\cap\mathbb{N}} f(tn)dn, \quad t \in \mathscr{H}_p(\mathbb{T}),\ t \in \mathbb{T}, \tag{6.5}$$

where $\mathbb{N}$ is a nilpotent subgroup of Δ_g.

Theorem 6.3. *The Satake's spherical maps* $S_{p,\mathbb{M}}$ *and* $S_{\mathbb{M},\mathbb{T}}$ *define the isomorphisms of* $\mathbb{Q}$*-algebras*

$$\mathscr{H}_{g,p}^{\mathbb{Q}} \cong \mathscr{H}_p(\mathbb{T})^{W_{\Delta_g}} \quad and \quad \mathscr{H}_p(\mathbb{M}) \cong \mathscr{H}_p(\mathbb{T})^{W_{\mathbb{M}}}. \tag{6.6}$$

We define the elements ϕ_k $(0 \leq k \leq g)$ in $\mathscr{H}_p(\mathbb{M})$ by

$$\phi_k := p^{-\frac{k(k+1)}{2}}\,\mathbb{M}(\mathbb{Z}_p)\begin{pmatrix} I_{g-k} & 0 & 0 \\ 0 & pI_g & 0 \\ 0 & 0 & I_k \end{pmatrix}\mathbb{M}(\mathbb{Z}_p), \quad i = 0, 1, \cdots, g.$$

Then we have the relation

$$S_{p,\mathbb{M}}(T(p)) = \sum_{k=0}^{g}\phi_k \tag{6.7}$$

and

$$S_{p,\mathbb{M}}\big(T_i(p^2)\big) = \sum_{j,k\geq 0,\, i+j\leq k} m_{k-j}(i)\, p^{-\binom{k-j+1}{2}} \phi_j\phi_k, \tag{6.8}$$

where

$$m_s(i) := \sharp\left\{\, A \in M(s,\mathbb{F}_p) \mid {}^tA = A, \quad \text{corank}(A) = i \,\right\}.$$

Moreover, for $k = 0, 1, \cdots, g$, we have

$$S_{\mathbb{M},\mathbb{T}}(\phi_k) = (v_1\cdots v_g)E_k(u_1/v_1,\cdots,u_g/v_g), \tag{6.9}$$

where E_k denotes the elementary symmetric function of degree k. The proof of (6.7)-(6.9) can be found in [2, pp. 142-145].

6.2. Action of the Hecke Algebra on Siegel Modular Forms

Let (ρ, V_ρ) be a finite dimensional irreducible representation of $GL(g,\mathbb{C})$ with highest weight $(k_1,\cdots,k_g)$. For a function $F : \mathbb{H}_g \longrightarrow V_\rho$ and $M \in \Delta_g^+$, we define

$$(f|_\rho M)(\Omega) = \rho(C\Omega + D)^{-1} f(M\cdot\Omega), \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Delta_g^+.$$

It is easily checked that $f|_\rho M_1M_2 = \big(f|_\rho M_1\big)|_\rho M_2$ for $M_1, M_2 \in \Delta_g^+$.

We now consider a subset $\mathscr{M}$ of Δ_g satisfying the following properties (M1) and (M2):

(M1) $\mathscr{M} = \cup_{j=1}^h \Gamma_g M_j$ (disjoint union);
(M2) $\mathscr{M}\,\Gamma_g \subset \mathscr{M}$.

For a Siegel modular form $f \in M_\rho(\Gamma_g)$, we define

$$T(\mathscr{M})f := \sum_{j=1}^h f|_\rho M_j. \tag{6.10}$$

This is well defined, i.e., is independent of the choice of representatives M_j because of the condition (M1). On the other hand, it follows from the condition (M2) that $T(\mathscr{M})f|_\rho\gamma = T(\mathscr{M})f$ for all $\gamma \in \Gamma_g$. Thus we get a linear operator

$$T(\mathscr{M}) : M_\rho(\Gamma_g) \longrightarrow M_\rho(\Gamma_g). \tag{6.11}$$

We know that each double coset $\Gamma_g M \Gamma_g$ with $M \in \Delta_g$ satisfies the condition $(M1)$ and $(M2)$. Thus a linear operator $T(\mathscr{M})$ defined in (6.10 induces naturally the action of the Hecke algebra $\mathscr{H}(\Gamma_g,\Delta_g)$ on $M_\rho(\Gamma_g)$. More precisely, if $\mathscr{N} = \sum_{j=1}^h c_j\Gamma_g M_j \Gamma_g \in \mathscr{H}(\Gamma_g,\Delta_g)$, we define

$$T(\mathscr{N}) = \sum_{j=1}^h c_j T(\Gamma_g M_j \Gamma_g).$$

Then $T(\mathscr{N})$ is an endomorphism of $M_\rho(\Gamma_g)$.

Now we fix a Siegel modular form F in $M_\rho(\Gamma_g)$ which is an eigenform of the Hecke algebra $\mathscr{H}(\Gamma_g,\Delta_g)$. Then we obtain an algebra homomorphism $\lambda_F:\mathscr{H}(\Gamma_g,\Delta_g)\longrightarrow\mathbb{C}$ determined by

$$T(F)=\lambda_F(T)F,\quad T\in\mathscr{H}(\Gamma_g,\Delta_g).$$

By Theorem 6.2 or Theorem 6.3, one has

$$\begin{aligned}\mathscr{H}(\Gamma_g,\Delta_{g,p}) &\cong \mathscr{H}_{g,p}^{\mathbb{Q}}\otimes\mathbb{C}\cong\mathbb{C}[Y]^{W_g}\\ &\cong \mathscr{H}_p(\mathbb{T})^{W_g}\otimes\mathbb{C}\\ &\cong \mathbb{C}\big[(u_1/v_1)^{\pm1},\cdots,(u_g/v_g)^{\pm1},(v_1\cdots v_g)^{\pm1}\big]^{W_g}\\ &\cong \mathbb{C}[Y_0,Y_0^{-1},Y_1,\cdots,Y_g],\end{aligned}$$

where $Y_0,Y_1,\cdots,Y_g$ are algebraically independent. Therefore one obtains an isomorphism

$$\mathrm{Hom}_{\mathbb{C}}\big(\mathscr{H}(\Gamma_g,\Delta_{g,p}),\mathbb{C}\big)\cong\mathrm{Hom}_{\mathbb{C}}\big(\mathscr{H}_{g,p}^{\mathbb{Q}}\otimes\mathbb{C},\mathbb{C}\big)\cong(\mathbb{C}^{\times})^{(g+1)}/W_g.$$

The algebra homomorphism $\lambda_F\in\mathrm{Hom}_{\mathbb{C}}\big(\mathscr{H}(\Gamma_g,\Delta_{g,p}),\mathbb{C}\big)$ is determined by the W_g-orbit of a certain $(g+1)$-tuple $\big(\alpha_{F,0},\alpha_{F,1},\cdots,\alpha_{F,g}\big)$ of nonzero complex numbers, called the *p-Satake parameters* of F. For brevity, we put $\alpha_i=\alpha_{F,i}$, $i=0,1,\cdots,g$. Therefore α_i is the image of u_i/v_i and α_0 is the image of $v_1\cdots v_g$ under the map Θ. Each generator $w_i\in W_{\Delta_g}\cong W_g$ acts by

$$w_j(\alpha_0)=\alpha_0\alpha_j^{-1}\quad w_j(\alpha_j)=\alpha_j^{-1},\quad w_j(\alpha_k)=0\text{ if }k\neq0,j.$$

These p-Satake parameters $\alpha_0,\alpha_1\cdots,\alpha_g$ satisfy the relation

$$\alpha_0^2\alpha_1\cdots\alpha_g=p^{\sum_{i=1}^g k_i-g(g+1)/2}.$$

Formula (6.12) follows from the fact that $T_g(p^2)=\Gamma_g(pI_{2g})\Gamma_g$ is mapped to

$$p^{-g(g+1)/2}\,(v_1\cdots v_g)^2\prod_{i=1}^g(u_i/v_i).$$

We refer to [34, p. 258] for more detail. According to Formula (6.7)-(6.9), the eigenvalues $\lambda_F\big(T(p)\big)$ and $\lambda_F\big(T_i(p^2)\big)$ with $1\le i\le g$ are given respectively by

$$\lambda_F\big(T(p)\big)=\alpha_0(1+E_1+E_2+\cdots+E_g)\tag{6.12}$$

and

(6.13)

$$\lambda_F\big(T_i(p^2)\big)=\sum_{j,k\ge0,\,j+i\le k}^{g}m_{k-j}(i)\,p^{-\binom{k-j+1}{2}}\,\alpha_0^2E_jE_k,\quad i=1,\cdots,g,$$

where E_j denotes the elementary symmetric function of degree j in the variables $\alpha_1,\cdots,\alpha_g$. The point is that the above eigenvalues $\lambda_F\big(T(p)\big)$

and $\lambda_F\big(T_i(p^2)\big)$ $(1 \leq i \leq g)$ are described in terms of the p-Satake parameters $\alpha_0, \alpha_1 \cdots, \alpha_g$.

Examples. (1) Suppose $g(\tau) = \sum_{n\geq 1} a(n)\, e^{2\pi i n\tau}$ is a normalized eigenform in $S_k(\Gamma_1)$. Let p be a prime. Let β be a complex number determined by the relation

$$(1-\beta X)(1-\bar{\beta}X) = 1 - a(p)X + p^{k-1}X^2.$$

Then

$$\beta + \bar{\beta} = a(p) \quad \text{and} \quad \beta\bar{\beta} = p^{k-1}.$$

The p-Satake parameters α_0 and α_1 are given by

$$(\alpha_0, \alpha_1) = \left(\beta, \frac{\bar{\beta}}{\beta}\right) \quad or \quad \left(\bar{\beta}, \frac{\beta}{\bar{\beta}}\right).$$

It is easily checked that $\alpha_0^2\alpha_1 = \beta\bar{\beta} = p^{k-1}$ (cf. Formula (6.12)).

(2) For a positive integer k with $k > g+1$, we let

$$G_k(\Omega) := \sum_{M \in \Gamma_{g,0}\backslash\Gamma_g} \det(C\Omega + D)^k, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be the Siegel Eisenstein series of weight k in $M_k(\Gamma_g)$, where

$$\Gamma_{g,0} := \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma_g \right\}$$

is a parabolic subgroup of Γ_g. It is known that G_k is an eigenform of all the Hecke operators (cf. [34, p. 268]). Let $S_1, \cdots, S_h$ be a complete system of representatives of positive definite even unimodular integral matrices of degree $2k$. If $k > g+1$, the Eisenstein series G_k can be expressed as the weighted mean of theta series $\theta_{S_1}, \cdots, \theta_{S_h}$:

$$G_k(\Omega) = \sum_{\nu=1}^{h} m_\nu\, \theta_{S_\nu}(\Omega), \quad \Omega \in \mathbb{H}_g, \tag{6.14}$$

where

$$m_\nu = \frac{A(S_\nu, S_\nu)^{-1}}{A(S_1,S_1)^{-1} + \cdots + A(S_h,S_h)^{-1}}, \quad 1 \leq \nu \leq h.$$

We recall that the theta series θ_{S_ν} is defined in Formula (5.14) and that for two symmetric integral matrices S of degree m and T of degree n, $A(S,T)$ is defined by

$$A(S,T) := \sharp\{\, G \in \mathbb{Z}^{(m,n)} \mid S[G] = {}^tGSG = T \,\}.$$

Formula (6.14) was obtained by Witt [148] as a special case of the analytic version of Siegel's Hauptsatz.

7. Jacobi Forms

In this section, we establish the notations and define the concept of Jacobi forms.

Let

$$Sp(g,\mathbb{R}) = \{M \in \mathbb{R}^{(2g,2g)} \mid {}^tMJ_gM = J_g\}$$

be the symplectic group of degree g, where

$$J_g := \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

For two positive integers g and h, we consider the *Heisenberg group*

$$H_{\mathbb{R}}^{(g,h)} := \{(\lambda,\mu,\kappa) \mid \lambda,\mu \in \mathbb{R}^{(h,g)},\ \kappa \in \mathbb{R}^{(h,h)},\ \kappa + \mu\,{}^t\lambda \text{ symmetric}\}$$

endowed with the following multiplication law

$$(\lambda,\mu,\kappa) \circ (\lambda',\mu',\kappa) := (\lambda+\lambda',\mu+\mu',\kappa+\kappa'+\lambda\,{}^t\mu'-\mu\,{}^t\lambda').$$

We recall that the Jacobi group $G_{g,h}^J := Sp(g,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$ is the semidirect product of the symplectic group $Sp(g,\mathbb{R})$ and the Heisenberg group $H_{\mathbb{R}}^{(g,h)}$ endowed with the following multiplication law

$$(M,(\lambda,\mu,\kappa))\cdot(M',(\lambda',\mu',\kappa')) := (MM',(\tilde{\lambda}+\lambda',\tilde{\mu}+\mu',\kappa+\kappa'+\tilde{\lambda}\,{}^t\mu'-\tilde{\mu}\,{}^t\lambda'))$$

with $M,M' \in Sp(g,\mathbb{R})$, $(\lambda,\mu,\kappa),(\lambda',\mu',\kappa') \in H_{\mathbb{R}}^{(g,h)}$ and $(\tilde{\lambda},\tilde{\mu}) := (\lambda,\mu)M'$. It is easy to see that $G_{g,h}^J$ acts on the Siegel-Jacobi space $\mathbb{H}_{g,h} := \mathbb{H}_g \times \mathbb{C}^{(h,g)}$ transitively by

$$(7.1) \qquad (M,(\lambda,\mu,\kappa))\cdot(\Omega,Z) := (M\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}),$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R})$, $(\lambda,\mu,\kappa) \in H_{\mathbb{R}}^{(g,h)}$ and $(\Omega,Z) \in \mathbb{H}_{g,h}$.

Let ρ be a rational representation of $GL(g,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbb{R}^{(h,h)}$ be a symmetric half-integral semi-positive definite matrix of degree h. Let $C^\infty(\mathbb{H}_{g,h},V_\rho)$ be the algebra of all C^∞ functions on $\mathbb{H}_{g,h}$ with values in V_ρ. For $f \in C^\infty(\mathbb{H}_{g,h},V_\rho)$, we define

$$\begin{aligned} &\left(f|_{\rho,\mathcal{M}}[(M,(\lambda,\mu,\kappa))]\right)(\Omega,Z) \\ &:= e^{-2\pi i\sigma(\mathcal{M}[Z+\lambda\Omega+\mu](C\Omega+D)^{-1}C)} \times e^{2\pi i\sigma(\mathcal{M}(\lambda\Omega{}^t\lambda+2\lambda{}^tZ+(\kappa+\mu{}^t\lambda)))} \\ &\quad \times\rho(C\Omega+D)^{-1}f(M\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}), \end{aligned}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R})$, $(\lambda,\mu,\kappa) \in H_{\mathbb{R}}^{(g,h)}$ and $(\Omega,Z) \in \mathbb{H}_{g,h}$.

Definition 7.1. Let ρ and $\mathcal{M}$ be as above. Let

$$H_{\mathbb{Z}}^{(g,h)} := \{(\lambda,\mu,\kappa) \in H_{\mathbb{R}}^{(g,h)} \mid \lambda,\mu \in \mathbb{Z}^{(h,g)},\ \kappa \in \mathbb{Z}^{(h,h)}\}.$$

Let Γ be a discrete subgroup of Γ_g of finite index. A *Jacobi form* of index $\mathcal{M}$ with respect to ρ on Γ is a holomorphic function $f \in C^\infty(\mathbb{H}_{g,h}, V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho,\mathcal{M}}[\tilde{\gamma}] = f$ for all $\tilde{\gamma} \in \Gamma^J := \Gamma \ltimes H_{\mathbb{Z}}^{(g,h)}$.

(B) f has a Fourier expansion of the following form :

$$f(\Omega, Z) = \sum_{\substack{T \geq 0 \\ \text{half-integral}}} \sum_{R \in \mathbb{Z}^{(g,h)}} c(T,R) \cdot e^{\frac{2\pi i}{\lambda_\Gamma} \sigma(T\Omega)} \cdot e^{2\pi i \sigma(RZ)}$$

with some nonzero integer $\lambda_\Gamma \in \mathbb{Z}$ and $c(T,R) \neq 0$ only if $\begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2} {}^tR & \mathcal{M} \end{pmatrix} \geq 0$.

If $g \geq 2$, the condition (B) is superfluous by the Köcher principle (cf. [165] Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ. Ziegler(cf. [165] Theorem 1.8 or [29] Theorem 1.1) proves that the vector space $J_{\rho,\mathcal{M}}(\Gamma)$ is finite dimensional. For more results on Jacobi forms with $g > 1$ and $h > 1$, we refer to [112], [149]-[153] and [165].

Definition 7.2. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma)$ is said to be a *cusp* (or *cuspidal*) form if $\begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2} {}^tR & \mathcal{M} \end{pmatrix} > 0$ for any T, R with $c(T,R) \neq 0$. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma)$ is said to be *singular* if it admits a Fourier expansion such that a Fourier coefficient $c(T,R)$ vanishes unless $\det \begin{pmatrix} \frac{1}{\lambda_\Gamma} T & \frac{1}{2} R \\ \frac{1}{2} {}^tR & \mathcal{M} \end{pmatrix} = 0$.

Example 7.3. Let $S \in \mathbb{Z}^{(2k,2k)}$ be a symmetric, positive definite, unimodular even integral matrix and $c \in \mathbb{Z}^{(2k,h)}$. We define the theta series

(7.2)
$$\vartheta_{S,c}^{(g)}(\Omega, Z) := \sum_{\lambda \in \mathbb{Z}^{(2k,g)}} e^{\pi i \{\sigma(S\lambda\Omega\, {}^t\lambda) + 2\sigma({}^tcS\lambda\, {}^tZ)\}}, \quad \Omega \in \mathbb{H}_g,\ Z \in \mathbb{C}^{(h,g)}.$$

We put $\mathcal{M} := \frac{1}{2} {}^t cSc$. We assume that $2k < g + rank\,(\mathcal{M})$. Then it is easy to see that $\vartheta_{S,c}^{(g)}$ is a singular Jacobi form in $J_{k,\mathcal{M}}(\Gamma_g)$(cf. [165] p.212).

Remark. Singular Jacobi forms are characterized by a special differential operator or the weight of the associated rational representation of the general linear group $GL(g, \mathbb{C})$ (cf. [152]).

Now we will make brief historical remarks on Jacobi forms. In 1985, the names Jacobi group and Jacobi forms got kind of standard by the classic book [29] by Eichler and Zagier to remind of Jacobi's "Fundamenta nova theoriae functionum ellipticorum", which appeared in 1829 (cf. [68]). Before [29] these objects appeared more or less explicitly and under different names in the work of many authors. In 1966 Pyatetski-Shapiro [109] discussed the Fourier-Jacobi expansion of Siegel modular forms and the field of modular abelian functions. He gave the dimension of this field in the higher degree. About the same time Satake [119]-[120] introduced the notion of "groups of Harish-Chandra type" which are non reductive but still behave well enough so that he could determine their canonical automorphic factors and kernel functions. Shimura [127]-[128] gave a new foundation of the theory of complex multiplication of abelian functions using Jacobi theta functions. Kuznetsov [86] constructed functions which are almost Jacobi forms from ordinary elliptic modular functions. Starting 1981, Berndt [8]-[10] published some papers which studied the field of arithmetic Jacobi functions, ending up with a proof of Shimura reciprocity law for the field of these functions with arbitrary level. Furthermore he investigated the discrete series for the Jacobi group $G^J_{g,h}$ and developed the spectral theory for $L^2(\Gamma^J\backslash G^J_{g,h})$ in the case $g = h = 1$ (cf. [11]-[13]). The connection of Jacobi forms to modular forms was given by Maass, Andrianov, Kohnen, Shimura, Eichler and Zagier. This connection is pictured as follows. For k even, we have the following isomorphisms

$$M^*_k(\Gamma_2) \cong J_{k,1}(\Gamma_1) \cong M^+_{k-\frac{1}{2}}(\Gamma^{(1)}_0(4)) \cong M_{2k-2}(\Gamma_1).$$

Here $M^*_k(\Gamma_2)$ denotes Maass's Spezialschar or Maass space and $M^+_{k-\frac{1}{2}}(\Gamma^{(1)}_0(4))$ denotes the Kohnen plus space. These spaces shall be described in some more detail in the next section. For a precise detail, we refer to [93]-[95], [1], [29] and [76]. In 1982 Tai [134] gave asymptotic dimension formulae for certain spaces of Jacobi forms for arbitrary g and $h = 1$ and used these ones to show that the moduli $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g is *of general type* for $g \geq 9$. Feingold and Frenkel [31] essentially discussed Jacobi forms in the context of Kac-Moody Lie algebras generalizing the Maass correspondence to higher level. Gritsenko [44] studied Fourier-Jacobi expansions and a non-commutative Hecke ring in connection with the Jacobi group. After 1985 the theory of Jacobi forms for $g = h = 1$ had been studied more or less systematically by the Zagier school. A large part of the theory of Jacobi forms of higher degree was investigated by Kramer [82]-[83], [112], Yang [149]-[153]and Ziegler [165]. There were several attempts to establish L-functions in the context of the Jacobi group by Murase [104]-[105]and Sugano [106] using the so-called "Whittaker-Shintani functions". Kramer [82]-[83] developed an arithmetic theory of Jacobi forms of higher degree. Runge [112] discussed some part of the geometry of Jacobi forms for arbitrary g and $h = 1$. For a good survey on some motivation and background for the study of Jacobi forms, we refer to [14]. The theory of Jacobi forms has been extensively studied by many people until now and has many applications in other areas like geometry and physics.

8. Lifting of Elliptic Cusp forms to Siegel Modular Forms

In this section, we present some results about the liftings of elliptic cusp forms to Siegel modular forms. And we discuss the Duke-Imamoḡlu-Ikeda lift.

In order to discuss these lifts, we need two kinds of L-function or zeta functions associated to Siegel Hecke eigenforms. These zeta functions are defined by using the Satake parameters of their associated Siegel Hecke eigenforms.

Let $F \in M_\rho(\Gamma_g)$ be a nonzero Hecke eigenform on $\mathbb{H}_g$ of type ρ, where ρ is a finite dimensional irreducible representation of $GL(g, \mathbb{C})$ with highest weight $(k_1, \cdots, k_g)$. Let $\alpha_{p,0}, \alpha_{p,1}, \cdots, \alpha_{p,g}$ be the p-Satake parameters of F at a prime p. Using these Satake parameters, we define the *local spinor zeta function* $Z_{F,p}(s)$ of F at p by

$$Z_{F,p}(t) = (1 - \alpha_{p,0}t) \prod_{r=1}^{g} \prod_{1 \le i_1 < \cdots < i_r \le g} (1 - \alpha_{p,0}\alpha_{p,i_1} \cdots \alpha_{p,i_r} t).$$

Now we define the *spinor zeta function* $Z_F(s)$ by

$$Z_F(s) := \prod_{p\,:\,\text{prime}} Z_{F,p}(p^{-s})^{-1}, \quad \text{Re}\, s \gg 0. \tag{8.1}$$

For example, if $g = 1$, the spinor zeta function $Z_f(s)$ of a Hecke eigenform f is nothing but the Hecke L-function $L(f, s)$ of f.

Secondly one has the so-called *standard zeta function* $D_F(s)$ of a Hecke eigenform F in $S_\rho(\Gamma_g)$ defined by

$$D_F(s) := \prod_{p\,:\,\text{prime}} D_{F,p}(p^{-s})^{-1}, \tag{8.2}$$

where

$$D_{F,p}(t) = (1 - t) \prod_{i=1}^{g} (1 - \alpha_{p,i}t)(1 - \alpha_{p,i}^{-1}t).$$

For instance, if $g = 1$, the standard zeta function $D_f(s)$ of a Hecke eigenform $f(\tau) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n\tau}$ in $S_k(\Gamma_1)$ has the following

$$D_f(s - k + 1) = \prod_{p\,:\,\text{prime}} \left(1 + p^{-s+k-1}\right)^{-1} \cdot \sum_{n=1}^{\infty} a(n^2)n^{-s}.$$

For the present time being, we recall the Kohnen plus space and the Maass space. Let $\mathcal{M}$ be a positive definite, half-integral symmetric matrix of degree h. For a fixed element $\Omega \in \mathbb{H}_g$, we denote $\Theta^{(g)}_{\mathcal{M},\Omega}$ the vector space consisting of all the functions $\theta : \mathbb{C}^{(h,g)} \longrightarrow \mathbb{C}$ satisfying the condition:

$$(8.3) \qquad \theta(Z+\lambda\Omega+\mu) = e^{-2\pi i\,\sigma(\mathcal{M}[\lambda]\Omega+2\,{}^tZ\mathcal{M}\lambda)}\theta(Z), \quad Z \in \mathbb{C}^{(h,g)}$$

for all $\lambda, \mu \in \mathbb{Z}^{(h,g)}$. For brevity, we put $L := \mathbb{Z}^{(h,g)}$ and $L_{\mathcal{M}} := L/(2\mathcal{M})L$. For each $\gamma \in L_{\mathcal{M}}$, we define the theta series $\theta_\gamma(\Omega, Z)$ by

$$\theta_\gamma(\Omega, Z) = \sum_{\lambda\in L} e^{2\pi i\,\sigma(\mathcal{M}[\lambda+(2\mathcal{M})^{-1}\gamma]\Omega+2\,{}^tZ\mathcal{M}(\lambda+(2\mathcal{M})^{-1}\gamma))},$$

where $(\Omega, Z) \in \mathbb{H}_g \times \mathbb{C}^{(h,g)}$. Then $\{\,\theta_\gamma(\Omega,Z) \mid \gamma \in L_{\mathcal{M}}\,\}$ forms a basis for $\Theta^{(g)}_{\mathcal{M},\Omega}$. For any Jacobi form $\phi(\Omega, Z) \in J_{k,\mathcal{M}}(\Gamma_g)$, the function $\phi(\Omega,\cdot)$ with fixed Ω is an element of $\Theta^{(g)}_{\mathcal{M},\Omega}$ and $\phi(\Omega, Z)$ can be written as a linear combination of theta series $\theta_\gamma(\Omega, Z)\ (\gamma \in L_{\mathcal{M}})$:

$$(8.4) \qquad \phi(\Omega, Z) = \sum_{\gamma\in L_{\mathcal{M}}} \phi_\gamma(\Omega)\theta_\gamma(\Omega, Z).$$

We observe that $\phi = (\phi_\gamma(\Omega))_{\gamma\in L_{\mathcal{M}}}$ is a vector valued automorphic form with respect to a theta multiplier system.

We now consider the case: $h = 1,\ \mathcal{M} = I_h = 1,\ L = \mathbb{Z}^{(1,g)} \cong \mathbb{Z}^g$. We define the theta series $\theta^{(g)}(\Omega)$ by

$$(8.5) \qquad \theta^{(g)}(\Omega) = \sum_{\lambda\in L} e^{2\pi i\,\sigma(\lambda\Omega\,{}^t\lambda)} = \theta_0(\Omega, 0), \quad \Omega \in \mathbb{H}_g.$$

Let

$$\Gamma_0^{(g)}(4) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \,\middle|\, C \equiv 0 \pmod 4 \right\}$$

be the congruence subgroup of Γ_g. We define the automorphic factor $j : \Gamma_0^{(g)}(4) \times \mathbb{H}_g \longrightarrow \mathbb{C}^\times$ by

$$j(\gamma, \Omega) = \frac{\theta^{(g)}(\gamma\cdot\Omega)}{\theta^{(g)}(\Omega)}, \quad \gamma \in \Gamma_0^{(g)}(4),\ \Omega \in \mathbb{H}_g.$$

Thus one obtains the relation

$$j(\gamma,\Omega)^2 = \varepsilon(\gamma)\,\det(C\Omega + D), \quad \varepsilon(\gamma)^2 = 1,$$

for any $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(g)}(4)$.

Kohnen [76] introduced the so-called *Kohnen plus space* $M^+_{k-\frac{1}{2}}(\Gamma_0^{(g)}(4))$ consisting of holomorphic functions satisfying the following conditions $(K1)$ and $(K2)$:

(K1) $\quad f(\gamma\cdot\Omega) = j(\gamma,\Omega)^{2k-1} f(\Omega) \quad$ for all $\gamma \in \Gamma_0^{(g)}(4)$;

(K2) f has the Fourier expansion

$$f(\Omega) = \sum_{T \geq 0} a(T)\, e^{2\pi i\, \sigma(T\Omega)},$$

where T runs over the set of semi-positive half-integral symmetric matrices of degree g such that $a(T) = 0$ unless $T \equiv -\mu\, {}^t\mu \bmod 4S_g^*(\mathbb{Z})$ for some $\mu \in \mathbb{Z}^{(g,1)}$. Here we put

$$S_g^*(\mathbb{Z}) = \left\{ T \in \mathbb{R}^{(g,g)} \mid T = {}^tT,\ \sigma(TS) \in \mathbb{Z} \ \text{ for all } S = {}^tS \in \mathbb{Z}^{(g,g)} \right\}.$$

For a Jacobi form $\phi \in J_{k,1}(\Gamma_g)$, according to Formula (8.4), one has

$$\phi(\Omega, Z) = \sum_{\gamma \in L/2L} f_\gamma(\Omega)\, \theta_\gamma(\Omega, Z), \quad \Omega \in \mathbb{H}_g,\ Z \in \mathbb{C}^{(h,g)}. \tag{8.6}$$

Now we put

$$f_\phi(\Omega) := \sum_{\gamma \in L/2L} f_\gamma(4\Omega), \quad \Omega \in \mathbb{H}_g.$$

Then $f_\phi \in M^+_{k-\frac{1}{2}}\left(\Gamma_0^{(g)}(4)\right)$.

Theorem 8.1. ***(Kohnen-Zagier (g=1), Ibukiyama*** *$(g > 1)$) Suppose k is an even positive integer. Then there exists the isomorphism given by*

$$J_{k,1}(\Gamma_g) \cong M^+_{k-\frac{1}{2}}\left(\Gamma_0^{(g)}(4)\right), \quad \phi \mapsto f_\phi.$$

Moreover the isomorphism is compatible with the action of Hecke operators.

For a positive integer $k \in \mathbb{Z}^+$, H. Maass [93, 94, 95] introduced the so-called *Maass space* $M_k^*(\Gamma_2)$ consisting of all Siegel modular forms $F(\Omega) = \sum_{g \geq 0} a_F(T)\, e^{2\pi i\, \sigma(T\Omega)}$ on $\mathbb{H}_2$ of weight k satisfying the following condition

$$a_F(T) = \sum_{d \mid (n,r,m),\, d>0} d^{k-1}\, a_F\begin{pmatrix} \frac{dm}{d^2} & \frac{r}{2d} \\ \frac{r}{2d} & 1 \end{pmatrix} \tag{8.7}$$

for all $T = \begin{pmatrix} n & \frac{r}{2} \\ \frac{r}{2} & m \end{pmatrix} \geq 0$ with $n, r, m \in \mathbb{Z}$. For $F \in M_k(\Gamma_2)$, we let

$$F(\Omega) = \sum_{m \geq 0} \phi_m(\tau, z)\, e^{2\pi i m \tau'}, \quad \Omega = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathbb{H}_2$$

be the Fourier-Jacobi expansion of F. Then for any nonnegative integer m, we obtain the linear map

$$\rho_m : M_k(\Gamma_2) \longrightarrow J_{k,m}(\Gamma_1), \qquad F \mapsto \phi_m.$$

We observe that ρ_0 is nothing but the Siegel Φ-operator. Maass [93, 94, 95] showed that for k even, there exists a nontrivial map $V : J_{k,1}(\Gamma_1) \longrightarrow$

$M_k(\Gamma_2)$ such that $\rho_1 \circ V$ is the identity. More precisely, we let $\phi \in J_{k,1}(\Gamma_1)$ be a Jacobi form with Fourier coefficients $c(n,r)\,(n,r \in \mathbb{Z},\ r^2 \leq 4n)$ and define for any nonnegative integer $m \geq 0$

(8.8)
$$\big(V_m\phi\big)(\tau,z) = \sum_{n,r\in\mathbb{Z},\, r^2\leq 4mn} \left(\sum_{d|(n,r,m)} d^{k-1}\, c\left(\frac{mn}{d^2},\frac{r}{d}\right)\right) e^{2\pi i(n\tau+rz)}.$$

It is easy to see that $V_1\phi = \phi$ and $V_m\phi \in J_{k,m}(\Gamma_1)$. We define

$$(V\phi)(\Omega) = \sum_{m\geq 0} (V_m\phi)(\tau,z)\, e^{2\pi i m\tau'}, \quad \Omega = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathbb{H}_2. \tag{8.9}$$

We denote by $T_n\,(n \in \mathbb{Z}^+)$ the usual Hecke operators on $M_k(\Gamma_2)$ resp. $S_k(\Gamma_2)$. For instance, if p is a prime, $T_p = T(p)$ and $T_{p^2} = T_1(p^2)$. We denote by $T_{J,n}\,(m \in \mathbb{Z}^+)$ the Hecke operators on $J_{k,m}(\Gamma_1)$ resp. $J_{k,m}^{\rm cusp}(\Gamma_1)$ (cf. [29]).

Theorem 8.2. (***Maass***[92, 93, 94]***,Eichler-Zagier***[29]***, Theorem 6.3***) *Suppose k is an even positive integer. Then the map $\phi \mapsto V\phi$ gives an isomorphism of $J_{k,m}(\Gamma_1)$ onto $M_k^*(\Gamma_2)$ which sends cusp Jacobi forms to cusp forms and is compatible with the action of Hecke operators. If p is a prime, one has*

$$T_p \circ V = V \circ \big(T_{J,p} + p^{k-2}(p+1)\big)$$

and

$$T_{p^2} \circ V = V \circ \big(T_{J,p}^2 + p^{k-2}(p+1)T_{J,p} + p^{2k-2}\big).$$

In summary, we have the following isomorphisms

$$\begin{array}{rcccl} M_k^*(\Gamma_2) & \cong J_{k,m}(\Gamma_1) & \cong M_{k-\frac{1}{2}}^+\big(\Gamma_0^{(1)}(4)\big) \cong M_{2k-2}(\Gamma_1), \\ V_\phi & \longleftarrow \quad \phi & \longrightarrow f_\phi \end{array} \tag{8.10}$$

where the last isomorphism is the Shimura correspondence. All the above isomorphisms are compatible with the action of Hecke operators.

In 1978, providing some evidences, Kurokawa and Saito conjectured that there is a one-to-one correspondence between Hecke eigenforms in $S_{2k-2}(\Gamma_1)$ and Hecke eigenforms in $M_k(\Gamma_2)$ satisfying natural identity between their spinor zeta functions. This was solved mainly by Maass and then completely solved by Andrianov [1] and Zagier [164].

Theorem 8.3. *Suppose k is an even positive integer and let $F \in M_k^*(\Gamma_2)$ be a nonzero Hecke eigenform. Then there exists a unique normalized Hecke eigenform f in $M_{2k-2}(\Gamma_1)$ such that*

$$Z_F(s) = \zeta(s-k+1)\,\zeta(s-k+2)L(f,s), \tag{8.11}$$

where $L(f,s)$ is the Hecke L-function attached to f.

F is called the *Saito-Kurokawa lift* of f. Theorem 8.3 implies that $Z_F(s)$ has a pole at $s = k$ if F is an eigenform in $M_k^*(\Gamma_2)$. If $F \in S_k(\Gamma_2)$ is a Hecke eigenform, it was proved by Andrianov [2] that $Z_F(s)$ has an analytic continuation to the whole complex plane which is holomorphic everywhere if k is odd and is holomorphic except for a possible simple pole at $s = k$ if k is even. Moreover the global function

$$Z_F^*(s) := (2\pi)^{-s}\Gamma(s)\Gamma(s-k+2)Z_F(s)$$

is $(-1)^k$-invariant under $s \mapsto 2k-2-s$. It was proved that Evdokimov and Oda that $Z_F(s)$ is holomorphic if and only if F is contained in the orthogonal complement of $M_k^*(\Gamma_2)$ in $M_k(\Gamma_2)$. We remark that $M_k(\Gamma_2) = \mathbb{C}\, G_k \oplus S_k^*(\Gamma_2)$, where G_k is the Siegel Eisenstein series of degree 2 (cf. (6.14)) and $S_k^*(\Gamma_2) = S_k(\Gamma_2) \cap M_k^*(\Gamma_2)$.

Around 1996, Duke and Imamoğlu [26] conjectured a generalization of Theorem 7.3. More precisely, they formulated the conjecture that if f is a normalized Hecke eigenform in $S_{2k}(\Gamma_1)$ $(k \in \mathbb{Z}^+)$ and n is a positive integer with $n \equiv k \pmod 2$, then there exists a Hecke eigenform F in $S_{k+n}(\Gamma_{2n})$ such that the standard zeta function $D_F(s)$ of F equals

$$\zeta(s)\sum_{j=1}^{2n} L(f, s+k+n-j), \tag{8.12}$$

where $L(f,s)$ is the Hecke L-function of f. Later some evidence for this conjecture was given by Breulmann and Kuss [18]. In 1999, Ikeda [66] proved that the conjecture of Duke and Imamoğlu is true. Such a Hecke eigenform F in $S_{k+n}(\Gamma_{2n})$ is called the *Duke-Imamoğlu-Ikeda lift* of a normalized Hecke eigenform f in $S_{2k}(\Gamma_1)$.

Now we describe the work of Tamotsu Ikeda roughly. First we introduce some notations and recall some definitions. A symmetric square matrix A with entries a_{ij} in the quotient field of an integral domain R will be said to be *half integral* if $a_{ii} \in R$ for all i and $2a_{ij} \in R$ for all i, j with $i \neq j$. We denote by $\mathcal{S}_n(R)$ the set of all such symmetric half integral matrices of degree n. For a rational, half integral symmetric, non-degenerate matrix $T \in \mathcal{S}_{2n}(\mathbb{Q})$, we denote by

$$D_T := (-1)^n \det(2T)$$

the discriminant of T. We write

$$D_T = D_{T,0} f_T^2$$

with $D_{T,0}$ the corresponding fundamental discriminant and $f_T \in \mathbb{Z}^+$.

Fix a prime p. Let T be a non-degenerate matrix in $\mathcal{S}_{2n}(\mathbb{Z}_p)$. Then the local singular series of T at p is defined as

$$b_p(T;s) := \sum_R \nu_p(R)^{-s}\, e_p(\sigma(TR)), \qquad s \in \mathbb{C},$$

where R runs over all symmetric $2n \times 2n$ matrices with entries in $\mathbb{Q}_p/\mathbb{Z}_p$ and $\nu_p(R)$ is a power of p equal to the product of denominators of elementary divisors of R. Furthermore, for $x \in \mathbb{Q}_p$ we have put $e_p(x) = e^{2\pi i x'}$, where x' denotes the fractional part of x.

As is well known, $b_p(T;s)$ is a product of two polynomials in p^{-s} with coefficients in $\mathbb{Z}$. More precisely, we put

$$\gamma_p(T;X) := (1-X)\big(1-\xi_p(T)p^nX\big)^{-1}\prod_{j=1}^{n}\big(1-p^{2j}X^2\big),$$

where

$$\xi_p(T) := \chi_p\big((-1)^n \det T\big)$$

and for $a \in \mathbb{Q}_p^*$, $\chi_p(a)$ is defined by

$$\chi_p(a) = \begin{cases} 1 & \text{if } \ \mathbb{Q}_p(\sqrt{a}) = \mathbb{Q}_p, \\ -1 & \text{if } \ \mathbb{Q}_p(\sqrt{a})/\mathbb{Q}_p \text{ is unramified,} \\ 0 & \text{if } \ \mathbb{Q}_p(\sqrt{a})/\mathbb{Q}_p \text{ is ramified.} \end{cases}$$

Then we have

$$b_p(T;s) = \gamma_p(T;p^{-s})\,F_p(T;p^{-s}),$$

where $F_p(T;X)$ is a certain polynomial in $\mathbb{Z}[X]$ with constant term 1. A fundamental result of Katsurada [71] states that the Laurent polynomial

$$\widetilde{F}_p(T;X) := X^{-\mathrm{ord}_p f_T} F_p(T; p^{-n-1/2}X)$$

is symmetric, i.e.,

$$\widetilde{F}_p(T;X) = \widetilde{F}_p(T;X^{-1}),$$

where ord_p denotes the usual p-adic valuation on $\mathbb{Q}$. If p does not divide f_T, then $F_p(T;X) = \widetilde{F}_p(T;X) = 1$. We denote by $V = \big(\mathbb{F}_p^{2n}, q\big)$ the quadratic space over $\mathbb{F}_p$, where q is the quadratic form obtained from the quadratic form $x \mapsto T[x]$ $(x \in \mathbb{Z}_p^{2n})$ by reducing modulo p. We let $\langle\ ,\ \rangle$ be the associated bilinear form on $\mathbb{F}_p^{2n}$ given by

$$\langle x,y\rangle := q(x+y)-q(x)-q(y), \quad x,y \in \mathbb{F}_p^{2n}$$

and let

$$R(V) := \big\{ x \in \mathbb{F}_p^{2n} \mid \langle x,y\rangle = 0 \ \text{ for all } y \in \mathbb{F}_p^{2n},\ q(x)=0 \big\}$$

be the radical of V. We put $s_p := s_p(T) = \dim R(V)$ and denote by W an orthogonal complementary subspace of $R(V)$.

Following [72], one defines a polynomial $H_{n,p}(T;X)$ by

$$H_{n,p}(T;X) = \begin{cases} 1 & \text{if} s_p = 0, \\ \prod_{j=1}^{[(s_p-1)/2]}\big(1-p^{2j-1}X^2\big), & \text{if} s_p > 0,\, s_p \text{ odd}, \\ \big(1+\lambda_p(T)p^{(s_p-1)/2}X\big)\prod_{j=1}^{[(s_p-1)/2]}\big(1-p^{2j-1}X^2\big), & \text{if} s_p > 0,\, s_p \text{ even}, \end{cases}$$

where $[x]$ denotes the Gauss bracket of a real number x, and for s_p *even* we have put

$$\lambda_p(T) := \begin{cases} 1 & \text{if } W \text{ is a hyperbolic subspace or } s_p = 2n, \\ -1 & \text{otherwise.} \end{cases}$$

Following [77], for $\mu \in \mathbb{Z}$, $\mu \geq 0$, we define $\rho_T(p^\mu)$ by

$$\sum_{\mu \geq 0} \rho_T(p^\mu) X^\mu := \begin{cases} (1 - X^2)\, H_{n,p}(T; X) & \text{if } p \,|\, f_T, \\ 1 & \text{otherwise.} \end{cases}$$

We extend the function ρ_T multiplicatively to $\mathbb{Z}^+$ by defining

$$\sum_{a \geq 1} \rho_T(a)\, a^{-s} := \prod_{p \,|\, f_T} \left((1 - p^{-2s})\, H_{n,p}(T; p^{-s}) \right).$$

Let

$$\mathcal{D}(T) := GL_{2n}(\mathbb{Z}) \backslash \{ G \in M_{2n}(\mathbb{Z}) \cap GL_{2n}(\mathbb{Q}) \mid T[G^{-1}] \text{ half integral} \},$$

where $GL_{2n}(\mathbb{Z})$ acts by left multiplication. We see easily that $\mathcal{D}(T)$ is finite. For $a \in \mathbb{Z}^+$ with $a \,|\, f_T$, we define

$$\phi(a; T) := \sqrt{a} \sum_{d^2 \,|\, a} \; \sum_{G \in \mathcal{D}(T),\, |\det(G)| = d} \rho_{T[G^{-1}]} \left(\frac{a}{d^2} \right).$$

We observe that $\phi(a; T) \in \mathbb{Z}$ for all a.

Let f be a normalized Hecke eigenform in $S_{2k}(\Gamma_1)$. For a prime p, we let $\lambda(p)$ and α_p be the p-th Fourier coefficient and the Satake p-parameter of f respectively. Therefore one has

$$1 - \lambda(p) X + p^{2k-1} X^2 = \left(1 - p^{k-1/2} \alpha_p X\right)\left(1 - p^{k-1/2} \alpha_p^{-1} X\right).$$

Let

$$g(\tau) = \sum_{m \geq 1,\, (-1)^k m \equiv 0,1 (\mathrm{mod}\, 4)} c(m)\, e^{2\pi i m \tau}, \qquad \tau \in \mathbb{H}_1$$

be a Hecke eigenform in $S^+_{k+\frac{1}{2}}\left(\Gamma_0^{(1)}(4)\right)$ which corresponds to f under the Shimura isomorphism (8.10). Now we assume that n is a positive integer satisfying the condition $n \equiv k \pmod 2$. For a rational, half integral symmetric positive definite matrix T of degree $2n$, we define

$$a_f(T) := c(|D_{T,0}|)\, f_T^{k-\frac{1}{2}} \prod_{p | f_T} \tilde{F}_p(T; \alpha_p).$$

We consider the function $F(\Omega)$ defined by

$$F(\Omega) = \sum_{T > 0} a_f(T)\, e^{2\pi i\, \sigma(T\Omega)},$$

where T runs over all rational, half integral symmetric positive definite matrices of degree $2n$. Ikeda [66] proved that $F(\Omega)$ is a cuspidal Siegel-Hecke eigenform in $S_{k+n}(\Gamma_{2n})$ and the standard zeta function $D_F(s)$ of F is given by the formula (8.12). Therefore we have the mapping

$$I_{k,n} : S^+_{k+\frac{1}{2}}(\Gamma_0^{(1)}(4)) \longrightarrow S_{k+n}(\Gamma_{2n}) \tag{8.13}$$

defined by

$$g(\tau) = \sum_{(-1)^k m \equiv 0,1 (\mathrm{mod}\, 4)} c(m)e^{2\pi i m\tau} \longmapsto F(\Omega) = \sum_{T>0} A(T)\, e^{2\pi i\, \sigma(T\Omega)},$$

where T runs over all rational, half integral symmetric positive definite matrices of degree $2n$ and

$$A(T) = c(|D_{T,0}|)\, f_T^{k-\frac{1}{2}} \prod_{p|f_T} \tilde{F}_p(T;\alpha_p).$$

The mapping $I_{k,n}$ is called the *Ikeda's lift map*. Kohnen [77] showed the following identity

$$a_f(T) = \sum_{a|f_T} a^{k-1}\phi(a;T)\, c(|D_T|/a^2).$$

Kohnen and Kojima [78] characterized the image $S^*_{k+n}(\Gamma_{2n})$ of the Ikeda's lift map $I_{k,n}$ as follows:

Theorem 8.4. ***(Kohnen-Kojima** [78]**)*** *Suppose that $n \equiv 0,1$ (mod 4) and let $k \in \mathbb{Z}^+$ with $n \equiv k$ (mod 2). Let $F \in S_{k+n}(\Gamma_{2n})$ with Fourier coefficient $A(T)$. Then the following statements are equivalent:*

(a) $F \in S^*_{k+n}(\Gamma_{2n})$;

(b) there exist complex numbers $c(m)$ (with $m \in \mathbb{Z}^+$, and $(-1)^k m \equiv 0,1$ (mod 4)) such that

$$A(T) = \sum_{a|f_T} a^{k-1}\phi(a;T)\, c(|D_T|/a^2)$$

for all T.

They called the image of $I_{k,n}$ in $S_{k+n}(\Gamma_{2n})$ the *Maass space.* If $n = 1$, $M^*_k(\Gamma_2)$ coincides with the image of $I_{k,1}$. Thus this generalizes the original Maass space. Breulmann and Kuss [18] dealt with the special case of the lift map $I_{6,2} : S_{12}(\Gamma_1)(\cong S^+_{13/2}) \longrightarrow S_8(\Gamma_4)$. In the article [17], starting with the Leech lattice Λ, the authors constructed a nonzero Siegel cusp form of degree 12 and weight 12 which is the image of a cusp form $\Delta \in S_{12}(\Gamma_1)$ under the Ikeda lift map $I_{6,6}$. Here Δ is the cusp form in $S_{12}(\Gamma_1)$ defined by

$$\Delta(\tau) = (2\pi)^{12}\, q \prod_{n=1}^{\infty}(1-q^n)^{24}, \quad \tau \in \mathbb{H}_1,\ q = e^{2\pi i\tau}.$$

It is known that there exist 24 Niemeier lattices of rank 24, say, $L_1, \cdots, L_{24}$. The theta series

$$\theta_{L_i}(\Omega) = \sum_{G \in \mathbb{Z}^{(24,12)}} e^{2\pi i\, \sigma(L_i[G]\Omega)}, \quad \Omega \in \mathbb{H}_{12}, \quad i = 1, \cdots, 24$$

generate a subspace V_* of $M_{12}(\Gamma_{12})$. These θ_{L_i} $(1 \leq i \leq 24)$ are linearly independent. It can be seen that the intersection $V_* \cap S_{12}(\Gamma_{12})$ is one dimensional. This nontrivial cusp form in $V_* \cap S_{12}(\Gamma_{12})$ up to constant is just the Siegel modular form constructed by them. Under the assumption $n + r \equiv k \,(\mathrm{mod}\, 2)$ with $k, n, r \in \mathbb{Z}^+$, using the lift map $I_{k,n+r} : S^+_{k+\frac{1}{2}} \longrightarrow S_{k+n+r}(\Gamma_{2n+2r})$, recently Ikeda [67] constructed the following map

$$J_{k,n,r} : S^+_{k+\frac{1}{2}} \times S_{k+n+r}(\Gamma_r) \longrightarrow S_{k+n+r}(\Gamma_{2n+r}) \tag{8.14}$$

defined by

$$J_{k,n,r}(h, G)(\Omega) := \int_{\Gamma_r \backslash \mathbb{H}_r} I_{k,n+r}(h) \left(\begin{pmatrix} \Omega & 0 \\ 0 & \tau \end{pmatrix} \right) \overline{G^c(\tau)} \left(\det \operatorname{Im} \tau \right)^{k+n-1} d\tau,$$

where $h \in S^+_{k+\frac{1}{2}}$, $G \in S_{k+n+r}(\Gamma_r)$, $\Omega \in \mathbb{H}_{2n+r}$, $\tau \in \mathbb{H}_r$, $G^c(\tau) = \overline{G(-\overline{\tau})}$ and $(\det \operatorname{Im} \tau)^{-(r+1)} d\tau$ is an invariant volume element (cf. §2 (2.3)). He proved that the standard zeta function $D_{J_{k,n,r}(h,G)}(s)$ of $J_{k,n,r}(h, G)$ is equal to

$$D_{J_{k,n,r}(h,G)}(s) = D_G(s) \prod_{j=1}^{n} L(f, s + k + n - j),$$

where f is the Hecke eigenform in $S_{2k}(\Gamma_1)$ corresponding to $h \in S^+_{k+\frac{1}{2}}$ under the Shimura correspondence.

Question: Can you describe a geometric interpretation of the Duke-Imamoḡlu-Ikeda lift or the map $J_{k,n,r}$?

9. Holomorphic Differential Forms on Siegel Space

In this section, we describe the relationship between Siegel modular forms and holomorphic differential forms on the Siegel space. We also discuss the Hodge bundle. First of all we need to know the theory of toroidal compactifications of the Siegel space. We refer to [5, 107, 140] for the detail on toroidal compactifications of the Siegel space.

For a neat arithmetic subgroup Γ, e.g., $\Gamma = \Gamma_g(n)$ with $n \geq 3$, we can obtain a smooth projective toroidal compactification of $\Gamma \backslash \mathbb{D}_g$. The theory of toroidal compactifications of bounded symmetric domains was developed by Mumford's school (cf. [5] and [107]). We set

$$\mathcal{A}_g := \Gamma_g\backslash\mathbb{H}_g \quad \text{and} \quad \mathcal{A}_g^* := \Gamma_g\backslash\mathbb{H}_g^* = \bigcup_{0\leq i\leq g} \Gamma_i\backslash\mathbb{H}_i \quad \text{(disjoint union)}.$$

I. Satake [117] showed that $\mathcal{A}_g^*$ is a normal analytic space and W. Baily [6] proved that $\mathcal{A}_g^*$ is a projective variety. Let $\tilde{\mathcal{A}}_g$ be a toroidal compactification of $\mathcal{A}_g$. Then the boundary $\tilde{\mathcal{A}}_g - \mathcal{A}_g$ is a divisor with normal crossings and one has a universal semi-abelian variety over $\tilde{\mathcal{A}}_g$ in the orbifold. We refer to [59] for the geometry of $\mathcal{A}_g$.

Let θ be the second symmetric power of the standard representation of $GL(g,\mathbb{C})$. For brevity we set $N = \frac{1}{2}g(g+1)$. For an integer p with $0 \leq p \leq N$, we denote by $\theta^{[p]}$ the p-th exterior power of θ. For any integer q with $0 \leq q \leq N$, we let $\Omega^q(\mathbb{H}_g)^{\Gamma_g}$ be the vector space of all Γ_g-invariant holomorphic q-forms on $\mathbb{H}_g$. Then we obtain an isomorphism

$$\Omega^q(\mathbb{H}_g)^{\Gamma_g} \longrightarrow M_{\theta^{[q]}}(\Gamma_g).$$

Theorem 9.1. ***(Weissauer*** [143]***)*** *For an integer α with $0 \leq \alpha \leq g$, we let ρ_α be the irreducible representation of $GL(g,\mathbb{C})$ with the highest weight*

$$(g+1,\cdots,g+1,g-\alpha,\cdots,g-\alpha)$$

such that corank$(\rho_\alpha) = \alpha$ for $1 \leq \alpha \leq g$. If $\alpha = -1$, we let $\rho_\alpha = (g+1,\cdots,g+1)$. Then

$$\Omega^q(\mathbb{H}_g)^{\Gamma_g} = \begin{cases} M_{\rho_\alpha}(\Gamma_g) & \text{if } q = \frac{g(g+1)}{2} - \frac{\alpha(\alpha+1)}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Remark. If $2\alpha > g$, then any $f \in M_{\rho_\alpha}(\Gamma_g)$ is singular (cf. Theorem 5.4). Thus if $q < \frac{g(3g+2)}{8}$, then any Γ_g-invariant holomorphic q-form on $\mathbb{H}_g$ can be expressed in terms of vector valued theta series with harmonic coefficients. It can be shown with a suitable modification that the just mentioned statement holds for a sufficiently small congruence subgroup of Γ_g.

Thus the natural question is to ask how to determine the Γ_g-invariant holomorphic p-forms on $\mathbb{H}_g$ for the nonsingular range $\dfrac{g(3g+2)}{8} \leq p \leq \dfrac{g(g+1)}{2}$. Weissauer [144] answered the above question for $g = 2$. For $g > 2$, the above question is still open. It is well known that the vector space of vector valued modular forms of type ρ is finite dimensional. The computation or the estimate of the dimension of $\Omega^p(\mathbb{H}_g)^{\Gamma_g}$ is interesting because its dimension is finite even though the quotient space $\mathcal{A}_g$ is noncompact.

Example 1. Let

$$\varphi = \sum_{i \le j} f_{ij}(\Omega)\, d\omega_{ij} \tag{9.1}$$

be a Γ_g-invariant holomorphic 1-form on $\mathbb{H}_g$. We put

$$f(\Omega) = \big(f_{ij}(\Omega)\big) \text{ with } f_{ij} = f_{ji} \quad \text{and} \quad d\Omega = (d\omega_{ij}).$$

Then f is a matrix valued function on $\mathbb{H}_g$ satisfying the condition

$$f(\gamma\cdot\Omega) = (C\Omega+D) f(\Omega)\, {}^t(C\Omega+D) \text{ for all } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \text{ and } \Omega \in \mathbb{H}_g.$$

This implies that f is a Siegel modular form in $M_\theta(\Gamma_g)$, where θ is the irreducible representation of $GL(g,\mathbb{C})$ on $T_g = \mathrm{Symm}^2(\mathbb{C}^g)$ defined by

$$\theta(h)v = h\, v\, {}^t h, \quad h \in GL(g,\mathbb{C}),\ v \in T_g.$$

We observe that (9.6) can be expressed as $\varphi = \sigma(f\, d\Omega)$.

Example 2. Let

$$\omega_0 = d\omega_{11} \wedge d\omega_{12} \wedge \cdots \wedge d\omega_{gg}$$

be a holomorphic N-form on $\mathbb{H}_g$. If $\omega = f(\Omega)\,\omega_0$ is Γ_g-invariant, it is easily seen that

$$f(\gamma\cdot\Omega) = \det(C\Omega+D)^{g+1} f(\Omega) \quad \text{for all } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \text{ and } \Omega \in \mathbb{H}_g.$$

Thus $f \in M_{g+1}(\Gamma_g)$. It was shown by Freitag [34] that ω can be extended to a holomorphic N-form on $\tilde{\mathcal{A}}_g$ if and only if f is a cusp form in $S_{g+1}(\Gamma_g)$. Indeed, the mapping

$$S_{g+1}(\Gamma_g) \longrightarrow \Omega^N\big(\tilde{\mathcal{A}}_g\big) = H^0\big(\tilde{\mathcal{A}}_g, \Omega^N\big), \qquad f \mapsto f\,\omega_0$$

is an isomorphism. Let $\omega_k = F(\Omega)\,\omega_0^{\otimes k}$ be a Γ_g-invariant holomorphic form on $\mathbb{H}_g$ of degree kN. Then $F \in M_{k(g+1)}(\Gamma_g)$.

Example 3. We set

$$\eta_{ab} = \epsilon_{ab} \bigwedge_{\substack{1 \le \mu \le \nu \le g \\ (\mu,\nu) \ne (a,b)}} d\omega_{\mu\nu}, \quad 1 \le a \le b \le g,$$

where the signs ϵ_{ab} are determined by the relations $\epsilon_{ab}\,\eta_{ab} \wedge d\omega_{ab} = \omega_0$. We assume that

$$\eta_* = \sum_{1 \le a \le b \le g} F_{ab}\, \eta_{ab}$$

is a Γ_g-invariant holomorphic $(N-1)$-form on $\mathbb{H}_g$. Then the matrix valued function $F = \big(\epsilon_{ab}\, F_{ab}\big)$ with $\epsilon_{ab} = \epsilon_{ba}$ and $F_{ab} = F_{ba}$ is an element of $M_\tau(\Gamma_g)$, where τ is the irreducible representation of $GL(g,\mathbb{C})$ on T_g defined by

$$\tau(h)v = (\det h)^{g+1}\, {}^t h^{-1} v h^{-1}, \quad h \in GL(g,\mathbb{C}),\ v \in T_g.$$

We will mention the results due to Weissauer [144]. We let Γ be a congruence subgroup of Γ_2. According to Theorem 9.1, Γ-invariant holomorphic forms in $\Omega^2(\mathbb{H}_2)^\Gamma$ are corresponded to modular forms of type (3,1). We note that these invariant holomorphic 2-forms are contained in the *nonsingular range*. And if these modular forms are not cusp forms, they are mapped under the Siegel Φ-operator to cusp forms of weight 3 with respect to some congruence subgroup (dependent on Γ) of the elliptic modular group. Since there are finitely many cusps, it is easy to deal with these modular forms in the adelic version. Observing these facts, he showed that any 2-holomorphic form on $\Gamma\backslash\mathbb{H}_2$ can be expressed in terms of theta series with harmonic coefficients associated to binary positive definite quadratic forms. Moreover he showed that $H^2(\Gamma\backslash\mathbb{H}_2,\mathbb{C})$ has a pure Hodge structure and that the Tate conjecture holds for a suitable compactification of $\Gamma\backslash\mathbb{H}_2$. If $g \geq 3$, for a congruence subgroup Γ of Γ_g it is difficult to compute the cohomology groups $H^*(\Gamma\backslash\mathbb{H}_g,\mathbb{C})$ because $\Gamma\backslash\mathbb{H}_g$ is noncompact and highly singular. Therefore in order to study their structure, it is natural to ask if they have pure Hodge structures or mixed Hodge structures.

We now discuss the Hodge bundle on the Siegel modular variety $\mathcal{A}_g$. For simplicity we take $\Gamma = \Gamma_g(n)$ with $n \geq 3$ instead of Γ_g. We recall that $\Gamma_g(n)$ is a congruence subgroup of Γ_g consisting of matrices $M \in \Gamma_g$ such that $M \equiv I_{2g} \pmod{n}$. Let

$$\mathfrak{X}_g(n) := \Gamma_g(n) \ltimes \mathbb{Z}^{2g}\backslash\mathbb{H}_g \times \mathbb{C}^g$$

be a family of abelian varieties of dimension g over $\mathcal{A}_g(n) := \Gamma_g(n)\backslash\mathbb{H}_g$. We recall that $\Gamma_g(n) \ltimes \mathbb{Z}^{2g}$ acts on $\mathbb{H}_g \times \mathbb{C}^g$ freely by

$$\big(\gamma,(\lambda,\mu)\big)\cdot(\Omega,Z) = \big(\gamma\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}\big),$$

where $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(n),\ \lambda,\mu \in \mathbb{Z}^g,\ \Omega \in \mathbb{H}_g$ and $Z \in \mathbb{C}^g$. If we insist on using Γ_g, we need to work with orbifolds or stacks to have a universal family

$$\mathfrak{X}_g := \mathfrak{X}_g(n)/Sp(g,\mathbb{Z}/n\mathbb{Z})$$

available. We observe that $\Gamma_g(n)$ acts on $\mathbb{H}_g$ freely. Therefore we obtain a vector bundle $\mathbb{E} = \mathbb{E}_g$ over $\mathcal{A}_g(n)$ of rank g

$$\mathbb{E} = \mathbb{E}_g := \Gamma_g(n)\backslash\big(\mathbb{H}_g \times \mathbb{C}^g\big).$$

This bundle $\mathbb{E}$ is called the *Hodge bundle* over $\mathcal{A}_g(n)$. The finite group $Sp(g,\mathbb{Z}/n\mathbb{Z})$ acts on $\mathbb{E}$ and a $Sp(g,\mathbb{Z}/n\mathbb{Z})$-invariant section of $(\det\mathbb{E})^{\otimes k}$

with a positive integer k comes from a Siegel modular form of weight k in $M_k(\Gamma_g)$. The canonical line bundle $\kappa_g(n)$ of $\mathcal{A}_g(n)$ is isomorphic to $(\det \mathbb{E})^{\otimes(g+1)}$. A holomorphic section of $\kappa_g(n)$ corresponds to a Siegel modular form in $M_{g+1}(\Gamma_g(n))$ (cf. Example 2). We note that the sheaf $\Omega^1_{\mathcal{A}_g(n)}$ of holomorphic 1-forms on $\mathcal{A}_g(n)$ is isomorphic to $\mathrm{Symm}^2(\mathbb{E})$. This sheaf can be extended over a toroidal compactification $\tilde{\mathcal{A}}_g$ of $\mathcal{A}_g$ to an isomorphism

$$\Omega^1_{\tilde{\mathcal{A}}_g}(\log D) \cong \mathrm{Symm}^2(\mathbb{E}),$$

where the boundary $D = \tilde{\mathcal{A}}_g - \mathcal{A}_g$ is the divisor with normal crossings. Similarly to each finite dimensional representation (ρ, V_ρ) of $GL(g,\mathbb{C})$, we may associate the vector bundle

$$\mathbb{E}_\rho := \Gamma_g(n)\backslash(\mathbb{H}_g \times V_\rho)$$

by identifying (Ω, v) with $(\gamma\cdot\Omega, \rho(C\Omega + D)v)$, where $\Omega \in \mathbb{H}_g,\ v \in V_\rho$ and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(n)$. Obviously $\mathbb{E}_\rho$ is a holomorphic vector bundle over $\mathcal{A}_g(n)$ of rank $\dim V_\rho$.

10. Subvarieties of the Siegel Modular Variety

Here we assume that the ground field is the complex number field $\mathbb{C}$.

Definition 9.1. A nonsingular variety X is said to be *rational* if X is birational to a projective space $\mathbb{P}^n(\mathbb{C})$ for some integer n. A nonsingular variety X is said to be *stably rational* if $X \times \mathbb{P}^k(\mathbb{C})$ is birational to $\mathbb{P}^N(\mathbb{C})$ for certain nonnegative integers k and N. A nonsingular variety X is called *unirational* if there exists a dominant rational map $\varphi : \mathbb{P}^n(\mathbb{C}) \longrightarrow X$ for a certain positive integer n, equivalently if the function field $\mathbb{C}(X)$ of X can be embedded in a purely transcendental extension $\mathbb{C}(z_1, \cdots, z_n)$ of $\mathbb{C}$.

Remarks 9.2. (1) It is easy to see that the rationality implies the stably rationality and that the stably rationality implies the unirationality.

(2) If X is a Riemann surface or a complex surface, then the notions of rationality, stably rationality and unirationality are equivalent one another.

(3) Griffiths and Clemens [21] showed that most of cubic threefolds in $\mathbb{P}^4(\mathbb{C})$ are unirational but *not* rational.

The following natural questions arise :

QUESTION 1. Is a stably rational variety *rational*? Indeed, the question was raised by Bogomolov.

QUESTION 2. *Is a general hypersurface $X \subset \mathbb{P}^{n+1}(\mathbb{C})$ of degree $d \leq n+1$ unirational?*

Definition 9.3. Let X be a nonsingular variety of dimension n and let K_X be the canonical divisor of X. For each positive integer $m \in \mathbb{Z}^+$, we define the *m-genus* $P_m(X)$ of X by

$$P_m(X) := \dim_{\mathbb{C}} H^0(X, \mathcal{O}(mK_X)).$$

The number $p_g(X) := P_1(X)$ is called the *geometric genus* of X. We let

$$N(X) := \left\{ m \in \mathbb{Z}^+ \,|\, P_m(X) \geq 1 \right\}.$$

For the present, we assume that $N(X)$ is nonempty. For each $m \in N(X)$, we let $\{\phi_0, \cdots, \phi_{N_m}\}$ be a basis of the vector space $H^0(X, \mathcal{O}(mK_X))$. Then we have the mapping $\Phi_{mK_X} : X \longrightarrow \mathbb{P}^{N_m}(\mathbb{C})$ by

$$\Phi_{mK_X}(z) := (\phi_0(z) : \cdots : \phi_{N_m}(z)), \quad z \in X.$$

We define the *Kodaira dimension* $\kappa(X)$ of X by

$$\kappa(X) := \max \left\{ \dim_{\mathbb{C}} \Phi_{mK_X}(X) \,|\, m \in N(X) \right\}.$$

If $N(X)$ is empty, we put $\kappa(X) := -\infty$. Obviously $\kappa(X) \leq \dim_{\mathbb{C}} X$. A nonsingular variety X is said to be *of general type* if $\kappa(X) = \dim_{\mathbb{C}} X$. A singular variety Y in general is said to be rational, stably rational, unirational or of general type if any nonsingular model X of Y is rational, stably rational, unirational or of general type respectively. We define

$$P_m(Y) := P_m(X) \quad \text{and} \quad \kappa(Y) := \kappa(X).$$

A variety Y of dimension n is said to be *of logarithmic general type* if there exists a smooth compactification $\tilde{Y}$ of Y such that $D := \tilde{Y} - Y$ is a divisor with normal crossings only and the transcendence degree of the logarithmic canonical ring

$$\oplus_{m=0}^{\infty} H^0(\tilde{Y}, m(K_{\tilde{Y}} + [D]))$$

is $n+1$, i.e., the *logarithmic Kodaira dimension* of Y is n. We observe that the notion of being of logarithmic general type is weaker than that of being of general type.

Let $\mathcal{A}_g := \Gamma_g \backslash \mathbb{H}_g$ be the Siegel modular variety of degree g, that is, the moduli space of principally polarized abelian varieties of dimension g. It has been proved that $\mathcal{A}_g$ is of general type for $g \geq 6$. At first Freitag [32] proved this fact when g is a multiple of 24. Tai [134] proved this fact for $g \geq 9$ and Mumford [102] proved this fact for $g \geq 7$. Recently Grushevsky and Lehavi [45] announced that they proved that the Siegel modular variety $\mathcal{A}_6$ of genus 6 is of general type after constructing a series of new effective geometric divisors on $\mathcal{A}_g$. Before 2005 it had been known that $\mathcal{A}_g$ is of general type for $g \geq 7$. On the other hand, $\mathcal{A}_g$ is known to be unirational for $g \leq 5$: Donagi [25] for $g = 5$, Clemens [20] for $g = 4$ and classical for $g \leq 3$. For $g = 3$, using the moduli theory of curves, Riemann [111], Weber [142] and Frobenius [36] showed that $\mathcal{A}_3(2) := \Gamma_3(2)\backslash\mathbb{H}_3$ is a rational variety and moreover gave 6 generators of the modular function field $K(\Gamma_3(2))$ written explicitly in terms of derivatives of odd theta functions at the origin. So $\mathcal{A}_3$ is a unirational variety with a Galois covering of a rational variety of degree $[\Gamma_3 : \Gamma_3(2)] = 1,451,520$. Here $\Gamma_3(2)$ denotes the principal congruence subgroup of Γ_3 of level 2. Furthermore it was shown that $\mathcal{A}_3$ is stably rational(cf. [80], [16]). For a positive integer k; we let $\Gamma_g(k)$ be the principal congruence subgroup of Γ_g of level k. Let $\mathcal{A}_g(k)$ be the moduli space of abelian varieties of dimension g with k-level structure. It is classically known that $\mathcal{A}_g(k)$ is of logarithmic general type for $k \geq 3$ (cf. [101]). Wang [141] proved that $\mathcal{A}_2(k)$ is of general type for $k \geq 4$. On the other hand, van der Geer [37] showed that $\mathcal{A}_2(3)$ is rational. The remaining unsolved problems are summarized as follows:

Problem 1. Is $\mathcal{A}_3$ rational?

Problem 2. Are $\mathcal{A}_4$, $\mathcal{A}_5$ stably rational or rational?

Problem 3. What type of varieties are $\mathcal{A}_g(k)$ for $g \geq 3$ and $k \geq 2$?

We already mentioned that $\mathcal{A}_g$ is of general type if $g \geq 6$. It is natural to ask if the subvarieties of $\mathcal{A}_g$ $(g \geq 6)$ are of general type, in particular the subvarieties of $\mathcal{A}_g$ of codimension one. Freitag [35] showed that there exists a certain bound g_0 such that for $g \geq g_0$, each irreducible subvariety of $\mathcal{A}_g$ of codimension one is of general type. Weissauer [145] proved that every irreducible divisor of $\mathcal{A}_g$ is of general type for $g \geq 10$. Moreover he proved that every subvariety of codimension $\leq g - 13$ in $\mathcal{A}_g$ is of general type for $g \geq 13$. We observe that the smallest known codimension for which there exist subvarieties of $\mathcal{A}_g$ for large g which are not of general type is $g - 1$. $\mathcal{A}_1 \times \mathcal{A}_{g-1}$ is a subvariety of $\mathcal{A}_g$ of codimension $g - 1$ which is not of general type.

Remark. Let $\mathcal{M}_g$ be the coarse moduli space of curves of genus g over $\mathbb{C}$. Then $\mathcal{M}_g$ is an analytic subvariety of $\mathcal{A}_g$ of dimension $3g - 3$. It is known that $\mathcal{M}_g$ is unirational for $g \leq 10$. So the Kodaira dimension $\kappa(\mathcal{M}_g)$ of $\mathcal{M}_g$ is $-\infty$ for $g \leq 10$. Harris and Mumford [48] proved that $\mathcal{M}_g$ is of general type for odd g with $g \geq 25$ and $\kappa(\mathcal{M}_{23}) \geq 0$.

11. Proportionality Theorem

In this section we describe the proportionality theorem for the Siegel modular variety following the work of Mumford [101]. Historically F. Hirzebruch [55] first described a beautiful proportionality theorem for the case of a *compact* locally symmetric variety in 1956. We shall state his proportionality theorem roughly. Let D be a bounded symmetric domain and let Γ be a discrete torsion-free co-compact group of automorphisms of D. We assume that the quotient space $X_\Gamma := \Gamma\backslash D$ is a *compact* locally symmetric variety. We denote by $\check{D}$ the *compact dual* of D. Hirzebruch [55] proved that the Chern numbers of X_Γ are proportional to the Chern numbers of $\check{D}$, the constant of proportionality being the volume of X_Γ in a natural metric. Mumford [101] generalized Hirzebruch's proportionality theorem to the case of a noncompact arithmetic variety.

Before we describe the proportionality theorem for the Siegel modular variety, first of all we review the compact dual of the Siegel upper half plane $\mathbb{H}_g$. We note that $\mathbb{H}_g$ is biholomorphic to the generalized unit disk $\mathbb{D}_g$ of degree g through the Cayley transform (2.7). We suppose that $\Lambda = (\mathbb{Z}^{2g}, \langle\ ,\ \rangle)$ is a symplectic lattice with a symplectic form $\langle\ ,\ \rangle$. We extend scalars of the lattice Λ to $\mathbb{C}$. Let

$$\mathfrak{Y}_g := \left\{ L \subset \mathbb{C}^{2g} \,|\, \dim_{\mathbb{C}} L = g, \ \ \langle x, y\rangle = 0 \quad \text{for all } x, y \in L \right\}$$

be the complex Lagrangian Grassmannian variety parameterizing totally isotropic subspaces of complex dimension g. For the present time being, for brevity, we put $G = Sp(g,\mathbb{R})$ and $K = U(g)$. The complexification $G_{\mathbb{C}} = Sp(g,\mathbb{C})$ of G acts on $\mathfrak{Y}_g$ transitively. If H is the isotropy subgroup of $G_{\mathbb{C}}$ fixing the first summand $\mathbb{C}^g$, we can identify $\mathfrak{Y}_g$ with the compact homogeneous space $G_{\mathbb{C}}/H$. We let

$$\mathfrak{Y}_g^+ := \left\{ L \in \mathfrak{Y}_g \,|\, -i\langle x, \bar{x}\rangle > 0 \quad \text{for all } x(\neq 0) \in L \right\}$$

be an open subset of $\mathfrak{Y}_g$. We see that G acts on $\mathfrak{Y}_g^+$ transitively. It can be shown that $\mathfrak{Y}_g^+$ is biholomorphic to $G/K \cong \mathbb{H}_g$. A basis of a lattice $L \in \mathfrak{Y}_g^+$ is given by a unique $2g \times g$ matrix ${}^t(-I_g\ \Omega)$ with $\Omega \in \mathbb{H}_g$. Therefore we can identify L with Ω in $\mathbb{H}_g$. In this way, we embed $\mathbb{H}_g$ into $\mathfrak{Y}_g$ as an open subset of $\mathfrak{Y}_g$. The complex projective variety $\mathfrak{Y}_g$ is called the *compact dual* of $\mathbb{H}_g$.

Let Γ be an arithmetic subgroup of Γ_g. Let E_0 be a G-equivariant holomorphic vector bundle over $\mathbb{H}_g = G/K$ of rank n. Then E_0 is defined by the representation $\tau : K \longrightarrow GL(n,\mathbb{C})$. That is, $E_0 \cong G \times_K \mathbb{C}^n$ is a homogeneous vector bundle over G/K. We naturally obtain a holomorphic vector bundle E over $\mathcal{A}_{g,\Gamma} := \Gamma\backslash G/K$. E is often called an *automorphic* or *arithmetic* vector bundle over $\mathcal{A}_{g,\Gamma}$. Since K is compact, E_0 carries a G-equivariant Hermitian metric h_0 which induces a Hermitian metric h on E. According to Main Theorem in [101], E admits a *unique* extension $\tilde{E}$ to a smooth toroidal compactification $\tilde{\mathcal{A}}_{g,\Gamma}$ of $\mathcal{A}_{g,\Gamma}$ such that h is a singular Hermitian metric *good* on $\tilde{\mathcal{A}}_{g,\Gamma}$. For the precise definition of a *good metric* on $\mathcal{A}_{g,\Gamma}$ we refer to [101, p. 242]. According to Hirzebruch-Mumford's Proportionality Theorem (cf. [101, p. 262]), there is a natural metric on $G/K = \mathbb{H}_g$ such that the Chern numbers satisfy the following relation

$$c^{\alpha}(\tilde{E}) = (-1)^{\frac{1}{2}g(g+1)}\,\mathrm{vol}\,(\Gamma\backslash\mathbb{H}_g)\;c^{\alpha}(\check{E}_0) \tag{11.1}$$

for all $\alpha = (\alpha_1, \cdots, \alpha_n)$ with nonegative integers $\alpha_i\,(1 \leq i \leq n)$ and $\sum_{i=1}^{n}\alpha_i = \frac{1}{2}g(g+1)$, where $\check{E}_0$ is the $G_{\mathbb{C}}$-equivariant holomorphic vector bundle on the compact dual $\mathfrak{Y}_g$ of $\mathbb{H}_g$ defined by a certain representation of the stabilizer $\mathrm{Stab}_{G_{\mathbb{C}}}(e)$ of a point e in $\mathfrak{Y}_g$. Here $\mathrm{vol}\,(\Gamma\backslash\mathbb{H}_g)$ is the volume of $\Gamma\backslash\mathbb{H}_g$ that can be computed (cf. [131]).

Remark 11.1. Goresky and Pardon [41] investigated Chern numbers of an automorphic vector bundle over the Baily-Borel compactification $\overline{X}$ of a Shimura variety X. It is known that $\overline{X}$ is usually a highly singular complex projective variety. They also described the close relationship between the topology of X and the characteristic classes of the unique extension $\tilde{TX}$ of the tangent bundle TX of X to a smooth toroidal compactification $\tilde{X}$ of X.

12. Motives and Siegel Modular Forms

Assuming the existence of the hypothetical motive $M(f)$ attached to a Siegel modular form f of degree g, H. Yoshida [161] proved an interesting fact that $M(f)$ has at most $g+1$ period invariants. I shall describe his results in some detail following his papers [160, 161, 163].

First of all we start with listing major historical events concerning critical values of zeta functions.

Around 1670, Gottfried W. Leibniz (1646-1716) found the following identity

$$\sum_{k=0}^{\infty} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

In 1735, Leonhard Euler (1707-1783) discovered the following interesting identity

$$\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

experimentally and also in 1742 showed the following fact

$$\frac{\zeta(2n)}{\pi^{2n}} \in \mathbb{Q}, \quad n = 1, 2, 3, \cdots \in \mathbb{Z}^{+}.$$

In 1899, Adolf Hurwitz (1859-1919) showed the following fact

$$\sum_{z} z^{-4n}/\varpi^{4n} \in \mathbb{Q}, \quad n = 1, 2, 3, \cdots \in \mathbb{Z}^{+},$$

where z extends over all nonzero Gaussian integers and

$$\varpi = 2\int_0^1 \frac{dx}{\sqrt{1-x^4}}.$$

In 1959, Goro Shimura (1930-) proved that

$$\frac{L(n,\Delta)}{(2\pi i)^n c^{\pm}(\Delta)} \in \mathbb{Q}, \qquad 1 \leqq n \leqq 11, \quad \pm 1 = (-1)^n,$$

where

$$\Delta(\tau) = \sum_{n=1}^{\infty} \tau(n)\, q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24}, \qquad q = \exp(2\pi i \tau)$$

is the cusp form of weight 12 with respect to $SL(2,\mathbb{Z})$ and $c^{\pm}(\Delta) \in \mathbb{R}^{\times}$. Here

$$L(s,\Delta) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}$$

is the L-function of $\Delta(\tau)$ and $\tau(n)$ is the so-called Ramanujan tau function which has the following property

$$|\tau(p)| \leq 2\, p^{11/2} \quad \text{for all primes } p.$$

The above property was proved by Pierre Deligne (1944-) in 1974. For instance, $\tau(2) = -24$, $\tau(3) = 252$, $\tau(5) = 4830$, $\tau(7) = -16744$, $\tau(11) = 534612, \tau(13) = -577738$.

In 1977, Shimura [130] proved in a similar way that for a Hecke eigenform $f \in S_k(\Gamma_0(N), \psi)$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$,

$$\left(\frac{L(n,f)}{(2\pi i)^n c^{\pm}(f)}\right)^{\sigma} = \frac{L(n,f^{\sigma})}{(2\pi i)^n c^{\pm}(f^{\sigma})}, \quad 1 \leqq n \leqq k-1, \quad \pm 1 = (-1)^n,$$

where $c^{\pm}(f^{\sigma}) \in \mathbb{C}^{\times}$. By these results, it was expected that the critical values of zeta functions are related to periods of integrals. Here the notion of critical values, which is generally accepted now, can be defined as follows. Suppose that a zeta function $Z(s)$ multiplied by its gamma factor $G(s)$ satisfies a functional equation of standard type under the symmetry $s \longrightarrow v - s$. Then $Z(n)$, $n \in \mathbb{Z}$ is a critical value of $Z(s)$ if both of $G(n)$ and $G(v-n)$ are finite.

In 1979, Pierre Deligne [24] published a general conjecture which gives a prediction on critical values of the L-function of a motive. For a nice concise exposition of the theory of motives, we refer the reader to a paper of Jannsen [69]. For more comprehensive information, we refer to [70].

Let E be an algebraic number field with finite degree $l = [E:\mathbb{Q}]$. Let J_E be the set of all isomorphisms of E into $\mathbb{C}$. We put $R = E \otimes_{\mathbb{Q}} \mathbb{C}$. Let M be a motive over $\mathbb{Q}$ with coefficients in E. Roughly speaking motives arise as direct summands of the cohomology of a smooth projective algebraic variety defined over $\mathbb{Q}$. Naively they may be defined by a collection of realizations satisfying certain axioms. A motive M has at least three realizations : the Betti realization, the de Rham realization and the λ-adic realization.

First we let $H_B(M)$ be the Betti realization of M. Then $H_B(M)$ is a free module over E of rank $d := d(M)$. We put $H_B(M)_{\mathbb{C}} := H_B(M)\otimes_{\mathbb{Q}}\mathbb{C}$. We have the involution F_∞ acting on $H_B(M)_{\mathbb{C}}$ E-linearly. Therefore we obtain the the eigenspace decomposition

$$H_B(M)_{\mathbb{C}} = H_B^+(M) \oplus H_B^-(M), \tag{12.1}$$

where $H_B^+(M)$ (resp. $H_B^-(M)$) denotes the $(+1)$-eigenspace (resp. the (-1)-eigenspace) of $H_B(M)$. We let d^+ (resp. d^-) be the dimension $H_B^+(M)$ (resp. $H_B^-(M)$). Furthermore $H_B(M)_{\mathbb{C}}$ has the Hodge decomposition into $\mathbb{C}$-vector spaces:

$$H_B(M)_{\mathbb{C}} = \bigoplus_{p,q\in\mathbb{Z}} H^{p,q}(M), \tag{12.2}$$

where $H^{p,q}(M)$ is a free R-module. A motive M is said to be *of pure weight* $w := w(M)$ if $H^{p,q}(M) = \{0\}$ whenever $p+q \neq w$. From now on we shall assume that M is of pure weight.

Secondly we let $H_{\mathrm{DR}}(M)$ be the de Rham realization of M that is a free module over E of rank d. Let

$$H_{\mathrm{DR}}(M) = F^{i_1} \supsetneqq F^{i_2} \supsetneqq \cdots \supsetneqq F^{i_m} \supsetneqq F^{i_{m+1}} = \{0\} \tag{12.3}$$

be a decreasing Hodge filtration so that there are no different filtrations between successive members. The choice of members i_ν may not be unique for F^{i_ν}. For the sake of simplicity, we assume that i_ν is chosen for $1 \leq \nu \leq m$ so that it is the maxium number. We put

$$s_\nu = \mathrm{rank}\ H^{i_\nu, w-i_\nu}(M), \qquad 1 \leq \nu \leq m,$$

where rank means the rank as a free R-module. Let

$$I : H_B(M)_{\mathbb{C}} \longrightarrow H_{\mathrm{DR}}(M)_{\mathbb{C}} = H_{\mathrm{DR}}(M) \otimes_E \mathbb{C}$$

be the comparison isomorphism which satisfies the conditions

$$I\left(\bigoplus_{p'\geq p} H^{p',q}(M)\right) = F^p \otimes_{\mathbb{Q}} \mathbb{C}. \tag{12.4}$$

According to (12.4), we get

$$s_\nu = \dim_E F^{i_\nu} - \dim_E F^{i_{\nu+1}},\ \dim_E F^{i_\nu} = s_\nu + s_{\nu+1} + \cdots + s_m,\ 1 \leq \nu \leq m.$$

We choose a basis $\{w_1, \cdots, w_d\}$ of $H_{\mathrm{DR}}(M)$ over E so that $\{w_{s_1+s_2+\cdots+s_{\nu-1}+1}, \cdots, w_d\}$ is a basis of F^{i_ν} for $1 \le \nu \le m$. We observe that

$$d = s_1 + s_2 + \cdots + s_m \quad \text{all } s_\nu > 0 \text{ with } 1 \le \nu \le m. \tag{12.5}$$

We are in a position to describe the fundamental periods of M that Yoshida introduced. Let $\{v_1^+, v_2^+, \cdots, v_{d^+}^+\}$ (resp. $\{v_1^-, v_2^-, \cdots, v_{d^-}^-\}$) be a basis of $H_B^+(M)$ (resp. $H_B^-(M)$) over E. Writing

$$I(v_j^\pm) = \sum_{i=1}^{d} x_{ij}^\pm w_i, \quad x_{ij}^\pm \in R, \quad 1 \le j \le d^\pm, \tag{12.6}$$

we obtain a matrix $X^+ = (x_{ij}^+) \in R^{(d,d^+)}$ and a matrix $X^- = (x_{ij}^-) \in R^{(d,d^-)}$. We recall that $R^{(m,n)}$ denotes the set of all $m \times n$ matrices with entries in R. Let P_M be the lower parabolic subgroup of $GL(d)$ which corresponds to the partition (12.5). Let $P_M(E)$ be the group of E-rational points of P_M. Then the coset of X^+ (resp. X^-) in

$$P_M(E)\backslash R^{(d,d^+)}/GL(d^+, E) \quad (\text{resp. } P_M(E)\backslash R^{(d,d^-)}/GL(d^-, E))$$

is independent of the choice of a basis. We set $X_M = (X^+, X^-) \in R^{(d,d)}$. Then it is easily seen that the coset of X_M in

$$P_M(E)\backslash R^{(d,d)}/\big(GL(d^+, E) \times GL(d^-, E)\big)$$

is independent of the choice of a basis, i.e., well defined. A $d \times d$ matrix $X_M = (X^+, X^-)$ is called a *period matrix* of M.

For an m-tuple $(a_1, \cdots, a_m) \in \mathbb{Z}^m$ of integers, we define a character λ_1 of P_M by

$$\lambda_1\left(\begin{pmatrix} P_1 & 0 & \dots & 0 \\ * & P_2 & \dots & 0 \\ * & * & \ddots & \vdots \\ * & * & * & P_m \end{pmatrix}\right) = \prod_{j=1}^{m} \det(P_j)^{a_j}, \; P_j \in GL(s_j), \; 1 \le j \le m.$$

For a pair (k^+, k^-) of integers, we define a character λ_2 of $GL(d^+) \times GL(d^-)$ by

$$\lambda_2\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right) = (\det A)^{k^+} (\det B)^{k^-}, \qquad A \in GL(d^+), \; B \in GL(d^-).$$

A polynomial f on $R^{(d,d)}$ rational over $\mathbb{Q}$ is said to be *of the type* $\{(a_1, \cdots, a_m); (k^+, k^-)\}$ or *of the type* (λ_1, λ_2) if f satisfies the following condition

(12.7)

$$f(pxq) = \lambda_1(p)\lambda_2(q)f(x) \qquad \text{for all } \; p \in P_M, \; q \in GL(d^+) \times GL(d^-).$$

We now assume that f is a nonzero polynomial on $R^{(d,d)}$ of the type $\{(a_1, \cdots, a_m); (k^+, k^-)\}$. Let $X_M = (X^+, X^-)$ be a period matrix of a motive M as before. Then it is clear that $f(X_M)$ is uniquely determined up to multiplication by elements in $E^\times$. We call $f(X_M)$ a *period invariant* of M of the type $\{(a_1, \cdots, a_m); (k^+, k^-)\}$. Hereafter we understand the equality between period invariants mod $E^\times$.

We now consider the following special polynomials of the type (λ_1, λ_2) :

I. Let $f(x) = \det(x)$ for $x \in R^{(d,d)}$.

It is easily seen that $f(x)$ is of the type $\{(1, 1, \cdots, 1); (1, 1)\}$. Then $f(X_M)$ is nothing but Deligne's period $\delta(M)$.

II. Let $f^+(x)$ be the determinant of the upper left $d^+ \times d^+$-submatrix of $x \in R^{(d,d)}$. It is easily checked that $f^+(x)$ is of the type

$$\{(\overbrace{1, 1, \dots, 1}^{p^+}, 0, \dots, 0); (1, 0)\},$$

where p^+ is a positive integer such that $s_1 + s_2 + \cdots + s_{p^+} = d^+$. We note that $f^+(X_M)$ is Deligne's period $c^+(M)$.

III. Let $f^-(x)$ be the determinant of the upper right $d^- \times d^-$-submatrix of x. Then $f^-(x)$ is of the type

$$\{(\overbrace{1, 1, \dots, 1}^{p^-}, 0, \dots, 0); (0, 1)\}$$

and $f^-(X_M)$ is Deligne's period $c^-(M)$. Here p^- is a positive integer such that $s_1 + s_2 + \cdots + s_{p^-} = d^-$.

Either one of the above conditions is equivalent to that $F^\mp(M)$, hence also $c^\pm(M)$ can be defined (cf. [23], §1, [160], §2). We have $F^\mp(M) = F^{i_{p^\pm+1}}(M)$; $F^\pm(M)$ can be defined if M has a critical value. Let $\mathcal{P} = \mathcal{P}(M)$ denote the set of integers p such that $s_1 + s_2 + \cdots + s_p < \min(d^+, d^-)$. Yoshida (cf. [161], Theorem 3) showed that for every $p \in \mathcal{P}$, there exists a non-zero polynomial f_p of the type

$$\{(\overbrace{2, \dots, 2}^{p}, \overbrace{1, \dots, 1}^{m-2p}, \overbrace{0, \dots, 0}^{p}); (1, 1)\}$$

and that every polynomial satisfying (12.7) can be written uniquely as a monomial of $\det(x)$, $f^+(x)$, $f^-(x)$, $f_p(x)$, $p \in \mathcal{P}$. We put $c_p(M) = f_p(X_M)$. We call $\delta(M)$, $c^\pm(M)$, $c_p(M)$, $p \in \mathcal{P}$ the *fundamental periods* of M. Therefore any period invariant of M can be written as a monomial of the fundamental periods. Moreover Yoshida showed that if a motive M is constructed from motives $M_1, \cdots, M_t$ of pure weight by standard algebraic operations then the fundamental periods of M can be written as monomials of the fundamental periods of $M_1, \cdots, M_t$. He proved that a motive M has at most $\min(d^+, d^-) + 2$ fundamental periods including Deligne's periods $\delta(M)$ and $c^\pm(M)$.

Thirdly we let $H_\lambda(M)$ be the λ-adic realization of M. We note that $H_\lambda(M)$ is a free module over E_λ of rank d. We have a continuous λ-adic representation of the absolute Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $H_\lambda(M)$ for each prime λ. Also there is an isomorphism $I_\lambda : H_B(M) \otimes_E E_\lambda \longrightarrow H_\lambda(M)$ which transforms the involution F_∞ into the complex conjugation.

We recall that an integer $s = n$ is said to be *critical* for a motive M if both the infinite Euler factors $L_\infty(M, s)$ and $L_\infty(\check{M}, s)$ are holomorphic at $s = n$. Here $L(M, s)$ denotes the complex L-function attached to M and $\check{M}$ denotes the dual motive of M. Such values $L(M, n)$ are called *critical values* of $L(M, s)$. Deligne proposed the following.

Conjecture (Deligne [23]**).** *Let M be a motive of pure weight and $L(M, s)$ the L-function of M. Then for critical values $L(M, n)$, one has*

$$\frac{L(M,n)}{(2\pi i)^{d^\pm}\, c^\pm(M)} \in E, \quad d^\pm := d^\pm(M), \ \pm 1 = (-1)^n.$$

Indeed Deligne showed that $c^\pm(M) \in R^\times$ and Yoshida showed that other period invariants are elements of $R^\times$.

Remark 12.1. The Hodge decomposition (12.2) determines the gamma factors of the conjectural functional equation of $L(M, s)$. Conversely the gamma factor of the functional equation of $L(M, s)$ determines the Hodge decomposition if M is of pure weight.

Let $f \in S_k(\Gamma_g)$ be a nonzero Hecke eigenform on $\mathbb{H}_g$. Let $L_{\rm st}(s, f)$ and $L_{\rm sp}(s, f)$ be the standard zeta function and the spinor zeta function of f respectively. For the sake of simplicity we use the notations $L_{\rm st}(s, f)$ and $L_{\rm sp}(s, f)$ instead of $D_f(s)$ and $Z_f(s)$ (cf. §8) in this section. We put $w = kg - \frac{1}{2}g(g+1)$. We have a normalized Petersson inner product $\langle\ ,\ \rangle$ on $S_k(\Gamma_g)$ given by

$$\langle F, F\rangle = \mathrm{vol}\big(\Gamma_g\backslash\mathbb{H}_g\big)^{-1}\int_{\Gamma_g\backslash\mathbb{H}_g} |f(\Omega)|^2 \big(\det Y\big)^{k-g-1}[dX][dY], \ F \in S_k(\Gamma_g),$$

where $\Omega = X + iY \in \mathbb{H}_g$ with real $X = (x_{\mu\nu})$, $Y = (y_{\mu\nu})$, $[dX] = \bigwedge_{\mu\leq\nu} dx_{\mu\nu}$ and $[dY] = \bigwedge_{\mu\leq\nu} dy_{\mu\nu}$.

We assume the following (A1)-(A6) :

(A1) The Fourier coefficients of f are contained in a totally real algebraic number field E.

(A2) There exist motives $M_{\rm st}(f)$ and $M_{\rm sp}(f)$ over $\mathbb{Q}$ with coefficients in E satisfying the conditions

$$L\big(M_{\rm st}(f), s\big) = \big(L_{\rm st}(s, f^\sigma)\big)_{\sigma\in J_E} \text{ and } L\big(M_{\rm sp}(f), s\big) = \big(L_{\rm sp}(s, f^\sigma)\big)_{\sigma\in J_E}.$$

(A3) Both $M_{\rm st}(f)$ and $M_{\rm sp}(f)$ are of pure weight.

(A4) We assume

$$\bigwedge^{2g+1} M_{\mathrm{st}}(f) \cong T(0),$$

$$H_B(M_{\mathrm{st}}(f)) \otimes_{\mathbb{Q}} \mathbb{C} = H^{0,0}(M_{\mathrm{st}}(f)) \bigoplus_{i=1}^{g} \Big(H^{-k+i,k-i}(M_{\mathrm{st}}(f)) \oplus H^{k-i,-k+i}(M_{\mathrm{st}}(f)) \Big).$$

We also assume that the involution F_∞ acts on $H^{0,0}(M_{\mathrm{st}}(f))$ by $(-1)^g$.
(A5) We assume

$$\bigwedge^{2^g} M_{\mathrm{sp}}(f) \cong T(2^{g-1}w),$$

$$H_B(M_{\mathrm{sp}}(f)) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p,q} H^{p,q}(M_{\mathrm{sp}}(f)),$$

$$p=(k-i_1)+(k-i_2)+\cdots+(k-i_r),\quad q=(k-j_1)+(k-j_2)+\cdots+(k-j_s),$$
$$r+s=g, \qquad 1 \le i_1 < \cdots < i_r \le g, \qquad 1 \le j_1 < \cdots < j_s \le g,$$
$$\{i_1, \cdots, i_r\} \cup \{j_1, \cdots, j_s\} = \{1, 2, \ldots, g\},$$

including the cases $r=0$ or $s=0$.
(A6) If $w = kg - \frac{1}{2}g(g+1)$ is even, then the eigenvalues $+1$ and -1 of F_∞ on $H^{p,p}(M_{\mathrm{sp}}(f))$ occur with the equal multiplicities.

Let $J_E = \{\sigma_1, \sigma_2, \ldots, \sigma_l\}$, $l = [E : \mathbb{Q}]$ and write $x \in R \cong \mathbb{C}^{J_E}$ as $x = (x^{(1)}, x^{(2)}, \cdots, x^{(l)})$, $x^{(i)} \in \mathbb{C}$ so that $x^{(i)} = x^{\sigma_i}$ for $x \in E$. Yoshida showed that when $k > 2g$, assuming Deligne's conjecture, one has

$$c^{\pm}(M_{\mathrm{st}}(f)) = \pi^{kg}\big(\langle f^\sigma, f^\sigma \rangle\big)_{\sigma \in J_E}.$$

He proved the following interesting result (cf. Yoshida [161], Theorem 14).

Theorem 12.1. *Let the notation be the same as above. We assume that two motives over $\mathbb{Q}$ having the same L-function are isomorphic (Tate's conjecture). Then there exist p_1, p_2, $\cdots$, $p_r \in \mathbb{C}^\times$, $1 \le r \le g+1$ such that for any fundamental period $c \in R^\times$ of $M_{st}(f)$ or $M_{sp}(f)$, we have*

$$c^{(1)} = \alpha\, \pi^A p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$$

with $\alpha \in \overline{\mathbb{Q}}^\times$ and non-negative integers A, a_i, $1 \le i \le r$.

Remark 12.2. It is widely believed that the zeta function of the Siegel modular variety $\mathcal{A}_g := \Gamma_g \backslash \mathbb{H}_g$ can be expressed using the spinor zeta functions of (not necessarily holomorphic) Siegel modular forms:

$$\zeta(s, \mathcal{A}_g) \fallingdotseq \prod_f L_{\mathrm{sp}}(s, f).$$

Yoshida proposed the following conjecture.

Conjecture (Yoshida [161]**).** *If one of two motives $M_{st}(f)$ and $M_{sp}(f)$ is not of pure weight, then the associated automorphic representation to f is not tempered. Furthermore f can be obtained as a lifting from lower degree forms.*

References

[1] A. N. Andrianov, *Modular descent and the Saito-Kurokawa lift*, Invent. Math. **289**, Springer-Verlag (1987).

[2] A. N. Andrianov, *Quadratic forms and Hecke operators*, Grundlehren der Mathematik **53** (1979), 267–280.

[3] A. N. Andrianov and V. L. Kalinin, *On the anlaytic properties of standard zeta functions of Siegel modular forms*, Math. USSR. Sb. **35** (1979), 1–17.

[4] A. N. Andrianov and V. G. Zhuravlev, *Modular forms and Hecke operators*, Translations of Mathematical Monographs **145**, AMS, Providence, RI (1995).

[5] A. Ash, D. Mumford, M. Rapoport and Y. Tai, *Smooth compactification of locally symmetric varieties*, Math. Sci. Press (53 Jordan Rd., Brookline, MA. 02146, USA), 1975.

[6] W. Baily, *Satake's compactification of V_n^**, Amer. J. Math. **80** (1958), 348–364.

[7] W. Baily and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. Math. **84** (1966), 442–528.

[8] R. Berndt, *Zur Arithmetik der elliptischen Funktionenkörper höherer Stufe*, J. reine angew. Math., **326**(1981), 79-94.

[9] R. Berndt, *Meromorphic Funktionen auf Mumfords Kompaktifizierung der universellen elliptischen Kurve N-ter Stufe*, J. reine angew. Math., **326**(1981), 95-103.

[10] R. Berndt, *Shimuras Reziprozitätsgesetz für den Körper der arithmetischen elliptischen Funktionen beliebiger Stufe* , J. reine angew. Math., **343**(1983), 123-145.

[11] R. Berndt, *Die Jacobigruppe und die Wärmeleitungsgleichung* , Math. Z., **191**(1986), 351-361.

[12] R. Berndt, *The Continuous Part of $L^2(\Gamma^J\backslash G^J)$ for the Jacobi Group*, Abh. Math. Sem. Univ. Hamburg., **60**(1990), 225-248.

[13] R. Berndt and S. Böcherer, *Jacobi Forms and Discrete Series Representations of the Jacobi Group*, Math. Z., **204**(1990), 13-44.

[14] R. Berndt, *On Automorphic Forms for the Jacobi Group* , Jb. d. Dt. Math.-Verein., **97**(1995), 1-18.

[15] R. Berndt and R. Schmidt, *Elements of the Representation Theory of the Jacobi Group*, Birkhäuser, 1998.

[16] F. A. Bogomolov and P. I. Katsylo, *Rationality of some quotient varieties*, Math. USSR Sbornik., **54**(1986), 571-576.

[17] R. Borcherds, E. Freitag and R. Weissauer, *A Siegel cusp form of degree* 12 *and weight* 12, J. reine angew. Math. **494** (1998), 141–153.

[18] S. Breulmann and M. Kuss, *On a conjecture of Duke-Imamoğlu*, Proc. Ameri. Math. Soc. **128** (2000), 1595–1604.

[19] D. Bump and Y. J. Choie, *Derivatives of modular forms of negative weight*, Pure Appl. Math. Q. **2** (2006), no. 1, 111-133.

[20] H. Clemens, *Double Solids*, Adv. Math. **47**(1983), 107-230.

[21] H. Clemens and P. Griffiths, , Ann. of Math. **95**(1972), 281–356.

[22] C. Consani and C. Faber, *On the cusp forms motives in genus 1 and level 1*, arXiv:math.AG/0504418.

[23] P. Deligne, *Formes modulaires et représentations ℓ-adiques*: Sém. Bourbaki 1968/9, no. 355; Lecture Notes in Math. **179**, Springer-Verlag, Berlin-Heidelberg-New York (1971), 139–172.
[24] P. Deligne, *Valeurs de fonctions L, et périods d'intégrals,* Proc. Symp. Pure Math., Amer. Math. Soc., Providence, RI, **33**, Part 2 (1979), 313–346.
[25] R. Donagi, *The unirationality of A_5*, Ann. of Math., **119**(1984), 269-307.
[26] W. Duke and Ö. Imamoḡlu, *A converse theorem and the Saito-Kurokawa lift*, IMRN **310** (1996), 347–355.
[27] W. Duke and Ö. Imamoḡlu, *Siegel modular forms of small weight*, Math. Ann. **310** (1998), 73–82.
[28] N. Dummigan, *Period ratios of modular forms*, Math. Ann. **318** (2000), 621–636.
[29] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progress in Mathematics **55**, Birkhäuser, Boston, Basel and Stuttgart, 1985.
[30] G. Faltings and C.-L. Chai, *Degeneration of abelian varieties*, Ergebnisse der Math. **22**, Springer-Verlag, Berlin-Heidelberg-New York (1990).
[31] A. J. Feingold and I. B. Frenkel, *A Hyperbolic Kac-Moody Algebra and the Theory of Siegel Modular Forms of genus* 2, Math. Ann., **263**(1983), 87-144.
[32] E. Freitag, *Die Kodairadimension von Körpern automorpher Funktionen*, J. reine angew. Math. **296** (1977), 162–170.
[33] E. Freitag, *Stabile Modulformen*, Math. Ann. **230** (1977), 162–170.
[34] E. Freitag, *Siegelsche Modulfunktionen*, Grundlehren de mathematischen Wissenschaften **55**, Springer-Verlag, Berlin-Heidelberg-New York (1983).
[35] E. Freitag, *Holomorphic tensors on subvarieties of the Siegel modular variety*, Birkhäuser, Prog. Math. Boston, **46**(1984), 93-113.
[36] G. Frobenius, *Über die Beziehungen zwischen* 28 *Doppeltangenten einer eben Curve vierte Ordnung* , J. reine angew. Math., **99**(1886), 285-314.
[37] G. van der Geer, *Note on abelian schemes of level three*, Math. Ann., **278**(1987), 401-408.
[38] G. van der Geer, *Hilbert Modular Surfaces*, Springer-Verlag, Berlin-Heidelberg-New York (1987).
[39] G. van der Geer, *Siegel Modular Forms*, arXiv:math.AG/0605346.
[40] S. Gelbart and H. Jacquet, *A relation between automorphic representations of $GL(2)$ and $GL(3)$*, Ann. Sci. École Norm Sup. **11** (1978), 471–542.
[41] M. Goresky and W. Pardon, *Chern classes of automorphic vector bundles*, Invent. Math. **147** (2002), 561-612.
[42] M. Goresky and Y.-S. Tai, *Toroidal and reductive Borel-Serre compactifications of locally symmetric spaces*, Amer. J. Math. **121** (1999), no. 5, 1095-1151.
[43] D. Grenier, *An analogue of Siegel's ϕ-operator for automorphic forms for $GL(n, \mathbb{Z})$,* Trans. Amer. Math. Soc. **331**, No. 1 (1992), 463-477.
[44] V. A. Gritsenko, *The action of modular operators on the Fourier-Jacobi coefficients of modular forms*, Math. USSR Sbornik, **74**(1984), 237-268.
[45] S. Grushevsky and D. Lehavi, *$\mathcal{A}_g$ is of general type*, arXiv:math.AG/0512530.
[46] Harish-Chandra, *Representations of a semisimple Lie group on a Banach space. I.,* Trans. Amer. Math. Soc. **75** (1953), 185-243.
[47] Harish-Chandra, *The characters of semisimple Lie groups,* Trans. Amer. Math. Soc. **83** (1956), 98-163.
[48] J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math., **67**(1982), 23-88.
[49] M. Harris, *Arithmetic vector bundles and automorphic forms on Shimura varieties I*, Invent. Math. **82** (1985), 151–189.

[50] M. Harris, *Arithmetic vector bundles and automorphic forms on Shimura varieties II*, Compositio Math. **60** (1986), no. 3, 323–378.

[51] M. Harris and S. Zucker, *Boundary cohomology of Shimura varieties I*, Ann. Sci. École Norm Sup. **27** (1994), 249–344.

[52] M. Harris and S. Zucker, *Boundary cohomology of Shimura varieties II*, Invent. Math. **116** (1994), 243–308.

[53] S. Helgason, *Differential operators on homogeneous spaces,* Acta Math. **102** (1959), 239-299.

[54] S. Helgason, *Groups and geometric analysis,* Academic Press, New York (1984).

[55] F. Hirzebruch, *Automorphe Formen und der Satz von Riemann-Roch*, Symposium Internacional de Topologia, Unesco (1958).

[56] J. W. Hoffman and S. H. Weintraub, *Cohomology of the Siegel modular group of degree 2 and level 4* , Mem. Amer. Math. Soc. **631** (1998), 59–75.

[57] J. W. Hoffman and S. H. Weintraub, *Cohomology of the boundary of the Siegel modular varieties of degree 2, with applications*, Fund. Math. **178** (2003), 1-47.

[58] R. Howe, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond,* The Schur lectures (1992) (Tel Aviv), Israel Math. Conf. Proceedings, **vol. 8** (1995), 1–182.

[59] K. Hulek and G. K. Sankaran, *The Geometry of Siegel modular varieties*, Advanced Studies in Pure Mathematics, Higher Dimensional Birational Geometry **35** (2002), 89-156 *or* ArXiv:math.AG/9810153.

[60] J. Igusa, *On Siegel modular forms of genus two*, Amer. J. Math. **84** (1962), 175–200.

[61] J. Igusa, *On Siegel modular forms of genus two II*, Amer. J. Math. **86** (1964), 392–412.

[62] J. Igusa, *On the grades ring of theta constants*, Amer. J. Math. **86** (1964), 219–246.

[63] J. Igusa, *On the grades ring of theta constants II*, Amer. J. Math. **88** (1966), 221–236.

[64] J. Igusa, *Modular forms and projective invariants*, Amer. J. Math. **89** (1967), 817–855.

[65] J. Igusa, *Theta Functions,* Springer-Verlag, Berlin-Heidelberg-New York (1971).

[66] T. Ikeda, *On the lifting of elliptic cusp forms to Siegel cusp forms of degree $2n$*, Ann. Math. **154** (2001), 641–681.

[67] T. Ikeda, *Pullback of the lifting of elliptic cusp forms and Miyawaki's conjecture*, Duke Math. J. **131** (2006), no. 3, 469–497.

[68] C. G. J. Jacobi, *Fundamenta nova theoriae functionum ellipticum*, Königsberg, (1829).

[69] U. Jannsen, *Motives, numerical equivalence, and semi-simplicity*, Invent. Math. **107** (1992), 447–452.

[70] U. Jannsen, S. Kleiman and J.-P. Serre, *Motives*, Proc. Symposia Pure Math. Amer. Math. Soc., Providence, RI, **55** (1994),Part 1 and 2.

[71] H. Katsurada, *An explicit form for Siegel series*, Amer. J. Math. **121** (1999), 415-452.

[72] Y. Kitaoka, *Dirichlet series in the theory of Siegel modular forms*, Nagoya Math. J. **95** (1984), 73-84.

[73] H. Klingen, *Zum Darstellungssatz für Siegelsche Modulformen*, Math. Zeitschr. **102** (1967), 30–43.

[74] H. Klingen, *Introductory lectures on Siegel modular forms*, Cambridge Studies in advanced mathematics **20**, Cambridge University Press (1990).

[75] A. W. Knapp, *Representation Theory of Semisimple Groups,* Princeton University Press, Princeton, New Jersey (1986).
[76] W. Kohnen, *Modular forms of half-integral weight on* $\Gamma_0(4)$, Math. Ann. **248** (1980), 249–266.
[77] W. Kohnen, *Lifting modular forms of half-integral weight to Siegel modular forms of even degree*, Math. Ann. **322** (2003), 787–809.
[78] W. Kohnen and H. Kojima, *A Maass space in higher genus*, Compositio Math. **141** (2005), 313–322.
[79] W. Kohnen and D. Zagier, *Modular forms with rational periods*, in: Modular forms (Durham, 1983), 197–249, Ellis Horwood Ser. Math. Appl. (1984).
[80] J. Kollar and F. O. Schreyer, *The moduli of curves is stably rational for* $g \leq 6$, Duke Math. J., **51**(1984), 239-242.
[81] A. Korányi and J. Wolf, *Generalized Cayley transformations of bounded symmetric domains,* Amer. J. Math. **87** (1965), 899-939.
[82] J. Kramer, *A geometrical approach to the theory of Jacobi forms*, Compositio Math., **79**(1991), 1-19.
[83] J. Kramer, *An arithmetic theory of Jacobi forms in higher dimensions* , J. reine angew. Math., **458**(1995), 157-182.
[84] S. Kudla and S. Rallis, *A regularized Siegel-Weil formula : the first term identity*, Ann. Math. **140** (1994), 1–80.
[85] N. Kurokawa, *Examples of eigenvalues of Hecke operators on Siegel cusp forms of degree two*, Invent. Math. **49** (1978), 149–165.
[86] N. V. Kuznetsov, *A new class of identities for the Fourier coefficients of modular forms*, Acta Arith. (1975), 505–519.
[87] G. Laumon, *Sur la cohomologie à supports compacts des variétés de Shimura* $GSp(4)_{\mathbb{Q}}$, Compositio Math. **105** (1997), 267–359.
[88] R. Lee and S. H. Weintraub, *Cohomology of a Siegel modular variety of degree* 2, Contemp. Math. **36**, Amer. Math. Soc. (1985), 433–488.
[89] J.-S. Li and J. Schwermer, *On the Eisenstein cohomology of arithmetic groups*, Duke Math. J. **123** (2004), 141–169.
[90] G. Lion and M. Vergne, *The Weil representation, Maslov index and Theta series*, Progress in Mathematics, **6**, Birkhäuser, Boston, Basel and Stuttgart, 1980.
[91] H. Maass, *Die Differentialgleichungen in der Theorie der Siegelschen Modulfunktionen*, Math. Ann. **126** (1953), 44–68.
[92] H. Maass, *Siegel modular forms and Dirichlet series,* Lecture Notes in Math. **216**, Springer-Verlag, Berlin-Heidelberg-New York (1971).
[93] H. Maass, *Über eine Spezialschar von Modulformen zweiten Grades I*, Invent. Math. **52** (1979), 95-104.
[94] H. Maass, *Über eine Spezialschar von Modulformen zweiten Grades II*, Invent. Math. **53** (1979), 249–253.
[95] H. Maass, *Über eine Spezialschar von Modulformen zweiten Grades III*, Invent. Math. **53** (1979), 255–265.
[96] R. MacPherson, *Chern classes for singular varieties*, Ann. of Math. **100** (1974), 423–432.
[97] H. Minkowski, *Gesammelte Abhandlungen:* Chelsea, New York (1967).
[98] S. Mizumoto, *Poles and residues of standard L-functions attached to Siegel modular forms*, Math. Ann. **289** (1991), 589–612.
[99] D. Mumford, *On the equations defining Abelian varieties I, II, III*, Invent. math. **1** (1966), 287–354; **3** (1967), 75–135; **3** (1967), 215–244.
[100] D. Mumford, *Abelian Varieties*, Oxford University Press (1970): Reprinted (1985).

[101] D. Mumford, *Hirzebruch's Proportionality Theorem in the Non-Compact Case*, Invent. Math. **42** (1977), 239–272.

[102] D. Mumford, *On the Kodaira dimension of the Siegel modular variety,* Lecture Notes in Math. **997**, Springer-Verlag, Berlin (1983), 348–375.

[103] D. Mumford, *Tata Lectures on Theta I*, Progress in Mathematics, **28**, Birkhäuser, Boston, Basel and Stuttgart, 1983.

[104] A. Murase, *L-functions attached to Jacobi forms of degree n. Part I : The Basic Identity*, J. reine angew. Math., **401** (1989), 122-156.

[105] A. Murase, *L-functions attached to Jacobi forms of degree n. Part II : Functional Equation*, Math. Ann., **290** (1991), 247-276.

[106] A. Murase and T. Sugano, *Whittaker-Shintani Functions on the Symplectic Group of Fourier-Jacobi Type*, Compositio Math., **79** (1991), 321-349.

[107] Y. Namikawa, *Toroidal compactification of Siegel spaces*, Lecture Notes in Math. **812**, Springer-Verlag, Berlin-Heidelberg-New York (1980).

[108] T. Oda and J. Schwermer, *Mixed Hodge structures and automorphic forms for Siegel modular varieties of degree two*, Math. Ann. **286** (1990), 481–509.

[109] I. Piateski-Sharpiro, *Automorphic Functions and the Geometry of Classical Domains,* Gordan-Breach, New York (1966).

[110] C. Poor, *Schottky's form and the hyperelliptic locus*, Proc. Amer. Math. Soc. **124** (1996), 1987–1991.

[111] B. Riemann, *Zur Theorie der Abel'schen Funktionen für den* $p = 3$, Math. Werke, Teubener, Leipzig, (1876), 456-476.

[112] B. Runge, *Theta functions and Siegel-Jacobi functions*, Acta Math., **175** (1995), 165-196.

[113] N. C. Ryan, *Computing the Satake p-parameters of Siegel modular forms*, arXiv:math.NT/0411393.

[114] L. Saper, *L_2-cohomology of arithmetic varieties*, Ann. Math.(2) **132** (1990), no. 1, 1–69.

[115] L. Saper, *On the cohomology of locally symmetric spaces and of their compactifications*, Current developments in mathematics, 2002, 219–289, Int. Press, Somerville, MA, 2003.

[116] L. Saper, *Geometric rationality of equal-rank Satake compactifications*, Math. Res. Lett. **11** (2004), n0. 5-6, 653–671.

[117] I. Satake, *On the compactification of the Siegel space*, J. Indian Math. Soc. **20** (1956), 259–281.

[118] I. Satake, *Theory of spherical functions on reductive algebraic groups over p-adic fields*, Publ. Math. IHES, Nr. **20** (1963).

[119] I. Satake, *Fock Representations and Theta Functions*, Ann. Math. Studies, **66** (1971), 393-405.

[120] I. Satake, *Algebraic Structures of Symmetric Domains,* Kano Memorial Lectures 4, Iwanami Shoton, Publishers and Princeton University Press (1980).

[121] T. Satoh, *On certain vector valued Siegel modular forms of degree two*, Math. Ann. **274** (1986), 335–352.

[122] A. J. Scholl, *Motives for modular forms*, Invent. Math. **100** (1990), 419–430.

[123] J. Schwermer, *On arithmetic quotients of the Siegel upper half space of degree two*, Compositio Math., **58** (1986), 233–258.

[124] J. Schwermer, *On Euler products and residual Eisenstein cohomology classes for Siegel modular varieties*, Forum Math., **7** (1995), 1–28.

[125] G. Shimura, *Introduction to the arithmetic theory of automorphic functions,* Kano Memorial Lectures 1, Iwanami Shoton, Publishers and Princeton University Press (1971).

[126] G. Shimura, *On modular correspondence for $Sp(n,\mathbb{Z})$ and their congruence relations,* Proc. Acad. Sci. USA **49** (1963), 824–828.

[127] G. Shimura, *On modular forms of half integral weight* , Ann. of Math. **97** (1973), 440-481.

[128] G. Shimura, *On certain reciprocity laws for theta functions and modular forms,* Acta Math. **141** (1979), 35-71.

[129] G. Shimura, *Invariant differential operators on hermitian symmetric spaces,* Ann. of Math. **132** (1990), 237-272.

[130] G. Shimura, *On the periods of modular forms*, Math. Ann. **229** (1977), 211-221.

[131] C. L. Siegel, *Symplectic Geometry,* Amer. J. Math. **65** (1943), 1-86; Academic Press, New York and London (1964); Gesammelte Abhandlungen, no. 41, vol. II, Springer-Verlag (1966), 274-359.

[132] C. L. Siegel, *Gesammelte Abhandlungen I-IV*, Springer-Verlag(I-III: 1966; IV: 1979).

[133] C. L. Siegel, *Topics in Complex Function Theory : Abelian Functions and Modular Functions of Several Variables*, vol. III, Wiley-Interscience, 1973.

[134] Y.-S. Tai, *On the Kodaira Dimension of the Moduli Space of Abelian Varieties*, Invent. Math. **68** (1982), 425–439.

[135] R. Taylor, *On the ℓ-adic cohomology of Siegel threefolds*, Invent. Math. **114** (1993), 289–310.

[136] B. Totaro, *Chern numbers for singular varieties and elliptic homology*, Ann. Math.(2) **151** (2000), no. 2, 757–791.

[137] R. Tsushima, *A formula for the dimension of spaces of Siegel cusp forms of degree three*, Amer. J. Math. **102** (1980), 937–977.

[138] S. Tsuyumine, *On Siegel modular forms of degree three*, Amer. J. Math. **108** (1986), 755–862: Appendum. Amer. J. Math. **108** (1986), 1001–1003.

[139] T. Veenstra, *Siegel modular forms, L-functions and Satake parameters*, Journal of Number Theory **87** (2001), 15–30.

[140] W. Wang, *On the Smooth Compactification of Siegel Spaces* , J. Diff. Geometry, **38**(1993), 351-386.

[141] W. Wang, *On the moduli space of principally polarized abelian varieties*, Contemporary Math., **150**(1993), 361-365.

[142] H. Weber, *Theorie der Abel'schen Funktionen von Geschlecht* 3, Berlin, (1876).

[143] R. Weissauer, *Vektorwertige Modulformen kleinen Gewichts*, J. Reine Angew. Math. **343** (1983), 184–202.

[144] R. Weissauer, *Stabile Modulformen und Eisensteinreihnen*, Lecture Notes in Math. **1219**, Springer-Verlag, Berlin (1986).

[145] R. Weissauer, *Untervarietäten der Siegelschen Modulmannigfatigkeiten von allgemeinen Typ*, Math. Ann., **343**(1983), 209-220.

[146] R. Weissauer, *Differentialformen zu Untergruppen der Siegelschen Modulgruppe zweiten Grades*, J. reine angew Math., **391**(1988), 100-156.

[147] H. Weyl, *The classical groups: Their invariants and representations,* Princeton Univ. Press, Princeton, New Jersey, second edition (1946).

[148] E. Witt, *Eine Identität zwischen Modulformen zweiten Grades*, Math. Sem. Hansisch Univ. **14** (1941), 323-337.

[149] J.-H. Yang, *The Siegel-Jacobi Operator*, Abh. Math. Sem. Univ. Hamburg **63** (1993), 135–146.

[150] J.-H. Yang, *Vanishing theorems on Jacobi forms of higher degree*, J. Korean Math. Soc., **30**(1)(1993), 185-198.

[151] J.-H. Yang, *Remarks on Jacobi forms of higher degree*, Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, edited by Jin-Woo Son and

Jae-Hyun Yang, the Pyungsan Institute for Mathematical Sciences, (1993), 33-58.

[152] J.-H. Yang, *Singular Jacobi Forms,* Trans. Amer. Math. Soc. **347 (6)** (1995), 2041-2049.

[153] J.-H. Yang, *Construction of vector valued modular forms from Jacobi forms,* Canadian J. of Math. **47 (6)** (1995), 1329-1339 or arXiv:math.NT/0612502.

[154] J.-H. Yang, *Kac-Moody algebras, the monstrous moonshine, Jacobi forms and infinite products,* Proceedings of the 1995 Symposium on Number theory, geometry and related topics, the Pyungsan Institute for Mathematical Sciences (1996), 13–82 or arXiv:math.NT/0612474.

[155] J.-H. Yang, *A geometrical theory of Jacobi forms of higher degree,* Proceedings of Symposium on Hodge Theory and Algebraic Geometry (edited by Tadao Oda), Sendai, Japan (1996), 125-147 *or* Kyungpook Math. J. **40 (2)** (2000), 209-237 *or* arXiv:math.NT/0602267.

[156] J.-H. Yang, *The Method of Orbits for Real Lie Groups,* Kyungpook Math. J. **42 (2)** (2002), 199-272 *or* arXiv:math.RT/0602056.

[157] J.-H. Yang, *A note on a fundamental domain for Siegel-Jacobi space,* Houston Journal of Mathematics **32 (3)** (2006), 701–712.

[158] J.-H. Yang, *Invariant metrics and Laplacians on Siegel-Jacobi space,* arXiv:math.NT/0507215 v1 or Journal of Number Theory **127** (2007), 83–102.

[159] J.-H. Yang, *Invariant metrics and Laplacians on Siegel-Jacobi disk,* arXiv:math.NT/0507217 v1 or revised version (2006), to appear in Chinese Annals of Mathematics, Ser. B.

[160] H. Yoshida, *On the zeta functions of Shimura varieties and periods of Hilbert modular forms*, Duke Math. J. **75** (1994), 121–191.

[161] H. Yoshida, *Motives and Siegel modular forms*, Amer. J. Math. **123** (2001), no. 6, 1171–1197.

[162] H. Yoshida, *Hiroyuki Yoshida's letter to Shalika:* April 16, 2001.

[163] H. Yoshida, *Motives and Siegel modular forms*, Proceedings of Japanese-German Seminar: Explicit structure of modular forms and zeta functions, edited by T. Ibukiyama and W. Kohnen (2002), 197–115.

[164] D. Zagier, *Sur la conjecture de Saito-Kurokawa (d'après H. Maass)*: Seminaire Delange-Pisot-Poitou, Paris, 1979-80, Progress in Mathematics **12**, Birkhäuser, Boston, Basel and Stuttgart (1981), 371–394.

[165] C. Ziegler, *Jacobi Forms of Higher Degree*, Abh. Math. Sem. Hamburg **59** (1989), 191–224.

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MR2547502 (2010m:11062) 11F46 (11F50 14K10)
Yang, Jae-Hyun (KR-INHA)
Theory of the Siegel modular variety. (English summary)
Number theory and applications, 219–278, *Hindustan Book Agency, New Delhi,* 2009.

This article gives an overview of a large amount of results in the theory of Siegel and Jacobi forms and Hecke theory. It contains a lot of useful citations and historical remarks. A bit more precisely, it treats the following topics:

- Invariant metrics, Laplacians and differential operators on Siegel space.
- Siegel's fundamental domain.
- Basic properties of Siegel modular forms.
- The Siegel operator.
- Construction of Siegel modular forms via Klingen Eisenstein series and theta series.
- Singular modular forms.
- Structure of the Hecke algebra and its action on Siegel modular forms.
- Jacobi forms.
- Lifting of elliptic cusp forms to Siegel modular forms, in particular some details on the Ikeda lift.
- Holomorphic differential forms and their relation to Siegel modular forms.
- Types of the modular varieties $\mathcal{A}_g = \Gamma_g \backslash \mathfrak{H}_g$.
- Mumford's extension of Hirzebruch's proportionality theorem.
- Motives $M(f)$ associated to Siegel modular forms f, in particular some details from three papers by Yoshida culminating in the result that $M(f)$ has at most $g+1$ period invariants.

{For the entire collection see MR2547486 (2010f:11006)}

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Yang, Jae-Hyun

Theory of the Siegel modular variety. (English)

Adhikari, Sukumar Das (ed.) et al., Number theory and applications. Proceedings of the international conferences on number theory and cryptography, Allahabad, India, December 2006 and February 2007. New Delhi: Hindustan Book Agency. 219-278 (2009). ISBN 978-81-85931-97-5/hbk

This survey covers the theory of Siegel modular forms and the Hecke algebra and therefore takes a different point of view from the earlier survey [The geometry of Siegel modular varieties. Mathematical Society of Japan. Adv. Stud. Pure Math. 35, 89–156 (2002; Zbl 1074.14021)] of *K. Hulek* and the reviewer. Nevertheless there is, inevitably, considerable overlap. A significant amount of material is not covered by either article: the author mentions harmonic analysis and the Galois representations associated with Siegel modular forms as examples of this.

The paper under review covers invariant metrics and Laplacians; the fundamental domain for the action of the symplectic group on the period domain; Siegel modular forms and the action of the Hecke algebra; Jacobi forms and lifting; differential forms on Siegel modular varieties; Hirzebruch-Mumford proportionality; and some work of Yoshida on motives.

While the parts on the Hecke algebra and on motives in particular are helpful, it must unfortunately be noted that some places contain misinterpretations of the literature, historical inaccuracies or outright mistakes. One of the most important of these relates to the Kodaira dimension of $\mathcal{A}_6$. In 2005 a claim had been made of a proof that $\mathcal{A}_6$ was of general type. This claim was never accepted by the experts and was withdrawn in July 2007, before the publication of this paper but presumably after its acceptance in final form. Here it is treated as a theorem, and consequently all the statements of the current state of knowledge on this question are wrong. (As of August 2012, nothing whatever is known about the Kodaira dimension of $\mathcal{A}_6$.)

There are many other problems. Let us mention a few. It is stated on page 220 that "Siegel's fundamental domain $\mathcal{F}_g$ is now known as the Siegel modular variety and is usually known as $\mathcal{A}_g$", but of course $\mathcal{A}_g$ is the quotient, not the fundamental domain. Problem 1 on page 263 asks whether $\mathcal{A}_3$ is rational, but this was proved by *P. Katsylo* in 1996 [Comment. Math. Helv. 71, No. 4, 507–524 (1996; Zbl 0885.14013)]. Problem 3 on the same page asks "what type of varieties are $\mathcal{A}_g(k)$ for $g \geq 3$ and $k \geq 2$?": the answer (except for the unknown case $g = 6$, $k = 1$) is given in Section II.2 of [*K. Hulek* and *G. K. Sankaran*, Higher dimensional birational geometry. Papers from the international conference on higher dimensional algebraic varieties held at Kyoto University, Kyoto, Japan, June 2–6 and 9–13, 1997. Tokyo: Mathematical Society of Japan. Adv. Stud. Pure Math. 35, 89–156 (2002; Zbl 1074.14021)], with full attributions. In particular the fact that $\mathcal{A}_2(k)$ is of general type for $k \geq 4$, here attributed to *X. Wang* [Contemp. Math. 150, 361–365 (1993; Zbl 0811.14039)], actually follows immediately from *D. Mumford*'s paper [Lect. Notes Math. 997, 348–375 (1983; Zbl 0527.14036)] of a decade earlier: Wang gives a different proof and some other results. On the other hand $\mathcal{A}_2(3)$ is

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rational, but this has been known not since *G. van der Geer*'s paper [Math. Ann. 278, 401–408 (1987; Zbl 0688.14034)] as stated here but at least since 1936, when *J. A. Todd* [Proc. Lond. Math. Soc., II. Ser. 42, 316–323 (1936; Zbl 0016.04001)] proved that the Burkhardt quartic is rational. The relation between the Burkhardt quartic and level 3 abelian surfaces was established by *H. Burkhardt* himself [Math. Ann. XXXVIII, 161–224 (1891; JFM 23.0490.01] in 1891. van der Geer gives a modern proof and stronger results. Question 1 on page 261 asks whether every stably rational variety is rational, saying that the question was raised by Bogomolov. This famous problem of Zariski was solved by famous counterexamples of *A. Beauville*, et al. [Ann. Math. (2) 121, 283–318 (1985; Zbl 0589.14042)] in 1985. Finally the paragraph on the Kodaira dimension of $\mathcal{M}_g$ states, correctly, that $\mathcal{M}_g$ was shown to be of general type for odd $g \geq 25$, but omits to mention the equally strong results of *D. Eisenbud* and *J. Harris* [Invent. Math. 90, 359–387 (1987; Zbl 0631.14023)] for g even.

G. K. Sankaran (Bath)

Keywords : Siegel modular variety; Siegel modular forms; abelian varieties; Satake parameters; lifting invariant holomorphic differential forms; proportionality theorem; motives
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∗14K10 Algebraic moduli, classification (abelian varieties)
11F46 Siegel modular groups and their modular and automorphic forms
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[I] Topics on the Langlands Program

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A NOTE ON THE ARTIN CONJECTURE

JAE-HYUN YANG

ABSTRACT. In this paper, we survey some recent results on the Artin conjecture and discuss some aspects for the Artin conjecture.

1. INTRODUCTION

Let $K/\mathbb{Q}$ be a Galois extension of $\mathbb{Q}$ and $\rho : \mathrm{Gal}(K/\mathbb{Q}) \longrightarrow GL(n,\mathbb{C})$ a nontrivial irreducible representation of its Galois group. E. Artin [1] associated to this data an L-function $L(s,\rho)$, defined for $\mathrm{Re}\, s > 1$, which he conjectured to continue analytically to an entire function on the whole complex plane $\mathbb{C}$ satisfying a functional equation. In 1947, R. Brauer [6] showed that the Artin L-function $L(s,\rho)$ has a meromorphic continuation to a meromorphic function on $\mathbb{C}$ and satisfies a functional equation.

Artin established his conjecture for the *monomial* representations, those induced from one-dimensional representation of a subgroup. His conjecture has not been solved yet in any dimension ≥ 2. More evidence is provided in dimension 2 by R. Langlands, J. Tunnell, R. Taylor et al. In the case of two dimensional icosahedral representations, his conjecture still remains open. When ρ is an odd icosahedral representation, infinitely many examples of the Artin conjecture are known by the work of R. Taylor and others.

This article is organized as follows. In Section 2, we review the Galois representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ roughly. In Section 3, we describe the definition of the Artin L-function. In Section 4, we explain the connection between the Artin conjecture and the Langlands Functoriality Conjecture. In Section 5, we survey some known results on the Artin conjecture in the two dimensional case. In the final section we discuss some aspects for the Artin conjecture.

NOTATIONS: Throughout this paper, F denotes a number field, $\overline{F}$ an algebraic closure of F, and $\mathrm{Gal}(\overline{F}/F)$ the absolute Galois group of F. We regard $\mathrm{Gal}(\overline{F}/F)$ as a topological group relative to the Krull topology. We write $\mathbb{A}_F$ and I_F for the adele ring and the idele group attached to F respectively. For each place v of F, we let F_v be the completion of F relative to v. We also fix an algebraic closure $\overline{F_v}$ of F_v for each place v. For a square matrix A, $\mathrm{tr}(A)$ denotes the trace of A.

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2. Galois Representations

R. Taylor published a good survey paper [23] about Galois representations. The content of this section is a brief description of Section 1 in [23].

Let $\mathbb{Q}$ be the field of rational numbers and $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$. We let $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of $\mathbb{Q}$. We see that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is a profinite topological group, a basis of open neighborhoods of the identity being given by the subgroups $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ as K runs over subextensions of $\overline{\mathbb{Q}}/\mathbb{Q}$ which is finite over $\mathbb{Q}$. Let $\mathbb{Q}_p$ be the field of p-adic numbers, which is a totally disconnected locally compact topological field. $\overline{\mathbb{Q}}_p/\mathbb{Q}_p$ is an infinite extension of $\mathbb{Q}_p$ and $\overline{\mathbb{Q}}_p$ is not complete. We shall denote its completion by $\mathbb{C}_p$. Let $\mathbb{Z}_p$ (resp. $\mathcal{O}_{\overline{\mathbb{Q}}_p}$) be the ring of integers in $\mathbb{Q}_p$ (resp. $\overline{\mathbb{Q}}_p$). These are local rings with maximal ideals $p\mathbb{Z}_p$ and $\mathfrak{m}_{\overline{\mathbb{Q}}_p}$ respectively. Then it is easy to see that the field $\overline{\mathbb{F}}_p := \mathcal{O}_{\overline{\mathbb{Q}}_p}/\mathfrak{m}_{\overline{\mathbb{Q}}_p}$ is an algebraic closure of the field $\mathbb{F}_p := \mathbb{Z}_p/p\mathbb{Z}_p$. Thus we obtain a continuous map

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \longrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$$

which is surjective. Its kernel is called the *inertia subgroup* of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, and is denoted by $I_{\mathbb{Q}_p}$. The Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ is procyclic and has a canonical generator Fr_p called the Frobenius element defined by

$$\mathrm{Fr}_p(x) = x^p, \quad x \in \overline{\mathbb{F}}_p.$$

I want to describe $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ via its representations. We have two natural representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, which are

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL(n, \mathbb{C}), \quad \text{the Artin representations}$$

and

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL(n, \overline{\mathbb{Q}}_l), \quad \text{the } l\text{-adic representations.}$$

Here $GL(n, \overline{\mathbb{Q}}_l)$ is a group with l-adic topology. These representations are continuous.

The l-adic representations are closely related to an arithmetic geometry.

- A choice of embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l$ establishes a bijection between isomorphism classes of Artin representations and isomorphism classes of l-adic representations with open kernel.
- There is a unique character
$$\chi_l : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathbb{Z}_l^\times \subset \overline{\mathbb{Q}}_l^\times$$
such that
$$\sigma\zeta = \zeta^{\chi_l(\sigma)}$$
for all l-power roots of unity ζ. This is called the *l-adic cyclotomic character*.
- If $X/\mathbb{Q}$ is a smooth projective variety, then the natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the cohomology
$$H^i(X(\mathbb{C}), \overline{\mathbb{Q}}_l) \cong H^i_{et}(X \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_l)$$
is an l-adic representation.

We now discuss l-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Let $W_{\mathbb{Q}_p}$ be the subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ consisting of elements $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ such that σ maps to $\mathrm{Fr}_p^{\mathbb{Z}} \subset \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. We endow $W_{\mathbb{Q}_p}$ with a topology by decreeing that $I_{\mathbb{Q}_p}$ with its usual topology should be an open subgroup of $W_{\mathbb{Q}_p}$. We first consider the case $l \neq p$. We define a *WD-representation* of $W_{\mathbb{Q}_p}$ over a field E to be a pair

$$r : W_{\mathbb{Q}_p} \longrightarrow GL(V), \quad \text{a continuous representation of } W_{\mathbb{Q}_p} \text{ with open kernel}$$

and

$$N \in \mathrm{End}(V), \quad \text{a nilpotent endomorphism of } V$$

such that

$$r(\phi)\, N\, r(\phi^{-1}) = p^{-1}N$$

for every lift $\phi \in W_{\mathbb{Q}_p}$ of Fr_p, where V is a finite dimensional E-vector space. A WD-representation (r, N) is said to be *unramified* if $N = 0$ and $r(I_{\mathbb{Q}_p}) = \{1\}$. In the case $E = \overline{\mathbb{Q}}_l$, we call a WD-representation (r, N) *l-integral* if all eigenvalues of $r(\phi)$ has the absolute value 1. If $l \neq p$, then there is an equvalence of categories between l-integral WD-representations of $W_{\mathbb{Q}_p}$ over $\overline{\mathbb{Q}}_l$ and l-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. We will write $WD_p(R)$ for the WD-representation associated to an l-adic representation R of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. An l-adic representation R is said to be *unramified* if $WD_p(R)$ is unramified. The case $l = p$ is much more complicated because there are many more p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. These have been extensively studied by J.-M. Fontaine et al. They single out certain special p-adic representations which are called *de Rham*. Indeed most p-adic representations are not de Rham. To any de Rham representation R of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on a $\overline{\mathbb{Q}}_p$-vector space V they associate the following pair:

- A WD-representation $\mathrm{WD}_p(R)$ of $W_{\mathbb{Q}_p}$ over $\overline{\mathbb{Q}}_p$.
- A multiset $\mathrm{HT}(R)$ of $\dim V$ integers, called the Hodge-Tate numbers of R. The multiplicity of i in $\mathrm{HT}(R)$ is

$$\dim_{\overline{\mathbb{Q}}_p}(V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(i))^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)},$$

  where $\mathbb{C}_p(i)$ denotes $\mathbb{C}_p$ with $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-action and $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts on $\mathbb{C}_p$ via $\chi_p(\sigma)^i$ times its usual Galois action on $\mathbb{C}_p$.

We refer to [10, 11, 12] and [2] for more details on de Rham representations and their related materials.

We now discuss a so-called *geometric* l-adic representations. Fontaine and Mazur [13] proposed the following conjecture.

Conjecture A (Fontaine-Mazur) *Suppose that*

$$R : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL(V)$$

is an irreducible l-adic representation which is unramified at all but finitely many primes and with $R|_{Gal(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)}$ de Rham. Then there is a smooth projective variety $X/\mathbb{Q}$ and integers $i \geq 0$ and j such that V is a subquotient of $H^i(X(\mathbb{C}), \overline{\mathbb{Q}}_l(j))$. In particular R is pure of some wight $w \in \mathbb{Z}$.

Tate formulated the following conjecture.

Conjecture B (Tate). *Suppose that $X/\mathbb{Q}$ is a smooth projective variety. Then there is a decomposition*

$$H^i(X(\mathbb{C}),\overline{\mathbb{Q}}) = \oplus_j M_j$$

with the following properties:

1. *For each prime l and for each embeddings $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l$, $M_j \otimes_{\overline{\mathbb{Q}},l} \overline{\mathbb{Q}}_l$ is an irreducible subrepresentation of $H^i(X(\mathbb{C}),\overline{\mathbb{Q}}_l)$.*
2. *For all indices j and for all primes p there is a WD-representation $WD_p(M_j)$ of $W_{\mathbb{Q}_p}$ over $\overline{\mathbb{Q}}_l$ such that*
$$WD_p(M_j)\otimes_{\overline{\mathbb{Q}},\iota}\overline{\mathbb{Q}}_l \cong WD_p(M_j\otimes_{\overline{\mathbb{Q}},\iota}\overline{\mathbb{Q}}_l)$$
for all primes l and all embeddings $\iota : \overline{\mathbb{Q}} \longrightarrow \overline{\mathbb{Q}}_l$.
3. *There is a multiset of integers $HT(M_j)$ such that*
   (a) *for all primes l and all embeddings $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l$*
$$\mathrm{HT}(M_j\otimes_{\overline{\mathbb{Q}},\iota}\overline{\mathbb{Q}}_l) = \mathrm{HT}(M_j)$$
   (b) *and for all $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$*
$$\dim_{\mathbb{C}}(M_j\otimes_{\overline{\mathbb{Q}},\iota}\mathbb{C}) \cap H^{a,i-a}(X(\mathbb{C}),\mathbb{C})$$
   is the multiplicity of a in $HT(M_j)$.

If one believes conjecture A and B, then geometric l-adic representations should come in compatible families as l varies. There are many ways to make precise the notion of such a compatible family. See [23] for one of such family.

3. Artin L-Functions

Let F be a number field. We let

$$\sigma : \mathrm{Gal}(\overline{F}/F) \longrightarrow GL(V)$$

be a finite dimensional Galois representation over F, where V is a finite dimensional complex vector space. The *Artin L-function* $L(s,\sigma)$ attached to the Galois representation σ is defined to be an Euler product

$$L(s,\sigma) = \prod_v L(s,\sigma_v),$$

where v runs over all places of F. The local factor $L(s,\sigma_v)$ is defined as follows. First we choose an embedding $i_v : \overline{F} \longrightarrow \overline{F_v}$, which gives rise to an embedding of Galois groups

$$j_v : \mathrm{Gal}(\overline{F_v}/F_v) \longrightarrow \mathrm{Gal}(\overline{F}/F)$$

via *restriction*. The composition $\sigma_v = \sigma \circ j_v$ is a continuous representation of $\mathrm{Gal}(\overline{F_v}/F_v)$. It depends on the choice of an embedding i_v, but different choices of i_v lead to conjugate embeddings j_v. So the equivalence class of σ_v is well defined and depends *only* on v.

In the nonarchimedean case, we let k_v and $\overline{k_v}$ denote the residue fields of F_v and $\overline{F_v}$ respectively. $\mathrm{Gal}(\overline{F_v}/F_v)$ acts on $\overline{k_v}$ and we have an exact sequence

$$1 \longrightarrow I_v \longrightarrow \mathrm{Gal}(\overline{F_v}/F_v) \longrightarrow \mathrm{Gal}(\overline{k_v}/k_v) \longrightarrow 1,$$

where I_v is the inertia subgroup. We set $q_v = |k_v|$. A *Frobenius element* Fr_v is an element of $\mathrm{Gal}(\overline{F_v}/F_v)$ whose image in $\mathrm{Gal}(\overline{k_v}/k_v)$ is the automorphism

$$x \longmapsto x^{q_v}, \quad x \in \overline{k_v}.$$

We note that the action of $\sigma(\mathrm{Fr}_v)$ on the subspace V^{I_v} of inertial invariants in V is independent of the choice of Fr_v. We define the local factor $L(s, \sigma_v)$ at v by

$$L(s, \sigma_v) = \det\left(1 - q_v^{-s}\sigma_v(\mathrm{Fr}_v)|_{V^{I_v}}\right)^{-1}.$$

The Galois representation σ is said to be *unramified* at v if $\sigma_v(I_v) = 1$. In this case, the element $\sigma_v(\mathrm{Fr}_v)$ is independent of the choice of Fr_v. The *Frobenius class* attached to v is the conjugacy class $\{\sigma_v(\mathrm{Fr}_v)\}$ of $\sigma_v(\mathrm{Fr}_v)$ in $GL(V)$. The Frobenius class is independent of the choice of an embedding j_v and thus depends only on v. We note that it is a semisimple conjugacy class, that is, it consists of diagonalizable elements.

If v is archimedean, then $F_v \cong \mathbb{R}$ or $\mathbb{C}$. In case $F_v \cong \mathbb{R}$, we get $\mathrm{Gal}(\overline{F_v}/F_v) \cong \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, c\}$, where c denotes the complex conjugation. The eigenvalues of $\sigma_v(c)$ are ± 1. Let m_+ (resp. m_-) be the number of $+1$ (resp. -1) eigenvalues of $\sigma_v(c)$. In this case, we define the local factor $L(s, \sigma_v)$ by

$$L(s, \sigma_v) = \left(\pi^{-s/2}\Gamma(s/2)\right)^{m_+}\left(\pi^{-(s+1)/2}\Gamma((s+1)/2)\right)^{m_-}.$$

If $F_v \cong \mathbb{C}$, then $\mathrm{Gal}(\overline{F_v}/F_v) \cong \{1\}$. In this case, we define

$$L(s, \sigma_v) = \left(2(2\pi)^{-s}\Gamma(s)\right)^n,$$

where $n = \dim_{\mathbb{C}} V$.

It is easy to see that

$$L(s, \sigma \oplus \tau) = L(s, \sigma)L(s, \tau)$$

for any two Galois representations σ and τ. For any finite set S of places, we define the *partial* L-function $L_S(s, \sigma)$ by

$$L_S(s, \sigma) = \prod_{v \notin S} L(s, \sigma_v).$$

We observe that if σ is the trivial representation and S is the archimedean places, then

$$L_S(s, \sigma) = \prod_{v < \infty}\left(1 - q_v^{-s}\right)^{-1}$$

is nothing but the so-called Dedekind zeta function $\zeta_F(s)$of F.

4. The Artin Conjecture and Functoriality

Since the eigenvalues of $\sigma_v(\mathrm{Fr}_v)$ at each place v are roots of unity, it is easy to see that the Euler product for $L(s,\sigma)$ converges absolutely for $\mathrm{Re}\, s > 1$. According to the works of E. Hecke [17], E. Artin [1] and R. Brauer [6], we obtain the following theorem.

Theorem 4.1. *Let F be a number field. Let $\sigma : Gal(\overline{F}/F) \longrightarrow GL(V)$ be a finite dimensional complex Galois representation over F. Then the Artin L-function $L(s,\sigma)$ has a meromorphic continuation to a meromorphic function on $\mathbb{C}$. Moreover $L(s,\sigma)$ satisfies a functional equation*

$$L(s,\sigma) = \epsilon(s,\sigma)L(1-s,\sigma^*),$$

where $\epsilon(s,\sigma)$ is the so-called epsilon factor (cf. [22]) and σ^* denotes the contragredient representation of σ.

Artin Conjecture. *If $\sigma : Gal(\overline{F}/F) \longrightarrow GL(V)$ is a nontrivial irreducible finite dimensional complex Galois representation over F, then the Artin L-function $L(s,\sigma)$ can be analytically continued to an entire function on $\mathbb{C}$.*

Let F be a number field. For a cuspidal representation π of $GL(n,\mathbb{A}_F)$, we can define the *automorphic L-function* $L(s,\pi)$ of π given by

$$L(s,\pi) = \prod_v L(s,\pi_v),$$

where v runs over all places of F. The precise definition of $L(s,\pi_v)$ can be found in [5], [14], [15] and [19]. Jacquet and Langlands [18] proved that if $n=2$, $L(s,\pi)$ can be analytically continued to an entire function on the whole complex plane $\mathbb{C}$. Godement and Jacquet [16] proved that for any positive integer n, the L-function $L(s,\pi)$ can be analytically continued to an entire function on the whole complex plane $\mathbb{C}$.

R. Langlands proposed the following conjecture in order to attack the Artin conjecture.

Langlands Functoriality Conjecture. *Let F be a number field. Let*

$$\sigma : \mathrm{Gal}(\overline{F}/F) \longrightarrow GL(n,\mathbb{C})$$

be a nontrivial irreducible finite dimensional complex Galois representation over F. Then there exists a cuspidal representation $\pi(\sigma)$ of $GL(n,\mathbb{A}_F)$ such that

$$L(s,\sigma) = L(s,\pi(\sigma)).$$

We observe that if Langlands Functoriality Conjecture is true, by the work of Jacquet, Langlands and Godement, the Artin conjecture is true. The Langlands Functoriality Conjecture gave rise to some solutions of the Artin conjecture for an irreducible two-dimensional Galois representation.

I want to introduce the recent work of Andrew Booker. Let

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL(2, \mathbb{C})$$

be an irreducible two-dimensional complex Galois representation over $\mathbb{Q}$. Booker [3] proved the following result.

Theorem 4.2. *If $L(s, \rho)$ is not automorphic, then it has infinitely many poles. In particular, the Artin conjecture for ρ implies the Langlands Functoriality Conjecture for ρ.*

5. Special Cases of the Artin Conjecture

Let F be a number field. Let

$$\rho : \mathrm{Gal}(\overline{F}/F) \longrightarrow GL(2, \mathbb{C})$$

be an irreducible two-dimensional complex Galois representation over F. The adjoint representation of the group $GL(2, \mathbb{C})$ on the Lie algebra $\mathfrak{gl}(2, \mathbb{C})$ induces the adjoint action of $GL(2, \mathbb{C})$ on the three dimensional Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ of 2×2 complex matrices of trace zero. We denote this representation by

$$\mathrm{Ad} : GL(2, \mathbb{C}) \longrightarrow GL(3, \mathbb{C}).$$

The symmetric bilinear form $\mathrm{tr}(AB)$ is invariant under the adjoint action of $GL(2, \mathbb{C})$, and the image of Ad is isomorphic to the complex orthogonal group $SO(3, \mathbb{C})$ defined by this bilinear form $\mathrm{tr}(AB)$. Irreducible two-dimensional representations are classified according to the image of $\mathrm{Ad} \circ \rho$ in $SO(3, \mathbb{C})$. It is known that a finite subgroup of $SO(3, \mathbb{C})$ is either *cyclic, dihedral* or isomorphic to one of the symmetry groups of the Platonic solids:

(1) *tetrahedral group* $\cong A_4$;
(2) *octahedral group* $\cong S_4$;
(3) *icosahedral group* $\cong A_5$.

We shall say that ρ is of *cyclic, dihedral, tetrahedral, octahedral, icosahedra type* if the image of $\mathrm{Ad} \circ \rho$ in $SO(3, \mathbb{C})$ is of the corresponding type. The Artin conjecture was solved by E. Artin for the cyclic and dihedral type, by R. Langlands [20] for the tetrahedral type, and was solved completely by J. Tunnell [25, 26] for the octahedral type. Indeed we can show that $\pi(\rho)$ exists if ρ is of cyclic, dihedral, tetrahedral or octahedral type. We refer to [21] for sketchy proofs in such these types. The Artin conjecture for the *icosahedral type* has not been solved yet, although it has been verified in few very special cases [7, 8, 9, 24].

Recently Taylor et al gave some evidences to the Artin conjecture for the icosahedral type. K. Buzzard, M. Dickinson, N. Sheherd-Baron and R. Taylor [8] proved that the Artin conjecture is true for certain special odd icosahedral representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by showing that they are modular. I describe this content explicitly.

Theorem 5.1. *Suppose that $\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL(2,\mathbb{C})$ is a continuous representation and that ρ is odd, i.e., the determinant of $\rho(c)$ is -1, where c is the complex conjugation. If ρ is of icosahedral type, we assume that*

- *the projectivised representation $proj(\rho) : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow PGL(2,\mathbb{C})$ is unramified at 2*

  and that image of a Frobenius element at 2 under $proj(\rho)$ has order 3,
- *and $proj(\rho)$ is unramified at 5.*

Then there is a new form of weight one such that for all primes p the p-th Fourier coefficient of f equals the trace of Frobenius at p on the inertia at p covariants of ρ. In particular the Artin L-function for ρ is the Mellin transform of a newform of weight one and is an entire function.

Moreover R. Taylor [24] proved the following.

Theorem 5.2. *Let $\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL(2,\mathbb{C})$ is a continuous representation and that ρ is odd, i.e., the determinant of $\rho(c)$ is -1, where c is the complex conjugation. If ρ is of icosahedral type, we assume that the projective image of the inertia group at 3 has odd order and the projective image of the decomposition group at 5 is unramified at 2. Then ρ is modular and its Artin L-function $L(s,\rho)$ is entire.*

6. Final Remarks

As mentioned before, we still have no idea of verifying the Artin conjecture for n-dimensional Galois representations with $n \geq 3$. In the case of two dimensional icosahedral representations, the Artin conjecture still remains open. If the Artin conjecture is true for a certain Galois representation ρ, it might be interesting to find methods for locating zeros of the Artin L-function $L(s,\rho)$. In [4], A. Booker discusses two methods for locating zeros of $L(s,\rho)$. He also presents a group-theoretic criterion under which one may verify the Artin conjecture for some non-monomial Galois representations, up to finite height in the complex plane.

References

[1] E. Artin, *Zur Theorie der L-Reihen mit allgemeinen Gruppencharakteren*, Abh. Math. Sem. Univ. Hamburg **8** (1930), 292-306.

[2] L. Berger, *Représentations p-adiques et équations différentielles*, Invent. Math. **148** (2002), 219-284.

[3] A. Booker, *Poles of Artin L-functions and the strong Artin conjecture*, Ann. of Math. **158** (2003), 1089-1098.

[4] A. Booker, *Artin's Conjecture, Turing's Method and the Riemann Hypothesis*, arXiv:math.NT/0507502 v1, 25 July (2005).

[5] A. Borel, *Automorphic L-functions*, Proceedings of Symposia in Pure Mathematics, Part II, American Math. Soc., Volume **33** (1979), 27-61.

[6] R. Brauer, *On Artin's L-series with general group characters*, Ann. of Math. **48** (1971), 502-514.

[7] J. P. Buhler, *Icosahedral Galois representations,* Lecture Notes in Mathematics **654**, Springer-Verlag, New York, 1978.

[8] K. Buzzard, M. Dickinson, N. Shepard-Barron and R. Taylor, *On Icosaheral Artin representations,* Duke Math. J. **109** (2001), 283-318.

[9] K. Buzzard and W. Stein, *A mod five approach to modularity of icosaheral Galois representations,* Pacific Jour. Math. **203** (2002), 265-282.

[10] J.-M. Fontaine, *Le corps des périods p-adiques,* Astérisque **223**, Soc. Math. de France, Paris (1994), 59-111.

[11] J.-M. Fontaine, *Représentations p-adiques semi-stables,* Astérisque **223**, Soc. Math. de France, Paris (1994), 113-184.

[12] J.-M. Fontaine, *Représentations l-adiques potentiellement semi-stables,* Astérisque **223**, Soc. Math. de France, Paris (1994), 321-347.

[13] J.-M. Fontaine and B. Mazur, *Geometric Galois Representations,* in Elliptic Curves, Modular Forms and Fermat's Last Theorem (J. Coates and S.-T. Yau eds.), International Press (1995), 41-78.

[14] S. Gelbart, *An elementary introduction to the Langlands program,* Bull. Amer. Math. Soc. **10** (1984), 177-219.

[15] S. Gelbart and H. Jacquet, *A relation between automorphic representations of* $GL(2)$ *and* $GL(3)$, Ann. Sci. École Norm. Sup. (4) **11** (1978), 471-542.

[16] R. Godement and H. Jacquet, *Zeta-functions of Simple Algebras,* Lecture Notes in Mathematics **260**, Springer-Verlag, New York, 1972.

[17] E. Hecke, *Mathematische Werke,* 3rd ed., Vandenhoeck and Ruprecht, Göttingen, 1983 (German).

[18] H. Jacquet and R. Langlands, *Automorphic Forms on* $GL(2)$, Lecture Notes in Mathematics **114**, Springer-Verlag, New York, 1970.

[19] A. W. Knapp, *Introduction to the Langlands Program,* Proceedings of Symposia in Pure Mathematics, Volume **61** (1997), 245-302.

[20] R. Langlands, *Base Change for* $GL(2)$, Annals of Math. Studies **96**, Princeton University Press, Princeton, 1980.

[21] J. D. Rogawski, *Functoriality and the Artin Conjecture,* Proceedings of Symposia in Pure Mathematics, Volume **61** (1997), 331-353.

[22] J. Tate, *Number theoretic background,* Proceedings of Symposia in Pure Mathematics, Part II, American Math. Soc., Volume **33** (1979), 3-26.

[23] R. Taylor, *Galois Representations,* Proceedings of ICM, Beijing, Vol. I (2002), 449-474.

[24] R. Taylor, *On icosahedral Artin representations II,* Amer. J. Math. **125 (3)** (2003), 549-566.

[25] J. Tunnell, *Artin's conjecture for representations of octahedral type,* Bull. Amer. Math. Soc. **5** (1981), 173-175.

[26] J. Tunnell, *On the local Langlands conjecture for* $GL(2)$, Invent. Math. **46** (1978), 179-200.

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Langlands Functoriality Conjecture

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To the memory of my parents

Abstract. Functoriality conjecture is one of the central and influential subjects of the present day mathematics. Functoriality is the profound lifting problem formulated by Robert Langlands in the late 1960s in order to establish nonabelian class field theory. In this expository article, I describe the Langlands-Shahidi method, the local and global Langlands conjectures and the converse theorems which are powerful tools for the establishment of functoriality of some important cases, and survey the interesting results related to functoriality conjecture.

1. Introduction

Functoriality Conjecture or the Principle of Functoriality is the profound question that was raised and formulated by Robert P. Langlands in the late 1960s to establish nonabelian class field theory and its reciprocity law. Functoriality conjecture describes deep relationships among automorphic representations on different groups. This conjecture can be described in a rough form as follows: To every L-homomorphism $\varphi : {}^LH \longrightarrow {}^LG$ between the L-groups of H and G that are quasi-split reductive groups, there exists a *natural* lifting or transfer of automorphic representations of H to those of G. In 1978, Gelbart and Jacquet [22] established an example of the functoriality for the symmetric square Sym^2 of $GL(2)$ using the converse theorem on $GL(3)$. In 2002 after 24 years, Kim and Shahidi [46] established the functoriality for the symmetric cube Sym^3 of $GL(2)$ and thereafter Kim [43] proved the validity of the functoriality for the symmetric fourth Sym^4 of $GL(2)$. These results have led to breakthroughs toward certain important conjectures in number theory, those of Ramanujan, Selberg and Sato-Tate conjectures. We refer to [81] for more detail on applications to the progress made toward to the conjectures just mentioned. Recently the Sato-Tate conjecture for an important class of cases related to elliptic curves has been verified by Clozel, Harris, Shepherd-Barron

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and Taylor [11], [27], [69], [83]. Past ten years the functoriality for the tensor product $GL(2) \times GL(2) \longrightarrow GL(4)$ by Ramakrishnan [71], for the tensor product $GL(2) \times GL(3) \longrightarrow GL(6)$ by Kim and Shahidi [46], for the exterior square of $GL(4)$ by Kim [43] and the weak functoriality to $GL(N)$ for generic cuspidal representations of split classical groups $SO(2n+1)$, $Sp(2n)$ and $SO(2n)$ by Cogdell, Kim, Piatetski-Shapiro and Shahidi [13], [14] were established by applying appropriate converse theorems of Cogdell and Piatetski-Shapiro [15], [16] to analytical properties of certain automorphic L-functions arising from the Langlands-Shahidi method. In fact, the Langlands functoriality was established only for very special L-homomorphisms between the L-groups. It is natural to ask how to find a larger class of certain L-homomorphisms for which the functoriality is valid. It is still very difficult to answer this question.

The Arthur-Selberg trace formula has also provided some instances of Langlands functoriality (see [4], [6], [61], [62]). In a certain sense, it seems that the trace formula is a useful and powerful tool to tackle the functoriality conjecture. Nevertheless the incredible power of Langlands functoriality seems beyond present technology and knowledge. We refer the reader to [66] for Langlands' comments on the limitations of the trace formula. Special cases of functoriality arises naturally from the conjectural theory of endoscopy (cf. [50]), in which a comparison of trace formulas would be used to characterize the internal structure of automorphic representations of a given group. I shall not deal with the trace formula, the base change and the theory of endoscopy in this article. Nowadays local and global Langlands conjectures are believed to be encompassed in the functoriality (cf. [65], [66]). Quite recently Khare, Larsen and Savin [41], [42] made a use of the functorial lifting from $SO(2n+1)$ to $GL(2n)$, from $Sp(2n)$ to $GL(2n+1)$ and the theta lifting of the exceptional group G_2 to $Sp(6)$ to prove that certain finite simple groups $PSp_n(\mathbb{F}_{\ell^k})$, $G_2(\mathbb{F}_{\ell^k})$ and $SO_{2n+1}(\mathbb{F}_{\ell^k})$ with some mild restrictions appear as Galois groups over $\mathbb{Q}$.

This paper is organized as follows. In Section 2, we review the notion of automorphic L-functions and survey the Langlands-Shahidi method briefly following closely the article of Shahidi [77]. I would like to recommend to the reader two lecture notes which were very nicely written by Cogdell [12] and Kim [44] for more information on automorphic L-functions. In Section 3, we review the Weil-Deligne group briefly and formulate the local Langlands conjecture. We describe the recent results about the local Langlands conjecture for $GL(n)$ and $SO(2n+1)$. In Section 4, we discuss the global Langlands conjecture which is still not well formulated in the number field case. The work on the global Langlands conjecture for $GL(2)$ over a function field done by Drinfeld was extended by Lafforgue ten years ago to give a proof of the global Langlands conjecture for $GL(n)$ over a function field. We will not deal with the function field case in this article. We refer to [52] for more detail. Unfortunately there is very little known of the global Langlands conjecture in the number field case. I have an audacity to mention the Langlands hypothetical group and the hypothetical motivic Galois group following the line of Arthur's argument in [3]. In Section 5, I formulate the Langlands functoriality conjecture in several ways and describe the striking examples of Langlands functoriality established past ten years. I want to mention that there is a descent method of studying the opposite direction of the lift initiated by Ginzburg, Rallis and Soudry (see [25], [39]). I shall not deal with the descent method here. In the appendix, I describe a brief history of the converse theorems obtained by Hamburger, Hecke, Weil, Cogdell, Piatetski-Shapiro, Jinag and Soudry. I present the more or less exact formulations of the converse theorems. As mentioned earlier, the converse theorems play a crucial role in establishing the functoriality for the examples discussed in Section 5.

Notations: We denote by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We denote by $\mathbb{R}_+^*$ the multiplicative group of positive real numbers. $\mathbb{C}^*$ (resp. $\mathbb{R}^*$) denotes the multiplicative group of nonzero complex(resp. real) numbers. We denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers respectively. For a number field F, we denote by $\mathbb{A}_F$ and $\mathbb{A}_F^*$ the ring of adeles of F and the multiplicative group of ideles of F respectively. If there is no confusion, we write simply $\mathbb{A}$ and $\mathbb{A}^*$ instead of $\mathbb{A}_F$ and $\mathbb{A}_F^*$. For a field k we denote by Γ_k the Galois group $Gal(\overline{k}/k)$, where $\overline{k}$ is a separable algebraic closure of k. We denote by $\mathbb{G}_m$ the multiplicative group in one variable. $\mathbb{G}_a$ denotes the additive group in one variable.

2. Automorphic L-functions

Let G be a connected, quasi-split reductive group over a number field F. For each place v of F, we let F_v be the completion of F, $\mathfrak{o}_v$ the rings of integers of F_v, $\mathfrak{p}_v$ the maximal ideal of $\mathfrak{o}_v$, and let q_v be the order of the residue field $k_v = \mathfrak{o}_v/\mathfrak{p}_v$. We denote by $\mathbb{A} = \mathbb{A}_F$ the ring of adeles of F. We fix a Borel subgroup B of G over F. Write $B = TU$, where T is a maximal torus and U is the unipotent radical, both over F. Let P be a parabolic subgroup of G. Assume $P \supset B$. Let $P = MN$ be a Levi decomposition of P with Levi factor M and its unipotent radical N. Then $N \subset U$. For each place v of F, we let $G_v = G(F_v)$. Similarly we use B_v, T_v, U_v, P_v, M_v and N_v to denote the corresponding groups of F_v-rational points. Let $G(\mathbb{A}), B(\mathbb{A}), \cdots, N(\mathbb{A})$ be the corresponding adelic groups for the subgroups defined before. When G is unramified over a place v in the sense that G is quasi-split over F_v and that G is split over a finite unramified extension of F_v, we let $K_v = G(\mathfrak{o}_v)$. Otherwise we shall fix a special maximal compact subgroup $K_v \subset G_v$. We set $K_{\mathbb{A}} = \otimes_v K_v$. Then $G(\mathbb{A}) = P(\mathbb{A})K_{\mathbb{A}}$.

Let $\Pi = \otimes_v \Pi_v$ be a cuspidal automorphic representation of $G(\mathbb{A})$. We refer to [36], [37], [59] for the notion of automorphic representations. Let S be a finite set of places including all archimedean ones such that both Π_v and G_v are unramified for any place $v \notin S$. Then for each $v \notin S$, Π_v determines uniquely a semi-simple conjugacy class $c(\Pi_v)$ in the L-group ${}^LG_v$ of G_v as a group defined over F_v. We refer to [9], [48] for the definition and construction of the L-group. We note that there exists a natural homomorphism $\xi_v : {}^LG_v \longrightarrow {}^LG$. For a finite dimensional representation r of ${}^LG$, putting $r_v = r \circ \xi_v$, the local Langlands L-function $L(s, \Pi_v, r_v)$

associated to Π_v and r_v is defined to be (cf. [9], [54])

$$L(s,\Pi_v,r_v)=\det\Big(I-r_v\big(c(\Pi_v)\big)q_v^{-s}\Big)^{-1}. \tag{2.1}$$

We set

$$L_S(s,\Pi,r)=\prod_{v\notin S}L(s,\Pi_v,r_v). \tag{2.2}$$

Langlands (cf. [54]) proved that $L_S(s,\Pi,r)$ converges absolutely for sufficiently large $\mathrm{Re}(s)>0$ and defines a holomorphic function there. Furthermore he proposed the following question.

Conjecture A(Langlands, [54]). $L_S(s,\Pi,r)$ has a meromorphic continuation to the whole complex plane and satisfies a standard functional equation.

F. Shahidi (cf. [77], [78]) gave a partial answer to the above conjecture using the so-called Langlands-Shahidi method. I shall describe Shahidi's results briefly following his article [77].

Let P be a maximal parabolic subgroup of G and $P=MN$ its Levi decomposition with its Levi factor M and its unipotent radical N. Let A be the split torus in the center of M. For every group H defined over F, we let $X(H)_F$ be the group of F-rational characters of H. We set

$$\mathfrak{a}=\mathrm{Hom}\big(X(M)_F,\mathbb{R}\big).$$

Then

$$\mathfrak{a}^*=X(M)_F\otimes_{\mathbb{Z}}\mathbb{R}\cong X(A)_F\otimes_{\mathbb{Z}}\mathbb{R}.$$

We set $\mathfrak{a}^*_{\mathbb{C}}:=\mathfrak{a}^*\otimes_{\mathbb{R}}\mathbb{C}$. Let $\mathfrak{z}$ be the real Lie algebra of the split torus in the center $C(G)$ of G. Then $\mathfrak{z}\subset\mathfrak{a}$ and $\mathfrak{a}/\mathfrak{z}$ is of dimension 1. The imbedding $X(M)_F\hookrightarrow X(M)_{F_v}$ induces an imbedding $\mathfrak{a}_v\hookrightarrow\mathfrak{a}$, where $\mathfrak{a}_v=\mathrm{Hom}\big(X(M)_{F_v},\mathbb{R}\big)$. The Harish-Chandra homomorphism $H_P:M(\mathbb{A})\longrightarrow\mathfrak{a}$ is defined by

$$\exp\langle\chi,H_P(m)\rangle=\prod_v|\chi(m_v)|_v,\quad \chi\in X(M)_F,\ m=\otimes_v m_v\in M(\mathbb{A}).$$

We may extend H_P to $G(\mathbb{A})$ by letting it trivial on $N(\mathbb{A})$ and $K_{\mathbb{A}}$. We define $H_{P_v}:M_v\longrightarrow\mathfrak{a}_v$ by

$$q_v^{\langle\chi_v,H_{P_v}(m)\rangle}=|\chi_v(m_v)|_v,\quad \chi_v\in X(M)_{F_v},\ m_v\in M_v$$

for a finite place v, and define

$$\exp\langle\chi_v,H_{P_v}(m)\rangle=|\chi_v(m_v)|_v,\quad \chi_v\in X(M)_{F_v},\ m_v\in M_v$$

for an infinite place v. Then we have

$$\exp\langle\chi, H_P(m)\rangle = \prod_{v=\infty} \exp\langle\chi, H_{P_v}(m)\rangle \cdot \prod_{v<\infty} q_v^{\langle\chi, H_{P_v}(m)\rangle}, \tag{2.3}$$

where $\chi \in X(M)_F$ and $m = \otimes_v m_v \in M(\mathbb{A})$. We observe that the product in (2.3) is finite.

Let A_0 be the maximal F-split torus in T. We denote by Φ the set of roots of A_0. Then $\Phi = \Phi^+ \cup \Phi^-$, where Φ^+ is the set of roots generating U and $\Phi^- = -\Phi^+$. Let $\Delta \subset \Phi^+$ be the set of simple roots. The unique reduced root of A in N can be identified by an element $\alpha \in \Delta$. Let ρ_P be half the sum of roots generating N. We set

$$\widetilde{\alpha} = \langle\rho_P, \alpha\rangle^{-1}\rho_P.$$

Here, for any pair of roots α and β in Φ^+, the pairing $\langle\alpha, \beta\rangle$ is defined as follows. Let $\widetilde{\Phi}^+$ be the set of non-restricted roots of T in U. We see that the set of simple roots $\widetilde{\Delta}$ in $\widetilde{\Phi}^+$ restricts to Δ. Identifying α and β with roots in $\widetilde{\Phi}^+$, we set

$$\langle\alpha, \beta\rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)},$$

where $(\ ,\ )$ is the standard inner product on $\mathbb{R}^l$ with $l = |\widetilde{\Delta}|$.

Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $M(\mathbb{A})$. Given a $K_{\mathbb{A}} \cap M(\mathbb{A})$-finite function ϕ in the representation space V_π of π, we may extend ϕ to a function $\widetilde{\phi}$ on $G(\mathbb{A})$ (cf. [75]). Then the Eisenstein series $E(s, \widetilde{\phi}, g, P)$ is defined by

$$E(s, \widetilde{\phi}, g, P) = \sum_{\gamma \in P(F)\backslash G(F)} \widetilde{\phi}(\gamma g) \exp\langle s\widetilde{\alpha} + \rho_P, H_P(\gamma g)\rangle, \quad g \in G(\mathbb{A}). \tag{2.4}$$

The Eisensetein series $E(s, \widetilde{\phi}, g, P)$ converges absolutely for sufficiently large $\mathrm{Re}(s) \gg 0$ and extends to a meromorphic function of s on $\mathbb{C}$, with a finite number of poles in the plane $\mathrm{Re}(s) > 0$, all simple and on the real axis (See [58]).

Let W be the Weyl group of A_0 in G. We denote the subset of Δ which generates M by θ. Then $\Delta = \theta \cup \{\alpha\}$. Then there exists a unique element $\widetilde{w} \in W$ such that $\widetilde{w}(\theta) \subset \Delta$ and $\widetilde{w}(\alpha) \in \Phi^-$. Fix a representative $w \in K_{\mathbb{A}} \cap G(F)$ for $\widetilde{w}$. We shall also denote every component of w by w again.

Let

$$I(s, \pi) = \mathrm{Ind}_{M(\mathbb{A})N(\mathbb{A})}^{G(\mathbb{A})} \pi \otimes \exp\langle s\tilde{\alpha}, H_P(\,\cdot\,)\rangle \otimes 1$$

be the representation of $G(\mathbb{A})$ induced from $P(\mathbb{A})$. Then $I(s, \pi) = \otimes_v I(s, \pi_v)$ with

$$I(s, \pi_v) = \mathrm{Ind}_{M_v N_v}^{G_v} \pi_v \otimes q_v^{\langle s\tilde{\alpha}, H_P(\,\cdot\,)\rangle} \otimes 1,$$

where q_v should be replaced by exp if v is archimedean. We let M' be the subgroup of G generated by $\widetilde{w}(\theta)$. Then there is a parabolic subgroup $P' \supset B$ with $P' = M'N'$.

Here M' is a Levi factor of P' and N' is the unipotent radical of P'. For $f \in I(s,\pi)$ and sufficiently large $\mathrm{Re}(s) \gg 0$, we define

$$M(s,\pi)f(g) = \int_{N'(\mathbb{A})} f(w^{-1}ng)dn, \quad g \in G(\mathbb{A}). \tag{2.5}$$

At each place v, for sufficiently large $\mathrm{Re}(s) \gg 0$, we define a local intertwining operator by

$$A(s,\pi_v,w)f_v(g) = \int_{N'_v} f_v(w^{-1}ng)dn, \quad g \in G_v, \tag{2.6}$$

where $f_v \in I(s,\pi_v)$. Then

$$M(s,\pi) = \otimes_v A(s,\pi_v,w). \tag{2.7}$$

It follows from the theory of Eisenstein series that for $\mathrm{Re}(s) \gg 0$, $M(s,\pi)$ extends to a meromorphic function of $s \in \mathbb{C}$ with only a finite number of simple poles (cf. [58]).

Let ${}^LM$ and ${}^LN$ be the Levi factor and the unipotent radical of the parabolic subgroup ${}^LP = {}^LM\,{}^LN$ of the L-group ${}^LG$. Then we have the representation r : ${}^LM \longrightarrow \mathrm{End}({}^L\mathfrak{n})$ given by the adjoint action of ${}^LM$ on the Lie algebra ${}^L\mathfrak{n}$ of ${}^LN$. Let

$$r = r_1 \oplus r_2 \oplus \cdots \oplus r_m$$

be the decomposition of r into irreducible constituents. Each irreducible constituent (r_i, V_i) with $1 \le i \le m$ is characterized by

$$V_i = \left\{ X_{\beta^\vee} \in {}^L\mathfrak{n} \,|\, \langle \tilde{\alpha}, \beta \rangle = i \right\}, \quad i = 1, \cdots, m.$$

We refer to [55] and [77, Proposition 4.1] for more detail.

According to [53] and [55], one has

$$M(s,\pi)f = \Big(\otimes_{v\in S} A(s,\pi_v,w)f_v \Big) \bigotimes \Big(\otimes_{v\notin S} \tilde{f}_v \Big) \times \prod_{i=1}^m \frac{L_S(is,\pi,r_i)}{L_S(1+is,\pi,r_i)}, \tag{2.8}$$

where $f = \otimes_v f_v$ is an element in $I(s,\pi)$ such that for each $v \notin S$, f_v is the unique K_v-fixed vector with $f_v(e_v) = 1$, $\tilde{f}_v$ is the K_v-fixed vector in $I(-s, \widetilde{w}(\pi_v))$ with $\tilde{f}_v(e_v) = 1$, and $\widetilde{r}_i$ denotes the contragredient of r_i $(1 \le i \le m)$.

For every archimedean place v of F, let $\varphi_v : W_{F_v} \longrightarrow {}^LM_v$ be the corresponding homomorphism (cf. [63]) attached to π_v. One has a natural homomorphism η_v : ${}^LM_v \longrightarrow {}^LM$. We put

$$r_{i,v} = r_i \circ \eta_v, \quad i = 1, 2, \cdots, m.$$

Then $r_{i,v}\circ\varphi_v = r_i\circ\eta_v\circ\varphi_v$ is a finite dimensional representation of the Weil group W_{F_v} on V_i. Let $L(s, r_{i,v}\circ\varphi_v)$ be the corresponding Artin L-function attached to $r_{i,v}\circ\varphi_v$ (cf. [57]). We set

$$L^S(s,\pi,r_i) = \prod_{v=\infty} L(s, r_{i,v}\circ\varphi_v)\cdot \prod_{\substack{v\notin S\\ v<\infty}} L(s,\pi_v, r_i\circ\eta_v). \tag{2.9}$$

Let $\rho : M(\mathbb{A}) \longrightarrow \overline{M}(\mathbb{A})$ be the projection of $M(\mathbb{A})$ onto its adjoint group.

Shahidi showed the following.

Theorem 2.1(Shahidi [77]). *Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $\overline{M}(\mathbb{A})$. Then every L-function $L^S(s,\pi, r_i\circ {}^L\rho)$, $1\le i\le m$, extends to a meromorphic function of s to the whole complex plane. Moreover, if π is generic, then each $L^S(s,\pi,r_i\circ {}^L\rho)$ satisfies a standard functional equation, that is,*

$$L^S(s,\pi,r_i\circ {}^L\rho) = \varepsilon_S(s,\pi,r_i\circ {}^L\rho)\, L^S(1-s,\pi,\widetilde{r_i\circ {}^L\rho}),$$

where $\varepsilon_S(1s,\pi,r_i\circ {}^L\rho)$ is the root number attached to π and $r_i\circ {}^L\rho$, and $\widetilde{\tau}$ denotes the contragredient of a representation τ.

Furthermore, for a given generic cuspidal automorphic representation $\pi = \otimes_v\pi_v$ of $M(\mathbb{A})$, Shahidi defined the local L-functions $L(s,\pi_v,r_i)$ and the local root numbers $\varepsilon(s,\pi_v,r_i,\psi)$ with $1\le i\le m$ at *bad places v* so that the completed L-function $L(s,\pi,r_i)$ and the completed root number $\varepsilon(s,\pi,r_i)$ defined by

$$L(s,\pi,r_i) = \prod_{\text{all } v} L(s,\pi_v,r_i),\quad \varepsilon(s,\pi,r_i) = \prod_{\text{all } v}\varepsilon(s,\pi_v,r_i,\psi),\quad i=1,\cdots,m$$

satisfy a standard functional equation

$$L(s,\pi,r_i) = \varepsilon(s,\pi,r_i)\, L(1-s,\pi,\widetilde{r_i}),\quad i=1,\cdots,m. \tag{2.10}$$

Example 2.2(Kim-Shahidi [45]). Let F be a number field and let G be a simply connected semisimple split group of type G_2 over F. We set $\mathbb{A}_\infty = \prod_{v=\infty} F_v$. Let K_∞ be the standard maximal compact subgroup of $G(\mathbb{A}_\infty)$ and $K_v = G(\mathfrak{o}_v)$ for a finite place v. Then $K_{\mathbb{A}} = K_\infty \times \prod_{v<\infty} K_v$ is a maximal compact subgroup of $G(\mathbb{A})$. Fix a split maximal torus T in G and let $B=TU$ be a Borel subgroup of G. In what follows the roots are those of T in U. Let $\Delta = \{\beta_1,\beta_6\}$ be a basis of the root system Φ with respect to (T,B) with the long simple root β_1 and the short one β_6. Then the other roots are given by

$$\beta_2 = \beta_1+\beta_6,\quad \beta_3 = 2\beta_1+3\beta_6,\quad \beta_4=\beta_1+2\beta_6,\quad \beta_5=\beta_1+3\beta_6.$$

Let P be the maximal parabolic subgroup of G generated by β_1 with Levi decomposition $P=MN$, where $M\simeq GL(2)$. See [80, Lemma 2.1]. Thus one has

$$\mathfrak{a}^* = \mathbb{R}\,\beta_4,\quad \mathfrak{a} = \mathbb{R}\,\beta_4^\vee\quad\text{and}\quad \rho_P = \frac{5}{2}\,\beta_4.$$

Let $\widetilde{\alpha} = \beta_4$. Then $s\widetilde{\alpha}\,(s \in \mathbb{C})$ corresponds to the character $|\det(m)|^s$. We note that $\mathbb{A}^* = \mathbb{A}_1^* \cdot \mathbb{R}_+^*$, where $\mathbb{A}_1^*$ is the group of ideles of norm 1. Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $M(\mathbb{A}) = GL(2,\mathbb{A})$. We may and will assume that the central character ω_π of π is trivial on $\mathbb{R}_+^*$. For a K-finite function φ in the representation space of π, the Eisenstein series $E(s,\widetilde{\varphi},g) = E(s,\widetilde{\varphi},g,P)$ defined by Formula (2.4) converges absolutely for sufficiently large $\mathrm{Re}(s) \gg 0$ and extends to a meromorphic function of s on $\mathbb{C}$, with a finite number of poles in the plane $\mathrm{Re}(s) > 0$, all simple and on the real axis. The discrete spectrum $L^2_{disc}\big(G(F)\backslash G(\mathbb{A})\big)_{(M,\pi)}$ is spanned by the residues of Eisenstein series for $\mathrm{Re}(s) > 0$ (See [58]). We know that the poles of Eisenstein series coincide with those of its constant terms. So it is enough to consider term along P. For each $f \in I(s,\pi)$, the constant term of $E(s,f,g)$ along P is given by

$$E_0(s,f,g) = \sum_{w \in \Omega} M(s,\pi,w)f(g), \quad \Omega = \{1, s_6 s_1 s_6 s_1 s_6\},$$

where s_i is the reflection along β_i defined by

$$s_i(\beta) = \beta - \frac{2(\beta_i,\beta)}{(\beta_i,\beta_i)}\beta_i, \quad 1 \le i \le 6,\ \beta \in \Phi.$$

Weyl group representatives are all chosen to lie in $K_{\mathbb{A}} \cap G(F)$. Here

$$M(s,\pi,w)f(g) = \int_{N_w^-(\mathbb{A})} f(w^{-1}ng)dn = \prod_v \int_{N_w^-(F_v)} f_v(w_v^{-1} n_v g_v)dn_v,$$

where $g = \otimes_v g_v \in G(\mathbb{A})$, $f = \otimes_v f_v$ is an element of $I(s,\pi)$ such that f_v is the unique K_v-fixed function normalized by $f_v(e_v) = 1$ for almost all v, and

$$N_w^- = \prod_{\substack{\alpha>0 \\ w^{-1}\alpha<0}} U_\alpha, \quad U_\alpha = \text{the one parameter unipotent subgroup.}$$

Let $\mathrm{St} : GL(2,\mathbb{C}) \longrightarrow GL(2,\mathbb{C})$ be the standard representation of $GL(2,\mathbb{C})$ and

$$\mathrm{Sym}^3 : GL(2,\mathbb{C}) = {}^L M \longrightarrow GL(4,\mathbb{C})$$

be the third symmetric power representation of $GL(2,\mathbb{C})$. Let

$$\big(\mathrm{Sym}^3\big)^0 = \mathrm{Sym}^3 \bigotimes \big(\wedge^2\, \mathrm{St}\big)^{-1}$$

be the adjoint cube representation of $GL(2,\mathbb{C})$ (cf. [80, p. 249]). Then the adjoint representation r of ${}^L M = GL(2,\mathbb{C})$ on the Lie algebra ${}^L\mathfrak{n}$ of ${}^L N$ is given by

$$r = \big(\mathrm{Sym}^3\big)^0 \oplus \wedge^2 \mathrm{St}.$$

According to Formula (2.8), one obtain, for $w = s_6 s_1 s_6 s_1 s_6$, the formula

$$\begin{aligned} M(s,\pi)f = \quad & \Big(\otimes_{v\in S} M(s,\pi_v,w)f_v\Big) \bigotimes \Big(\otimes_{v\notin S}\tilde{f}_v\Big) \\ & \times L_S\big(s,\widetilde{\pi},(\mathrm{Sym}^3)^0\big)\, L_S\big(2s,\widetilde{\pi},\wedge^2\mathrm{St}\big) \\ & \times L_S\big(1+s,\widetilde{\pi},(\mathrm{Sym}^3)^0\big)^{-1} L_S\big(1+2s,\widetilde{\pi},\wedge^2\mathrm{St}\big)^{-1}, \end{aligned}$$

where S is a finite set of places of F including all the archimedean places such that π_v is unramifies for every $v \notin S$. Here $L_S\big(s,\pi,\wedge^2\mathrm{St}\big)$ is the partial Hecke L-function. Kim and Shahidi [45] proved that if π is a non-monomial cuspidal representation of $M(\mathbb{A}) = GL(2,\mathbb{A})$ in the sense that $\pi \ncong \pi\otimes\eta$ for any nontrivial grossencharacter η of $F^*\backslash\mathbb{A}_F^+$, the symmetric cube L-function $L\big(s,\pi,\mathrm{Sym}^3\big)$ and the adjoint cube L-function $L\big(s,\pi,(\mathrm{Sym}^3)^0\big)$ are both *entire* and satisfy the standard functional equations

$$L\big(s,\pi,\mathrm{Sym}^3\big) = \varepsilon\big(s,\pi,\mathrm{Sym}^3\big)\, L\big(1-s,\widetilde{\pi},\mathrm{Sym}^3\big)$$

and

$$L\big(s,\pi,(\mathrm{Sym}^3)^0\big) = \varepsilon\big(s,\pi,(\mathrm{Sym}^3)^0\big)\, L\big(1-s,\widetilde{\pi},(\mathrm{Sym}^3)^0\big).$$

It follows from this fact that if π is not monomial, the partial Rankin triple L-function $L_S(s,\pi\times\pi\times\pi)$ is entire. Ikeda [32] calculated the poles of the Rankin triple L-function $L_S(s,\pi\times\pi\times\pi)$ for $GL(2)$. And we have the following relations

$$L_S(s,\pi\times\pi\times\pi) = L\big(s,\pi,\mathrm{Sym}^3\big)\,\big(L_S(s,\pi\otimes\omega_\pi)\big)^2 \tag{2.11}$$

and

$$L\big(s,\pi,\mathrm{Sym}^3\big) = L_S\big(s,\pi\otimes\omega_\pi,(\mathrm{Sym}^3)^0\big). \tag{2.12}$$

According to Formula (2.10), $L_S(s,\pi\times\pi\times\pi)$ could have double zeros at $s = 1/2$.

In [47], Kim and Shahidi studied the cuspidality of the symmetric fourth power $\mathrm{Sym}^4(\pi)$ of a cuspidal representation π of $GL(2,\mathbb{A})$ and the partial symmetric m-th power L-functions $L_S\big(s,\pi,\mathrm{Sym}^m\big)$ $(1\leq m\leq 9)$. For the definition of $\mathrm{Sym}^m(\pi)$, we refer to Example 5.6 in this article. We summarize their results.

Theorem 2.3(Kim-Shahidi [47]). *Let π be a cuspidal automorphic representation of $GL(2,\mathbb{A})$ with ω_π its central character. Then $\mathrm{Sym}^4(\pi)\otimes\omega_\pi^{-1}$ is a cuspidal representation of $GL(5,\mathbb{A})$ except for the following three cases:*
(1) π is monomial in the sense that $\pi\cong\pi\otimes\eta$ for some nontrivial Grössencharacter η of F.
(2) π is not monomial and $\mathrm{Sym}^3(\pi)\otimes\omega_\pi^{-1}$ is not cuspidal.
(3) $\mathrm{Sym}^3(\pi)\otimes\omega_\pi^{-1}$ is cuspidal and there exists a nontrivial quadratic character χ such that
$\mathrm{Sym}^3(\pi)\otimes\omega_\pi^{-1}\cong\mathrm{Sym}^3(\pi)\otimes\omega_\pi^{-1}\otimes\chi.$

As applications of Theorem 2.3, they obtained the following.

Proposition 2.4(Kim-Shahidi [47]). *Let* π *be a cuspidal automorphic representation of* $GL(2,\mathbb{A})$ *with* ω_π *its central character such that* $\mathrm{Sym}^3(\pi)$ *is cuspidal. Then the following statements hold:*
(a) *Each partial symmetric m-th power L-functions* $L_S\big(s,\pi,\mathrm{Sym}^m\big)$ $(m = 6,7,8,9)$ *has a meromorphic continuation and satisfies a standard functional equation.*
(b) $L_S\big(s,\pi,\mathrm{Sym}^5\big)$ *and* $L_S\big(s,\pi,\mathrm{Sym}^7\big)$ *are holomorphic and nonzero for* $\mathrm{Re}(s) \geq 1$.
(c) *If* $\omega_\pi^3 = 1$, $L_S\big(s,\pi,\mathrm{Sym}^6\big)$ *is holomorphic and nonzero for* $\mathrm{Re}(s) \geq 1$.
(d) *If* $\mathrm{Sym}^4(\pi)$ *is cuspidal,* $L_S\big(s,\pi,\mathrm{Sym}^6\big)$ *is holomorphic and nonzero for* $\mathrm{Re}(s) \geq 1$.
(e) *If* $\mathrm{Sym}^4(\pi)$ *is cuspidal and* $\omega_\pi^4 = 1$, $L_S\big(s,\pi,\mathrm{Sym}^8\big)$ *is holomorphic and nonzero for* $\mathrm{Re}(s) \geq 1$.
(f) *If* $\mathrm{Sym}^4(\pi)$ *is cuspidal,* $L_S\big(s,\pi,\mathrm{Sym}^9\big)$ *has a most a simple pole or a simple zero at* $s = 1$.
(g) *If* $\mathrm{Sym}^4(\pi)$ *is not cuspidal,* $L_S\big(s,\pi,\mathrm{Sym}^9\big)$ *is holomorphic and nonzero for* $\mathrm{Re}(s) \geq 1$.

Proposition 2.5(Kim-Shahidi [47]). *Let* $\pi = \otimes_v \pi_v$ *be a cuspidal automorphic representation of* $GL(2,\mathbb{A})$ *such that* $\mathrm{Sym}^3(\pi)$ *is cuspidal. Let* $\mathrm{diag}(\alpha_v,\beta_v)$ *be the Satake parameter for* π_v. *Then* $|\alpha_v|, |\beta_v| < q_v^{1/9}$. *If* $\mathrm{Sym}^4(\pi)$ *is not cuspidal, then* $|\alpha_v| = |\beta_v| = 1$, *that is, the full Ramanujan conjecture holds.*

Proposition 2.6(Kim-Shahidi [47]). *Let* $\pi = \otimes_v \pi_v$ *be a nonmonomial cuspidal automorphic representation of* $GL(2,\mathbb{A})$ *with a trivial central character. Suppose* $m \leq 9$. *Then the following statements hold:*
(1) *Suppose* $\mathrm{Sym}^3(\pi)$ *is not cuspidal. Then* $L_S\big(s,\pi,\mathrm{Sym}^m\big)$ *is holomorphic and nonzero at* $s = 1$, *except for* $m = 6, 8$*; the L-functions* $L_S\big(s,\pi,\mathrm{Sym}^6\big)$ *and* $L_S\big(s,\pi,\mathrm{Sym}^8\big)$ *each have a simple pole at* $s = 1$.
(2) *Suppose* $\mathrm{Sym}^3(\pi)$ *is cuspidal but* $\mathrm{Sym}^4(\pi)$ *is not cuspidal. Then* $L_S\big(s,\pi,\mathrm{Sym}^m\big)$ *is holo-morphic and nonzero at* $s = 1$ *for* $m = 1,\cdots,7$ *and* $m = 9$*; the L-function* $L_S\big(s,\pi,\mathrm{Sym}^8\big)$ *has a simple pole at* $s = 1$.

We are still far from solving Conjecture A. We have two known methods to study analytic properties of automorphic L-functions. The first is the method of constructing explicit zeta integrals that is called the Rankin-Selberg method. The second is the so-called Langlands-Shahidi method I just described briefly. In the late 1960s Langlands [55] recognized that many automorphic L-functions occur in the constant terms of the Eisenstein series associated to cuspidal automorphic representations of the Levi subgroups of maximal parabolic subgroups of split reductive groups through his intensively deep work on the theory of Eisenstein series. He obtained some analytic properties of certain automorphic L-functions using the meromorphic continuation and the functional equation of Eisenstein series. As mentioned above, he proved the meromorphic continuation of certain class of L-functions but did not gave an answer to the functional equation. Shahidi [77] generalized Langlands' recognition to quasi-split groups, and calculated non-constant terms of the Eisenstein series and hence obtained the functional equations of a more broader class of many automorphic L-functions. In fact, those L-functions dealt with by Shahidi include most of the L-functions studied by other mathematicians (cf. [23], [24]). The first method has some advantage to provide more precise information on the location of poles and the special values of automorphic L-functions. On the other hand, the Langlands-Shahidi method has been applied to a large class of automorphic L-functions, and is likely to be more suited to the theory of harmonic analysis on a reductive group. Moreover the second method plays an important role in investigating the non-vanishing of automorphic L-functions on the line $\mathrm{Re}(s) = 1$. One of the main contributions of Shahidi to the Langlands-Shahidi method is to define local L-functions even at bad places in such a way that the completed L-function satisfies the functional equation. We are in need of new methods to have more knowledge on the analytic and arithmetic properties of automorphic L-functions.

3. Local Langlands conjecture

Let k be a local field and let W_k be its Weil group. We review the definition of the Weil group W_k following the article of Tate (cf. [82]). If $k = \mathbb{C}$, then $W_{\mathbb{C}} = \mathbb{C}^{\times}$. If $k = \mathbb{R}$, then

$$W_{\mathbb{R}} = \mathbb{C}^* \cup \tau\mathbb{C}^*, \quad \tau z \tau^{-1} = \overline{z},$$

where $z \mapsto \overline{z}$ is the nontrivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Then $W_{\mathbb{R}}^{\mathrm{ab}} = \mathbb{R}^*$. Here if Y^c denotes the closure of the commutator subgroup of a topological group Y, we set $Y^{\mathrm{ab}} = Y/Y^c$.

Suppose k is a nonarchimedean local field and $\overline{k}$ a separable algebraic closure of k. Let q be the order of the residue field κ of k. We set $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$ and $\Gamma_\kappa = \mathrm{Gal}(\overline{\kappa}/\kappa)$. Let $\Phi_\kappa : x \mapsto x^q$ be the Frobenius automorphism in Γ_κ. We set $\langle \Phi_\kappa \rangle = \{ \Phi_\kappa^n \mid n \in \mathbb{Z} \}$. Let $\varphi : \Gamma_k \longrightarrow \Gamma_\kappa$ be the canonical surjective homomorphism given by $\sigma \mapsto \sigma|_{\overline{\kappa}}$. The Weil group W_k is defined to be the set $W_k = \varphi^{-1}(\langle \Phi_\kappa \rangle)$. Obviously one has an exact sequence

$$1 \longrightarrow I_k \longrightarrow W_k \longrightarrow \langle \Phi_\kappa \rangle \longrightarrow 1,$$

where $I_k = \ker \varphi$ is the inertia group of k. We recall that W_k is topologized such that I_k has the induced topology from Γ_k, I_k is open in Γ_k and multiplication by Φ is a homeomorphism. Here Φ denotes a choice of a geometric Frobenius element in $\varphi^{-1}(\Phi_\kappa) \subset \Gamma_k$. We note that we have a continuous homomorphism $W_k \longrightarrow \Gamma_k$ with dense image. According to the local class field theory, one has the isomorphism

$$k^* \cong W_k^{\mathrm{ab}}. \tag{3.1}$$

In order to generalize the isomorphism (3.1) for $GL(1)$ to $GL(2)$, P. Deligne [18] introduced the so-called *Weil-Deligne group* W_k'. It is defined to be the group scheme over $\mathbb{Q}$ which is the semidirect product of W_k by the additive group $\mathbb{G}_a$ on which W_k acts by the rule $wxw^{-1} = ||w||x$. We refer to [82, p. 19] for the definition of $||w||$. Namely, $W_k' = W_k \ltimes \mathbb{G}_a$ is the group scheme over $\mathbb{Q}$ with the multiplication

$$(w_1, a_1)(w_2, a_2) = (w_1 w_2, a_1 + ||w_1||\, a_2), \quad w_1, w_2 \in W_k,\ a_1, a_2 \in \mathbb{G}_a.$$

Definition 3.1(Deligne [19]). Let E be field of characteristic 0. A representation of W_k' over E is a pair $\rho' = (\rho, N)$ consisting of
(a) A finite dimensional vector space V over E and a homomorphism $\rho : W_k \longrightarrow GL(V)$ whose kernel contains an open subgroup of I_k, i.e., which is continuous for the discrete topology on $GL(V)$.
(b) A nilpotent endomorphism N of V such that

$$\rho(w) N \rho(w)^{-1} = ||w||\, N, \quad w \in W_k.$$

We see that a homomorphism of group schemes over E

$$\rho' : W_k \times_{\mathbb{Q}} E \longrightarrow GL(V)$$

determines, and is determined by a pair (ρ, N) as in Definition 3.1 such that

$$\rho'((w, a)) = \exp(aN)\, \rho(w), \quad w \in W_k,\ a \in \mathbb{G}_a.$$

Let $\rho' = (\rho, N)$ be a representation of W_k' over E. Define $\nu : W_k \longrightarrow \mathbb{Z}$ by $||w|| = q^{-\nu(w)}$, $w \in W_k$. Then according to [19, (8.5)], there is a *unique unipotent* automorphism u of V such that

$$uN = Nu, \quad u\rho(w) = \rho(w)u, \quad w \in W_k$$

and such that

$$\exp(aN)\, \rho(w)\, u^{-\nu(w)}$$

is a semisimple automorphism of V for all $a \in E$ and $w \in W_k - I_k$. Then $\rho' = (\rho u^{-\nu}, N)$ is called the Φ-semisimplication of ρ'. And ρ' is called Φ-semisimple if and only if $\rho' = \rho'_{ss}$, $u = 1$, i.e., the Frobeniuses acts semisimply.

Let $\rho' = (\rho, N, V)$ be a representation of W_k' over E. We let $V_N^I := (\ker N)^{I_k}$ be the subspace of I_k-invariants in $\ker N$. We define a local L-factor by

$$(3.2)\quad Z(t, V) = \det\left(1 - t\, \rho(\Phi)\big|_{V_N^I}\right)^{-1}, \ \text{ and } \ L(s, V) = Z(q^{-s}, V), \ \text{when } E \subset \mathbb{C}.$$

We note that if $\rho' = (\rho, N)$ is a representation of W_k', then ρ' is irreducible if and only if $N = 0$ and ρ is irreducible.

Let G be a connected reductive group over a local field. A homomorphism $\alpha : W_k' \longrightarrow {}^L G$ is said to be *admissible* [9, p. 40] if the following conditions (i)-(iii)

are satisfied
(i) α is a homomorphism over Γ_k, i.e., the following diagram is commutative:

$$\begin{array}{ccc} W_k' & \xrightarrow{\alpha} & {}^LG \\ & \searrow \quad \swarrow & \\ & \Gamma_k & \end{array}$$

(ii) α is continuous, $\alpha(\mathbb{G}_a)$ are unipotent in ${}^LG^0$, and α maps semisimple elements into semisimple elements in ${}^LG$. Here an element x is said to be *semisimple* if $x \in I_k$, and an element $g \in {}^LG$ is called *semisimple* if $r(g)$ is semisimple for any finite dimensional representation r of ${}^LG$.
(iii) If $\alpha(W_k')$ is contained in a Levi subgroup of a proper parabolic subgroup P of ${}^LG$, then P is relevant. See [9, p. 32].

Let $\mathcal{G}_k(G)$ be the set of all admissible homomorphisms $\phi : W_k' \longrightarrow {}^LG$ modulo inner automorphisms by elements of ${}^LG^0$. We observe that we can associate canonically to $\phi \in \mathcal{G}_k(G)$ a character χ_ϕ of the center $C(G)$ of G (cf. [9, p. 43], [63]). Let $Z_L^0 = C({}^LG^0)$ be the center of ${}^LG^0$. Following [9, pp. 43-44] and [63], we can construct a character ω_α of $G(k)$ associated to a cohomology class $\alpha \in H^1(W_k', Z_L^0)$. If we write $\phi \in \mathcal{G}_k(G)$ as $\phi = (\phi_1, \phi_2)$ with $\phi : W_k' \longrightarrow {}^LG^0$ and $\phi : W_k' \longrightarrow \Gamma_k$, then ϕ_1 defines a cocycle of W_k' in ${}^LG^0$, and the map $\phi \mapsto \phi_1$ yields an embedding $\mathcal{G}_k(G) \hookrightarrow H^1(W_k', {}^LG^0)$. Then $H^1(W_k', Z_L^0)$ acts on $H^1(W_k', {}^LG^0)$ and this action leaves $\mathcal{G}_k(G)$ stable [9, p. 40].

Let $\prod(G(k))$ be the set of all equivalence classes of irreducible admissible representations of $G(k)$. The following conjecture gives an arithmetic parametrization of irreducible admissible representations of $G(k)$.

Local Langlands Conjecture[LLC]. Let k be a local field. Let $\mathcal{G}_k(G)$ and $\prod(G(k))$ be as above. Then there is a surjective map $\prod(G(k)) \longrightarrow \mathcal{G}_k(G)$ with finite fibres which partitions $\prod(G(k))$ into disjoint finite sets $\prod_\phi(G(k))$, simply $\prod_\phi$ called *L-packets* satisfying the following (i)-(v):
(i) If $\pi \in \prod_\phi$, then the central character χ_π of π is equal to χ_ϕ.
(ii) If $\alpha \in H^1(W_k', Z_L^0)$ and ω_α is its associated character of $G(k)$, then

$$\prod\nolimits_{\alpha\cdot\phi} = \left\{ \pi\omega_\alpha \mid \pi \in \prod\nolimits_\phi \right\}.$$

(iii) One element of $\prod_\phi$ is square integrable modulo the center $C(G)$ of G if and only if all elements are square integrable modulo the center $C(G)$ of G if and only if $\phi(W_k')$ is not contained in any proper Levi subgroup of ${}^LG$.
(iv) One element of $\prod_\phi$ is tempered if and only if all elements of $\prod_\phi$ are tempered if and only if $\phi(W_k)$ is bounded.
(v) If H is a connected reductive group over k and $\eta : H(k) \longrightarrow G(k)$ is a k-morphism with commutative kernel and cokernel, then there is a required compatibilty between decompositions for $G(k)$ and $H(k)$. More precisely, η induces a canonical map ${}^L\eta : {}^LG \longrightarrow {}^LH$, and if we set $\phi' = {}^L\eta \circ \phi$ for $\phi \in \mathcal{G}_k(G)$, then any $\pi \in \prod_\phi(G(k))$, viewed as an $H(k)$-module, decomposes into a direct sum of finitely

many irreducible admissible representations belonging to $\prod_{\phi'}(H(k))$.

Remark 3.2. (a) If k is archimedean, i.e., $k = \mathbb{R}$ or $\mathbb{C}$, [LLC] was solved by Langlands [63]. We also refer the reader to [1], [2], [49].
(b) In case k is non-archimedean, Kazhdan and Lusztig [40] had shown how to parametrize those irreducible admissible representations of $G(k)$ having an Iwahori fixed vector in terms of admissible homomorphisms of W'_k.
(c) For a local field k of positive characteristic $p > 0$, [LLC] was established by Laumon, Rapoport and Stuhler [67].
(d) In case $G = GL(n)$ for a non-archimedean local field k, [LLC] was established by Harris and Taylor [28], and by Henniart [31]. In both cases, the correspondence was established at the level of a correspondence between irreducible Galois representations and supercuspidal representations.
(e) Let k be a a non-archimedean local field of characteristic 0 and let $G = SO(2n+1)$ the split special orthogonal group over k. In this case, Jiang and Soudry [38], [39] gave a parametrization of *generic supercuspidal* representations of $SO(2n+1)$ in terms of admissible homomorphisms of W'_k. More precisely, there is a unique bijection of the set of conjugacy classes of all admissible, completely reducible, multiplicity-free, symplectic complex representations $\phi : W'_k \longrightarrow {}^L SO(2n+1) = Sp(2n, \mathbb{C})$ onto the set of all equivalence classes of irreducible generic supercuspidal representations of $SO(2n+1, k)$.

For $\pi \in \prod_\phi(G)$ with $\phi \in \mathcal{G}_k(G)$, if r is a finite dimensional complex representation of ${}^L G$, we define the L- and ε-factors

$$L(s, \pi, r) = L(s, r \circ \phi) \quad \text{and} \quad \varepsilon(s, \pi, r, \psi) = \varepsilon(s, r \circ \phi, \psi), \tag{3.3}$$

where $L(s, r \circ \phi)$ is the Artin-Weil L-function.

Remark 3.3. For a non-archimedean local field k, Deligne [18] gave the complete formulation of [LLC] for $GL(2)$. In [18], he utilized for the first time the Weil-Deligne group W'_k, which was introduced by him in [19], in the context of ℓ-adic representations, in order to obtain a correct formulation in the case of $GL(2)$ over a non-archimedean local field.

Remark 3.4. The representations in the L-packet $\prod_\phi$ are parametrized by the component group

$$C_\phi := S_\phi / Z_L S_\phi^0,$$

where S_ϕ is the centralizer of the image of ϕ in ${}^L G$, S_ϕ^0 is the identity component of S_ϕ, and Z_L is the center of ${}^L G$. We refer the reader to [5], [51] for more information on the L-packets.

Example 3.5. Let π be a spherical or unramified representation of $G(k)$ with a non-archimedean local field k. It is known that $\pi \hookrightarrow I(\chi)$ for a unique unramified quasi-character χ of a maximal torus $T(k)$ of $G(k)$. Then π determines a semi-simple conjugate class $c(\pi) = \{t_\pi\} \subset {}^LT \subset {}^LG$. Then the Langlands' parameter ϕ_π for π is

$$\phi_\pi : k^* \longrightarrow {}^LT \subset {}^LG, \qquad \phi_\pi(\widetilde{\omega}) = t_\pi$$

such that

$$\phi_\pi(\widetilde{\omega}) = t_\pi \quad \text{and } \phi_\pi \text{ is trivial on } \mathcal{O}^*,$$

where $\widetilde{\omega}$ denotes a uniformizer in k.
If $\pi = \pi(\mu_1, \cdots, \mu_n)$ is a spherical representation of $GL_n(k)$ with unramified characters μ_i $(i = 1, \cdots, n)$ of k^*, the semi-simple conjugacy class $c(\pi)$ is given by

$$c(\pi) = \{\, \mathrm{diag}(\mu_1(\widetilde{\omega}), \cdots, \mu_n(\widetilde{\omega}))\}$$

and the Langlands' parameter ϕ_π for π is

$$\phi_\pi : k^* \longrightarrow {}^LT \subset {}^LG = GL(n, \mathbb{C}), \qquad \phi_\pi(\widetilde{\omega}) = \mathrm{diag}(\mu_1(\widetilde{\omega}), \cdots, \mu_n(\widetilde{\omega})).$$

4. Global Langlands conjecture

Let k be a global field and $\mathbb{A}$ its ring of adeles. This section is based on Arthur's article [3].

As in the local case of Section 3, the global Langlands conjecture should be a nonabelian generalization of abelian global class field theory. When Deligne [19] recognized the need to introduce the Weil-Deligne group W_k' for the local Langlands correspondence for $GL(2)$, it was realized that there seemed to be no natural global version of W_k'. In fact, Γ_k, W_k and W_k' are too small to parameterize all automorphic representations of a reductive group. Thus in the 1970s Langlands [60] attempted to discover a *hypothetical* group L_k to replace the Weil-Deligne group W_k'. Nowadays it is believed by experts that this group L_k should be related to the equally *hypothetical motivic Galois* group $\mathcal{M}_k$ of k. The group L_k is often called the *hypothetical* (or *conjectual*) Langlands group or the *automorphic* Langlands group. The notion of L_k and $\mathcal{M}_k$ is still not clear.

The global Langlands conjecture can be written as follows.

Global Langlands Conjecture[GLC]. Automorphic representations of $G(\mathbb{A})$ can be parametrized by admissible homomorphisms $\phi : L_k \longrightarrow {}^LG$ required to have the following properties (1)-(4):
(1) There is an L-packet $\prod_\phi$ which consists of automorphic representations of $G(\mathbb{A})$ attached to ϕ.
(2) Each L-packet $\prod_\phi$ is a finite set.
(3) Any automorphic representation of $G(\mathbb{A})$ belongs to $\prod_\phi$ for a unique homomorphism ϕ.
(4) The $\prod_\phi$'s are disjoint.

We first consider the case that k is a function field. Drinfeld [21] formulated a version of the global Langlands conjecture for function fields relating the irreducible two dimensional representations of the Galois group Γ_k with irreducible cuspidal representations of $GL(2,\mathbb{A})$, and established the global Langlands conjecture. In the early 2000s Lafforgue [52] had extended the work of Drinfeld mentioned above to $GL(n)$ to give a proof of the global Langlands conjecture for $GL(n)$ over a function field. The formulation of the global Langlands conjecture made by Drinfeld and Lafforgue is essentially the same as that in the local non-archimedean case discussed in Section 3 with a few modification. So we omit the details for the global Langlands conjecture over a function field here. We refer to [20], [21], [52] for more detail.

Next we consider the case k is a number field. We first recall that according to the global class field theory, for $n = 1$, there is a canonical bijection between the continuous characters of Γ_k and characters of finite order of the idele group $k^*\backslash\mathbb{A}^*$. We should replace Γ_k by the Weil group W_k in order to obtain all the characters of $k^*\backslash\mathbb{A}^*$. For $n \geq 2$, by analogy with the local Langlands conjecture, we need a global analogue of the Weil-Deligne group W_k'. However no such analogue is available at this moment. We hope that the hypothetical Langlands group L_k plays a role as W_k' and fits into an exact sequence

$$1 \longrightarrow L_k^c \longrightarrow L_k \longrightarrow \Gamma_k \longrightarrow 1, \tag{4.1}$$

where L_k^c is a complex pro-reductive group. L_k should be a locally compact group equipped with an embedding $i_v : L_{k_v} \longrightarrow L_k$ for each completion k_v of k. Let G be a connected, quasi-split reductive group over k. We set $G_v := G(k_v)$ for every place v of k. Let $\mathfrak{L}_k(G)$ be the set of all equivalence classes of continuous, completely reducible homomorphisms ϕ of L_k into ${}^LG$, and $\mathcal{A}_k(G)$ the set of equivalence classes of all automorphic representations of $G(\mathbb{A})$. For each place v of k, let $\mathfrak{L}_{k_v}(G)$ be the set of equivalence classes of continuous, completely reducible homomorphisms $\phi_v : L_{k_v} \longrightarrow {}^LG_v$ and $\mathcal{A}_{k_v}(G)$ the set of continuous irreducible admissible representations of G_v. We would hope to have a bijection

$$\mathfrak{L}_k(G) \longrightarrow \mathcal{A}_k(G), \qquad \phi \mapsto \pi_\phi. \tag{4.2}$$

Moreover the set $\mathfrak{L}_k^0(G)$ of equivalence classes of irreducible representations in $\mathfrak{L}_k(G)$ should be in bijective correspondence with the set $\mathcal{A}_k^0(G)$ of all *cuspidal* automorphic representations in $\mathcal{A}_k(G)$. This would be supplemented by local bijection

$$\mathfrak{L}_{k_v}(G) \longrightarrow \mathcal{A}_{k_v}(G) \quad \text{for any place } v\ of\ k. \tag{4.3}$$

The local and global bijections should be *compatible* in the sense that for any $\phi : L_k \longrightarrow {}^LG$, there is an automorphic representation $\pi_\phi = \otimes_v \pi_{\phi,v}$ of $G(\mathbb{A})$ with the correspondence $\phi \mapsto \pi_\phi$ such that for each place v of k, the restriction $\phi_v = \phi \circ i_v$ of ϕ to L_{k_v} corresponds to the local component $\pi_{\phi,v}$ of π_ϕ. Of course, one expects that all of these correspondences (4.2) and (4.3) would satisfy properties similar to those in the local Langlands conjecture, e.g., the preservation of L- and ε-factors with twists, etc.

The local Langlands groups are elementary. They are given by

$$L_{k_v} = \begin{cases} W_{k_v} & \text{if } v \text{ is archimedean,} \\ W_{k_v} \times SU(2,\mathbb{R}) & \text{if } v \text{ is nonarchimedean,} \end{cases}$$

where W_{k_v} is again the Weil group of k_v. Thus the local Langlands group L_{k_v} is a split extension of W_{k_v} by compact simply connected Lie group. But the hypothetical Langlands group will be much larger. It would be an infinite fibre product of nonsplit extension

$$1 \longrightarrow K_c \longrightarrow L_c \longrightarrow W_k \longrightarrow 1 \tag{4.4}$$

of the Weil group W_k by a compact, semsimple, simply connected Lie group K_c. However one would have to establish something in order to show that L_k has all the desired properties.

Grothendieck's conjectural theory of motives introduces the so-called *motivic Galois* group $\mathcal{M}_k$, which is a reductive proalgebraic group over $\mathbb{C}$ and comes with a proalgebraic projection $\mathcal{M}_k \longrightarrow \Gamma_k$. A *motive* of rank n is to be defined as a proalgebraic representation

$$\mathbb{M} : \mathcal{M}_k \longrightarrow GL(n, \mathbb{C}).$$

We observe that any continuous representation of Γ_k pulls back to $\mathcal{M}_k$ and can be viewed as a motive in the above sense. It is conjectured that the arithmetic information in any motive $\mathbb{M}$ is directly related to analytic information from some automorphic representations of $G(\mathbb{A})$. The conjectural theory of motives also applies to any completion k_v of k. It produces a proalgebraic group $\mathcal{M}_{k_v}$ over $\mathbb{C}$ that fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{k_v} & \longrightarrow & \mathcal{M}_k \\ \downarrow & & \downarrow \\ \Gamma_{k_v} & \hookrightarrow & \Gamma_k \end{array}$$

of proalgebraic homomorphisms.

In 1979, Langlands [60] speculated the following:

Conjecture B(Langlands [60, Section 2]). There is a commutative diagram

$$\begin{array}{ccc} L_k & \xrightarrow{\phi} & \mathcal{M}_k \\ & \searrow \quad \swarrow & \\ & \Gamma_k & \end{array}$$

together with a compatible commutative diagram

$$\begin{array}{ccc} L_{k_v} & \xrightarrow{\phi_v} & \mathcal{M}_{k_v} \\ & \searrow \quad \swarrow & \\ & \Gamma_k & \end{array}$$

for each completion k_v of k, in which ϕ and ϕ_v are continuous homeomorphisms. There should be the analogue of the notion of *admissibility* of the maps ϕ and ϕ_v as in the local case (cf. [9, p. 40]).

The above conjecture implies that we can attach to any proalgebraic homomorphism μ from $\mathcal{M}_k$ to ${}^LG$ over Γ_k, its associated automorphic representation of $G(\mathbb{A})$. In particular, if we take $G = GL(n)$, it means that we can attach to any motive $\mathbb{M}$ of rank n, an automorphic representation $\pi_{\mathbb{M}} = \otimes_v \pi_{\mathbb{M},v}$ of $GL(n,\mathbb{A})$ with the following property: The family of semi-simple conjugacy classes $c(\pi_{\mathbb{M}}) = \{c(\pi_{\mathbb{M},v})\}$ in $GL(n,\mathbb{C})$ associated to $\pi_{\mathbb{M}}$ is equal to the family of conjugacy classes $c(\mathbb{M}) = \{c_v(\mathbb{M})\}$ obtained from $\mathbb{M}$, and the local homomorphism $\mathcal{M}_{k_v} \longrightarrow \mathcal{M}_k$ at places v that are unramified for $\mathbb{M}$. In fact, $c_v(\mathbb{M})$ is the image of the Frobenius class Fr_v under a different kind of Γ_k, namely a compatible family

$$\Gamma_k \longrightarrow \prod_{\ell \notin S(\mathbb{M}) \cup \{v\}} GL(n, \overline{\mathbb{Q}}_\ell)$$

of ℓ-adic representations attached to $\mathbb{M}$. Our task now is to find some natural ways to construct an explicit candidate for $\mathcal{M}_k$ and then to clarify the structure of $\mathcal{M}_k$. It is suggested by experts [70] that $\mathcal{M}_k$ be a proalgebraic fibre product of certain extensions

$$1 \longrightarrow \mathcal{D}_c \longrightarrow \mathcal{M}_c \longrightarrow \mathcal{T}_k \longrightarrow 1 \tag{4.5}$$

of a fixed group $\mathcal{T}_k$ by complex, semisimple, simply connected algebraic groups $\mathcal{D}_c$. The group $\mathcal{T}_k$ is an extension

$$1 \longrightarrow \mathcal{S}_k \longrightarrow \mathcal{T}_k \longrightarrow \Gamma_k \longrightarrow 1 \tag{4.6}$$

of Γ_k by a complex proalgebraic torus $\mathcal{S}_k$ (cf. [73, Chapter II], [60, Section 5], [74, Section 7]). The contribution to $\mathcal{M}_k$ of any $\mathcal{M}_c$ is required to match the contribution to L_k of a corresponding L_c, in which K_c is a compact real form of $\mathcal{D}_c$. This construction should have to come with the following diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & L_k^c & \longrightarrow & L_k & \longrightarrow & \Gamma_k & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{M}_k^c & \longrightarrow & \mathcal{M}_k & \longrightarrow & \Gamma_k & \longrightarrow & 1 \end{array}$$

where $\mathcal{M}_k^c$ is a complex pro-reductive group.

Let $\Psi_k(G)$ be the set of equivalence classes of continuous, completely reducible homomorphisms of $\mathcal{M}_k$ into ${}^LG$, and for each place v of k, let $\Psi_{k_v}(G)$ be the set of equivalence classes of continuous, completely reducible homomorphisms of $\mathcal{M}_{k_v}$ into ${}^LG_v$. Then one should have to obtain a bijective correspondence

$$\Psi_k(G) \longrightarrow \mathcal{A}_k(G). \tag{4.7}$$

This would be supplemented by local bijective correspondences

$$\mathcal{M}_{k_v}(G) \longrightarrow \mathcal{A}_{k_v}(G) \tag{4.8}$$

for all places v of k.

5. Langlands functoriality

As we see in section 2, Shahidi [77] gave a partially affirmative answer to Conjecture A, which is a question raised by Langlands for a larger class of automorphic L-functions $L(s,\pi,r)$ obtained from cuspidal automorphic representations π of a Levi subgroup M of a quasi-split reductive group and the adjoint representation of ${}^LM$ on the real Lie algebra ${}^L\mathfrak{n}$ of ${}^LN$. This suggest trying, given a L-function and a quasi-split reductive group G, to see whether G has an automorphic representation with the given L-function. Many instances of such questions can be regarded as special cases of the *lifting problem*, nowadays called the *Principle of Functoriality*, with respect to morphisms of L-groups. The motivation of this problem stems from a global side. There is also a local version for this problem. These questions were raised and also formulated by Langlands [54] in the late 1960s. Roughly speaking, the principle of functoriality describes profound relationships among automorphic forms on different groups.

Let k be a local or global field, and let H, G two connected reductive groups defined over k. A homomorphism $\sigma : {}^LH \longrightarrow {}^LG$ is said to an *L-homomorphism* if it satisfies the following conditions (1)-(3):
(1) σ is a homomorphism over the absolute Galois group Γ_k, namely, the following diagram is commutative;

$$\begin{array}{ccc} {}^LH & \xrightarrow{\sigma} & {}^LG \\ & \searrow \quad \swarrow & \\ & \Gamma_k & \end{array}$$

(2) σ is continuous;
(3) The restriction of σ to ${}^LH^0$ is a complex analytic homomorphism of ${}^LH^0$ into ${}^LG^0$.

Let $\mathcal{G}_k(H)$ (resp. $\mathcal{G}_k(G)$) be the set of all admissible homomorphisms $\phi : W_k' \longrightarrow {}^LH$ (resp. ${}^LG$) modulo inner automorphisms by elements of ${}^LH^0$ (resp. ${}^LG^0$). Suppose G is quasi-split. Given a fixed L-homomorphism $\sigma : {}^LH \longrightarrow {}^LG$, if ϕ is any element in $\mathcal{G}_k(H)$, then the composition $\sigma\circ\phi$ is an element in $\mathcal{G}_k(G)$. It is easily seen that the correspondence $\phi\mapsto\sigma\circ\phi$ yields the canonical map

$$\mathcal{G}_k(\sigma) : \mathcal{G}_k(H) \longrightarrow \mathcal{G}_k(G).$$

If k is a global field and v is a place of k, then ${}^LG_v$ can be viewed as a subgroup of ${}^LG$ because Γ_{k_v} is regarded as a subgroup of Γ_k. Therefore σ induces the L-homomorphism $\sigma_v : {}^LH_v \longrightarrow {}^LG_v$ and hence a local map

$$\mathcal{G}_k(\sigma_v) : \mathcal{G}_k(H_v) \longrightarrow \mathcal{G}_k(G_v).$$

We refer to [9, pp. 54-58] for more detail on these stuffs.

Langlands Functoriality Conjecture(Langlands [54]). Let k be a global field, and let H, G two connected reductive groups over k with G quasi-split. Suppose $\sigma : {}^L H \longrightarrow {}^L G$ is an L-homomorphism. Then for any automorphic representation $\pi = \otimes_v \pi_v$ of $H(\mathbb{A})$, there exists an automorphic representation $\Pi = \otimes_v \Pi_v$ of $G(\mathbb{A})$ such that

$$c(\Pi_v) = \sigma(c(\pi_v)), \quad v \notin S(\pi) \cup S(\Pi), \tag{5.1}$$

where $S(\pi)$ (resp. $S(\Pi)$) denotes a finite set of ramified places of k for π (resp. Π) so that π_v (resp. Π_v) is unramified for every place $v \notin S(\pi)$ (resp. $v \notin S(\Pi)$). We note that the condition (5.1) is equivalent to the condition

$$L_S(s, \Pi, r) = L_S(s, \pi, r \circ \sigma), \quad S = S(\pi) \cup S(\Pi) \tag{5.2}$$

for every finite dimensional complex representation r of ${}^L G$.

Remark 5.1. For a nonarchimedean local field k, we can formulate a local version of Langlands Functoriality Conjecture replacing the word "automorphic" by "admissible" and modifying some facts of an L-homomorphism.

Remark 5.2. Suppose k is a nonarchimedean local field with the ring of integers $\mathcal{O}$. Suppose H and G are quasi-split and there is a finite extension E of k such that both H and G split over E, and have an $\mathcal{O}$-structure so that both $H(\mathcal{O})$ and $G(\mathcal{O})$ are special maximal compact subgroups. Let π be an unramified representation of $H(k)$ in the sense that π has a nonzero $H(\mathcal{O})$-fixed vector, and let $\phi = \phi_\pi \in \mathcal{G}_k(H)$ be the unramified parameter of π. Then for any L-homomorphism $\sigma : {}^L H \longrightarrow {}^L G$, the parameter $\widetilde{\phi} = \sigma \circ \phi$ is unramified and defines an L-packet $\prod_{\widetilde{\phi}}(G)$ which contains exactly one unramified representation Π of $G(k)$ to be called the *natural lift* of π (cf. [9, p. 55]).

If we assume that the Local Langlands Conjecture, briefly [LLC] is valid, we can reformulate the Langlands Functoriality Conjecture using [LLC] in the following way. Let $\pi = \otimes_v \pi_v$ be an automorphic representation of $H(\mathbb{A})$. According to [LLC], we can attach to each π_v, an element $\phi_v : W'_k \longrightarrow {}^L H_v$ in $\mathcal{G}_{k_v}(H)$. The composition $\sigma \circ \phi_v$ is an element in $\mathcal{G}_{k_v}(G)$. By [LLC] again, one has an irreducible admissible representation Π_v of G_v attached to $\sigma \circ \phi_v$. Then $\Pi = \otimes_v \Pi_v$ is an irreducible admissible representation of $G(\mathbb{A})$. Therefore Langlands Functoriality Conjecture is equivalent to the statement that Π must be an automorphic representation of $G(\mathbb{A})$.

If we assume that Global Langlands Conjecture, briefly [GLC] is valid, we can also reformulate Langlands Functoriality Conjecture using [GLC] as follows: Given an automorphic representation π of $H(\mathbb{A})$ with its associated parameter $\phi_\pi : L_k \longrightarrow {}^L H$, there must be an L-packet $\Pi_{\sigma \circ \phi_\pi}(G)$ attached to $\sigma \circ \phi_\pi$.

Example 5.3. Suppose $H = \{1\}$ and $G = GL(n)$. Clearly an automorphic representation π of $H(\mathbb{A})$ is trivial. The choice of σ amounts to that of an admissible homomorphism

$$\sigma : \mathrm{Gal}(E/k) \longrightarrow GL(n, \mathbb{C}) = {}^L G$$

for a finite Galois extension E of k. Therefore Langlands Functoriality Conjecture reduces to the following assertion.

Strong Artin Conjecture(Langlands [54]). Let k be a number field. For an n-dimensional complex representation σ of $\mathrm{Gal}(E/k)$, there is an automorphic representation π of $GL(n, \mathbb{A})$ such that

$$c(\pi_v) = \sigma(Fr_v), \quad v \notin S(E),$$

where $S(E)$ is a finite set of places including all the ramified places of E. The above conjecture was established partially but remains unsettled for the most part. We summarize the cases that have been established until now chronically. (a) The case $n = 1$: This is the Artin reciprocity law, namely, $k^* \cong W_k^{\mathrm{ab}}$, which is the essential part of abelian class field theory. The image of σ is of cyclic type or of dihedral type.
(b) The case where $n = 2$ and $\mathrm{Gal}(E/k)$ is solvable: The conjecture was solved by Langlands [61] when the image of σ in $PSL(2, \mathbb{C})$ is of tetrahedral type, that is, isomorphic to A_4, and by Tunnell [84] when the image of σ in $PSL(2, \mathbb{C})$ is of octahedral type, i.e., isomorphic to S_4. It is a consequence of cyclic base change for $GL(2)$. These cases were used by A. Wiles [86] in his proof of Fermat's Last Theorem.
(c) The case where n is arbitrary and $\mathrm{Gal}(E/k)$ is nilpotent: The conjecture was established by Arthur and Clozel [6] as an application of cyclic base change for $GL(n)$.
(d) The case where $n = 2$ and the image of σ is of icosahedral type: Partial results were obtained by Taylor et al. (cf. [10]) C. Khare proved this case.
(e) The case where $n = 4$ and $\mathrm{Gal}(E/k)$ is solvable: The conjecture was established by Ramakrishnan [72] for representations σ that factor through the group $GO(4, \mathbb{C})$ of orthogonal similitudes.

Example 5.4. Let k be a number field. Let $H = Sp(2n)$, $SO(2n+1)$, $SO(2n)$ be the split form, and $G = GL(N)$, where $N = 2n+1$ or $2n$. Then ${}^L Sp(2n) = SO(2n+1, \mathbb{C})$, ${}^L SO(2n+1) = Sp(2n, \mathbb{C})$, ${}^L SO(2n) = SO(2n, \mathbb{C})$ and ${}^L GL(N) = GL(N, \mathbb{C})$. As an L-homomorphism $\sigma : {}^L H \longrightarrow {}^L G$, we take the embeddings

$${}^L Sp(2n) \hookrightarrow GL(2n+1, \mathbb{C}), \quad {}^L SO(2n+1) \hookrightarrow GL(2n, \mathbb{C}), \quad {}^L SO(2n) \hookrightarrow GL(2n, \mathbb{C}).$$

In each of these cases, the Langlands weak functorial lift for irreducible *generic* cuspidal automorphic representations of $H(\mathbb{A})$ was established by Cogdell, Kim, Piateski-Shapiro and Shahidi [13], [14]. Here the notion of "weak" automorphy means that an automorphic representation of $GL(n)$ exists whose automorphic L-function matches the desired Euler product except for a finite number of factors. The proof is based on the converse theorems for $GL(n)$ established by Cogdell and Piateski-Shapiro [15], [16]. It is still an open problem to establish the Langlands functorial lift from irreducible *non-generic* cuspidal automorphic representations of

$H(\mathbb{A})$ to $G(\mathbb{A})$. Let $H = \mathrm{GSpin}_m$ be the general spin group of semisimple rank $[\frac{m}{2}]$, i.e., a group whose derived group is Spin_m. Then the L-group of G is given by

$$^L\mathrm{GSpin}_m = \begin{cases} GSO_m & \text{if } m \text{ is even;} \\ GSp_{2[\frac{m}{2}]} & \text{if } m \text{ is odd.} \end{cases}$$

In each case we have an embedding

$$i : {}^LH^0 \hookrightarrow GL(N,\mathbb{C}), \quad N = m \text{ or } 2\Big[\frac{m}{2}\Big]. \tag{5.3}$$

Asgari and Shahidi [7], [8] proved that if π is a generic cuspidal representation of GSpin_m, then the functoriality is valid for the embedding (5.3).
If $H = SO(2n+1)$, for generic cuspidal representations, Jiang and Soudry [38] proved that the Langlands functorial lift from $SO(2n+1)$ to $GL(2n)$ is *injective* up to isomorphism. Using the functorial lifting from $SO(2n+1)$ to $GL(2n)$, Khare, Larsen and Savin [41] proved that for any prime ℓ and any even positive integer n, there are infinitely many exponents k for which the finite simple group $PSp_n(\mathbb{F}_{\ell^k})$ appears as a Galois group over $\mathbb{Q}$. Furthermore, in their recent paper [42] they extended their earlier work to prove that for a positive integer t, assuming that t is even if $\ell = 3$ in the first case (1) below, the following statements (1)-(3) hold:
(1) Let ℓ be a prime. Then there exists an integer k divisible by t such that the simple group $G_2(\mathbb{F}_{\ell^k})$ appears as a Galois group over $\mathbb{Q}$.
(2) Let ℓ be an odd prime. Then there exists an integer k divisible by t such that the simple finite group $SO_{2n+1}(\mathbb{F}_{\ell^k})^{\mathrm{der}}$ or the finite classical group $SO_{2n+1}(\mathbb{F}_{\ell^k})$ appears as a Galois group over $\mathbb{Q}$.
(3) If $\ell \equiv 3, 5 \pmod 8$ and ℓ is a prime, then there exists an integer k divisible by t such that the simple finite group $SO_{2n+1}(\mathbb{F}_{\ell^k})^{\mathrm{der}}$ appears as a Galois group over $\mathbb{Q}$. The construction of Galois groups in (1)-(3) is based on the functorial lift from $Sp(2n)$ to $GL(2n+1)$, and the backward lift from $GL(2n+1)$ to $Sp(2n)$ plus the theta lift from G_2 to $Sp(6)$.

Example 5.5. Let k be a number field. For two positive integers m and n, we let

$$H = GL(m) \times GL(n) \qquad \text{and} \qquad G = GL(mn).$$

Then ${}^LH = GL(m,\mathbb{C}) \times GL(n,\mathbb{C})$ and ${}^LG = GL(mn,\mathbb{C})$. We take the L-homomorphism

$$\sigma : GL(m) \times GL(n) \longrightarrow GL(mn,\mathbb{C})$$

given by the tensor product. Suppose $\pi = \otimes_v \pi_v$ and $\tau = \otimes_v \tau_v$ are two cuspidal automorphic representations of $GL(m,\mathbb{A})$ and $GL(n,\mathbb{A})$ respectively. By [LLC] for $GL(N)$ [28], [31], [63], one has the parametrizations

$$\phi_v : W_k' \longrightarrow GL(m,\mathbb{C}) \qquad \text{and} \qquad \psi_v : W_k' \longrightarrow GL(n,\mathbb{C}).$$

Let

$$[\phi_v, \psi_v] : W_k' \longrightarrow GL(m,\mathbb{C}) \times GL(n,\mathbb{C}) \hookrightarrow GL(mn,\mathbb{C}) = {}^LG$$

be the admissible homomorphism of W_k' into ${}^LH$ defined by

$$[\phi_v, \psi_v](w) = \big(\phi_v(w), \psi_v(w)\big), \quad w \in W_k'.$$

The composition $\theta_v = \sigma \circ [\phi_v, \psi_v]$ is an admissible homomorphism of W_k' into ${}^LG$. According to [LLC] for $GL(N)$, one has an irreducible admissible representation of $GL(mn, k_v)$ attached to θ_v, denoted by $\pi_v \boxtimes \tau_v$. We set

$$\pi \boxtimes \tau = \bigotimes_v \pi_v \boxtimes \tau_v.$$

The validity of the Langlands Functoriality Conjecture for the L-homomorphism $\sigma : {}^LH \longrightarrow {}^LG$ implies that $\pi \boxtimes \tau$ is an automorphic representation of $GL(mn, \mathbb{A})$. Ramakrishnan [71] used the converse theorem for $GL(4)$ of Cogdell and Piatetski-Shapiro to establish the functoriality for $GL(2) \times GL(2)$. Kim and Shahidi [46] established the functoriality for $GL(2) \times GL(3)$.

Example 5.6. Let $H = GL(2)$. For a positive integer $m \geq 2$, let $G = GL(m+1)$. Suppose $\pi = \otimes_v \pi_v$ is an automorphic representation of $H(\mathbb{A})$. According to [LLC] for $GL(n)$ [28], [31], [63], for each place v of k, we have a semisimple conjugacy class $c(\pi_v) = \{\text{diag}(\alpha_v, \beta_v)\} \subset GL(2, \mathbb{C})$. By [LLC] for $GL(n)$ again, for each place v of k, there is an irreducible admissible representation of $GL(m+1, k_v)$, denoted by $\text{Sym}^m(\pi_v)$ attached to the semisimple conjugacy class

$$\big\{\text{diag}\big(\alpha_v^m, \alpha_v^{m-1}\beta_v, \cdots, \beta_v^m\big)\big\} \subset GL(m+1, \mathbb{C}).$$

We set

$$\text{Sym}^m(\pi) := \bigotimes_v \text{Sym}^m(\pi_v).$$

Then $\text{Sym}^m(\pi)$ is an irreducible admissible representation of $GL(m+1, \mathbb{A})$. The validity of the Langlands Functoriality Conjecture for the L-homomorphism $\text{Sym}^m : GL(2, \mathbb{C}) \longrightarrow GL(m+1, \mathbb{C})$ implies that $\text{Sym}^m(\pi)$ is an automorphic representation of $GL(m+1, \mathbb{A})$. As a consequence, we obtain a complete resolution of the Ramanujan-Petersson conjecture for Maass forms, the Selberg conjecture for eigenvalues and the Sato-Tate conjecture. In 1978, Gelbart and Jacquet [22] established the functoriality for Sym^2 using the converse theorem on $GL(3)$. In 2002, Kim and Shahidi [46] established the functoriality for Sym^3 using the functoriality for $GL(2) \times GL(3)$. Thereafter Kim [43] established the functoriality for Sym^4. The proof is based on the converse theorems for $GL(n)$ established by Cogdell and Piateski-Shapiro [15, 16]. We refer to [47] for more results on this topic.

Example 5.7. For a positive integer $n \geq 2$, we let

$$H = GL(n) \qquad \text{and} \qquad G = GL(N), \quad N = \frac{(n-1)n}{2}.$$

Let

$$\wedge^2 : {}^LH = GL(n, \mathbb{C}) \longrightarrow {}^LG = GL(N, \mathbb{C})$$

be the L-homomorphism of ${}^L H$ into ${}^L G$ given by the exterior square. Suppose $\pi = \otimes_v \pi_v$ is a cuspidal automorphic representation of $GL(n,\mathbb{A})$. According to [LLC] for $GL(m)$, for each place v of k, one has the admissible homomorphism

$$\phi_v : W_k' \longrightarrow {}^L H = GL(n,\mathbb{C})$$

parameterizing π_v. The composition $\psi_v = \wedge^2 \circ \phi_v$ again yields an irreducible admissible representation $\wedge^2 \pi_v$ of $GL(N,k_v)$ for every unramified representation π_v. We set

$$\wedge^2 \pi = \bigotimes_v \wedge^2 \pi_v .$$

Then $\wedge^2\pi$ is an irreducible admissible representation of $GL(N,\mathbb{A})$. The validity of the Langlands functoriality implies that $\wedge^2\pi$ is an automorphic representation of $GL(N,\mathbb{A})$. Kim [43] established the functoriality for the case $n = 4$, that is a functorial lift from $GL(4)$ to $GL(6)$.

Remark 5.8. As we see in Example 5.3, 5.4 and 5.5, the converse theorem for $GL(n)$ obtained by Cogdell and Piateski-Shapiro plays a crucial role in establishing the functoriality for $GL(2) \times GL(3)$, Sym^3 and Sym^4. There are widely known three methods in establishing the Langlands functoriality which are based on the theory of the Selberg-Arthur trace formula [4], [6], [61], [62], the converse theorems for $GL(n)$ [15], [16], [22] and the theta correspondence or theta lifting method (R. Howe, J. -S. Li, S. Kudla et al.).

According to the above examples and the converse theorems for $GL(n)$, we see that the importance of the Langlands Functoriality Conjecture is that automorphic L-functions of any automorphic representations of any group should be the L-functions of automorphic representations of $GL(n,\mathbb{A})$. In this sense we can say that $GL(n,\mathbb{A})$ is speculated to be the mother of all automorphic representations, and their offspring L-functions are already supposed to have meromorphic continuations and the standard functional equation.

Appendix : Converse theorems

We have seen that the converse theorems have been effectively applied to the establishment of the Langlands functoriality in certain special interesting cases (cf. Example 5.4, 5.5, 5.6 and 5.7). We understand that the converse theorems give a criterion for a given irreducible representation of $GL(n,\mathbb{A})$ to be *automorphic* in terms of the analytic properties of its associated automorphic L-functions. In this appendix, we give a brief survey of the history of the converse theorems and survey the recent results in the local converse theorems.

The first converse theorem was established by Hamburger [26] in 1921. This theorem states that any Dirichlet series satisfying the functional equation of the Riemann zeta function $\zeta(s)$ and suitable regularity conditions must be a multiple of $\zeta(s)$. More precisely, this theorem can be formulated:

Theorem A(Hamburger [26], 1921). *Let two Dirichlet series* $g(s) = \sum_{n\geq 1} a_n n^{-s}$ *and* $h(s) = \sum_{n\geq 1} b_n n^{-s}$ *converge absolutely for* $\mathrm{Re}(s) > 1$. *Suppose that both* $(s-1)h(s)$ *and* $(s-1)g(s)$ *are entire functions of finite order. Assume we have the following functional equation:*

$$\pi^{-\frac{s}{2}}\,\Gamma(s/2)\,h(s) = \pi^{-\frac{1-s}{2}}\,\Gamma((1-s)/2)\,g(1-s).$$

Then $g(s) = h(s) = a_1\zeta(s)$. *Here* $\Gamma(s)$ *is the usual Gamma function, and an entire function* $f(s)$ *is said to be of order* ρ *if*

$$f(s) = O\big(|s|^{\rho+\epsilon}\big) \qquad \textit{for any} \qquad \epsilon > 0.$$

Unfortunately Hamburger's converse theorem was not well recognized until the generalization to L-functions attached to holomorphic modular forms was done by Hecke [29] in 1936. Hecke proved his converse theorem connecting certain L-functions which satisfy a certain functional equation with holomorphic modular forms with respect to the full modular group $SL(2,\mathbb{Z})$. For a good understanding of Hecke's converse theorem, we need to describe Hecke's idea and argument roughly. Let

$$f(\tau) = \sum_{n\geq 1} a_n e^{2\pi i n\tau}$$

be a holomorphic modular form of weight d with respect to $SL(2,\mathbb{Z})$. To such a function f Hecke attached an L-function $L(s,f)$ via the Mellin transform

$$(2\pi)^{-s}\,\Gamma(s)\,L(s,f) = \int_0^\infty f(iy)\,y^s\,\frac{dy}{y}$$

and derived the functional equation for $L(s,f)$. He inverted this process by taking a Dirichlet series

$$D(s) = \sum_{n\geq 1} \frac{a_n}{n^s}$$

and assuming that it converges absolutely in some half plane, has an analytic continuation to an entire function of finite order, and satisfies the same functional equation as $L(s,f)$. In his masterpiece [29], he could reconstruct a holomorphic modular form from $D(s)$ by Mellin inversion

$$f(iy) = \sum_{n\geq 1} a_n e^{-2\pi n y} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} (2\pi)^{-s}\,\Gamma(s)\,D(y)\,y^s\,ds$$

and obtain the modular transformation law for $f(\tau)$ under $\tau \mapsto -\tau^{-1}$ from the functional equation for $D(s)$ under $s \mapsto d-s$. This is Hecke's converse theorem! You might agree that Hecke's original idea and argument are remarkably beautiful. In 1949, in his seminal paper [68], Maass, a student of Hecke, extended Hecke's method to non-holomorphic forms for $SL(2,\mathbb{Z})$. In 1967, the next very important step was made by Weil in his paper [85] dedicated to C. L. Siegel (1896-1981) celebrating Siegel's seventieth birthday. Weil showed how to work with Dirichlet series attached to holomorphic modular forms with respect to congruence subgroups of $SL(2,\mathbb{Z})$. He proved that if a Dirichlet series together with a sufficient number of twists satisfies *nice* analytic properties and functional equations with reasonably suitable regularity, then it stems from a holomorphic modular form with respect to a congruence subgroup of $SL(2,\mathbb{Z})$. More precisely his converse theorem can be formulated as follows.

Theorem B(Weil [85], 1967). *Fix two positive integers d and N. Suppose the Dirichlet series*

$$D(s)=\sum_{n\geq 1}\frac{a_n}{n^s}$$

satisfies the following properties:
$(W1)$ *$D(s)$ converges absolutely for sufficiently large* $\mathrm{Re}(s)\gg 0$;
$(W2)$ *For every primitive character χ of modulus r with $(r,N)=1$, the function*

$$\Lambda(s,\chi):=(2\pi)^{-s}\,\Gamma(s)\sum_{n\geq 1}\frac{a_n\chi(n)}{n^s}$$

has an analytic continuation to an entire function of s to the whole complex plane, and is bounded in vertical strips of finite width;
$(W3)$ *Every such a function $\Lambda(s,\chi)$ satisfies the functional equation*

$$\Lambda(s,\chi)=w_\chi\, r^{-1}\,(r^2N)^{\frac{d}{2}-s}\,\Lambda(d-s,\overline{\chi}),$$

where

$$w_\chi=i^d\,\chi(N)\,g(\chi)^2$$

and

$$g(\chi)=\sum_{n\,(\mathrm{mod}\ r)}\chi(n)\,e^{2\pi i n/r}.$$

Then the function

$$f(\tau)=\sum_{n\geq 1}a_n e^{2\pi i n\tau},\qquad \tau\in\mathbb{H}=\{\tau\in\mathbb{C}\mid \mathrm{Im}\,\tau>0\}$$

is a holomorphic cusp form of weight d with respect to the congruence subgroup $\Gamma_0(N)$, where

$$\Gamma_0(N)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})\,\middle|\, c\equiv 0\ (\mathrm{mod}\ N)\right\}.$$

Weil's proof follows closely Hecke's idea and argument. We see that his converse theorem provides a condition for modularity of the Dirichlet series $D(s)$ under $\Gamma_0(N)$ in terms of the functional equations of Dirichlet series *twisted by primitive characters*. In fact, Weil's converse theorem influenced the complete proof of the Shimura-Taniyama conjecture given by Wiles [86], Taylor et al. So we can say that the work of Weil marks the beginning of the modern era in the study of the connection between L-functions and automorphic forms.

In 1970 Jacquet and Langlands [33] established the converse theorem for $GL(2)$ in the adelic context of automorphic representations of $GL(2,\mathbb{A})$ based on Hecke's original idea. In 1979 Jacquet, Piatetski-Shapiro and Shalika [34] established the converse theorem for $GL(3)$ in the adelic context. Finally in 1994, generalizing the work on the converse theorems on $GL(2)$ and $GL(3)$, Cogdell and Piatetski-Shapiro [12], [15], [16] proved the converse theorem for $GL(n)$ with arbitrary $n \geq 1$ in the context of automorphic representations. The idea and technique in the proof of Cogdell and Piatetski-Shapiro are surprisingly almost the same as Hecke's. We now describe the converse theorems formulated and proved by them.

Let k be a global field, $\mathbb{A}$ its adele ring, and let ψ be a fixed nontrivial continuous additive character of $\mathbb{A}$ which is trivial on k. Let $\pi = \otimes_v \pi_v$ be an irreducible admissible representation of $GL(n,\mathbb{A})$, and let $\tau = \otimes_v \tau_v$ be a cuspidal automorphic representation of $GL(m,\mathbb{A})$ with $m < n$. We define formally

$$L(s,\pi\times\tau) = \prod_v L(s,\pi_v\times\tau_v) \quad \text{and} \quad \varepsilon(s,\pi\times\tau) = \prod_v \varepsilon(s,\pi_v\times\tau_v,\psi_v).$$

We say that $L(s,\pi\times\tau)$ is *nice* if it satisfies the following properties:
(N1) $L(s,\pi\times\tau)$ and $L(s,\widetilde{\pi}\times\widetilde{\tau})$ have analytic continuations to entire functions, where $\widetilde{\pi}$ (resp. $\widetilde{\tau}$) denotes the contragredient of π (resp. τ);
(N2) $L(s,\pi\times\tau)$ and $L(s,\widetilde{\pi}\times\widetilde{\tau})$ are bounded in vertical strips of finite width;
(N3) These entire functions satisfy the standard functional equation

$$L(s,\pi\times\tau) = \varepsilon(s,\pi\times\tau)\, L(1-s,\widetilde{\pi}\times\widetilde{\tau}).$$

Theorem C(Cogdell and Piatetski [15], [16], 1994). *Let π be an irreducible admissible representation of $GL(n,\mathbb{A})$ whose central character is trivial on k^* and whose L-function $L(s,\pi)$ converges absolutely in some half plane. Assume that $L(s,\pi\times\tau)$ is nice for every cuspidal automorphic representation τ of $GL(m,\mathbb{A})$ for $1 \leq m \leq n-2$. Then π is a cuspidal automorphic repesentation of $GL(n,\mathbb{A})$.*

Furthermore they proved the following theorem.

Theorem D(Cogdell and Piatetski [16], 1999). *Let π be an irreducible admissible representation of $GL(n,\mathbb{A})$ whose central character is trivial on k^* and whose L-function $L(s,\pi)$ converges absolutely in some half plane. Let S be a finite set of finite places. Assume that $L(s,\pi\times\tau)$ is nice for every cuspidal automorphic representation τ of $GL(m,\mathbb{A})$ for $1 \leq m \leq n-2$, which is unramified at the places in S. Then π*

is quasi-automorphic in the sense that there is an automorphic representation π' of $GL(n,\mathbb{A})$ such that $\pi_v \cong \pi'_v$ for all $v \notin S$.

The local converse theorem for $GL(n)$ was first formulated by Piatetski-Shapiro in his unpublished Maryland notes (1976) with his idea of deducing the local converse theorem from his global converse theorem. It was proved by Henniart [30] using a local approach. The local converse theorem is a basic ingredient in the proof of [LCC] for $GL(n)$ by Harris and Taylor [28] and by Henniart [31].

The local converse theorem for $GL(n)$ can be formulated as follows.

Theorem E(Henniart [30], 1993). *Let k be a nonarchimedean local field of characteristic 0. Let τ and τ' be irreducible admissible generic representations of $GL(n,k)$ with the same central character. Assume the twisted local gamma factors (cf. [35]) are the same, i.e.,*

$$\gamma(s,\tau\times\rho,\psi)=\gamma(s,\tau'\times\rho,\psi)$$

for all irreducible supercuspidal representations ρ of $GL(m,k)$ with $1\le m\le n-1$. Then τ is isomorphic to τ'.

Remark 1. It is known that the twisting condition on m reduces from $n-1$ to $n-2$. It is expected as a conjecture of H. Jacquet [16, Conjecture 8.1] that the twisting condition on m should be reduced from $n-1$ to $[\frac{n}{2}]$.

Remark 2. The local converse theorem for generic representations of $U(2,1)$ and for $GSp(4)$ was established by E. M. Baruch in his Ph. D. thesis (Yale Univ., 1995).

Jiang and Soudry [38] proved the local converse theorem for irreducible admissible generic representations of $SO(2n+1,k)$.

Theorem F(Jiang and Soudry [38], 2003). *Let σ and σ' be irreducible admissible generic representations of $SO(2n+1,k)$. Assume the twisted local gamma factors are the same, i.e.*

$$\gamma(s,\sigma\times\rho,\psi)=\gamma(s,\sigma'\times\rho,\psi)$$

for all irreducible supercuspidal representations ρ of $GL(m,k)$ with $1\le m\le 2n-1$. Then σ is isomorphic to σ'.

I shall give a brief sketch of the idea of their proof. They first reduce the proof of Theorem F to the case where both σ and σ' are supercuspidal by studying the existence of poles of twisted local gamma factors and related properties. Developing the explicit local Howe duality for irreducible admissible generic representations of $SO(2n+1,k)$ and the metaplectic group $\widetilde{Sp}(2n,k)$, and using the global weak Langlands functorial lifting form $SO(2n+1)$ to $GL(2n)$ (cf. Example 5.4, [13], [14]) and the local backward lifting from $GL(2n,k)$ to $\widetilde{Sp}(2n,k)$, they relate the local converse theorem for $SO(2n+1)$ with that for $GL(2n)$ which is well known now.

As an application of Theorem F, I repeat again that Jiang and Soudry [38], [39] proved the Local Langlands Reciprocity Law for $SO(2n+1)$. More precisely, there exists a *unique* bijective correspondence between the set of conjugacy classes of all

$2n$-dimensional, admissible, completely reducible, multiplicity-free, symplectic complex representations of the Weil group W_k and the set of all equivalence classes of irreducible generic supercuspidal representations of $SO(2n+1, k)$, which preserves the relevant local factors. As an application of Theorem F to the global theory, they proved that the weak Langlands functorial lifting from irreducible *generic* cuspidal automorphic representations of $SO(2n+1)$ to irreducible automorphic representations of $GL(2n)$ is *injective* up to isomorphism. It is still an open problem to establish the Langlands functorial lift from irreducible *non-generic* cuspidal automorphic representations of $SO(2n+1)$ to $GL(2n)$. As another application to the global theory, they proved the *rigidity theorem* in the sense that if $\pi = \otimes_v \pi_v$ and $\tau = \otimes_v \tau_v$ are irreducible generic cuspidal automorphic representations of $SO(2n+1, \mathbb{A})$ such that π_v is isomorphic to τ_v for almost all local places v, then π is isomorphic to τ.

References

[1] J. Adams, D. Barbasch abd D. Vogan, The Langlands Classification and Irreducible Characters for real Reductive Groups, Progr. Math. 104, Birkhäuser, Boston, 1992.

[2] J. Adams and D. Vogan, *L-groups, projective representations, and the Langlands classification*, Amer. J. Math., **114**(1992), 45-138.

[3] J. Arthur, *The principle of functoriality*, Bull. Amer. Math. Soc., **40(1)**(2003), 39-53.

[4] J. Arthur, *An Introduction to the Trace Formula, In Harmonic Analysis, the Trace Formula, and Shimura Varieties*, Proceedings of the Clay Mathematics Institute, 2003 Summer School, The Fields institute, Toronto, Canada, June 2-27, 2003, edited by J. Arthur, D. Ellwood and R. Kottwitz, Amer. Math. Soc. and CMI, 2005, 1-263.

[5] J. Arthur, *A notes on L-packets*, Pure and Applied Mathematics Quarterly, **2**(2006), 199-217.

[6] J. Arthur and L. Clozel, Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula, Ann. of Math. Studies 120, Princeton Univ. Press, N.J., 1989.

[7] M. Asgari and F. Shahidi, *Generic transfer for general spin groups*, Duke Math. J., **132**(2006), 137-190.

[8] M. Asgari and F. Shahidi, *Generic transfer for* $GSp(4)$ *to* $GL(4)$, Compositio Math., **142**(2006), 541-550.

[9] A. Borel, *Automorphic L-functions*, Proc. Sympos. Pure Math., **33**, Part 2, Amer. Math. Soc., 1979, 27-62.

[10] K. Buzzard, M. Dickinson, N. Shepard-Barron and R. Taylor, *On Icosaheral Artin representations,* Duke Math. J., **109**(2001), 283-318.

[11] L. Clozel, M. Harris and R. Taylor, *Automorphy for some ℓ-adic lifts of automorphic mod ℓ representations* (preprint), http://www.math.harvard.edu/~rtaylor/.

[12] J. Cogdell, Lectures on L-functions, Converse Theorems, and Functoriality for $GL(n)$, in Lectures in Automorphic L-functions, Fields institute Monographs, Amer. Math. Soc., 2004, 3-96.

[13] J. Cogdell, H. Kim, I. Piatetski-Shapiro, and F. Shahidi, *On lifting from classical groups to GL_N*, Publ. Math. Inst. Hautes Études Sci., **93**(2001), 5-30.

[14] J. Cogdell, H. Kim, I. Piatetski-Shapiro, and F. Shahidi, *Functoriality for the classical groups*, Publ. Math. Inst. Hautes Études Sci., **99**(2004), 163-233.

[15] J. Cogdell and I. Piatetski-Shapiro, *Converse theorems for GL_n*, Publ. Math. Inst. Hautes Études Sci., **79**(1994), 157-204.

[16] J. Cogdell and I. Piatetski-Shapiro, *Converse theorems for GL_n, II*, J. reine angew. Math., **507**(1999), 165-188.

[17] P. Deligne, Formes modulaires et représentations ℓ-adiques: Sém. Bourbaki 1968/9, no. 355; Lecture Notes in Math. 179, Springer-Verlag, Berlin-Heidelberg-New York, 1971, 139-172.

[18] P. Deligne, Formes modulaires et représentations of $GL(2)$, Antwerp II, in Modular Forms of One Variable II, Lecture Notes in Math. 349, Springer-Verlag, New York, 1973, 55-106.

[19] P. Deligne, Les constants des équation fontionnelle des fonctions L , in Modular Forms of One Variable II, Lecture Notes in Math., 349, Springer-Verlag, New York, 1973, 501-595.

[20] V. Drinfeld, *Proof of the Petersson conjecture for $GL(2)$ over a global field of characteristic p*, Funct. Anal. and its Appl., **22**(1988), 28-43.

[21] V. Drinfeld, *Cohomology of compactified modules of F-sheaves of rank* 2, J. Soviet Math., **46**(1989), 1789-1821.

[22] S. Gelbart and H. Jacquet, *A relation between automorphic representation of $GL(2)$ and $GL(3)$*, Ann. Scient. Éc. Normale Sup., **11**(1978), 471-552.

[23] S. Gelbart and F. Shahidi, *Analytic Properties of Automorphic L-Functions*, Perspective in Mathematics, **6**(1988), Academic Press.

[24] S. Gelbart, I. Piatetski-Shapiro and S. Rallis, Explicit Constructions of Automorphic L-Functions, Lecture Notes in Math. 1254, Springer-Verlag, New York, 1987.

[25] D. Ginzburg, S. Rallis and D. Soudry, *Generic automorphic forms on $SO(2n+1)$: functorial lift on $GL(2n)$, endoscopy, and the base change*, Internat. Math. Res. Notices, no. **14**(2001), 729-764.

[26] H. Hamburger, *Über die Riemannsche Funktionalgleichung der ζ-Funktion*, Math. Zeit., **10**(1921), 240-254; **11**(1922), 224-245; **13**(1922), 283-311.

[27] M. Harris, N. Shepherd-Barron and R. Taylor, *Ihara's lemma and potential automorphy* (preprint), http://www.math.harvard.edu/~rtaylor/.

[28] M. Harris and R. Taylor, On the Geonmetry and Cohomology of Some Simple Shimura Varieties, Ann. of Math. Studies 151, Princeton Univ. Press, Princeton, N. J., 2001.

[29] E. Hecke, *Über die Bestimmung Dirichletscher Reihnen durch Funktionalgleichungen*, Math. Ann., **112**(1936), 664-699.

[30] G. Henniart, *Caractérisation de la correspondance de Langlands locale par les facteurs ε de paires*, Invent. Math., **113**(1993), 339-350.

[31] G. Henniart, *Une preuve simple des conjectures de Langlands de $GL(n)$ sur un corps p-adique*, Invent. Math., **139**(2000), 439-455.

[32] T. Ikeda, *On the location of poles of the triple L-functions*, Compositio Math., **83**(1992), 187-237.

[33] H. Jacquet and R. Langlands, Automorphic Forms on $GL(2)$, Lecture Notes in Math. 114, Springer-Verlag, New York, 1970.

[34] H. Jacquet, I. Piatetski-Shapiro and J. Shalika, *Automorphic forms on* $GL(3)$, I and II, Ann. of Math., **109**(1979), 169-258.

[35] H. Jacquet, I. Piatetski-Shapiro and J. Shalika, *Rankin-Selberg convolutions*, Amer. J. Math., **105**(1983), 367-464.

[36] H. Jacquet and J. Shalika, *On Euler products and the classification of automorphic representations* I, Amer. J. Math., **103**(1981), 499-558.

[37] H. Jacquet and J. Shalika, *On Euler products and the classification of automorphic representations* II, Amer. J. Math., **103**(1981), 777-815.

[38] D. Jiang and D. Soudry, *The local converse theorem for* $SO(2n+1)$ *and applications*, Ann. Math., **157**(2003), 743-806.

[39] D. Jiang and D. Soudry, Generic representations and local langlands reciprocity law for p-adic SO_{2n+1}, in Contributions to Automorphic Forms, Geometry and Number Theory, edited by H. Hida, D. Ramakrishnan and F. Shahidi, the John Hopkins University Press, Baltimore, 2004, 457-519.

[40] D. Kazhdan and G. Lusztig, *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Invent. Math., **95**(1987), 153-215.

[41] C. Khare, M. Larsen and G. Savin, *Functoriality and the inverse Galois problem*, Compositio Math., **144**(2008), 541-564.

[42] C. Khare, M. Larsen and G. Savin, *Functoriality and the inverse Galois problem II: Groups of type* B_n *and* G_2, arXiv:0807.0861v1, 2008.

[43] H. Kim, *Functoriality for the exterior square of* GL_4 *and the symmetric fourth of* GL_2, J. of Amer. Math. Soc., **16(1)**(2002), 139-183.

[44] H. Kim, Automorphic L-functions, in Lectures in Automorphic L-functions, Fields institute Monographs, Amer. Math. Soc., 2004, 97-201.

[45] H. Kim and F. Shahidi, *Symmetric cube L-functions for* GL_2 *are entire*, Ann. of Math., **150**(1999), 645-662.

[46] H. Kim and F. Shahidi, *Functional products for* $GL_2 \times GL_3$ *and functorial symmetric cube for* GL_2, Ann. of Math., **155**(2002), 837-893.

[47] H. Kim and F. Shahidi, *Cuspidality of symmetric powers with applications*, Duke Math. J., **112**(2002), 177-197.

[48] A. Knapp and G. Zuckerman, Normalizing factors and L-groups, Proc. Symp. Pure Math., Amer. Math. Soc., Providence, RI, 33, Part 1, 1979, 93-106.

[49] A. Knapp and G. Zuckerman, *Classification of irreducible tempered representations of semisimple groups*, Ann. of Math., **116**(1982), 389-455.

[50] R. Kottwitz and D. Shelstad, *Foundations of Twisted Endoscopy*, Astérisque, **255**(1999).

[51] J. -P. Labesse and R. Langlands, *L-indistinguishabilty for* $SL(2)$, Canad. J. Math., **31**(1979), 726-785.

[52] L. Lafforgue, *Chtoucas de Drinfeld et correspondance de Langlands*, Invent. Math., **147**(2002), 1-241.

[53] K. F. Lai, On the Tamagawa number of quasi-split groups, Thesis, Yale University, 1974.

[54] R. Langlands, Problems in the theory of automorphic forms, in Lectures in modern analysis and Applications, Lecture Notes in Math. 170, Springer-Verlag, New York, 1970, 18-61.

[55] R. Langlands, Euler Products, Yale University Press, New Haven 310, 1971.

[56] R. Langlands, Modular forms and ℓ-adic representations, in Modular Forms of One Variable II, Lecture Notes in Math. 349, Springer-Verlag, New York, 1973, 361-500.

[57] R. Langlands, On Artin's L-functions, Rice University Studies 56, 23-28.

[58] R. Langlands, On the Functional Equations Satisfied by Eisenstein Series, Lecture Notes in Math. 544, Springer-Verlag, New York, 1976.

[59] R. Langlands, *On the notion of automorphic representations. A supplement to the preceding paper*, Proc. Sympos. Pure Math. **33**, Part 1, Amer. Math. Soc., 1979, 203-208.

[60] R. Langlands, *Automorphic representations, Shimura varieties, and motives*, Proc. Sympos. Pure Math., **33**, Part 2, Amer. Math. Soc., 1979, 205-246.

[61] R. Langlands, Base Change for $GL(2)$, Ann. of Math. Studies 96, Princeton Univ. Press, Princeton, N. J. 1980.

[62] R. Langlands, *Les débuts d'une formule des traces stables*, Publ. Math. Ubiv. Paris VII., **13**(1983).

[63] R. Langlands, *On the classiciation of irreducible representations of real algebraic groups*, in *Representation Theory and Harmonic Analysis on Semisimple Lie Groups*, AMS Mathematical Surveys and Monographs, **31**(1989), 101-170.

[64] R. Langlands, *Representations of abelian algebraic groups*, Pacific J. Math., 1997, 231-250.

[65] R. Langlands, Where stands functoriality today?, in Representation Theory and Automorphic Forms (March 1996, Edinburgh, Scotland), edited by T. N. Bailey and A. W. Knapp, 1997, 457-471.

[66] R. Langlands, Beyond endoscopy, in Contributions to Automorphic Forms, Geometry and Number Theory, edited by H. Hida, D. Ramakrishnan and F. Shahidi, the John Hopkins University Press, Baltimore 2004, 611-698.

[67] G. Laumon, M. Rapoport and U. Stuhler, *$\mathcal{D}$-eliptic sheaves and the Langlands correspondence*, Invent. Math., **113**(1993), 217-338.

[68] H. Maass, *Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihnen durch Funktionalgleichungen*, Math. Ann., **121**(1949), 141-183.

[69] B. Mazur, *Finding Meaning in Error Terms*, Bull. Amer. Math. Soc., **45(2)**April(2008), 185-228.

[70] D. Ramakrishnan, *Pure motives and automorphic forms*, in Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., **55**, Part 1, Amer. Math. Soc., 1994, 377-400.

[71] D. Ramakrishnan, *Modularity of the Rankin-Selberg L-series, and multiplicity one for* $SL(2)$, Ann. of Math., **152**(2000), 45-111.

[72] D. Ramakrishnan, Modularity of solvable Artin representations of $GO(4)$-type, Int. Math. Research Notices, no. 1 (2002), 1-54.

[73] J.-P. Serre, Abelian ℓ-adic Representations and Elliptic Curves, Benjamin, New York, 1968.

[74] J.-P. Serre, *Propriétés conjecturales des groupes de Galois motiviques et des representations* ℓ*-adiques*, in Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math. **55**, Part 1, Amer. Math. Soc., 1994, 377-400.

[75] F. Shahidi, *Functional equations satisfied by certain L-functions*, Compositio Math., **37**(1978), 171-208.

[76] F. Shahidi, *On certain L-functions*, Amer. J. Math., **103**(1981), 297-356.

[77] F. Shahidi, *On the Ramanujan conjecture and finiteness of poles for certain L-functions*, Ann. of Math., **127**(1988), 547-584.

[78] F. Shahidi, *A proof of Langlands' conjecture on Plancherel measures; Complementary series for p-adic groups*, Ann. of Math., **132**(1988), 273-330.

[79] F. Shahidi, *Local coefficients as Mellin transforms of Bessel functions; towards a general stability*, IMRN, **39**(2002), 2075-2119.

[80] F. Shahidi, *Third symmetric powe L-functions for* $GL(2)$, Compositio Math., **70**(1989), 245-273.

[81] F. Shahidi, *Langlands Functoriality Conjecture and Number Theory, In Representation Theory and Automorphic Forms*, edited by T. Kobayashi, W. Schmid and J. -H. Yang, Progress in Mathematics, Birkhäser, **255**(2008), 151-173.

[82] J. Tate, *Number theoretic background, in Automorphic Forms, representation Theory and L-functions*, Proc. Sympos. Pure Math., **33**, Part 2, Amer. Math. Soc., 1979, 3-26.

[83] R. Taylor, *Automorphy for some* ℓ*-adic lifts of automorphic mod* ℓ *representations* II (preprint), http://www.math.harvard.edu/~rtaylor/.

[84] J. Tunnell, *Artin's conjecture for representations of octahedral type,* Bull. Amer. Math. Soc., **5**(1981), 173-175.

[85] A. Weil, *Über die Bestimmung Dirichletscher Reihnen durch Funktionalgleichungen*, Math. Ann., **168**(1967), 149-156.

[86] A. Wiles, *Modular elliptic curves and Fermat's Last Theorem*, Ann. of Math., **141**(1995), 443-551.

[J] Harmonic Analysis on the Minkowski-Euclid Space

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Harmonic Analysis on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$, II : Unitary Representations of the Group $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$

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Abstract. In this paper, we study unitary representations of the group $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$.

1. Introduction

This paper is a continuation of "Harmonic Analysis on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$, I". The aim of this paper is to study the unitary representations of the group $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$ in detail.

The motivation for studying the group $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$ can be explained as follows. We consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \left\{ (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\, {}^t\lambda \text{ symmetric} \right\}$$

endowed with the following multiplication law

$$(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda').$$

We define the semidirect product of $Sp(n,\mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$

$$Sp_{n,m} = Sp(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

endowed with the following multiplication law

$$\begin{aligned} & (M, (\lambda, \mu; \kappa)) \cdot (M', (\lambda', \mu'; \kappa')) \\ = \ & (MM', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu', ; \kappa + \kappa' + \tilde{\lambda}\, {}^t\mu' - \tilde{\mu}\, {}^t\lambda')), \end{aligned}$$

where $M,\ M' \in Sp(n,\mathbb{R})$ and $(\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)M'$. It is easy to see that the Jacobi group $Sp_{n,m}$ acts on the homogeneous space $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ transitively by

(1.1) $$(M, (\lambda, \mu; \kappa)) \cdot (Z, W) := (M\langle Z\rangle, (W + \lambda Z + \mu)(CZ + D)^{-1}),$$

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where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(Z,W) \in \mathbb{H}_n \times \mathbb{C}^{(m,n)}$. We let

$$GL_{n,m} = GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$$

be the semidirect product of $GL(n,\mathbb{R})$ and the commutative additive group $\mathbb{R}^{(m,n)}$ equipped with the following multiplication law

$$(g,a)\cdot(h,b) = (gh, a\,{}^t h^{-1} + b), \tag{1.2}$$

where $g,\ h \in GL(n,\mathbb{R})$ and $a,b \in \mathbb{R}^{(m,n)}$. Then the action (1.1) of $Sp_{n,m}$ on $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ gives a *canonical* action of $GL_{n,m}$ on the nonsymmetric homogeneous space $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ given by

$$(g,a)\cdot(Y,V) := (gY\,{}^t g,\ (V+a)\,{}^t g), \tag{1.3}$$

where $g \in GL(n,\mathbb{R}),\ a \in \mathbb{R}^{(m,n)},\ Y \in \mathcal{P}_n$ and $V \in \mathbb{R}^{(m,n)}$. In [15], we developed the theory of automorphic forms on $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$ generalizing automorphic forms on $GL(n,\mathbb{R})$.

This paper is organized as follows. In Section 2, we survey the unitary representations of the general linear group $GL(n,\mathbb{R})$. The unitary dual of $GL(n,\mathbb{R})$ was completely determined by E. Stein [10], B. Speh [6]-[8], D. Vogan [14] and other people. We also review certain principal series of $GL(n,\mathbb{R})$ investigated by R. Howe and S. T. Lee [2]. In Section 3, we study the unitary representations of $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$. Using the Mackey's method, we compute the unitary dual of $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$ explicitly in the cases of $n = 2, 3,\ m$ arbitrary. We also deal with certain unitary representations of $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$ (cf. (3.8)) and discuss their irreducibility.

Notations. We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers, and the field of complex numbers respectively. $\mathbb{R}^\times$ denotes the multiplicative group consisting of nonzero real numbers. The symbol $\mathbb{C}_1^\times$ denotes the multiplicative group consisting of all complex numbers z with $|z| = 1$. The symbol ":=" means that the expression on the right hand side is the definition of that on the left. We denote by $\mathbb{Z}^+$ the set of all positive integers. We denote by $F^{(k,l)}$ the set of all $k \times l$ matrices with entries in a commutative ring F. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. For a Lie group G, we denote by $\hat{G}$ the unitary dual of G.

2. A survey on the unitary dual of $GL(n,\mathbb{R})$

In this section, we survey the unitary dual of $GL(n,\mathbb{R})$. The references are [12]-[14], [5] and [6]-[8].

First we define the Stein's complimentary series (cf. [10]). Assume $n = 2m$ with m a positive integer. We write

$$P = LN = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \,\middle|\, A, D \in GL(m,\mathbb{R}),\ B \in \mathbb{R}^{(m,m)} \right\},$$

where

$$L = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \,\Big|\, A, D \in GL(m,\mathbb{R}) \right\} \cong GL(m,\mathbb{R}) \times GL(m,\mathbb{R})$$

and

$$N = \left\{ \begin{pmatrix} I_m & B \\ 0 & I_m \end{pmatrix} \,\Big|\, B \in \mathbb{R}^{(m,m)} \right\} \cong \mathbb{R}^{(m,m)}.$$

Then P is a maximal parabolic subgroup of $GL(2m,\mathbb{R})$ and N is the unipotent radical of P.

Let $\delta_m : GL(m,\mathbb{R}) \longrightarrow \mathbb{R}$ be the modular function. That is, $\delta_m(g) = \det g$ for $g \in GL(m,\mathbb{R})$. We *fix* a one-dimensional unitary character j of $GL(m,\mathbb{R})$ and a complex number t. We let $\phi_{2m}(j,t) : P \longrightarrow \mathbb{C}^\times$ be the (generally non-unitary) character of P defined by

$$\phi_{2m}(j,t)((g,h),n) := j(gh) \cdot [\delta_m(gh^{-1})]^t,$$

where $g,\ h \in GL(m,\mathbb{R})$ and $n \in N$. We put

$$\sigma_{2m}(j,t) = \mathrm{Ind}_P^{GL(n,\mathbb{R})} \phi_{2m}(j,t). \tag{2.1}$$

According to Stein [10], we see that $\sigma_{2m}(j,t)$ is unitary and irreducible for $t \in i\mathbb{R}$, and that $\sigma_{2m}(j,t)$ is irreducible for $|t| < \frac{1}{2}$. We call the representations $\sigma_{2m}(j,t)$ for $0 < t < \frac{1}{2}$ the *Stein complimentary series* of $GL(2m,\mathbb{R})$.

We observe that the characters of $GL(m,\mathbb{R})$ may be identified in a natural way with the characters of $GL(1,\mathbb{R})$, and hence j extends to a character of $GL(2m,\mathbb{R})$.

Now we fix a unitary character

$$j_1 : \mathbb{R}^\times \longrightarrow \mathbb{C}^\times. \tag{2.2}$$

This corresponds naturally to a family of characters

$$j_m : GL(m,\mathbb{R}) \longrightarrow \mathbb{C}^\times \tag{2.3}$$

characterized by the property that for $m \le m'$,

$$j_{m'}|_{GL(m,\mathbb{R})} = j_m.$$

We refer to the collection $\{j_m\}$ loosely as j. We recall that a representation σ of $GL(m,\mathbb{R})$ is called *spherical* if the trivial representation of $O(m)$ is contained in the restriction of σ to $O(m)$.

Definition 2.1. Let j be a family of characters as in (2.3). Define one-dimensional representations μ_m of $O(m)$ by

$$\mu_m = j_m|_{O(m)}. \tag{2.4}$$

Write μ for the collection $\{\mu_m\}$. We call μ_m a *special one dimensional representation* of $O(m)$. A representation σ of $GL(m,\mathbb{R})$ is called *almost spherical* of type μ_m if μ_m occurs in the restriction of σ to $O(m)$, in other words, if $j_m^{-1}\otimes\sigma$ is spherical.

Definition 2.2. An (ordered) partition of a positive integer n is a sequence

$$\pi=(n_1,n_2,\cdots,n_r),\quad n_i\in\mathbb{Z}^+,\ \sum_{i=1}^r n_i=n.$$

We define

$$\begin{aligned} GL(\pi) &:= GL(n_1,\mathbb{R})\times\cdots\times GL(n_r,\mathbb{R})\subset GL(n,\mathbb{R}),\\ O(\pi) &:= O(n_1)\times\cdots\times O(n_r)=O(n)\cap GL(\pi).\end{aligned}$$

We let $P(\pi)$ be the parabolic subgroup of $GL(n,\mathbb{R})$ generated by $GL(\pi)$ and the Borel subgroup B of $GL(n,\mathbb{R})$ consisting of upper triangular matrices. We let $N(\pi)$ the unipotent radical of $P(\pi)$.

We fix $\mu=\{\mu_m\}$ as in Definition 2.1. The data are a partition $\pi=(n_i)$ of n, and a collection

$$\tau=(\tau_i),\quad \tau_i\in\widehat{GL(n_i,\mathbb{R})},$$

such that

(a) τ_i is almost spherical of type μ_{n_i}, and

(b) τ_i is either a unitary character or a Stein complimentary series.

We call the following induced representation

$$\sigma_\pi(\tau):=\mathrm{Ind}_{P(\pi)}^{GL(n,\mathbb{R})}\otimes\tau_i$$

a *basic almost spherical representation* of type μ.

Theorem 2.3.

(1) *$\sigma_\pi(\tau)$ and $\sigma_{\pi'}(\tau')$ are equivalent if and only if (π',τ') is a permutation of (π,τ).*

(2) *The basic almost spherical representations are unitary.*

(3) *The basic almost spherical representations are irreducible.*

(4) *Any irreducible unitary almost spherical representation of $GL(n,\mathbb{R})$ is basic.*

The outline of proof can be found in [14], p. 455.

Definition 2.4. Let G be a real Lie group with Lie algebra $\mathfrak{g}$. Let K be a compact subgroup of G. Let V be a $\mathfrak{g}$-module that is also a module for K. We say that V is a $(\mathfrak{g},K)$-module if the following conditions (1)-(3) are satisfied:

(1) The action of $\mathfrak{g}$ on V is compatible with that of K on V. That means that $k \cdot X \cdot v = Ad(k)X \cdot k \cdot v$ for $v \in V,\ k \in K,\ X \in \mathfrak{g}$.

(2) If $v \in V$, then Kv spans a finite dimensional vector space W_v of V such that the action of K on W_v is continuous.

(3) If $Y \in \mathfrak{k}$ and if $v \in V$, then $\frac{d}{dt}\Big|_{t=0} \exp(tY)v = Yv$.

A $(\mathfrak{g}, K)$-module is said to be *finitely generated* if it is finitely generated as a $U(\mathfrak{g})$-module. V is said to be *irreducible* if V and 0 are the only $\mathfrak{g}$ and K-invariant subspaces of V.

Definition 2.5. Suppose G is a reductive Lie group with K a maximal compact subgroup of G. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{k}$, and T the corresponding Cartan subgroup. Write $2\rho_c$ for the sum of the roots of $\mathfrak{t}$ in $\mathfrak{b}$. Fix an irreducible representation μ of K of highest weight γ in $\hat{T}$. Let $\gamma_0 \in \mathfrak{t}^*$ be a weight of γ. We define the norm $\|\mu\|$ of μ by

$$\|\mu\| = \langle \mu + 2\rho_c,\ \mu + 2\rho_c \rangle. \tag{2.5}$$

If X is any $(\mathfrak{g}, K)$-module, we say that μ is a *lowest K-type* of X if

(a) μ occurs in the restriction of X to K; and

(b) $\|\mu\|$ is minimal subject to (a).

Theorem 2.6. *Let X be an irreducible $(\mathfrak{g}, K)$-module for $G = GL(n, \mathbb{R})$. Then X has a unique lowest K-type. It occurs with multiplicity one in X.*

Remark 2.7. Representations of general reductive groups may have several lowest K-types. For more detail, we refer to [12].

For a positive integer n, we let $m = [n/2]$ and $\epsilon = n - 2m$. Then $n = 2m + \epsilon$. We set

$$T_0 = SO(2) \times \cdots \times SO(2) \quad (m \text{ copies}).$$

Embedding T_0 in $O(n)$ and identifying $SO(2)$ with the circle, we obtain

$$\widehat{T_0} \cong \mathbb{Z}^m.$$

The Cartan subgroup T of $O(n)$ is given by

$$T = T_0 \rtimes \{E_n,\ r_n\},$$

where E_n denotes the identity matrix of degree n and $r_n = \mathrm{diag}(1, \cdots, 1, -1)$ is the diagonal matrix of degree n.

Proposition 2.8. *The irreducible representations of $O(n)$ are parametrized by pairs (γ, η), subject to the following conditions.*

(a) *γ is a decreasing sequence of m non-negative integers, in other words, a weight of T_0.*

(b) *If n is even and γ_m is not zero, then η is $\frac{1}{2}$; otherwise $\eta = 0$ or 1.*

Let μ be the irreducible representation of $O(n)$ of highest weight (γ, η). If $\eta = 0$ or 1, the restriction of μ to $SO(n)$ is the irreducible representation of highest weight γ. If $\eta = \frac{1}{2}$, the restriction of μ to $SO(n)$ is the sum of the irreducible representations of highest weight $\gamma = (\gamma_1, \cdots, \gamma_{m-1}, \gamma_m)$ and $(\gamma_1, \cdots, \gamma_{m-1}, -\gamma_m)$.

Let μ be an irreducible representation of $O(n)$ of highest weight (γ, η) as in Proposition 2.8. Let p be the largest integer such that γ_p is at least 2. Define

$$\lambda(\mu) = (\gamma_1 - 1, \cdots, \gamma_p - 1, 0, \cdots, 0). \tag{2.6}$$

Let $\pi = (p_1, \cdots, p_r)$ be the coarsest ordered partition of p such that γ is constant on the parts of π. Then the centralizer $L_\theta := L_\theta(\mu)$ of $\lambda(\mu)$ in $GL(n, \mathbb{R})$ is given by

$$L_\theta = GL(\pi, \mathbb{C}) \times GL(n - 2p, \mathbb{R}), \tag{2.7}$$

where

$$GL(\pi, \mathbb{C}) = \prod_{i=1}^{r} GL(p_i, \mathbb{C}).$$

We let μ_{L_θ} be the representation of $L_\theta \cap K$ of highest weight

$$((\gamma_1 - 1, \cdots, \gamma_p - 1, \gamma_{p+1}, \cdots, \gamma_m), \eta).$$

Let μ_f be the representation of $O(n - 2p)$ parametrized by $((\gamma_{p+1}, \cdots, \gamma_m), \eta)$. Let $\gamma(j)$ denote the constant value of γ on the j-th part of π. Then we get

$$\mu_{L_\theta} = \left[\otimes_{j=1}^{r} \det{}^{\gamma(j)-1}\right] \otimes \mu_f. \tag{2.8}$$

We write $q = n - 2p$. The last $[n/2] - p$ terms of γ are zeros and ones; say there are q' ones. Define q_0 and q_1 as follows:

$$\text{if } \eta = 0 \text{ or } \frac{1}{2}, \text{ then } q_1 = q' \text{ and } q_0 = q - q_1; \tag{2.9}$$

and

$$\text{if } \eta = 1 \text{ or } \frac{1}{2}, \text{ then } q_0 = q' \text{ and } q_1 = q - q_0. \tag{2.10}$$

Let

$$L := GL(\pi, \mathbb{C}) \times GL(q_0, \mathbb{R}) \times GL(q_1, \mathbb{R}). \tag{2.11}$$

We define

$$\mu_L := \left[\otimes_{j=1}^{r} \det{}^{\gamma(j)-1}\right] \otimes 1 \otimes \det. \tag{2.12}$$

It is clear that μ_L is an almost spherical representation of $L \cap O(n)$.

Lemma 2.9. *Suppose q_0 and q_1 are non-negative integers, and $q = q_0 + q_1$. Write $q = 2r + \epsilon$ with $r = [q/2]$. Then there is a unique decreasing sequence γ of r ones and zeros, and an η equal to 0, $\frac{1}{2}$ or 1, with the following properties (1) and (2) :*

(1) *η is $\frac{1}{2}$ if and only if q is even and $\gamma_r = 1$;*

(2) *(2.9) and (2.10) hold, where q' is the number of ones in γ.*

Write $\mu_f = \mu_f(q_0, q_1)$ for the irreducible representation of $O(q)$ of highest weight (γ, η) with $q = q_0 + q_1$. Then μ_f is the lowest $O(q)$-type of

$$\mathrm{Ind}\,{}_{O(q_0)\times O(q_1)}^{O(q)}(1 \otimes \det).$$

D. Vogan proved the following important theorem.

Theorem 2.10 (Vogan, [14]). *Let $G = GL(n,\mathbb{R})$ and $K = O(n)$. Let (L, μ_L) be the one defined by (2.11) and (2.12). Then L is a product of various $GL(m_i,\mathbb{R})$, and μ_L is a special one dimensional representation of $L \cap K$ (see Definition 2.1). And there is a functor Ω defining a bijection from the set of irreducible unitary representations of L, almost spherical of type μ_L onto the set of irreducible unitary representations of G of lowest K-type μ. In particular, Ω has the following properties*:

(a) *If Y is a basic almost spherical representation of L of type μ_L, then ΩY is unitary and irreducible.*

(b) *If X is any irreducible unitary representation of G of lowest K-type μ, then there is a unitary almost spherical representation Y of L such that X is a subquotient of ΩY.*

Now we describe the functor Ω in a rough way. The main idea in the proof of Theorem 2.10 is to reduce irreducible unitary representations to the case of spherical representations. Together with Theorem 2.3, the above theorem parametrizes the unitary dual of $GL(n,\mathbb{R})$.

For brevity, we set $G = GL(n,\mathbb{R})$ and $K = O(n)$ for the time being. Fix an element μ on the unitary dual $\hat{K}$ of K. Define $\lambda := \lambda(\mu)$ as in (2.6). Then λ belongs to a fixed Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$. We may find a θ-stable parabolic subalgebra

$$\mathfrak{q}_\theta = \mathfrak{l}_\theta + \mathfrak{u}_\theta$$

of $\mathfrak{g}$, where L_θ is defined as in (2.7). $\mathfrak{u}_\theta$ is characterized by the properties:

$$\Delta(\mathfrak{u}_\theta, \mathfrak{t}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}) |\ \langle \alpha, \lambda \rangle > 0\,\}$$

and

$$\Delta(\mathfrak{l}_\theta, \mathfrak{t}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}) |\ \langle \alpha, \lambda \rangle = 0\,\}\,.$$

We define a functor

$$\Omega_\theta = \mathcal{L}^S((\mathfrak{q}_\theta, L_\theta \cap K) \uparrow (\mathfrak{g}, K))$$

from $(\mathfrak{l}_\theta, L_\theta \cap K)$-modules to $(\mathfrak{g}, K)$-modules. The definition of $\mathcal{L}^S$ is explained in Section 5 of [12]. We define a functor

$$(\Omega^K)_\theta = (\mathcal{L}^K)^S$$

from representations of $L_\theta \cap K$ to representations of K.

Fix a real parabolic subgroup P of L_θ with Levi factor L, where L is defined as in (2.11). Let

$$P = LN$$

be the Levi decomposition of P. We define a functor

$$\Omega_\mathbb{R} = \mathrm{Ind}\,(L \uparrow L_\theta)$$

from $(\mathfrak{l}, L \cap K)$-modules to $(\mathfrak{l}_\theta, L_\theta \cap K)$-modules. We also define a functor

$$(\Omega^K)_\mathbb{R} = \mathrm{Ind}((L \cap K) \uparrow (L_\theta \cap K))$$

from representations of $L \cap K$ to representations of $L_\theta \cap K$.

We define a functor

$$\Omega = \Omega_\theta \circ \Omega_\mathbb{R} \tag{2.13}$$

from $(\mathfrak{l}, L \cap K)$-modules to $(\mathfrak{g}, K)$-modules. We set

$$\Omega^K = (\Omega^K)_\theta \circ (\Omega^K)_\mathbb{R}$$

a functor from representations of $L \cap K$ to representations of K. The functor Ω in (2.13) is nothing but the functor mentioned in Theorem 2.9. The complete description of the unitary dual of $GL(n, \mathbb{R})$ was given by V. Bargman [1] for $n = 2$, B. Speh [7] for $n = 3$, 4 and D. Vogan [14] for the general case. For the case $n = 2$, we pass from $SL(2, \mathbb{R})$ to the group $SL(2, \mathbb{R})^\pm$ of matrices of determinant ± 1. Then we pass from $SL(2, \mathbb{R})^\pm$ to $GL(2, \mathbb{R})$ pasting on a character of a group $\mathbb{R} \cdot I_2 \subset GL(2, \mathbb{R})$ (cf. [1], [3]). For the general case, first we let B be the Borel subgroup of $GL(n, \mathbb{R})$ consisting of the upper triangular matrices with nonzero determinant. We let U be the unipotent radical of B and T a split Cartan subgroup of B. Let

$$\underline{\chi} = (\chi_1, \chi_2, \cdots, \chi_n)$$

be a character of T, that is, a collection of n characters of $\mathbb{R}^\times$. We extend $\underline{\chi}$ to a character of B trivial on U. Then the induced representation

$$I(\underline{\chi}) = \mathrm{Ind}_B^{GL(n,\mathbb{R})} \underline{\chi}$$

has a unique irreducible quotient

$$J(\underline{\chi}) = I(\underline{\chi})/I(\underline{\chi})_0, \tag{2.14}$$

where $I(\underline{\chi})_0$ is the only maximal proper closed invariant subspace of $I(\underline{\chi})$. It can be shown that for a character $\underline{\chi} = (\chi_1, \chi_2, \cdots, \chi_n)$ of T such that $\mathrm{Re}\,(s_i - s_j) \in \mathbb{Z}^+$ for all $i,\ j$ with $1 \le i < j \le n$, the necessary and sufficient condition on the unitarity of $J(\underline{\chi})$ is that there exist a partition $n = n_1 + n_2 + \cdots + n_r$ $(r \in \mathbb{Z}^+)$ and unitary characters η_i of $\mathbb{R}^\times$ for $i = 1, \cdots, r$ such that

$$J(\underline{\chi}) \cong \mathrm{Ind}_{\prod_{i=1}^r GL(n_i,\mathbb{R})}^{GL(n,\mathbb{R})} \otimes_{i=1}^r \eta_i(\det_{GL(n_i,\mathbb{R})}).$$

Vogan [14] proved that the unitary dual of $GL(n,\mathbb{R})$ consists of

(UD1) unitarily induced representation;

(UD2) complimentary series;

(UD3) the one-dimensional representations;

(UD4) a family $J(\underline{\chi})$ in (2.14) which are not induced from any parabolic subgroups of $GL(n,\mathbb{R})$.

Now we discuss certain principal series of $GL(n,\mathbb{R})$. Let $\pi = (n_1, \cdots, n_r)$ be a partition of n. We recall that $P(\pi)$ is the parabolic subgroup of $GL(n,\mathbb{R})$ generated by $GL(\pi)$ and the Borel subgroup B (cf. Definition 2.2). Obviously

$$P(\pi) = \{g = (g_{ij}) |\ g_{ij} \in M(n_i, n_j; \mathbb{R}),\ g_{ij} = 0\,(1 \le j < i \le r)\}\,. \tag{2.15}$$

If $n = r$, i.e., $n_1 = \cdots = n_r = 1$, then $P(\pi)$ is called a *minimal* parabolic subgroup of $GL(n,\mathbb{R})$. If $r = 2$, that is, if $n_1 + n_2 = n$, then $P(\pi)$ is said to be a *maximal* parabolic subgroup of $GL(n,\mathbb{R})$.

For multi-indices $\epsilon = (\epsilon_1, \cdots, \epsilon_r) \in (\mathbb{Z}/2\mathbb{Z})^r$ and $\nu = (\nu_1, \cdots, \nu_r) \in \mathbb{C}^r$, we define the character $\chi_{\epsilon,\nu}$ of $P(\pi)$ by

$$\chi_{\epsilon,\nu}(g) = \prod_{i=1}^r |\det g_{ii}|^{\nu_i}\,(\mathrm{sgn}(\det g_{ii}))^{\epsilon_i}\,, \tag{2.16}$$

where $g = (g_{ij}) \in P(\pi)$ (cf. (2.15)). It is known that for any $\epsilon = (\epsilon_1, \cdots, \epsilon_r) \in (\mathbb{Z}/2\mathbb{Z})^r$ and $\nu = (\nu_1, \cdots, \nu_r) \in (\sqrt{-1}\mathbb{R})^r$, the induced representation

$$\tau_{\epsilon,\nu}(\pi) = \mathrm{Ind}_{P(\pi)}^{GL(n,\mathbb{R})} \chi_{\epsilon,\nu} \tag{2.17}$$

is an irreducible unitary representation of $GL(n,\mathbb{R})$. If $P(\pi)$ is a minimal parabolic subgroup, $\tau_{\epsilon,\nu}(\pi)$ in (2.17) is called a unitary *principal series* of $GL(n,\mathbb{R})$. If $r < n$, that is, if one of n_j's is larger than 1, $\tau_{\epsilon,\nu}(\pi)$ in (2.17) is called a unitary *degenerate series* of $GL(n,\mathbb{R})$. If $\nu \notin (\sqrt{-1}\mathbb{R})^r$, the principal series $\tau_{\epsilon,\nu}(\pi)$ is not unitary in general.

For a positive integer k with $1 \le k \le [n/2]$, we let

$$P_k = \left\{ \begin{pmatrix} c & b \\ 0 & a \end{pmatrix} \in GL(n,\mathbb{R}) \,\middle|\, a \in GL(k,\mathbb{R}),\ c \in GL(n-k,\mathbb{R}),\ b \in M(n-k,k;\mathbb{R}) \right\}$$

be a maximal parabolic subgroup of $GL(n,\mathbb{R})$. For $\alpha \in \mathbb{C}$, we define the character $\chi_\alpha^\pm : P_k \longrightarrow \mathbb{C}$ by

$$\chi_\alpha^\pm \left(\begin{pmatrix} c & b \\ 0 & a \end{pmatrix} \right) = \begin{cases} (\det a)^\alpha & \text{if } \det a > 0, \\ \pm |\det a|^\alpha & \text{if } \det a < 0. \end{cases}$$

Howe and Lee [2] investigated the irreducibility and the unitarity of the following degenerate series $\tau_{k,\alpha}$ of $GL(n,\mathbb{R})$ defined by

$$\tau_{k,\alpha}^\pm := \operatorname{Ind}_{P_k}^{GL(n,\mathbb{R})} \chi_\alpha^\pm. \tag{2.18}$$

The representation space $\tau_{k,\alpha}^\pm$ is the space consisting of functions $f : GL(n,\mathbb{R}) \longrightarrow \mathbb{C}$ satisfying the condition

$$f(gp) = [\chi_\alpha^\pm(p)]^{-1} f(g), \quad g \in GL(n,\mathbb{R}),\ p \in P_k.$$

$GL(n,\mathbb{R})$ acts on the space $\tau_{k,\alpha}^\pm$ by left translation:

$$(g \cdot f)(h) = f(g^{-1}h), \quad g,h \in GL(n,\mathbb{R}),\ f \in \tau_{k,\alpha}^\pm.$$

Howe and Lee [2] proved the irreducibility of $\tau_{k,\alpha}^\pm$ as follows:

(a) If $\alpha \notin \mathbb{Z}$, then $\tau_{k,\alpha}^\pm$ are irreducible.

(b) If α is an even integer such that $-n/2 \le \alpha \le -2[(k+1)/2]$, then $\tau_{k,\alpha}^+$ is irreducible. If α is an even integer such that $\alpha \ge 2 - 2[(k+1)/2]$, then $\tau_{k,\alpha}^+$ is reducible.

(c) If α is an even integer such that $-n/2 \le \alpha \le -1 - 2[k/2]$, then $\tau_{k,\alpha}^-$ is irreducible. If α is an even integer such that $\alpha \ge 2 - 2[k/2]$, then $\tau_{k,\alpha}^-$ is reducible.

(d) If α is an odd integer such that $-n/2 \le \alpha \le -1 - 2[k/2]$, then $\tau_{k,\alpha}^+$ is irreducible. If α is an odd integer such that $\alpha \ge 1 - 2[k/2]$, then $\tau_{k,\alpha}^+$ is reducible.

(e) If α is an odd integer such that $-n/2 \le \alpha \le -1 - 2[(k+1)/2]$, then $\tau_{k,\alpha}^-$ is irreducible. If α is an odd integer such that $\alpha \ge 3 - 2[(k+1)/2]$, then $\tau_{k,\alpha}^-$ is reducible.

For the unitarity of $\tau_{k,\alpha}^{\pm}$, we refer to [2], pp. 306-308. We realize the degenerate series $\tau_{k,\alpha}^{\pm}$ in another way. We consider the following action σ of $GL(n,\mathbb{R})$ on $\mathbb{R}^{(n,k)}$ defined by

$$\sigma(g)(x) := {}^tg^{-1}x, \quad g \in GL(n,\mathbb{R}),\ x \in \mathbb{R}^{(n,k)}. \tag{2.19}$$

We let $M(n,k;\mathbb{R})^0$ be the set of all $n \times k$ real matrices of rank k. For $\alpha \in \mathbb{C}$, we let $\mathcal{L}_{k,\alpha}^{\pm}$ be the space consisting of functions $f : M(n,k;\mathbb{R})^0 \longrightarrow \mathbb{C}$ satisfying the following condition

$$f(xa) = \begin{cases} (\det a)^{\alpha} f(x) & \text{if } \det a > 0, \\ \pm|\det a|^{\alpha} f(x) & \text{if } \det a < 0 \end{cases}$$

for $x \in M(n,k;\mathbb{R})^0$ and $a \in GL(k,\mathbb{R})$. Then the action σ in (2.19) induces the representation $\sigma_{k,\alpha}^{\pm}$ of $GL(n,\mathbb{R})$ on $\mathcal{L}_{k,\alpha}^{\pm}$ defined by

$$\left(\sigma_{k,\alpha}^{\pm}(g)f\right)(x) = f(\sigma(g^{-1})x) = f({}^tgx), \quad g \in GL(n,\mathbb{R}),\ x \in M(n,k;\mathbb{R})^0.$$

Then we can show that $\tau_{k,\alpha}^{\pm}$ is isomorphic to $\sigma_{k,\alpha}^{\pm}$.

3. Unitary representations of $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$

In this section, we find the unitary dual of $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$ using the Mackey's method and deal with certain unitary representations of $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$.

For brevity, we put

$$A := \mathbb{R}^{(m,n)}, \quad GL_n := GL(n,\mathbb{R}) \text{ and } GL_{n,m} := GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}.$$

The multiplication on $GL_{n,m}$ is given by

$$(g,a)\cdot(h,b) = (gh,\ a\,{}^th^{-1} + b), \quad (g,a),\,(h,b) \in GL_{n,m}. \tag{3.1}$$

We may identify A with the subgroup $\{(I_n,a)|\ a \in A\}$ of $GL_{n,m}$. It is clear that A is a commutative normal subgroup of $GL_{n,m}$ and the center of $GL_{n,m}$ consists only of the identity element $(I_n,0)$. Moreover we have the split exact sequence

$$0 \longrightarrow A \longrightarrow GL_{n,m} \longrightarrow GL_n \longrightarrow 1.$$

We see that the unitary dual $\hat{A}$ of A is isomorphic to A. Indeed, the unitary character ρ_λ of A corresponding to $\lambda \in A$ is defined by

$$\rho_\lambda(a) := e^{2\pi i\sigma({}^t\lambda a)}, \quad a \in A. \tag{3.2}$$

For the time being, we write $g_a = (g,a) \in GL_{n,m}$ for $g \in GL_n$ and $a \in A$, and we identify an element g of GL_n with an element $(g,0)$ in $GL_{n,m}$. The group

$GL_{n,m}$ acts on A by conjugation because A is a normal subgroup of $GL_{n,m}$. This induces the action of $GL_{n,m}$ on $\hat{A}$ as follows:

$$GL_{n,m} \times \hat{A} \longrightarrow \hat{A}, \quad (g_a, \rho) \mapsto \rho^{g_a}, \tag{3.3}$$

where $g_a \in GL_{n,m},\ \rho \in \hat{A}$ and the unitary character ρ^{g_a} of A is defined by

$$\rho_{g_a}(b) := \rho(g_a^{-1} b g_a), \quad b \in A.$$

Since

$$g_a^{-1} b g_a = (g,a)^{-1} b (g,a) = (I_n, b\,{}^t g^{-1}) = g^{-1} b g$$

for any $g \in GL_n$ and $a, b \in A$, we obtain

$$\rho^{g_a}(b) = \rho^g(b) = \rho(b\,{}^t g^{-1}). \tag{3.4}$$

In particular, $\rho^a = \rho$ for every element $a \in A$.

Lemma 3.1. *The action of an element $g_a = (g,a)$ on an element ρ_λ of $\hat{A}$ (cf. (3.2)) is given by*

$$\rho_\lambda^{g_a} = \rho_\lambda^g = \rho_{\lambda g^{-1}}, \quad \lambda \in A. \tag{3.5}$$

Proof. If $b \in A$, then

$$\begin{aligned} \rho_\lambda^{g_a}(b) = \rho_\lambda^g(b) &= \rho_\lambda(b\,{}^t g^{-1}) \\ &= e^{2\pi i \sigma({}^t\lambda b\,{}^t g^{-1})} \\ &= e^{2\pi i \sigma({}^t(\lambda g^{-1}) b)} \\ &= \rho_{\lambda g^{-1}}(b) \quad \text{(according to (3.2))}. \end{aligned}$$

If $\rho \in \hat{A}$, we denote by Ω_ρ the $GL_{n,m}$-orbit of ρ and let

$$GL_{n,m}(\rho) = \{ g_a \in GL_{n,m} \mid \rho^{g_a} = \rho \}$$

be the stabilizer or isotrophy subgroup of $GL_{n,m}$ at ρ. Then the mapping defined by

$$GL_{n,m}/GL_{n,m}(\rho) \longrightarrow \Omega_\rho, \quad g_a \cdot GL_{n,m}(\rho) \longrightarrow \rho^{g_a}$$

is a homeomorphism, in other words, A is regularly embedded. Obviously A is a subgroup of $GL_{n,m}(\rho)$. We define the subset $\widehat{GL_{n,m}(\rho)}_*$ of the unitary dual $\widehat{GL_{n,m}(\rho)}$ of $GL_{n,m}(\rho)$ by

$$\widehat{GL_{n,m}(\rho)}_* = \left\{ \tau \in \widehat{GL_{n,m}(\rho)} \mid \tau|_A \text{ is a multiple of } \rho \right\}.$$

According to G. Mackey [4], we obtain the following.

Theorem 3.2. *For any $\tau \in \widehat{GL_{n,m}(\rho)}_*$, the induced representation*

$$\text{Ind}\,_{GL_{n,m}(\rho)}^{GL_{n,m}}\tau$$

is an irreducible unitary representation of $GL_{n,m}$. And the unitary dual $\widehat{GL_{n,m}(\rho)}$ of $GL_{n,m}$ is given by

$$\widehat{GL_{n,m}} = \bigcup_{[\rho]\in GL_{n,m}\backslash \hat{A}} \left\{ \text{Ind}\,_{GL_{n,m}(\rho)}^{GL_{n,m}}\tau \,\middle|\, \tau \in \widehat{GL_{n,m}(\rho)}_* \right\}.$$

We deal with the special cases $n = 3,\ 4$ explicitly. The other cases $n \geq 4$ may be dealt with similarly.

Case I. $n = 2$.

(I-1) $m = 1$.

In this case, $A = \mathbb{R}^{(1,2)} \cong \mathbb{R}^2$. We identify the unitary dual $\hat{A}$ of A with $\mathbb{R}^2$. From (3.5), we see that $GL_{2,1}$-orbits in $\hat{A}$ consists of two orbits Ω_0, Ω_1 given by

$$\Omega_{[21];0} = \{(0,0)\}, \quad \Omega_{[21];1} = \mathbb{R}^2 - \{(0,0)\}.$$

We observe that $\Omega_{[21];0}$ is the $GL_{2,1}$-orbit of $(0,0)$ and $\Omega_{[22];1}$ is a $GL_{2,1}$-orbit of any element $(\lambda,\mu) \neq (0,0)$.

Now we choose the element $\delta = \rho_{(1,0)}$ of $\hat{A}$. That is, $\delta(x,y) = e^{2\pi i x}$ for $x, y \in \mathbb{R}^2$. It is easily checked that the stabilizer of $\rho_{(0,0)}$ is $GL_{2,1}$ and the stabilizer $GL_{2,1}(\delta)$ of δ is given by

$$GL_{2,1}(\delta) = \left\{ \left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix},\ \alpha \right) \in GL_{2,1} \,\middle|\, c \in \mathbb{R},\ d \in \mathbb{R}^{\times},\ \alpha \in \mathbb{R}^{(1,2)} \right\}.$$

According to Theorem 3.2, we obtain

Theorem 3.3. *Let $n = 2$ and $m = 1$. Then the irreducible unitary representations of $GL_{2,1}$ are the following:*

(a) *The irreducible unitary representation π, where the restriction of π to A is trivial and the restriction of π to GL_2 is an irreducible unitary representation of GL_2.*

(b) *The representation*

$$\pi_\lambda = \text{Ind}\,_{GL_{2,1}(\delta)}^{GL_{2,1}}\tau_\lambda \quad (\lambda \in \mathbb{R})$$

induced from the irreducible unitary representation τ_λ of $GL_{2,1}(\delta)$ such that $\tau_\lambda|_A$ is a multiple of δ.

(I-2) $m=2$.

In this case, $\hat{A}\cong\mathbb{R}^{(2,2)}$. From now on, we identify $\hat{A}$ with $\mathbb{R}^{(2,2)}$.

Lemma 3.4. *Let $n=2$ and $m=2$. Then the $GL_{2,2}$-orbits in $\hat{A}$ consist of the following orbits*

$$\begin{aligned}\Omega_{[22];0} &= \left\{\begin{pmatrix}0&0\\0&0\end{pmatrix}\right\},\\ \Omega_{[22];1} &= \left\{\begin{pmatrix}A\\0\end{pmatrix}\in\mathbb{R}^{(2,2)}\,\Big|\, A\in\mathbb{R}^{(1,2)},\ A\neq 0\right\},\\ \Omega_{[22];2} &= \left\{\begin{pmatrix}0\\A\end{pmatrix}\in\mathbb{R}^{(2,2)}\,\Big|\, A\in\mathbb{R}^{(1,2)},\ A\neq 0\right\},\\ \Omega_{[22];3}(\delta) &= \left\{\begin{pmatrix}A\\\delta A\end{pmatrix}\in\mathbb{R}^{(2,2)}\,\Big|\, A\in\mathbb{R}^{(1,2)},\ A\neq 0\right\}\quad(\,\delta\in\mathbb{R}^{\times}\,)\end{aligned}$$

and

$$\Omega_{[22];4}=GL(2,\mathbb{R}).$$

$\Omega_{[22];0}$ is the $GL_{2,2}$-orbit of $\mathbf{0}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$, $\Omega_{[22];1}$ is the $GL_{2,2}$-orbit of $\begin{pmatrix}\alpha\\0\end{pmatrix}$ with $0\neq\alpha\in\mathbb{R}^{(1,2)}$, $\Omega_{[22];2}$ is the $GL_{2,2}$-orbit of $\begin{pmatrix}0\\\beta\end{pmatrix}$ with $0\neq\beta\in\mathbb{R}^{(1,2)}$, $\Omega_{[22];3}(\delta)$ is the $GL_{2,2}$-orbit of $\begin{pmatrix}\alpha\\\delta\alpha\end{pmatrix}$ with $0\neq\alpha\in\mathbb{R}^{(1,2)}$ and $\Omega_{[22];4}$ is the $GL_{2,2}$-orbit of any invertible matrix $M\in GL(2,\mathbb{R})$.

Proof. Without difficulty we may prove the above lemma. We note that $\Omega_{[22];3}(\delta_1)=\Omega_{[22];3}(\delta_2)$ if and only if $\delta_1=\delta_2$. So we leave the detail to the reader. □

We put

$$e=\begin{pmatrix}1&0\\0&0\end{pmatrix}\quad\text{and}\quad f=\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

Obviously $e\in\Omega_{[22];1}$ and $f\in\Omega_{[22];2}$.

Then we may prove the following lemma.

Lemma 3.5.

(a) *The stabilizer of $\mathbf{0}$ is $GL_{2,2}$.*

(b) *The stabilizer $GL_{2,2}(e)$ of e is given by*

$$GL_{2,2}(e)=\left\{\left(\begin{pmatrix}1&0\\c&d\end{pmatrix},\alpha\right)\,\Big|\,c\in\mathbb{R},\ d\in\mathbb{R}^{\times},\ \alpha\in\mathbb{R}^{(2,2)}\right\}.$$

For each $x\in\Omega_{[22];1}$, the stabilizer $GL_{2,2}(x)$ of x is conjugate to $GL_{2,2}(e)$. Precisely if $x=eg_0$ with $g_0\in GL(2,\mathbb{R})$, then $GL_{2,2}(x)=(g_0,0)^{-1}GL_{2,2}(e)(g_0,0)$.

(c) *The stabilizer* $GL_{2,2}(f)$ *of* f *is given by*

$$GL_{2,2}(f) = \left\{ \left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \alpha \right) \Big| \ c \in \mathbb{R},\ d \in \mathbb{R}^{\times},\ \alpha \in \mathbb{R}^{(2,2)} \right\}.$$

For each $y \in \Omega_{[22];2}$, *the stabilizer* $GL_{2,2}(y)$ *of* y *is conjugate to* $GL_{2,2}(f)$.

(d) *The stabilizer* $GL_{2,2}(\delta)$ *of* $\begin{pmatrix} 1 & 0 \\ \delta & 0 \end{pmatrix}$ $(\delta \in \mathbb{R}^{\times})$ *is given by*

$$GL_{2,2}(\delta) = \left\{ \left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \alpha \right) \Big| \ c \in \mathbb{R},\ d \in \mathbb{R}^{\times},\ \alpha \in \mathbb{R}^{(2,2)} \right\}.$$

For each $z \in \Omega_{[22];3}(\delta)$, *the stabilizer* $GL_{2,2}(z)$ *of* z *is conjugate to* $GL_{2,2}(\delta)$.

(e) *The stabilizer* $GL_{2,2}(M)$ *of* $M \in \Omega_{[22];4}$ *is given by*

$$GL_{2,2}(M) = \left\{ (I_2, \alpha) \,|\, \alpha \in \mathbb{R}^{(2,2)} \right\} \cong \mathbb{R}^{(2,2)}.$$

Therefore A *is regularly embedded.*

For $\lambda \in \mathbb{R}$, *we let* χ_λ *be the unitary character of* $\mathbb{R}$ *defined by* $\chi_\lambda(a) := e^{2\pi i \lambda a}$ $(a \in \mathbb{R})$ *and for* $M \in \mathbb{R}^{(2,2)}$, *we let* τ_M *be the unitary character of* $A = \mathbb{R}^{(2,2)}$ *defined by*

$$\tau_M(X) := e^{2\pi i ({}^tMX)}, \quad X \in A. \tag{3.6}$$

According to Theorem 3.2, we obtain the following

Theorem 3.6. *Let* $n = 2$ *and* $m = 2$. *Then the irreducible unitary representations of* $GL_{2,2}$ *are the following*:

(a) *The irreducible unitary representations* π, *where the restriction of* π *to* A *is trivial and the restriction of* π *to* $GL(2,\mathbb{R})$ *is an irreducible unitary representation of* $GL(2,\mathbb{R})$.

(b) *The representations* $\pi_{\lambda;e} := \text{Ind}\,_{GL_{2,2}(\text{e})}^{GL_{2,2}} \tau_{\lambda,e}$ $(\lambda \in \mathbb{R})$ *induced from the irreducible unitary representation* $\tau_{\lambda,e}$ *of* $GL_{2,2}(e)$ *whose restriction to* A *is a multiple of* τ_e *(cf.* (3.6)*). In fact,* $\tau_{\lambda,e}$ *of the form*

$$\tau_{\lambda,e}\left(\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \right) \right) = e^{2\pi\alpha_1} \left(\text{Ind}\,_{\mathbb{R}}^{\mathbb{R}\rtimes\mathbb{R}^{\times}} \chi_\lambda \right) \left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \right),$$

where $c,\ \alpha_1,\ \cdots,\ \alpha_4 \in \mathbb{R},\ d \in \mathbb{R}^{\times}$.

(c) *The representations* $\pi_{\lambda;f} := \text{Ind}\,_{GL_{2,2}(\text{f})}^{GL_{2,2}} \theta_{\lambda,f}$ $(\lambda \in \mathbb{R})$ *induced from the irreducible unitary representation* $\theta_{\lambda,f}$ *of* $GL_{2,2}(f)$ *whose restriction to* A *is a multiple of* τ_f *(cf.* (3.6)*). Indeed,* $\theta_{\lambda,f}$ *is of the form*

$$\theta_{\lambda,f}\left(\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \right) \right) = e^{2\pi i\alpha_3} \left(\text{Ind}\,_{\mathbb{R}}^{\mathbb{R}\rtimes\mathbb{R}^{\times}} \chi_\lambda \right) \left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \right),$$

where $c,\ \alpha_1,\ \cdots,\ \alpha_4 \in \mathbb{R},\ d \in \mathbb{R}^{\times}$.

(d) *The representations* $\pi_{\lambda;\delta} := \text{Ind}\,_{GL_{2,2}(\delta)}^{GL_{2,2}} \theta_{\lambda,\delta}\,(\lambda \in \mathbb{R},\ \delta \in \mathbb{R}^\times,\ \ r \in \mathbb{R})$ *induced from the irreducible unitary representation* $\sigma_{\lambda,\delta}$ *of* $GL_{2,2}(\delta)$. *Indeed,* $\sigma_{\lambda,\delta}$ *is of the form*

$$\sigma_{\lambda,\delta}\left(\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}\right)\right) = e^{2\pi i(\alpha_1+\alpha_3\delta)} \left(\text{Ind}\,_{\mathbb{R}}^{\mathbb{R}\rtimes\mathbb{R}^\times} \chi_\lambda\right)\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}\right),$$

where $c,\ \alpha_1,\ \cdots,\ \alpha_4 \in \mathbb{R},\ d \in \mathbb{R}^\times$.

(e) *The representations* $\pi_M := \text{Ind}\,_A^{GL_{2,2}} \tau_M\ (M \in GL(2,\mathbb{R}))$ *of* $GL_{2,2}$ *induced from the unitary character* τ_M *of* A *defined by* $\tau_M(X) = e^{2\pi i\sigma(MX)},\ X \in A$.

Proof. We leave the detail of the proof to the reader. □

(I-3) $m > 2$.

This case is more complicated than the above cases. Here we consider only the case $m = 3$. The other case $m \geq 4$ may be dealt similarly.

Lemma 3.7. *Let* $n = 2$ *and* $m = 3$. *That is,* $A = \mathbb{R}^{(3,2)}$. *Then the* $GL_{2,3}$*-orbits in* $\hat{A}$ *are given by*

$$\begin{aligned}
\Omega_{[23];0} &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \\
\Omega_{[23];1} &= \left\{ \begin{pmatrix} A \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^{(3,2)} \,\middle|\, A \in \mathbb{R}^{(1,2)},\ A \neq 0 \right\}, \\
\Omega_{[22];2} &= \left\{ \begin{pmatrix} 0 \\ A \\ 0 \end{pmatrix} \in \mathbb{R}^{(3,2)} \,\middle|\, A \in \mathbb{R}^{(1,2)},\ A \neq 0 \right\}, \\
\Omega_{[23];3} &= \left\{ \begin{pmatrix} 0 \\ 0 \\ A \end{pmatrix} \in \mathbb{R}^{(3,2)} \,\middle|\, A \in \mathbb{R}^{(1,2)},\ A \neq 0 \right\},
\end{aligned}$$

$$\begin{aligned}
\Omega_{[23]}(1;\delta) &= \left\{ \begin{pmatrix} 0 \\ A \\ \delta A \end{pmatrix} \in \mathbb{R}^{(3,2)} \,\middle|\, A \in \mathbb{R}^{(1,2)},\ A \neq 0 \right\} \quad (\delta \in \mathbb{R}^\times), \\
\Omega_{[23]}(2;\delta) &= \left\{ \begin{pmatrix} A \\ 0 \\ \delta A \end{pmatrix} \in \mathbb{R}^{(3,2)} \,\middle|\, A \in \mathbb{R}^{(1,2)},\ A \neq 0 \right\} \quad (\delta \in \mathbb{R}^\times), \\
\Omega_{[23]}(3;\delta) &= \left\{ \begin{pmatrix} A \\ \delta A \\ 0 \end{pmatrix} \in \mathbb{R}^{(3,2)} \,\middle|\, A \in \mathbb{R}^{(1,2)},\ A \neq 0 \right\} \quad (\delta \in \mathbb{R}^\times),
\end{aligned}$$

$$\Omega_{[23]}(\lambda,\mu) = \left\{ \begin{pmatrix} A \\ \lambda A \\ \mu A \end{pmatrix} \in \mathbb{R}^{(3,2)} \,\middle|\, A \in \mathbb{R}^{(1,2)},\ A \neq 0 \right\} \quad (\lambda,\ \mu \in \mathbb{R}^\times)$$

and

$$\Omega_{12}(\lambda,\mu) = \left\{ \begin{pmatrix} A \\ B \\ \lambda A + \mu B \end{pmatrix} \in \mathbb{R}^{(3,2)} \,\middle|\, \begin{pmatrix} A \\ B \end{pmatrix} \in GL_2 \right\} \quad (\lambda, \mu \in \mathbb{R}),$$

$$\Omega_{13}(\lambda,\mu) = \left\{ \begin{pmatrix} A \\ \lambda A + \mu B \\ B \end{pmatrix} \in \mathbb{R}^{(3,2)} \,\middle|\, \begin{pmatrix} A \\ B \end{pmatrix} \in GL_2 \right\} \quad (\lambda, \mu \in \mathbb{R}),$$

$$\Omega_{23}(\lambda,\mu) = \left\{ \begin{pmatrix} \lambda A + \mu B \\ A \\ B \end{pmatrix} \in \mathbb{R}^{(3,2)} \,\middle|\, \begin{pmatrix} A \\ B \end{pmatrix} \in GL_2 \right\} \quad (\lambda, \mu \in \mathbb{R}).$$

Proof. It is easy to prove the above lemma. We leave the proof to the reader. □

We put

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and for each $\delta,\ \lambda,\ \mu \in \mathbb{R}^\times$

$$f_{1,\delta} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ \delta & 0 \end{pmatrix}, \quad f_{2,\delta} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \delta & 0 \end{pmatrix}, \quad f_{3,\delta} = \begin{pmatrix} 1 & 0 \\ \delta & 0 \\ 0 & 0 \end{pmatrix},$$

$$f_{\lambda,\mu} = \begin{pmatrix} 1 & 0 \\ \lambda & 0 \\ \mu & 0 \end{pmatrix}.$$

We also set for each $(\lambda,\mu) \in \mathbb{R}^2$,

$$h_{12}(\lambda,\mu) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \lambda & \mu \end{pmatrix}, \quad h_{13}(\lambda,\mu) = \begin{pmatrix} 1 & 0 \\ \lambda & \mu \\ 0 & 1 \end{pmatrix}$$

and

$$h_{23}(\lambda,\mu) = \begin{pmatrix} \lambda & \mu \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We note that $\mathbf{0} \in \Omega_{[23];0}$, $e_i \in \Omega_{[23];i}\,(i = 1,2,3)$, $f_{j,\delta} \in \Omega_{[23]}(j;\delta)\,(j = 1,2,3)$, $f_{\lambda,\mu} \in \Omega_{[23]}(\lambda,\mu)$, $h_{12}(\lambda,\mu) \in \Omega_{12}(\lambda,\mu)$, $h_{13}(\lambda,\mu) \in \Omega_{13}(\lambda,\mu)$, $h_{23}(\lambda,\mu) \in \Omega_{23}(\lambda,\mu)$.

Then we may prove the following lemma without difficulty.

Lemma 3.8.

(a) *The stabilizer of* $\mathbf{0}$ *is* $GL_{2,3}$.

(b) *Let* $GL_{2,3}(i)$ *be the stabilizer of* e_i $(i=1,2,3)$. *Then*

$$GL_{2,3}(i)=\left\{\left(\begin{pmatrix}1 & 0\\ c & d\end{pmatrix},\ \alpha\right)\mid c\in\mathbb{R},\ d\in\mathbb{R}^{\times},\ \alpha\in\mathbb{R}^{(3,2)}\right\},\quad i=1,2,3.$$

(c) *For* $\delta\in\mathbb{R}^{\times}$, *we let* $GL_{2,3}(i;\delta)$ *be the stabilizer of* $f_{i,\delta}$ $(i=1,2,3)$. *Then*

$$GL_{2,3}(i,\delta)=\left\{\left(\begin{pmatrix}1 & 0\\ c & d\end{pmatrix},\ \alpha\right)\mid c\in\mathbb{R},\ d\in\mathbb{R}^{\times},\ \alpha\in\mathbb{R}^{(3,2)}\right\},\quad i=1,2,3.$$

(d) *For any* $\lambda,\ \mu\in\mathbb{R}^{\times}$, *we let* $GL_{2,3}(\lambda,\mu)$ *be the stabilizer of* $f_{\lambda,\mu}$. *Then*

$$GL_{2,3}(\lambda,\mu)=\left\{\left(\begin{pmatrix}1 & 0\\ c & d\end{pmatrix},\ \alpha\right)\mid c\in\mathbb{R},\ d\in\mathbb{R}^{\times},\ \alpha\in\mathbb{R}^{(3,2)}\right\}\quad(\lambda,\mu\in\mathbb{R}^{\times}).$$

(e) *For any* $\lambda,\ \mu\in\mathbb{R}^{\times}$, *we let* $GL_{2,3}(12;\lambda,\mu)$, $GL_{2,3}(13;\lambda,\mu)$, $GL_{2,3}(23;\lambda,\mu)$ *be the stabilizers of* $h_{12}(\lambda,\mu)$, $h_{13}(\lambda,\mu)$, $h_{23}(\lambda,\mu)$ *respectively. Then*

$$GL_{2,3}(12;\lambda,\mu)=GL_{2,3}(13;\lambda,\mu)=GL_{2,3}(23;\lambda,\mu)=\left\{(I_2,\alpha)\mid\alpha\in\mathbb{R}^{(3,2)}\right\}.$$

Therefore we see easily that A *is regularly embedded.*

According to Theorem 3.2, we obtain the following.

Theorem 3.9. *Let* $n=2$ *and* $m=3$. *Then the irreducible unitary representations of* $GL_{2,3}$ *are the following*:

(a) *The irreducible unitary representations* π, *where the restriction of* π *to* A *is trivial and the restriction of* π *to* $GL(2,\mathbb{R})$ *is an irreducible unitary representation of* $GL(2,\mathbb{R})$.

(b) *The representations* $\pi_{1,\lambda}:=\mathrm{Ind}\,_{GL_{2,3}(1)}^{GL_{2,3}}\tau_{1,\lambda}$ $(\lambda\in\mathbb{R})$ *induced from the unitary representation* $\tau_{1,\lambda}$ *of* $GL_{2,3}(1)$ *defined by*

$$\tau_{1,\lambda}\left(\left(\begin{pmatrix}1 & 0\\ c & d\end{pmatrix},\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\\ \alpha_5 & \alpha_6\end{pmatrix}\right)\right)=e^{2\pi i\alpha_1}\left(\mathrm{Ind}_{\mathbb{R}}^{\mathbb{R}\rtimes\mathbb{R}^{\times}}\chi_\lambda\right)\left(\begin{pmatrix}1 & 0\\ c & d\end{pmatrix}\right),$$

where $c,\alpha_1,\cdots,\alpha_6\in\mathbb{R}$ *and* $d\in\mathbb{R}^{\times}$.

(c) *The representations* $\pi_{2,\lambda}:=\mathrm{Ind}\,_{GL_{2,3}(2)}^{GL_{2,3}}\tau_{2,\lambda}$ $(\lambda\in\mathbb{R})$ *induced from the unitary representation* $\tau_{2,\lambda}$ *of* $GL_{2,3}(2)$ *defined by*

$$\tau_{2,\lambda}\left(\left(\begin{pmatrix}1 & 0\\ c & d\end{pmatrix},\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\\ \alpha_5 & \alpha_6\end{pmatrix}\right)\right)=e^{2\pi i\alpha_3}\cdot\left(\mathrm{Ind}_{\mathbb{R}}^{\mathbb{R}\rtimes\mathbb{R}^{\times}}\chi_\lambda\right)\left(\begin{pmatrix}1 & 0\\ c & d\end{pmatrix}\right),$$

where $c,\alpha_1,\cdots,\alpha_6\in\mathbb{R}$ *and* $d\in\mathbb{R}^{\times}$.

(d) *The representations* $\pi_{3,\lambda} := \mathrm{Ind}\,_{GL_{2,3}(3)}^{GL_{2,3}}\tau_{3,\lambda}\ (\lambda \in \mathbb{R})$ *induced from the unitary representation* $\tau_{3,\lambda}$ *of* $GL_{2,3}(3)$ *defined by*

$$\tau_{3,\lambda}\left(\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix}\right)\right) = e^{2\pi i \alpha_5} \cdot \left(\mathrm{Ind}_{\mathbb{R}}^{\mathbb{R}\rtimes\mathbb{R}^\times}\chi_\lambda\right)\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}\right),$$

where $c, \alpha_1, \cdots, \alpha_6 \in \mathbb{R}$ *and* $d \in \mathbb{R}^\times$.

(e) *The representations* $\pi_{(1,\delta),\lambda} := \mathrm{Ind}\,_{GL_{2,3}(1;\delta)}^{GL_{2,3}}\tau_{(1,\delta),\lambda}\ (\delta \in \mathbb{R}^\times,\ \lambda \in \mathbb{R})$ *induced from the unitary representation* $\tau_{(1,\delta),\lambda}$ *of* $GL_{2,3}(1;\delta)$ *defined by*

$$\tau_{(1,\delta),\lambda}\left(\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix}\right)\right) = e^{2\pi i (\alpha_3 + \delta\alpha_5)} \cdot \left(\mathrm{Ind}_{\mathbb{R}}^{\mathbb{R}\rtimes\mathbb{R}^\times}\chi_\lambda\right)\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}\right),$$

where $c, \alpha_1, \cdots, \alpha_6 \in \mathbb{R}$ *and* $d \in \mathbb{R}^\times$.

(f) *The representations* $\pi_{(2,\delta),\lambda} := \mathrm{Ind}\,_{GL_{2,3}(2;\delta)}^{GL_{2,3}}\tau_{(2,\delta),\lambda}\ (\delta \in \mathbb{R}^\times,\ \lambda \in \mathbb{R})$ *induced from the unitary representation* $\tau_{(2,\delta),\lambda}$ *of* $GL_{2,3}(2;\delta)$ *defined by*

$$\tau_{(2,\delta),\lambda}\left(\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix}\right)\right) = e^{2\pi i (\alpha_1 + \delta\alpha_5)} \cdot \left(\mathrm{Ind}_{\mathbb{R}}^{\mathbb{R}\rtimes\mathbb{R}^\times}\chi_\lambda\right)\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}\right),$$

where $c, \alpha_1, \cdots, \alpha_6 \in \mathbb{R}$ *and* $d \in \mathbb{R}^\times$.

(g) *The representations* $\pi_{(3,\delta),\lambda} := \mathrm{Ind}\,_{GL_{2,3}(3;\delta)}^{GL_{2,3}}\tau_{(3,\delta),\lambda}\ (\delta \in \mathbb{R}^\times,\ \lambda \in \mathbb{R})$ *induced from the unitary representation* $\tau_{(3,\delta),\lambda}$ *of* $GL_{2,3}(3;\delta)$ *defined by*

$$\tau_{(3,\delta),\lambda}\left(\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix}\right)\right) = e^{2\pi i (\alpha_1 + \delta\alpha_3)} \cdot \left(\mathrm{Ind}_{\mathbb{R}}^{\mathbb{R}\rtimes\mathbb{R}^\times}\chi_\lambda\right)\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}\right),$$

where $c, \alpha_1, \cdots, \alpha_6 \in \mathbb{R}$ *and* $d \in \mathbb{R}^\times$.

(h) *The representations* $\pi_{(r;\lambda,\mu)} := \mathrm{Ind}\,_{GL_{2,3}(\lambda,\mu)}^{GL_{2,3}}\tau_{(\lambda,\mu),r}\ (r \in \mathbb{R},\ \lambda, \mu \in \mathbb{R}^\times)$ *induced from the unitary representation* $\tau_{(\lambda,\mu),r}$ *of* $GL_{2,3}(\lambda,\mu)$ *defined by*

$$\tau_{(\lambda,\mu),r}\left(\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix}\right)\right) = e^{2\pi i (\alpha_1 + \lambda\alpha_3 + \mu\alpha_5)} \cdot \left(\mathrm{Ind}_{\mathbb{R}}^{\mathbb{R}\rtimes\mathbb{R}^\times}\chi_\lambda\right)\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}\right),$$

where $c, \alpha_1, \cdots, \alpha_6 \in \mathbb{R}$ *and* $d \in \mathbb{R}^\times$.

(i) *The representations* $\pi_{(12;\lambda,\mu)} := \text{Ind}\,{}_{GL_{2,3}(12;\lambda,\mu)}^{GL_{2,3}}\tau_{(12;\lambda,\mu)}\ (\lambda,\mu \in \mathbb{R})$ *induced from the unitary representation* $\tau_{(12;\lambda,\mu)}$ *of* $GL_{2,3}(12;\lambda,\mu)$ *defined by*

$$\tau_{(12;\lambda,\mu)}\left(\left(I_2,\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\\ \alpha_5 & \alpha_6\end{pmatrix}\right)\right) = e^{2\pi i(\alpha_1+\lambda\alpha_5+(\alpha_4+\mu\alpha_6))},\quad \alpha_1,\cdots,\alpha_6\in\mathbb{R}.$$

(j) *The representations* $\pi_{(13;\lambda,\mu)} := \text{Ind}\,{}_{GL_{2,3}(13;\lambda,\mu)}^{GL_{2,3}}\tau_{(13;\lambda,\mu)}\ (\lambda,\mu \in \mathbb{R})$ *induced from the unitary representation* $\tau_{(13;\lambda,\mu)}$ *of* $GL_{2,3}(13;\lambda,\mu)$ *defined by*

$$\tau_{(13;\lambda,\mu)}\left(\left(I_2,\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\\ \alpha_5 & \alpha_6\end{pmatrix}\right)\right) = e^{2\pi i(\alpha_1+\lambda\alpha_3+(\alpha_6+\mu\alpha_4))},\quad \alpha_1,\cdots,\alpha_6\in\mathbb{R}.$$

(k) *The representations* $\pi_{(23;\lambda,\mu)} := \text{Ind}\,{}_{GL_{2,3}(23;\lambda,\mu)}^{GL_{2,3}}\tau_{(23;\lambda,\mu)}\ (\lambda,\mu \in \mathbb{R})$ *induced from the unitary representation* $\tau_{(23;\lambda,\mu)}$ *of* $GL_{2,3}(23;\lambda,\mu)$ *defined by*

$$\tau_{(23;\lambda,\mu)}\left(\left(I_2,\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\\ \alpha_5 & \alpha_6\end{pmatrix}\right)\right) = e^{2\pi i(\alpha_3+\lambda\alpha_1+(\alpha_6+\mu\alpha_2))},\quad \alpha_1,\cdots,\alpha_6\in\mathbb{R}.$$

Proof. We leave the detail of the proof to the reader. □

Case II. $n = 3$.

(II-1) $m = 1$.

In this case, $A \cong \mathbb{R}^{(1,3)} = \mathbb{R}^3$. We identify the unitary dual $\hat{A}$ of A with $\mathbb{R}^3$. According to (3.5), we see that $GL_{3,1}$-orbits in $\hat{A}$ consists of two orbits $\Omega_{[31];0}$, $\Omega_{[31];1}$ given by

$$\Omega_0 = \{(0,0,0)\},\quad \Omega_1 = \mathbb{R}^3 - \{(0,0,0)\}.$$

We note that Ω_0 is the $GL_{3,1}$-orbit of $(0,0,0)$ and Ω_1 is a $GL_{3,1}$-orbit of any element different from $(0,0,0)$. We put $e = (1,0,0)$. Then the stabilizer $GL_{3,1}(e)$ of e is given by

$$GL_{3,1}(e) = \left\{\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix},\alpha\right)\in GL_{3,1}\,\middle|\, a\in\mathbb{R}^{(2,1)},\ g\in GL_2,\ \alpha\in\mathbb{R}^{(1,3)}\right\}.$$

According to Theorem 3.2, we obtain the following.

Theorem 3.10. *Let* $n = 3$ *and* $m = 1$. *Then the irreducible unitary representations of* $GL_{3,1}$ *are the following*:

(a) *The irreducible unitary representation* π, *where the restriction of* π *to* A *is trivial and the restriction of* π *to* GL_3 *is an irreducible unitary representation of* GL_3.

(b) *The representation* $\pi_\nu := \mathrm{Ind}_{GL_{3,1}(e)}^{GL_{3,1}} \sigma_\nu$ *induced from the unitary representation* σ_ν *of* $GL_{3,1}(e)$ *defined by*

$$\sigma\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix}, (\alpha_1, \alpha_2, \alpha_3)\right) = e^{2\pi i\alpha_1}\left(\mathrm{Ind}_{\mathbb{R}^2}^{\mathbb{R}^2 \ltimes GL_2}\theta_\nu\right)\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix}\right),$$

where $\theta_\nu\,(\nu \in \mathbb{R}^2)$ *is the unitary character of* $\mathbb{R}^2$ *defined by* $\theta_\nu(a) = e^{2\pi i({}^t\nu a)}\,(a \in \mathbb{R}^2)$. *We note that* $GL_{3,1}(e)$ *is isomorphic to the group* $\mathbb{R}^2 \ltimes GL_2 \cong GL_{2,1}$. *We already dealt with the unitary representations of* $GL_{2,1}$.

(II-2) $m = 2$.

In this case, $A \cong \mathbb{R}^{(2,3)} \cong \hat{A}$.

Lemma 3.11. *Let* $n = 3$ *and* $m = 2$. *Then the* $GL_{3,2}$*-orbits in* $\hat{A}$ *consist of the following orbits*:

$$\begin{aligned}
\Omega_{[32];0} &= \left\{\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right\},\\
\Omega_{[32];1} &= \left\{\begin{pmatrix}A\\ 0\end{pmatrix} \in \mathbb{R}^{(2,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0\right\},\\
\Omega_{[32];2} &= \left\{\begin{pmatrix}0\\ A\end{pmatrix} \in \mathbb{R}^{(2,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0\right\},\\
\Omega_{[32];3}(\delta) &= \left\{\begin{pmatrix}A\\ \delta A\end{pmatrix} \in \mathbb{R}^{(2,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0\right\} \quad (\delta \in \mathbb{R}^\times)
\end{aligned}$$

and

$$\Omega_{[32];4} = \left\{M \in \mathbb{R}^{(2,3)} \mid \mathrm{rank}\, M = 2\right\}.$$

$\Omega_{[32];0}$ *is the* $GL_{3,2}$*-orbit of* $\mathbf{0}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, $\Omega_{[32];1}$ *is the* $GL_{3,2}$*-orbit of* $\begin{pmatrix}\alpha\\ 0\end{pmatrix}$ *with* $0 \neq \alpha \in \mathbb{R}^{(1,3)}$, $\Omega_{[32];2}$ *is the* $GL_{3,2}$*-orbit of* $\begin{pmatrix}0\\ \beta\end{pmatrix}$ *with* $0 \neq \beta \in \mathbb{R}^{(1,3)}$, $\Omega_{[32];3}(\delta)$ *is the* $GL_{3,2}$*-orbit of* $\begin{pmatrix}\alpha\\ \delta\alpha\end{pmatrix}$ *with* $0 \neq \alpha \in \mathbb{R}^{(1,3)}$ *and* $\Omega_{[32];4}$ *is the* $GL_{3,2}$*-orbit of any invertible matrix* $M \in \mathbb{R}^{(2.3)}$ *with* $\mathrm{rank}\, M = 2$.

Proof. Without difficulty we may prove the above lemma. We note that $\Omega_{[32];3}(\delta_1) = \Omega_{[32];3}(\delta_1)$. So we leave the detail to the reader. □

We put

$$e^* = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \quad \text{and} \quad f^* = \begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}.$$

Obviously $e^* \in \Omega_{[32];1}$ and $f^* \in \Omega_{[32];2}$.

Then we may prove the following lemma.

Lemma 3.12.

(a) *The stabilizer of* $\mathbf{0}$ *is* $GL_{3,2}$.

(b) *The stabilizer* $GL_{3,2}(e^*)$ *of* e *is given by*

$$GL_{3,2}(e^*) = \left\{ \left(\begin{pmatrix} 1 & 0 \\ \alpha & g \end{pmatrix}, \alpha \right) \,\Big|\, a \in \mathbb{R}^{(2,1)},\ g \in GL_2,\ \alpha \in \mathbb{R}^{(2,3)} \right\}. \tag{3.7}$$

For each $x \in \Omega_{[32];1}$, *the stabilizer* $GL_{3,2}(x)$ *of* x *is conjugate to* $GL_{3,2}(e^*)$. *Precisely if* $x = e^* g_0$ *with* $g_0 \in GL_3$, *then* $GL_{3,2}(x) = (g_0, 0)^{-1} GL_{3,2}(e^*)(g_0, 0)$.

(c) *The stabilizer* $GL_{3,2}(f^*)$ *of* f^* *is given by (3.6).*

(d) *The stabilizer* $GL_{3,2}(\delta)$ *of* $\begin{pmatrix} 1 & 0 & 0 \\ \delta & 0 & 0 \end{pmatrix}$ $(\delta \in \mathbb{R}^\times)$ *is given by (3.6).*

(e) *The stabilizer* $GL_{3,2}(M)$ *of* $M \in \Omega_{[32];4}$ *is given by*

$$GL_{3,2}(M) = \left\{ \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix}, \alpha \right) \,\Bigg|\, a, b \in \mathbb{R},\ c \in \mathbb{R}^\times,\ \alpha \in \mathbb{R}^{(2,3)} \right\}.$$

Therefore A *is regularly embedded.*

According to Theorem 3.2, we obtain the following.

Theorem 3.13. *Let* $n = 3$ *and* $m = 2$. *Then the irreducible unitary representations of* $GL_{3,2}$ *are the following*:

(a) *The irreducible unitary representations* ρ, *where the restriction of* ρ *to* A *is trivial and the restriction of* ρ *to* GL_3 *is an irreducible unitary representation of* GL_3.

(b) *The representations* $\rho_{e^*} := \mathrm{Ind}_{GL_{3,2}(e^*)}^{GL_{3,2}} \tau_{e^*}$ *induced from the irreducible unitary representation* τ_{e^*} *of* $GL_{3,2}(e^*)$. *Here* τ_{e^*} *is of the form*

$$\tau_{e^*}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \end{pmatrix} \right) = e^{2\pi\alpha_1} \cdot \pi\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix} \right),$$

where π *is an irreducible unitary representation of* $\mathbb{R}^2 \rtimes GL_2$ *given by Theorem 3.3.*

(c) *The representations* $\rho_{f^*} := \mathrm{Ind}_{GL_{3,2}(f^*)}^{GL_{3,2}} \tau_{f^*}$ *induced from the irreducible unitary representation* τ_{f^*} *of* $GL_{3,2}(f^*)$. *Here* τ_{f^*} *is of the form*

$$\tau_{f^*}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \end{pmatrix} \right) = e^{2\pi\alpha_4} \cdot \pi\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix} \right),$$

where π *is an irreducible unitary representation of* $\mathbb{R}^2 \rtimes GL_2$ *given by Theorem 3.3.*

(d) *The representations* $\rho_\delta := \mathrm{Ind}_{GL_{3,2}(\delta)}^{GL_{3,2}} \tau_\delta$ *induced from the irreducible unitary representation* τ_δ *of* $GL_{3,2}(\delta)$ *defined by*

$$\tau_\delta\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix}, \begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \alpha_4 & \alpha_5 & \alpha_6\end{pmatrix}\right) = e^{2\pi\alpha_1+\delta\alpha_4}\cdot \pi\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix}\right),$$

where π *is an irreducible unitary representation of* $\mathbb{R}^2 \rtimes GL_2$ *given by Theorem 3.3.*

(e) *The representations* $\rho_M := \mathrm{Ind}_{GL_{3,2}(M)}^{GL_{3,2}} \tau_M$ $(M \in \mathbb{R}^{(2,3)}$ *with rank* $M = 2)$ *of* $GL_{3,2}$ *induced from the unitary character* τ_M *of* $GL_{3,2}(M)$ *defined by*

$$\tau_M(X)\left(\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ a & b & c\end{pmatrix}, \begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \alpha_4 & \alpha_5 & \alpha_6\end{pmatrix}\right) = e^{2\pi i(\alpha_1+\alpha_5)}\cdot \pi_M\left(\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ a & b & c\end{pmatrix}\right),$$

where π_M *is an irreducible unitary representation of* $\mathbb{R}^2 \rtimes GL_1$.

Proof. We leave the detail of the proof to the reader. □

(II-3) $m = 3$.

In this case, $A = \mathbb{R}^{(3,3)}$.

Lemma 3.14. *Let* $n = 3$ *and* $m = 3$. *That is,* $A = \mathbb{R}^{(3,3)}$. *Then the* $GL_{3,3}$*-orbits in* $\hat{A}$ *consist of the following orbits*:

$$\begin{aligned}
\Omega_{[33];0} &= \left\{\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right\},\\
\Omega_{[33];1} &= \left\{\begin{pmatrix}A\\ 0\\ 0\end{pmatrix} \in \mathbb{R}^{(3,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0\right\},\\
\Omega_{[33];2} &= \left\{\begin{pmatrix}0\\ A\\ 0\end{pmatrix} \in \mathbb{R}^{(3,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0\right\},\\
\Omega_{[33];3} &= \left\{\begin{pmatrix}0\\ 0\\ A\end{pmatrix} \in \mathbb{R}^{(3,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0\right\},
\end{aligned}$$

$$\begin{aligned}
\Omega_{[33]}(1;\delta) &= \left\{\begin{pmatrix}0\\ A\\ \delta A\end{pmatrix} \in \mathbb{R}^{(3,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0\right\} \quad (\delta \in \mathbb{R}^\times),\\
\Omega_{[33]}(2;\delta) &= \left\{\begin{pmatrix}A\\ 0\\ \delta A\end{pmatrix} \in \mathbb{R}^{(3,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0\right\} \quad (\delta \in \mathbb{R}^\times),
\end{aligned}$$

$$\Omega_{[33]}(3;\delta) = \left\{ \begin{pmatrix} A \\ \delta A \\ 0 \end{pmatrix} \in \mathbb{R}^{(3,3)} \,\Big|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \quad (\delta \in \mathbb{R}^{\times}),$$

$$\Omega_{[33]}(\lambda,\mu) = \left\{ \begin{pmatrix} A \\ \lambda A \\ \mu A \end{pmatrix} \in \mathbb{R}^{(3,3)} \,\Big|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \quad (\lambda,\ \mu \in \mathbb{R}^{\times})$$

and

$$\Omega_{12;\lambda,\mu} = \left\{ \begin{pmatrix} A \\ B \\ \lambda A + \mu B \end{pmatrix} \in \mathbb{R}^{(3,3)} \,\Big|\, \text{rank} \begin{pmatrix} A \\ B \end{pmatrix} = 2 \right\} \quad (\lambda, \mu \in \mathbb{R}),$$

$$\Omega_{13;\lambda,\mu} = \left\{ \begin{pmatrix} A \\ \lambda A + \mu B \\ B \end{pmatrix} \in \mathbb{R}^{(3,3)} \,\Big|\, \text{rank} \begin{pmatrix} A \\ B \end{pmatrix} = 2 \right\} \quad (\lambda, \mu \in \mathbb{R}),$$

$$\Omega_{23;\lambda,\mu} = \left\{ \begin{pmatrix} \lambda A + \mu B \\ A \\ B \end{pmatrix} \in \mathbb{R}^{(3,3)} \,\Big|\, \text{rank} \begin{pmatrix} A \\ B \end{pmatrix} = 2 \right\} \quad (\lambda, \mu \in \mathbb{R}),$$

$$\Omega_{[33];*} = GL_3.$$

Proof. It is easy to prove the above lemma. We leave the proof to the reader. □

We put

$$\theta_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \theta_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and for each $\delta,\ \lambda,\ \mu \in \mathbb{R}^{\times}$

$$\theta_{1,\delta} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \delta & 0 & 0 \end{pmatrix}, \quad \theta_{2,\delta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \delta & 0 & 0 \end{pmatrix}, \quad \theta_{3,\delta} = \begin{pmatrix} 1 & 0 & 0 \\ \delta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\theta_{\lambda,\mu} = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 0 & 0 \\ \mu & 0 & 0 \end{pmatrix}.$$

We also set for each $\lambda,\ \mu \in \mathbb{R}^{\times}$,

$$\phi_{12;\lambda,\mu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & \mu & 0 \end{pmatrix}, \quad \phi_{13;\lambda,\mu} = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & \mu & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \quad \phi_{23;\lambda,\mu} = \begin{pmatrix} \lambda & \mu & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We note that $\theta_i \in \Omega_{[34];i}\,(i = 1,2,3)$, $\theta_{j,\delta} \in \Omega_{[33]}(j;\delta)\,(j = 1,2,3)$, $\theta_{\lambda,\mu} \in \Omega_{[33]}(\lambda,\mu)$, $\phi_{12;\lambda,\mu} \in \Omega_{12;\lambda,\mu}$, $\phi_{13;\lambda,\mu} \in \Omega_{13;\lambda,\mu}$, $\phi_{23;\lambda,\mu} \in \Omega_{23;\lambda,\mu}$.

Then by a simple calculation, we may prove the following lemma without difficulty.

Lemma 3.15.

(a) *The stabilizer of* $\mathbf{0}$ *is* $GL_{3,3}$.

(b) *The stabilizer* $GL_{3,3}(1)$ *of* θ_1 *is given by*

$$GL_{3,3}(1) = \left\{ \left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \alpha \right) \,\middle|\, a \in \mathbb{R}^{(2,1)},\ g \in GL_2,\ \alpha \in \mathbb{R}^{(3,3)} \right\}.$$

(c) *The stabilizer* $GL_{3,3}(2)$ *of* θ_2 *is given by*

$$GL_{3,3}(2) = \left\{ \left(\begin{pmatrix} * & * & * \\ 0 & 1 & 0 \\ * & * & * \end{pmatrix}, \alpha \right) \in GL_{3,3} \,\middle|\, \alpha \in \mathbb{R}^{(3,3)} \right\}.$$

(d) *The stabilizer* $GL_{3,3}(3)$ *of* θ_3 *is given by*

$$GL_{3,3}(3) = \left\{ \left(\begin{pmatrix} g & a \\ 0 & 1 \end{pmatrix}, \alpha \right) \,\middle|\, a \in \mathbb{R}^{(2,1)},\ g \in GL_2,\ \alpha \in \mathbb{R}^{(3,3)} \right\}.$$

(e) *The stabilizer* $GL_{3,3}(i;\delta)$ *of* $\theta_{i;\delta} (i = 1, 2, 3)$ *is given by*

$$GL_{3,3}(i;\delta) = \left\{ \left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \alpha \right) \,\middle|\, a \in \mathbb{R}^{(2,1)},\ g \in GL_2,\ \alpha \in \mathbb{R}^{(3,3)} \right\}.$$

(f) *The stabilizer* $GL_{3,3}(\lambda,\mu)$ *of* $\theta_{\lambda,\mu}$ *is given by*

$$GL_{3,3}(\lambda,\mu) = \left\{ \left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \alpha \right) \,\middle|\, a \in \mathbb{R}^{(2,1)},\ g \in GL_2,\ \alpha \in \mathbb{R}^{(3,3)} \right\}.$$

(g) *The stabilizers* $GL_{3,3}(12;\lambda,\mu)$, $GL_{3,3}(13;\lambda,\mu)$, $GL_{3,3}(23;\lambda,\mu)$ *of* $\phi_{12;\lambda,\mu}$, $\phi_{13;\lambda,\mu}$, $\phi_{23;\lambda,\mu}$ *respectively are given by*

$$\begin{aligned} GL_{3,3}(12;\lambda,\mu) &= GL_{3,3}(13;\lambda,\mu) = GL_{3,3}(23;\lambda,\mu) \\ &= \left\{ \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix}, \alpha \right) \in GL_{3,3} \,\middle|\, a, b \in \mathbb{R},\ c \in \mathbb{R}^{\times},\ \alpha \in \mathbb{R}^{(3,3)} \right\}. \end{aligned}$$

(h) *The stabilizer of* I_3 *is* $\{(I_3, 0)|\ \alpha \in \mathbb{R}^{(3,3)}\} \cong A$.

According to Theorem 3.2, we obtain the following.

Theorem 3.16. *Let* $n = 3$ *and* $m = 3$. *Then irreducible unitary representations of* $GL_{3,3}$ *are the following.*

(a) *The irreducible unitary representations ρ, where the restriction of ρ to A is trivial and the restriction of ρ to GL_3 is an irreducible unitary representation of GL_3.*

(b) *The representation $\rho_{\theta_1} := \text{Ind}\,{}^{GL_{3,3}}_{GL_{3,3}(1)}\tau_{\theta_1}$ induced from the unitary representation τ_{θ_1} of $GL_{3,3}(1)$. Here τ_{θ_1} is of the form*

$$\tau_{\theta_1}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}\right) = e^{2\pi i \alpha_1} \cdot \pi_{\theta_1}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where π_{θ_1} is the irreducible unitary representation of $\mathbb{R}^2 \ltimes GL_2$ given by Theorem 3.3.

(c) *The representation $\rho_{\theta_2} := \text{Ind}\,{}^{GL_{3,3}}_{GL_{3,3}(2)}\tau_{\theta_2}$ induced from the unitary representation τ_{θ_2} of $GL_{3,3}(2)$. Here τ_{θ_2} is of the form*

$$\tau_{\theta_2}\left(\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ 0 & 1 & 0 \\ g_{31} & g_{32} & g_{33} \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}\right) = e^{2\pi i \alpha_5} \cdot \pi_{\theta_2}\left(\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ 0 & 1 & 0 \\ g_{31} & g_{32} & g_{33} \end{pmatrix}\right),$$

where π_{θ_2} is the irreducible unitary representation of $\mathbb{R}^2 \ltimes GL_2$ given by Theorem 3.3.

(d) *The representation $\rho_{\theta_3} := \text{Ind}\,{}^{GL_{3,3}}_{GL_{3,3}(3)}\tau_{\theta_3}$ induced from the unitary representation τ_{θ_3} of $GL_{3,3}(3)$. Here τ_{θ_3} is of the form*

$$\tau_{\theta_3}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}\right) = e^{2\pi i \alpha_9} \cdot \pi_{\theta_3}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where π_{θ_3} is the irreducible unitary representation of $\mathbb{R}^2 \ltimes GL_2$ given by Theorem 3.3.

(e) *The representation $\rho_{1,\delta} := \text{Ind}\,{}^{GL_{3,3}}_{GL_{3,3}(1;\delta)}\tau_{1,\delta}$ induced from the unitary representation $\tau_{1,\delta}$ of $GL_{3,3}(1,\delta)$. Here $\tau_{1,\delta}$ is of the form*

$$\tau_{1,\delta}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}\right) = e^{2\pi i (\alpha_4 + \delta\alpha_7)} \cdot \pi_{1,\delta}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where $\pi_{1,\delta}$ is the irreducible unitary representation of $\mathbb{R}^2 \ltimes GL_2$ given by Theorem 3.3.

(f) *The representation $\rho_{2,\delta} := \text{Ind}\,{}^{GL_{3,3}}_{GL_{3,3}(2;\delta)}\tau_{2,\delta}$ induced from the unitary representation $\tau_{2,\delta}$ of $GL_{3,3}(2,\delta)$. Here $\tau_{1,\delta}$ is of the form*

$$\tau_{2,\delta}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}\right) = e^{2\pi i (\alpha_1 + \delta\alpha_7)} \cdot \pi_{2,\delta}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where $\pi_{2,\delta}$ *is the irreducible unitary representation of* $\mathbb{R}^2 \ltimes GL_2$ *given by Theorem 3.3.*

(g) *The representation* $\rho_{3,\delta} := \operatorname{Ind}\,{}_{GL_{3,3}(3,\delta)}^{GL_{3,3}}\tau_{1,\delta}$ *induced from the unitary representation* $\tau_{1,\delta}$ *of* $GL_{3,3}(3,\delta)$. *Here* $\tau_{3,\delta}$ *is of the form*

$$\tau_{3,\delta}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}\right) = e^{2\pi i(\alpha_1+\delta\alpha_4)} \cdot \pi_{3,\delta}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where $\pi_{3,\delta}$ *is the irreducible unitary representation of* $\mathbb{R}^2 \ltimes GL_2$ *given by Theorem 3.3.*

(h) *The representation* $\rho_{\lambda,\mu} := \operatorname{Ind}\,{}_{GL_{3,3}(\mu)}^{GL_{3,3}}\tau_{\lambda,\mu}$ *induced from the unitary representation* $\tau_{\lambda,\mu}$ *of* $GL_{3,3}(\lambda,\mu)$. *Here* $\tau_{\lambda,\mu}$ *is of the form*

$$\tau_{\lambda,\mu}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}\right) = e^{2\pi i(\alpha_1+\lambda\alpha_4+\mu\alpha_7)} \cdot \pi_{\lambda,\mu}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where $\pi_{\lambda,\mu}$ *is the irreducible unitary representation of* $\mathbb{R}^2 \ltimes GL_2$ *given by Theorem 3.3.*

(i) *The representation* $\rho_{12;\lambda,\mu} := \operatorname{Ind}\,{}_{GL_{3,3}(12;\lambda,\mu)}^{GL_{3,3}}\tau_{12;\lambda,\mu}$ *induced from the unitary representation* $\tau_{12;\lambda,\mu}$ *of* $GL_{3,3}(12;\lambda,\mu)$. *Here* $\tau_{12;\lambda,\mu}$ *is of the form*

$$\begin{aligned} &\tau_{12;\lambda,\mu}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}\right) \\ = \;& e^{2\pi i(\alpha_1+\alpha_5+\lambda\alpha_7+\mu\alpha_8)} \cdot \pi_{12;\lambda,\mu}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix}\right), \end{aligned}$$

where $\pi_{12;\lambda,\mu}$ *is the irreducible unitary representation of* $\mathbb{R}^2 \ltimes GL_1$ *given by Theorem 3.3.*

(j) *The representation* $\rho_{13;\lambda,\mu} := \operatorname{Ind}\,{}_{GL_{3,3}(13;\lambda,\mu)}^{GL_{3,3}}\tau_{13;\lambda,\mu}$ *induced from the unitary representation* $\tau_{13;\lambda,\mu}$ *of* $GL_{3,3}(13;\lambda,\mu)$. *Here* $\tau_{13;\lambda,\mu}$ *is of the form*

$$\begin{aligned} &\tau_{13;\lambda,\mu}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & b & c \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}\right) \\ = \;& e^{2\pi i(\alpha_1+\alpha_8+\lambda\alpha_4+\mu\alpha_5)} \cdot \pi_{13;\lambda,\mu}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & b & c \end{pmatrix}\right), \end{aligned}$$

where $\pi_{13;\lambda,\mu}$ *is the irreducible unitary representation of* $\mathbb{R}^2 \ltimes GL_1$.

(k) *The representation* $\rho_{23;\lambda,\mu} := \text{Ind}\,{}^{GL_{3,3}}_{GL_{3,3}(23;\lambda,\mu)}\tau_{23;\lambda,\mu}$ *induced from the unitary representation* $\tau_{23;\lambda,\mu}$ *of* $GL_{3,3}(23;\lambda,\mu)$. *Here* $\tau_{23;\lambda,\mu}$ *is of the form*

$$\begin{aligned}
&\tau_{23;\lambda,\mu}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}\right) \\
= \ & e^{2\pi i(\alpha_4+\alpha_8+\lambda\alpha_1+\mu\alpha_2)}\cdot \pi_{23;\lambda,\mu}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix}\right),
\end{aligned}$$

where $\pi_{23;\lambda,\mu}$ *is the irreducible unitary representation of* $\mathbb{R}^2 \ltimes GL_1$.

(l) *The representation* $\rho_{I_3} := \text{Ind}_A^{GL_{3,3}}\chi_{I_3}$ *induced from the unitary character* χ_{I_3} *of* A *given by*

$$\chi_{I_3}\left(\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}\right) = e^{2\pi i(\alpha_1+\alpha_5+\alpha_9)}.$$

(II-4) $m = 4$.

In this case, $A = \mathbb{R}^{(4,3)}$.

Lemma 3.17. *Let* $n = 3$ *and* $m = 4$. *Then the* $GL_{3,4}$*-orbits in* $\hat{A}$ *consists of the following orbits*:

$$\begin{aligned}
\Omega_{[34];0} &= \{0\}, \\
\Omega_{[34];1} &= \left\{ \begin{pmatrix} A \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\}, \\
\Omega_{[34];2} &= \left\{ \begin{pmatrix} 0 \\ A \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\}, \\
\Omega_{[34];3} &= \left\{ \begin{pmatrix} 0 \\ 0 \\ A \\ 0 \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\}, \\
\Omega_{[34];4} &= \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ A \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\},
\end{aligned}$$

$$\Omega_{12;\delta} = \left\{ \begin{pmatrix} 0 \\ 0 \\ A \\ \delta A \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \ (\delta \in \mathbb{R}^{\times}),$$

$$\Omega_{13;\delta} = \left\{ \begin{pmatrix} 0 \\ A \\ 0 \\ \delta A \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \ (\delta \in \mathbb{R}^{\times}),$$

$$\Omega_{14;\delta} = \left\{ \begin{pmatrix} 0 \\ A \\ \delta A \\ 0 \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \ (\delta \in \mathbb{R}^{\times}),$$

$$\Omega_{23;\delta} = \left\{ \begin{pmatrix} A \\ 0 \\ 0 \\ \delta A \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \ (\delta \in \mathbb{R}^{\times}),$$

$$\Omega_{24;\delta} = \left\{ \begin{pmatrix} A \\ 0 \\ \delta A \\ 0 \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \ (\delta \in \mathbb{R}^{\times}),$$

$$\Omega_{34;\delta} = \left\{ \begin{pmatrix} A \\ \delta A \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \ (\delta \in \mathbb{R}^{\times}),$$

$$\Omega_{1;\lambda,\mu} = \left\{ \begin{pmatrix} 0 \\ A \\ \lambda A \\ \mu A \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \ (\lambda, \mu \in \mathbb{R}^{\times}),$$

$$\Omega_{2;\lambda,\mu} = \left\{ \begin{pmatrix} A \\ 0 \\ \lambda A \\ \mu A \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \ (\lambda, \mu \in \mathbb{R}^{\times}),$$

$$\Omega_{3;\lambda,\mu} = \left\{ \begin{pmatrix} A \\ \lambda A \\ 0 \\ \mu A \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \ (\lambda, \mu \in \mathbb{R}^{\times}),$$

$$\Omega_{4;\lambda,\mu} = \left\{ \begin{pmatrix} A \\ \lambda A \\ \mu A \\ 0 \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \ (\lambda, \mu \in \mathbb{R}^{\times}),$$

$$\Omega_{\lambda,\mu,\kappa} = \left\{ \begin{pmatrix} A \\ \lambda A \\ \mu A \\ \kappa A \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \ (\lambda, \mu, \kappa \in \mathbb{R}^{\times})$$

and for any $\lambda,\ \mu,\ \kappa,\ \delta \in \mathbb{R}$,

$$\Omega_{12;\lambda,\mu,\kappa,\delta} = \left\{ \begin{pmatrix} A \\ B \\ \lambda A + \mu B \\ \kappa A + \delta B \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, \text{rank} \begin{pmatrix} A \\ B \end{pmatrix} = 2 \right\},$$

$$\Omega_{14;\lambda,\mu,\kappa,\delta} = \left\{ \begin{pmatrix} A \\ \lambda A + \mu B \\ \kappa A + \delta B \\ B \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, \text{rank} \begin{pmatrix} A \\ B \end{pmatrix} = 2 \right\},$$

$$\Omega_{23;\lambda,\mu,\kappa,\delta} = \left\{ \begin{pmatrix} \lambda A + \mu B \\ A \\ B \\ \kappa A + \delta B \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, \text{rank} \begin{pmatrix} A \\ B \end{pmatrix} = 2 \right\},$$

$$\Omega_{24;\lambda,\mu,\kappa,\delta} = \left\{ \begin{pmatrix} \lambda A + \mu B \\ A \\ \kappa A + \delta B \\ B \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, \text{rank} \begin{pmatrix} A \\ B \end{pmatrix} = 2 \right\},$$

$$\Omega_{34;\lambda,\mu,\kappa,\delta} = \left\{ \begin{pmatrix} \lambda A + \mu B \\ \kappa A + \delta B \\ A \\ B \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, \text{rank} \begin{pmatrix} A \\ B \end{pmatrix} = 2 \right\},$$

$$\Omega_{123;\lambda,\mu,\kappa} = \left\{ \begin{pmatrix} A \\ B \\ C \\ \lambda A + \mu B + \kappa C \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, \text{rank} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 3 \right\},$$

$$\Omega_{124;\lambda,\mu,\kappa} = \left\{ \begin{pmatrix} A \\ B \\ \lambda A + \mu B + \kappa C \\ C \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, \text{rank} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 3 \right\},$$

$$\Omega_{134;\lambda,\mu,\kappa} = \left\{ \begin{pmatrix} A \\ \lambda A + \mu B + \kappa C \\ B \\ C \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, \text{rank} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 3 \right\},$$

$$\Omega_{234;\lambda,\mu,\kappa} = \left\{ \begin{pmatrix} \lambda A + \mu B + \kappa C \\ A \\ B \\ C \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, \text{rank} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 3 \right\}.$$

We put

$$\xi_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\xi_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and for each $\delta \in \mathbb{R}^\times$, we set

$$\xi_{12;\delta} = \begin{pmatrix} 1 & 0 & 0 \\ \delta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi_{13;\delta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \delta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\xi_{14;\delta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \delta & 0 & 0 \end{pmatrix}, \quad \xi_{23;\delta} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \delta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\xi_{24;\delta} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ \delta & 0 & 0 \end{pmatrix}, \quad \xi_{34;\delta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ \delta & 0 & 0 \end{pmatrix}.$$

For any $\lambda,\ \mu,\ \kappa \in \mathbb{R}^\times$, we put

$$\xi_{1;\lambda,\mu} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \lambda & 0 & 0 \\ \mu & 0 & 0 \end{pmatrix}, \quad \xi_{2;\lambda,\mu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \\ \mu & 0 & 0 \end{pmatrix},$$

$$\xi_{3;\lambda,\mu} = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \\ \mu & 0 & 0 \end{pmatrix}, \quad \xi_{4;\lambda,\mu} = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 0 & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\xi_{\lambda,\mu,\kappa} = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 0 & 0 \\ \mu & 0 & 0 \\ \kappa & 0 & 0 \end{pmatrix}.$$

We also put for any $\lambda,\ \mu,\ \kappa,\ \delta \in \mathbb{R}$,

$$\xi_{12;\lambda,\mu,\kappa,\delta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & \mu & 0 \\ \kappa & \delta & 0 \end{pmatrix}, \quad \xi_{13;\lambda,\mu,\kappa,\delta} = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & \mu & 0 \\ \kappa & \delta & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\xi_{14;\lambda,\mu,\kappa,\delta} = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & \mu & 0 \\ \kappa & \delta & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \xi_{23;\lambda,\mu,\kappa,\delta} = \begin{pmatrix} \lambda & \mu & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ \kappa & \delta & 0 \end{pmatrix}$$

and

$$\xi_{24;\lambda,\mu,\kappa,\delta} = \begin{pmatrix} \lambda & \mu & 0 \\ 1 & 0 & 0 \\ \kappa & \delta & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Lemma 3.18.

(a) *The stabilizer of* $\mathbf{0}$ *is* $GL_{3,4}$.

(b) *The stabilizer* $GL_{3,4}(i)$ *of* $\xi_i\ (i = 1, 2, 3, 4)$ *is given by*

$$\left\{ \left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \alpha \right) \in GL_{3,4} \,\middle|\, a \in \mathbb{R}^{(2,1)},\ g \in GL_2,\ \alpha \in \mathbb{R}^{(4,3)} \right\}. \tag{3.8}$$

(c) *The stabilizer* $GL_{3,4}(ij;\delta)$ *of* $\xi_{ij;\delta}\ (1 \le i \le j \le 4)$ *is given by* (3.7).

(d) *The stabilizer* $GL_{3,4}(i;\lambda,\mu)$ *of* $\xi_{i;\lambda,\mu}\ (1 \le i \le 4)$ *is given by* (3.7).

(e) *The stabilizer* $GL_{3,4}(\lambda,\mu,\kappa)$ *of* $\xi_{\lambda,\mu,\kappa}$ *is given by* (3.7).

(f) *The stabilizer* $GL_{3,4}(ij;\lambda,\mu,\kappa,\delta)$ *of* $\xi_{ij;\lambda,\mu,\kappa,\delta}\ (1 \le i \le j \le 4)$ *is given by*

$$\left\{ \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix}, \alpha \right) \in GL_{3,4} \,\middle|\, a, b, c (\neq 0) \in \mathbb{R},\ \alpha \in \mathbb{R}^{(4,3)} \right\}.$$

According to Theorem 3.2, we obtain the following.

Theorem 3.19. *Let* $n = 3$ *and* $m = 4$. *We put*

$$\alpha = (\alpha_{ij}) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} \end{pmatrix} \in \mathbb{R}^{(4,3)}.$$

Then the irreducible unitary representations of $GL_{3,4}$ *are the following:*

(a) *The irreducible unitary representation* ρ, *where the restriction of* ρ *to* A *is trivial and the restriction of* ρ *to* GL_3 *is an irreducible unitary representation of* GL_3.

(b) *The representation* $\rho_{\xi_i} := \mathrm{Ind}\,_{GL_{3,4}(i)}^{GL_{3,4}} \tau_{\xi_i}$ $(1 \leq i \leq 4)$ *induced from the unitary irreducible representation* τ_{ξ_i} *of* $GL_{3,4}(i)$. *Here* τ_{ξ_i} *is of the form*

$$\tau_{\xi_i}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, (\alpha_{ij})\right) = e^{2\pi i \alpha_{31}} \cdot \pi_{\xi_i}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where π_{ξ_i} *is the irreducible unitary representation of* $\mathbb{R}^2 \ltimes GL_2$.

(c) *The representation* $\rho_{\xi_{ij;\delta}} := \mathrm{Ind}\,_{GL_{3,4}(ij;\delta)}^{GL_{3,4}} \tau_{\xi_{ij;\delta}}$ $(1 \leq i \leq j \leq 4)$ *induced from the unitary irreducible representation* $\tau_{\xi_{ij;\delta}}$ *of* $GL_{3,4}(ij;\delta)$. *Here* $\tau_{\xi_{ij;\delta}}$ *is of the form*

$$\tau_{\xi_{ij;\delta}}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, (\alpha_{ij})\right) = e^{2\pi i(\alpha_{i1} + \delta\alpha_{j1})} \cdot \pi_{\xi_{34;\delta}}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where $\pi_{\xi_{34;\delta}}$ *is the irreducible unitary representation of* $\mathbb{R}^2 \ltimes GL_2$.

(d) *The representation* $\rho(\xi_{1;\lambda,\mu}) := \mathrm{Ind}\,_{GL_{3,4}(1;\lambda,\mu)}^{GL_{3,4}} \tau(\xi_{1;\lambda,\mu})$ *induced from the unitary irreducible representation* $\tau(\xi_{1;\lambda,\mu})$ *of* $GL_{3,4}(1;\lambda,\mu)$. *Here* $\tau(\xi_{1;\lambda,\mu})$ *is of the form*

$$\tau(\xi_{1;\lambda,\mu})\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, (\alpha_{ij})\right) = e^{2\pi i(\alpha_{21} + \lambda\alpha_{31} + \mu\alpha_{41})} \cdot \pi(\xi_{1;\lambda,\mu})\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where $\pi(\xi_{1;\lambda,\mu})$ *is the irreducible unitary representation of* $\mathbb{R}^2 \ltimes GL_2$.

(e) *The representation* $\rho(\xi_{2;\lambda,\mu}) := \mathrm{Ind}\,_{GL_{3,4}(2;\lambda,\mu)}^{GL_{3,4}} \tau(\xi_{2;\lambda,\mu})$ *induced from the unitary irreducible representation* $\tau(\xi_{2;\lambda,\mu})$ *of* $GL_{3,4}(2;\lambda,\mu)$. *Here* $\tau(\xi_{2;\lambda,\mu})$ *is of the form*

$$\tau(\xi_{2;\lambda,\mu})\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, (\alpha_{ij})\right) = e^{2\pi i(\alpha_{11} + \lambda\alpha_{31} + \mu\alpha_{41})} \cdot \pi(\xi_{2;\lambda,\mu})\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where $\pi(\xi_{2;\lambda,\mu})$ *is the irreducible unitary representation of* $\mathbb{R}^2 \ltimes GL_2$.

(f) *The representation* $\rho(\xi_{3;\lambda,\mu}) := \mathrm{Ind}\,_{GL_{3,4}(3;\lambda,\mu)}^{GL_{3,4}} \tau(\xi_{3;\lambda,\mu})$ *induced from the unitary irreducible representation* $\tau(\xi_{3;\lambda,\mu})$ *of* $GL_{3,4}(3;\lambda,\mu)$. *Here* $\tau(\xi_{3;\lambda,\mu})$ *is of the form*

$$\tau(\xi_{3;\lambda,\mu})\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, (\alpha_{ij})\right) = e^{2\pi i(\alpha_{11} + \lambda\alpha_{21} + \mu\alpha_{41})} \cdot \pi(\xi_{3;\lambda,\mu})\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where $\pi(\xi_{3;\lambda,\mu})$ *is the irreducible unitary representation of* $\mathbb{R}^2 \ltimes GL_2$.

(g) *The representation* $\rho(\xi_{4;\lambda,\mu}) := \mathrm{Ind}\,_{GL_{3,4}(4;\lambda,\mu)}^{GL_{3,4}}\tau(\xi_{4;\lambda,\mu})$ *induced from the unitary irreducible representation* $\tau(\xi_{4;\lambda,\mu})$ *of* $GL_{3,4}(4;\lambda,\mu)$. *Here* $\tau(\xi_{4;\lambda,\mu})$ *is of the form*

$$\tau(\xi_{4;\lambda,\mu})\left(\begin{pmatrix}1&0\\a&g\end{pmatrix},(\alpha_{ij})\right)=e^{2\pi i(\alpha_{11}+\lambda\alpha_{21}+\mu\alpha_{31})}\cdot\pi(\xi_{1;\lambda,\mu})\left(\begin{pmatrix}1&0\\a&g\end{pmatrix}\right),$$

where $\pi(\xi_{1;\lambda,\mu})$ *is the irreducible unitary representation of* $\mathbb{R}^2\ltimes GL_2$.

(h) *The representation* $\rho(\xi_{\lambda,\mu,\kappa}) := \mathrm{Ind}\,_{GL_{3,4}(\lambda,\mu,\kappa)}^{GL_{3,4}}\tau(\xi_{\lambda,\mu,\kappa})$ *induced from the unitary irreducible representation* $\tau(\xi_{\lambda,\mu,\kappa})$ *of* $GL_{3,4}(\lambda,\mu,\kappa)$. *Here* $\tau(\xi_{\lambda,\mu,\kappa})$ *is of the form*

$$\tau(\xi_{\lambda,\mu,\kappa})\left(\begin{pmatrix}1&0\\a&g\end{pmatrix},(\alpha_{ij})\right)=e^{2\pi i(\alpha_{11}+\lambda\alpha_{21}+\mu\alpha_{31}+\kappa\alpha_{41})}\cdot\pi(\xi_{\lambda,\mu,\kappa})\left(\begin{pmatrix}1&0\\a&g\end{pmatrix}\right),$$

where $\pi(\xi_{\lambda,\mu,\kappa})$ *is the irreducible unitary representation of* $\mathbb{R}^2\ltimes GL_2$.

(i) *The representation* $\rho(\xi_{12;\lambda,\mu,\delta}) := \mathrm{Ind}\,_{GL_{3,4}(12;\lambda,\mu,\kappa,\delta)}^{GL_{3,4}}\tau(\xi_{12;\lambda,\mu,\kappa,\delta})$ *induced from the unitary irreducible representation* $\tau(\xi_{12;\lambda,\mu,\kappa,\delta})$ *of* $GL_{3,4}(12;\lambda,\mu,\kappa,\delta)$. *Here* $\tau(\xi_{12;\lambda,\mu,\kappa,\delta})$ *is of the form*

$$\begin{aligned}&\tau(\xi_{12;\lambda,\mu,\kappa,\mu})\left(\begin{pmatrix}1&0\\a&g\end{pmatrix},(\alpha_{ij})\right)\\=\;&e^{2\pi i(\alpha_{11}+\alpha_{22}+\lambda\alpha_{31}+\mu\alpha_{32}+\kappa\alpha_{41}+\delta\alpha_{42})}\cdot\pi(\xi_{12;\lambda,\mu,\kappa,\delta})\left(\begin{pmatrix}1&0\\a&g\end{pmatrix}\right),\end{aligned}$$

where $\pi(\xi_{12;\lambda,\mu,\kappa,\delta})$ *is the irreducible unitary representation of* $\mathbb{R}^2\ltimes GL_2$.

(j) *The representation* $\rho(\xi_{13;\lambda,\mu,\delta}) := \mathrm{Ind}\,_{GL_{3,4}(13;\lambda,\mu,\kappa,\delta)}^{GL_{3,4}}\tau(\xi_{13;\lambda,\mu,\kappa,\delta})$ *induced from the unitary irreducible representation* $\tau(\xi_{13;\lambda,\mu,\kappa,\delta})$ *of* $GL_{3,4}(13;\lambda,\mu,\kappa,\delta)$. *Here* $\tau(\xi_{13;\lambda,\mu,\kappa,\delta})$ *is of the form*

$$\begin{aligned}&\tau(\xi_{13;\lambda,\mu,\kappa,\mu})\left(\begin{pmatrix}1&0\\a&g\end{pmatrix},(\alpha_{ij})\right)\\=\;&e^{2\pi i(\alpha_{11}+\alpha_{32}+\lambda\alpha_{21}+\mu\alpha_{22}+\kappa\alpha_{41}+\delta\alpha_{42})}\cdot\pi(\xi_{13;\lambda,\mu,\kappa,\delta})\left(\begin{pmatrix}1&0\\a&g\end{pmatrix}\right),\end{aligned}$$

where $\pi(\xi_{13;\lambda,\mu,\kappa,\delta})$ *is the irreducible unitary representation of* $\mathbb{R}^2\ltimes GL_2$.

(k) *The representation* $\rho(\xi_{14;\lambda,\mu,\delta}) := \mathrm{Ind}\,_{GL_{3,4}(14;\lambda,\mu)}^{GL_{3,4}}\tau(\xi_{14;\lambda,\mu,\kappa,\delta})$ *induced from the unitary irreducible representation* $\tau(\xi_{14;\lambda,\mu,\kappa,\delta})$ *of* $GL_{3,4}(14;\lambda,\mu,\kappa,\delta)$. *Here* $\tau(\xi_{14;\lambda,\mu,\kappa,\delta})$ *is of the form*

$$\begin{aligned}&\tau(\xi_{14;\lambda,\mu,\kappa,\mu})\left(\begin{pmatrix}1&0\\a&g\end{pmatrix},(\alpha_{ij})\right)\\=\;&e^{2\pi i(\alpha_{11}+\alpha_{42}+\lambda\alpha_{21}+\mu\alpha_{22}+\kappa\alpha_{31}+\delta\alpha_{32})}\cdot\pi(\xi_{14;\lambda,\mu,\kappa,\delta})\left(\begin{pmatrix}1&0\\a&g\end{pmatrix}\right),\end{aligned}$$

where $\pi(\xi_{14;\lambda,\mu,\kappa,\delta})$ *is the irreducible unitary representation of* $\mathbb{R}^2 \ltimes GL_2$.

(l) *The representation* $\rho(\xi_{23;\lambda,\mu,\delta}) := \text{Ind}\,_{GL_{3,4}(23;\lambda,\mu)}^{GL_{3,4}}\tau(\xi_{23;\lambda,\mu,\kappa,\delta})$ *induced from the unitary irreducible representation* $\tau(\xi_{23;\lambda,\mu,\kappa,\delta})$ *of* $GL_{3,4}(23;\lambda,\mu,\kappa,\delta)$. *Here* $\tau(\xi_{23;\lambda,\mu,\kappa,\delta})$ *is of the form*

$$\begin{aligned}&\tau(\xi_{23;\lambda,\mu,\kappa,\mu})\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix},(\alpha_{ij})\right)\\ = \;& e^{2\pi i(\alpha_{21}+\alpha_{32}+\lambda\alpha_{11}+\mu\alpha_{12}+\kappa\alpha_{41}+\delta\alpha_{42})}\cdot\pi(\xi_{23;\lambda,\mu,\kappa,\delta})\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix}\right),\end{aligned}$$

where $\pi(\xi_{23;\lambda,\mu,\kappa,\delta})$ *is the irreducible unitary representation of* $\mathbb{R}^2 \ltimes GL_2$.

(m) *The representation* $\rho(\xi_{24;\lambda,\mu,\delta}) := \text{Ind}\,_{GL_{3,4}(24;\lambda,\mu)}^{GL_{3,4}}\tau(\xi_{24;\lambda,\mu,\kappa,\delta})$ *induced from the unitary irreducible representation* $\tau(\xi_{24;\lambda,\mu,\kappa,\delta})$ *of* $GL_{3,4}(24;\lambda,\mu,\kappa,\delta)$. *Here* $\tau(\xi_{24;\lambda,\mu,\kappa,\delta})$ *is of the form*

$$\begin{aligned}&\tau(\xi_{24;\lambda,\mu,\kappa,\mu})\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix},(\alpha_{ij})\right)\\ = \;& e^{2\pi i(\alpha_{21}+\alpha_{42}+\lambda\alpha_{11}+\mu\alpha_{12}+\kappa\alpha_{31}+\delta\alpha_{32})}\cdot\pi(\xi_{14;\lambda,\mu,\kappa,\delta})\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix}\right),\end{aligned}$$

where $\pi(\xi_{14;\lambda,\mu,\kappa,\delta})$ *is the irreducible unitary representation of* $\mathbb{R}^2 \ltimes GL_2$.

Remark 3.20. The other cases $n \geq 4$ are more complicated than the previous cases $n = 2,\ 3$ but can be dealt with in a similar way.

We note that $GL_{n,m}$ acts on $\mathbb{R}^{(m,n)}$ on the right transitively by

$$x\cdot(g,a) := x\,{}^t g^{-1} + a,\quad x,\, a\in\mathbb{R}^{(m,n)},\ g\in GL(n,\mathbb{R}).$$

For $\lambda\in\mathbb{C}$, we define the representation π_λ of $GL_{n,m}$ on $L^2(\mathbb{R}^{(m,n)})$ by

$$(\pi_\lambda((g,a))f)(x) := |\det g|^{-\lambda} f(x\cdot(g,a)), \tag{3.9}$$

where $(g,a)\in GL_{n,m},\ f\in L^2(\mathbb{R}^{(m,n)})$. Then π_λ is unitary if and only if $\lambda\in\frac{1}{2}+i\mathbb{R}$. In fact,

$$\begin{aligned}\|\pi_\lambda((g,a))f\|^2_{L^2(\mathbb{R}^{(m,n)})} &= \int_{\mathbb{R}^{(m,n)}} |\det g|^{-\lambda} f(x\,{}^t g^{-1}+a)\overline{|\det g|^{-\lambda}}\,\overline{f(x\,{}^t g^{-1}+a)}dx\\ &= \int_{\mathbb{R}^{(m,n)}} |\det g|^{1-2\operatorname{Re}\lambda}|f(x)|^2 dx\\ &= |\det g|^{1-2\operatorname{Re}\lambda}\|f\|^2_{L^2(\mathbb{R}^{(m,n)})}.\end{aligned}$$

We recall the following fact.

Theorem 3.21. *Suppose H is a subgroup of GL_n and let $H_{m,n} := H \ltimes \mathbb{R}^{(m,n)}$. Then $\big(\pi_\lambda|_{H_{m,n}}, L^2(\mathbb{R}^{(m,n)})\big)$ is irreducible if and only if the action of $H_{m,n}$ on $\mathbb{R}^{(m,n)}$ is ergodic.*

According to the above theorem, if $\lambda \in \frac{1}{2} + i\mathbb{R}$, then π_λ is irreducible because $GL_{n,m}$ acts on $\mathbb{R}^{(m,n)}$ ergodically.

References

[1] V. Bargmann, *Irreducible unitary representations of the Lorentz group*, Ann. Math., **48**(1947), 568-640.

[2] R. Howe and S. T. Lee, *Degenerate principal series representations of $GL_n(\mathbb{C})$ and $GL_n(\mathbb{R})$*, Jour. Funct. Analysis, **166**(1999), 244-309.

[3] A. W. Knapp, *Representations of $GL_2(\mathbb{R})$*, Proceedings of Symposia in Pure Math.; Automorphic Forms, Representations, and L-Functions, **33(I)**(1979), 87-91.

[4] G. Mackey, *Unitary representations of Group Extensions 1*, Acta Math., **99**(1958), 265-311.

[5] C. Moeglin, Representations of $GL(n)$ over the Real Field, Proceedings of Symposia in Pure Math.; Representation Theory and Automorphic Forms, **61**(1997), 157-166.

[6] B. Speh, *Unitary representations of $SL(n,\mathbb{R})$ and the cohomology of congruence subgroups; in Noncommutative Harmonic Analysis and Lie Groups, J. Carmona and M. Vergne (eds.)*, Lecture Notes Math., Springer-Verlag, **880**(1981), 483-505.

[7] B. Speh, *The unitary dual of $GL(3,\mathbb{R})$ and $GL(4,\mathbb{R})$*, Math. Ann., **258**(1981), 113-133.

[8] B. Speh, *Unitary representations of $GL(n,\mathbb{R})$ with Non-trivial $(\mathfrak{g}, K)$-Cohomology*, Invent. Math., **71**(1983), 443-465.

[9] B. Speh and D. Vogan, *Reducibility of generalized series representations*, Acta Math., **145**(1980), 227-299.

[10] E. M. Stein, *Analysis in matrix spaces and some new representations of $SL(n,\mathbb{C})$*, Ann. Math., **86**(1967), 461-490.

[11] M. Tadic, *Unitary dual of p-adic $GL(n)$*, Bull. AMS.(N.S.), **13**(1985), 39-42.

[12] D. Vogan, Representations of Real Reductive Lie Groups, Birkhäuser, Boston-Basel-Stuttgart, 1981.

[13] D. Vogan, *Unitarizability of certain series of representations*, Ann. Math., **120**(1984), 141-187.

[14] D. Vogan, *The unitary dual of $GL(n)$ over an archimedian field*, Invent. Math., **83**(1986), 449-505.

[15] J.-H. Yang, *Harmonic analysis on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$, I: Harmonic analysis of the theory of automorphic forms on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$*, preprint.

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Jae-Hyun Yang

Abstract. For two positive integers m and n, let $\mathcal{P}_n$ be the open convex cone in $\mathbb{R}^{n(n+1)/2}$ consisting of positive definite $n \times n$ real symmetric matrices and let $\mathbb{R}^{(m,n)}$ be the set of all $m \times n$ real matrices. In this paper, we investigate differential operators on the non-reductive homogeneous space $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ that are invariant under the natural action of the semidirect product group $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$ on the Minkowski-Euclid space $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$. These invariant differential operators play an important role in the theory of automorphic forms on $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$ generalizing that of automorphic forms on $GL(n,\mathbb{R})$.

1. Introduction

Let

$$\mathcal{P}_n = \left\{ Y \in \mathbb{R}^{(n,n)} \mid Y = {}^tY > 0 \right\}$$

be the open convex cone of positive definite symmetric real matrices of degree n in the Euclidean space $\mathbb{R}^{n(n+1)/2}$, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l and ${}^tM$ denotes the transpose matrix of a matrix M. Then the general linear group $GL(n,\mathbb{R})$ acts on $\mathcal{P}_n$ transtively by

$$g \cdot Y = gY\,{}^tg, \qquad g \in GL(n,\mathbb{R}),\ Y \in \mathcal{P}_n. \tag{1.1}$$

Therefore, $\mathcal{P}_n$ is a symmetric space which is diffeomorphic to the quotient space $GL(n,\mathbb{R})/O(n)$, where $O(n)$ denotes the orthogonal group of degree n. A. Selberg [10] investigated differential operators on $\mathcal{P}_n$ invariant under the action (1.1) of $GL(n,\mathbb{R})$ (cf. [7, 8]).

Let

$$GL_{n,m} = GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$$

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be the semidirect product of $GL(n,\mathbb{R})$ and the abelian additive group $\mathbb{R}^{(m,n)}$ equipped with the following multiplication law

$$(g,\lambda)\cdot(h,\mu) = \big(gh, \lambda\,{}^t h^{-1} + \mu\big),$$

where $g,h \in GL(n,\mathbb{R})$ and $\lambda,\mu \in \mathbb{R}^{(m,n)}$. Then we have the *natural action* of $GL_{n,m}$ on the non-reductive homogeneous space $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ given by

$$(g,\lambda)\cdot(Y,V) = \big(gY\,{}^t g,\ (V+\lambda)\,{}^t g\big), \tag{1.2}$$

where $g \in GL(n,\mathbb{R}),\ \lambda \in \mathbb{R}^{(m,n)},\ Y \in \mathcal{P}_n$ and $V \in \mathbb{R}^{(m,n)}$.

For brevity, we set $\mathcal{P}_{n,m} = \mathcal{P}_n \times \mathbb{R}^{(m,n)}$ and $K = O(n)$. Since the action (1.2) of $GL_{n,m}$ is transitive, $\mathcal{P}_{n,m}$ is diffeomorphic to $GL_{n,m}/K$. We observe that the action (1.2) of $GL_{n,m}$ generalizes the action (1.1) of $GL(n,\mathbb{R})$.

The significance in studying the non-reductive homogeneous space $\mathcal{P}_{n,m}$ may be explained as follows. Let

$$\Gamma_{n,m} = GL(n,\mathbb{Z}) \ltimes \mathbb{Z}^{(m,n)}$$

be the arithmetic subgroup of $GL_{n,m}$, where $\mathbb{Z}$ is the ring of integers. The arithmetic quotient $\Gamma_{n,m}\backslash\mathcal{P}_{n,m}$ may be regarded as the universal family of principally polarized real tori of dimension mn (cf. [14]). We propose to name the space $\mathcal{P}_{n,m}$ the *Minkowski-Euclid space* since it was H. Minkowski [9] who found a fundamental domain for $\mathcal{P}_n$ with respect to the arithmetic subgroup $GL(n,\mathbb{Z})$ by means of the reduction theory. In this setting, using the invariant differential operators on $\mathcal{P}_{n,m}$, we can develop a theory of automorphic forms on $\mathcal{P}_{n,m}$ generalizing that on $\mathcal{P}_n$.

The aim of this paper is to study differential operators on $\mathcal{P}_{n,m}$ that are invariant under the action (1.2) of $GL_{n,m}$. This paper is organized as follows. In Section 2, we review differential operators on $\mathcal{P}_n$ invariant under the action (1.1) of $GL(n,\mathbb{R})$. In Section 3, we investigate differential operators on $\mathcal{P}_{n,m}$ invariant under the action (1.2) of $GL_{n,m}$. For two positive integers m and n, let

$$S_{n,m} = \Big\{(X,Z)\mid X = {}^tX \in \mathbb{R}^{(n,n)},\ Z \in \mathbb{R}^{(m,n)}\Big\}$$

be the real vector space of dimension $\frac{n(n+1)}{2} + mn$. From the adjoint action of the group $GL_{n,m}$, we have the *natural action* of the orthogonal group $O(n)$ on $S_{n,m}$ given by

$$k\cdot(X,Z) = (k\,X\,{}^tk,\, Z\,{}^tk),\quad k \in O(n),\ (X,Z) \in S_{n,m}. \tag{1.3}$$

The action (1.3) of $K = O(n)$ induces canonically the representation σ of $O(n)$ on the polynomial algebra $\mathrm{Pol}(S_{n,m})$ consisting of complex-valued polynomial functions on $S_{n,m}$. Let $\mathrm{Pol}(S_{n,m})^K$ denote the subalgebra of $\mathrm{Pol}(S_{n,m})$ consisting of all polynomials on $S_{n,m}$ invariant under the representation σ of $O(n)$, and $\mathbb{D}(\mathcal{P}_{n,m})$ denote the algebra of all differential operators on $\mathcal{P}_{n,m}$ invariant under the action (1.2) of $GL_{n,m}$. We see that there is a canonically defined linear bijection of $\text{Pol}(S_{n,m})^K$ onto $\mathbb{D}(\mathcal{P}_{n,m})$ which is not multiplicative. We will see that $\mathbb{D}(\mathcal{P}_{n,m})$ is *not* commutative. The most important problem here is in finding a complete list of explicit generators of $\text{Pol}(S_{n,m})^K$ and a complete list of explicit generators of $\mathbb{D}(\mathcal{P}_{n,m})$. We propose several natural problems. We present some explicit invariant differential operators which may be useful. In Section 4, we deal with the case when $n = 1$. In Section 5, we deal with the case when $n = 2$ and $m = 1, 2$. In Section 6, we deal with the case when $n = 3$ and $m = 1, 2$. In Section 7, we deal with the case when $n = 4$ and $m = 1, 2$. In the final section, we present some open problems and discuss a notion of automorphic forms on $\mathcal{P}_{n,m}$ using $GL_{n,m}$-invariant differential operators on the Minkowski-Euclid space $\mathcal{P}_{n,m}$.

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Notations. Denote by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the field of rational numbers, the field of real numbers and the field of complex numbers, respectively. Denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers, respectively. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\text{tr}(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transposed matrix of M. For a positive integer n, I_n denotes the identity matrix of degree n.

2. Review on invariant differential operators on $\mathcal{P}_n$

For a variable $Y = (y_{ij}) \in \mathcal{P}_n$, set

$$dY = (dy_{ij}) \quad \text{and} \quad \frac{\partial}{\partial Y} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial y_{ij}} \right),$$

where δ_{ij} denotes the Kronecker delta symbol.

For a fixed element $g \in GL(n, \mathbb{R})$, put

$$Y_* = g \cdot Y = gY\,{}^tg, \quad Y \in \mathcal{P}_n.$$

Then

$$dY_* = g\,dY\,{}^tg \quad \text{and} \quad \frac{\partial}{\partial Y_*} = {}^tg^{-1} \frac{\partial}{\partial Y}\, g^{-1}. \tag{2.1}$$

Consider the following differential operators

$$D_i = \text{tr}\left(\left(Y \frac{\partial}{\partial Y} \right)^i \right), \quad i = 1, 2, \ldots, n, \tag{2.2}$$

where $\mathrm{tr}(A)$ denotes the trace of a square matrix A. By Formula (2.1), we get

$$\left(Y_*\frac{\partial}{\partial Y_*}\right)^i = g\left(Y\frac{\partial}{\partial Y}\right)^i g^{-1}$$

for any $g \in GL(n,\mathbb{R})$. Hence each D_i is invariant under the action (1.1) of $GL(n,\mathbb{R})$.

Selberg [10] proved the following.

Theorem 2.1. *The algebra $\mathbb{D}(\mathcal{P}_n)$ of all differential operators on $\mathcal{P}_n$ invariant under the action* (1.1) *of $GL(n,\mathbb{R})$ is generated by $D_1, D_2, \ldots, D_n$. Furthermore, $D_1, D_2, \ldots, D_n$ are algebraically independent and $\mathbb{D}(\mathcal{P}_n)$ is isomorphic to the commutative ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$ with n indeterminates $x_1, x_2, \ldots, x_n$.*

Proof. The proof can be found in [4], p. 337, [8], pp. 64–66 and [11], pp. 29–30. The last statement follows immediately from the work of Harish-Chandra [1, 2] or [4], p. 294. ☐

Let $\mathfrak{g} = \mathbb{R}^{(n,n)}$ be the Lie algebra of $GL(n,\mathbb{R})$. The adjoint representation Ad of $GL(n,\mathbb{R})$ is given by

$$\mathrm{Ad}(g) = gXg^{-1}, \quad g \in GL(n,\mathbb{R}),\ X \in \mathfrak{g}.$$

The Killing form B of $\mathfrak{g}$ is given by

$$B(X,Y) = 2n\,\mathrm{tr}(XY) - 2\,\mathrm{tr}(X)\,\mathrm{tr}(Y), \quad X, Y \in \mathfrak{g}.$$

Since $B(aI_n, X) = 0$ for all $a \in \mathbb{R}$ and $X \in \mathfrak{g}$, B is degenerate. So the Lie algebra $\mathfrak{g}$ of $GL(n,\mathbb{R})$ is not semi-simple.

The Lie algebra $\mathfrak{k}$ of K is

$$\mathfrak{k} = \left\{ X \in \mathfrak{g} \mid X + {}^tX = 0 \right\}.$$

Let $\mathfrak{p}$ be the subspace of $\mathfrak{g}$ defined by

$$\mathfrak{p} = \left\{ X \in \mathfrak{g} \mid X = {}^tX \in \mathbb{R}^{(n,n)} \right\}.$$

Then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

is the direct sum of $\mathfrak{k}$ and $\mathfrak{p}$ with respect to the Killing form B. Since $\mathrm{Ad}(k)\mathfrak{p} \subset \mathfrak{p}$ for any $k \in K$, K acts on $\mathfrak{p}$ via the adjoint representation by

$$k \cdot X = \mathrm{Ad}(k)X = kX\,{}^tk, \quad k \in K,\ X \in \mathfrak{p}. \tag{2.3}$$

The action (2.3) induces the action of K on the polynomial algebra $\mathrm{Pol}(\mathfrak{p})$ of $\mathfrak{p}$ and the symmetric algebra $S(\mathfrak{p})$. Denote by $\mathrm{Pol}(\mathfrak{p})^K$ (resp., $S(\mathfrak{p})^K$) the subalgebra of $\mathrm{Pol}(\mathfrak{p})$ (resp., $S(\mathfrak{p})$) consisting of all K-invariants. The following inner product $(\ ,\ )$ on $\mathfrak{p}$ defined by

$$(X,Y) = B(X,Y), \quad X, Y \in \mathfrak{p}$$

gives an isomorphism as vector spaces

$$\mathfrak{p} \cong \mathfrak{p}^*, \quad X \mapsto f_X, \quad X \in \mathfrak{p}, \tag{2.4}$$

where $\mathfrak{p}^*$ denotes the dual space of $\mathfrak{p}$ and f_X is the linear functional on $\mathfrak{p}$ defined by

$$f_X(Y) = (Y, X), \quad Y \in \mathfrak{p}.$$

It is known that there is a canonical linear bijection of $S(\mathfrak{p})^K$ onto $\mathbb{D}(\mathcal{P}_n)$. Identifying $\mathfrak{p}$ with $\mathfrak{p}^*$ by the above isomorphism (2.4), we get a canonical linear bijection

$$\Theta_n : \mathrm{Pol}(\mathfrak{p})^K \longrightarrow \mathbb{D}(\mathcal{P}_n) \tag{2.5}$$

of $\mathrm{Pol}(\mathfrak{p})^K$ onto $\mathbb{D}(\mathcal{P}_n)$. The map Θ_n is described explicitly as follows. Put $N = n(n+1)/2$. Let $\{\xi_\alpha \mid 1 \le \alpha \le N\}$ be a basis of $\mathfrak{p}$. If $P \in \mathrm{Pol}(\mathfrak{p})^K$, then

$$\Big(\Theta_n(P)f\Big)(gK) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g\exp\left(\sum_{\alpha=1}^{N} t_\alpha \xi_\alpha\right)K\right)\right]_{(t_\alpha)=0}, \tag{2.6}$$

where $f \in C^\infty(\mathcal{P}_n)$. We refer the reader to [3, 4] for more detail. In general, it is difficult to express $\Theta_n(P)$ explicitly for a polynomial $P \in \mathrm{Pol}(\mathfrak{p})^K$.

Let

$$q_i(X) = \mathrm{tr}\big(X^i\big), \quad i = 1, 2, \ldots, n \tag{2.7}$$

be the polynomials on $\mathfrak{p}$. Here we take coordinates $x_{11}, x_{12}, \ldots, x_{nn}$ in $\mathfrak{p}$ given by

$$X = \begin{pmatrix} x_{11} & \frac{1}{2}x_{12} & \ldots & \frac{1}{2}x_{1n} \\ \frac{1}{2}x_{12} & x_{22} & \ldots & \frac{1}{2}x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}x_{1n} & \frac{1}{2}x_{2n} & \ldots & x_{nn} \end{pmatrix}.$$

For any $k \in K$,

$$(k \cdot q_i)(X) = q_i(k^{-1}Xk) = \mathrm{tr}(k^{-1}X^ik) = q_i(X), \quad i = 1, 2, \ldots, n.$$

Thus $q_i \in \mathrm{Pol}(\mathfrak{p})^K$ for $i = 1, 2, \ldots, n$. By a classical invariant theory (cf. [5, 12]), we can prove that the algebra $\mathrm{Pol}(\mathfrak{p})^K$ is generated by the polynomials $q_1, q_2, \ldots, q_n$ and that $q_1, q_2, \ldots, q_n$ are algebraically independent. Using Formula (2.6), we can show without difficulty that

$$\Theta_n(q_1) = \mathrm{tr}\left(2Y\frac{\partial}{\partial Y}\right).$$

However, $\Theta_n(q_i)$ $(i = 2, 3, \ldots, n)$ are yet known explicitly.

We propose the following conjecture.

Conjecture 1. For any n,

$$\Theta_n(q_i) = \operatorname{tr}\left(\left(2Y\frac{\partial}{\partial Y}\right)^i\right), \quad i = 1, 2, \ldots, n.$$

Remark. The author has verified that the above conjecture is true for $n = 1, 2$.

For a positive real number A,

$$ds^2_{n;A} = A \cdot \operatorname{tr}\big(Y^{-1}dY\, Y^{-1}dY\big)$$

is a Riemannian metric on $\mathcal{P}_n$ invariant under the action (1.1). The Laplacian $\Delta_{n;A}$ of $ds^2_{n;A}$ is given by

$$\Delta_{n;A} = \frac{1}{A}\operatorname{tr}\left(\left(Y\frac{\partial}{\partial Y}\right)^2\right).$$

For instance, consider the case when $n = 2$ and $A > 0$. If we write for $Y \in \mathcal{P}_2$,

$$Y = \begin{pmatrix} y_1 & y_3 \\ y_3 & y_2 \end{pmatrix} \quad \text{and} \quad \frac{\partial}{\partial Y} = \begin{pmatrix} \frac{\partial}{\partial y_1} & \frac{1}{2}\frac{\partial}{\partial y_3} \\ \frac{1}{2}\frac{\partial}{\partial y_3} & \frac{\partial}{\partial y_2} \end{pmatrix},$$

then

$$\begin{aligned} ds^2_{2;A} &= A\ \operatorname{tr}\big(Y^{-1}dY\, Y^{-1}dY\big) \\ &= \frac{A}{\left(y_1y_2 - y_3^2\right)^2}\ \Big\{ y_2^2\, dy_1^2\ +\ y_1^2\, dy_2^2\ +\ 2\left(y_1y_2 + y_3^2\right) dy_3^2 \\ &\qquad +\ 2\, y_3^2\ dy_1dy_2 - 4\, y_2\, y_3\ dy_1dy_3 - 4\, y_1y_3\ dy_2dy_3 \Big\} \end{aligned}$$

and its Laplacian $\Delta_{2;A}$ on $\mathcal{P}_2$ is

$$\begin{aligned} \Delta_{2;A} &= \frac{1}{A}\ \operatorname{tr}\left(\left(Y\frac{\partial}{\partial Y}\right)^2\right) \\ &= \frac{1}{A}\left\{ y_1^2\,\frac{\partial^2}{\partial y_1^2} + y_2^2\,\frac{\partial^2}{\partial y_2^2} + \frac{1}{2}(y_1y_2 + y_3^2)\frac{\partial^2}{\partial y_3^2} \right. \\ &\qquad + 2\left(y_3^2\frac{\partial^2}{\partial y_1\partial y_2} + y_1y_3\frac{\partial^2}{\partial y_1\partial y_3} + y_2y_3\frac{\partial^2}{\partial y_2\partial y_3}\right) \\ &\qquad \left. + \frac{3}{2}\left(y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} + y_3\frac{\partial}{\partial y_3}\right)\right\}. \end{aligned}$$

3. Invariant differential operators on $\mathcal{P}_{n,m}$

For a variable $(Y, V) \in \mathcal{P}_{n,m}$ with $Y \in \mathcal{P}_n$ and $V \in \mathbb{R}^{(m,n)}$, put

$$Y = (y_{ij}) \text{ with } y_{ij} = y_{ji}, \quad V = (v_{kl}),$$

$$dY = (dy_{ij}), \quad dV = (dv_{kl}),$$

$$[dY] = \wedge_{i\le j} dy_{ij}, \qquad [dV] = \wedge_{k,l} dv_{kl},$$

and

$$\frac{\partial}{\partial Y} = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial y_{ij}} \right), \quad \frac{\partial}{\partial V} = \left(\frac{\partial}{\partial v_{kl}} \right),$$

where $1 \le i, j, l \le n$ and $1 \le k \le m$.

For a fixed element $(g, \lambda) \in GL_{n,m}$, write

$$(Y_\star, V_\star) = (g, \lambda) \cdot (Y, V) = \big(g\, Y\, {}^t g, (V + \lambda)\, {}^t g \big),$$

where $(Y, V) \in \mathcal{P}_{n,m}$. Then we get

$$Y_\star = g\, Y\, {}^t g, \quad V_\star = (V + \lambda)\, {}^t g \tag{3.1}$$

and

$$\frac{\partial}{\partial Y_\star} = {}^t g^{-1} \frac{\partial}{\partial Y}\, g^{-1}, \quad \frac{\partial}{\partial V_\star} = \frac{\partial}{\partial V}\, g^{-1}. \tag{3.2}$$

Lemma 3.1. *For any two positive real numbers A and B, the following metric $ds^2_{n,m;A,B}$ on $\mathcal{P}_{n,m}$ defined by*

$$ds^2_{n,m;A,B} = A\,\sigma(Y^{-1} dY\, Y^{-1} dY) + B\,\sigma(Y^{-1}\, {}^t(dV)\, dV) \tag{3.3}$$

is a Riemannian metric on $\mathcal{P}_{n,m}$ which is invariant under the action (1.2) *of $GL_{n,m}$. The Laplacian $\Delta_{n,m;A,B}$ of $(\mathcal{P}_{n,m},\, ds^2_{n,m;A,B})$ is given by*

$$\Delta_{n,m;A,B} = \frac{1}{A}\,\sigma\left(\left(Y \frac{\partial}{\partial Y} \right)^2 \right) - \frac{m}{2\,A} \sigma \left(Y \frac{\partial}{\partial Y} \right) + \frac{1}{B} \sum_{k \le p} \left(\left(\frac{\partial}{\partial V} \right) Y \,{}^t\!\left(\frac{\partial}{\partial V} \right) \right)_{kp}.$$

Moreover, $\Delta_{n,m;A,B}$ is a differential operator of order 2 which is invariant under the action (1.2) *of $GL_{n,m}$.*

Proof. The proof can be found in [14]. □

Lemma 3.2. *The following volume element $dv_{n,m}(Y, V)$ on $\mathcal{P}_{n,m}$ defined by*

$$dv_{n,m}(Y, V) = (\det\, Y)^{-\frac{n+m+1}{2}} [dY][dV] \tag{3.4}$$

is invariant under the action (1.2) *of $GL_{n,m}$.*

Proof. The proof can be found in [14]. □

Theorem 3.1. *Any geodesic through the origin $(I_n, 0)$ for the Riemannian metric $ds^2_{n,m;1,1}$ is of the form*

$$\gamma(t) = \left(\lambda(2t)[k],\; Z \left(\int_0^t \lambda(t-s) ds \right) [k] \right),$$

where k is a fixed element of $O(n)$, Z is a fixed $h \times g$ real matrix, t is a real variable, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are fixed real numbers not all zero and

$$\lambda(t) := \mathrm{diag}\,(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}).$$

Furthermore, the tangent vector $\gamma'(0)$ of the geodesic $\gamma(t)$ at $(I_n, 0)$ is $(D[k], Z)$, where $D = \mathrm{diag}\,(2\lambda_1, \ldots, 2\lambda_n)$.

Proof. The proof can be found in [14]. □

Theorem 3.2. *Let (Y_0, V_0) and (Y_1, V_1) be two points in $\mathcal{P}_{n,m}$. Let g be an element in $GL(n,\mathbb{R})$ such that $Y_0[\,{}^tg] = I_n$ and $Y_1[\,{}^tg]$ is diagonal. Then the length $s\big((Y_0, V_0), (Y_1, V_1)\big)$ of the geodesic joining (Y_0, V_0) and (Y_1, V_1) for the $GL_{n,m}$-invariant Riemannian metric $ds^2_{n,m;A,B}$ is given by*

(3.5)
$$s\big((Y_0, V_0), (Y_1, V_1)\big) = A \left\{ \sum_{j=1}^{n} (\ln t_j)^2 \right\}^{1/2} + B \int_0^1 \left(\sum_{j=1}^{n} \Delta_j \, e^{-(\ln t_j)\, t} \right)^{1/2} dt,$$

where $\Delta_j = \sum_{k=1}^m \widetilde{v}_{kj}^2 \ (1 \leq j \leq n)$ with $(V_1 - V_0)\,{}^tg = (\widetilde{v}_{kj})$ and $t_1, \ldots, t_n$ denotes the zeros of $\det(t\,Y_0 - Y_1)$.

Proof. The proof can be found in [14]. □

The Lie algebra $\mathfrak{g}_\star$ of $GL_{n,m}$ is given by

$$\mathfrak{g}_\star = \Big\{ (X, Z) \mid X \in \mathbb{R}^{(n,n)},\ Z \in \mathbb{R}^{(m,n)} \Big\}$$

equipped with the following Lie bracket

$$\big[(X_1, Z_1),\, (X_2, Z_2)\big] = \big([X_1, X_2]_0,\, Z_2\,{}^tX_1 - Z_1\,{}^tX_2\big),$$

where $[X_1, X_2]_0 = X_1X_2 - X_2X_1$ denotes the usual matrix bracket and (X_1, Z_1), $(X_2, Z_2) \in \mathfrak{g}_\star$. The adjoint representation $\mathrm{Ad}_\star$ of $GL_{n,m}$ is given by

$$\mathrm{Ad}_\star\,((g,\lambda))(X, Z) = \big(\, gXg^{-1},\, (Z - \lambda\,{}^tX)\,{}^tg \,\big), \tag{3.6}$$

where $(g, \lambda) \in GL_{n,m}$ and $(X, Z) \in \mathfrak{g}_\star$. Also, the adjoint representation $\mathrm{ad}_\star$ of $\mathfrak{g}_\star$ on $\mathrm{End}\,(\mathfrak{g}_\star)$ is given by

$$\mathrm{ad}_\star\,\big((X, Z)\big)\big((X_1, Z_1)\big) = \big[(X, Z), (X_1, Z_1)\big].$$

We see that the Killing form $B_\star$ of $\mathfrak{g}_\star$ is given by

$$B_\star\big((X_1, Z_1), (X_2, Z_2)\big) = (2\,n + m)\,\mathrm{tr}(X_1X_2) - 2\,\mathrm{tr}(X_1)\,\mathrm{tr}(X_2).$$

The Lie algebra $\mathfrak{k}$ of K is

$$\mathfrak{k} = \Big\{ (X, 0) \in \mathfrak{g}_\star \,\Big|\, X + {}^tX = 0 \Big\}.$$

Let $\mathfrak{p}_\star$ be the subspace of $\mathfrak{g}_\star$ defined by

$$\mathfrak{p}_\star = \Big\{ (X, Z) \in \mathfrak{g}_\star \,\Big|\, X = {}^tX \in \mathbb{R}^{(n,n)},\ Z \in \mathbb{R}^{(m,n)} \Big\}.$$

Then we have the following relations

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k} \qquad \text{and} \qquad [\mathfrak{k}, \mathfrak{p}_\star] \subset \mathfrak{p}_\star.$$

In addition, we have

$$\mathfrak{g}_\star = \mathfrak{k} \oplus \mathfrak{p}_\star \qquad (\,\text{the direct sum}\,).$$

K acts on $\mathfrak{p}_\star$ via the adjoint representation $\mathrm{Ad}_\star$ of $GL_{n,m}$ by

$$k \cdot (X, Z) = \big(kX\,{}^t k, Z\,{}^t k\big), \quad k \in K,\ (X, Z) \in \mathfrak{p}_\star. \tag{3.7}$$

The action (3.7) induces the action of K on the polynomial algebra $\mathrm{Pol}(\mathfrak{p}_\star)$ of $\mathfrak{p}_\star$ and the symmetric algebra $S(\mathfrak{p}_\star)$. Denote by $\mathrm{Pol}(\mathfrak{p}_\star)^K$ (resp., $S(\mathfrak{p}_\star)^K$) the subalgebra of $\mathrm{Pol}(\mathfrak{p}_\star)$ (resp., $S(\mathfrak{p}_\star)$) consisting of all K-invariants. The following inner product $(\ ,\ )_\star$ on $\mathfrak{p}_\star$ defined by

$$\big((X_1, Z_1), (X_2, Z_2)\big)_\star = \mathrm{tr}(X_1 X_2) + \mathrm{tr}(Z_1\,{}^t Z_2), \quad (X_1, Z_1), (X_2, Y_2) \in \mathfrak{p}_\star$$

gives an isomorphism as vector spaces

$$\mathfrak{p}_\star \cong \mathfrak{p}_\star^*, \quad (X, Z) \mapsto f_{X,Z}, \quad (X, Z) \in \mathfrak{p}_\star, \tag{3.8}$$

where $\mathfrak{p}_\star^*$ denotes the dual space of $\mathfrak{p}_\star$ and $f_{X,Z}$ is the linear functional on $\mathfrak{p}_\star$ defined by

$$f_{X,Z}\big((X_1, Z_1)\big) = \big((X, Z), (X_1, Z_1)\big)_\star, \quad (X_1, Z_1) \in \mathfrak{p}_\star.$$

Let $\mathbb{D}(\mathcal{P}_{n,m})$ be the algebra of all differential operators on $\mathcal{P}_{n,m}$ that are invariant under the action (1.2) of $GL_{n,m}$. It is known that there is a canonical linear bijection of $S(\mathfrak{p}_\star)^K$ onto $\mathbb{D}(\mathcal{P}_{n,m})$. Identifying $\mathfrak{p}_\star$ with $\mathfrak{p}_\star^*$ by the above isomorphism (3.5), we get a canonical linear bijection

$$\Theta_{n,m} : \mathrm{Pol}(\mathfrak{p}_\star)^K \longrightarrow \mathbb{D}(\mathcal{P}_{n,m}) \tag{3.9}$$

of $\mathrm{Pol}(\mathfrak{p}_\star)^K$ onto $\mathbb{D}(\mathcal{P}_{n,m})$. The map $\Theta_{n,m}$ is described explicitly as follows. Put $N_\star = n(n+1)/2 + mn$. Let $\{\eta_\alpha \mid 1 \le \alpha \le N_\star\}$ be a basis of $\mathfrak{p}_\star$. If $P \in \mathrm{Pol}(\mathfrak{p}_\star)^K$, then

$$\Big(\Theta_{n,m}(P) f\Big)(gK) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g \exp\left(\sum_{\alpha=1}^{N_\star} t_\alpha \eta_\alpha \right) K \right) \right]_{(t_\alpha)=0}, \tag{3.10}$$

where $f \in C^\infty(\mathcal{P}_{n,m})$. We refer the reader to [4], pp. 280–289. In general, it is very hard to express $\Theta_{n,m}(P)$ explicitly for a polynomial $P \in \mathrm{Pol}(\mathfrak{p}_\star)^K$.

Take a coordinate (X, Z) in $\mathfrak{p}_\star$ such that

$$X = \begin{pmatrix} x_{11} & \frac{1}{2}x_{12} & \dots & \frac{1}{2}x_{1n} \\ \frac{1}{2}x_{12} & x_{22} & \dots & \frac{1}{2}x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}x_{1n} & \frac{1}{2}x_{2n} & \dots & x_{nn} \end{pmatrix} \in \mathfrak{p} \quad \text{and} \quad Z = (z_{kl}) \in \mathbb{R}^{(m,n)}.$$

Define the polynomials α_j, $\beta_{pq}^{(k)}$, R_{jp} and S_{jp} on $\mathfrak{p}_\star$ by

$$\alpha_j(X, Z) = \mathrm{tr}\big(X^j\big), \quad 1 \le j \le n, \tag{3.11}$$

$$\beta_{pq}^{(k)}(X, Z) = \big(Z\,X^k\,{}^t Z\big)_{pq}, \quad 0 \le k \le n-1,\ 1 \le p \le q \le m, \tag{3.12}$$

$$R_{jp}(X, Z) = \mathrm{tr}\big(X^j({}^t ZZ)^p\big), \quad 0 \le j \le n-1,\ 1 \le p \le m, \tag{3.13}$$

$$S_{jp}(X, Z) = \det\big(X^j({}^t ZZ)^p\big), \quad 0 \le j \le n-1,\ 1 \le p \le m, \tag{3.14}$$

where $\left(Z^tZ\right)_{pq}$ $\left(\text{resp.,}\left(ZX^tZ\right)_{pq}\right)$ denotes the (p,q)-entry of Z^tZ $\left(\text{resp., } ZX^tZ\right)$.

For any $m\times m$ real matrix S, define the polynomials $M_{j;S}$, $Q_{p;S}$, $\Omega_{i,p,j;S}$ and $\Theta_{i,p,j;S}$ on $\mathfrak{p}_\star$ by

$$M_{j;S}(X,Z) = \operatorname{tr}\big((X+{}^tZSZ)^j\big),\quad 1\le j\le n, \tag{3.15}$$

$$Q_{p;S}(X,Z) = \operatorname{tr}\big(({}^tZ\,S\,Z)^p\big),\quad 1\le p\le n, \tag{3.16}$$

$$\Omega_{i,p,j;S}(X,Z) = \operatorname{tr}\Big(X^i({}^tZSZ)^p(X+{}^tZSZ)^j\Big), \tag{3.17}$$

$$\Theta_{i,p,j;S}(X,Z) = \det\Big(X^i({}^tZSZ)^p(X+{}^tZSZ)^j\Big), \tag{3.18}$$

where $0\le i,j\le n-1,\ 1\le p\le n$. We see that all α_j, $\beta_{pq}^{(k)}$, R_{jp}, S_{jp}, $M_{j;S}$, $Q_{p;S}$, $\Omega_{i,p,j;S}$ and $\Theta_{i,p,j;S}$ are elements of $\mathrm{Pol}(\mathfrak{p}_\star)^K$.

We propose the following natural problems.

Problem 1. Find a complete list of explicit generators of $\mathrm{Pol}(\mathfrak{p}_\star)^K$.

Problem 2. Find all relations among a set of generators of $\mathrm{Pol}(\mathfrak{p}_\star)^K$.

Problem 3. Find an easy or an effective way to express explicitly the images of the above invariant polynomials under the Helgason map $\Theta_{n,m}$.

Problem 4. Decompose $\mathrm{Pol}(\mathfrak{p}_\star)^K$ into $O(n)$-irreducibles.

Problem 5. Find a complete list of explicit generators of the algebra $\mathbb{D}(\mathcal{P}_{n,m})$ or construct explicit $GL_{n,m}$-invariant differential operators on $\mathcal{P}_{n,m}$.

Problem 6. Find all relations among a set of generators of $\mathbb{D}(\mathcal{P}_{n,m})$.

Problem 7. Is $\mathrm{Pol}(\mathfrak{p}_\star)^K$ finitely generated? Is $\mathbb{D}(\mathcal{P}_{n,m})$ finitely generated?

M. Itoh [6] proved the following theorem.

Theorem 3.3. *$Pol(\mathfrak{p}_\star)^K$ is generated by α_j $(1\le j\le n)$ and $\beta_{pq}^{(k)}$ $(0\le k\le n-1,\ 1\le p\le q\le m)$.*

Proof. We refer the reader to Theorem 3.1 in [6]. □

M. Itoh solved Problem 2 in [6], Theorem 3.2.

We present some invariant differential operators on $\mathcal{P}_{n,m}$. Define the differential operators D_j, Ω_{pq} and L_p on $\mathcal{P}_{n,m}$ by

$$D_j = \operatorname{tr}\left(\left(2Y\frac{\partial}{\partial Y}\right)^j\right),\quad 1\le j\le n, \tag{3.19}$$

(3.20)

$$\Omega_{pq}^{(k)} = \left\{\frac{\partial}{\partial V}\left(2Y\frac{\partial}{\partial Y}\right)^k Y\, {}^t\!\left(\frac{\partial}{\partial V}\right)\right\}_{pq},\quad 0\le k\le n-1,\ 1\le p\le q\le m,$$

and

$$L_p = \operatorname{tr}\left(\left\{Y\,{}^t\!\left(\frac{\partial}{\partial V}\right)\frac{\partial}{\partial V}\right\}^p\right), \quad 1 \le p \le m. \tag{3.21}$$

Here, for a matrix A, we denote by A_{pq} the (p, q)-entry of A.

Also, define the differential operators S_{jp} by

$$S_{jp} = \operatorname{tr}\left(\left(2Y\frac{\partial}{\partial Y}\right)^j \left\{Y\,{}^t\!\left(\frac{\partial}{\partial V}\right)\frac{\partial}{\partial V}\right\}^p\right), \tag{3.22}$$

where $1 \le j \le n$ and $1 \le p \le m$.

For any real matrix S of degree m, define the differential operators $\Phi_{j;S},\ L_{p;S}$ and $\Phi_{i,p,j;S}$ by

$$\Phi_{j;S} = \operatorname{tr}\left(\left\{Y\left(2\frac{\partial}{\partial Y} + {}^t\!\left(\frac{\partial}{\partial V}\right)S\left(\frac{\partial}{\partial V}\right)\right)\right\}^j\right), \quad 1 \le j \le n, \tag{3.23}$$

$$L_{p;S} = \operatorname{tr}\left(\left\{Y\,{}^t\!\left(\frac{\partial}{\partial V}\right)S\left(\frac{\partial}{\partial V}\right)\right\}^p\right), \quad 1 \le p \le m \tag{3.24}$$

and

(3.25)

$$\begin{aligned}&\Phi_{i,p,j;S}(X,Z)\\ &= \operatorname{tr}\left(\left(2\,Y\frac{\partial}{\partial Y}\right)^i \left(Y\,{}^t\!\left(\frac{\partial}{\partial V}\right)S\left(\frac{\partial}{\partial V}\right)\right)^p \left\{Y\left(2\frac{\partial}{\partial Y} + {}^t\!\left(\frac{\partial}{\partial V}\right)S\left(\frac{\partial}{\partial V}\right)\right)\right\}^j\right).\end{aligned}$$

We want to mention a special invariant differential operator on $\mathcal{P}_{n,m}$. In [13], the author studied the following differential operator $M_{n,m,\mathcal{M}}$ on $\mathcal{P}_{n,m}$ defined by

$$M_{n,m,\mathcal{M}} = \det(Y)\cdot\det\left(\frac{\partial}{\partial Y} + \frac{1}{8\pi}\,{}^t\!\left(\frac{\partial}{\partial V}\right)\mathcal{M}^{-1}\left(\frac{\partial}{\partial V}\right)\right), \tag{3.26}$$

where $\mathcal{M}$ is a positive definite, symmetric half-integral matrix of degree m. This differential operator characterizes *singular Jacobi forms*. For more detail, we refer the reader to [13]. From (3.1) and (3.2), we can easily see that the differential operator $M_{n,m,\mathcal{M}}$ is invariant under the action (1.2) of $GL_{n,m}$.

Question. Calculate the inverse of $M_{n,m,\mathcal{M}}$ under the Helgason map $\Theta_{n,m}$.

4. The case when $n = 1$

In this section, we consider the case when $n = m = 1$ and the case when $n = 1$ and $m \ge 2$ separately.

4.1. The case when $n = 1$ and $m = 1$

In this case,

$$GL_{1,1} = \mathbb{R}^\times \ltimes \mathbb{R}, \quad K = O(1), \quad \mathcal{P}_{1,1} = \mathbb{R}^+ \times \mathbb{R},$$

where $\mathbb{R}^\times = \{a \in \mathbb{R} \,|\, a \neq 0\}$ and $\mathbb{R}^+ = \{a \in \mathbb{R} \,|\, a > 0\}$. Clearly, $\mathfrak{k} = 0$ and $\mathfrak{p}_\star = \mathfrak{g}_\star = \{(x,z) \,|\, x, z \in \mathbb{R}\}$. Then $e = (1,0)$ and $f = (0,1)$ form the standard basis for $\mathfrak{p}_\star$. Using this basis, we take a coordinate (x,z) in $\mathfrak{p}_\star$; that is, if $w \in \mathfrak{p}_\star$, then we write $w = xe + zf$. We can show that $\mathrm{Pol}(\mathfrak{p}_\star)^K$ is generated by the following polynomials

$$\alpha(x,z) = x \quad \text{and} \quad \beta(x,z) = z^2.$$

The generators α and β are *algebraically independent*. Let (y,v) be a coordinate in $\mathcal{P}_{1,1}$ with $y > 0$ and $v \in \mathbb{R}$. Then using Formula (3.10), we can show that

$$\Theta_{1,1}(\alpha) = 2y\,\frac{\partial}{\partial y} \quad \text{and} \quad \Theta_{1,1}(\beta) = y\frac{\partial^2}{\partial v^2}.$$

We see that $\Theta_{1,1}(\alpha)$ and $\Theta_{1,1}(\beta)$ generate the algebra $\mathbb{D}(\mathcal{P}_{1,1})$ and are *algebraically dependent*. Indeed, we have the following noncommutation relation

$$\Theta_{1,1}(\alpha)\Theta_{1,1}(\beta) - \Theta_{1,1}(\beta)\Theta_{1,1}(\alpha) = 2\,\Theta_{1,1}(\beta).$$

Hence the algebra $\mathbb{D}(\mathcal{P}_{1,1})$ is *not* commutative. The unitary dual $\widehat{K}$ of K consists of two elements. Let

$$\mathrm{Pol}(\mathfrak{p}_\star) = \sum_{\tau \in \widehat{K}} m_\tau \tau$$

be the decomposition of $\mathrm{Pol}(\mathfrak{p}_\star)$ into K-irreducibles. It is easy to see that the multiplicity m_τ of τ is infinite for all $\tau \in \widehat{K}$. So the action of K on $\mathrm{Pol}(\mathfrak{p}_\star)$ is not multiplicity-free. In this case, the seven problems proposed in Section 3 are completely solved.

4.2. The case when $n = 1$ and $m \geq 2$

Consider the case when $n = 1$ and $m \geq 2$. In this case,

$$GL_{1,m} = \mathbb{R}^\times \ltimes \mathbb{R}^{(m,1)}, \quad K = O(1), \quad \mathcal{P}_{1,m} = \mathbb{R}^+ \times \mathbb{R}^{(m,1)},$$

where $\mathbb{R}^\times = \{a \in \mathbb{R} \,|\, a \neq 0\}$ and $\mathbb{R}^+ = \{a \in \mathbb{R} \,|\, a > 0\}$. Clearly, $\mathfrak{k} = 0$ and $\mathfrak{p}_\star = \mathfrak{g}_\star = \big\{(x,z) \,|\, x \in \mathbb{R},\ z \in \mathbb{R}^{(m,1)}\big\}$. Let $\{e_1, \ldots, e_m\}$ be the standard basis of $\mathbb{R}^{(m,1)}$. Then

$$\eta_0 = (1,0),\ \eta_1 = (0,e_1),\ \eta_2 = (0,e_2), \ldots,\ \eta_m = (0,e_m)$$

form a basis of $\mathfrak{p}_\star$. Using this basis, we take a coordinate $(x, z_1, z_2, \ldots, z_m)$ in $\mathfrak{p}_\star$; that is, if $w \in \mathfrak{p}_\star$, then we write $w = x\eta_0 + \sum_{k=1}^m z_k \eta_k$. We can show that $\mathrm{Pol}(\mathfrak{p}_\star)^K$ is generated by the following polynomials

$$\alpha(x,z) = x \quad \text{and} \quad \beta_{kl}(x,z) = z_k\, z_l, \quad 1 \leq k \leq l \leq m,$$

where $z = (z_1, z_2, \ldots, z_m)$. We see easily that one has the following relations

$$\beta_{kk}\beta_{ll} = \beta_{kl}^2 \qquad \text{for } 1 \leq k < l \leq m$$

and

$$\beta_{kk}\beta_{ll}^2\beta_{pp} = \beta_{kl}^2\beta_{lp}^2 \qquad \text{for } 1 \le k < l < p \le m.$$

Therefore, the generators α and $\beta_{kl}\,(1 \le k \le l \le m)$ are *algebraically dependent.*

Let (y, v) be a coordinate in $\mathcal{P}_{1,m}$ with $y > 0$ and $v = {}^t(v_1, v_2, \ldots, v_m) \in \mathbb{R}^{(m,1)}$. Then using Formula (3.10), we can show that

$$\Theta_{1,m}(\alpha) = 2y\,\frac{\partial}{\partial y} \quad \text{and} \quad \Theta_{1,m}(\beta_{kl}) = y\,\frac{\partial^2}{\partial v_k \partial v_l}, \quad 1 \le k \le l \le m.$$

We see that $\Theta_{1,m}(\alpha)$ and $\Theta_{1,m}(\beta_{kl})\,(1 \le k \le l \le m)$ generate the algebra $\mathbb{D}(\mathcal{P}_{1,m})$. Although $\Theta_{1,m}(\beta_{kl})\,(1 \le k \le l \le m)$ commute with each other, $\Theta_{1,m}(\alpha)$ does not commute with any $\Theta_{1,m}(\beta_{kl})$. Indeed, we have the noncommutation relation

$$\Theta_{1,m}(\alpha)\Theta_{1,m}(\beta_{kl}) - \Theta_{1,m}(\beta_{kl})\Theta_{1,m}(\alpha) = 2\,\Theta_{1,m}(\beta_{kl}).$$

Hence the algebra $\mathbb{D}(\mathcal{P}_{1,m})$ is *not* commutative. It is easily seen that the action of K on $\mathrm{Pol}(\mathfrak{p}_\star)$ is *not* multiplicity-free.

5. The case when $n = 2$

In this section, we deal with the case when $n = 2,\ m = 1$ and the case when $n = m = 2$.

5.1. The case when $n = 2$ and $m = 1$

In this case,

$$GL_{2,1} = GL(2,\mathbb{R}) \ltimes \mathbb{R}^{(1,2)}, \quad K = O(2) \ \text{ and } \ GL_{2,1}/K = \mathcal{P}_2 \times \mathbb{R}^{(1,2)} = \mathcal{P}_{2,1}.$$

We see easily that

$$\mathfrak{p}_\star = \left\{ (X, Z) \mid X = {}^tX \in \mathbb{R}^{(2,2)}, \quad Z \in \mathbb{R}^{(1,2)} \right\}.$$

Put

$$e_1 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0 \right), \quad e_2 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 0 \right), \quad e_3 = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0 \right)$$

and

$$f_1 = (0, (1, 0)), \qquad f_2 = (0, (0, 1)).$$

Then $\{\, e_1,\, e_2,\, e_3,\, f_1,\, f_2 \,\}$ forms a basis for $\mathfrak{p}_\star$. For variables $(X, Z) \in \mathfrak{p}_\star$, write

$$X = \begin{pmatrix} x_1 & \frac{1}{2}x_3 \\ \frac{1}{2}x_3 & x_2 \end{pmatrix} \quad \text{and} \quad Z = (z_1, z_2).$$

The following polynomials

$$\alpha_1(X, Z) = \mathrm{tr}(X) = x_1 + x_2, \qquad \alpha_2(X, Z) = \mathrm{tr}(X^2) = x_1^2 + x_2^2 + \frac{1}{2}\,x_3^2,$$

$$\xi(X, Z) = Z\,{}^tZ = z_1^2 + z_2^2$$

and

$$\varphi(X,Z) = ZX\,{}^tZ = x_1\,z_1^2 + x_2\,z_2^2 + x_3\,z_1z_2$$

generate the algebra $\mathrm{Pol}(\mathfrak{p}_\star)^K$. We can show that the invariants α_1, α_2, ξ and φ are *algebraically independent*. We omit the detail.

Now we compute the $GL_{2,1}$-invariant differential operators D_1, D_2, Ψ, Δ on $\mathcal{P}_{2,1}$ corresponding to the K-invariants α_1, α_2, ξ, φ, respectively, under a canonical linear bijection

$$\Theta_{2,1}\,:\,\mathrm{Pol}(\mathfrak{p}_\star)^K \longrightarrow \mathbb{D}(\mathcal{P}_{2,1}).$$

For real variables $t=(t_1,t_2,t_3)$ and $s=(s_1,s_2)$, we have

$$\begin{aligned}&\exp\left(t_1e_1+t_2e_2+t_3e_3+s_1f_1+s_2f_2\right)\\ &=\left(\begin{pmatrix} a_1(t,s) & a_3(t,s)\\ a_3(t,s) & a_2(t,s)\end{pmatrix},\,(\,b_1(t,s),\,b_2(t,s)\,)\right),\end{aligned}$$

where

$$\begin{aligned}
a_1(t,s) &= 1+t_1+\frac{1}{2!}(t_1^2+t_3^2)+\frac{1}{3!}(t_1^3+2t_1t_3^2+t_2t_3^2)+\cdots,\\
a_2(t,s) &= 1+t_2+\frac{1}{2!}(t_2^2+t_3^2)+\frac{1}{3!}(t_1t_3^2+2t_2t_3^2+t_2^3)+\cdots,\\
a_3(t,s) &= t_3+\frac{1}{2!}(t_1+t_2)t_3+\frac{1}{3!}(t_1t_2+t_1^2+t_2^2+t_3^2)t_3+\cdots,\\
b_1(t,s) &= s_1-\frac{1}{2!}(s_1t_1+s_2t_3)+\frac{1}{3!}\left\{\,s_1(t_1^2+t_3^2)+s_2(t_1t_3+t_2t_3)\right\}-\cdots,\\
b_2(t,s) &= s_2-\frac{1}{2!}(s_1t_3+s_2t_2)+\frac{1}{3!}\left\{\,s_1(t_1+t_2)t_3+s_2(t_2^2+t_3^2)\right\}-\cdots.
\end{aligned}$$

For brevity, we write a_i, b_k for $a_i(t,s)$, $b_k(t,s)$ ($i=1,2,3,\ k=1,2$), respectively. We now fix an element $(g,c)\in GL_{2,1}$ and write

$$g=\begin{pmatrix} g_1 & g_{12}\\ g_{21} & g_2\end{pmatrix}\quad\text{and}\quad c=(c_1,c_2).$$

Put

$$\big(Y(t,s),\,V(t,s)\big)=\left((g,c)\cdot\exp\left(\sum_{i=1}^{3}t_ie_i+\sum_{k=1}^{2}s_kf_k\right)\right)\cdot(I_2,0)$$

with

$$Y(t,s)=\begin{pmatrix} y_1(t,s) & y_3(t,s)\\ y_3(t,s) & y_2(t,s)\end{pmatrix}\quad\text{and}\quad V(t,s)=(v_1(t,s),\,v_2(t,s)).$$

By an easy computation, we obtain

$$\begin{aligned}
y_1 &= (g_1a_1+g_{12}a_3)^2+(g_1a_3+g_{12}a_2)^2,\\
y_2 &= (g_{21}a_1+g_2a_3)^2+(g_{21}a_3+g_2a_2)^2,\\
y_3 &= (g_1a_1+g_{12}a_3)(g_{21}a_1+g_2a_3)+(g_1a_3+g_{12}a_2)(g_{21}a_3+g_2a_2),
\end{aligned}$$

$$v_1 = (c_1 + b_1 a_1 + b_2 a_3) g_1 + (c_2 + b_1 a_3 + b_2 a_2) g_{12},$$
$$v_2 = (c_1 + b_1 a_1 + b_2 a_3) g_{21} + (c_2 + b_1 a_3 + b_2 a_2) g_2.$$

Using the chain rule, we can easily compute the $GL_{2,1}$-invariant differential operators $D_1 = \Theta_{2,1}(\alpha_1)$, $D_2 = \Theta_{2,1}(\alpha_2)$, $\Psi = \Theta_{2,1}(\xi)$ and $\Delta = \Theta_{2,1}(\varphi)$. They are given by

$$\begin{aligned}
D_1 &= 2\,\mathrm{tr}\left(Y \frac{\partial}{\partial Y}\right) = 2\left(y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} + y_3 \frac{\partial}{\partial y_3}\right),\\
D_2 &= \mathrm{tr}\left(\left(2Y \frac{\partial}{\partial Y}\right)^2\right)\\
&= 3\,D_1 + 8\left(y_3^2 \frac{\partial^2}{\partial y_1 \partial y_2} + y_1 y_3 \frac{\partial^2}{\partial y_1 \partial y_3} + y_2 y_3 \frac{\partial^2}{\partial y_2 \partial y_3}\right)\\
&\quad + 4\left\{ y_1^2 \frac{\partial^2}{\partial y_1^2} + y_2^2 \frac{\partial^2}{\partial y_2^2} + \frac{1}{2}(y_1 y_2 + y_3^2)\frac{\partial^2}{\partial y_3^2}\right\},\\
\Psi &= \mathrm{tr}\left(Y\, {}^t\!\left(\frac{\partial}{\partial V}\right)\left(\frac{\partial}{\partial V}\right)\right)\\
&= y_1 \frac{\partial^2}{\partial v_1^2} + 2y_3 \frac{\partial^2}{\partial v_1 \partial v_2} + y_2 \frac{\partial^2}{\partial v_2^2}
\end{aligned}$$

and

$$\begin{aligned}
\Delta &= \frac{\partial}{\partial V}\left(2Y \frac{\partial}{\partial Y}\right) Y\, {}^t\!\left(\frac{\partial}{\partial V}\right)\\
&= 2\left(y_1^2 \frac{\partial^3}{\partial y_1 \partial v_1^2} + 2\,y_1 y_3 \frac{\partial^3}{\partial y_1 \partial v_1 \partial v_2} + y_3^2 \frac{\partial^3}{\partial y_1 \partial v_2^2}\right)\\
&\quad + 2\left(y_3^2 \frac{\partial^3}{\partial y_2 \partial v_1^2} + 2\,y_2\, y_3 \frac{\partial^3}{\partial y_2 \partial v_1 \partial v_2} + y_2^2 \frac{\partial^3}{\partial y_2 \partial v_2^2}\right)\\
&\quad + 2\left\{ y_1 y_3 \frac{\partial^3}{\partial y_3 \partial v_1^2} + (y_1 y_2 + y_3^2)\frac{\partial^3}{\partial y_3 \partial v_1 \partial v_2} + y_2 y_3 \frac{\partial^3}{\partial y_3 \partial v_2^2}\right\}\\
&\quad + 3\left(y_1 \frac{\partial^2}{\partial v_1^2} + 2y_3 \frac{\partial^2}{\partial v_1 \partial v_2} + y_2 \frac{\partial^2}{\partial v_2^2}\right).
\end{aligned}$$

Clearly, D_1 commutes with D_2 but Ψ does not commute with D_1 nor with D_2. Indeed, we have the following noncommutation relations

$$[D_1,\, \Psi] = D_1 \Psi - \Psi D_1 = 2\,\Psi$$

and

$$\begin{aligned}
[D_2,\, \Psi] &= D_2 \Psi - \Psi D_2\\
&= 2\,(\,2\,D_1\, - 1\,)\,\Psi - \,8 \det(Y) \cdot \det\left(\frac{\partial}{\partial Y} + {}^t\!\left(\frac{\partial}{\partial V}\right)\frac{\partial}{\partial V}\right)
\end{aligned}$$

$$+\ 8\det(Y)\cdot\det\left(\frac{\partial}{\partial Y}\right)\ -\ 4\left(y_1y_2+y_3^2\right)\frac{\partial^3}{\partial y_3\partial v_1\partial v_2}.$$

Hence the algebra $\mathbb{D}(\mathcal{P}_{2,1})$ is *not* commutative.

5.2. The case when $n=2$ and $m=2$

In this case,

$$GL_{2,2}=GL(2,\mathbb{R})\ltimes\mathbb{R}^{(2,2)},\quad K=O(2)\ \text{ and }\ GL_{2,2}/K=\mathcal{P}_2\times\mathbb{R}^{(2,2)}=\mathcal{P}_{2,2}.$$

We see easily that

$$\mathfrak{p}_\star=\left\{(X,Z)\mid X={}^tX\in\mathbb{R}^{(2,2)},\quad Z\in\mathbb{R}^{(2,2)}\right\}.$$

Let O_2 be the 2×2 zero matrix. Put

$$e_1=\left(\begin{pmatrix}1&0\\0&0\end{pmatrix},O_2\right),\quad e_2=\left(\begin{pmatrix}0&0\\0&1\end{pmatrix},O_2\right),\quad e_3=\left(\begin{pmatrix}0&\frac{1}{2}\\\frac{1}{2}&0\end{pmatrix},O_2\right)$$

and

$$f_1=\left(O_2,\begin{pmatrix}1&0\\0&0\end{pmatrix}\right),\quad f_2=\left(O_2,\begin{pmatrix}0&1\\0&0\end{pmatrix}\right),$$
$$f_3=\left(O_2,\begin{pmatrix}0&0\\1&0\end{pmatrix}\right),f_4=\left(O_2,\begin{pmatrix}0&0\\0&1\end{pmatrix}\right).$$

Then $\{\,e_1,\,e_2,\,e_3,\,f_1,\,f_2,\,f_3,\,f_4\,\}$ forms a basis for $\mathfrak{p}_\star$. For variables $(X,Z)\in\mathfrak{p}_\star$, write

$$X=\begin{pmatrix}x_1&\frac{1}{2}x_3\\\frac{1}{2}x_3&x_2\end{pmatrix}\quad\text{and}\quad Z=\begin{pmatrix}z_{11}&z_{12}\\z_{21}&z_{22}\end{pmatrix}.$$

From Theorem 3.3, the algebra $\mathrm{Pol}(\mathfrak{p}_\star)^K$ is generated by the following polynomials

$$\alpha_1(X,Z)=\mathrm{tr}(X)=x_1\,+\,x_2,$$
$$\alpha_2(X,Z)=\mathrm{tr}(X^2)=x_1^2\,+\,x_2^2\,+\,\frac{1}{2}\,x_3^2,$$
$$\beta_{11}^{(0)}(X,Z)=(Z\,{}^tZ)_{11}=\,z_{11}^2\,+\,z_{12}^2,$$
$$\beta_{12}^{(0)}(X,Z)=(Z\,{}^tZ)_{12}=\,z_{11}z_{21}+z_{12}z_{22},$$
$$\beta_{22}^{(0)}(X,Z)=(Z\,{}^tZ)_{22}=\,z_{21}^2\,+\,z_{22}^2,$$
$$\beta_{11}^{(1)}(X,Z)=(ZX\,{}^tZ)_{11}=\,x_1z_{11}^2\,+\,x_2z_{12}^2+x_3z_{11}z_{12},$$
$$\beta_{12}^{(1)}(X,Z)=(ZX\,{}^tZ)_{12}=\,x_1z_{11}z_{21}+x_2z_{12}z_{22}+\frac{1}{2}x_3(z_{11}z_{22}+z_{12}z_{21}),$$
$$\beta_{22}^{(1)}(X,Z)=(ZX\,{}^tZ)_{22}=\,x_1z_{21}^2\,+\,x_2z_{22}^2+x_3z_{21}z_{22}.$$

Set

$$\Delta_{ab} := \det \begin{pmatrix} \beta_{11}^{(a)} & \beta_{12}^{(b)} \\ \beta_{12}^{(a)} & \beta_{22}^{(b)} \end{pmatrix} \qquad \text{for } a, b = 0, 1.$$

By a direct computation, we can show that the following equation

$$\alpha_1 \Delta_{00} - \Delta_{01} - \Delta_{10} = 0 \tag{5.1}$$

holds.

We take a coordinate (Y, V) in $\mathcal{P}_{2,2}$, that is,

$$Y = \begin{pmatrix} y_1 & y_3 \\ y_3 & y_2 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}.$$

Put

$$\frac{\partial}{\partial Y} = \begin{pmatrix} \frac{\partial}{\partial y_1} & \frac{1}{2}\frac{\partial}{\partial y_3} \\ \frac{1}{2}\frac{\partial}{\partial y_3} & \frac{\partial}{\partial y_2} \end{pmatrix} \quad \text{and} \quad \frac{\partial}{\partial V} = \begin{pmatrix} \frac{\partial}{\partial v_{11}} & \frac{\partial}{\partial v_{12}} \\ \frac{\partial}{\partial v_{21}} & \frac{\partial}{\partial v_{22}} \end{pmatrix}.$$

Consider the following differential operators

$$D_i := \operatorname{tr}\left(\left(2\,Y \frac{\partial}{\partial Y} \right)^i \right), \quad i = 1, 2$$

and

$$\Omega_{pq}^{(k)} = \left\{ \frac{\partial}{\partial V} \left(2Y \frac{\partial}{\partial Y} \right)^k Y \, {}^t\!\left(\frac{\partial}{\partial V} \right) \right\}_{pq}, \quad k = 0, 1, \ 1 \le p \le q \le 2.$$

Note that D_1, D_2, $\Omega_{11}^{(0)}, \ldots, \Omega_{22}^{(1)}$ are $GL_{2,2}$-invariant. For brevity, we put

$$\partial_{ij} = \frac{\partial}{\partial v_{ij}}, \quad i, j = 1, 2.$$

It is easily seen that

$$\begin{aligned}
D_1 &= \operatorname{tr}\left(2\,Y \frac{\partial}{\partial Y} \right) = 2 \sum_{i=1}^{3} y_i \frac{\partial}{\partial y_i}, \\
D_2 &= 3\,D_1 + 8 \left(y_3^2 \frac{\partial^2}{\partial y_1 \partial y_2} + y_1 y_3 \frac{\partial^2}{\partial y_1 \partial y_3} + y_2 y_3 \frac{\partial^2}{\partial y_2 \partial y_3} \right) \\
&\quad + 4 \left\{ y_1^2 \frac{\partial^2}{\partial y_1^2} + y_2^2 \frac{\partial^2}{\partial y_2^2} + \frac{1}{2}\big(y_1 y_2 + y_3^2\big) \frac{\partial^2}{\partial y_3^2} \right\}, \\
\Omega_{11}^{(0)} &= y_1\, \partial_{11}^2 + y_2\, \partial_{12}^2 + 2\, y_3\, \partial_{11} \partial_{12}, \\
\Omega_{12}^{(0)} &= y_1\, \partial_{11}\partial_{21} + y_2\, \partial_{12}\partial_{22} + y_3\, \left(\partial_{11}\partial_{22} + \partial_{12}\partial_{21} \right), \\
\Omega_{22}^{(0)} &= y_1\, \partial_{21}^2 + y_2\, \partial_{22}^2 + 2\, y_3\, \partial_{21}\partial_{22}.
\end{aligned}$$

Then by a direct computation, we have the following relations

$$[D_1, D_2] = 0, \tag{5.2}$$

$$(5.3) \qquad [\Omega_{kl}^{(0)}, \Omega_{pq}^{(0)}] = 0, \quad 1 \leq k \leq l \leq 2,\ 1 \leq p \leq q \leq 2,$$

$$(5.4) \qquad [D_1, \Omega_{11}^{(0)}] = 2\,\Omega_{11}^{(0)}, \quad [D_1, \Omega_{12}^{(0)}] = 2\,\Omega_{12}^{(0)}, \quad [D_1, \Omega_{22}^{(0)}] = 2\,\Omega_{22}^{(0)}.$$

Therefore, $\mathbb{D}(\mathcal{P}_{2,2})$ is not commutative.

6. The case when $n = 3$

6.1. The case when $n = 3$ and $m = 1$

In this case,

$GL_{3,1} = GL(3,\mathbb{R}) \ltimes \mathbb{R}^{(1,3)}, \quad K = O(3)$ and $GL_{3,1}/K = \mathcal{P}_3 \times \mathbb{R}^{(1,3)} = \mathcal{P}_{3,1}.$

We see easily that

$$\mathfrak{p}_\star = \left\{ (X, Z) \mid X = {}^tX \in \mathbb{R}^{(3,3)}, \quad Z \in \mathbb{R}^{(1,3)} \right\}.$$

Put

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_5 = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}, \quad E_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

Let O_3 be the 3×3 zero matrix and let $O_{1,3} = (0,0,0) \in \mathbb{R}^{(1,3)}$. Put

$$e_i = (E_i, O_{1,3}), \quad 1 \leq i \leq 6,$$
$$f_1 = (O_3, (1,0,0)), \quad f_2 = (O_3, (0,1,0)), \quad f_3 = (O_3, (0,0,1)).$$

Then $\{\, e_i,\ f_j \,|\, 1 \leq i \leq 6,\ 1 \leq j \leq 3 \,\}$ forms a basis for $\mathfrak{p}_\star$. Using this basis, we write for variables $(X, Z) \in \mathfrak{p}_\star$,

$$X = \begin{pmatrix} x_1 & \frac{1}{2}x_4 & \frac{1}{2}x_5 \\ \frac{1}{2}x_4 & x_2 & \frac{1}{2}x_6 \\ \frac{1}{2}x_5 & \frac{1}{2}x_6 & x_3 \end{pmatrix} \quad \text{and} \quad Z = (z_1, z_2, z_3).$$

From Theorem 3.3, the algebra $\mathrm{Pol}(\mathfrak{p}_\star)^K$ is generated by the following polynomials

$$\alpha_1(X,Z) = x_1 + x_2 + x_3,$$
$$\alpha_2(X,Z) = x_1^2 + x_2^2 + x_3^2 + \frac{1}{2}\left(x_4^2 + x_5^2 + x_6^2\right),$$
$$\alpha_3(X,Z) = x_1^3 + x_2^3 + x_3^3 + \frac{3}{4}\ \left\{ (x_1 + x_2)x_4^2 + (x_1 + x_3)x_5^2 + (x_2 + x_3)x_6^2 \right\}$$
$$+\ \frac{3}{4}\,x_4 x_5 x_6,$$
$$\beta_0(X,Z) = z_1^2 + z_2^2 + z_3^2,$$

$$\beta_1(X,Z) = x_1 z_1^2 + x_2 z_2^2 + x_3 z_3^2 + x_4 z_1 z_2 + x_5 z_1 z_3 + x_6 z_2 z_3,$$

$$\begin{aligned}\beta_2(X,Z) = x_1^2 z_1^2 + x_2^2 z_2^2 + \frac{1}{4}\left\{(x_4^2+x_5^2)z_1^2 + (x_4^2+x_6^2)z_2^2 + (x_5^2+x_6^2)z_3^2\right\} \\ + \left(x_1x_4 + x_2x_4 + \frac{1}{2}x_5x_6\right) z_1z_2 + \left(x_1x_5 + x_3x_5 + \frac{1}{2}x_4x_6\right) z_1z_3 \\ + \left(x_2x_6 + x_3x_6 + \frac{1}{2}x_4x_5\right) z_2z_3.\end{aligned}$$

We take a coordinate (Y,V) in $\mathcal{P}_{3,1}$, that is,

$$Y = \begin{pmatrix} y_1 & y_4 & y_5 \\ y_4 & y_2 & y_6 \\ y_5 & y_6 & y_3 \end{pmatrix} \quad \text{and} \quad V = (v_1, v_2, v_3).$$

Put

$$\frac{\partial}{\partial Y} = \begin{pmatrix} \frac{\partial}{\partial y_1} & \frac{1}{2}\frac{\partial}{\partial y_4} & \frac{1}{2}\frac{\partial}{\partial y_5} \\ \frac{1}{2}\frac{\partial}{\partial y_4} & \frac{\partial}{\partial y_2} & \frac{1}{2}\frac{\partial}{\partial y_6} \\ \frac{1}{2}\frac{\partial}{\partial y_5} & \frac{1}{2}\frac{\partial}{\partial y_6} & \frac{\partial}{\partial y_3} \end{pmatrix} \quad \text{and} \quad \frac{\partial}{\partial V} = \left(\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}, \frac{\partial}{\partial v_3}\right).$$

Consider the following differential operators

$$D_i := \operatorname{tr}\left(\left(2\,Y\frac{\partial}{\partial Y}\right)^i\right), \quad i = 1,2,3$$

and

$$\Omega_k = \frac{\partial}{\partial V}\left(2Y\frac{\partial}{\partial Y}\right)^k Y\,{}^t\!\left(\frac{\partial}{\partial V}\right), \quad k = 0,1,2.$$

Note that D_1, D_2, D_3, Ω_0, Ω_1 and Ω_2 are $GL_{2,2}$-invariant. It is easily seen that

$$D_1 = \operatorname{tr}\left(2\,Y\frac{\partial}{\partial Y}\right) = 2\sum_{i=1}^{6} y_i \frac{\partial}{\partial y_i},$$

$$\begin{aligned}\Omega_0 = y_1\,\frac{\partial^2}{\partial v_1^2} + y_2\,\frac{\partial^2}{\partial v_2^2} + y_3\,\frac{\partial^2}{\partial v_3^2} \\ + 2\,y_4\,\frac{\partial^2}{\partial v_1 \partial v_2} + 2\,y_5\,\frac{\partial^2}{\partial v_1 \partial v_3} + 2\,y_6\,\frac{\partial^2}{\partial v_2 \partial v_3}.\end{aligned}$$

Then we have the following relations

(6.1) $$[D_i, D_j] = 0 \quad \text{for all } i,j = 1,2,3$$

and

(6.2) $$[D_1, \Omega_0] = 2\,\Omega_0.$$

Therefore, $\mathbb{D}(\mathcal{P}_{3,1})$ is not commutative.

6.2. The case when $n = 3$ and $m = 2$

In this case,

$$GL_{3,2} = GL(3,\mathbb{R}) \ltimes \mathbb{R}^{(2,3)}, \quad K = O(3) \ \text{ and } \ GL_{3,2}/K = \mathcal{P}_3 \times \mathbb{R}^{(2,3)} = \mathcal{P}_{3,2}.$$

We see easily that

$$\mathfrak{p}_\star = \left\{ (X,Z) \mid X = {}^tX \in \mathbb{R}^{(3,3)}, \quad Z \in \mathbb{R}^{(2,3)} \right\}.$$

Put

$$\begin{aligned} E_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ E_4 &= \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_5 = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}, \quad E_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}. \end{aligned}$$

and

$$\begin{aligned} F_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\ F_4 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad F_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad F_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Let O_3 be the 3×3 zero matrix and let

$$O_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{(2,3)}.$$

Put

$$e_i = (E_i, O_{2,3}), \quad f_j = (O_3, F_j) \quad 1 \le i,j \le 6.$$

Then $\{ e_i,\ f_j \,|\, 1 \le i,j \le 6 \}$ forms a basis for $\mathfrak{p}_\star$. Using this basis, we write for variables $(X,Z) \in \mathfrak{p}_\star$,

$$X = \begin{pmatrix} x_1 & \frac{1}{2}x_4 & \frac{1}{2}x_5 \\ \frac{1}{2}x_4 & x_2 & \frac{1}{2}x_6 \\ \frac{1}{2}x_5 & \frac{1}{2}x_6 & x_3 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \end{pmatrix}.$$

From Theorem 3.3, the algebra $\mathrm{Pol}(\mathfrak{p}_\star)^K$ is generated by the following polynomials

$$\begin{aligned} \alpha_1(X,Z) &= x_1 + x_2 + x_3, \\ \alpha_2(X,Z) &= x_1^2 + x_2^2 + x_3^2 + \frac{1}{2}\big(x_4^2 + x_5^2 + x_6^2\big), \\ \alpha_3(X,Z) &= x_1^3 + x_2^3 + x_3^3 + \frac{3}{4}\ \big\{(x_1+x_2)x_4^2 + (x_1+x_3)x_5^2 + (x_2+x_3)x_6^2\big\} \\ &\quad + \frac{3}{4}\, x_4x_5x_6, \\ \beta_{11}^{(0)}(X,Z) &= z_{11}^2 + z_{12}^2 + z_{13}^2, \\ \beta_{12}^{(0)}(X,Z) &= z_{11}z_{21} + z_{12}z_{22} + z_{13}z_{23}, \end{aligned}$$

$$\beta_{22}^{(0)}(X,Z) = z_{21}^2 + z_{22}^2 + z_{23}^2,$$
$$\beta_{11}^{(1)}(X,Z) = x_1 z_{11}^2 + x_2 z_{12}^2 + x_3 z_{13}^2 + x_4 z_{11} z_{12} + x_5 z_{11} z_{13} + x_6 z_{12} z_{13},$$
$$\begin{aligned}\beta_{12}^{(1)}(X,Z) = {} & x_1 z_{11} z_{21} + x_2 z_{12} z_{22} + x_3 z_{13} z_{23} + \frac{1}{2} x_4 (z_{11} z_{22} + z_{12} z_{21}) \\ & + \frac{1}{2} x_5 (z_{11} z_{23} + z_{13} z_{21}) + \frac{1}{2} x_6 (z_{12} z_{23} + z_{13} z_{22}),\end{aligned}$$
$$\beta_{22}^{(1)}(X,Z) = x_1 z_{21}^2 + x_2 z_{22}^2 + x_3 z_{23}^2 + x_4 z_{21} z_{22} + x_5 z_{21} z_{23} + x_6 z_{22} z_{23},$$
$$\begin{aligned}\beta_{11}^{(2)}(X,Z) = {} & x_1^2 z_{11}^2 + x_2^2 z_{12}^2 + x_3^2 z_{13}^2 \\ & + \frac{1}{4} \left\{ x_4^2 (z_{11}^2 + z_{12}^2) + x_5^2 (z_{11}^2 + z_{13}^2) + x_6^2 (z_{12}^2 + z_{13}^2) \right\} \\ & + (x_1 + x_2) x_4 z_{11} z_{12} + (x_1 + x_3) x_5 z_{11} z_{13} + (x_2 + x_3) x_6 z_{12} z_{13} \\ & + \frac{1}{2} \left(x_4 x_5 z_{12} z_{13} + x_4 x_6 z_{11} z_{13} + x_5 x_6 z_{11} z_{12} \right),\end{aligned}$$
$$\begin{aligned}\beta_{12}^{(2)}(X,Z) = {} & x_1^2 z_{11} z_{21} + x_2^2 z_{12} z_{22} + x_3^2 z_{13} z_{23} \\ & + \frac{1}{4} \left\{ \left(x_4^2 + x_5^2\right) z_{11} z_{21} + \left(x_4^2 + x_6^2\right) z_{12} z_{22} + \left(x_5^2 + x_6^2\right) z_{13} z_{23} \right\} \\ & + \frac{1}{2} \left(x_1 x_4 + x_2 x_4 + \frac{1}{2} x_5 x_6 \right) (z_{11} z_{22} + z_{12} z_{21}) \\ & + \frac{1}{2} \left(x_1 x_5 + x_3 x_5 + \frac{1}{2} x_4 x_6 \right) (z_{11} z_{23} + z_{13} z_{21}) \\ & + \frac{1}{2} \left(x_2 x_6 + x_3 x_6 + \frac{1}{2} x_4 x_5 \right) (z_{12} z_{23} + z_{13} z_{22}),\end{aligned}$$
$$\begin{aligned}\beta_{22}^{(2)}(X,Z) = {} & x_1^2 z_{21}^2 + x_2^2 z_{22}^2 + x_3^2 z_{23}^2 \\ & + \frac{1}{4} \left\{ x_4^2 (z_{21}^2 + z_{22}^2) + x_5^2 (z_{21}^2 + z_{23}^2) + x_6^2 (z_{22}^2 + z_{23}^2) \right\} \\ & + (x_1 + x_2) x_4 z_{21} z_{22} + (x_1 + x_3) x_5 z_{21} z_{23} + (x_2 + x_3) x_6 z_{22} z_{23} \\ & + \frac{1}{2} \left(x_4 x_5 z_{22} z_{23} + x_4 x_6 z_{21} z_{23} + x_5 x_6 z_{21} z_{22} \right).\end{aligned}$$

Set

$$\Delta_{ab} := \det \begin{pmatrix} \beta_{11}^{(a)} & \beta_{12}^{(b)} \\ \beta_{12}^{(a)} & \beta_{22}^{(b)} \end{pmatrix} \qquad \text{for } a, b = 0, 1, 2.$$

By a direct computation, we can show that

$$(\alpha_1^2 - \alpha_2)\,\Delta_{00} - 2\,\alpha_1 \left(\Delta_{01} + \Delta_{10}\right) + 2 \left(\Delta_{02} + \Delta_{11} + \Delta_{20}\right) = 0. \tag{6.3}$$

We take a coordinate (Y, V) in $\mathcal{P}_{3,2}$, that is,

$$Y = \begin{pmatrix} y_1 & y_4 & y_5 \\ y_4 & y_2 & y_6 \\ y_5 & y_6 & y_3 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \end{pmatrix}.$$

Put

$$\frac{\partial}{\partial Y}=\begin{pmatrix} \frac{\partial}{\partial y_1} & \frac{1}{2}\frac{\partial}{\partial y_4} & \frac{1}{2}\frac{\partial}{\partial y_5} \\ \frac{1}{2}\frac{\partial}{\partial y_4} & \frac{\partial}{\partial y_2} & \frac{1}{2}\frac{\partial}{\partial y_6} \\ \frac{1}{2}\frac{\partial}{\partial y_5} & \frac{1}{2}\frac{\partial}{\partial y_6} & \frac{\partial}{\partial y_3} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{\partial}{\partial v_{11}} & \frac{\partial}{\partial v_{12}} & \frac{\partial}{\partial v_{13}} \\ \frac{\partial}{\partial v_{21}} & \frac{\partial}{\partial v_{22}} & \frac{\partial}{\partial v_{23}} \end{pmatrix}.$$

Consider the following differential operators

$$D_i := \operatorname{tr}\left(\left(2\,Y\frac{\partial}{\partial Y}\right)^i\right), \quad i=1,2,3$$

and

$$\Omega_{pq}^{(k)}=\left\{\frac{\partial}{\partial V}\left(2Y\frac{\partial}{\partial Y}\right)^k\,{}^tY\left(\frac{\partial}{\partial V}\right)\right\}_{pq}, \quad k=0,1,2,\ 1\le p\le q\le 2.$$

Note that D_1, D_2, D_3, $\Omega_{11}^{(0)},\dots,\Omega_{22}^{(2)}$ are $GL_{3,2}$-invariant. For brevity, we put

$$\partial_{ij}=\frac{\partial}{\partial v_{ij}}, \quad i=1,2,\ j=1,2,3.$$

It is easily seen that

$$\begin{aligned} D_1 &= \operatorname{tr}\left(2\,Y\frac{\partial}{\partial Y}\right)=2\sum_{i=1}^{6} y_i\frac{\partial}{\partial y_i}, \\ \Omega_{11}^{(0)} &= y_1\,\partial_{11}^2+\,y_2\,\partial_{12}^2+\,y_3\,\partial_{13}^2+2\,y_4\,\partial_{11}\partial_{12}+2\,y_5\,\partial_{11}\partial_{13}+2\,y_6\,\partial_{12}\partial_{13}, \\ \Omega_{12}^{(0)} &= y_1\,\partial_{11}\partial_{21}+y_2\,\partial_{12}\partial_{22}+y_3\,\partial_{13}\partial_{23}+y_4\left(\partial_{11}\partial_{22}+\partial_{12}\partial_{21}\right) \\ &\quad +y_5\left(\partial_{11}\partial_{23}+\partial_{13}\partial_{21}\right)+y_6\left(\partial_{12}\partial_{23}+\partial_{13}\partial_{22}\right), \\ \Omega_{22}^{(0)} &= y_1\,\partial_{21}^2+\,y_2\,\partial_{22}^2+\,y_3\,\partial_{23}^2+2\,y_4\,\partial_{21}\partial_{22}+2\,y_5\,\partial_{21}\partial_{23}+2\,y_6\,\partial_{22}\partial_{23}. \end{aligned}$$

Then we have the following relations

$$[D_i,D_j]=0 \quad \text{for all } i,j=1,2,3, \tag{6.4}$$

$$[\Omega_{kl}^{(0)},\Omega_{pq}^{(0)}]=0, \quad 1\le k\le l\le 2,\ 1\le p\le q\le 2 \tag{6.5}$$

and

$$[D_1,\Omega_{11}^{(0)}]=2\,\Omega_{11}^{(0)}, \quad [D_1,\Omega_{12}^{(0)}]=2\,\Omega_{12}^{(0)}, \quad [D_1,\Omega_{22}^{(0)}]=2\,\Omega_{22}^{(0)}. \tag{6.6}$$

Therefore, $\mathbb{D}(\mathcal{P}_{3,2})$ is not commutative.

7. The case when $n=4$

6.1. The case when $n=4$ and $m=1$

In this case,

$$GL_{4,1}=GL(4,\mathbb{R})\ltimes\mathbb{R}^{(1,4)}, \quad K=O(4) \ \text{ and } \ GL_{4,1}/K=\mathcal{P}_4\times\mathbb{R}^{(1,4)}=\mathcal{P}_{4,1}.$$

We see easily that

$$\mathfrak{p}_\star = \left\{ (X,Z) \mid X = {}^tX \in \mathbb{R}^{(4,4)}, \quad Z \in \mathbb{R}^{(1,4)} \right\}.$$

Put

$$\begin{aligned}
E_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
E_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ E_5 = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ E_6 = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
E_7 &= \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}, \ E_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
E_9 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}, \ E_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}.
\end{aligned}$$

Let O_4 be the 4×4 zero matrix and let $O_{1,4} = (0,0,0,0) \in \mathbb{R}^{(1,4)}$. Put

$$\begin{aligned}
&e_i = (E_i, O_{1,4}), \quad 1 \le i \le 10, \\
&f_1 = (O_4, (1,0,0,0)), \ f_2 = (O_4, (0,1,0,0)), \\
&f_3 = (O_4, (0,0,1,0)), \ f_4 = (O_4, (0,0,0,1)).
\end{aligned}$$

Then $\{ e_i, \ f_j \,|\, 1 \le i \le 10, \ 1 \le j \le 4 \}$ forms a basis for $\mathfrak{p}_\star$. Using this basis, we write for variables $(X,Z) \in \mathfrak{p}_\star$,

$$X = \begin{pmatrix} x_1 & \frac{1}{2}x_5 & \frac{1}{2}x_6 & \frac{1}{2}x_7 \\ \frac{1}{2}x_5 & x_2 & \frac{1}{2}x_8 & \frac{1}{2}x_9 \\ \frac{1}{2}x_6 & \frac{1}{2}x_8 & x_3 & \frac{1}{2}x_{10} \\ \frac{1}{2}x_7 & \frac{1}{2}x_9 & \frac{1}{2}x_{10} & x_4 \end{pmatrix} \quad \text{and} \quad Z = (z_1, z_2, z_3, z_4).$$

Put

$$A = x_1^2 + \frac{1}{4}\,x_5^2 + \frac{1}{4}\,x_6 + \frac{1}{4}\,x_7^2, \tag{7.1}$$

$$B = x_2^2 + \frac{1}{4}\,x_5^2 + \frac{1}{4}\,x_8 + \frac{1}{4}\,x_9^2, \tag{7.2}$$

$$C = x_3^2 + \frac{1}{4}\,x_6^2 + \frac{1}{4}\,x_8 + \frac{1}{4}\,x_{10}^2, \tag{7.3}$$

$$D = x_4^2 + \frac{1}{4}\,x_7^2 + \frac{1}{4}\,x_9 + \frac{1}{4}\,x_{10}^2, \tag{7.4}$$

$$E = \frac{1}{2}(x_1 + x_2)\,x_5 + \frac{1}{4}(x_6\,x_8 + x_7\,x_9), \tag{7.5}$$

$$F = \frac{1}{2}(x_1 + x_3)\,x_6 + \frac{1}{4}(x_3\,x_6 + x_5\,x_8), \tag{7.6}$$

$$G = \frac{1}{2}(x_1 + x_4)\,x_7 + \frac{1}{4}(x_5\,x_9 + x_6\,x_{10}), \tag{7.7}$$

$$H = \frac{1}{2}(x_2 + x_3)\,x_8 + \frac{1}{4}(x_5\,x_6 + x_9\,x_{10}), \tag{7.8}$$

$$I = \frac{1}{2}(x_2 + x_4)\,x_9 + \frac{1}{4}(x_5\,x_7 + x_8\,x_{10}), \tag{7.9}$$

$$J = \frac{1}{2}(x_3 + x_4)\,x_{10} + \frac{1}{4}(x_6\,x_{10} + x_6\,x_7). \tag{7.10}$$

From Theorem 3.3, the algebra $\mathrm{Pol}(\mathfrak{p}_\star)^K$ is generated by the following polynomials

$$\begin{aligned}
\alpha_1(X,Z) &= x_1 + x_2 + x_3 + x_4,\\
\alpha_2(X,Z) &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + \frac{1}{2}\big(x_5^2 + x_6^2 + x_7^2 + x_8^2 + x_9^2 + x_{10}^2\big),\\
\alpha_3(X,Z) &= x_1^3 + x_2^3 + x_3^3 + x_4^3\\
&\quad + \frac{3}{4}\,x_1\big(x_5^2 + x_6^2 + x_7^2\big) + \frac{3}{4}\,x_2\big(x_5^2 + x_8^2 + x_9^2\big)\\
&\quad + \frac{3}{4}\,x_3\big(x_6^2 + x_8^2 + x_{10}^2\big) + \frac{3}{4}\,x_4\big(x_7^2 + x_9^2 + x_{10}^2\big)\\
&\quad + \frac{3}{4}\big(x_5\,x_6\,x_8 + x_5\,x_7\,x_9 + x_6\,x_7\,x_{10} + x_8\,x_9\,x_{10}\big),\\
\alpha_4(X,Z) &= A^2 + B^2 + C^2 + D^2 + 2\big(E^2 + F^2 + G^2 + H^2 + I^2 + J^2\big),\\
\beta_0(X,Z) &= z_1^2 + z_2^2 + z_3^2 + z_4^2,\\
\beta_1(X,Z) &= x_1 z_1^2 + x_2 z_2^2 + x_3 z_3^2 + x_4 z_4^2,\\
&\quad + x_5 z_1 z_2 + x_6 z_1 z_3 + x_7 z_1 z_4 + x_8 z_2 z_3 + x_9 z_2 z_4 + x_{10} z_3 z_4,\\
\beta_2(X,Z) &= A z_1^2 + B z_2^2 + C z_3^2 + D z_4^2,\\
&\quad + 2\big(E z_1 z_2 + F z_1 z_3 + G z_1 z_4 + H z_2 z_3 + I z_2 z_4 + J z_3 z_4\big),\\
\beta_3(X,Z) &= \frac{1}{2}\big(2A x_1 + E x_5 + F x_6 + G x_7\big) z_1^2\\
&\quad + \frac{1}{2}\big(2B x_2 + E x_5 + H x_8 + I x_9\big) z_2^2\\
&\quad + \frac{1}{2}\big(2C x_3 + F x_6 + H x_8 + J x_{10}\big) z_3^2\\
&\quad + \frac{1}{2}\big(2D x_4 + G x_7 + I x_9 + J x_{10}\big) z_4^2\\
&\quad + \frac{1}{2}\left\{2E(x_1 + x_2) + (A + B)x_5 + H x_6 + I x_7 + F x_8 + G x_9\right\} z_1 z_2
\end{aligned}$$

$$+\frac{1}{2}\{2F(x_1+x_3)+Hx_5+(A+C)x_6+Jx_7+Ex_8+Gx_{10}\}\,z_1z_3$$
$$+\frac{1}{2}\{2G(x_1+x_4)+Ix_5+Jx_6+(A+D)x_7+Ex_9+Fx_{10}\}\,z_1z_4$$
$$+\frac{1}{2}\{2H(x_2+x_3)+Fx_5+Ex_6+(B+C)x_8+Jx_9+Ix_{10}\}\,z_2z_3$$
$$+\frac{1}{2}\{2I(x_2+x_4)+Gx_5+Ex_7+Jx_8+(B+D)x_9+Hx_{10}\}\,z_2z_4$$
$$+\frac{1}{2}\{2J(x_3+x_4)+Gx_6+Fx_7+Ix_8+Hx_9+(C+D)x_{10}\}\,z_3z_4.$$

We take a coordinate (Y,V) in $\mathcal{P}_{4,1}$, that is,

$$Y=\begin{pmatrix} y_1 & y_5 & y_6 & y_7 \\ y_5 & y_2 & y_8 & y_9 \\ y_6 & y_8 & y_3 & y_{10} \\ y_7 & y_9 & y_{10} & y_4 \end{pmatrix} \quad \text{and} \quad V=(v_1,v_2,v_3,v_4).$$

Put

$$\frac{\partial}{\partial Y}=\begin{pmatrix} \frac{\partial}{\partial y_1} & \frac{1}{2}\frac{\partial}{\partial y_5} & \frac{1}{2}\frac{\partial}{\partial y_6} & \frac{1}{2}\frac{\partial}{\partial y_7} \\ \frac{1}{2}\frac{\partial}{\partial y_5} & \frac{\partial}{\partial y_2} & \frac{1}{2}\frac{\partial}{\partial y_8} & \frac{1}{2}\frac{\partial}{\partial y_9} \\ \frac{1}{2}\frac{\partial}{\partial y_6} & \frac{1}{2}\frac{\partial}{\partial y_8} & \frac{\partial}{\partial y_3} & \frac{1}{2}\frac{\partial}{\partial y_{10}} \\ \frac{1}{2}\frac{\partial}{\partial y_7} & \frac{1}{2}\frac{\partial}{\partial y_9} & \frac{1}{2}\frac{\partial}{\partial y_{10}} & \frac{\partial}{\partial y_4} \end{pmatrix} \quad \text{and} \quad \frac{\partial}{\partial V}=\left(\frac{\partial}{\partial v_1},\frac{\partial}{\partial v_2},\frac{\partial}{\partial v_3},\frac{\partial}{\partial v_4}\right).$$

Let

$$D_i=\mathrm{tr}\left(\left(2\,Y\frac{\partial}{\partial Y}\right)^i\right),\quad i=1,2,3,4$$

and

$$\Omega_j=\frac{\partial}{\partial V}\left(2\,Y\frac{\partial}{\partial Y}\right)^j Y\,{}^t\!\left(\frac{\partial}{\partial V}\right),\quad j=0,1,2,3.$$

It is easily seen that

$$D_1=\mathrm{tr}\left(2\,Y\frac{\partial}{\partial Y}\right)=2\sum_{i=1}^{10} y_i\frac{\partial}{\partial y_i}.$$

For brevity, we put

$$\partial_i=\frac{\partial}{\partial v_i},\quad i=1,2,3,4.$$

Then we get

$$\Omega_0=y_1\,\partial_1^2+y_2\,\partial_2^2+y_3\,\partial_3^2+y_4\,\partial_4^2+2\,y_5\,\partial_1\partial_2$$
$$+2\,y_6\,\partial_1\partial_3+2\,y_7\,\partial_1\partial_4+2\,y_8\,\partial_2\partial_3+2\,y_9\,\partial_2\partial_4+2\,y_{10}\,\partial_3\partial_4.$$

We observe that D_1, D_2, D_3, D_4, Ω_0, $\Omega_1,\Omega_2,\Omega_3$ are invariant differential operators in $\mathbb{D}(\mathcal{P}_{4,1})$. Then we have the following relations

(7.11) $$[D_i,D_j]=0\quad \text{for all } i,j=1,2,3,4$$

and

$$(7.12) \qquad [D_1, \Omega_0] = 2\,\Omega_0.$$

Therefore, $\mathbb{D}(\mathcal{P}_{4,1})$ is not commutative.

6.2. The case when $n = 4$ and $m = 2$

In this case,

$$GL_{4,2} = GL(4,\mathbb{R}) \ltimes \mathbb{R}^{(2,4)}, \quad K = O(4) \ \text{ and } \ \mathcal{P}_{4,2} = GL_{4,2}/K = \mathcal{P}_4 \times \mathbb{R}^{(2,4)}.$$

We see easily that

$$\mathfrak{p}_\star = \left\{ (X,Z) \mid X = {}^tX \in \mathbb{R}^{(4,4)}, \quad Z \in \mathbb{R}^{(2,4)} \right\}.$$

Put

$$E_1 = \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \ E_2 = \begin{pmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \ E_3 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix},$$

$$E_4 = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1 \end{pmatrix}, \ E_5 = \begin{pmatrix} 0&\frac12&0&0\\\frac12&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad E_6 = \begin{pmatrix} 0&0&\frac12&0\\0&0&0&0\\\frac12&0&0&0\\0&0&0&0 \end{pmatrix},$$

$$E_7 = \begin{pmatrix} 0&0&0&\frac12\\0&0&0&0\\0&0&0&0\\\frac12&0&0&0 \end{pmatrix}, \ E_8 = \begin{pmatrix} 0&0&0&0\\0&0&\frac12&0\\0&\frac12&0&0\\0&0&0&0 \end{pmatrix},$$

$$E_9 = \begin{pmatrix} 0&0&0&0\\0&0&0&\frac12\\0&0&0&0\\0&\frac12&0&0 \end{pmatrix}, \ E_{10} = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&\frac12\\0&0&\frac12&0 \end{pmatrix}.$$

Let O_4 be the 4×4 zero matrix and let

$$O_{2,4} = \begin{pmatrix} 0&0&0&0\\0&0&0&0 \end{pmatrix} \in \mathbb{R}^{(2,4)}.$$

Put

$$e_i = (E_i, O_{2,4}), \quad 1 \le i \le 10,$$

$$f_1 = \left(O_4, \begin{pmatrix} 1&0&0&0\\0&0&0&0 \end{pmatrix} \right), \ f_2 = \left(O_4, \begin{pmatrix} 0&1&0&0\\0&0&0&0 \end{pmatrix} \right),$$

$$f_3 = \left(O_4, \begin{pmatrix} 0&0&1&0\\0&0&0&0 \end{pmatrix} \right), \ f_4 = \left(O_4, \begin{pmatrix} 0&0&0&1\\0&0&0&0 \end{pmatrix} \right),$$

$$f_5 = \left(O_4, \begin{pmatrix} 0&0&0&0\\1&0&0&0 \end{pmatrix} \right), \ f_6 = \left(O_4, \begin{pmatrix} 0&0&0&0\\0&1&0&0 \end{pmatrix} \right),$$

$$f_7 = \left(O_4, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right), \ f_8 = \left(O_4, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right).$$

Then $\{ e_i, f_j \,|\, 1 \leq i \leq 10, \ 1 \leq j \leq 8 \}$ forms a basis for $\mathfrak{p}_\star$. Using this basis, we write for variables $(X, Z) \in \mathfrak{p}_\star$,

$$X = \begin{pmatrix} x_1 & \frac{1}{2}x_5 & \frac{1}{2}x_6 & \frac{1}{2}x_7 \\ \frac{1}{2}x_5 & x_2 & \frac{1}{2}x_8 & \frac{1}{2}x_9 \\ \frac{1}{2}x_6 & \frac{1}{2}x_8 & x_3 & \frac{1}{2}x_{10} \\ \frac{1}{2}x_7 & \frac{1}{2}x_9 & \frac{1}{2}x_{10} & x_4 \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{21} & z_{22} & z_{23} & z_{24} \end{pmatrix}.$$

Set

$$\square_{11} = \frac{1}{2} \big(2A\, x_1 + E\, x_5 + F\, x_6 + G\, x_7\big),$$

$$\square_{22} = \frac{1}{2} \big(2B\, x_2 + E\, x_5 + H\, x_8 + I\, x_9\big),$$

$$\square_{33} = \frac{1}{2} \big(2C\, x_3 + F\, x_6 + H\, x_8 + J\, x_{10}\big),$$

$$\square_{44} = \frac{1}{2} \big(2D\, x_4 + G\, x_7 + I\, x_9 + J\, x_{10}\big),$$

$$\square_{12} = \frac{1}{2} \left\{2E(x_1 + x_2) + (A+B)x_5 + Hx_6 + Ix_7 + Fx_8 + Gx_9\right\},$$

$$\square_{13} = \frac{1}{2} \left\{2F(x_1 + x_3) + Hx_5 + (A+C)x_6 + Jx_7 + Ex_8 + Gx_{10}\right\},$$

$$\square_{14} = \frac{1}{2} \left\{2G(x_1 + x_4) + Ix_5 + Jx_6 + (A+D)x_7 + Ex_9 + Fx_{10}\right\},$$

$$\square_{23} = \frac{1}{2} \left\{2H(x_2 + x_3) + Fx_5 + Ex_6 + (B+C)x_8 + Jx_9 + Ix_{10}\right\},$$

$$\square_{24} = \frac{1}{2} \left\{2I(x_2 + x_4) + Gx_5 + Ex_7 + Jx_8 + (B+D)x_9 + Hx_{10}\right\},$$

$$\square_{34} = \frac{1}{2} \left\{2J(x_3 + x_4) + Gx_6 + Fx_7 + Ix_8 + Hx_9 + (C+D)x_{10}\right\}.$$

From Theorem 3.3, the algebra $\mathrm{Pol}(\mathfrak{p}_\star)^K$ is generated by the following 16 polynomials

$$\alpha_1(X,Z) = x_1 + x_2 + x_3 + x_4,$$

$$\alpha_2(X,Z) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + \frac{1}{2}\big(x_5^2 + x_6^2 + x_7^2 + x_8^2 + x_9^2 + x_{10}^2\big),$$

$$\begin{aligned} \alpha_3(X,Z) = x_1^3 + x_2^3 + x_3^3 + x_4^3 \\ &+ \frac{3}{4}\, x_1\big(x_5^2 + x_6^2 + x_7^2\big) + \frac{3}{4}\, x_2\big(x_5^2 + x_8^2 + x_9^2\big) \\ &+ \frac{3}{4}\, x_3\big(x_6^2 + x_8^2 + x_{10}^2\big) + \frac{3}{4}\, x_4\big(x_7^2 + x_9^2 + x_{10}^2\big) \\ &+ \frac{3}{4}\big(x_5\, x_6\, x_8 + x_5\, x_7\, x_9 + x_6\, x_7\, x_{10} + x_8\, x_9\, x_{10}\big), \end{aligned}$$

$$\begin{aligned}
\alpha_4(X,Z) &= A^2+B^2+C^2+D^2+2\left(E^2+F^2+G^2+H^2+I^2+J^2\right),\\
\beta_{11}^{(0)}(X,Z) &= z_{11}^2+z_{12}^2+z_{13}^2+z_{14}^2,\\
\beta_{12}^{(0)}(X,Z) &= z_{11}z_{21}+z_{12}z_{22}+z_{13}z_{23}+z_{14}z_{24},\\
\beta_{22}^{(0)}(X,Z) &= z_{21}^2+z_{22}^2+z_{23}^2+z_{24}^2,\\
\beta_{11}^{(1)}(X,Z) &= x_1z_{11}^2+x_2z_{12}^2+x_3z_{13}^2+x_4z_{14}^2+x_5z_{11}z_{12}\\
&\quad +x_6z_{11}z_{13}+x_7z_{11}z_{14}+x_8z_{12}z_{13}+x_9z_{12}z_{14}+x_{10}z_{13}z_{14},\\
\beta_{12}^{(1)}(X,Z) &= x_1z_{11}z_{21}+x_2z_{12}z_{22}+x_3z_{13}z_{23}+x_4z_{14}z_{24}\\
&\quad +\frac{1}{2}x_5\left(z_{11}z_{22}+z_{12}z_{21}\right)+\frac{1}{2}x_6\left(z_{11}z_{23}+z_{13}z_{21}\right)\\
&\quad +\frac{1}{2}x_7\left(z_{11}z_{24}+z_{14}z_{21}\right)+\frac{1}{2}x_8\left(z_{12}z_{23}+z_{13}z_{22}\right)\\
&\quad +\frac{1}{2}x_9\left(z_{12}z_{24}+z_{14}z_{22}\right)+\frac{1}{2}x_{10}\left(z_{13}z_{24}+z_{14}z_{23}\right),\\
\beta_{22}^{(1)}(X,Z) &= x_1z_{21}^2+x_2z_{22}^2+x_3z_{23}^2+x_4z_{24}^2+x_5z_{21}z_{22}\\
&\quad +x_6z_{21}z_{23}+x_7z_{21}z_{23}++x_8z_{22}z_{23}+x_9z_{22}z_{24}+x_{10}z_{23}z_{24},\\
\beta_{11}^{(2)}(X,Z) &= A\,z_{11}^2+B\,z_{12}^2+C\,z_{13}^2+D\,z_{14}^2+2\,E\,z_{11}z_{12}+2\,F\,z_{11}z_{13}\\
&\quad +\ 2\,G\,z_{11}z_{14}+2\,H\,z_{12}z_{13}+2\,I\,z_{12}z_{14}+2\,J\,z_{13}z_{14},\\
\beta_{12}^{(2)}(X,Z) &= A\,z_{11}z_{21}+B\,z_{12}z_{22}+C\,z_{13}z_{23}+D\,z_{14}z_{24}\\
&\quad +\ E\left(z_{11}z_{22}+z_{12}z_{21}\right)+\ F\left(z_{11}z_{23}+z_{13}z_{21}\right)\\
&\quad +\ G\left(z_{11}z_{24}+z_{14}z_{21}\right)+\ H\left(z_{12}z_{23}+z_{13}z_{22}\right)\\
&\quad +\ I\left(z_{12}z_{24}+z_{14}z_{22}\right)+\ J\left(z_{13}z_{24}+z_{14}z_{23}\right),\\
\beta_{22}^{(2)}(X,Z) &= A\,z_{21}^2+B\,z_{22}^2+C\,z_{23}^2+D\,z_{24}^2+2\,E\,z_{21}z_{22}+2\,F\,z_{21}z_{23}\\
&\quad +\ 2\,G\,z_{21}z_{24}+2\,H\,z_{22}z_{23}+2\,I\,z_{22}z_{24}+2\,J\,z_{23}z_{24},\\
\beta_{11}^{(3)}(X,Z) &= \Box_{11}\,z_{11}^2+\Box_{22}\,z_{12}^2+\Box_{33}\,z_{13}^2+\Box_{44}\,z_{14}^2+\Box_{12}\,z_{11}z_{12}\\
&\quad +\ \Box_{13}\,z_{11}z_{13}+\Box_{14}\,z_{11}z_{14}+\Box_{23}\,z_{12}z_{13}\\
&\quad +\Box_{24}\,z_{12}z_{14}+\Box_{34}\,z_{13}z_{14},\\
\beta_{12}^{(3)}(X,Z) &= \Box_{11}\,z_{11}z_{21}+\Box_{22}\,z_{12}z_{22}+\Box_{33}\,z_{13}z_{23}+\Box_{44}\,z_{14}z_{24}\\
&\quad +\Box_{12}\,z_{11}z_{22}+\ \Box_{13}\,z_{11}z_{23}+\Box_{14}\,z_{11}z_{24}+\Box_{23}\,z_{12}z_{23}\\
&\quad +\Box_{24}\,z_{12}z_{24}+\Box_{34}\,z_{13}z_{24},\\
\beta_{22}^{(3)}(X,Z) &= \Box_{11}\,z_{21}^2+\Box_{22}\,z_{22}^2+\Box_{33}\,z_{23}^2+\Box_{44}\,z_{24}^2+\Box_{12}\,z_{21}z_{22}\\
&\quad +\ \Box_{13}\,z_{21}z_{23}+\Box_{14}\,z_{21}z_{24}+\Box_{23}\,z_{22}z_{23}\\
&\quad +\Box_{24}\,z_{22}z_{24}+\Box_{34}\,z_{23}z_{24}.
\end{aligned}$$

Here, $A,B,C,\dots,J$ are defined as in (7.1)-(7.10).

Set

$$\Delta_{ab} := \det \begin{pmatrix} \beta_{11}^{(a)} & \beta_{12}^{(b)} \\ \beta_{12}^{(a)} & \beta_{22}^{(b)} \end{pmatrix} \qquad \text{for } a, b = 0, 1, 2, 3.$$

By a tedious direct computation, we can show that

$$\begin{aligned} (7.13) \quad & \left(\alpha_1^3 - 3\,\alpha_1\alpha_2 + 2\,\alpha_3\right)\Delta_{00} - 3\left(\alpha_1^2 - \alpha_2\right)\left(\Delta_{01} + \Delta_{10}\right) \\ & + 6\,\alpha_1\left(\Delta_{02} + \Delta_{11} + \Delta_{20}\right) + 6\left(\Delta_{03} + \Delta_{12} + \Delta_{21} + \Delta_{30}\right) = 0. \end{aligned}$$

Take a coordinate (Y, V) in $\mathcal{P}_{4,2}$, that is,

$$Y = \begin{pmatrix} y_1 & y_5 & y_6 & y_7 \\ y_5 & y_2 & y_8 & y_9 \\ y_6 & y_8 & y_3 & y_{10} \\ y_7 & y_9 & y_{10} & y_4 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \end{pmatrix}.$$

Put

$$\frac{\partial}{\partial Y} = \begin{pmatrix} \frac{\partial}{\partial y_1} & \frac{1}{2}\frac{\partial}{\partial y_5} & \frac{1}{2}\frac{\partial}{\partial y_6} & \frac{1}{2}\frac{\partial}{\partial y_7} \\ \frac{1}{2}\frac{\partial}{\partial y_5} & \frac{\partial}{\partial y_2} & \frac{1}{2}\frac{\partial}{\partial y_8} & \frac{1}{2}\frac{\partial}{\partial y_9} \\ \frac{1}{2}\frac{\partial}{\partial y_6} & \frac{1}{2}\frac{\partial}{\partial y_8} & \frac{\partial}{\partial y_3} & \frac{1}{2}\frac{\partial}{\partial y_{10}} \\ \frac{1}{2}\frac{\partial}{\partial y_7} & \frac{1}{2}\frac{\partial}{\partial y_9} & \frac{1}{2}\frac{\partial}{\partial y_{10}} & \frac{\partial}{\partial y_4} \end{pmatrix} \quad \text{and} \quad \frac{\partial}{\partial V} = \begin{pmatrix} \frac{\partial}{\partial v_{11}} & \frac{\partial}{\partial v_{12}} & \frac{\partial}{\partial v_{13}} & \frac{\partial}{\partial v_{14}} \\ \frac{\partial}{\partial v_{21}} & \frac{\partial}{\partial v_{22}} & \frac{\partial}{\partial v_{23}} & \frac{\partial}{\partial v_{24}} \end{pmatrix}.$$

Let

$$D_i = \operatorname{tr}\left(\left(2\,Y\frac{\partial}{\partial Y}\right)^i\right), \quad i = 1, 2, 3, 4$$

and

$$\Omega_{pq}^{(k)} = \left\{ \frac{\partial}{\partial V}\left(2Y\frac{\partial}{\partial Y}\right)^k Y\, {}^t\!\left(\frac{\partial}{\partial V}\right) \right\}_{pq}, \quad k = 0, 1, 2, 3,\ 1 \le p \le q \le 2.$$

Note that D_1, D_2, D_3, D_4, $\Omega_{11}^{(0)}, \ldots, \Omega_{22}^{(3)}$ are $GL_{4,2}$-invariant. It is easily seen that

$$D_1 = \operatorname{tr}\left(2\,Y\frac{\partial}{\partial Y}\right) = 2\sum_{i=1}^{10} y_i \frac{\partial}{\partial y_i}.$$

For brevity, we put

$$\partial_{ij} = \frac{\partial}{\partial v_{ij}}, \quad i = 1, 2,\ 1 \le j \le 4.$$

Then we get

$$\begin{aligned} \Omega_{11}^{(0)} &= y_1\,\partial_{11}^2 + y_2\,\partial_{12}^2 + y_3\,\partial_{13}^2 + y_4\,\partial_{14}^2 + 2\,y_5\,\partial_{11}\partial_{12} + 2\,y_6\,\partial_{11}\partial_{13} \\ &\quad + 2\,y_7\,\partial_{11}\partial_{14} + 2\,y_8\,\partial_{12}\partial_{13} + 2\,y_9\,\partial_{12}\partial_{14} + 2\,y_{10}\,\partial_{13}\partial_{14}, \\ \Omega_{12}^{(0)} &= y_1\,\partial_{11}\partial_{21} + y_2\,\partial_{12}\partial_{22} + y_3\,\partial_{13}\partial_{23} + y_4\,\partial_{14}\partial_{24} \\ &\quad + y_5\left(\partial_{11}\partial_{22} + \partial_{12}\partial_{21}\right) + y_6\left(\partial_{11}\partial_{23} + \partial_{13}\partial_{21}\right) \\ &\quad + y_7\left(\partial_{11}\partial_{24} + \partial_{14}\partial_{21}\right) + y_8\left(\partial_{12}\partial_{23} + \partial_{13}\partial_{22}\right) \end{aligned}$$

$$+\, y_9\,(\partial_{12}\partial_{24}+\partial_{14}\partial_{22}) + y_{10}\,(\partial_{13}\partial_{24}+\partial_{14}\partial_{23}),$$

$$\Omega^{(0)}_{22} = y_1\,\partial^2_{21}+y_2\,\partial^2_{22}+y_3\,\partial^2_{23}+y_4\,\partial^2_{24}+2\,y_5\,\partial_{21}\partial_{22}+2\,y_6\,\partial_{21}\partial_{23} + 2\,y_7\,\partial_{21}\partial_{24}+2\,y_8\,\partial_{22}\partial_{23}+2\,y_9\,\partial_{22}\partial_{24}+2\,y_{10}\,\partial_{23}\partial_{24}.$$

Then we have the following relations

$$[D_i, D_j] = 0 \quad \text{for all } i,j=1,2,3,4, \tag{7.14}$$

$$[\Omega^{(0)}_{kl}, \Omega^{(0)}_{pq}] = 0, \quad 1\le k\le l\le 2,\ 1\le p\le q\le 2, \tag{7.15}$$

and

$$[D_1, \Omega^{(0)}_{11}] = 2\,\Omega^{(0)}_{11}, \quad [D_1, \Omega^{(0)}_{12}] = 2\,\Omega^{(0)}_{12}, \quad [D_1, \Omega^{(0)}_{22}] = 2\,\Omega^{(0)}_{22}. \tag{7.16}$$

Therefore, $\mathbb{D}(\mathcal{P}_{4,2})$ is not commutative.

8. Final remarks

In this section, we present some open problems and discuss a notion of automorphic forms on $\mathcal{P}_{n,m}$ using $GL_{n,m}$-invariant differential operators on the Minkowski-Euclid space $\mathcal{P}_{n,m}$.

Recall the invariant polynomials $\alpha_j\,(1\le j\le n)$ from (3.11) and $\beta^{(k)}_{pq}$ $(0\le k\le n-1,\ 1\le p\le q\le m)$ from (3.12). Also recall the invariant differential operators $D_j\,(1\le j\le n)$ from (3.19) and $\Omega^{(k)}_{pq}$ $(0\le k\le n-1,\ 1\le p\le q\le m)$ from (3.20).

Theorem 8.1. *The following relations hold:*

$$[D_i, D_j] = 0 \quad \textit{for all } 1\le i,j\le n, \tag{8.1}$$

$$[\Omega^{(0)}_{kl}, \Omega^{(0)}_{pq}] = 0, \quad 1\le k\le l\le m,\ 1\le p\le q\le m, \tag{8.2}$$

and

$$[D_1, \Omega^{(0)}_{pq}] = 2\,\Omega^{(0)}_{pq} \quad \textit{for all } \ 1\le p\le q\le m. \tag{8.3}$$

Proof. The relation (8.1) follows from the work of Atle Selberg (cf. [8, 10, 11]). Take a coordinate (Y,V) in $\mathcal{P}_{n,m}$ with $Y=(y_{ij})$ and $V=(v_{kl})$. Put

$$\frac{\partial}{\partial Y} = \left(\frac{1+\delta_{ij}}{2}\frac{\partial}{\partial y_{ij}}\right) \quad \text{and} \quad \frac{\partial}{\partial V} = \left(\frac{\partial}{\partial v_{kl}}\right),$$

where $1\le i,j,l\le n$ and $1\le k\le m$. Then we get

$$D_1 = 2\sum_{1\le i\le j\le n} y_{ij}\,\frac{\partial}{\partial y_{ij}},$$

$$\Omega^{(0)}_{pq} = \sum_{a=1}^{n} y_{aa}\,\frac{\partial^2}{\partial v_{pa}\partial v_{qa}} + \sum_{1\le a<b\le n} y_{ab}\left(\frac{\partial^2}{\partial v_{pa}\partial v_{qb}}+\frac{\partial^2}{\partial v_{pb}\partial v_{qa}}\right).$$

By a direct calculation, we obtain the desired relations (8.2) and (8.3). □

Conjecture 2.

$$\Theta_{n,m}(\alpha_j) = D_j \quad \text{for all } 1 \leq j \leq n, \tag{8.4}$$

$$\Theta_{n,m}\big(\beta_{pq}^{(k)}\big) = \Omega_{pq}^{(k)} \quad \text{for all } 0 \leq k \leq n-1,\ 1 \leq p \leq q \leq m. \tag{8.5}$$

We refer to Conjecture 1 in Section 2.

Conjecture 3. The invariants $D_j\,(1 \leq j \leq n)$ and $\Omega_{pq}^{(k)}$ $(0 \leq k \leq n-1,\ 1 \leq p \leq q \leq m)$ generate the noncommutative algebra $\mathbb{D}(\mathcal{P}_{n,m})$.

Conjecture 4. The above relations (8.1), (8.2) and (8.3) generate all relations among the set

$$\Big\{ D_j,\, \Omega_{pq}^{(k)} \mid 1 \leq j \leq n,\ 0 \leq k \leq n-1,\ 1 \leq p \leq q \leq m \Big\}.$$

Problem 8. Find a natural way to construct generators of $\mathbb{D}(\mathcal{P}_{n,m})$.

Using $GL_{n,m}$-invariant differential operators on the Minkowski-Euclid space $\mathcal{P}_{n,m}$, we introduce a notion of automorphic forms on $\mathcal{P}_{n,m}$ (cf. [11]).

Let

$$\Gamma_{n,m} := GL(n,\mathbb{Z}) \ltimes \mathbb{Z}^{(m,n)}$$

be the arithmetic subgroup of $GL_{n,m}$. Let $\mathcal{Z}_{n,m}$ be the center of $\mathbb{D}(\mathcal{P}_{n,m})$.

Definition 8.1. A smooth function $f : \mathcal{P}_{n,m} \longrightarrow \mathbb{C}$ is said to be an automorphic form for $\Gamma_{n,m}$ if it satisfies the following conditions:

(A1) f is $\Gamma_{n,m}$-invariant.

(A2) f is an eigenfunction of any differential operator in the center $\mathcal{Z}_{n,m}$ of $\mathbb{D}(\mathcal{P}_{n,m})$.

(A3) f has a growth condition.

We define another notion of automorphic forms as follows.

Definition 8.2. Let $\mathbb{D}_\spadesuit$ be a commutative subalgebra of $\mathbb{D}(\mathcal{P}_{n,m})$ containing the Laplacian $\Delta_{n,m;A,B}$. A smooth function $f : \mathcal{P}_{n,m} \longrightarrow \mathbb{C}$ is said to be an automorphic form for $\Gamma_{n,m}$ with respect to $\mathbb{D}_\spadesuit$ if it satisfies the following conditions:

(A1) f is $\Gamma_{n,m}$-invariant.

(A2) f is an eigenfunction of any differential operator in $\mathbb{D}_\spadesuit$.

(A3) f has a growth condition.

References

[1] Harish-Chandra, *Representations of semisimple Lie groups. I*, Trans. Amer. Math. Soc. **75** (1953), 185–243.

[2] ———, *The characters of semisimple Lie groups*, Trans. Amer. Math. Soc. **83** (1956), 98–163.

[3] S. Helgason, *Differential operators on homogeneous spaces*, Acta Math. **102** (1959), 239–299.

[4] ———, *Groups and Geometric Analysis*, Academic Press, New York, 1984.

[5] R. Howe, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond*, The Schur lectures (1992) (Tel Aviv), 1-82, Israel Math. Conf. Proc., 8, Bar-Ilan Univ., Ramat Gan, 1995.
[6] M. Itoh, *On the Yang Problem (SFT)*, Max-Planck Institut für Mathematik, Bonn, 2011.
[7] H. Maass, *Die Bestimmung der Dirichletreihnen mit Grössencharakteren zu den Modulformen n-ten Grades*, J. Indian Math. Soc. **9** (1955), 1–23.
[8] ______, *Siegel modular forms and Dirichlet series*, Lecture Notes in Math., **vol. 216**, Springer-Verlag, Berlin-Heidelberg-New York, 1971.
[9] H. Minkowski, *Gesammelte Abhandlungen*, Chelsea, New York, 1967.
[10] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. B. **20** (1956), 47–87.
[11] A. Terras, *Harmonic Analysis on Symmetric Spaces and Applications II*, Springer-Verlag, 1988.
[12] H. Weyl, *The classical groups: Their invariants and representations*, Princeton Univ. Press, Princeton, New Jersey, second edition, 1946.
[13] J.-H. Yang, *Singular Jacobi forms*, Trans. Amer. Math. Soc. **347** (1995), no. 6, 2041–2049.
[14] ______, *Polarized Real Tori*, arXiv:0912.5084v1 [math.AG] (2009) or a revised version (2012).

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POLARIZED REAL TORI

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Abstract. For a fixed positive integer g, we let $\mathcal{P}_g = \left\{ Y \in \mathbb{R}^{(g,g)} \mid Y = {}^tY > 0 \right\}$ be the open convex cone in the Euclidean space $\mathbb{R}^{g(g+1)/2}$. Then the general linear group $GL(g,\mathbb{R})$ acts naturally on $\mathcal{P}_g$ by $A \star Y = AY\,{}^tA$ ($A \in GL(g,\mathbb{R}),\ Y \in \mathcal{P}_g$). We introduce a notion of polarized real tori. We show that the open cone $\mathcal{P}_g$ parametrizes principally polarized real tori of dimension g and that the Minkowski domain $\mathfrak{R}_g = GL(g,\mathbb{Z})\backslash\mathcal{P}_g$ may be regarded as a moduli space of principally polarized real tori of dimension g. We also study smooth line bundles on a polarized real torus by relating them to holomorphic line bundles on its associated polarized real abelian variety.

1. Introduction

For a given fixed positive integer g, we let

$$\mathbb{H}_g = \{\, \Omega \in \mathbb{C}^{(g,g)} \mid \Omega = {}^t\Omega, \quad \operatorname{Im}\Omega > 0 \,\}$$

be the Siegel upper half plane of degree g and let

$$Sp(g,\mathbb{R}) = \{ M \in \mathbb{R}^{(2g,2g)} \mid {}^tMJ_gM = J_g \}$$

be the symplectic group of degree g, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l, ${}^tM$ denotes the transpose matrix of a matrix M and

$$J_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

Then $Sp(g,\mathbb{R})$ acts on $\mathbb{H}_g$ transitively by

$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{1.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R})$ and $\Omega \in \mathbb{H}_g$. Let

$$\Gamma_g = Sp(g,\mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R}) \,\Big|\, A,B,C,D \text{ integral} \right\}$$

be the Siegel modular group of degree g. This group acts on $\mathbb{H}_g$ properly discontinuously. C. L. Siegel investigated the geometry of $\mathbb{H}_g$ and automorphic forms on $\mathbb{H}_g$ systematically. Siegel [23] found a fundamental domain $\mathcal{F}_g$ for $\Gamma_g\backslash\mathbb{H}_g$ and described it explicitly. Moreover

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he calculated the volume of $\mathcal{F}_g$. We also refer to [10], [14], [23] for some details on $\mathcal{F}_g$. Siegel's fundamental domain is now called the Siegel modular variety and is usually denoted by $\mathcal{A}_g$. In fact, $\mathcal{A}_g$ is one of the important arithmetic varieties in the sense that it is regarded as the moduli of principally polarized abelian varieties of dimension g. Suggested by Siegel, I. Satake [18] found a canonical compactification, now called the Satake compactification of $\mathcal{A}_g$. Thereafter W. Baily [3] proved that the Satake compactification of $\mathcal{A}_g$ is a normal projective variety. This work was generalized to bounded symmetric domains by W. Baily and A. Borel [4] around the 1960s. Some years later a theory of smooth compactification of bounded symmetric domains was developed by Mumford school [2]. G. Faltings and C.-L. Chai [7] investigated the moduli of abelian varieties over the integers and could give the analogue of the Eichler-Shimura theorem that expresses Siegel modular forms in terms of the cohomology of local systems on $\mathcal{A}_g$. I want to emphasize that Siegel modular forms play an important role in the theory of the arithmetic and the geometry of the Siegel modular variety $\mathcal{A}_g$.

We let

$$\mathcal{P}_g = \left\{ Y \in \mathbb{R}^{(g,g)} \mid Y = {}^tY > 0 \right\}$$

be an open convex cone in $\mathbb{R}^N$ with $N = g(g+1)/2$. The general linear group $GL(g,\mathbb{R})$ acts on $\mathcal{P}_g$ transitively by

$$A \circ Y := AY\,{}^tA, \qquad A \in GL(g,\mathbb{R}),\ Y \in \mathcal{P}_g. \tag{1.2}$$

We observe that the action (1.2) is naturally induced from the symplectic action (1.1). Thus $\mathcal{P}_g$ is a symmetric space diffeomorphic to $GL(g,\mathbb{R})/O(g)$. Let

$$GL(g,\mathbb{Z}) = \{ \gamma \in GL(g,\mathbb{R}) \mid \gamma \text{ is integral} \}$$

be an arithmetic discrete subgroup of $GL(g,\mathbb{R})$. Using the reduction theory Minkowski [16] found a fundamental domain $\mathfrak{R}_g$, the so-called Minkowski domain for the action (1.2) of $GL(g,\mathbb{Z})$ on $\mathcal{P}_g$. In fact, using the Minkowski domain $\mathfrak{R}_g$ Siegel found his fundamental domain $\mathcal{F}_g$. As in the case of $\mathbb{H}_g$, automorphic forms on $\mathcal{P}_g$ for $GL(g,\mathbb{Z})$ and geometry on $\mathcal{P}_g$ have been studied by many people, e.g., Selberg [20], Maass [14] et al.

The aim of this article is to study arithmetic-geometric meaning of the Minkowski domain $\mathfrak{R}_g$. First we introduce a notion of polarized real tori by relating special real tori to polarized real abelian varieties. We realize that $\mathcal{P}_g$ parametrizes principally polarized real tori of dimension g and also that $\mathfrak{R}_g$ may be regarded as a moduli space of principally polarized real tori of dimension g. We also study smooth line bundles over a polarized real torus by relating to holomorphic line bundles over the associated polarized abelian variety. Those line bundles over a polarized real torus play an important role in investigating some geometric properties of a polarized real torus.

We let

$$G^M := GL(g,\mathbb{R}) \ltimes \mathbb{R}^g$$

be the semidirect product of $GL(g,\mathbb{R})$ and $\mathbb{R}^g$ with multiplication law

$$(A,a)\cdot(B,b) := (AB, a\,{}^tB^{-1} + b), \qquad A,B \in GL(g,\mathbb{R}), \quad a,b \in \mathbb{R}^g.$$

Then we have the *natural action* of G^M on the Minkowski-Euclid space $\mathcal{P}_g \times \mathbb{R}^g$ defined by

$$(A,a)\cdot(Y,\zeta) := \left(AY\,{}^tA,\ (\zeta + a)\,{}^tA\right), \qquad (A,a) \in G^M,\ Y \in \mathcal{P}_g,\ \zeta \in \mathbb{R}^g. \tag{1.3}$$

We let

$$G^M(\mathbb{Z}) = GL(g,\mathbb{Z}) \ltimes \mathbb{Z}^g$$

be the discrete subgroup of G^M. Then $G^M(\mathbb{Z})$ acts on $\mathcal{P}_g \times \mathbb{R}^g$ properly discontinuously. We show that by associating a principally polarized real torus of dimension g to each equivalence class in $\mathfrak{R}_g$, the quotient space

$$G^M(\mathbb{Z})\backslash(\mathcal{P}_g \times \mathbb{R}^g)$$

may be regarded as a family of principally polarized real tori of dimension g. To each equivalence class $[Y] \in GL(g,\mathbb{Z})\backslash\mathcal{P}_g$ with $Y \in \mathcal{P}_g$ we associate a principally polarized real torus $T_Y = \mathbb{R}^g/\Lambda_Y$, where $\Lambda_Y = Y\mathbb{Z}^g$ is a lattice in $\mathbb{R}^g$.

Let Y_1 and Y_2 be two elements in $\mathcal{P}_g$ with $[Y_1] \neq [Y_2]$, that is, $Y_2 \neq A Y_1 {}^t\!A$ for all $A \in GL(g,\mathbb{Z})$. We put $\Lambda_i = Y_i\mathbb{Z}^g$ for $i = 1, 2$. Then a torus $T_1 = \mathbb{R}^g/\Lambda_1$ is diffeomorphic to $T_2 = \mathbb{R}^g/\Lambda_2$ as smooth manifolds but T_1 is not isomorphic to T_2 as polarized real tori.

The Siegel modular variety $\mathcal{A}_g$ has three remarkable properties: (a) it is the moduli space of principally polarized abelian varieties of dimension g, (b) it has the structure of a quasi-projective complex algebraic variety which is defined over $\mathbb{Q}$, and (c) it has a canonical compactification, the so-called Satake-Baily-Borel compactification which is defined over $\mathbb{Q}$. Unfortunately the Minkowski domain $\mathfrak{R}_g$ does not admit the structure of a real algebraic variety. Moreover $\mathfrak{R}_g$ does not admit a compactification which is defined over $\mathbb{Q}$. Silhol [26] constructs the moduli space of real principally poarized abelian varieties and he shows that it is a topological ramified covering of $\mathfrak{R}_g$. Furthermore Silhol constructs a compactification of this moduli space analogous to the Satake-Baily-Borel compactification. However, neither the moduli space nor this compactification has an algebraic structure. On the other hand, by considering real abelian varieties with a suitable level structure Goresky and Tai [9] shows that the moduli space of real principally polarized abelian varieties with level $4m$ structure ($m \geq 1$) coincides with the set of real points of a quasi-projective algebraic variety defined over $\mathbb{Q}$ and consists of finitely many copies of the quotient $\mathfrak{G}_g(4m)\backslash\mathcal{P}_g$ with a discrete subgroup $\mathfrak{G}_g(4m)$ of $GL(g,\mathbb{Z})$, where $\mathfrak{G}_g(4m) = \{\gamma \in GL(g,\mathbb{Z}) \mid \gamma \equiv I_g \pmod{4m}\}$.

This paper is organized as follows. In Section 2, we collect some basic properties about the symplectic group $Sp(g,\mathbb{R})$ to be used frequently in the subsequent sections. In Section 3, we give basic definitions concerning real abelian varieties and review some properties of real abelian varieties. In Section 4, we discuss a moduli space for real abelian varieties and recall some basic properties of a moduli for real abelian varieties. In Section 5 we discuss compactifications of the moduli space for real abelian varieties and review some results on this moduli space obtained by Silhol [26], Goresky and Tai [9]. In Section 6 we introduce a notion of polarized real tori and investigate some properties of polarized real tori. We give several examples of polarized real tori. In Section 7 we study smooth line bundles over a real torus, in particular a polarized real torus by relating those smooth line bundles to holomorphic line bundles over the associated complex torus. To each smooth line bundle on a real torus we naturally attach a holomorphic line bundle over the associated complex torus. Conversely to a holomorphic line bundle over a polarized abelian variety we associate a smooth line bundle over the associated polarized real torus. Using these results on line bundles, we embed a real torus in a complex projective space and hence in a real projective space smoothly. We also review briefly holomorphic line bundles over a complex torus. In Section 8 we study the moduli space for polarized real tori. We first review basic geometric properties on the Minkowski domain $\mathfrak{R}_g$. We show that $\mathcal{P}_g$ parameterizes principally polarized real tori of dimension g and that $\mathfrak{R}_g$ can be regarded as the moduli space of principally polarized real tori of dimension g. We show that the quotient space $G^M(\mathbb{Z})\backslash(\mathcal{P}_g\times\mathbb{R}^g)$ may be considered as a family of principally polarized tori of dimension g. In Section 9 we discuss real semi-abelian varieties corresponding to the boundary points of a compactification of a moduli space for real abelian varieties. We recall that a semi-abelian variety is defined to be an extension of an abelian variety by a group of multiplicative type. In Section 10 we discuss briefly real semi-tori corresponding to the boundary points of a moduli space for polarized real tori. In the final section we present some problems related to real polarized tori which should be investigared in the near future. In the appendix we collect and review some results on non-abelian cohomology to be needed necessarily in this article. We give some sketchy proofs for the convenience of the reader.

Finally I would like to mention that this work was motivated and initiated by the works of Silhol [26] and Goresky-Tai [9].

Notations: We denote by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers respectively. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k\times l$ matrices with entries in a commutative ring F. For a square matrix $A\in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. For any $M\in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. I_n denotes the identity matrix of degree n. For a matrix Z, we denote by $\mathrm{Re}\,Z$ (resp. $\mathrm{Im}\,Z$) the real (resp. imaginary) part of Z. For $A\in F^{(k,l)}$ and $B\in F^{(k,k)}$, we set $B[A]={}^tABA$. For a complex matrix A, $\overline{A}$ denotes the complex *conjugate* of A. For $A\in\mathbb{C}^{(k,l)}$ and $B\in\mathbb{C}^{(k,k)}$, we use the abbreviation $B\{A\}={}^t\overline{A}BA$. We denote $\mathbb{C}_1^*=\{\,\xi\in\mathbb{C}\,|\,|\xi|=1\,\}$. Let

$$\Gamma_g=\left\{\gamma\in\mathbb{Z}^{(2g,2g)}\;|\;{}^t\gamma\,J_g\gamma=J_g\right\}$$

denote the Siegel modular group of degree g, where

$$J_g=\begin{pmatrix}0 & I_g\\ -I_g & 0\end{pmatrix}$$

is the symplectic matrix of degree $2g$. For a positive integer N, we let

$$\Gamma_g(N)=\{\,\gamma\in\Gamma_g\,|\,\gamma\equiv I_{2g}\,(\mathrm{mod}\;N)\,\}$$

denote the principal congruence subgroup of Γ_g of level N and for a positive integer m, we let

$$\Gamma_g(2,2m)=\left\{\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma_g\;\middle|\;A,D\equiv I_g\;(\mathrm{mod}\;2),\quad B,C\equiv 0\;(\mathrm{mod}\;2m)\right\}. \tag{1.4}$$

Let $\mathfrak{G}_g:=GL(g,\mathbb{Z})$ and for a positive integer N let

$$\mathfrak{G}_g(N)=\{\,\gamma\in GL(g,\mathbb{Z})\,|\,\gamma\equiv I_g\,(\mathrm{mod}\;N)\,\}. \tag{1.5}$$

2. The Symplectic Group

For a given fixed positive integer g, we let let

$$Sp(g,\mathbb{R}) = \left\{ M \in \mathbb{R}^{(2g,2g)} \mid {}^tMJ_gM = J_g \right\}$$

be the symplectic group of degree g.

If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R})$ with $A,B,C,D \in \mathbb{R}^{(g,g)}$, then it is easily seen that

$$A\,{}^tD - B\,{}^tC = I_g, \quad A\,{}^tB = B\,{}^tA, \quad C\,{}^tD = D\,{}^tC \tag{2.1}$$

or

$${}^tAD - {}^tCB = I_g, \quad {}^tAC = {}^tCA, \quad {}^tBD = {}^tDB. \tag{2.2}$$

The inverse of such a symplectic matrix M is given by

$$M^{-1} = M = \begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix}.$$

We identify $GL(g,\mathbb{R}) \hookrightarrow Sp(g,\mathbb{R})$ with its image under the embedding

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}, \quad A \in GL(g,\mathbb{R}).$$

A Cartan involution θ of $Sp(g,\mathbb{R})$ is given by $\theta(x) = J_g\,x\,J_g^{-1}$, $x \in Sp(g,\mathbb{R})$, in other words,

$$\theta\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R}). \tag{2.3}$$

The fixed point set K of θ is given by

$$K = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in Sp(g,\mathbb{R}) \right\}.$$

We may identify K with the unitary group $U(g)$ of degree g via

$$K \ni \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \longmapsto A + iB \in U(g).$$

Let

$$\mathbb{H}_g = \left\{ \Omega \in \mathbb{C}^{(g,g)} \mid \Omega = {}^t\Omega, \quad \operatorname{Im}\Omega > 0 \right\}$$

be the Siegel upper half plane of degree g. Then $Sp(g,\mathbb{R})$ acts on $\mathbb{H}_g$ transitively by

$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}, \tag{2.4}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R})$ and $\Omega \in \mathbb{H}_g$. The stabilizer at iI_g is given by the compact subgroup $K \cong U(g)$ of $Sp(g,\mathbb{R})$. Thus $\mathbb{H}_g$ is biholomorphic to the Hermitian symmetric space $Sp(g,\mathbb{R})/K$ via

$$Sp(g,\mathbb{R})/K \longrightarrow \mathbb{H}_g, \qquad xK \longmapsto x\cdot(iI_g), \ x \in Sp(g,\mathbb{R}).$$

We note that the Siegel modular group Γ_g of degree g acts on $\mathbb{H}_g$ properly discontinuously.

Now we let

$$I_* := \begin{pmatrix} -I_g & 0 \\ 0 & I_g \end{pmatrix}. \tag{2.5}$$

We define the involution $\tau : Sp(g,\mathbb{R}) \longrightarrow Sp(g,\mathbb{R})$ by

$$\tau(x) := I_* \, x \, I_*, \qquad x \in Sp(g,\mathbb{R}). \tag{2.6}$$

Precisely τ is given by

$$\tau \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R}). \tag{2.7}$$

Lemma 2.1. *(1) $\tau(x) = x$, $x \in Sp(g,\mathbb{R})$ if and only if $x \in GL(g,\mathbb{R})$.*
(2) $\tau\theta = \theta\tau$. So $\tau(K) = K$.
(3) If $A + iB \in U(g)$ with $A, B \in \mathbb{R}^{(g,g)}$, then $\tau(A + iB) = A - iB$.

Proof. It is easy to prove the above lemma. We leave the proof to the reader. □

We note that $\tau : Sp(g,\mathbb{R}) \longrightarrow Sp(g,\mathbb{R})$ passes to an involution (which we denote by the same letter) $\tau : \mathbb{H}_g \longrightarrow \mathbb{H}_g$ such that

$$\tau(x \cdot \Omega) = \tau(x)\, \tau(\Omega) \qquad \text{for all } x \in Sp(g,\mathbb{R}),\ \Omega \in \mathbb{H}_g. \tag{2.8}$$

In fact, we can see easily that the involution $\tau : \mathbb{H}_g \longrightarrow \mathbb{H}_g$ is the antiholomorphic involution given by

$$\tau(\Omega) = -\overline{\Omega}, \qquad \Omega \in \mathbb{H}_g. \tag{2.9}$$

Its fixed point set is the orbit

$$i\,\mathcal{P}_g = GL(g,\mathbb{R}) \cdot (iI_g) \subset \mathbb{C}^{(g,g)}$$

of $GL(g,\mathbb{R})$, where

$$\mathcal{P}_g = \left\{ Y \in \mathbb{R}^{(g,g)} \mid Y = {}^tY > 0 \right\}$$

is the open convex cone of positive definite symmetric real matrices of degree g in the Euclidean space $\mathbb{R}^{g(g+1)/2}$.

For $x \in Sp(g,\mathbb{R})$ and $\Omega \in \mathbb{H}_g$, we define the set

$$\mathbb{H}_g^{\tau x} := \left\{ \Omega \in \mathbb{H}_g \mid x \cdot \Omega = \tau(\Omega) = -\overline{\Omega} \right\} \tag{2.10}$$

be the locus of x-real points. If $\Gamma \subset Sp(g,\mathbb{R})$ is an arithmetic subgroup of $Sp(g,\mathbb{R})$ such that $\tau(\Gamma) = \Gamma$, we define

$$\mathbb{H}_g^{\tau\Gamma} := \bigcup_{\gamma \in \Gamma} \mathbb{H}_g^{\tau\gamma}. \tag{2.11}$$

Lemma 2.2. *Let $x \in Sp(g,\mathbb{R})$ and $\mathbb{H}_g^x$ be the set of points in $\mathbb{H}_g$ which are fixed under the action of x. Then the set $\mathbb{H}_g^x \cap i\,\mathcal{P}_g$ is a proper real algebraic variety of $i\,\mathcal{P}_g$ if $x \neq \pm I_g \in GL(g,\mathbb{R})$.*

Proof. It is easy to prove the above lemma. We omit the proof. □

3. Real Abelian Varieties

In this section we review basic notions and some results on real principally polarized abelian varieties (cf. [9, 21, 24, 25, 26]).

Definition 3.1. *A pair $(\mathfrak{A}, S)$ is said to be a* real abelian variety *if $\mathfrak{A}$ is a complex abelian variety and S is an anti-holomorphic involution of $\mathfrak{A}$ leaving the origin of $\mathfrak{A}$ fixed. The set of all fixed points of S is called the real point of $(\mathfrak{A}, S)$ and denoted by $(\mathfrak{A}, S)(\mathbb{R})$ or simply $\mathfrak{A}(\mathbb{R})$. We call S a* real structure *on $\mathfrak{A}$.*

Definition 3.2. *(1) A* polarization *on a complex abelian variety $\mathfrak{A}$ is defined to be the Chern class $c_1(D) \in H^2(\mathfrak{A}, \mathbb{Z})$ of an ample divisor D on $\mathfrak{A}$. We can identify $H^2(\mathfrak{A}, \mathbb{Z})$ with $\bigwedge^2 H^1(\mathfrak{A}, \mathbb{Z})$. We write $\mathfrak{A} = V/L$, where V is a finite dimensional complex vector space and L is a lattice in V. So a polarization on $\mathfrak{A}$ can be defined as an alternating form E on $L \cong H_1(\mathfrak{A}, \mathbb{Z})$ satisfying the following conditions (E1) and (E2):*

(E1) The Hermitian form $H : V \times V \longrightarrow \mathbb{C}$ defined by

$$H(u, v) = E(i\,u, v) + i\,E(u, v), \qquad u, v \in V \tag{3.1}$$

is positive definite. Here E can be extended $\mathbb{R}$-linearly to an alternating form on V.

(E2) $E(L \times L) \subset \mathbb{Z}$, i.e., E is integral valued on $L \times L$.

(2) Let $(\mathfrak{A}, S)$ be a real abelian variety with a polarization E of dimension g. A polarization E is said to be real *or* S-real *if*

$$E(S_*(a), S_*(b)) = -E(a, b), \qquad a, b \in H_1(\mathfrak{A}, \mathbb{Z}). \tag{3.2}$$

Here $S_* : H_1(\mathfrak{A}, \mathbb{Z}) \longrightarrow H_1(\mathfrak{A}, \mathbb{Z})$ is the map induced by a real structure S. If a polarization E is real, the triple $(\mathfrak{A}, E, S)$ is called a real polarized abelian variety. *A polarization E on $\mathfrak{A}$ is said to be* principal *if for a suitable basis (i.e., a symplectic basis) of $H_1(\mathfrak{A}, \mathbb{Z}) \cong L$, it is represented by the symplectic matrix J_g (cf. see Notations in the introduction). A real abelian variety $(\mathfrak{A}, S)$ with a principal polarization E is called a* real principally polarized abelian variety.

(3) Let $(\mathfrak{A}, E)$ be a principally polarized abelian variety of dimension g and let $\{\alpha_i \mid 1 \le i \le 2g\}$ be a symplectic basis of $H_1(\mathfrak{A}, \mathbb{Z})$. It is known that there is a basis $\{\omega_1, \cdots, \omega_g\}$ of the vector space $H^0(\mathfrak{A}, \Omega^1)$ of holomorphic 1-forms on $\mathfrak{A}$ such that

$$\left(\int_{\alpha_j} \omega_i \right) = (\Omega, I_g) \qquad \text{for some } \Omega \in \mathbb{H}_g.$$

The $g \times 2g$ matrix (Ω, I_g) or simply Ω is called a period matrix *for $(\mathfrak{A}, E)$.*

The definition of a *real polarized abelian variety* is motivated by the following theorem.

Theorem 3.1. *Let $(\mathfrak{A}, S)$ be a real abelian variety and let E be a polarization on $\mathfrak{A}$. Then there exists an ample S-invariant (or S-real) divisor with Chern class E if and only if E satisfies the condition (3.2).*

Proof. The proof can be found in [25, Theorem 3.4, pp. 81-84]. □

Now we consider a principally polarized abelian variety of dimension g with a level structure. Let N be a positive integer. Let $(\mathfrak{A} = \mathbb{C}^g/L, E)$ be a principally polarized abelian variety of dimension g. From now on we write $\mathfrak{A} = \mathbb{C}^g/L$, where L is a lattice in $\mathbb{C}^g$. A level N structure on $\mathfrak{A}$ is a choice of a basis $\{U_i, V_j\}\,(1 \le i,j \le g)$ for a N-torsion points of $\mathfrak{A}$ which is symplectic, in the sense that there exists a symplectic basis $\{u_i, v_j\}$ of L such that

$$U_i \equiv \frac{u_i}{N} \pmod{L} \qquad \text{and} \qquad V_j \equiv \frac{v_j}{N} \pmod{L}, \qquad 1 \le i,j \le g.$$

For a given level N structure, such a choice of a symplectic basis $\{u_i, v_j\}$ of L determines a mapping

$$F : \mathbb{R}^g \oplus \mathbb{R}^g \longrightarrow \mathbb{C}^g$$

such that $F(\mathbb{Z}^g \oplus \mathbb{Z}^g) = L$ by $F(e_i) = u_i$ and $F(f_j) = v_j$, where $\{e_i, f_j\}\,(1 \le i,j \le g)$ is the standard basis of $\mathbb{R}^g \oplus \mathbb{R}^g$. The choice $\{u_i, v_j\}$ (or equivalently, the mapping F) will be referred to as a *lift* of the level N structure. Such a mapping F is well defined modulo the principal congruence subgroup $\Gamma_g(N)$, that is, if F' is another lift of the level structure, then $F' \circ F^{-1} \in \Gamma_g(N)$. A level N structure $\{U_i, V_j\}$ is said to be compatible with a real structure S on $(\mathfrak{A}, E)$ if, for some (and hence for any) lift $\{u_i, v_j\}$ of the level structure,

$$S\left(\frac{u_i}{N}\right) \equiv -\frac{u_i}{N} \pmod{L} \qquad \text{and} \qquad S\left(\frac{v_j}{N}\right) \equiv \frac{v_j}{N} \pmod{L}, \qquad 1 \le i,j \le g.$$

Definition 3.3. *A real principally polarized abelian variety of dimension g with a level N structure is a quadruple $\mathcal{A} = (\mathfrak{A}, E, S, \{U_i, V_j\})$ with $\mathfrak{A} = \mathbb{C}^g/L$, where $(\mathfrak{A}, E, S)$ is a real principally polarized abelian variety and $\{U_i, V_j\}$ is a level N structure compatible with a real structure S. An isomorphism*

$$\mathcal{A} = (\mathfrak{A}, E, S, \{U_i, V_j\}) \cong (\mathfrak{A}', E', S', \{U_i', V_j'\}) = \mathcal{A}'$$

is a complex linear mapping $\phi : \mathbb{C}^g \longrightarrow \mathbb{C}^g$ such that

$$\phi(L) = L', \tag{3.3}$$

$$\phi_*(E) = E', \tag{3.4}$$

$$\phi_*(S) = S', \ \textit{that is}, \quad \phi \circ S \circ \phi^{-1} = S', \tag{3.5}$$

$$\phi\left(\frac{u_i}{N}\right) \equiv \frac{u_i'}{N} \ (\mathit{mod}\, L') \qquad \textit{and} \qquad \phi\left(\frac{v_j}{N}\right) \equiv \frac{v_j'}{N} \ (\mathit{mod}\, L'), \qquad 1 \le i,j \le g. \tag{3.6}$$

for some lift $\{u_i, v_j\}$ and $\{u_i', v_j'\}$ of the level structures.

Now we show that a given positive integer N and a given $\Omega \in \mathbb{H}_g$ determine naturally a principally polarized abelian variety $(\mathfrak{A}_\Omega, E_\Omega)$ of dimension g with a level N structure. Let E_0 be the standard alternating form on $\mathbb{R}^g \oplus \mathbb{R}^g$ with the symplectic matrix J_g with respect to the standard basis of $\mathbb{R}^g \oplus \mathbb{R}^g$. Let $F_\Omega : \mathbb{R}^g \oplus \mathbb{R}^g \longrightarrow \mathbb{C}^g$ be the real linear mapping with matrix (Ω, I_g), that is,

$$F_\Omega \begin{pmatrix} x \\ y \end{pmatrix} := \Omega\, x + y, \qquad x, y \in \mathbb{R}^g. \tag{3.7}$$

We define $E_\Omega := (F_\Omega)_*(E_0)$ and $L_\Omega := F_\Omega(\mathbb{Z}^g \oplus \mathbb{Z}^g)$. Then $(\mathfrak{A}_\Omega = \mathbb{C}^g/L_\Omega, E_\Omega)$ is a principally polarized abelian variety. The Hermitian form H_Ω on $\mathbb{C}^g$ corresponding to E_Ω is given by

$$H_\Omega(u,v) = {}^t u\, (\mathrm{Im}\,\Omega)^{-1}\, \overline{v}, \quad E_\Omega = \mathrm{Im}\, H_\Omega, \qquad u, v \in \mathbb{C}^g. \tag{3.8}$$

If $z_1, \cdots, z_g$ are the standard coordinates on $\mathbb{C}^g$, then the holomorphic 1-forms $dz_1, \cdots, dz_g$ have the period matrix (Ω, I_g). If $\{e_i, f_j\}$ is the standard basis of $\mathbb{R}^g \oplus \mathbb{R}^g$, then $\{F_\Omega(e_i/N),\ F_\Omega(f_j/N)\}$

(mod L_Ω) is a level N structure on $(\mathfrak{A}_\Omega, E_\Omega)$, which we refer to as the *standard N structure.* Assume that Ω_1 and Ω_2 are two elements of $\mathbb{H}_g$ such that

$$\psi : (\mathfrak{A}_{\Omega_1} = \mathbb{C}^g/L_{\Omega_1}, E_{\Omega_1}) \longrightarrow (\mathfrak{A}_{\Omega_2} = \mathbb{C}^g/L_{\Omega_2}, E_{\Omega_2})$$

is an isomorphism of the corresponding principally polarized abelian varieties, i.e., $\psi(L_{\Omega_1}) = L_{\Omega_2}$ and $\psi_*(E_{\Omega_1}) = E_{\Omega_2}$. We set

$$h = {}^t\big(F_{\Omega_2}^{-1} \circ \psi \circ F_{\Omega_1}\big) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then we see that $h \in \Gamma_g$. And we have

$$\Omega_1 = h \cdot \Omega_2 = (A\Omega_2 + B)(C\Omega_2 + D)^{-1} \tag{3.9}$$

and

$$\psi(Z) = {}^t(C\Omega_2 + D)Z, \qquad Z \in \mathbb{C}^g. \tag{3.10}$$

Let $\Omega \in \mathbb{H}_g$ such that $\gamma\cdot\Omega = \tau(\Omega) = -\overline{\Omega}$ for some $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$. We define the mapping $S_{\gamma,\Omega} : \mathbb{C}^g \longrightarrow \mathbb{C}^g$ by

$$S_{\gamma,\Omega}(Z) := {}^t(C\Omega + D)\,\overline{Z}, \qquad Z \in \mathbb{C}^g. \tag{3.11}$$

Then we can show that $S_{\gamma,\Omega}$ is a real structure on $(\mathfrak{A}_\Omega, E_\Omega)$ which is compatible with the polarization E_Ω (that is, $E_\Omega(S_{\gamma,\Omega}(u), S_{\gamma,\Omega}(v)) = -E_\Omega(u,v)$ for all $u, v \in \mathbb{C}^g$). Indeed according to Comessatti's Theorem (see Theorem 3.1), $S_{\gamma,\Omega}(Z) = \overline{Z}$, i.e., $S_{\gamma,\Omega}$ is a complex conjugation. Therefore we have

$$E_\Omega(S_{\gamma,\Omega}(u), S_{\gamma,\Omega}(v)) = E_\Omega(\overline{u}, \overline{v}) = -E_\Omega(u,v)$$

for all $u, v \in \mathbb{C}^g$. From now on we write simply $\sigma_\Omega = S_{\gamma,\Omega}$.

Theorem 3.2. *Let $(\mathfrak{A}, E, S)$ be a real principally polarized abelian variety of dimension g. Then there exists $\Omega = X + iY \in \mathbb{H}_g$ such that $2X \in \mathbb{Z}^{(g,g)}$ and there exists an isomorphism of real principally polarized abelian varieties*

$$(\mathfrak{A}, E, S) \cong (\mathfrak{A}_\Omega, E_\Omega, \sigma_\Omega),$$

where σ_Ω is a real structure on $\mathfrak{A}_\Omega$ induced by a complex conjugation $\sigma : \mathbb{C}^g \longrightarrow \mathbb{C}^g$.

The above theorem is essentially due to Comessatti [6]. We refer to [24, 25] for the proof of Theorem 3.2.

Theorem 3.2 leads us to define the subset $\mathscr{H}_g$ of $\mathbb{H}_g$ by

$$\mathscr{H}_g := \left\{ \Omega \in \mathbb{H}_g \mid 2\,\mathrm{Re}\,\Omega \in \mathbb{Z}^{(g,g)} \right\}. \tag{3.12}$$

Assume $\Omega = X + iY \in \mathscr{H}_g$. Then according to Theorem 3.2, $(\mathfrak{A}_\Omega, E_\Omega, \sigma_\Omega)$ is a real principally polarized abelian variety of dimension g. The matrix M_σ for the action of a complex conjugation σ on the lattice $L_\Omega = \Omega\mathbb{Z}^g + \mathbb{Z}^g$ with respect to the basis given by the columns of (Ω, I_g) is given by

$$M_\sigma = \begin{pmatrix} -I_g & 0 \\ 2X & I_g \end{pmatrix}. \tag{3.13}$$

Since

$$^tM_\sigma\, J_g\, M_\sigma = \begin{pmatrix} -I_g & 2X \\ 0 & I_g \end{pmatrix} J_g \begin{pmatrix} -I_g & 0 \\ 2X & I_g \end{pmatrix} = -J_g,$$

the canonical polarization J_g is σ-real.

Theorem 3.3. *Let Ω and Ω_* be two elements in $\mathscr{H}_g$. Then Ω and Ω_* represent (real) isomorphic triples $(\mathfrak{A}, E, \sigma)$ and $(\mathfrak{A}_*, E_*, \sigma_*)$ if and only if there exists an element $A \in GL(g, \mathbb{Z})$ such that*

$$2\,Re\,\Omega_* = 2\,A\,(Re\,\Omega)\,{}^tA \quad (mod\,2) \tag{3.14}$$

and

$$Im\,\Omega_* = A\,(Im\,\Omega)\,{}^tA. \tag{3.15}$$

Proof. Suppose $(\mathfrak{A}, E, \sigma)$ and $(\mathfrak{A}_*, E_*, \sigma_*)$ are real isomorphic. Then we can find an element $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$ such that

$$\Omega_* = (A\Omega + B)(C\Omega + D)^{-1}.$$

The map

$$\varphi : \mathbb{C}^g / L_{\Omega_*} = \mathfrak{A}_{\Omega_*} \longrightarrow \mathfrak{A}_\Omega = \mathbb{C}^g / L_\Omega$$

induced by the map

$$\widetilde{\varphi} : \mathbb{C}^g \longrightarrow \mathbb{C}^g, \qquad Z \longmapsto {}^t(C\Omega + D)\, Z$$

is a real isomorphism. Since $\widetilde{\varphi} \circ \sigma_* = \sigma \circ \widetilde{\varphi}$, i.e., $\widetilde{\varphi}$ commutes with complex conjugation on $\mathbb{C}^g$, we have $C = 0$. Therefore

$$\Omega_* = (A\Omega + B)\,{}^tA = (AX\,{}^tA + B\,{}^tA) + i\,AY\,{}^tA,$$

where $\Omega = X + iY$. Hence we obtain the desired results (3.14) and (3.15).

Conversely we assume that there exists $A \in GL(g, \mathbb{Z})$ satisfying the conditions (3.14) and (3.15). Then

$$\Omega_* = \gamma \cdot \Omega = (A\Omega + B)\,{}^tA$$

for some $\gamma = \begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix} \in \Gamma_g$ with $B \in \mathbb{Z}^{(g,g)}$ with $B\,{}^tA = A\,{}^tB$. The map $\psi : \mathfrak{A}_\Omega \longrightarrow \mathfrak{A}_{\Omega_*}$ induced by the map

$$\widetilde{\psi} : \mathbb{C}^g \longrightarrow \mathbb{C}^g, \qquad Z \longmapsto A^{-1} Z$$

is a complex isomorphism commuting complex conjugation σ. Therefore ψ is a real isomorphism of $(\mathfrak{A}, E, \sigma)$ onto $(\mathfrak{A}_*, E_*, \sigma_*)$. □

According to Theorem 3.3, we are led to define the subgroup $\Gamma_g^\star$ of Γ_g by

$$\Gamma_g^\star := \left\{ \begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix} \in \Gamma_g \;\middle|\; B \in \mathbb{Z}^{(g,g)}, \quad A\,{}^tB = B\,{}^tA \right\}. \tag{3.16}$$

It is easily seen that $\Gamma_g^\star$ acts on $\mathscr{H}_g$ properly discontinuously by

$$\gamma \cdot \Omega = A\Omega\,{}^tA + B\,{}^tA, \tag{3.17}$$

where $\gamma = \begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix} \in \Gamma_g^\star$ and $\Omega \in \mathscr{H}_g$.

4. Moduli Spaces for Real Abelian Varieties

In Section 3, we knew that $\Gamma_g^\star$ acts on $\mathscr{H}_g$ properly discontinuously by the formula (3.17). So the quotient space

$$\mathscr{X}_{\mathbb{R}}^g := \Gamma_g^\star \backslash \mathscr{H}_g$$

inherits a structure of stratified real analytic space from the real analytic structure on $\mathscr{H}_g$. The stratified real analytic space $\mathscr{X}_{\mathbb{R}}^g$ classifies, up to real isomorphism, real principally polarized abelian varieties $(\mathfrak{A}, E, S)$ of dimension g. Thus $\mathscr{X}_{\mathbb{R}}^g$ is called the (real) moduli space of real principally polarized abelian varieties $(\mathfrak{A}, E, S)$ of dimension g.

To study the structure of $\mathscr{X}_{\mathbb{R}}^g$, we need the following result of A. A. Albert [1].

Lemma 4.1. *Let $S_g(\mathbb{Z}/2)$ be the set of all $g \times g$ symmetric matrices with coefficients in $\mathbb{Z}/2$. We note that $GL(g, \mathbb{Z}/2)$ acts on $S_g(\mathbb{Z}/2)$ by $N \longmapsto AN\,{}^tA$ with $A \in GL(g, \mathbb{Z}/2)$ and $N \in S_g(\mathbb{Z}/2)$. We put*

$$\pi(N) := \prod_{k=1}^{g} (1 - n_{kk}) \qquad \text{for } N = (n_{ij}) \in S_g(\mathbb{Z}/2).$$

Then $N \in S_g(\mathbb{Z}/2)$ is equivalent mod $GL(g, \mathbb{Z}/2)$ to a matrix of the form :

$$\text{(I)} \quad \begin{pmatrix} I_\lambda & 0 \\ 0 & 0 \end{pmatrix} \quad \text{if } \ \pi(N) = 0 \text{ and } \operatorname{rank}(N) = \lambda$$

or

$$\text{(II)} \quad \begin{pmatrix} H_\lambda & 0 \\ 0 & 0 \end{pmatrix} \text{ with } H_\lambda := \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix} \in \mathbb{Z}^{(\lambda,\lambda)} \quad \text{if } \ \pi(N) = 1 \text{ and } \operatorname{rank}(N) = \lambda.$$

$N \in S_g(\mathbb{Z}/2)$ is said to be *diasymmetric* in Case (I) and to be *orthosymmetric* in Case (II).

Theorem 4.1. *Let $(\mathfrak{A}, E)$ be a principally polarized abelian variety of dimension g. Then there exists a real structure S on $\mathfrak{A}$ such that E is S-real if and only if $(\mathfrak{A}, E)$ admits a period matrix of the following form*

$$\left(I_g, \frac{1}{2} M + iY\right), \qquad Y \in \mathcal{P}_g,$$

where M is one of the forms (I) and (II) in Lemma 4.1.

The above theorem is essentially due to Comessatti [6]. We refer to [24] or [25, Theorem 2.3, pp. 78–80 and Theorem 4.1, pp. 86–88] for the proof of the above theorem.

Lemma 4.2. *Let Ω_1 and Ω_2 be two elements of $\mathscr{H}_g$ such that*

$$\Omega_i = \frac{1}{2} X_i + iY_i, \quad M_i \in \mathbb{Z}^{(g,g)}, \ Y_i \in \mathcal{P}_g, \ i = 1, 2.$$

Then Ω_1 and Ω_2 have images, under the natural projection $\pi_g : \mathscr{H}_g \longrightarrow \mathscr{X}_{\mathbb{R}}^g$, in the same connected component of $\mathscr{X}_{\mathbb{R}}^g$, if and only if $\operatorname{rank}(M_1 \bmod 2) = \operatorname{rank}(M_2 \bmod 2)$ and $\pi(M_1 \bmod 2) = \pi(M_2 \bmod 2)$.

Theorem 4.2. *$\mathscr{X}_{\mathbb{R}}^g$ is a real analytic manifold of dimension $g(g+1)/2$ and has $g+1+\left[\frac{g}{2}\right]$ connected components. Moreover $\mathscr{X}_{\mathbb{R}}^g$ is semi-algebraic, i.e., $\mathscr{X}_{\mathbb{R}}^g$ is defined by a finite number of polynomial equalities and inequalities.*

Proof. The proof can be found in [21, Theorem 6.1, p. 161]. □

Remark 4.1. *Let $\Omega = \frac{1}{2}M + iY \in \mathscr{H}_g$ with $M = {}^tM \in \mathbb{Z}^{(g,g)}$. If $\text{rank}\,(M\, mod\ 2) = \lambda$, then $\mathfrak{A}_\Omega(\mathbb{R})$ has $2^{g-\lambda}$ connected components (cf. [21, 24]). The other invariant $\pi(M_2\, mod\ 2)$ is an invariant related to the polarization.*

Recall that by Lemma 4.1, the connected components of $\mathscr{X}_{\mathbb{R}}^g$ correspond to the different possible values of $(\lambda, i) = \big(\text{rank}\,(M\ mod\ 2), \pi(M\ mod\ 2)\big)$ on which we have the restriction :

$$0 \le \lambda \le g, \quad i = 0 \text{ or } 1, \quad \text{and} \quad i = 0 \text{ if } \lambda \text{ is odd}, \quad i = 1 \text{ if } \lambda = 0. \tag{4.1}$$

We denote by $\mathscr{X}_{(\lambda,i)}^g$ the connected components of $\mathscr{X}_{\mathbb{R}}^g$ corresponding to the invariants (λ, i).

Definition 4.1. *Let $M \in \mathbb{Z}^{(g,g)}$ be a $g \times g$ symmetric integral matrix. We say that M is of the standard form if M is of one of the forms in Lemma 4.1 (we observe that for fixed (λ, i) this form is unique).*

Now we can prove the following.

Lemma 4.3. *Let $M \in \mathbb{Z}^{(g,g)}$ be a symmetric integral matrix which is of the standard form with invariants (λ, i). Let*

$$\Gamma_{(\lambda,i)}^g := \left\{ A \in GL(g, \mathbb{Z}) \,|\; AM\,{}^tA \equiv M \ \ (mod\ 2) \ \right\}.$$

Then

$$\mathscr{X}_{(\lambda,i)}^g \cong \Gamma_{(\lambda,i)}^g \backslash \mathcal{P}_g.$$

Proof. Let $[\Omega]$ be a class in $\mathscr{X}_{(\lambda,i)}^g$. By Lemma 4.1 and Lemma 4.2, there exist a symmetric integral matrix $M \in \mathbb{Z}^{(g,g)}$ with invariants (λ, i) of the standard form and an element $Y \in \mathcal{P}_g$ such that $\frac{1}{2}M + iY$ is a representative for the class $[\Omega]$. If $Y_* \in \mathcal{P}_g$ is such that $\frac{1}{2}M + iY_*$ is also a representative for the class $[\Omega]$, according to Theorem 3.2,

$$M \equiv AM\,{}^tA \ \ (mod\ 2) \qquad \text{and} \qquad Y_* = AY\,{}^tA$$

for some $A \in GL(g, \mathbb{Z})$. □

Theorem 4.3. *$\mathscr{X}_{(\lambda,i)}^g$ is a connected semi-algebraic set with a real analytic structure.*

Proof. The proof can be found in [21, p. 160]. □

Let $(\mathfrak{A}, E, S)$ be a real polarized abelian variety and $-S$ be the real structure obtained by composing S with the involution $z \longmapsto -z$ of $\mathfrak{A}$. We see that $(\mathfrak{A}, E, -S)$ is also a real polarized abelian variety. In general $(\mathfrak{A}, E, -S)$ is not real isomorphic to $(\mathfrak{A}, E, S)$. Therefore the following correspondence

$$\Sigma : \mathscr{X}_{\mathbb{R}}^g \longrightarrow \mathscr{X}_{\mathbb{R}}^g, \qquad (\mathfrak{A}, E, S) \longmapsto (\mathfrak{A}, E, -S) \tag{4.2}$$

defines a non-trivial involution of $\mathscr{X}_{\mathbb{R}}^g$.

Let $M \in \mathbb{Z}^{(g,g)}$ be a symmetric integral matrix which is of the standard form with invariants (λ, i). It is easily checked that $M^3 = M$. We put

$$\Sigma_M := \begin{pmatrix} -M & I_g \\ -(I_g + M^2) & M \end{pmatrix}. \tag{4.3}$$

It is easy to see the following facts (4.4) and (4.5).

$$\Sigma_M \in \Gamma_g \qquad \text{and} \qquad \left(\Sigma_M\right)^{-1} = -\Sigma_M. \tag{4.4}$$

$$\left({}^t\Sigma_M\right)^{-1} \begin{pmatrix} -I_g & 0 \\ M & I_g \end{pmatrix} {}^t\Sigma_M = \begin{pmatrix} I_g & 0 \\ -M & -I_g \end{pmatrix}. \tag{4.5}$$

Now we assume that $\Omega = \frac{1}{2}M + i\,Y \in \mathscr{H}_g$ represents $(\mathfrak{A}, E, S)$. By (3.13), the matrices of S and $-S$ are given by

$$M_S = \begin{pmatrix} -I_g & 0 \\ M & I_g \end{pmatrix} \qquad \text{and} \qquad M_{-S} = \begin{pmatrix} I_g & 0 \\ -M & -I_g \end{pmatrix} \tag{4.6}$$

respectively with respect to the $\mathbb{R}$-basis given by the columns of (Ω, I_g). By the formulas (4.5) and (4.6) we see that $\Sigma_M(\Omega)$ represents the real polarized abelian variety.

Lemma 4.4. *Let $M \in \mathbb{Z}^{(g,g)}$ be a symmetric integral matrix which is of the standard form with invariants (λ, i) and $Y \in \mathcal{P}_g$. Then we have*

$$\Sigma_M\left(\frac{1}{2}M + i\,Y\right) = \frac{1}{2}M + i \begin{pmatrix} \frac{1}{2}I_\lambda & 0 \\ 0 & I_{g-\lambda} \end{pmatrix} Y^{-1} \begin{pmatrix} \frac{1}{2}I_\lambda & 0 \\ 0 & I_{g-\lambda} \end{pmatrix}^{-1}. \tag{4.7}$$

Proof. Using the fact that $M^3 = M$, by a direct computation, we get

$$\Sigma_M\left(\frac{1}{2}M + i\,Y\right) = M\left(I_g + M^2\right)^{-1} + i\left(I_g - \frac{1}{2}M^2\right)Y^{-1}\left(I_g + M^2\right)^{-1}. \tag{4.8}$$

It is easily checked that

$$\left(I_g + M^2\right)^{-1} = I_g - \frac{1}{2}M^2 = \begin{pmatrix} \frac{1}{2}I_\lambda & 0 \\ 0 & I_{g-\lambda} \end{pmatrix}. \tag{4.9}$$

The formula (4.7) follows immediately from (4.8) and (4.9). □

Proposition 4.1. *The map $\Sigma : \mathscr{X}_{\mathbb{R}}^g \longrightarrow \mathscr{X}_{\mathbb{R}}^g$ defined by*

$$\Sigma\left([(\mathfrak{A}, E, S)]\right) := [(\mathfrak{A}, E, -S)], \qquad [(\mathfrak{A}, E, S)] \in \mathscr{X}_{\mathbb{R}}^g \tag{4.10}$$

is a real analytic involution of $\mathscr{X}_{\mathbb{R}}^g$. For each connected component $\mathscr{X}_{(\lambda,i)}^g$, we have

$$\Sigma\left(\mathscr{X}_{(\lambda,i)}^g\right) = \mathscr{X}_{(\lambda,i)}^g.$$

Hence Σ leaves the connected components of $\mathscr{X}_{\mathbb{R}}^g$ globally fixed.

Proof. Let $M \in \mathbb{Z}^{(g,g)}$ be a symmetric integral matrix which is of the standard form with invariants (λ, i). We denote by $\mathscr{H}_g(M)$ the connected component of $\mathscr{H}_g$ containing the matrices of the form $\frac{1}{2}M + i\,Y \in \mathscr{H}_g$ with $Y \in \mathcal{P}_g$. According to (4.5) and Lemma 4.4, we see that Σ_M defines an involution of $\mathscr{H}_g(M)$. Since $\mathscr{H}_g(M)$ is mapped onto $\mathscr{X}_{(\lambda,i)}^g$, we obtain the desired result. □

5. Compactifications of the Moduli Space $\mathscr{X}_{\mathbb{R}}^g$

In this section we review the compactification $\overline{\mathscr{X}_{\mathbb{R}}^g}$ of $\mathscr{X}_{\mathbb{R}}^g$ obtained by R. Silhol [26] and the Baily-Borel compactification of $\Gamma_g(4m)\backslash\mathbb{H}_g$ which is related to the moduli space of real abelian varieties with level $4m$ structure.

First of all we recall the Satake compactification of the Siegel modular variety $\mathcal{A}_g := \Gamma_g\backslash\mathbb{H}_g$. Let

$$\mathbb{D}_g := \left\{ W \in \mathbb{C}^{(g,g)} \mid W = {}^tW,\ I_g - \overline{W}W > 0 \right\} \tag{5.1}$$

be the generalized unit disk of degree g which is a bounded realization of $\mathbb{H}_g$. In fact, the Cayley transform $\Phi_g : \mathbb{D}_g \longrightarrow \mathbb{H}_g$ defined by

$$\Phi_g(W) := i\,(I_g + W)(I_g - W)^{-1}, \qquad W \in \mathbb{D}_g \tag{5.2}$$

is a biholomorphic mapping of $\mathbb{D}_g$ onto $\mathbb{H}_g$ which gives the bounded realization of $\mathbb{H}_g$ by $\mathbb{D}_g$ [23, pp. 281-283]. The inverse Ψ_g of Φ_g is given by

$$\Psi_g(\Omega) = (\Omega - i\,I_g)(\Omega + i\,I_g)^{-1}, \qquad \Omega \in \mathbb{H}_g. \tag{5.3}$$

We let

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} I_g & I_g \\ iI_g & -iI_g \end{pmatrix}$$

be the $2g \times 2g$ matrix represented by Φ_g. Then

$$T^{-1}Sp(g,\mathbb{R})\,T = \left\{ \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in \mathbb{C}^{(2g,2g)} \,\Big|\, {}^tP\overline{P} - {}^t\overline{Q}Q = I_g,\ {}^tP\overline{Q} = {}^t\overline{Q}P \right\}.$$

Indeed, if $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R})$, then

$$T^{-1}MT = \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix},$$

where

$$P = \frac{1}{2}\left\{(A + D) + i\,(B - C)\right\} \tag{5.4}$$

and

$$Q = \frac{1}{2}\left\{(A - D) - i\,(B + C)\right\}. \tag{5.5}$$

For brevity, we set

$$G_* = T^{-1}Sp(g,\mathbb{R})T.$$

Then G_* is a subgroup of $SU(g,g)$, where

$$SU(g,g) = \left\{ h \in \mathbb{C}^{(2g,2g)} \mid {}^thI_{g,g}\overline{h} = I_{g,g},\ \det h = 1 \right\}, \quad I_{g,g} = \begin{pmatrix} I_g & 0 \\ 0 & -I_g \end{pmatrix}.$$

In the case $g = 1$, we observe that

$$T^{-1}Sp(1,\mathbb{R})T = T^{-1}SL_2(\mathbb{R})T = SU(1,1).$$

If $g > 1$, then G_* is a *proper* subgroup of $SU(g,g)$. In fact, since ${}^tTJ_gT = -\,i\,J_g$, we get

$$G_* = \left\{ h \in SU(g,g) \mid {}^thJ_gh = J_g \right\}.$$

Let

$$P^+ = \left\{ \begin{pmatrix} I_g & Z \\ 0 & I_g \end{pmatrix} \,\Big|\, Z = {}^tZ \in \mathbb{C}^{(g,g)} \right\}$$

be the P^+-part of the complexification of $G_* \subset SU(g,g)$.

Since the Harish-Chandra decomposition of an element $\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix}$ in G_*^J is

$$\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} = \begin{pmatrix} I_g & Q\overline{P}^{-1} \\ 0 & I_g \end{pmatrix} \begin{pmatrix} P - Q\overline{P}^{-1}\overline{Q} & 0 \\ 0 & \overline{P} \end{pmatrix} \begin{pmatrix} I_g & 0 \\ \overline{P}^{-1}\overline{Q} & I_g \end{pmatrix},$$

the P^+-component of the following element

$$\begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \cdot \begin{pmatrix} I_g & W \\ 0 & I_g \end{pmatrix}, \quad W \in \mathbb{D}_g$$

of the complexification of G_*^J is given by

$$\left(\begin{pmatrix} I_g & (PW+Q)(\overline{Q}W + \overline{P})^{-1} \\ 0 & I_g \end{pmatrix} \right).$$

We note that $Q\overline{P}^{-1} \in \mathbb{D}_g$. We get the Harish-Chandra embedding of $\mathbb{D}_g$ into P^+ (cf. [12, p. 155] or [19, pp. 58-59]). Therefore we see that G_* acts on $\mathbb{D}_g$ transitively by

$$(5.6) \qquad \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \cdot W = (PW+Q)(\overline{Q}W + \overline{P})^{-1}, \quad \begin{pmatrix} P & Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in G_*, \; W \in \mathbb{D}_g.$$

The isotropy subgroup at the origin o is given by

$$K = \left\{ \begin{pmatrix} P & 0 \\ 0 & \overline{P} \end{pmatrix} \,\Big|\, P \in U(g) \right\}.$$

Thus G_*/K is biholomorphic to $\mathbb{D}_g$. The action (2.4) is compatible with the action (5.6) via the Cayley transform (5.2).

In summary, $Sp(g,\mathbb{R})$ acts on $\mathbb{D}_g$ transitively by

$$(5.7) \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot W = (PW+Q)(\overline{Q}W + \overline{P})^{-1}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{R}), \; W \in \mathbb{D}_g,$$

where P and Q are given by (5.4) and (5.5). This action extends to the closure $\overline{\mathbb{D}}_g$ of $\mathbb{D}_g$ in $\mathbb{C}^{g(g+1)/2}$.

For an integer s with $0 \le s \le g$, we let

$$(5.8) \qquad \mathscr{F}_s := \left\{ W = \begin{pmatrix} W_1 & 0 \\ 0 & I_{g-s} \end{pmatrix} \,\Big|\, W_1 \in \mathbb{D}_s \right\} \subset \overline{\mathbb{D}}_g.$$

We say that $\mathscr{F}_s$ is the *standard boundary component* of degree s. If there exists an element $\gamma \in Sp(g,\mathbb{Q})$ (equivalently $\gamma \in \Gamma_g$) with $\mathscr{F} = \gamma(\mathscr{F}_s) \subset \overline{\mathbb{D}}_g$, then $\mathscr{F}$ is said to be a *rational boundary component* of degree s. The Siegel upper half plane $\mathbb{H}_s$ is attached to $\mathbb{H}_g$ as a limit of matrices in $\mathbb{C}^{(g,g)}$ by

$$\Omega_1 \longmapsto \lim_{Y \longrightarrow \infty} \begin{pmatrix} \Omega_1 & 0 \\ 0 & iY \end{pmatrix}, \qquad \Omega_1 \in \mathbb{H}_s, \; Y \in \mathcal{P}_{g-s},$$

meaning that all the eigenvalues of Y converge to ∞.

For a rational boundary component $\mathscr{F} \subset \overline{\mathbb{D}}_g$, we let

$$P(\mathscr{F}) = \{\, \alpha \in Sp(g,\mathbb{Q}) \mid \alpha(\mathscr{F}) = \mathscr{F} \,\}$$

be the normalizer in $Sp(g,\mathbb{Q})$ of $\mathscr{F}$ (or the parabolic subgroup of $Sp(g,\mathbb{Q})$ associated to $\mathscr{F}$) and let

$$Z(\mathscr{F}) = \{\, \alpha \in Sp(g,\mathbb{Q}) \mid \alpha(W) = W \ \text{ for all } W \in \mathscr{F} \,\}$$

be the centralizer of $\mathscr{F}$. We put

$$G(\mathscr{F}) := P(\mathscr{F})/Z(\mathscr{F}) \cong Sp(s,\mathbb{Q}).$$

Obviously $G(\mathscr{F})$ acts on $\mathscr{F}$. We choose the standard boundary component $\mathscr{F} = \mathscr{F}_s$. An element γ of $P(\mathscr{F})$ is of the form

$$\gamma = \begin{pmatrix} A_1 & 0 & B_1 & * \\ * & u & * & * \\ C_1 & 0 & D_1 & * \\ 0 & 0 & 0 & {}^t u^{-1} \end{pmatrix} \in Sp(g,\mathbb{Q}), \tag{5.9}$$

where

$$\gamma_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in Sp(s,\mathbb{Q}) \qquad \text{and} \qquad u \in GL(g-s,\mathbb{Q}).$$

The unipotent radical $U(\mathscr{F})$ of $P(\mathscr{F})$ is given by

$$U(\mathscr{F}) = \left\{ \begin{pmatrix} I_s & 0 & 0 & {}^t\mu \\ \lambda & I_{g-s} & \mu & \kappa \\ 0 & 0 & I_s & -{}^t\lambda \\ 0 & 0 & 0 & I_{g-s} \end{pmatrix} \;\middle|\; \lambda,\mu \in \mathbb{Q}^{(g-s,s)},\ \kappa \in \mathbb{Q}^{(g-s,g-s)} \right\} \tag{5.10}$$

and the centralizer $Z_U(\mathscr{F})$ of $U(\mathscr{F})$ is given by

$$Z_U(\mathscr{F}) = \left\{ \begin{pmatrix} I_s & 0 & 0 & 0 \\ 0 & I_{g-s} & 0 & \kappa \\ 0 & 0 & I_s & 0 \\ 0 & 0 & 0 & I_{g-s} \end{pmatrix} \;\middle|\; \kappa \in \mathbb{Q}^{(g-s,g-s)} \right\}. \tag{5.11}$$

We have inclusions of normal subgroups

$$Z_U(\mathscr{F}) \subset U(\mathscr{F}) \subset P(\mathscr{F}).$$

The Levi factor $L(\mathscr{F})$ of $P(\mathscr{F})$ is given by

$$L(\mathscr{F}) = G_h(\mathscr{F})\, G_l(\mathscr{F}) \tag{5.12}$$

with

$$G_h(\mathscr{F}) = \left\{ \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & I_{g-s} & 0 & 0 \\ C_1 & 0 & D_1 & 0 \\ 0 & 0 & 0 & I_{g-s} \end{pmatrix} \in P(\mathscr{F}) \;\middle|\; \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in Sp(s,\mathbb{Q}) \right\}. \tag{5.13}$$

and

$$G_l(\mathscr{F}) = \left\{ \begin{pmatrix} I_s & 0 & 0 & 0 \\ 0 & S & 0 & 0 \\ 0 & 0 & I_s & 0 \\ 0 & 0 & 0 & {}^t S^{-1} \end{pmatrix} \in P(\mathscr{F}) \;\middle|\; S \in GL(g-s,\mathbb{Q}) \right\}. \tag{5.14}$$

The subgroup $U(\mathscr{F})G_h(\mathscr{F})$ is normal in $P(\mathscr{F})$. The map $P(\mathscr{F}) \longrightarrow Sp(s,\mathbb{Q})$, $\gamma \mapsto \gamma_1$ is surjective and induces the isomorphism $G_h(\mathscr{F}_s) \cong Sp(s,\mathbb{Q})$. We note that the map f : $P(\mathscr{F}_s) \cap Sp(g,\mathbb{Z}) \longrightarrow Sp(s,\mathbb{Z})$, $\gamma \mapsto \gamma_1$ is obtained via $\begin{pmatrix} W & 0 \\ 0 & I_{g-s} \end{pmatrix} \longmapsto W$, in the sense that if $\gamma \in P(\mathscr{F}_s)$, then

$$\gamma \cdot \begin{pmatrix} W & 0 \\ 0 & I_{g-s} \end{pmatrix} = \begin{pmatrix} \gamma_1(W) & 0 \\ 0 & I_{g-s} \end{pmatrix}.$$

We define

$$\mathbb{D}_g^{st} := \coprod_{0 \le s \le g} \mathscr{F}_s$$

and

$$\mathbb{D}_g^{*} := \coprod_{\mathscr{F}:rational} \mathscr{F},$$

where $\mathscr{F}$ runs over all rational boundary components. Via the Cayley transform Φ_g (cf. (5.2)), we identify

$$\mathbb{D}_g^{st} = \mathbb{H}_g^{st} = \coprod_{0 \le s \le g} \mathbb{H}_s.$$

Definition 5.1. *Let $u > 1$. We denote by $\mathfrak{W}_g(u)$ the set of all matrices $\Omega = X + iY$ in $\mathbb{H}_g$ with $X = (x_{ij}) \in \mathbb{R}^{(g,g)}$ satisfying the conditions* $(\Omega 1)$ *and* $(\Omega 2)$ *:*

$(\Omega 1)$ $|x_{ij}| < u$;

$(\Omega 2)$ *if $Y = {}^tWDW$ is the Jacobi decomposition of Y with $W = (w_{ij})$ strictly upper triangular and $D = diag(d_1, \cdots, d_g)$ diagonal, then we have*

$$|w_{ij}| < u, \quad 1 < u\, d_1, \quad d_i < u\, d_{i+1}, \quad i = 1, \cdots, g-1.$$

It is well known that for sufficiently large $u > 0$, the set $\mathfrak{W}_g(u)$ is a fundamental set for the action of Γ_g on $\mathbb{H}_g$, that is, $\Gamma_g \cdot \mathfrak{W}_g(u) = \mathbb{H}_g$, and

$$\{\, \gamma \in \Gamma_g \mid \gamma \cdot \mathfrak{W}_g(u) \cap \mathfrak{W}_g(u) \neq \varnothing \,\}$$

is a finite set. We observe that if $\Omega = \begin{pmatrix} \Omega_1 & \Omega_3 \\ {}^t\Omega_3 & \Omega_2 \end{pmatrix} \in \mathfrak{W}_g(u)$ with $\Omega_1 \in \mathbb{C}^{(s,s)}$, then $\Omega_1 \in \mathfrak{W}_s(u)$.

Definition 5.2. *We can choose a sufficiently large $u_0 > 0$ such that for all $0 \le s \le g$, $\mathfrak{W}_s(u_0)$ is a fundamental set for the action of Γ_s on $\mathbb{H}_s$. In this case we simply write $\mathfrak{W}_s = \mathfrak{W}_s(u_0)$ with $0 \le s \le g$. We define*

$$\mathfrak{W}_g^{*} := \coprod_{0 \le s \le g} \mathfrak{W}_s.$$

For $\Omega_* \in \mathfrak{W}_{g-r}$, we let U be a neighborhood of Ω_* in $\mathfrak{W}_{g-r}$ and v a positive real number. For $0 \le s \le r$, we let $W_s(U,v)$ be the set of all

$$\Omega = \begin{pmatrix} \Omega_1 & \Omega_3 \\ {}^t\Omega_3 & \Omega_2 \end{pmatrix} \in \mathfrak{W}_{g-s} \quad \text{with} \quad \Omega_1 \in \mathbb{C}^{(g-r,g-r)}$$

satisfying the conditions $(\mathfrak{W}1)$ *and* $(\mathfrak{W}1)$ *:*

$(\mathfrak{W}1)$ $\Omega_1 \in U$;

$(\mathfrak{W}2)$ *if $Y = {}^tWDW$ is the Jacobi decomposition of Y with W strictly upper triangular and $D = diag(d_1, \cdots, d_g)$ diagonal, then we have $d_{g-r+1} > v$.*

A fundamental set of neighborhoods of $\Omega_* \in \mathfrak{W}_{g-r}$ for the Satake topology on $\mathfrak{W}_g^*$ is given by the collection $\{\bigcup_{0\le s\le r} W_s(U,v)\}$'s, where U runs through neighborhoods of Ω_* in $\mathfrak{W}_{g-r}$ and v ranges in $\mathbb{R}^+$. We regard

$$\mathfrak{W}_g^* \subset \mathbb{H}_g^{st} \cong \mathbb{D}_g^{st}$$

as a subset of $\mathbb{D}_g^*$.

The Satake topology on $\mathbb{D}_g^*$ is characterized as the unique topology $\mathscr{T}$ extending the ordinary matrix topology on $\mathbb{D}_g$ and satisfying the following properties (ST1)–(ST4) :

(ST1) $\mathscr{T}$ induces on $\mathfrak{W}_g^*$ the topology defined in Definition 5.2

(ST2) $Sp(g,\mathbb{Q})$ acts continuously on $\mathbb{D}_g^*$;

(ST3) $\mathcal{A}_g^* = \Gamma_g\backslash\mathbb{D}_g^*$ is a compact Hausdorff space ;

(ST4) For any $\Omega \in \mathbb{D}_g^*$, there exists a fundamental set of neighborhoods $\{U\}$ of Ω such that $\gamma\cdot U = U$ if $\gamma \in \Gamma_g(\Omega) := \{\gamma\in\Gamma_g \mid \gamma\cdot\Omega = \Omega\}$, and $\gamma\cdot U\cap U = \varnothing$ if $\gamma\notin\Gamma_g(\Omega)$.

For a proof of these above facts we refer to [4].

Now we are ready to investigate the compactification of the moduli space $\mathscr{X}_{\mathbb{R}}^g$ of real principally polarized abelian varieties of dimension g obtained by R. Silhol.

Definition 5.3. *Let $u > 1$. We let $F_g(u)$ be the set of all $\Omega = X + iY \in \mathscr{H}_g$ with $X = Re\,\Omega = (x_{ij})$ satisfying the following conditions (a) and (b) :*

(a) $x_{ij} = 0$ or $\frac{1}{2}$;

(b) if $Y = {}^tWDW$ is the Jacobi decomposition of Y with W strictly upper triangular and $D = diag(d_1,\cdots,d_g)$ diagonal, then we have

$$|w_{ij}| \le u \qquad \text{and} \qquad 0 < d_i \le u\, d_{i+1}.$$

We define $F_g'(u)$ to be the set of matrices in $\mathscr{H}_g$ satisfying the condition $|x_{ij}| \le \frac{1}{2}$ and the above condition (b). Let $u_0 >$ be as in Definition 5.2. We put

$$F_g := F_g(u_0).$$

It is well known that F_g is a fundamental set for the action of Γ_g^* on $\mathscr{H}_g$.

For two nonnegative integers s and t, we define two subsets $\mathscr{F}_{s,t}$ and $F_{s,t}$ of $\mathbb{D}_g^*$ as follows.

$$\mathscr{F}_{s,t} := \left\{ \begin{pmatrix} -I_s & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & I_t \end{pmatrix} \in \mathbb{D}_g^* \;\middle|\; W \in \mathbb{D}_{g-(s+t)} \right\} \tag{5.15}$$

and

$$F_{s,t} := \left\{ \begin{pmatrix} -I_s & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & I_t \end{pmatrix} \in \mathscr{F}_{s,t} \;\middle|\; W \in F_{g-(s+t)} \right\}. \tag{5.16}$$

For $M \in \mathbb{Z}^{(g,g)}$, we set

$$F_M := \{\,\Omega\in F_g \mid 2\,\mathrm{Re}\,\Omega = M\,\}.$$

In particular, $F_{\mathbf{0}} = \{\,\Omega\in F_g \mid \mathrm{Re}\,\Omega = 0\,\}$, where $\mathbf{0}$ denotes the $g\times g$ zero matrix. We let

$$\mathcal{M} := \left\{ M = (m_{ij}) \in \mathbb{Z}^{(g,g)} \mid M = {}^tM,\ m_{ij} = 0 \text{ or } 1 \right\}.$$

For any $M \in \mathcal{M}$, we set

$$B_M := \begin{pmatrix} I_g & \frac{1}{2}M \\ 0 & I_g \end{pmatrix} \in Sp(g, \mathbb{Q}).$$

By the definition we have

$$F_g = \bigcup_{M \in \mathcal{M}} B_M(F_{\mathbf{0}}) \qquad \text{and} \qquad \overline{F}_g = \bigcup_{M \in \mathcal{M}} B_M(\overline{F}_{\mathbf{0}}).$$

We can show that $\overline{\mathscr{H}_g} = \Gamma_g^\star \cdot \overline{F}_g$.

Now we embed $\mathscr{H}_g$ into $\mathbb{D}_g^*$ via the Cayley transform (5.3). We let $\overline{\mathscr{H}_g}$ be the closure of $\mathscr{H}_g$ in $\mathbb{D}_g^*$. Then the action of $\Gamma_g^\star$ extends to an action of $\Gamma_g^\star$ on $\overline{\mathscr{H}_g}$ (see (3.16), (3.17), (ST2)). R. Silhol proved that the quotient space $\Gamma_g^\star \backslash \overline{\mathscr{H}_g}$ is a connected, compact Hausdorff space (cf. [26, pp. 173-177]). Let $\pi : \overline{\mathscr{H}_g} \longrightarrow \Gamma_g^\star \backslash \overline{\mathscr{H}_g}$ be the canonical projection. For $M \in \mathcal{M}$, we define

$$\mathscr{H}_M = \left\{ \frac{1}{2} M + iY \in \mathscr{H}_g \right\}.$$

We let $\overline{\mathscr{H}}_M$ be the closure of $\mathscr{H}_M$ in $\overline{\mathscr{H}_g}$. Then without difficulty we can see that

$$\Gamma_g^\star \backslash \overline{\mathscr{H}_g} = \bigcup_{0 \le s+t \le g} \bigcup_{M \in \mathcal{M}} \left(\pi \left(B_M(\mathscr{F}_{s,t}) \cup \overline{\mathscr{H}}_{\mathbf{0}} \right) \right), \tag{5.17}$$

Let $\{ \mathscr{X}_i \mid 1 \le i \le N \}$ with $N = g + 1 + \left[\frac{g}{2} \right]$ be the connected components of $\mathscr{X}_{\mathbb{R}}^g \subset \Gamma_g^\star \backslash \overline{\mathscr{H}_g}$ and let Σ_i be the restriction to $\mathscr{X}_i$ of the fundamental involution Σ (cf. Proposition 4.1). We note that Σ does not extend to a global involution of $\Gamma_g^\star \backslash \overline{\mathscr{H}_g}$. But Σ_i extends to an involution of the closure $\overline{\mathscr{X}}_i$ of $\mathscr{X}_i$ in $\Gamma_g^\star \backslash \overline{\mathscr{H}_g}$. We observe that for each $1 \le i \le N$, we have $\overline{\mathscr{X}}_i = \Gamma_g^\star(M_i) \backslash \overline{\mathscr{H}}_{M_i}$ for some $M_i \in \mathcal{M}$. Here $\Gamma_g^\star(M_i) = \left\{ \gamma \in \Gamma_g^\star \mid \gamma\left(\mathscr{H}_{M_i}\right) = \mathscr{H}_{M_i} \right\}$.

Definition 5.4. *Let $z_1 \in \overline{\mathscr{X}}_i$ and $z_2 \in \overline{\mathscr{X}}_j$. We say that z_1 and z_2 are Σ-equivalent and write $z_1 \sim z_2$ if $\Sigma_i(z_1) = \Sigma_j(z_2)$.*

Silhol [26, p. 185] showed that $\sim$ defines an equivalence relation in $\Gamma_g^\star \backslash \overline{\mathscr{H}_g}$.

By a direct computation, we obtain

$$P(\mathscr{F}_{s,t}) = \left\{ \begin{pmatrix} v_1 & 0 & 0 & 0 & 0 & v_{21} \\ * & A_1 & 0 & 0 & B_1 & * \\ * & * & u_2 & -u_{21} & * & * \\ * & * & -u_{12} & u_1 & * & * \\ * & C_1 & 0 & 0 & D_1 & * \\ v_{12} & 0 & 0 & 0 & 0 & v_2 \end{pmatrix} \in Sp(g, \mathbb{Q}) \right\},$$

where

$$\gamma_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in Sp(g - r, \mathbb{Q}) \quad \text{with} \quad r = s + t$$

and

$$U = \begin{pmatrix} u_1 & u_{12} \\ u_{21} & u_2 \end{pmatrix} \in GL(r, \mathbb{Q}), \qquad V = \begin{pmatrix} v_1 & v_{21} \\ v_{12} & v_2 \end{pmatrix} = {}^tU^{-1}.$$

Now we define

$$\mathscr{X}_{\mathbb{R}}^g(s,t) := (\Gamma_g^\star \cap P(\mathscr{F}_{s,t})) \backslash (\mathscr{F}_{s,t} \cap \overline{\mathscr{H}_g}). \tag{5.18}$$

It is easily checked that

$$\Gamma_g^\star \cap P(\mathscr{F}_{s,t}) \cong \Gamma_{g-(s+t)}^\star \quad \text{and} \quad \mathscr{F}_{s,t} \cap \overline{\mathscr{H}_g} \cong \overline{\mathscr{H}}_{g-(s+t)}.$$

We define

$$\overline{\mathscr{X}_{\mathbb{R}}^g} := \Gamma_g^\star \backslash \overline{\mathscr{H}_g} / \sim . \tag{5.19}$$

Silhol [26, Theorem 8.17] proved the following theorem.

Theorem 5.1. *$\overline{\mathscr{X}_{\mathbb{R}}^g}$ is a connected compact Hausdorff space containing $\mathscr{X}_{\mathbb{R}}^g$ as a dense open subset. As a set,*

$$\overline{\mathscr{X}_{\mathbb{R}}^g} = \coprod_{0 \le s+t \le g} \mathscr{X}_{\mathbb{R}}^g(s,t).$$

We recall that $\mathbb{H}_g^*$ denotes the Satake partial compactification of $\mathbb{H}_g$ that is obtained by attaching all rational boundary components with the Satake topology. We know that $Sp(g,\mathbb{Q})$ acts on $\mathbb{H}_g^*$, the involution $\tau : \mathbb{H}_g \longrightarrow \mathbb{H}_g$ (cf. (2.9)) extends to $\mathbb{H}_g^*$ and $\tau(\alpha \cdot x) = \tau(\alpha)\,\tau(x)$ for all $\alpha \in Sp(g,\mathbb{Q})$ and $x \in \mathbb{H}_g^*$.

Let $N = 4m$ with m a positive integer. We write

$$X(N) := \Gamma_g(N)\backslash \mathbb{H}_g \qquad \text{and} \qquad V(N) := \Gamma_g(N)\backslash \mathbb{H}_g^*.$$

We let

$$\pi_{BB} : \mathbb{H}_g^* \longrightarrow V(N) = \Gamma_g(N)\backslash \mathbb{H}_g^* \tag{5.20}$$

be the canonical projection of $\mathbb{H}_g^*$ to the Baily-Borel compactification of $X(N)$. The involution τ passes to complex conjugation $\tau : V(N) \longrightarrow V(N)$, whose fixed points we denote by $V(N)_{\mathbb{R}}$. Obviously the τ-fixed set

$$X(N)_{\mathbb{R}} := \{\, x \in X(N) \mid \tau(x) = x \,\}$$

is a subset of $V(N)_{\mathbb{R}}$. We let $\overline{X(N)}_{\mathbb{R}}$ denote the closure of $X(N)_{\mathbb{R}}$ in $V(N)_{\mathbb{R}}$.

Theorem 5.2. *There exists a natural rational structure on $V(N)$ which is compatible with the real structure defined by τ.*

Proof. It follows from Shimura's result [22] that the $\Gamma_g(N)$-automorphic forms on $\mathbb{H}_g$ are generated by those automorphic forms with rational Fourier coefficients. □

If $\gamma \in \Gamma_g(N)$ and $\mathscr{F}$ is a rational boundary component of $\mathbb{H}_g^*$ such that $\tau(\mathscr{F}) = \mathscr{F}$, we define the set of γ-real points of $\mathscr{F}$ to be

$$\mathscr{F}^{\tau\gamma} := \{\, x \in \mathscr{F} \mid \tau(x) = \gamma \cdot x \,\}. \tag{5.21}$$

Then $\pi_{BB}\big(\mathscr{F}^{\tau\gamma}\big) \subset V(N)_{\mathbb{R}}$.

Definition 5.5. *Let $N = 4m$. A $\Gamma_g(N)$-real boundary pair $(\mathscr{F},\gamma)$ of degree s consists of a rational boundary component $\mathscr{F}$ of degree s and an element $\gamma \in \Gamma_g(N)$ such that $\mathscr{F}^{\tau\gamma} \neq \varnothing$. We say that two $\Gamma_g(N)$-real boundary components $(\mathscr{F},\gamma)$ and $(\mathscr{F}_*,\gamma_*)$ are equivalent if the resulting loci of real points $\pi_{BB}(\mathscr{F}^{\tau\gamma}) = \pi_{BB}(\mathscr{F}_*^{\tau\gamma})$ coincide.*

We observe that if $(\mathscr{F},\gamma)$ is a $\Gamma_g(N)$-real boundary pair and if $\alpha\in\Gamma_g(N)$, we see that $\tau(\mathscr{F})=\gamma(\mathscr{F})$ and $(\alpha(\mathscr{F}),\tau(\alpha)\gamma\,\alpha^{-1})$ is an equivalent $\Gamma_g(N)$-real boundary pair.

Fix a positive integer s with $1\le s\le g$. We define the map $\Phi:\mathbb{H}_s\longrightarrow\mathbb{H}_g^*$ by

$$\Phi(\Omega_1)=\lim_{Y\longrightarrow\infty}\begin{pmatrix}\Omega_1 & 0\\ 0 & iY\end{pmatrix},\qquad \Omega_1\in\mathbb{H}_s,\ Y\in\mathcal{P}_{g-s}. \tag{5.22}$$

Obviously $\Phi(\mathbb{H}_s)=\mathscr{F}_s$ is the standard boundary component of degree s (cf. (5.8)).

Let

$$\nu_s:P(\mathscr{F}_s)\longrightarrow G_h(\mathscr{F}_s)$$

be the projection to the quotient. It is easily seen that ν_s commutes with τ. Therefore $\mathscr{F}_s$ is preserved by τ. The set

$$\mathscr{F}_s^\tau=\{\,\Phi(iY)\,|\,Y\in\mathcal{P}_s\,\}$$

is the set of τ-fixed points in $\mathscr{F}_s$ and may be canonically identified with $\mathcal{P}_s$. We denote by $i\,I_s$ its canonical base point. Then $\mathscr{F}_s$ is attached to $\mathbb{H}_g$ so that the cone $\Phi(i\,\mathcal{P}_s)$ is contained in the closure of the cone $i\,\mathcal{P}_g$.

Proposition 5.1. *Let $(\mathscr{F},\gamma)$ be a $\Gamma_g(N)$-real boundary pair of degree s. Then there exists $\gamma_*\in\Gamma_g$ such that $\gamma_*(\mathscr{F}_s)=\mathscr{F}$ and*

$$\tau(\gamma_*)^{-1}\gamma\,\gamma_*=\begin{pmatrix}A & B\\ 0 & {}^tA^{-1}\end{pmatrix}\in\ker(\nu_s).$$

Moreover, we may take $B=0$, i.e., there exist $\gamma'\in\Gamma_g(4m)$ and $\gamma_0\in\Gamma_g$ so that $\mathscr{F}^{\tau\gamma'}=\mathscr{F}^{\tau\gamma}$, $\gamma_0(\mathscr{F}_s)=\mathscr{F}$, and so that

$$\tau(\gamma_0)^{-1}\gamma'\,\gamma_0=\begin{pmatrix}A & 0\\ 0 & {}^tA^{-1}\end{pmatrix}\in\ker(\nu_s).$$

Proof. The proof can be found in [9, pp. 19-21]. □

As an application of Proposition 5.1, we get the following theorem.

Theorem 5.3. *Let $m\ge 1$ be a positive integer. Let $\mathscr{F}$ be a proper rational boundary component of $\mathbb{H}_g$ of degree $g-1$. Let $\gamma\in\Gamma_g(4m)$ such that*

$$\mathscr{F}^{\tau\gamma}=\{\,x\in\mathscr{F}\,|\,\tau(x)=\gamma\cdot x\,\}\neq\varnothing.$$

Then $\mathscr{F}^{\tau\gamma}$ is contained in the closure of $\mathbb{H}_g^{\tau\Gamma_g(4m)}$ in $\mathbb{H}_g^*$, where

$$\mathbb{H}_g^{\tau\Gamma_g(4m)}=\left\{\Omega\in\mathbb{H}_g\,|\,\tau(\Omega)=-\overline{\Omega}=\gamma\cdot\Omega\quad \textit{for some } \gamma\in\Gamma_g(4m)\right\}$$

denotes the set of $\Gamma_g(4m)$-real points of $\mathbb{H}_g$.

Proof. The proof can be found in [9, pp. 23]. □

Theorem 5.4. *Let k be a positive integer with $k\ge 2$. Let $(\mathscr{F},\gamma)$ be a $\Gamma_g(2^k)$-real boundary pair. Then there exists $\gamma_1\in\Gamma_g(2^k)$ such that $\mathscr{F}^{\tau\gamma}=\mathscr{F}^{\tau\gamma_1}$ and $\mathscr{F}^{\tau\gamma}$ is contained in the closure $\overline{\mathbb{H}_g^{\tau\gamma_1}}$ of $\mathbb{H}_g^{\tau\gamma_1}$ in $\mathbb{H}_g^*$.*

Proof. The proof can be found in [9, pp. 23-26]. □

We may summarize the above results as follows. The Baily-Borel compactification $V(N) = \Gamma_g(N)\backslash\mathbb{H}_g^*$ with $N = 4m$ is stratified by finitely many strata of the form $\pi_{BB}(\mathscr{F})$, where $\mathscr{F}$ is a rational boundary component. Each such strata is isomorphic to the standard rational boundary component $\mathscr{F}_s \cong \mathbb{H}_s$. The stratum $\pi_{BB}(\mathscr{F})$ is called a *boundary* stratum of degree s. Let $V(N)^r$ denote the union of all boundary strata of rank $g-r$. We define

$$V(N)^r_{\mathbb{R}} := V(N)^r \cap V(N)_{\mathbb{R}}.$$

According to Theorem 5.4, we have

$$V(N)^0_{\mathbb{R}} \cup V(N)^1_{\mathbb{R}} \subset \overline{X(N)_{\mathbb{R}}} \subset V(N)_{\mathbb{R}},$$

where $\overline{X(N)_{\mathbb{R}}}$ denotes the closure of $X(N)_{\mathbb{R}}$ in $V(N)$.

6. **Polarized Real Tori**

In this section we introduce the notion of polarized real tori.

First we review the properties of real tori briefly. We fix a positive integer g in this section. Let $T = \mathbb{R}^g/\Lambda$ be a real torus of dimension g, where Λ is a lattice in $\mathbb{R}^g$. T has a unique structure of a smooth (or real analytic) manifold such that the canonical projection $p : \mathbb{R}^g \longrightarrow T$ is smooth (or real analytic). We fix the standard basis $\{e_1, \cdots, e_g\}$ for $\mathbb{R}^g$. We see that $\Lambda = \Pi\mathbb{Z}^g$ for some $\Pi \in GL(g,\mathbb{R})$. A matrix Π is called a *period matrix* for T. Let $\mathbb{C}_1^* = \{z \in \mathbb{C} \mid |z| = 1\}$ be a circle. Since T is homeomorphic to $\mathbb{C}_1^* \times \cdots \times \mathbb{C}_1^*$ (g-times), the fundamental group is

$$\pi_1(T) \cong \pi_1(\mathbb{C}_1^*) \times \cdots \times \pi_1(\mathbb{C}_1^*) \cong \mathbb{Z}^g.$$

We see that

$$H_k(T,\mathbb{Z}) \cong \mathbb{Z}^{{}_gC_k} \cong H^k(T,\mathbb{Z}), \quad k = 0, 1, \cdots, g$$

and

$$H^*(T,\mathbb{Z}) \cong \bigwedge H^1(T,\mathbb{Z}) \cong \bigwedge \mathbb{Z}^g.$$

Thus the Euler characteristic of T is zero. The mapping class group $MCG(T)$ is

$$MCG(T) = \mathrm{Aut}(\pi_1(T)) = \mathrm{Aut}(\mathbb{Z}^g) = GL(g,\mathbb{Z}).$$

It is known that any connected compact real manifold can be embedded into the Euclidean space $\mathbb{R}^d$ with large d. Thus a torus T can be embedded in a real projective space $\mathbb{P}^d(\mathbb{R})$. Any connected compact abelian real Lie group is a real torus. Any two real tori of dimension g are isomorphic as real Lie groups. We easily see that if S is a connected closed subgroup of a real torus T, then S and T/S are real tori and $T \cong S \times T/S$.

Let $T = V/\Lambda$ and $T' = V'/\Lambda'$ be two real tori. A *homomorphism* $\phi : T \longrightarrow T'$ is a real analytic map compatible with the group structures. It is easily seen that a homomorphism $\phi : T \longrightarrow T'$ can be lifted to a uniquely determined $\mathbb{R}$-linear map $\Phi : V \longrightarrow V'$. This yields an injective homomorphism of abelian groups

$$\tau_a : \mathrm{Hom}(T,T') \longrightarrow \mathrm{Hom}_{\mathbb{R}}(V,V'), \qquad \phi \longmapsto \Phi,$$

where $\mathrm{Hom}(T,T')$ is the abelian group of all homomorphisms of T into T' and $\mathrm{Hom}_{\mathbb{R}}(V,V')$ is the abelian group of all $\mathbb{R}$-linear maps of V into V'. The above τ_a is called a real analytic representation of $\mathrm{Hom}(T,T')$. The restriction Φ_Λ of Φ to Λ is $\mathbb{Z}$-linear. Φ_Λ determines Φ and ϕ completely. Thus we get an injective homomorphism

$$\tau_r : \mathrm{Hom}(T,T') \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\Lambda,\Lambda'), \qquad \phi \longmapsto \Phi_\Lambda,$$

called the rational representation of $\mathrm{Hom}(T,T')$.

Lemma 6.1. *Let $\phi : T \longrightarrow T'$ be a homomorphism of real tori. Then*

(1) the image $Im\phi$ is a real subtorus of T';

(2) the kernel $ker\phi$ of ϕ is a closed subgroup of T and the identity component $(ker\phi)_0$ of $ker\phi$ is a real subtorus of T of finite index in $ker\phi$.

Proof. It follows from the fact that a connected compact abelian real Lie group is a real torus. Since $\ker\phi$ is compact, $\ker\phi$ has only a finite number of connected components. □

A surjective homomorphism $\phi : T \longrightarrow T'$ of real tori with finite kernel is called a *real isogeny* or simply an *isogeny*. The *exponent* $e(\phi)$ of an isogeny ϕ is defined to be the exponent of the finite group $\ker\phi$, that is, the smallest positive integer e such that $e\cdot x = 0$ for all $x \in \ker\phi$. Two real tori are said to be *isogenous* if there is an isogeny between them. It is clear that a homomorphism $\phi : T \longrightarrow T'$ is an isogeny if and only if it is surjective and $\dim T = \dim T'$. We can see that if $\Gamma \subset T$ is a finite subgroup, the quotient space T/Γ is a real torus and the natural projection $p_\Gamma : T \longrightarrow T/\Gamma$ is an isogeny.

For a homomorphism $\phi : T \longrightarrow T'$ of real tori, we define the *degree* of ϕ to be

$$\deg\phi := \begin{cases} \mathrm{ord}\,(\ker\phi) & if\ \ker\phi\ is\ finite; \\ 0 & otherwise. \end{cases}$$

Let $T = V/\Lambda$ be a real torus of dimension g. For any nonzero integer $n \in \mathbb{Z}$, we define the isogeny $n_T : T \longrightarrow T$ by $n_T(x) := n\cdot x$ for all $x \in T$. The kernel $T(n)$ of n_T is called the group of *n-division points* of T. It is easily seen that $T(n) \cong (\mathbb{Z}/n\mathbb{Z})^g$ because $\ker n_T = \frac{1}{n}\Lambda/\Lambda \cong \Lambda/n\Lambda \cong (\mathbb{Z}/n\mathbb{Z})^g$. So $\deg n_T = n^g$.

We put

$$\mathrm{Hom}_{\mathbb{Q}}(T,T') := \mathrm{Hom}(T,T') \otimes_{\mathbb{Z}} \mathbb{Q}$$

and

$$\mathrm{End}(T) := \mathrm{Hom}(T,T), \qquad \mathrm{End}_{\mathbb{Q}}(T) := \mathrm{End}(T) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

For any $\alpha \in \mathbb{Q}$ and $\phi \in \mathrm{Hom}(T,T')$, we define the degree of $\alpha\phi \in \mathrm{Hom}_{\mathbb{Q}}(T,T')$ by

$$\deg(\alpha\phi) := \alpha^g \deg\phi.$$

Lemma 6.2. *For any isogeny $\phi : T \longrightarrow T'$ of real tori with exponent e, there exists an isogeny $\psi : T' \longrightarrow T$, unique up to isomorphisms, such that $\psi\circ\phi = e_T$ and $\phi\circ\psi = e_{T'}$.*

Proof. Since $\ker\phi \subseteq \ker e_T$, there exists a unique map $\psi : T' \longrightarrow T$ such that $\psi\circ\phi = e_T$. It is easy to see that ψ is also an isogeny and that $\ker\psi \subseteq \ker e_{T'}$. Therefore there is a unique isogeny $\phi' : T' \longrightarrow T$ such that $\phi'\circ\psi = e_{T'}$. Since

$$\phi'\circ e_T = \phi'\circ\psi\circ\phi = e_{T'}\circ\phi = \phi\circ e_T$$

and e_T is surjective, we have $\phi' = \phi$. Hence we obtain $\psi\circ\phi = e_T$ and $\phi\circ\psi = e_{T'}$. □

According to Lemma 6.2, we see that isogenies define an equivalence relation on the set of real tori, and that an element in $\mathrm{End}(T)$ is an isogeny if and only if it is invertible in $\mathrm{End}_{\mathbb{Q}}(T)$.

For a real torus $T = V/\Lambda$ of dimension g, we put $V^* := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. Then the following canonical $\mathbb{R}$-bilinear form

$$\langle\ ,\ \rangle_T : V^* \times V \longrightarrow \mathbb{R}, \qquad \langle \ell, v\rangle_T := \ell(v),\ \ \ell \in V^*,\ \ v \in V$$

is non-degenerate. Thus the set

$$\widehat{\Lambda} := \{\, \ell \in V^* \mid \langle \ell, \Lambda\rangle_T \subseteq \mathbb{Z}\,\}$$

is a lattice in V^*. The quotient

$$\widehat{T} := V^*/\widehat{\Lambda}$$

is a real torus of dimension g which is called the dual real torus of T. Identifying V with the space of $\mathbb{R}$-linear forms $V^* \longrightarrow \mathbb{R}$ by double duality, the non-degeneracy of $\langle\ ,\ \rangle_T$ implies that Λ is the lattice in V dual to $\widehat{\Lambda}$. Therefore we get

$$\widehat{\widehat{T}} = T.$$

Let $\phi : T_1 \longrightarrow T_2$ be a homomorphism of real tori with $T_i = V_i/\Lambda_i\,(i = 1, 2)$ and with real analytic representation $\Phi : V_1 \longrightarrow V_2$. Since the dual map $\Phi^* : V_2^* \longrightarrow V_1^*$ satisfies the condition $\Phi^*\big(\widehat{\Lambda}_2\big) \subseteq \widehat{\Lambda}_1$, Φ^* induces a homomorphism, called the dual map

$$\widehat{\phi} : \widehat{T}_2 \longrightarrow \widehat{T}_1.$$

If $\psi : T_2 \longrightarrow T_3$ is another homomorphism of real tori, then we get

$$\widehat{\psi \circ \phi} = \widehat{\phi} \circ \widehat{\psi}.$$

If $\phi : T_1 \longrightarrow T_2$ is an isogeny of real tori, then dual map $\widehat{\phi} : \widehat{T}_2 \longrightarrow \widehat{T}_1$ is also an isogeny.

Definition 6.1. *A real torus $T = \mathbb{R}^g/\Lambda$ with a lattice Λ in $\mathbb{R}^g$ is said to be polarized if the the associated complex torus $\mathfrak{A} = \mathbb{C}^g/L$ is a polarized real abelian variety, where $L = \mathbb{Z}^g + i\,\Lambda$ is a lattice in $\mathbb{C}^g$. Moreover if $\mathfrak{A}$ is a principally polarized real abelian variety, T is said to be principally polarized. Let $\Phi : T \longrightarrow \mathfrak{A}$ be the smooth embedding of T into $\mathfrak{A}$ defined by*

$$\Phi(v + \Lambda) := i\,v + L, \qquad v \in \mathbb{R}^g. \tag{6.1}$$

Let $\mathfrak{L}$ be a polarization of $\mathfrak{A}$, that is, an ample line bundle over $\mathfrak{A}$. The pullback $\Phi^*\mathfrak{L}$ is called a polarization of T. We say that a pair $(T, \Phi^*\mathfrak{L})$ is a polarized real torus.

Example 6.1. Let $Y \in \mathcal{P}_g$ be a $g \times g$ positive definite symmetric real matrix. Then $\Lambda_Y = Y\mathbb{Z}^g$ is a lattice in $\mathbb{R}^g$. Then the g-dimensional torus $T_Y = \mathbb{R}^g/\Lambda_Y$ is a principally polarized real torus. Indeed,

$$\mathfrak{A}_Y = \mathbb{C}^g/L_Y, \qquad L_Y = \mathbb{Z}^g + i\,\Lambda_Y$$

is a princially polarized real abelian variety. Its corresponding hermitian form H_Y is given by

$$H_Y(x, y) = E_Y(i\,x, y) + i\,E_Y(x, y) = {}^t x\, Y^{-1}\, \overline{y}, \qquad x, y \in \mathbb{C}^g,$$

where E_Y denotes the imaginary part of H_Y. It is easily checked that H_Y is positive definite and $E_Y(L_Y \times L_Y) \subset \mathbb{Z}$ (cf. [17, pp. 29–30]). The real structure σ_Y on $\mathfrak{A}_Y$ is a complex conjugation.

Example 6.2. Let $Q = \begin{pmatrix} \sqrt{2} & \sqrt{3} \\ \sqrt{3} & -\sqrt{5} \end{pmatrix}$ be a 2×2 symmetric real matrix of signature $(1,1)$. Then $\Lambda_Q = Q\mathbb{Z}^2$ is a lattice in $\mathbb{R}^2$. Then the real torus $T_Q = \mathbb{R}^2/\Lambda_Q$ is not polarized because the associated complex torus $\mathfrak{A}_Q = \mathbb{C}^2/L_Q$ is not an abelian variety, where $L_Q = \mathbb{Z}^2 + i\,\Lambda_Q$ is a lattice in $\mathbb{C}^2$.

Definition 6.2. *Two polarized tori $T_1 = \mathbb{R}^g/\Lambda_1$ and $T_2 = \mathbb{R}^g/\Lambda_2$ are said to be isomorphic if the associated polarized real abelian varieties $\mathfrak{A}_1 = \mathbb{C}^g/L_1$ and $\mathfrak{A}_2 = \mathbb{C}^g/L_2$ are isomorphic, where $L_i = \mathbb{Z}^g + i\,\Lambda_i$ $(i = 1,2)$, more precisely, if there exists a linear isomorphism $\varphi : \mathbb{C}^g \longrightarrow \mathbb{C}^g$ such that*

$$\varphi(L_1) = L_2, \tag{6.2}$$

$$\varphi_*(E_1) = E_2, \tag{6.3}$$

$$\varphi_*(\sigma_1) = \varphi \circ \sigma_1 \circ \varphi^{-1} = \sigma_2, \tag{6.4}$$

where E_1 and E_2 are polarizations of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively, and σ_1 and σ_2 denotes the real structures (in fact complex conjugations) on $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively.

Example 6.3. Let Y_1 and Y_2 be two $g \times g$ positive definite symmetric real matrices. Then $\Lambda_i := Y_i\,\mathbb{Z}^g$ is a lattice in $\mathbb{R}^g$ $(i = 1,2)$. We let

$$T_i := \mathbb{R}^g/\Lambda_i, \qquad i = 1,2$$

be real tori of dimension g. Then according to Example 6.1, T_1 and T_2 are principally polarized real tori. We see that T_1 is isomorphic to T_2 as polarized real tori if and only if there is an element $A \in GL(g,\mathbb{Z})$ such that $Y_2 = A\,Y_1\,{}^tA$.

Example 6.4. Let $Y = \begin{pmatrix} \sqrt{2} & \sqrt{3} \\ \sqrt{3} & \sqrt{5} \end{pmatrix}$. Let $T_Y = \mathbb{R}^2/\Lambda_Y$ be a two dimensional principally polarized torus, where $\Lambda_Y = Y\mathbb{Z}^2$ is a lattice in $\mathbb{R}^2$. Let T_Q be the torus in Example 6.2. Then T_Y is diffeomorphic to T_Q. But T_Q is not polarized. T_Y admits a differentiable embedding into a complex projective space but T_Q does not.

Let $Y \in \mathcal{P}_g$ be a $g \times g$ positive definite symmetric real matrix. Then $\Lambda_Y = Y\mathbb{Z}^g$ is a lattice in $\mathbb{R}^g$. We already showed that the g-dimensional torus $T_Y = \mathbb{R}^g/\Lambda_Y$ is a principally polarized real torus (cf. Example 6.1). We know that the following complex torus

$$\mathfrak{A}_Y = \mathbb{C}^g/L_Y, \qquad L_Y = \mathbb{Z}^g + i\,\Lambda_Y$$

is a princially polarized real abelian variety. We define a map $\Phi_Y : T_Y \longrightarrow \mathfrak{A}_Y$ by

$$\Phi_Y(a + \Lambda_Y) := i\,a + L_Y, \qquad a \in \mathbb{R}^g.$$

Then Φ_Y is well defined and is an injective smooth map. Therefore T_Y is smoothly embedded into a complex projective space and hence into a real projective space because $\mathfrak{A}_Y$ can be holomorphically embedded into a complex projective space (cf. [17, pp. 29–30]).

Let $\mathfrak{A} = \mathbb{C}^g/L$ and $\mathfrak{A}' = \mathbb{C}^{g'}/L'$ be two abelian complex tori of dimension g and dimension g' respectively, where L (resp. L') is a lattice in $\mathbb{C}^g$ (resp. $\mathbb{C}^{g'}$). A homomorphism $f : \mathfrak{A} \longrightarrow \mathfrak{A}'$ lifts to a uniquely determined $\mathbb{C}$-linear map $F : \mathbb{C}^g \longrightarrow \mathbb{C}^{g'}$. This yields an injective homomorphism

$$\rho_a : \mathrm{Hom}(\mathfrak{A},\mathfrak{A}') \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^g,\mathbb{C}^{g'}) = \mathbb{C}^{(g',g)}, \qquad f \longmapsto F = \rho_a(f).$$

Its restriction $F|_L$ to the lattice L is $\mathbb{Z}$-linear and determines F and f completely. Therefore we get an injective homomorphism

$$\rho_r : \mathrm{Hom}(\mathfrak{A}, \mathfrak{A}') \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(L, L'), \qquad f \longmapsto F|_L.$$

Let $\widetilde{\Pi} \in \mathbb{C}^{(g,2g)}$ and $\widetilde{\Pi}' \in \mathbb{C}^{(g',2g')}$ be period matrices for $\mathfrak{A}$ and $\mathfrak{A}'$ respectively. With respect to the chosen bases, $\rho_a(f)$ (resp. $\rho_r(f)$) can be considered as a matrix in $\mathbb{C}^{(g',g)}$ $\big($resp. $\mathbb{Z}^{(2g',2g)}\big)$. We have the following diagram:

$$\begin{array}{ccc} \mathbb{Z}^{2g} & \xrightarrow{\ \widetilde{\Pi}\ } & \mathbb{C}^{g} \\ \Big\downarrow \rho_r(f) & & \Big\downarrow \rho_a(f) \\ \mathbb{Z}^{2g'} & \xrightarrow{\ \widetilde{\Pi}'\ } & \mathbb{C}^{g'}, \end{array}$$

that is, by the equation

$$\rho_a(f)\,\widetilde{\Pi} = \widetilde{\Pi}'\,\rho_r(f).$$

Conversely any two matrices $A \in \mathbb{C}^{(g',g)}$ and $R \in \mathbb{Z}^{(2g',2g)}$ satisfying the equation $A\widetilde{\Pi} = \widetilde{\Pi}'R$ define a homomorphism $\mathfrak{A} \longrightarrow \mathfrak{A}'$.

For two real tori T_1 and T_2 of dimension g_1 and dimension g_2 respectively, we let $\mathrm{Ext}(T_2, T_1)$ be the set of all isomorphism classes of extensions of T_2 by T_1 up to real analytic isomorphism. Since any two real tori of dimension $g_1 + g_2$ are isomorphic as real analytic real Lie groups, $\mathrm{Ext}(T_2, T_1)$ is trivial. This leads us to consider polarized real tori T_1 and T_2 with $T_i = \mathbb{R}^{g_i}/\Lambda_i$ $(i = 1, 2)$. Here Λ_i is a lattice in $\mathbb{R}^{g_i}$ for $i = 1, 2$. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be the polarized real abelian varieties associated to T_1 and T_2 respectively, that is,

$$\mathfrak{A}_i = \mathbb{C}^{g_i}/L_i, \qquad L_i = \mathbb{Z}^{g_i} + \Lambda_i \mathbb{Z}^{g_i}, \quad i = 1, 2.$$

Let $\mathrm{Ext}(T_2, T_1)_{\mathrm{pt}}$ be the set of all isomorphism classes of extensions of $\mathfrak{A}_2$ by $\mathfrak{A}_1$. We can show that a homomorphism $\phi : \mathfrak{A}_2' \longrightarrow \mathfrak{A}_2$ such that $\mathfrak{A}_2'$ is the real abelian variety associated to a polarized real torus T_2' induces a map

$$\phi^* : \mathrm{Ext}(T_2, T_1)_{\mathrm{pt}} \longrightarrow \mathrm{Ext}(T_2', T_1)_{\mathrm{pt}} \tag{6.5}$$

and that a homomorphism $\psi : \mathfrak{A}_1 \longrightarrow \mathfrak{A}_1'$ such that $\mathfrak{A}_1'$ is the real abelian variety associated to a polarized real torus T_1' induces a map

$$\psi_* : \mathrm{Ext}(T_2, T_1)_{\mathrm{pt}} \longrightarrow \mathrm{Ext}(T_2, T_1')_{\mathrm{pt}}. \tag{6.6}$$

Indeed, if

$$e \ : \quad 0 \longrightarrow \mathfrak{A}_1 \xrightarrow{\ \iota\ } \mathfrak{A} \xrightarrow{\ p\ } \mathfrak{A}_2 \longrightarrow 0 \tag{6.7}$$

is an extension in $\mathrm{Ext}(T_2, T_1)_{\mathrm{pt}}$, the image $\phi^*(e)$ is defined to be the identity component of the kernel of the homomorphism $C_{p,\phi} : \mathfrak{A} \times \mathfrak{A}_2' \longrightarrow \mathfrak{A}_2$ defined by

$$C_{p,\phi}(x, y) := p(x) - \phi(y), \qquad x \in \mathfrak{A},\ y \in \mathfrak{A}_2'.$$

The dualization of the exact sequence (6.7) gives an element $\hat{e} \in \mathrm{Ext}(\widehat{\mathfrak{A}}_1, \widehat{\mathfrak{A}}_2)$. We define

$$\psi_*(e) := \widehat{\widehat{\psi}^*(\hat{e})} \ \in \mathrm{Ext}(T_2, T_1')_{\mathrm{pt}} = \mathrm{Ext}(\mathfrak{A}_2, \mathfrak{A}_1'). \tag{6.8}$$

Therefore $\mathrm{Ext}(\ ,\ )_{\mathrm{pt}}$ is a functor which is contravariant in the first and covariant in the second argument.

We can equip the set $\mathrm{Ext}(T_2,T_1)_{\mathrm{pt}}$ with the canonical group structure as follows : Let e and $e_\diamond$ be the extensions in $\mathrm{Ext}(T_2,T_1)_{\mathrm{pt}}$ which are represented by the exact sequence (6.7) and the following exact sequence

$$e_\diamond \ : \quad 0 \longrightarrow \mathfrak{A}_1 \longrightarrow \mathfrak{A}_\diamond \longrightarrow \mathfrak{A}_2 \longrightarrow 0.$$

The product $e \times e_\diamond$ is represented by the exact sequence

$$e \times e_\diamond \ : \quad 0 \longrightarrow \mathfrak{A}_1 \times \mathfrak{A}_1 \longrightarrow \mathfrak{A} \times \mathfrak{A}_\diamond \longrightarrow \mathfrak{A}_2 \times \mathfrak{A}_2 \longrightarrow 0.$$

If $\Delta : \mathfrak{A}_2 \longrightarrow \mathfrak{A}_2 \times \mathfrak{A}_2$ is the diagonal map, $x \longmapsto (x,x),\ x \in \mathfrak{A}_2$ and $\mu : T_1 \times T_1 \longrightarrow T_1$ is the addition map, $(s,t) \longmapsto s+t,\ s,t \in \mathfrak{A}_1$, the sum $e + e_\diamond$ is defined to be the image of $e \times e_\diamond$ under the composition

$$\mathrm{Ext}(T_2 \times T_2, T_1 \times T_1)_{\mathrm{pt}} \xrightarrow{\Delta^*} \mathrm{Ext}(T_2, T_1 \times T_1)_{\mathrm{pt}} \xrightarrow{\mu_*} \mathrm{Ext}(T_2,T_1)_{\mathrm{pt}},$$

that is,

$$e + e_\diamond := \mu_* \Delta^* (e \times e_\diamond). \tag{6.9}$$

We can show that $\mathrm{Ext}(T_2,T_1)_{\mathrm{pt}}$ is an abelian group with respect to the addition (6.9) (cf. [5]).

Now we describe the group $\mathrm{Ext}(T_2,T_1)_{\mathrm{pt}}$ in terms of period matrices. First we fix period matrices Π_1 and Π_2 for T_1 and T_2 respectively, that is, $\Lambda_i = \Pi_i \mathbb{Z}^{g_i}$ for $i = 1,2$. We know that $\Pi_i \in GL(g_i, \mathbb{R})$ for $i = 1,2$. To each extension

$$e \ : \quad 0 \longrightarrow \mathfrak{A}_1 \longrightarrow \mathfrak{A} \longrightarrow \mathfrak{A}_2 \longrightarrow 0$$

in $\mathrm{Ext}(T_2,T_1)_{\mathrm{pt}}$, there is associated a period matrix for $\mathfrak{A}$ of the form

$$\begin{pmatrix} \widetilde{\Pi}_1 & \sigma \\ 0 & \widetilde{\Pi}_2 \end{pmatrix}, \qquad \widetilde{\Pi}_i = (I_{g_i}, \Pi_i) \text{ for } i = 1,2, \quad \sigma \in \mathbb{C}^{(g_1, 2g_2)}. \tag{6.10}$$

Conversely it is obvious that for any $\sigma \in \mathbb{C}^{(g_1,2g_2)}$, the matrix of the form (6.10) is a period matrix defining an extension of $\mathfrak{A}_2$ by $\mathfrak{A}_1$ in $\mathrm{Ext}(T_2,T_1)_{\mathrm{pt}}$.

Lemma 6.3. *Let σ and σ' be elements in $\mathbb{C}^{(g_1,2g_2)}$. Suppose that Π_1 and Π_2 are period matrices for polarized real tori T_1 and T_2 respectively. Then the period matrices*

$$\widetilde{\Pi}_\sigma = \begin{pmatrix} \widetilde{\Pi}_1 & \sigma \\ 0 & \widetilde{\Pi}_2 \end{pmatrix} \qquad \text{and} \qquad \widetilde{\Pi}_{\sigma'} = \begin{pmatrix} \widetilde{\Pi}_1 & \sigma' \\ 0 & \widetilde{\Pi}_2 \end{pmatrix}, \quad \widetilde{\Pi}_i = (I_{g_i}, \Pi_i) \text{ for } i = 1,2$$

define isomorphic extensions of $\mathfrak{A}_2$ by $\mathfrak{A}_1$ in $\mathrm{Ext}(T_2,T_1)_{\mathrm{pt}}$ if and only if

$$\sigma' = \sigma + \widetilde{\Pi}_1 M + A \widetilde{\Pi}_2 \tag{6.11}$$

with some $M \in \mathbb{Z}^{(2g_1,2g_2)}$ and $A \in \mathbb{C}^{(g_1,g_2)}$.

Proof. Let $\widetilde{\Pi}_\sigma$ and $\widetilde{\Pi}_{\sigma'}$ define isomorphic extensions e and e' of $\mathfrak{A}_2$ by $\mathfrak{A}_1$:

$$\begin{array}{ccccccccccc} e \ : & 0 & \longrightarrow & \mathfrak{A}_1 & \longrightarrow & \mathfrak{A} & \longrightarrow & \mathfrak{A}_2 & \longrightarrow & 0 \\ & & & \| & & \downarrow f & & \| & & \\ e' \ : & 0 & \longrightarrow & \mathfrak{A}_1 & \longrightarrow & \mathfrak{A}' & \longrightarrow & \mathfrak{A}_2 & \longrightarrow & 0 \end{array}$$

Then we have the following commutative diagram :

$$\begin{array}{ccc} \mathbb{Z}^{2g_1+2g_2} & \xrightarrow{\widetilde{\Pi}_\sigma} & \mathbb{C}^{g_1+g_2} \\ \Big\downarrow \rho_r(f) & & \Big\downarrow \rho_a(f) \\ \mathbb{Z}^{2g_1+2g_2} & \xrightarrow{\widetilde{\Pi}_{\sigma'}} & \mathbb{C}^{g_1+g_2} \end{array}$$

Therefore there are $A \in \mathbb{C}^{(g_1,g_2)}$ and $M \in \mathbb{Z}^{(2g_1,2g_2)}$ that satisfy the following equation

$$\begin{pmatrix} I_{g_1} & A \\ 0 & I_{g_2} \end{pmatrix} \widetilde{\Pi}_\sigma = \widetilde{\Pi}_{\sigma'} \begin{pmatrix} I_{g_1} & -M \\ 0 & I_{g_2} \end{pmatrix}. \tag{6.12}$$

We obtain the equation (6.11) from the equation (6.12).

Conversely if we have σ and σ' in $\mathbb{C}^{(g_1,g_2)}$ satisfying the equation (6.11), then we see easily that $\widetilde{\Pi}_\sigma$ and $\widetilde{\Pi}_{\sigma'}$ define isomorphic extensions of $\mathfrak{A}_2$ by $\mathfrak{A}_1$. □

Proposition 6.1. *Let σ and σ' be elements in $\mathbb{C}^{(g_1,2g_2)}$. Suppose that Π_1 and Π_2 are period matrices for real tori T_1 and T_2 respectively. Assume that the following period matrices*

$$\widetilde{\Pi}_\sigma = \begin{pmatrix} \widetilde{\Pi}_1 & \sigma \\ 0 & \widetilde{\Pi}_2 \end{pmatrix} \qquad \textit{and} \qquad \widetilde{\Pi}_{\sigma'} = \begin{pmatrix} \widetilde{\Pi}_1 & \sigma' \\ 0 & \widetilde{\Pi}_2 \end{pmatrix}, \quad \widetilde{\Pi}_i = (I_{g_i}, \Pi_i) \text{ for } i = 1, 2$$

define extensions e and e' of $\mathfrak{A}_2$ by $\mathfrak{A}_1$ in Ext$(T_2, T_1)_{\rm pt}$. Then the period matrix

$$\widetilde{\Pi}_{\sigma+\sigma'} = \begin{pmatrix} \widetilde{\Pi}_1 & \sigma+\sigma' \\ 0 & \widetilde{\Pi}_2 \end{pmatrix}$$

defines the extension $e + e'$ in Ext$(T_2, T_1)_{\rm pt}$.

Proof. We denote

$$\mathfrak{A} = \mathbb{C}^{g_1+g_2}/\widetilde{\Pi}_\sigma \mathbb{Z}^{2g_1+2g_2} \qquad \text{and} \qquad \mathfrak{A}' = \mathbb{C}^{g_1+g_2}/\widetilde{\Pi}_{\sigma'} \mathbb{Z}^{2g_1+2g_2}.$$

Then we have the extensions

$$e \; : \quad 0 \longrightarrow \mathfrak{A}_1 \longrightarrow \mathfrak{A} \longrightarrow \mathfrak{A}_2 \longrightarrow 0$$

and

$$e' \; : \quad 0 \longrightarrow \mathfrak{A}_1 \longrightarrow \mathfrak{A}' \longrightarrow \mathfrak{A}_2 \longrightarrow 0$$

in $\mathrm{Ext}(T_2, T_1)_{\rm pt}$. The complex torus $\mathfrak{A} \times \mathfrak{A}'$ defined by the extension $e \times e'$ in $\mathrm{Ext}(\mathfrak{A}_2 \times \mathfrak{A}_2, \mathfrak{A}_1 \times \mathfrak{A}_1)$ is given by the period matrix

$$\Box_1 = \begin{pmatrix} \widetilde{\Pi}_1 & 0 & \sigma & 0 \\ 0 & \widetilde{\Pi}_1 & 0 & \sigma' \\ 0 & 0 & \widetilde{\Pi}_2 & 0 \\ 0 & 0 & 0 & \widetilde{\Pi}_2 \end{pmatrix}.$$

Let $\Delta : \mathfrak{A}_2 \longrightarrow \mathfrak{A}_2 \times \mathfrak{A}_2$ be the diagonal map. Then we have the induced map $\Delta^* : \mathrm{Ext}(\mathfrak{A}_2 \times \mathfrak{A}_2, \mathfrak{A}_1 \times \mathfrak{A}_1) \longrightarrow \mathrm{Ext}(\mathfrak{A}_2, \mathfrak{A}_1 \times \mathfrak{A}_1)$. If

$$\Delta^*(e \times e') \; : \; 0 \longrightarrow \mathfrak{A}_1 \times \mathfrak{A}_1 \longrightarrow S \longrightarrow \mathfrak{A}_2 \longrightarrow 0$$

is given, the complex torus S is given by a period matrix of the form

$$\Box_2 = \begin{pmatrix} \widetilde{\Pi}_1 & 0 & \alpha \\ 0 & \widetilde{\Pi}_1 & \beta \\ 0 & 0 & \widetilde{\Pi}_2 \end{pmatrix}$$

with $\alpha \in \mathbb{C}^{(g_1,2g_2)}$ and $\beta \in \mathbb{C}^{(g_1,2g_2)}$. The homomorphism

$$\mathrm{Ext}(\mathfrak{A}_2 \times \mathfrak{A}_2, \mathfrak{A}_1 \times \mathfrak{A}_1) \longrightarrow \mathrm{Ext}(\mathfrak{A}_2, \mathfrak{A}_1 \times \mathfrak{A}_1), \qquad e \times e' \longmapsto \Delta^*(e \times e')$$

corresponds to a homomorphism $S \longmapsto \mathfrak{A} \times \mathfrak{A}'$ of real tori given by the equation

$$\Box_1 \cdot \begin{pmatrix} I_{2g_1} & 0 & M_1 \\ 0 & I_{2g_1} & M_2 \\ 0 & 0 & I_{2g_2} \\ 0 & 0 & I_{2g_2} \end{pmatrix} = \begin{pmatrix} I_{g_1} & 0 & A_1 \\ 0 & I_{g_1} & A_2 \\ 0 & 0 & I_{g_2} \\ 0 & 0 & I_{g_2} \end{pmatrix} \cdot \Box_2.$$

Thus we have the equations

$$\alpha = \sigma + \widetilde{\Pi}_1 M_1 - A_1 \widetilde{\Pi}_2 \qquad \text{and} \qquad \beta = \sigma' + \widetilde{\Pi}_1 M_2 - A_2 \widetilde{\Pi}_2.$$

According to Lemma 6.3,

$$\begin{pmatrix} \widetilde{\Pi}_1 & 0 & \sigma \\ 0 & \widetilde{\Pi}_1 & \sigma' \\ 0 & 0 & \widetilde{\Pi}_2 \end{pmatrix}$$

is also a period matrix for S respectively $\Delta^*(e \times e')$. We denote

$$e + e' \ : \quad 0 \longrightarrow \mathfrak{A}_1 \longrightarrow \mathfrak{B} \longrightarrow \mathfrak{A}_2 \longrightarrow 0.$$

A period matrix for $\mathfrak{B}$ is of the form

$$\Pi_{\mathfrak{B}} := \begin{pmatrix} \widetilde{\Pi}_1 & \tau \\ 0 & \widetilde{\Pi}_2 \end{pmatrix}, \qquad \tau \in \mathbb{R}^{(g_1,g_2)}.$$

The homomorphism $\mu_* : \Delta^*(e \times e') \longmapsto e + e'$ defines a homomorphism $S \longmapsto \mathfrak{B}$ which is given by the equation

$$\begin{pmatrix} I_{g_1} & I_{g_1} & A \\ 0 & 0 & I_{g_2} \end{pmatrix} \begin{pmatrix} \widetilde{\Pi}_1 & 0 & \sigma \\ 0 & \widetilde{\Pi}_1 & \sigma' \\ 0 & 0 & \widetilde{\Pi}_2 \end{pmatrix} = \begin{pmatrix} \widetilde{\Pi}_1 & \tau \\ 0 & \widetilde{\Pi}_2 \end{pmatrix} \begin{pmatrix} I_{2g_1} & I_{2g_1} & M \\ 0 & 0 & I_{2g_2} \end{pmatrix} \tag{6.13}$$

with $A \in \mathbb{C}^{(g_1,g_2)}$ and $M \in \mathbb{Z}^{(2g_1,2g_2)}$. Comparing both sides in the equation (6.13), we obtain

$$\tau = \sigma + \sigma' - \widetilde{\Pi}_1 M + A \widetilde{\Pi}_2.$$

According to Lemma 6.3, we see that

$$\widetilde{\Pi}_{\sigma+\sigma'} = \begin{pmatrix} \widetilde{\Pi}_1 & \sigma + \sigma' \\ 0 & \widetilde{\Pi}_2 \end{pmatrix}$$

is a period matrix for $\mathfrak{B}$, respectively $e + e'$. □

Let $T_1, T_2, \Pi_1, \Pi_2, \widetilde{\Pi}_1, \widetilde{\Pi}_2, \sigma, \sigma', \widetilde{\Pi}_\sigma$ and $\widetilde{\Pi}_{\sigma'}$ be as above in Proposition 6.1. We note that the assignment

$$\sigma \longmapsto \mathbb{C}^{g_1+g_2} / \widetilde{\Pi}_\sigma \mathbb{Z}^{2g_1+2g_2}, \qquad \sigma \in \mathbb{C}^{(g_1,2g_2)}$$

induces a surjective homomorphism of abelian groups

$$\Phi_{\Pi_1,\Pi_2} : \mathbb{C}^{(g_1,2g_2)} \longrightarrow \mathrm{Ext}(T_2, T_1)_{\mathrm{pt}}. \tag{6.14}$$

According to Lemma 6.3, we see that the kernel of Φ_{Π_1,Π_2} is given by

$$\ker \Phi_{\Pi_1,\Pi_2} = \widetilde{\Pi}_1 \mathbb{Z}^{(2g_1,2g_2)} + \mathbb{C}^{(g_1,g_2)} \widetilde{\Pi}_2.$$

Obviously the homomorphism Φ_{Π_1,Π_2} depends on the choice of the period matrices Π_1 and Π_2.

Proposition 6.2. *Let T_1 and T_1' be polarized real tori of dimension g_1 and dimension g_1' with period matrices Π_1 and Π_1' respectively. Let T_2 and T_2' be polarized real tori of dimension g_2 and dimension g_2' with period matrices Π_2 and Π_2' respectively. Then*

(a) for a homomorphism $f : \mathfrak{A}_2' \longrightarrow \mathfrak{A}_2$ such that $\mathfrak{A}_2'$ is the polarized real abelian variety associated to a polarized real torus T_2', the following diagram

$$\begin{array}{ccc} \mathbb{C}^{(g_1,2g_2)} & \xrightarrow{\Phi_{\Pi_1,\Pi_2}} & \mathrm{Ext}(T_2,T_1)_{\mathrm{pt}} \\ \downarrow \cdot\rho_r(f) & & \downarrow f^* \\ \mathbb{C}^{(g_1,2g_2')} & \xrightarrow{\Phi_{\Pi_1,\Pi_2'}} & \mathrm{Ext}(T_2',T_1)_{\mathrm{pt}} \end{array}$$

commutes and

(b) for a homomorphism $h : \mathfrak{A}_1 \longrightarrow \mathfrak{A}_1'$ such that $\mathfrak{A}_1'$ is the polarized real abelian variety associated to a polarized real torus T_1', the following diagram

$$\begin{array}{ccc} \mathbb{C}^{(g_1,2g_2)} & \xrightarrow{\Phi_{\Pi_1,\Pi_2}} & \mathrm{Ext}(T_2,T_1)_{\mathrm{pt}} \\ \rho_a(h)\cdot \downarrow & & \downarrow h_* \\ \mathbb{C}^{(g_1',2g_2)} & \xrightarrow{\Phi_{\Pi_1',\Pi_2}} & \mathrm{Ext}(T_2,T_1')_{\mathrm{pt}} \end{array}$$

commutes.

Proof. (a) For an extension $e \in \mathrm{Ext}(T_2,T_1)_{\mathrm{pt}}$ we choose $\sigma \in \mathbb{C}^{(g_1,2g_2)}$ with $\Phi_{\Pi_1,\Pi_2}(\sigma) = e$ and $\sigma' \in \mathbb{C}^{(g_1,2g_2')}$ with $\Phi_{\Pi_1,\Pi_2'}(\sigma') = f^*(e)$. We see that the following diagram with exact rows

$$\begin{array}{ccccccccccc} f^*(e) : & 0 & \longrightarrow & \mathfrak{A}_1 & \longrightarrow & \mathfrak{A}' & \longrightarrow & \mathfrak{A}_2' & \longrightarrow & 0 \\ & & & \| & & \downarrow f^* & & \downarrow f & & \\ e : & 0 & \longrightarrow & \mathfrak{A}_1 & \longrightarrow & \mathfrak{A} & \longrightarrow & \mathfrak{A}_2 & \longrightarrow & 0 \end{array}$$

commutes. Thus σ and σ' are related by the equation

$$\begin{pmatrix} I_{g_1} & A \\ 0 & \rho_a(f) \end{pmatrix} \begin{pmatrix} \widetilde{\Pi}_1 & \sigma' \\ 0 & \widetilde{\Pi}_2' \end{pmatrix} = \begin{pmatrix} \widetilde{\Pi}_1 & \sigma \\ 0 & \widetilde{\Pi}_2 \end{pmatrix} \begin{pmatrix} I_{2g_1} & M \\ 0 & \rho_r(f) \end{pmatrix} \tag{6.15}$$

with $A \in \mathbb{C}^{(g_1,g_2')}$ and $M \in \mathbb{Z}^{(2g_1,2g_2')}$. Comparing both sides in the equation (6.15). we get

$$\sigma' = \sigma \cdot \rho_r(f) + \widetilde{\Pi}_1 M - A\widetilde{\Pi}_2'.$$

According to Lemma 6.3, we have

$$\Phi_{\Pi_1,\Pi_2'}(\sigma') = \Phi_{\Pi_1,\Pi_2'}(\sigma \cdot \rho_r(f)) = f^*(e).$$

This completes the proof of (a).

(b) For an extension $e_\diamond \in \mathrm{Ext}(T_2,T_1)_{\mathrm{pt}}$ we choose $\sigma_\diamond \in \mathbb{C}^{(g_1,2g_2)}$ with $\Phi_{\Pi_1,\Pi_2}(\sigma_\diamond) = e_\diamond$ and $\sigma'_\diamond \in \mathbb{C}^{(g'_1,2g_2)}$ with $\Phi_{\Pi'_1,\Pi_2}(\sigma'_\diamond) = h_*(e_\diamond)$. We see that the following diagram with exact rows

$$\begin{array}{ccccccccccc} e_\diamond & : & 0 & \longrightarrow & \mathfrak{A}_1 & \longrightarrow & \mathfrak{A}_\diamond & \longrightarrow & \mathfrak{A}_2 & \longrightarrow & 0 \\ & & & & \downarrow h & & \downarrow h_* & & \| & & \\ e & : & 0 & \longrightarrow & \mathfrak{A}'_1 & \longrightarrow & \mathfrak{A}'_\diamond & \longrightarrow & \mathfrak{A}_2 & \longrightarrow & 0 \end{array}$$

commutes. Thus $\sigma_\diamond$ and $\sigma'_\diamond$ are related by the equation

$$\begin{pmatrix} \widetilde{\Pi}'_1 & \sigma'_\diamond \\ 0 & \widetilde{\Pi}_2 \end{pmatrix}\begin{pmatrix} \rho_r(h) & M_\diamond \\ 0 & I_{2g_2} \end{pmatrix} = \begin{pmatrix} \rho_a(h) & A_\diamond \\ 0 & I_{g_2} \end{pmatrix}\begin{pmatrix} \widetilde{\Pi}_1 & \sigma_\diamond \\ 0 & \widetilde{\Pi}_2 \end{pmatrix}. \tag{6.16}$$

with $A_\diamond \in \mathbb{C}^{(g'_1,g'_2)}$ and $M_\diamond \in \mathbb{Z}^{(2g'_1,2g'_2)}$. Comparing both sides in the equation (6.16). we get

$$\sigma'_\diamond = \rho_a(h)\cdot\sigma_\diamond + A_\diamond\widetilde{\Pi}_2 - \widetilde{\Pi}'_1 M_\diamond.$$

According to Lemma 6.3, we get

$$h_*(e_\diamond) = h_*(\Phi_{\Pi_1,\Pi_2}(\sigma_\diamond)) = \Phi_{\Pi'_1,\Pi_2}(\rho_a(h)\cdot\sigma_\diamond).$$

This completes the proof of (b). □

Corollary 6.1. *For $e \in \mathrm{Ext}(T_2,T_1)_{\mathrm{pt}}$ and $n \in \mathbb{Z}$, we have*

$$n^*_{\mathfrak{A}_2}(e) = n\cdot e = (n_{\mathfrak{A}_1})_*(e).$$

Proof. We consider the following commutative diagram :

$$\begin{array}{ccc} \mathbb{C}^{(g_1,2g_2)} & \xrightarrow{\Phi_{\Pi_1,\Pi_2}} & \mathrm{Ext}(T_2,T_1)_{\mathrm{pt}} \\ \cdot\rho_r(n_{\mathfrak{A}_2})\downarrow & & \downarrow (n_{\mathfrak{A}_2})^* \\ \mathbb{C}^{(g_1,2g_2)} & \xrightarrow{\Phi_{\Pi_1,\Pi_2}} & \mathrm{Ext}(T_2,T_1)_{\mathrm{pt}} \end{array}$$

Since $\rho_r(n_{\mathfrak{A}_2}) = n\,I_{2g_2}$, we get

$$(n_{\mathfrak{A}_2})^*(e) = \Phi_{\Pi_1,\Pi_2}(n\sigma) = n\cdot\Phi_{\Pi_1,\Pi_2}(\sigma) = n\cdot e.$$

By a similar argument, we get

$$(n_{\mathfrak{A}_1})_*(e) = n\cdot e.$$

□

Proposition 6.3. *We have an isomorphism of abelian groups*

$$\mathbb{C}^{(g_1,g_2)}/(I_{g_1},\Pi_1)\mathbb{Z}^{(2g_1,2g_2)}\begin{pmatrix}\Pi_2\\ I_{g_2}\end{pmatrix} \cong \mathrm{Ext}(T_2,T_1)_{\mathrm{pt}}.$$

Proof. Let $\sigma = (\sigma_1,\sigma_2) \in \mathbb{C}^{(g_1,2g_2)}$ with $\sigma_1,\sigma_2 \in \mathbb{C}^{(g_1,g_2)}$ corresponding to the extension $e = \Phi_{\Pi_1,\Pi_2}(\sigma) \in \mathrm{Ext}(T_2,T_1)_{\mathrm{pt}}$. By Lemma 6.3, the matrix

$$\sigma - \sigma_1\widetilde{\Pi}_2 = (\sigma_1,\sigma_2) - \sigma_1(I_{g_2},\Pi_2) = (0,\sigma_2 - \sigma_1\mathrm{H}_2)$$

corresponds to the same extension e. This shows that every extension in $\mathrm{Ext}(T_2,T_1)_{\mathrm{pt}}$ can be represented by a matrix $\sigma = (0,\alpha)$ with $\alpha \in \mathbb{C}^{(g_1,g_2)}$. Hence we get a surjective homomorphism of abelian groups

$$\mathbb{C}^{(g_1,g_2)} \longrightarrow \mathrm{Ext}(T_2,T_1)_{\mathrm{pt}}.$$

According to Lemma 6.3, the matrices α and $\alpha' \in \mathbb{C}^{(g_1,g_2)}$ define the same extension if and only if

$$(6.17) \qquad (0, \alpha - \alpha') = \widetilde{\Pi}_1 \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} + A\,\widetilde{\Pi}_2$$

with $\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \in \mathbb{Z}^{(2g_1,2g_2)}$ and $A \in \mathbb{C}^{(g_1,g_2)}$. From the equation (6.17) we get

$$A = -M_1 - \Pi_1 M_3.$$

Thus we have

$$\begin{aligned} \alpha - \alpha' &= \Pi_1 M_4 - \Pi_1 M_3 \Pi_2 + M_2 - M_1 \Pi_2 \\ &= (I_{g_1}, \Pi_1) \begin{pmatrix} -M_1 & M_2 \\ -M_3 & M_4 \end{pmatrix} \begin{pmatrix} \Pi_2 \\ I_{g_2} \end{pmatrix}. \end{aligned}$$

This completes the proof of the above proposition. □

7. Line Bundles over a Polarized Real Torus

Before we investigate complex line bundles over a real torus, we need a knowledge of holomorphic line bundles on a complex torus. We briefly review some results on holomorphic line bundles on a complex torus (cf. [13], [17]).

Let $X = \mathbb{C}^g/L$ be a complex torus, where L is a lattice in $\mathbb{C}^g$. The exponential sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 1$ induces the long exact sequence

$$\cdots \longrightarrow H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \longrightarrow \cdots$$

We recall that the Néron-Severi group $NS(X)$ (resp. $Pic^0(X)$) is defined to be the image of c_1 (resp. the kernel of c_1). For a hermitian form H on $\mathbb{C}^g$ whose imaginary part $E_H := \mathrm{Im}\,(H)$ is integral on $L \times L$, a semi-character for H is defined to be a map $\alpha : L \longrightarrow \mathbb{C}_1^*$ is defined to be a map such that

$$\alpha(\ell_1 + \ell_2) = \alpha(\ell_1)\,\alpha(\ell_2)\,e^{\pi\, i\, E_H(\ell_1,\ell_2)}, \qquad \ell_1, \ell_2 \in L.$$

We let $\mathrm{Her}(L)$ be the set of all hermitian forms on $\mathbb{C}^g$ whose imaginary parts are integral on $L \times L$. For any $H \in \mathrm{Her}(L)$, we denote by $\mathrm{SC}(H)$ the set of all semi-characters for H. To each pair (H, α) with $H \in \mathrm{Her}(L)$ and $\alpha \in \mathrm{SC}(H)$, we associate the automorphic factor $J_{H,\alpha} : L \times \mathbb{C}^g \longrightarrow \mathbb{C}^*$ defined by

$$(7.1) \qquad J_{H,\alpha}(\ell, z) := \alpha(\ell)\, e^{\frac{\pi}{2} H(\ell,\ell) + \pi H(z,\ell)}, \qquad \ell \in L,\ z \in \mathbb{C}^g.$$

A lattice L acts on the trivial line bundle $\mathbb{C}^g \times \mathbb{C}$ on $\mathbb{C}^g$ freely by

$$(7.2) \qquad \ell \cdot (z, \xi) = (z + \ell, J_{H,\alpha}(\ell, z)\xi), \qquad \ell \in L,\ z \in \mathbb{C}^g,\ \xi \in \mathbb{C}.$$

The quotient

$$(7.3) \qquad \mathfrak{L}(H, \alpha) := (\mathbb{C}^g \times \mathbb{C})/L$$

obtained by the action (7.2) of L has a natural structure of a holomorphic line bundle over X. We note that for each such pairs (H_1, α_1) and (H_2, α_2), we have

$$J_{H_1,\alpha_1} \cdot J_{H_2,\alpha_2} = J_{H_1+H_2,\alpha_1\alpha_2} \quad \text{and} \quad \mathfrak{L}(H_1, \alpha_1) \otimes \mathfrak{L}(H_2, \alpha_2) = \mathfrak{L}(H_1 + H_2, \alpha_1\,\alpha_2).$$

Let $\mathfrak{B}(L)$ be the set of all pairs (H,α) with $H \in \mathrm{Her}(L)$ and $\alpha \in \mathrm{SC}(H)$. Then $\mathfrak{B}(L)$ has a group structure equipped with multiplication law

$$(H_1,\alpha_1)\cdot(H_2,\alpha_2) = (H_1+H_2,\alpha_1\,\alpha_2), \quad H_i \in \mathrm{Her}(L),\ \alpha_i \in \mathrm{SC}(H_i),\ i=1,2.$$

Appell-Humbert Theorem says that we have the following canonical isomorphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(L,\mathbb{C}_1{}^*) & \longrightarrow & \mathfrak{B}(L) & \longrightarrow & NS(X) & \longrightarrow & 0 \\ & & \big\downarrow c_L & & \big\downarrow \beta_L & & \big\downarrow \mathrm{id} & & \\ 0 & \longrightarrow & Pic^0(X) & \longrightarrow & Pic(X) & \longrightarrow & NS(X) & \longrightarrow & 0 \end{array}$$

Here $\beta_L : \mathfrak{B}(L) \longrightarrow Pic(X) = H^1(X,\mathcal{O}_X^*)$ is the group isomorphism defined by

$$\beta_L((H,\alpha)) := \mathfrak{L}(H,\alpha), \qquad (H,\alpha) \in \mathfrak{B}(L)$$

and c_L is the isomorphism induced by β_L. It is known that $NS(X)$ is a free abelian group of rank $\rho(X) \le g^2$, where $\rho(X)$ is the Picard number of X. By Appell-Humbert Theorem, $NS(X)$ is realized in several ways as follows:

$$\begin{aligned} NS(X) &= Pic(X)/Pic^0(X) = c_1\big(H^1(X,\mathcal{O}_X^*)\big) \\ &= \{\, H : \mathbb{C}^g \times \mathbb{C}^g \longrightarrow \mathbb{C} \text{ hermitian},\ \mathrm{Im}\,(H)(L\times L) \subseteq \mathbb{Z} \,\} \\ &= \{\, E : \mathbb{C}^g \times \mathbb{C}^g \longrightarrow \mathbb{R} \text{ alternating},\ E(L\times L) \subseteq \mathbb{Z},\ E(i\cdot,\cdot) \text{ symmetric} \,\}. \end{aligned}$$

Let $\widehat{X} = Pic^0(X)$ be the dual complex torus of X. There exists the holomorphic line bundle $\mathscr{P}$ over $X \times \widehat{X}$ uniquely determined up to isomorphism, the so-called *Poincaré bundle* satisfying the following properties (PB1) and (PB2):

(PB1) $\mathscr{P}|_{X\times L} \cong L$ for all $L \in \widehat{X}$, and

(PB2) $\mathscr{P}|_{\{0\}\times\widehat{X}}$ is trivial on $\widehat{X}$.

We can see that $H^g(X,\mathscr{P}) = \mathbb{C}$ and $H^q(X,\mathscr{P}) = 0$ for all $q \neq g$.

Let $T_\Lambda = V/\Lambda$ be a real torus of dimension g, where $V \cong \mathbb{R}^g$ is a real vector space of dimension g and Λ is a lattice in V. Let $\rho : \Lambda \longrightarrow \mathbb{C}^*$ be a character of Λ. Let $B : V\times V \longrightarrow \mathbb{R}$ be a real valued symmetric bilinear form on V. We define the map $I_{B,\rho} : \Lambda \times V \longrightarrow \mathbb{C}^*$ by

$$(7.4) \qquad I_{B,\rho}(\lambda,v) = \rho(\lambda)\, e^{\pi B(\lambda,\lambda) + 2\pi B(v,\lambda)}, \qquad \lambda \in \Lambda,\ v \in V,\ \eta \in \mathbb{C}.$$

It is easily checked that $I_{B,\rho}$ satisfies the following equation

$$I_{B,\rho}(\lambda_1+\lambda_2,v) = I_{B,\rho}(\lambda_1,\lambda_2+v)I_{B,\rho}(\lambda_2,v), \qquad \lambda_1,\lambda_2 \in \Lambda,\ v \in V.$$

Then Λ acts on the trivial line bundle $V\times\mathbb{C}$ over V freely by

$$(7.5) \qquad \lambda\cdot(v,\eta) = (v+\lambda,\ I_{B,\rho}(\lambda,v)\eta), \qquad \lambda \in \Lambda,\ v \in V,\ \eta \in \mathbb{C}.$$

Thus the quotient space

$$(7.6) \qquad L(B,\rho) = (V\times\mathbb{C})/\Lambda$$

has a natural structure of a smooth (or real analytic) line bundle over a real torus T_Λ.

Lemma 7.1. *Suppose $B: V\times V \longrightarrow \mathbb{R}$ is a positive definite bilinear form on V. We define the function $\theta_{B,\rho}: V \longrightarrow \mathbb{C}$ by*

$$\theta_{B,\rho}(v) = \sum_{\lambda\in\Lambda} \rho(\lambda)^{-1}\, e^{-\pi\, B(\lambda,\lambda)\,-\,2\,\pi\, B(v,\lambda)}, \qquad v\in V. \tag{7.7}$$

Then map $\Theta_{B,\rho}: V \longrightarrow V\times\mathbb{C}$ defined by

$$\Theta_{B,\rho}(v) = (v, \theta_{B,\rho}(v)), \qquad v\in V \tag{7.8}$$

defines a smooth (or real analytic) global section of the line bundle $L(B,\rho)$.

Proof. For any $\lambda\in\Lambda$ and $v\in V$, we have

$$\begin{aligned}\theta_{B,\rho}(\lambda+v) &= \sum_{\mu\in\Lambda}\rho(\mu)^{-1}\, e^{-\pi\, B(\mu,\mu)\,-\,2\,\pi\, B(\lambda+v,\mu)}\\ &= \rho(\lambda) e^{\pi\, B(\lambda,\lambda)\,+\,2\,\pi\, B(v,\lambda)} \sum_{\mu\in\Lambda}\rho(\lambda+\mu)^{-1}\, e^{-\pi\, B(\lambda+\mu,\lambda+\mu)\,-\,2\,\pi\, B(v,\lambda+\mu)}\\ &= I_{B,\rho}(\lambda,v)\,\theta_{B,\rho}(v).\end{aligned}$$

Therefore $\Theta_{B,\rho}$ is a smooth global section of $L(B,\rho)$. □

Lemma 7.2. *Suppose $B: V\times V \longrightarrow \mathbb{R}$ is a positive definite bilinear form on V. Assume B is integral on $\Lambda\times\Lambda$, that is, $B(\Lambda\times\Lambda)\subset\mathbb{Z}$. Then for any character $\rho:\Lambda\longrightarrow\mathbb{C}$, the function $f_{B,\rho}: V\longrightarrow\mathbb{C}$ defined by*

$$f_{B,\rho}(v) = \sum_{\lambda\in\Lambda}\rho(\lambda)\, e^{-\pi\, B(\lambda,\lambda)\,+\,2\,\pi\, i\, B(v,\lambda)}, \qquad v\in V \tag{7.9}$$

is invariant under the action of Λ. Therefore $f_{B,\rho}$ may be regarded as a function on T_Λ.

Proof. It follows immediately from the definition. □

We see that

$$L_\Lambda = \mathbb{Z}^g + i\,\Lambda \subset \mathbb{C}^g$$

is a lattice in $\mathbb{C}^g$. We consider the complex torus

$$\mathfrak{T}_\Lambda = \mathbb{C}^g/L_\Lambda.$$

We define the $\mathbb{R}$-linear map $S_B: \mathbb{C}^g\times\mathbb{C}^g\longrightarrow\mathbb{R}$ and $E_B: \mathbb{C}^g\times\mathbb{C}^g\longrightarrow\mathbb{R}$ by

$$S_B(x,y) = B(x_1,y_1) + B(x_2,y_2) \tag{7.10}$$

and

$$E_B(x,y) = B(x_2,y_1) - B(x_1,y_2), \tag{7.11}$$

where $x = x_1 + i\,x_2\in\mathbb{C}^g$ and $y = y_1 + i\,y_2\in\mathbb{C}^g$ with $x_1,x_2,y_1,y_2\in\mathbb{R}^g$. It is easily seen that S_B is symmetric and E_B is alternating. We note that $S_B(x,y) = E_B(i\,x,y)$ for all $x,y\in\mathbb{C}^g$. We define the hermitian form $H_B: \mathbb{C}^g\times\mathbb{C}^g\longrightarrow\mathbb{C}$ by

$$H_B(x,y) := S_B(x,y) + i\,E_B(x,y), \qquad x,y\in\mathbb{C}^g. \tag{7.12}$$

Moreover we assume that E_B is integral on $L_\Lambda\times L_\Lambda$. Let $\alpha: L_\Lambda\longrightarrow\mathbb{C}_1^*$ be a semi-character of L_Λ for H_B such that

$$\alpha(\ell_1+\ell_2) = \alpha(\ell_1)\,\alpha(\ell_2)\, e^{\pi\, i\, E_B(\ell_1,\ell_2)}, \qquad \ell_1,\,\ell_2\in L_\Lambda.$$

Then the mapping $J_{B,\alpha} : L_\Lambda \times \mathbb{C}^g \longrightarrow \mathbb{C}^*$ defined by

$$J_{B,\alpha}(\ell, z) = \alpha(\ell)\, e^{\frac{\pi}{2} H_B(\ell,\ell) + \pi H_B(z,\ell)}, \qquad \ell \in L_\Lambda,\ z \in \mathbb{C}^g \tag{7.13}$$

is an automorphic factor for L_Λ on $\mathbb{C}^g$. Clearly L_Λ acts on the trivial line bundle $\mathbb{C}^g \times \mathbb{C}$ over $\mathbb{C}^g$ freely by

$$\ell \cdot (z, \xi) = (\ell + z, J_{B,\alpha}(\ell, z)\xi\,), \qquad \ell \in L_\Lambda,\ z \in \mathbb{C}^g,\ \xi \in \mathbb{C}. \tag{7.14}$$

The quotient

$$\mathfrak{L}(B,\alpha) := (\mathbb{C}^g \times \mathbb{C})/L_\Lambda$$

of $\mathbb{C}^g \times \mathbb{C}$ by L_Λ has a natural structure of a holomorphic line bundle over a complex torus $\mathfrak{T}_\Lambda$.

In summary, to each pair (B,α) with s symmetric $\mathbb{R}$-bilinear form B on V such that E_B is integral on $L_\Lambda \times L_\Lambda$ and a semi-character α for H_B there is associated the holomorphic line bundle $\mathfrak{L}(B,\alpha)$ over $\mathfrak{T}_\Lambda$.

We assume that B is non-degenerate of signature (r,s) with $r+s=g$. Then the hermitian form H_B is also non-degenerate of signature (r,s). Moreover we assume that E_B is integral on $L_\Lambda \times L_\Lambda$. Under these assumptions, Matsushima [15] proved that the cohomology group $H^q(\mathfrak{T}_\Lambda, \mathfrak{L}(B,\alpha)) = 0$ for all $q \neq s$ and that $H^s(\mathfrak{T}_\Lambda, \mathfrak{L}(B,\alpha))$ is identified with the complex vector space of all C^∞ functions f on $\mathbb{C}^g$ satisfying the following conditions:

(a) f is a differentiable theta functions for the automorphic factor $J_{B,\alpha}$; namely we have

$$f(\ell + z) = J_{B,\alpha}(\ell, z)\, f(z), \qquad \ell \in L_\Lambda,\ z \in \mathbb{C}^g,$$

(b) $\frac{\partial f}{\partial \overline{z}_i} = 0$ for all $i \in \{1, 2, \cdots, r\}$ and

$$\frac{\partial f}{\partial z_i} + \pi\, \overline{z}_i f = 0 \qquad \text{for all } i \in \{r+1, \cdots, g\},$$

where $(z_1, \cdots, z_g)$ is the coordinate of $\mathbb{C}^g$ determined by a privileged basis of $\mathbb{C}^g$ for the hermitian form H_B. We can show that the cohomology group $H^s\big(\mathfrak{T}_\Lambda, \mathfrak{L}(B,\alpha)^{\otimes 3}\big)$ defines a smooth embedding of $\mathfrak{T}_\Lambda$ into the projective space $\mathbb{P}^d(\mathbb{C})$ with $d+1 = \dim H^s\big(\mathfrak{T}_\Lambda, \mathfrak{L}(B,\alpha)^{\otimes 3}\big)$ which is holomorphic in $z_1, \cdots, z_r$ and anti-holomorphic in $z_{r+1}, \cdots, z_g$ (cf. [15] and [17]).

We consider the *canonical semi-character* $\gamma_{\Lambda,B}$ of L_Λ defined by

$$\gamma_{\Lambda,B}(\kappa + i\,\lambda) := e^{\pi\, i\, E_B(\kappa, i\lambda)}, \qquad \kappa \in \mathbb{Z}^g,\ \lambda \in \Lambda. \tag{7.15}$$

Then $\gamma_{\Lambda,B}$ defines the holomorphic line bundle $\mathfrak{L}(B, \gamma_{\Lambda,B})$ over a complex torus $\mathfrak{T}_\Lambda$. For any $z \in \mathfrak{T}_\Lambda$ we denote by T_z the translation of $\mathfrak{T}_\Lambda$ by z. Let $\pi_\Lambda : \mathbb{C}^g \longrightarrow \mathfrak{T}_\Lambda$ be the natural projection. Then there exists an element $c_{\Lambda,B,\alpha}$ of $\mathbb{C}^g$ such that

$$\mathfrak{L}(B,\alpha) = T^*_{\pi_\Lambda(c_{\Lambda,B,\alpha})} \mathfrak{L}(B, \gamma_{\Lambda,B}). \tag{7.16}$$

$c_{\Lambda,B,\alpha}$ is called a *characteristic* of the holomorphic line bundle $\mathfrak{L}(B,\alpha)$. We refer to [13] for detail.

Now we let $T_\Lambda = V/\Lambda$ be a polarized real torus of dimension g. Its associated polarized real abelian variety

$$\mathfrak{A}_\Lambda = \mathbb{C}^g/L_\Lambda, \qquad L_\Lambda = \mathbb{Z}^g + i\,\Lambda$$

admits a positive definite hermitian form H_Λ on $\mathbb{C}^g$ whose imaginary part $\mathrm{Im}\,(H_\Lambda)$ is integral on $\Lambda \times \Lambda$ (cf. [17, p. 35]). We write

$$H_\Lambda(x,y) = S_\Lambda(x,y) + i\,E_\Lambda(x,y), \qquad x,y \in \mathbb{C}^g,$$

where S_Λ and E_Λ are the real part (resp. imaginary part) of H_Λ respectively. We know that S_Λ is a real valued symmetric bilinear form on V and E_Λ is a real valued alternating bilinear form on V. Let $\alpha_\Lambda : L_\Lambda \longrightarrow \mathbb{C}_1^*$ be a canonical semi-character of L_Λ defined by

$$\alpha_\Lambda(\kappa + i\,\lambda) := e^{\pi\, i\, E_\Lambda(\kappa, i\lambda)}, \qquad \kappa \in \mathbb{Z}^g,\ \lambda \in \Lambda. \tag{7.17}$$

We let $J_{H_\Lambda,\alpha_\Lambda} : L_\Lambda \times \mathbb{C}^g \longrightarrow \mathbb{C}^*$ be the automorphic factor for Λ on V that is canonically given by

$$J_{H_\Lambda,\alpha_\Lambda}(\ell, z) = \alpha_\Lambda(\ell)\, e^{\frac{\pi}{2} H_\Lambda(\ell,\ell) + \pi H_\Lambda(z,\ell)}, \qquad \ell \in L_\Lambda,\ z \in \mathbb{C}^g. \tag{7.18}$$

Obviously L_Λ acts on $\mathbb{C}^g \times \mathbb{C}$ freely by

$$\ell \cdot (z,\xi) = \big(\ell + z,\ J_{H_\Lambda,\alpha_\Lambda}(\ell,z)\,\xi\,\big), \qquad \ell \in L_\Lambda,\ z \in \mathbb{C}^g,\ \xi \in \mathbb{C}.$$

So the quotient space

$$\mathfrak{L}(H_\Lambda, \alpha_\Lambda) := \big(\mathbb{C}^g \times \mathbb{C}\big)/L_\Lambda \tag{7.19}$$

has a natural structure of a holomorphic line bundle over an abelian variety $\mathfrak{A}_\Lambda$.

Now we define the map $\Phi_\Lambda : T_\Lambda \longrightarrow \mathfrak{A}_\Lambda$ by

$$\Phi_\Lambda(v + \Lambda) := i\,v + L_\Lambda, \qquad v \in \mathbb{R}^g. \tag{7.20}$$

Φ_Λ is a well defined injective mapping. It is well known that $H^q(\mathfrak{A}_\Lambda, \mathfrak{L}(H_\Lambda, \alpha_\Lambda)) = 0$ for all $q \neq 0$ and that the space of global holomorphic sections of $\mathfrak{L}(H_\Lambda,\alpha_\Lambda)^{\otimes n}$ for any positive integer $n \geq 3$ give a holomorphic embedding of $\mathfrak{A}_\Lambda$ as a closed complex manifold in a projective complex manifold $\mathbb{P}^d(\mathbb{C})$ (cf. [17, pp. 29–33]). Therefore we have a differentiable embedding of T_Λ into a complex projective space $\mathbb{P}^d(\mathbb{C})$ and hence into a real projective space $\mathbb{P}^N(\mathbb{R})$ with large enough $N > 0$.

We will characterize the pullback $L(\alpha_\Lambda) := \Phi_\Lambda^* \mathfrak{L}(H_\Lambda, \alpha_\Lambda)$. We first define the automorphic factor $I_{\alpha_\Lambda} : \Lambda \times \mathbb{R}^g \longrightarrow \mathbb{C}^*$ by

$$I_{\alpha_\Lambda}(\lambda, v) := \alpha_\Lambda(i\,\lambda)\, e^{\frac{\pi}{2} H_\Lambda(\lambda,\lambda) + \pi H_\Lambda(v,\lambda)}, \qquad \lambda \in \Lambda,\ v \in \mathbb{R}^g. \tag{7.21}$$

This automorphic factor I_{α_Λ} yields the smooth (or real analytic) line bundle over T_Λ which is nothing but the pullback $L(\alpha_\Lambda)$. We observe that if θ is a holomorphic theta function for $\mathfrak{L}(H_\Lambda, \alpha_\Lambda)$, then the function $f_\theta : \mathbb{R}^g \longrightarrow \mathbb{C}$ defined by $f_\theta(v) := \theta(i\,v)$, $v \in \mathbb{R}^g$ defines a global smooth (or real analytic) section of $L(\alpha_\Lambda)$.

Now we will show that a holomorphic line bundle $\mathfrak{L}(H_\Lambda, \alpha_\Lambda)$ over $\mathfrak{A}_\Lambda$ naturally yields a smooth line bundle over a polarized torus T_Λ. Let B_Λ be the restriction of S_Λ to $\mathbb{R}^g \times \mathbb{R}^g$. First we define the automorphic factor $I_{B_\Lambda,\alpha_\Lambda} : \Lambda \times \mathbb{R}^g \longrightarrow \mathbb{C}^*$ by

$$I_{B_\Lambda,\alpha_\Lambda}(\lambda, v) := \alpha_\Lambda(2\,i\,\lambda)\, e^{\pi B_\Lambda(\lambda,\lambda) + 2\pi B_\Lambda(v,\lambda)}, \qquad \lambda \in \Lambda,\ v \in \mathbb{R}^g. \tag{7.22}$$

This automorphic factor $I_{B_\Lambda,\alpha_\Lambda}(\lambda, v)$ yields a smooth line bundle

$$L(B_\Lambda, \alpha_\Lambda) := (\mathbb{R}^g \times \mathbb{C})/\Lambda \tag{7.23}$$

over a polarized real torus T_Λ. Since B_Λ is positive definite, according to Lemma 4.1, the space $\Gamma(T_\Lambda, L(B_\Lambda, \alpha_\Lambda))$ of smooth (or real analytic) global sections of $L(B_\Lambda, \alpha_\Lambda)$ is not zero.

If B_Λ is integral on $\Lambda\times\Lambda$, according to Lemma 4.2, we see that the function $f_{\Lambda,\alpha_\Lambda}:\mathbb{R}^g\longrightarrow\mathbb{C}$ defined by

$$f_{\Lambda,\alpha_\Lambda}(v)=\sum_{\lambda\in\Lambda}\alpha_\Lambda(2\,i\,\lambda)\,e^{-\pi\,B_\Lambda(\lambda,\lambda)+2\,\pi\,i\,B_\Lambda(v,\lambda)},\qquad v\in\mathbb{R}^g$$

is a function on T_Λ.

So far we have proved the following.

Theorem 7.1. *Let $T_\Lambda=V/\Lambda$ be a polarized real torus of dimension g. Then there is a smooth line bundle $L(B_\Lambda,\alpha_\Lambda)$ over T_Λ which is constructed canonically by (7.23).*

Example 7.1. Let $Y\in\mathcal{P}_g$ be a $g\times g$ positive definite symmetric real matrix. Then $\Lambda_Y=Y\mathbb{Z}^g$ is a lattice in $\mathbb{R}^g$. Then the g-dimensional torus $T_Y=\mathbb{R}^g/\Lambda_Y$ is a principally polarized real torus. Indeed,

$$\mathfrak{A}_Y=\mathbb{C}^g/L_Y,\qquad L_Y=\mathbb{Z}^g+i\,\Lambda_Y$$

is a princially polarized real abelian variety (cf. Example 6.1). Its corresponding hermitian form H_Y is given by

$$H_Y(x,y)=S_Y(x,y)+i\,E_Y(x,y)={}^t x\,Y^{-1}\,\overline{y},\qquad x,y\in\mathbb{C}^g,$$

where S_Y and E_Y denote the real part and the imaginary part of H_Y respectively. Let $\alpha:L_Y\longrightarrow\mathbb{C}_1^*$ be a semi-character of L_Y. To a pair (H_Y,α) the canonical automorphic factor $J_{Y,\alpha}:L_Y\times\mathbb{C}^g\longrightarrow\mathbb{C}$ is associated by

$$J_{Y,\alpha}(\ell,z)=\alpha(\ell)\,e^{\frac{1}{2}\pi\,i\,{}^t\ell\,Y^{-1}\,\overline{\ell}+\pi\,i\,{}^t z Y^{-1}\,\overline{\ell}},\qquad \ell\in L_Y,\ z\in\mathbb{C}^g.$$

The associated automorphic factor $I_{Y,\alpha}:\Lambda_Y\times\mathbb{R}^g\longrightarrow\mathbb{C}^*$ is given by

$$I_{Y,\alpha}(\lambda,v)=\alpha(2\,i\,\lambda)\,e^{\pi\,{}^t\lambda Y^{-1}\lambda+2\,\pi\,{}^t v\,Y^{-1}\lambda},\qquad \lambda\in\Lambda_Y,\ v\in\mathbb{R}^g.$$

We get the associated line bundle

$$L(B_Y,\alpha)=(\mathbb{R}^g\times\mathbb{C})/\Lambda_Y$$

given by $I_{Y,\alpha}$, where B_Y is the restriction of S_Y to $\mathbb{R}^g\times\mathbb{R}^g$. Then the function $\theta_{Y,\alpha}:\mathbb{R}^g\longrightarrow\mathbb{C}$ defined by

$$\theta_{Y,\alpha}(v)=\sum_{\lambda\in\Lambda_Y}\alpha(2\,i\,\lambda)\,e^{-\pi\,{}^t\lambda Y^{-1}\lambda-2\,\pi\,{}^t v\,Y^{-1}\lambda},\qquad v\in\mathbb{R}^g$$

yields a smooth global section of $L(B_Y,\alpha)$ over a real torus T_Y. The canonical semi-character α_Y of L_Y is given by

$$\alpha_Y(\kappa+i\,\lambda)=e^{-\pi\,i\,{}^t\kappa Y^{-1}\lambda},\qquad \kappa\in\mathbb{Z}^g,\ \lambda\in\Lambda_Y.$$

8. Moduli Space for Principally Polarized Real Tori

We have the natural action of $GL(g,\mathbb{R})$ on $\mathcal{P}_g$ given by

$$A*Y=AY\,{}^tA,\qquad A\in GL(g,\mathbb{R}),\ Y\in\mathcal{P}_g. \tag{8.1}$$

We put $\mathfrak{G}_g=GL(g,\mathbb{Z})$ (see Notations in the introduction). The fundamental domain $\mathfrak{R}_g$ for $\mathfrak{G}_g\backslash\mathcal{P}_g$ which was found by H. Minkowski [16] is defined as a subset of $\mathcal{P}_g$ consisting of $Y=(y_{ij})\in\mathcal{P}_g$ satisfying the following conditions (M.1)–(M.2) (cf. [10, p. 191] or [14, p. 123]):

(M.1) $aY\,{}^ta\ge y_{kk}$ for every $a=(a_i)\in\mathbb{Z}^g$ in which $a_k,\cdots,a_g$ are relatively prime for $k=1,2,\cdots,g$.

(M.2) $y_{k,k+1}\ge 0$ for $k=1,\cdots,g-1$.

We say that a point of $\mathfrak{R}_g$ is *Minkowski reduced* or simply *M-reduced.* $\mathfrak{R}_g$ has the following properties (R1)-(R6):

(R1) For any $Y\in\mathcal{P}_g$, there exist a matrix $A\in GL(g,\mathbb{Z})$ and $R\in\mathfrak{R}_g$ such that $Y=R[A]$ (cf. [10, p. 191] or [14, p. 139]). That is,

$$GL(g,\mathbb{Z})\circ\mathfrak{R}_g=\mathcal{P}_g.$$

(R2) $\mathfrak{R}_g$ is a convex cone through the origin bounded by a finite number of hyperplanes. $\mathfrak{R}_g$ is closed in $\mathcal{P}_g$ (cf. [14, p. 139]).

(R3) If Y and $Y[A]$ lie in $\mathfrak{R}_g$ for $A\in GL(g,\mathbb{Z})$ with $A\ne\pm I_g$, then Y lies on the boundary $\partial\mathfrak{R}_g$ of $\mathfrak{R}_g$. Moreover $\mathfrak{R}_g\cap(\mathfrak{R}_g[A])\ne\varnothing$ for only finitely many $A\in GL(g,\mathbb{Z})$ (cf. [14, p. 139]).

(R4) If $Y=(y_{ij})$ is an element of $\mathfrak{R}_g$, then

$$y_{11}\le y_{22}\le\cdots\le y_{gg}\quad\text{and}\quad |y_{ij}|<\frac{1}{2}y_{ii}\quad\text{for } 1\le i<j\le g.$$

We refer to [10, p. 192] or [14, pp. 123-124].

For $Y=(y_{ij})\in\mathcal{P}_g$, we put

$$dY=(dy_{ij})\qquad\text{and}\qquad \frac{\partial}{\partial Y}=\left(\frac{1+\delta_{ij}}{2}\frac{\partial}{\partial y_{ij}}\right).$$

For a fixed element $A\in GL(g,\mathbb{R})$, we put

$$Y_*=A\star Y=AY\,{}^tA,\quad Y\in\mathcal{P}_g.$$

Then

$$dY_*=A\,dY\,{}^tA\quad\text{and}\quad\frac{\partial}{\partial Y_*}={}^tA^{-1}\frac{\partial}{\partial Y}A^{-1}.\tag{8.2}$$

We consider the following differential operators

$$D_k=\sigma\left(\left(Y\frac{\partial}{\partial Y}\right)^k\right),\quad k=1,2,\cdots,g,\tag{8.3}$$

where $\sigma(M)$ denotes the trace of a square matrix M. By Formula (8.2), we get

$$\left(Y_*\frac{\partial}{\partial Y_*}\right)^i=A\left(Y\frac{\partial}{\partial Y}\right)^iA^{-1}$$

for any $A\in GL(g,\mathbb{R})$. So each D_i is invariant under the action (8.1) of $GL(g,\mathbb{R})$.

Selberg [20] proved the following.

Theorem 8.1. *The algebra $\mathbb{D}(\mathcal{P}_g)$ of all differential operators on $\mathcal{P}_g$ invariant under the action (8.1) of $GL(g,\mathbb{R})$ is generated by $D_1,D_2,\cdots,D_g$. Furthermore $D_1,D_2,\cdots,D_g$ are algebraically independent and $\mathbb{D}(\mathcal{P}_g)$ is isomorphic to the commutative ring $\mathbb{C}[x_1,x_2,\cdots,x_g]$ with g indeterminates $x_1,x_2,\cdots,x_g$.*

Proof. The proof can be found in [14, pp. 64-66]. □

We can see easily that

$$ds^2 = \sigma((Y^{-1}dY)^2)$$

is a $GL(g,\mathbb{R})$-invariant Riemannian metric on $\mathcal{P}_g$ and its Laplacian is given by

$$\Delta = \sigma\left(\left(Y\frac{\partial}{\partial Y}\right)^2\right).$$

We also can see that

$$d\mu_g(Y) = (\det Y)^{-\frac{g+1}{2}}\prod_{i\le j} dy_{ij}$$

is a $GL(g,\mathbb{R})$-invariant volume element on $\mathcal{P}_g$. The metric ds^2 on $\mathcal{P}_g$ induces the metric $ds^2_{\mathcal{R}}$ on $\mathfrak{R}_g$. Minkowski [16] calculated the volume of $\mathfrak{R}_g$ explicitly.

$\mathcal{P}_g$ parameterizes principally polarized real tori of dimension g. The Minkowski domain $\mathfrak{R}_g$ is the moduli space for isomorphism classes of principally polarized real tori of dimension g. According to (R2) we see that $\mathfrak{R}_g$ is a semi-algebraic set with real analytic structure. Unfortunately $\mathfrak{R}_g$ does not admit the structure of a real algebraic variety and does not admit a compactification which is defined over the rational number field $\mathbb{Q}$. We see that $\mathfrak{R}_g$ is real analytically isomorphic to the semi-algebraic subset $\mathscr{X}^g_{(0,1)}$ of $\mathscr{X}^g_{\mathbb{R}}$. We define the embedding $\Phi_g : \mathcal{P}_g \longrightarrow \mathscr{H}_g$ by

$$\Phi_g(Y) = iY, \qquad Y \in \mathcal{P}_g. \tag{8.4}$$

We have the following inclusions

$$\mathcal{P}_g \xrightarrow{\Phi_g} i\,\mathcal{P}_g \hookrightarrow \mathscr{H}_g \hookrightarrow \mathbb{H}_g \subset \mathbb{H}_g^*.$$

$\mathfrak{G}_g$ acts on $\mathcal{P}_g$ and $i\,\mathcal{P}_g$, $\Gamma_g^\star$ acts on $\mathscr{H}_g$, and Γ_g acts on $\mathbb{H}_g$ and $\mathbb{H}_g^*$. It might be interesting to characterize the boundary points of the closure of $i\,\mathcal{P}_g$ (or $\mathcal{P}_g$) in $\mathbb{H}_g^*$ explicitly. In Section 5 we reviewed Silhol's compactification $\overline{\mathscr{X}^g_{\mathbb{R}}}$ of $\mathscr{X}^g_{\mathbb{R}}$ which is analogous to the Satake-Baily-Borel compactification. The theory of automorphic forms on $\mathfrak{R}_g$ has been developed by Selberg [20], Maass [14] et al past a half century. According to Theorem 5.1, $\overline{\mathscr{X}^g_{\mathbb{R}}}$ is a connected compact Hausdorff space containing $\mathscr{X}^g_{\mathbb{R}}$ as an open dense subset of $\overline{\mathscr{X}^g_{\mathbb{R}}}$. But $\overline{\mathscr{X}^g_{\mathbb{R}}}$ does not admit an algebraic structure.

For any positive integer $h \in \mathbb{Z}^+$, we let

$$GL_{g,h} := GL(g,\mathbb{R}) \ltimes \mathbb{R}^{(h,g)} \tag{8.5}$$

be the semi-direct product of $GL(g,\mathbb{R})$ and $\mathbb{R}^{(h,g)}$ with the multiplication law

$$(A,a)\cdot(B,b) = (AB, a\,{}^tB^{-1} + b), \qquad A,B \in GL(g,\mathbb{R}),\quad a,b \in \mathbb{R}^{(h,g)}. \tag{8.6}$$

Then we have the *natural action* of $GL_{g,h}$ on the Minkowski-Euclid space $\mathcal{P}_g \times \mathbb{R}^{(h,g)}$ defined by

$$(A,a)\cdot(Y,\zeta) = \big(AY\,{}^tA,\ (\zeta + a)\,{}^tA\big), \qquad (A,a) \in GL_{g,h},\ Y \in \mathcal{P}_g,\ \zeta \in \mathbb{R}^{(h,g)}. \tag{8.7}$$

For a variable $(Y,V)\in \mathcal{P}_g\times\mathbb{R}^{(h,g)}$ with $Y\in\mathcal{P}_g$ and $V\in\mathbb{R}^{(h,g)}$, we put

$$Y=(y_{\mu\nu}) \text{ with } y_{\mu\nu}=y_{\nu\mu},\quad V=(v_{kl}),$$

$$dY=(dy_{\mu\nu}),\quad dV=(dv_{kl}),$$

$$[dY]=\prod_{\mu\le\nu}dy_{\mu\nu},\qquad [dV]=\prod_{k,l}dv_{kl},$$

and

$$\frac{\partial}{\partial Y}=\left(\frac{1+\delta_{\mu\nu}}{2}\frac{\partial}{\partial y_{\mu\nu}}\right),\quad \frac{\partial}{\partial V}=\left(\frac{\partial}{\partial v_{kl}}\right),$$

where $1\le\mu,\nu,\ l\le g$ and $1\le k\le h$.

Lemma 8.1. *For all two positive real numbers A and B, the following metric $ds^2_{g,h;A,B}$ on $\mathcal{P}_g\times\mathbb{R}^{(h,g)}$ defined by*

$$ds^2_{g,h;A,B}=A\,\sigma(Y^{-1}dY\,Y^{-1}dY)\,+\,B\,\sigma(Y^{-1}\,{}^t(dV)\,dV) \tag{8.8}$$

is a Riemannian metric on $\mathcal{P}_g\times\mathbb{R}^{(h,g)}$ which is invariant under the action (8.7) of $GL_{g,h}$. The Laplacian $\Delta_{g,h;A,B}$ of $(\mathcal{P}_g\times\mathbb{R}^{(h,g)},\,ds^2_{g,h;A,B})$ is given by

$$\Delta_{g,h;A,B}=\frac{1}{A}\,\sigma\left(\left(Y\frac{\partial}{\partial Y}\right)^2\right)-\frac{h}{2A}\,\sigma\left(Y\frac{\partial}{\partial Y}\right)+\frac{1}{B}\sum_{k\le p}\left(\left(\frac{\partial}{\partial V}\right)Y\,{}^t\!\left(\frac{\partial}{\partial V}\right)\right)_{kp}.$$

Moreover $\Delta_{g,h;A,B}$ is a differential operator of order 2 which is invariant under the action (8.7) of $GL_{g,h}$.

Proof. For a fixed element $(A,a)\in GL_{g,h}$, we set

$$(Y^*,V^*)=(A,a)\cdot(Y,V).$$

Then

$$Y^*=A\,Y\,{}^tA,\qquad V^*=(V+a)\,{}^tA.$$

The first statement follows immediately from the fact that

$$dY^*=A\,dY\,{}^tA\quad\text{ and }\quad dV^*=dV\,{}^tA.$$

Using the formula (13) in [8, p. 245], we can compute the Laplacian $\Delta_{g,h;A,B}$ of $(\mathcal{P}_g\times\mathbb{R}^{(h,g)},\,ds^2_{g,h;A,B})$. The last statement follows from the fact that

$$\frac{\partial}{\partial Y^*}={}^tA^{-1}\frac{\partial}{\partial Y}\,A^{-1},\quad \frac{\partial}{\partial V^*}=\frac{\partial}{\partial V}\cdot A^{-1}.$$

□

Lemma 8.2. *The following volume element $dv_{g,h}(Y,V)$ on $\mathcal{P}_g\times\mathbb{R}^{(h,g)}$ defined by*

$$dv_{g,h}(Y,V)=(\det Y)^{-\frac{g+h+1}{2}}[dY][dV] \tag{8.9}$$

is invariant under the action (8.7) of $GL_{g,h}$.

Proof. For a fixed element $(A,a)\in GL_{g,h}$, we set

$$(Y^*,V^*)=(A,a)\cdot(Y,V)=(\,AY\,{}^tA,(V+a)\,{}^tA).$$

Let $\frac{\partial(Y^*,V^*)}{\partial(Y,V)}$ be the Jacobian determinant of the action (8.7) of $GL_{g,h}$ on $\mathcal{P}_g\times\mathbb{R}^{(h,g)}$. It is known that the Jacobian determinant of the action $Y\longmapsto Y^*$ is given by

$$\frac{\partial(Y^*)}{\partial(Y)}=(\,\det A\,)^{g+1}.$$

Take the diagonal matrix $g=(d_1,\cdots,d_g)$ with distinct real numbers d_i. Obviously if $a=(a_{kl})$, $V=(v_{kl})$ and $V^*=(v^*_{kl})$, then $v^*_{kl}=(v_{kl}+a_{kl})d_l$ for all k,l. Thus we have

$$\frac{\partial(V^*)}{\partial(V)}=(d_1\cdots d_g)^h=(\,\det A\,)^h. \tag{8.10}$$

Since the set of all $g\times g$ real matrices whose eigenvalues are all distinct is everywhere dense in $GL(g,\mathbb{R})$, and $\frac{\partial(V^*)}{\partial(V)}$ is a rational function, the relation (8.10) holds for any $A\in GL(g,\mathbb{R})$. It is easy to see that

$$\frac{\partial(Y^*,V^*)}{\partial(Y,V)}=\frac{\partial(Y^*)}{\partial(Y)}\cdot\frac{\partial(V^*)}{\partial(V)}.$$

Thus we obtain

$$[dY^*][dV^*]=|\det A\,|^{g+h+1}[dY][dV].$$

Since $\det Y^*=(\,\det A\,)^2\det Y$, we have

$$(\,\det Y^*\,)^{-\frac{g+h+1}{2}}[dY^*][dV^*]=(\,\det Y\,)^{-\frac{g+h+1}{2}}[dY][dV].$$

Hence the volume element (8.9) is invariant under the action (8.7). □

It is known that

$$d\mu_g(Y):=(\det Y)^{-\frac{g+1}{2}}\,[dY]$$

is a volume element on $\mathcal{P}_g$ invariant under the action (8.1) of $GL(g,\mathbb{R})$ (cf. [14, p. 23]). Let r be a positive integer with $0<r<g$. We define a bijective transformation

$$\mathcal{P}_g\longrightarrow\mathcal{P}_r\times\mathcal{P}_s\times\mathbb{R}^{(s,r)},\quad r+s=g,\quad Y\longmapsto(F,G,H)$$

by

$$Y=\begin{pmatrix}F&0\\0&G\end{pmatrix}\left[\begin{pmatrix}I_r&0\\H&I_s\end{pmatrix}\right],\quad Y\in\mathcal{P}_n,\ F\in\mathcal{P}_r,\ G\in\mathcal{P}_s,\ H\in\mathbb{R}^{(s,r)}. \tag{8.11}$$

According to [14, pp. 24-26], we obtain

$$[dY]=(\det G)^r\,[dF][dH][dG], \tag{8.12}$$

equivalently

$$d\mu_g(Y)=(\det F)^{-\frac{s}{2}}\,(\det G)^{\frac{r}{2}}\,d\mu_r(F)\,d\mu_s(G)\,[dH]. \tag{8.13}$$

Therefore we get

$$dv_{g,h}(Y,V)=(\det F)^{-\frac{h+s}{2}}\,(\det G)^{\frac{r-h}{2}}\,d\mu_r(F)\,d\mu_s(G)\,[dH]\,[dV]. \tag{8.14}$$

Similarly if $Y \in \mathcal{P}_g,\ g = r + s$ with $0 < r < g$, we write

$$Y = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \left[\begin{pmatrix} I_r & R \\ 0 & I_s \end{pmatrix} \right], \quad Y \in \mathcal{P}_g,\ P \in \mathcal{P}_r,\ Q \in \mathcal{P}_s,\ R \in \mathbb{R}^{(r,s)}. \tag{8.15}$$

According to [14, pp. 27], we obtain

$$[dY] = (\det P)^s \, [dP][dQ][dR], \tag{8.16}$$

equivalently

$$d\mu_g(Y) = (\det P)^{\frac{s}{2}} \, (\det G)^{-\frac{r}{2}} \, d\mu_r(P) \, d\mu_s(Q) \, [dR]. \tag{8.17}$$

Therefore we get

$$dv_{g,h}(Y,V) = (\det P)^{\frac{s-h}{2}} \, (\det G)^{-\frac{r+h}{2}} \, d\mu_r(P) \, d\mu_s(Q) \, [dR] \, [dV]. \tag{8.18}$$

The coordinates (F,G,H) or (P,Q,R) are called the partial Iwasawa coordinates on $\mathcal{P}_g$.

Theorem 8.2. *Any geodesic through the origin* $(I_g, 0)$ *is of the form*

$$\gamma(t) = \left(\lambda(2t)[k],\ Z \left(\int_0^t \lambda(t-s) ds \right) [k] \right),$$

where k *is a fixed element of* $O(g)$, Z *is a fixed* $h \times g$ *real matrix,* t *is a real variable,* $\lambda_1, \lambda_2, \cdots, \lambda_g$ *are fixed real numbers but not all zero and*

$$\lambda(t) := diag\,(e^{\lambda_1 t}, \cdots, e^{\lambda_g t}).$$

Furthermore, the tangent vector $\gamma'(0)$ *of the geodesic* $\gamma(t)$ *at* $(I_g, 0)$ *is* $(D[k], Z)$*, where* $D = diag\,(2\lambda_1, \cdots, 2\lambda_g)$.

Proof. Let $W = (X, Z)$ be an element of $\mathfrak{p}$ with $X \neq 0$. Then the curve

$$\alpha(t) = \exp tW = \left(e^{tX},\ Z \left(\int_0^t e^{-sX} ds \right) \right), \quad t \in \mathbb{R}$$

is a geodesic in $GL_{g,h}$ with $\alpha'(0) = W$ passing through the identity of $GL_{g,h}$. Thus the curve

$$\gamma(t) = \alpha(t) \cdot (I_g, 0) = \left(e^{2tX},\ Z \left(\int_0^t e^{-sX} ds \right) e^{tX} \right)$$

is a geodesic in $\mathcal{P}_g \times \mathbb{R}^{(h,g)}$ passing through the origin $(I_g, 0)$. Since X is a symmetric real matrix, there is a diagonal matrix $\Lambda = \mathrm{diag}\,(\lambda_1, \cdots, \lambda_g)$ with $\lambda_1, \cdots, \lambda_g \in \mathbb{R}$ such that

$$X = {}^t k \Lambda k \quad \text{for some } k \in O(g),$$

where $\lambda_1, \cdots, \lambda_n$ are real numbers and not all zero. Thus we may write

$$\gamma(t) = \left((\delta_{kl} e^{2\lambda_k t})[k],\ Z \left(\int_0^t e^{(t-s)\Lambda} ds \right) [k] \right).$$

Hence this completes the proof. □

Theorem 8.3. *Let* (Y_0, V_0) *and* (Y_1, V_1) *be two points in* $\mathcal{P}_g \times \mathbb{R}^{(h,g)}$*. Let* g *be an element in* $GL(g, \mathbb{R})$ *such that* $Y_0[{}^t g] = I_g$ *and* $Y_1[{}^t g]$ *is diagonal. Then the length* $s\big((Y_0, V_0), (Y_1, V_1)\big)$

of the geodesic joining (Y_0,V_0) *and* (Y_1,V_1) *for the* $GL_{g,h}$*-invariant Riemannian metric* $ds^2_{g,h;A,B}$ *is given by*

$$s\big((Y_0,V_0),(Y_1,V_1)\big) = A\left\{\sum_{j=1}^{g}(\ln t_j)^2\right\}^{1/2} + B\int_0^1\left(\sum_{j=1}^{g}\Delta_j\, e^{-(\ln t_j)\,t}\right)^{1/2} dt, \tag{8.19}$$

where $\Delta_j = \sum_{k=1}^h \widetilde{v}_{kj}^2$ $(1\le j\le g)$ *with* $(V_1-V_0)\,{}^tg = (\widetilde{v}_{kj})$ *and* $t_1,\cdots,t_g$ *denotes the zeros of* $\det(t\,Y_0-Y_1)$.

Proof. Without loss of generality we may assume that $(Y_0,V_0) = (I_g,0)$ and $(Y_1,V_1) = (T,\widetilde{V})$ with $T=\mathrm{diag}(t_1,\cdots,t_n)$ diagoanl because the element $(g,-V_0)\in GL_{g,h}$ can be regarded as an isometry of $\mathcal{P}_g\times\mathbb{R}^{(h,g)}$ for the Riemannian metric $ds^2_{g,h;A,B}$ (cf. Lemma 8.1). Let$\gamma(t) = \big(\alpha(t),\beta(t)\big)$ with $0\le t\le 1$ be the geodesic in $\mathcal{P}_g\times\mathbb{R}^{(h,g)}$ joining two points $\gamma(0)=(Y_0,V_0)$ and $\gamma(1)=(Y_1,V_1)$, where $\alpha(t)$ is the uniquely determined curve in $\mathcal{P}_g$ and $\beta(t)$ is the uniquely determined curve in $\mathbb{R}^{(h,g)}$.

We now use the partial Iwasawa coordinates in $\mathcal{P}_g$. Then if $Y\in\mathcal{P}_g$, we write for any positive integer r with $0<r<g$, $r+s=g$,

$$Y = \begin{pmatrix} F & 0\\ 0 & G\end{pmatrix}\left[\begin{pmatrix} I_r & 0\\ H & I_s\end{pmatrix}\right],\qquad F\in\mathcal{P}_r,\ G\in\mathcal{P}_s,\ H\in\mathbb{R}^{(h,g)}.$$

For $V\in\mathbb{R}^{(h,g)}$, we write

$$V = (R,S),\qquad R\in\mathbb{R}^{(h,r)},\ S\in\mathbb{R}^{(h,s)}.$$

Now we express $ds^2_{g,h;A,B}$ in terms of F,G,H,R and S.

Lemma 8.3.

$$\begin{aligned} ds^2_{g,h;A,B} \;=\;& A\cdot\big\{\sigma\big((F^{-1}dF)^2\big)+\sigma\big((G^{-1}dG)^2\big)+2\,\sigma\big(F^{-1\,t}(dH)\,G\,dH\big)\big\}\\ &+B\cdot\big\{\sigma\big(F^{-1\,t}(dR)\,dR\big)+\sigma\big((G^{-1}+F^{-1}[\,{}^tH])\,{}^t(dS)\,dS\big)\big\}\\ &-2\,B\cdot\sigma\big(F^{-1\,t}H\,{}^t(dS)\,dR\big). \end{aligned}$$

Proof of Lemma 8.3. First we see that if $Y\in\mathcal{P}_g$, then

$$Y^{-1} = \begin{pmatrix} F^{-1} & 0\\ 0 & G^{-1}\end{pmatrix}\left[\begin{pmatrix} I_r & -{}^tH\\ 0 & I_s\end{pmatrix}\right] = \begin{pmatrix} F^{-1} & -F^{-1\,t}H\\ -HF^{-1} & G^{-1}+F^{-1}[\,{}^tH]\end{pmatrix},$$

$$dY = \begin{pmatrix} dF + dG\,[H] + {}^t(dH)\cdot GH + {}^tHG\cdot dH & {}^t(dH)\cdot G + {}^tH\cdot dG\\ dG\cdot H + G\cdot dH & dG\end{pmatrix}$$

and $dV = (dR,dS)$.

For brevity, we put

$$dY\cdot Y^{-1} = \begin{pmatrix} L_0 & L_1\\ L_2 & L_3\end{pmatrix}$$

and

$$Y^{-1\,t}(dV)\,dV = \begin{pmatrix} M_0 & M_1\\ M_2 & M_3\end{pmatrix}.$$

Here L_0, L_1, L_2 and L_3 denote the $r\times r$, $r\times s$, $s\times r$ and $s\times s$ matrix valued differential one forms respectively, and M_0, M_1, M_2 and M_3 denote the $r\times r$, $r\times s$, $s\times r$ and $s\times s$ matrix valued differential two forms respectively.

By an easy computation, we get

$$\begin{aligned} L_0 &= dF\cdot F^{-1} + {}^tHG\cdot dH\cdot F^{-1},\\ L_1 &= -dF\cdot F^{-1}\,{}^tH - {}^tHG\cdot dH\cdot F^{-1}{}^tH + {}^t(dH) + {}^tH\cdot dG\cdot G^{-1},\\ L_2 &= G\cdot dH\cdot F^{-1},\\ L_3 &= dG\cdot G^{-1} - G\cdot dH\cdot F^{-1}\,{}^tH,\\ M_0 &= F^{-1}\,{}^t(dR)\,dR - F^{-1}\,{}^tH\,{}^t(dS)\,dR,\\ M_3 &= -HF^{-1}\,{}^t(dR)\,dS + \big(G^{-1}+F^{-1}[\,{}^tH]\big)\,{}^t(dS)\,dS. \end{aligned}$$

Therefore we have

$$\begin{aligned} ds^2_{g,h;A,B} &= A\cdot\sigma\big((dY\cdot Y^{-1})^2\big) + B\cdot\sigma\big(Y^{-1}\,{}^t(dV)\,dV\,\big)\\ &= A\cdot\big\{\,\sigma\big(L_0^2 + L_1L_2\big) + \sigma\big(L_2L_1 + L_3^2\,\big)\big\}\\ &\quad + B\cdot\{\,\sigma(M_0) + \sigma(M_3)\,\}\\ &= A\cdot\big\{\sigma\big((F^{-1}dF)^2\big) + \sigma\big((G^{-1}dG)^2\big) + 2\,\sigma\big(F^{-1}{}^t(dH)\,G\,dH\big)\big\}\\ &\quad + B\cdot\big\{\sigma\big(F^{-1}\,{}^t(dR)\,dR\,\big) + \sigma\big((G^{-1}+F^{-1}[\,{}^tH])\,{}^t(dS)\,dS\big)\big\}\\ &\quad - 2\,B\cdot\sigma\big(F^{-1}\,{}^tH\,{}^t(dS)\,dR\big). \end{aligned}$$

□

Let $s\big((Y_0,V_0),(Y_1,V_1)\big)$ be the length of the geodesic $\gamma(t) = (\alpha(t),\beta(t))$ with $0\le t\le 1$. We put

$$\alpha(t) = \begin{pmatrix} F(t) & 0\\ 0 & G(t)\end{pmatrix}\left[\begin{pmatrix} I_r & 0\\ H(t) & I_s\end{pmatrix}\right],\quad \beta(t) = (R(t),S(t)),\quad 0\le t\le 1,$$

where $F(t)$, $G(t)$, $H(t)$, $R(t)$ and $S(t)$ are the uniquely determined curves in $\mathcal{P}_r$, $\mathcal{P}_s$, $\mathbb{R}^{(s,r)}$, $\mathbb{R}^{(h,r)}$ and $\mathbb{R}^{(h,s)}$ respectively.

then we have

$$\begin{aligned} &s\big((Y_0,V_0),(Y_1,V_1)\big)\\ = \;& A\cdot\int_0^1\left\{\sigma\left(\left(F^{-1}\frac{dF}{dt}\right)^2\right) + \sigma\left(\left(G^{-1}\frac{dG}{dt}\right)^2\right) + 2\,\sigma\left(F^{-1}\,{}^t\!\left(\frac{dH}{dt}\right)G\,\frac{dH}{dt}\right)\right\}^{1/2} dt\\ &+B\cdot\int_0^1\left\{\sigma\left(\gamma(t)^{-1}\,{}^t\!\left(\frac{dV}{dt}\right)\frac{dV}{dt}\right)\right\}^{1/2} dt\\ \ge\;& A\cdot\int_0^1\left\{\sigma\left(\left(F^{-1}\frac{dF}{dt}\right)^2\right) + \sigma\left(\left(G^{-1}\frac{dG}{dt}\right)^2\right)\right\}^{1/2} dt\\ &+B\cdot\int_0^1\left\{\sigma\left(F^{-1}\,{}^t\!\left(\frac{dR}{dt}\right)\frac{dR}{dt}\right) + \sigma\left(G^{-1}\,{}^t\!\left(\frac{dS}{dt}\right)\frac{dS}{dt}\right)\right\}^{1/2} dt. \end{aligned}$$

The reason is that the quadratic form $\sigma\big(F^{-1}\,{}^t(dH)G\,dH\big)$ is positive definite. Indeed, if $M, N \in GL(g,\mathbb{R})$ such that $F = {}^tMM$ and $G = {}^tNN$, then

$$\sigma\big(F^{-1}\,{}^t(dH)G\,dH\big) = \sigma\big({}^tWW\big), \quad W := N\cdot dH\cdot M^{-1}.$$

If

$$\sigma\left(F^{-1}\,{}^t\!\left(\frac{dH}{dt}\right)G\,\frac{dH}{dt}\right) = 0,$$

then $\frac{dH}{dt} = 0$ and hence $H(t)$ is constant in the interval $[0,1]$. Since $H(0)=0$, $H(t) = 0$ $(0 \le t \le 1)$.

Moreover the curve $\alpha(t)$ must be diagonal, that is,

$$\alpha(t) = \left(\delta_{\mu\nu}\,e^{\chi_\nu(t)}\right), \quad \chi_\nu(0) = 0,\ \chi_\nu(1) = \ln t_\nu, \quad 1 \le \nu \le g,$$

where $g_\nu(t)$ $(1 \le \nu \le g)$ are continuously differentiable in [0,1]. Thus we have

$$\frac{d\alpha}{dt} = \left(\delta_{\mu\nu}\,e^{\chi_\nu(t)}\,\frac{d\chi_\nu}{dt}\right)$$

and hence

$$\alpha(t)^{-1}\,\frac{d\alpha}{dt} = \left(\delta_{\mu\nu}\,\frac{d\chi_\nu}{dt}\right).$$

Therefore we have

$$\begin{aligned}
&\int_0^1 \left\{\sigma\left(\left(\alpha(t)^{-1}\,\frac{d\alpha}{dt}\right)^2\right)\right\}^{1/2} dt \\
=\ &\int_0^1 \left\{\sigma\left(\left(F^{-1}\frac{dF}{dt}\right)^2\right) + \sigma\left(\left(G^{-1}\frac{dG}{dt}\right)^2\right)\right\}^{1/2} dt \\
=\ &\int_0^1 \left\{\sum_{j=1}^n \left(\frac{d\chi_j}{dt}\right)^2\right\}^{1/2} dt.
\end{aligned}$$

The minimum value of $\sum_{j=1}^g \left(\frac{d\chi_j}{dt}\right)^2$ is obtained if the curve $\alpha(t)$ is the straight line, i.e., $\chi_j(t) = t\,\ln t_j$ $(1 \le j \le g)$, $0 \le t \le 1$ in the $(\chi_1,\cdots,\chi_g)$-space. Thus we get

$$\int_0^1 \left\{\sigma\left(\left(\alpha(t)^{-1}\,\frac{d\alpha}{dt}\right)^2\right)\right\}^{1/2} dt = \left\{\sum_{j=1}^n (\ln t_j)^2\right\}^{1/2}.$$

We put

$$\beta(t) = \big(\beta_{kj}(t)\big) \qquad \text{with} \quad 0 \le t \le 1,\ 1 \le k \le h,\ 1 \le j \le g.$$

Then we obtain

$$\begin{aligned}
&\int_0^1 \left\{ \sigma\left(\alpha(t)^{-1}\, {}^t\!\left(\frac{d\beta}{dt}\right) \frac{d\beta}{dt} \right) \right\}^{1/2} dt \\
=\ & \int_0^1 \left\{ \sigma\left(F^{-1}\, {}^t\!\left(\frac{dR}{dt}\right) \frac{dR}{dt} \right) + \sigma\left(G^{-1}\, {}^t\!\left(\frac{dS}{dt}\right) \frac{dS}{dt} \right) \right\}^{1/2} dt \\
=\ & \int_0^1 \left\{ \sum_{j=1}^{g} \sum_{k=1}^{h} e^{-t\,\ln t_j} \left(\frac{d\beta_{kj}}{dt}\right)^2 \right\}^{1/2} dt.
\end{aligned}$$

Each curve $\beta_{kj}(t)$ $(0 \le t \le 1)$ is a curve in $\mathbb{R}$ such that $\beta_k j(0) = 0$ and $\beta_{kj}(1) = \widetilde{v}_{kj}$. Thus each curve $\beta_{kj}(t)$ must be a straight line, that is, for all k, j with $1 \le k \le h$ and $1 \le j \le g$,

$$\beta_{kj}(t) = \widetilde{v}_{kj}\, t, \qquad 0 \le t \le 1.$$

Therefore we have

$$\begin{aligned}
&\int_0^1 \left\{ \sigma\left(\alpha(t)^{-1}\, {}^t\!\left(\frac{d\beta}{dt}\right) \frac{d\beta}{dt} \right) \right\}^{1/2} dt \\
=\ & \int_0^1 \left\{ \sum_{j=1}^{n} e^{-t\,\ln t_j} \left(\sum_{k=1}^{h} \widetilde{v}_{kj}^{\,2} \right) \right\}^{1/2} dt \\
=\ & \int_0^1 \left(\sum_{j=1}^{g} \Delta_j\, e^{-t\,\ln t_j} \right)^{1/2} dt.
\end{aligned}$$

Finally we obtain

$$s\big((Y_0,V_0),(Y_1,V_1)\big) = A \left\{ \sum_{j=1}^{g} (\ln t_j)^2 \right\}^{1/2} + B \int_0^1 \left(\sum_{j=1}^{g} \Delta_j\, e^{-(\ln t_j)\,t} \right)^{1/2} dt. \tag{8.20}$$

Hence we complete the proof. □

For a fixed element $(A,a) \in GL_{g,h}$, we let $\Theta_{A,a} : \mathcal{P}_g \times \mathbb{R}^{(h,g)} \longrightarrow \mathcal{P}_g \times \mathbb{R}^{(h,g)}$ be the mapping defined by

$$\Theta_{A,a}(Y,V) := (A,a)\cdot(Y,V), \qquad (Y,V) \in \mathcal{P}_g \times \mathbb{R}^{(h,g)}.$$

We consider the behaviour of the differential map $d\Theta_{A,a}$ of $\Theta_{A,a}$ at $(I_g, 0)$. Then $d\Theta_{A,a}$ is given by

$$d\Theta_{A,a}(u,v) = (Au\,{}^tA,\ v\,{}^tA),$$

where (u,v) is a tangent vector of $\mathcal{P}_g \times \mathbb{R}^{(h,g)}$ at $(I_g,0)$.

We let $\tilde{\theta}$ be the involution of $GL_{g,h}$ defined by

$$\tilde{\theta}((A,a)) := ({}^tA^{-1}, -a), \qquad (A,a) \in GL_{g,h}.$$

Then the differential map of $\tilde{\theta}$ at $(I_g,0)$, denoted by the same notation $\tilde{\theta}$ is given by

$$\tilde{\theta} : \mathfrak{g} \longrightarrow \mathfrak{g}, \quad \tilde{\theta}(X,Z) = (-\,{}^tX, -Z),$$

where $X \in \mathbb{R}^{(g,g)}$ and $Z \in \mathbb{R}^{(h,g)}$. We note that $\mathfrak{k}$ is the $(+1)$-eigenspace of $\tilde{\theta}$ and

$$\mathfrak{p} = \left\{ (X,Z) \mid X \in \mathbb{R}^{(g,g)},\ X = {}^tX,\ Z \in \mathbb{R}^{(h,g)} \right\}$$

is the (-1)-eigenspace of $\tilde{\theta}$.

Now we consider some differential forms on $\mathcal{P}_g \times \mathbb{R}^{(h,g)}$ which are invariant under the action of $GL(g,\mathbb{Z}) \ltimes \mathbb{Z}^{(h,g)}$. We let

$$\mathfrak{G}_{g,h} := GL(g,\mathbb{Z}) \ltimes \mathbb{Z}^{(h,g)}$$

be the discrete subgroup of $GL_{g,h}$. Let

$$\alpha_* = \sum_{\mu\le\nu} f_{\mu\nu}(Y,V)\, dy_{\mu\nu} + \sum_{k=1}^{h}\sum_{l=1}^{g} \phi_{kl}(Y,V)\, dv_{kl}$$

be a differential 1-form on $\mathcal{P}_g \times \mathbb{R}^{(h,g)}$ that is invariant under the action of $\mathfrak{G}_{g,h}$. We put

$$e_{\mu\nu} = \begin{cases} 1 & \text{if } \mu = \nu \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

We let

$$f(Y,V) = (e_{\mu\nu} f_{\mu\nu}(Y,V)) \quad \text{and} \quad \phi(Y,V) = {}^t(\phi_{kl}(Y,V)),$$

where $f(Y,V)$ is a $g\times g$ matrix with entries $f_{\mu\nu}(Y,V)$ and $\phi(Y,V)$ is a $g\times h$ matrix with entries $\phi_{kl}(Y,V)$. Then

$$\alpha_* = \sigma(f\,dY + \phi\, dV).$$

If $\tilde{\gamma} = (\gamma,\alpha) \in \mathfrak{G}_{g,h}$ with $\gamma \in GL(g,\mathbb{Z})$ and $\alpha \in \mathbb{Z}^{(h,g)}$, then we have the following transformation relation

$$f(\gamma Y\, {}^t\gamma,\ (V+\alpha)\,{}^t\gamma) = {}^t\gamma^{-1} f(Y,V)\,\gamma^{-1} \tag{8.21}$$

and

$$\phi(\gamma Y\, {}^t\gamma,\ (V+\alpha)\,{}^t\gamma) = {}^t\gamma^{-1}\phi(Y,V). \tag{8.22}$$

We let

$$\omega_0 = dy_{11}\wedge dy_{12}\wedge\cdots\wedge dy_{nn}\wedge dv_{11}\wedge\cdots\wedge dv_{hg}$$

be a differential form on $\mathcal{P}_g\times\mathbb{R}^{(h,g)}$ of degree $\tilde{N} := \frac{g(g+1)}{2} + gh$. If $\omega = h(Y,V)\omega_0$ is a differential form on $\mathcal{P}_g\times\mathbb{R}^{(h,g)}$ of degree $\tilde{N}$ that is invariant under the action of $\mathfrak{G}_{g,h}$. Then the function $h(Y,V)$ satisfies the transformation relation

$$h(\gamma Y\, {}^t\gamma,\ (V+\alpha)\,{}^t\gamma) = (\det\gamma)^{-(g+h+1)}\, h(Y,V) \tag{8.23}$$

for all $\gamma \in GL(g,\mathbb{Z})$ and $\alpha \in \mathbb{Z}^{(h,g)}$.

We write

$$\omega_1 = dy_{11}\wedge dy_{12}\wedge\cdots\wedge dy_{gg} \quad \text{and} \quad \omega_2 = dv_{11}\wedge\cdots\wedge dv_{hg}.$$

Now we define

$$\omega_{ab} = \epsilon_{ab} \bigwedge_{\substack{1\le\mu\le\nu\le g\\ (\mu,\nu)\ne(a,b)}} dy_{\mu\nu}\wedge\omega_2, \quad 1\le a\le b\le g$$

and

$$\tilde{\omega}_{cd} = \tilde{\epsilon}_{cd}\,\omega_1 \wedge \bigwedge_{\substack{1\le k\le h,\ 1\le l\le g\\ (k,l)\ne(c,d)}} dv_{kl}, \quad 1\le c\le h,\ 1\le d\le g.$$

Here the signs ϵ_{ab} and $\tilde{\epsilon}_{cd}$ are determined by the relations $\epsilon_{ab}\,\omega_{ab}\wedge dy_{ab}=\omega_0$ and $\tilde{\epsilon}_{cd}\,\omega_{cd}\wedge dv_{cd}=\omega_0$. We let

$$\beta_*=\sum_{\mu\le\nu}s_{\mu\nu}(Y,V)\,\omega_{\mu\nu}+\sum_{k=1}^{h}\sum_{l=1}^{g}\varphi_{kl}(Y,V)\,\tilde{\omega}_{kl}$$

be a differential form on $\mathcal{P}_g\times\mathbb{R}^{(h,g)}$ of degree $\widetilde{N}-1$ that is invariant under the action of $\mathfrak{G}_{g,h}$, where $s_{\mu\nu}(Y,V)$ and φ_{kl} are smooth functions on $\mathcal{P}_g\times\mathbb{R}^{(h,g)}$. We set

$$s=(\epsilon_{\mu\nu}s_{\mu\nu}),\ \ \epsilon_{\mu\nu}=\epsilon_{\nu\mu},\ \ s_{\mu\nu}=s_{\nu\mu}\quad\text{and}\quad \varphi=(\tilde{\epsilon}_{kl}\varphi_{kl}).$$

If we write

$$\Omega(Y,V)=\begin{pmatrix}s(Y,V)\\ \varphi(Y,V)\end{pmatrix},$$

then we obtain

$$\beta_*\wedge\begin{pmatrix}dY\\ dV\end{pmatrix}=\Omega\,\omega_0.$$

If $\tilde{\gamma}=(\gamma,\alpha)\in\mathfrak{G}_{g,h}$, then we have the following transformation relations :

$$s(\gamma Y\,{}^t\gamma,\ (V+\alpha)\,{}^t\gamma)=(\det\gamma)^{-(g+h+1)}\,\gamma\, s(Y,V)\,{}^t\gamma \tag{8.24}$$

and

$$\varphi(\gamma Y\,{}^t\gamma,\ (V+\alpha)\,{}^t\gamma)=(\det\gamma)^{-(g+h+1)}\,\varphi(Y,V)\,{}^t\gamma. \tag{8.25}$$

$\mathfrak{G}_{g,h}$ acts on $\mathcal{P}_g\times\mathbb{R}^{(h,g)}$ properly discontinuously. The quotient space

$$\mathfrak{G}_{g,h}\backslash\big(\mathcal{P}_g\times\mathbb{R}^{(h,g)}\big) \tag{8.26}$$

may be regarded as a family of principally polarized real tori of dimension gh. To each equivalence class $[Y]\in\mathfrak{G}_g\backslash\mathcal{P}_g$ with $Y\in\mathcal{P}_g$ we associate a principally polarized real torus $T_Y^{[h]}=T_Y\times\cdots\times T_Y$ with $T=\mathbb{R}^g/\Lambda_Y$, where $\Lambda_Y=Y\mathbb{Z}^g$ is a lattice in $\mathbb{R}^g$.

Let Y_1 and Y_2 be two elements in $\mathcal{P}_g$ with $[Y_1]\neq[Y_2]$, that is, $Y_2\neq A\,Y_1\,{}^tA$ for all $A\in\mathfrak{G}_g$. We put $\Lambda_i=Y_i\mathbb{Z}^g$ for $i=1,2$. Then a torus $T_1=\mathbb{R}^g/\Lambda_1$ is diffeomorphic to $T_2=\mathbb{R}^g/\Lambda_2$ as smooth manifolds but T_1 is not isomorphic to T_2 as polarized tori.

Lemma 8.4. *The following set*

$$\mathfrak{R}_{g,h}:=\Big\{(Y,V)\,|\,Y\in\mathfrak{R}_g,\ |v_{kj}|\le 1,\ V=(v_{kj})\in\mathbb{R}^{(h,g)}\Big\} \tag{8.27}$$

is a fundamental set for $\mathfrak{G}_{g,h}\backslash\mathcal{P}_g\times\mathbb{R}^{(h,g)}$.

Proof. It is easy to see that $\mathfrak{R}_{g,h}$ is a fundamental set for $\mathfrak{G}_{g,h}\backslash\mathcal{P}_g\times\mathbb{R}^{(h,g)}$. We leave the detail to the reader. □

For two positive integers g and h, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(g,h)}=\{\,(\lambda,\mu;\kappa)\,|\ \lambda,\mu\in\mathbb{R}^{(h,g)},\ \kappa\in\mathbb{R}^{(h,h)},\ \kappa+\mu\,{}^t\lambda\ \text{symmetric}\,\}$$

endowed with the following multiplication law

$$(\lambda,\mu;\kappa)\circ(\lambda',\mu';\kappa')=(\lambda+\lambda',\mu+\mu';\kappa+\kappa'+\lambda\,{}^t\mu'-\mu\,{}^t\lambda').$$

We define the semidirect product of $Sp(g,\mathbb{R})$ and $H_{\mathbb{R}}^{(g,h)}$

$$G^J=Sp(g,\mathbb{R})\ltimes H_{\mathbb{R}}^{(g,h)}$$

endowed with the following multiplication law

$$(M,(\lambda,\mu;\kappa))\cdot(M',(\lambda',\mu';\kappa'))=(MM',(\tilde{\lambda}+\lambda',\tilde{\mu}+\mu';\kappa+\kappa'+\tilde{\lambda}\,{}^t\mu'-\tilde{\mu}\,{}^t\lambda'))$$

with $M,M'\in Sp(g,\mathbb{R}),(\lambda,\mu;\kappa),\ (\lambda',\mu';\kappa')\in H_{\mathbb{R}}^{(g,h)}$ and $(\tilde{\lambda},\tilde{\mu})=(\lambda,\mu)M'$. Then G^J acts on the Siegel-Jacobi space $\mathbb{H}_g\times\mathbb{C}^{(h,g)}$ transitively by

$$(M,(\lambda,\mu;\kappa))\cdot(\Omega,Z)=(M\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}), \tag{8.28}$$

where $M=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in Sp(g,\mathbb{R}),\ (\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(g,h)}$ and $(\Omega,Z)\in\mathbb{H}_g\times\mathbb{C}^{(h,g)}$. We note that the Jacobi group G^J is *not* a reductive Lie group and also that the space $\mathbb{H}_g\times\mathbb{C}^{(h,g)}$ is not a symmetric space. We refer to [28, 29, 30, 31, 32, 33] for more detail on the Siegel-Jacobi space $\mathbb{H}_g\times\mathbb{C}^{(h,g)}$.

We let

$$\Gamma_{g,h}:=\Gamma_g\ltimes H_{\mathbb{Z}}^{(g,h)}$$

be the discrete subgroup of G^J, where

$$H_{\mathbb{Z}}^{(g,h)}=\{\,(\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(g,h)}\mid\lambda,\mu\in\mathbb{Z}^{(h,g)},\quad\kappa\in\mathbb{Z}^{(h,h)}\,\}.$$

We define the map $\Phi_{g,h}:\mathcal{P}_g\times\mathbb{R}^{(h,g)}\longrightarrow\mathbb{H}_g\times\mathbb{C}^{(h,g)}$ by

$$\Phi_{g,h}(Y,\zeta):=(i\,Y,\zeta),\qquad (Y,\zeta)\in\mathcal{P}_g\times\mathbb{R}^{(h,g)}. \tag{8.29}$$

We have the following inclusions

$$\mathcal{P}_g\times\mathbb{R}^{(h,g)}\xrightarrow{\Phi_{g,h}}\mathscr{H}_g\times\mathbb{C}^{(h,g)}\hookrightarrow\mathbb{H}_g\times\mathbb{C}^{(h,g)}\hookrightarrow\mathbb{H}_g^*\times\mathbb{C}^{(h,g)}.$$

$\mathfrak{G}_{g,h}$ acts on $\mathcal{P}_g\times\mathbb{R}^{(h,g)}$, $\Gamma_g^\star\ltimes H_{\mathbb{Z}}^{(g,h)}$ acts on $\mathscr{H}_g\times\mathbb{C}^{(h,g)}$ and $\Gamma_{g,h}$ acts on $\mathbb{H}_g\times\mathbb{C}^{(h,g)}$ and $\mathbb{H}_g^*\times\mathbb{C}^{(h,g)}$. It might be interesting to characterize the boundary points of the closure of the image of $\Phi_{g,h}$ in $\mathbb{H}_g^*\times\mathbb{C}^{(h,g)}$.

9. Real Semi-Abelian Varieties

In this section we review the work of Silhol on semi-abelian varieties [26] which is needed in the next section.

Definition 9.1. *A complex* semi-abelian variety *A is the extension of an abelian variety $\widetilde{A}$ by a group of multiplicative type. A semi-abelian variety is said to be* real *if it admits an anti-holomorphic involution which is a group homomorphism.*

Let T be a group of multiplicative type. We consider the exponential map $\exp:\mathfrak{t}\longrightarrow T$. The real structure S on T lifts to a real structure $S_{\mathfrak{t}}$ on $\mathfrak{t}$. Then $L_{\mathfrak{t}}:=\ker\exp$ is a free $\mathbb{Z}$-module and $S_{\mathfrak{t}}$ induces an involution on $L_{\mathfrak{t}}$. By standard results (cf. [25, I. (3.5.1)]), we can find a basis of L_t with respect to which the matrix for S_t is of the form

$$\begin{pmatrix} I_s & 0 & 0 & \cdots & 0 & 0 \\ 0 & B & 0 & \cdots & 0 & 0 \\ 0 & 0 & B & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & B & 0 \\ 0 & 0 & 0 & \cdots & 0 & -I_t \end{pmatrix}, \qquad B := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since fixing a basis of L_t is equivalent to fixing an isomorphism $T \cong (\mathbb{C}^*)^r$, we get

$$T = T_1 \times T_2 \times T_3, \qquad r = s' + 2p + t',$$

where

(i) $T_1 = \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ (s'-times) and S induces on each factor the involution $z \longmapsto \overline{z}$. In this case we write $T_1 = \mathbb{G}_m^0 \times \cdots \times \mathbb{G}_m^0$;

(ii) $T_2 = \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ (t'-times) and S induces on each factor the involution $z \longmapsto \overline{z}^{-1}$. In this case we write $T_1 = \mathbb{G}_m^\infty \times \cdots \times \mathbb{G}_m^\infty$;

(iii) $T_3 = (\mathbb{C}^* \times \mathbb{C}^*) \times \cdots \times (\mathbb{C}^* \times \mathbb{C}^*)$ (p-times) and S induces on each factor $(\mathbb{C}^* \times \mathbb{C}^*)$ the involution $(z_1, z_2) \longmapsto (\overline{z}_2, \overline{z}_1)$. In this case we write $T_3 = \mathbb{G}_m^{2*} \times \cdots \times \mathbb{G}_m^{2*}$.

Let $\Delta = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$ be the unit disk and let $\Delta^* = \{\zeta \in \mathbb{C} \mid 0 < |\zeta| < 1\}$ be a punctured unit disk. Let $\varphi : Z^* \longrightarrow \Delta^*$ be a holomorphic family of matrices $\varphi^{-1}(\zeta) = Z(\zeta)$ in $\mathbb{H}_g$. We have the natural action of the lattice $\mathbb{Z}^{2g}$ on $\Delta^* \times \mathbb{C}^g$ defined by

$$(9.1) \qquad (\lambda, \mu) \cdot (\zeta, z) := (\zeta, z + \lambda + Z(\zeta)\,\mu), \qquad \zeta \in \Delta^*,\ \lambda, \mu \in \mathbb{Z}^g,\ z \in \mathbb{C}^g.$$

Then the quotient space

$$(9.2) \qquad \mathbf{A}^* := \big(\Delta^* \times \mathbb{C}^g\big)/\mathbb{Z}^{2g}$$

is a holomorphic family of principally polarized abelian varieties associated to a holomorphic family $\varphi : Z^* \longrightarrow \Delta^*$.

Now we write

$$Z(\zeta) = X(\zeta) + i\,Y(\zeta) \in \mathbb{H}_g$$

and

$$Y(\zeta) = {}^tW(\zeta)D(\zeta)W(\zeta) \in \mathcal{P}_g \quad (\textit{the Jacobi decomposition})$$

with $\mathrm{diag}(d_1(\zeta), \cdots, d_g(\zeta)) \in \mathbb{R}^{(g,g)}$ is a diagonal matrix.

Now we assume the following conditions (F1)–(F3) : for any $\zeta \in \Delta_r^* := \{\, \zeta \in \mathbb{C} \mid 0 < |\zeta| < r \,\}$,

(F1) There exists a positive number $r > 0$ such that for any $\zeta \in \Delta_r^*$, $Z(\zeta) \in \mathfrak{W}_g(u)$ for some $u > 0$, where $\Delta_r^* := \{\, \zeta \in \mathbb{C} \mid 0 < |\zeta| < r \,\}$;

(F2) $X(\zeta)$ converges in $\mathbb{R}^{(g,g)}$ and $W(\zeta)$ converges in $GL(g, \mathbb{R})$ as $\zeta \to 0$;

(F3) $\lim_{\zeta \to 0} d_i(\zeta) = d_i$ converges for $1 \le i \le g - t$, and $\lim_{\zeta \to 0} d_i(\zeta) = \infty$ for $g - t < i \le g$.

Let

$$Z(0)=\begin{pmatrix} z_{11} & \cdots & z_{1,g-t} & 0 & \cdots & 0\\ \vdots & \ddots & \vdots & 0 & \ddots & \vdots\\ z_{g-t,1} & \cdots & z_{g-t,g-t} & 0 & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots\\ z_{g,1} & \cdots & z_{g,g-t} & 0 & \cdots & 0 \end{pmatrix},\qquad z_{ij}=\lim_{\zeta\to 0} z_{ij}(\zeta).$$

The action (9.1) extends to the action of $\mathbb{Z}^{2g}$ on $\Delta\times\mathbb{C}^g$ by letting $Z(0)$ be the fibre at $\zeta=0$. We take the quotient space

$$\mathbf{A} := \big(\Delta\times\mathbb{C}^g\big)/\mathbb{Z}^{2g}. \tag{9.3}$$

Then we see that $\mathbf{A}$ is an analytic variety fibred holomorphically over Δ, and the fibre at 0 is a semi-abelian variety

$$\mathbf{A}_0=\mathbb{C}^g/L_0,\qquad L_0:=\mathbb{Z}^g Z(0)+\mathbb{Z}^g\subset\mathbb{C}^g \tag{9.4}$$

of the abelian variety

$$\widetilde{\mathbf{A}}_0 := \mathbb{C}^{g-t}/L^\diamond,\qquad L^\diamond := \mathbb{Z}^{g-t}Z^\diamond(0)+\mathbb{Z}^{g-t}\subset\mathbb{C}^{g-t} \tag{9.5}$$

by $(\mathbb{C}^*)^t$, where

$$Z^\diamond(0)=\begin{pmatrix} z_{11} & \cdots & z_{1,g-t}\\ \vdots & \ddots & \vdots\\ z_{g-t,1} & \cdots & z_{g-t,g-t}\end{pmatrix}\in\mathbb{H}_{g-t}.$$

The extension

$$1\longrightarrow(\mathbb{C}^*)^t\longrightarrow\mathbf{A}_0\longrightarrow\widetilde{\mathbf{A}}_0\longrightarrow 0$$

is defined by the image of

$$z^\diamond_{g-k}=\big(z_{g-k,1},\cdots,z_{g-k,g-t}\big)\in\mathbb{C}^{g-t},\qquad k=0,\cdots,t-1$$

under the maps

$$\mathbb{C}^{g-t}\longrightarrow\widetilde{\mathbf{A}}_0\longrightarrow \mathrm{Pic}^0(\widetilde{\mathbf{A}}_0),$$

where the last map is the isomorphism defined by the polarization.

These above facts can be generalized as follows.

Proposition 9.1. *Let $\varphi: Z^*\longrightarrow\Delta^*$ be a holomorphic family of matrices $\varphi^{-1}(\zeta)=Z(\zeta)$ in $\mathbb{H}_g$ such that $\varphi^{-1}(\zeta)=Z(\zeta)$ converges in $\mathbb{H}_g^*$ as $\zeta\to 0$. Then there exists an analytic variety $\boldsymbol{A}(Z^*)\longrightarrow\Delta$ such that*

(i) the fibre at $\zeta(\neq 0)\in\Delta$ is the principally polarized abelian variety $\mathbb{C}^g/L_\zeta$ with the lattice $L_\zeta=\mathbb{Z}^gZ(\zeta)+\mathbb{Z}^g$;

(ii) the zero fibre $\boldsymbol{A}(Z^*)_0$ is a semi-abelian variety.

Proof. The proof can be found in [26, p. 189]. □

Theorem 9.1. *Let $\varphi: Z^*\longrightarrow\Delta^*$ be a holomorphic family of matrices $\varphi^{-1}(\zeta)=Z(\zeta)$ in $\mathbb{H}_g$ such that $\varphi^{-1}(\zeta)=Z(\zeta)$ converges in $\mathbb{H}_g^*$ as $\zeta\to 0$. We assume that $Z(\zeta)=\varphi^{-1}(\zeta)\in\mathscr{H}_g$ for $\zeta\in\mathbb{R}\cap\Delta^*$. Let (s,t) be such that*

$$\lim_{\zeta\to 0} Z(\zeta)\in\gamma\, B_M\big(\mathscr{F}_{s,t}\cap\overline{\mathscr{H}_0}\big)$$

for some $M\in\mathbb{Z}^{(g,g)}$ and some $\gamma\in\Gamma_g^*$. Then *(a)* $\boldsymbol{A}(Z^*)_0$ *has a natural real structure extending the real structures of the* $\boldsymbol{A}(Z^*)_\zeta{}'s$ *for* $\zeta\in\mathbb{R}\cap\Delta^*$ *;*

(b) As a real variety, $\boldsymbol{A}(Z^*)_0$ *is the extension of a real abelian variety* $\widetilde{\boldsymbol{A}}(Z^*)_0$ *by*

$$\big(\mathbb{G}_m^0\big)^{s'}\times\big(\mathbb{G}_m^{2*}\big)^p\times\big(\mathbb{G}_m^\infty\big)^{t'},\qquad s=s'+p,\ t=t'+p\,;$$

(c) Let $x\in\mathscr{X}_{\mathbb{R}}^g(s,t)\subset\overline{\mathscr{X}_{\mathbb{R}}^g}$ *be the image of* $\lim_{\zeta\to 0}Z(\zeta)$ *in* $\overline{\mathscr{X}_{\mathbb{R}}^g}$ *and let* $[x]$ *be the image of* x *under the isomorphism* $\mathscr{X}_{\mathbb{R}}^g(s,t)\cong\mathscr{X}_{\mathbb{R}}^{g-r}$ *with* $r=s+t$*. Then* $[x]$ *is the real isomorphism class of* $\widetilde{\boldsymbol{A}}(Z^*)_0$*.*

Proof. The proof can be found in [26, pp. 191–192]. □

Corollary 9.1. *Let* $\varphi:Z^*\longrightarrow\Delta^*$ *be as in Theorem 9.1. Assume*

$$\lim_{\zeta\longrightarrow 0}Z(\zeta)\in\mathscr{F}_{0,t}\ \ (resp.\ \mathscr{F}_{s,0}).$$

Then the class of the extension

$$0\longrightarrow\big(\mathbb{G}_m^\infty\big)^t\to\boldsymbol{A}(Z^*)_0\longrightarrow\widetilde{\boldsymbol{A}}(Z^*)_0\longrightarrow 0$$

$$\big(\,resp.\ \ 0\longrightarrow\big(\mathbb{G}_m^0\big)^s\longrightarrow\boldsymbol{A}(Z^*)_0\longrightarrow\widetilde{\boldsymbol{A}}(Z^*)_0\longrightarrow 0\,\big)$$

is defined by t *purely imaginary divisors on* $\boldsymbol{A}(Z^*)_0$ $\big($*resp.* s *real divisors* $\boldsymbol{A}(Z^*)_0\big)$*.*

10. Real Semi-Tori

A real semi-torus T of dimension g is defined to be an extension of a real torus $\widetilde{T}$ of dimension $g-t$ by a real group $(\mathbb{R}^*)^t$ of multiplicative type, where $\mathbb{R}^*=\mathbb{R}-\{0\}$.

Let $I=\{\,\xi\in\mathbb{R}\,|\,-1<\xi<1\,\}$ be the unit interval and $I^*=I-\{0\}$ be the punctured unit interval. Let $\varpi:\mathfrak{Y}^*\longrightarrow I^*$ be a real analytic family of matrices $\varpi^{-1}(\xi)=Y(\xi)\in\mathcal{P}_g$. We have the natural action of the lattice $\mathbb{Z}^g$ in $\mathbb{R}^g$ on $I^*\times\mathbb{R}^g$ defined by

$$\alpha\cdot(\xi,x)=(\xi,x+Y(\xi)\alpha),\quad \alpha\in\mathbb{Z}^g,\ \xi\in I^*,\ x\in\mathbb{R}^g. \tag{10.1}$$

The quotient space

$$\mathbf{T}^*:=(I^*\times\mathbb{R}^g)/\mathbb{Z}^g \tag{10.2}$$

is a real analytic family of real tori of dimension g associated to a real analytic family $\varpi:\mathfrak{Y}^*\longrightarrow I^*$. We let

$$Y(\xi)={}^tW(\xi)D(\xi)W(\xi)$$

be the Jacobi decomposition of $Y(\xi)$, where $D(\xi)=\mathrm{diag}(d_1(\xi),\cdots,d_g(\xi))$ is a real diagonal matrix and $W(\xi)$ is a strictly upper triangular real matrix of degree g. Now we assume the following conditions (T1)-(T4):

(T1) There exists a positive number r with $0<r,1$ such that for any $\xi\in I_r^*$, $i\,Y(\xi)\in\mathfrak{W}_g(u)$ for some $u>0$, where $I_r^*:=\{\,\xi\in\mathbb{R}\,|\,-r<\xi<r\,\}$;

(T2) $W(\xi)$ converges in $GL(g,\mathbb{R})$ as $\xi\to 0$;

(T3) $\lim_{\xi\to 0}d_i(\xi)=d_i$ converges for $1\le i\le g-t$, and $\lim_{\xi\to 0}d_i(\xi)=\infty$ for $g-t<i\le g$.

Let

$$Y(0)=\begin{pmatrix} y_{11} & \cdots & y_{1,g-t} & 0 & \cdots & 0\\ \vdots & \ddots & \vdots & 0 & \ddots & \vdots\\ y_{g-t,1} & \cdots & y_{g-t,g-t} & 0 & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots\\ y_{g,1} & \cdots & y_{g,g-t} & 0 & \cdots & 0 \end{pmatrix}, \qquad y_{ij}=\lim_{\xi\to 0} y_{ij}(\zeta).$$

The action (10.1) extends to the action of $\mathbb{Z}^g$ on $I\times\mathbb{R}^g$ by letting $Y(0)$ be the fibre at $\xi=0$. We take the quotient space

$$\mathbf{T} := \big(I\times\mathbb{R}^g\big)/\mathbb{Z}^g. \tag{10.3}$$

Then we see that $\mathbf{T}$ is a real analytic variety fibred real analytically over I, and the fibre at 0 is a real semi-torus

$$\mathbf{T}_0=\mathbb{R}^g/\Lambda_0,\qquad \Lambda_0:=\mathbb{Z}^gY(0)\subset\mathbb{R}^g \tag{10.4}$$

of the real torus

$$\widetilde{\mathbf{T}}_0 := \mathbb{R}^{g-t}/\Lambda^\diamond,\qquad \Lambda^\diamond:=\mathbb{Z}^{g-t}Y^\diamond(0) \text{ is a lattice in } \mathbb{R}^{g-t}$$

by $(\mathbb{C}^*)^t$, where

$$Y^\diamond(0)=\begin{pmatrix} y_{11} & \cdots & y_{1,g-t}\\ \vdots & \ddots & \vdots\\ y_{g-t,1} & \cdots & y_{g-t,g-t}\end{pmatrix}\in\mathcal{P}_{g-t}.$$

11. Open Problems and Remarks

In this final section we give some open problems related to polarized real tori to be studied in the future.

Problem 1. Characterize the boundary points of the closure of $i\mathcal{P}_g$ in $\mathbb{H}_g^*$ explicitly.

Problem 2. Find the explicit generators of the ring $\mathbb{D}(g,h)$ of differential operators on the Minkowski-Euclidean space $\mathcal{P}_g\times\mathbb{R}^{(h,g)}$ which are invariant under the action (8.7) of $GL_{g,h}=GL(g,\mathbb{R})\ltimes\mathbb{R}^{(h,g)}$.

Problem 3. Find all the relations among a complete explicit list of generators of $\mathbb{D}(g,h)$.

The orthogonal group $O(g)$ of degree g acts on the subspace

$$\mathfrak{p}=\left\{\,(X,Z)\,|\,X={}^tX\in\mathbb{R}^{(g,g)},\ Z\in\mathbb{R}^{(h,g)}\right\}$$

of the vector space $\mathbb{R}^{(g,g)}\times\mathbb{R}^{(h,g)}$ by

$$k\cdot(X,Z)=(kX\,{}^tk, Z\,{}^tk),\quad k\in O(g),\ (X,V)\in\mathfrak{p}. \tag{11.1}$$

The action (11.1) induces the action of $O(g)$ on the polynomial ring $\mathrm{Pol}(\mathfrak{p})$ on $\mathfrak{p}$. We denote by $I(\mathfrak{p})$ the subring of $\mathrm{Pol}(\mathfrak{p})$ consisting of polynomials on $\mathfrak{p}$ invariant under the action of $O(g)$. We see that there is a canonical linear bijection

$$\Theta: I(\mathfrak{p})\longrightarrow\mathbb{D}(g,h)$$

of $I(\mathfrak{p})$ onto $\mathbb{D}(g,h)$. We refer to [8] and [27] for more detail.

Remark 11.1. *M. Itoh* [11] *proved that* $I(\mathfrak{p})$ *is generated by* α_j $(1 \le j \le g)$ *and* $\beta_{pq}^{(k)}$ $(0 \le k \le g-1,\ 1 \le p \le q \le h)$, *where*

$$\alpha_j(X,Z) = tr\big(X^j\big), \quad 1 \le j \le g \tag{11.2}$$

and

$$\beta_{pq}^{(k)}(X,Z) = \big(Z\,X^k\,{}^tZ\,\big)_{pq}, \quad 0 \le k \le g-1,\ 1 \le p \le q \le h. \tag{11.3}$$

Here A_{pq} *denotes the* (p,q)*-entry of a matrix* A *of degree* h.

Remark 11.2. *M. Itoh* [11] *found all the relations among the above generators* α_j $(1 \le j \le g)$ *and* $\beta_{pq}^{(k)}$ $(0 \le k \le g-1,\ 1 \le p \le q \le h)$ *of* $I(\mathfrak{p})$.

Problem 4. Develop the theory of harmonic analysis on the Minkowski-Euclidean space $\mathcal{P}_g \times \mathbb{R}^{(h,g)}$ with respect to a discrete subgroup of $GL(g,\mathbb{Z}) \ltimes \mathbb{Z}^{(h,g)}$.

Problem 5. Characterize the boundary points of the closure of the image of $\Phi_{g,h}$ in $\mathbb{H}_g^* \times \mathbb{C}^{(h,g)}$ (cf. see (8.29)).

Problem 6. Find the explicit generators of the ring $\mathbb{D}\big(\mathbb{H}_g \times \mathbb{C}^{(h,g)}\big)$ of differential operators on the Siegel-Jacobi space $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ which are invariant under the action (8.28) of the Jacobi group $G^J = Sp(g,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(g,h)}$. We refer to [28] for more detail.

Problem 7. Find all the relations among a complete list of generators of $\mathbb{D}\big(\mathbb{H}_g \times \mathbb{C}^{(h,g)}\big)$.

Problem 8. Develop the theory of harmonic analysis on the Siegel-Jacobi space $\mathbb{H}_g \times \mathbb{C}^{(h,g)}$ with respect to a congruent subgroup of $\Gamma_{g,h} = Sp(g,\mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(g,h)}$. We refer to [29] for more detail.

Appendix : Non-Abelian Cohomology

In this section we review some results on the first cohomology set $H^1(<\tau>,\Gamma)$ obtained by Goresky and Tai [9], where $<\tau>= \{1,\tau\}$ is a group of order 2 and γ is a certain arithmetic subgroup. These results are often used in this article.

First of all we recall the basic definitions. Let S be a group. A group M is called a S-*group* if there exists an action of G on M, $S \times M \longrightarrow M$, $(\sigma,a) \longmapsto \sigma(a)$ such that $\sigma(ab) = \sigma(a)\sigma(b)$ for all $\sigma \in S$ and $a,b \in M$. From now on we let 1_S (resp. 1_M) be the identity element of S (resp. M). We observe that if M is a S-group, then $\sigma(1_M) = 1_M$ for all $\sigma \in S$.

Definition. *Let M be a S-group, where S is a group. We define*

$$H^0(S,M) := \{\, a \in M \mid \sigma(a) = a \quad \text{for all } \sigma \in S \,\}.$$

A map $f : S \longrightarrow M$ is called a 1-cocycle with values in M if $f(\sigma\tau) = f(\sigma)\,\sigma(f(\tau))$ for all $\sigma, \tau \in S$. We observe that if f is a 1-cocycle, then $f(1_S) = 1_M$. We denote by $Z^1(S,M)$ the set of all 1-cocycles of S with values in M. Let f_1 and f_2 be two 1-cocycles in $Z^1(S,M)$. We say that f_1 is cohomologous to f_2, denoted $f_1 \sim f_2$, if there exists an element $h \in M$ such that

$$f_2(\sigma) = h^{-1} f_1(\sigma)\,\sigma(h) \qquad \text{for all } \sigma \in S.$$

Let $f_\flat : S \longrightarrow M$ be the trivial map, i.e., $f_\flat(\sigma) = 1_M$ for all $\sigma \in S$. A map $f : S \longrightarrow M$ is called a 1-coboundary if $f \sim f_\flat$, i.e., if there exists $h \in M$ such that $f(\sigma) = h^{-1}\sigma(h)$ for all $\sigma \in S$.

Obviously a 1-coboundary is a 1-cocycle. It is easy to see that $\sim$ is an equivalence relation on $Z^1(S,M)$. So we define the first cohomology set

$$H^1(S,M) := Z^1(S,M)/\sim.$$

Remark. In general, $H^1(S,M)$ does not admit a group structure. But $H^1(S,M)$ has an identity, that is, the cohomologous class containing the trivial 1-cocycle $f_\flat$.

Example. Let L be a Galois extension of a number field K with Galois group G. A linear algebraic group defined over K has naturally the structure of G-group. It is known that $H^1(G, GL(n,L))$ is trivial for all $n \geq 1$. Using the following exact sequence of G-groups

$$1 \longrightarrow SL(n,L) \longrightarrow GL(n,L) \longrightarrow L^* \longrightarrow 1, \qquad L^* = L - \{0\},$$

we can show that $H^1(G, SL(n,L))$ is trivial.

We put $G = Sp(g,\mathbb{R})$ and $K = U(g)$. Then $\mathbf{D} = G/K$ is biholomorphic to $\mathbb{H}_g$. Let $S_\tau = \{1, \tau\}$ be a group of order 2 as before. We define the S_τ-group structure on G via the action (2.7) of S_τ on G. Let Γ be an arithmetic subgroup of $Sp(g,\mathbb{Q})$. We let

$$X_\Gamma := \Gamma\backslash G/K \cong \Gamma\backslash\mathbb{H}_g$$

and let $\pi_\Gamma : \mathbf{D} \longrightarrow X_\Gamma$ be the natural projection. For any $\gamma \in \Gamma$, we define the map $f : S_\tau \longrightarrow \Gamma$ by

$$f_\gamma(1) = 1_\Gamma \qquad \text{and} \qquad f_\gamma(\tau) = \gamma, \tag{1}$$

where 1_Γ denotes the identity element of Γ.

Lemma 1. *Let $\gamma \in \Gamma$. Then*

(a) f_γ is a 1-cocycle if and only if $\gamma\,\tau(\gamma) = 1_\Gamma$, equivalently, $\tau(\gamma)\,\gamma = 1_\Gamma$.

(b) A cocycle f_γ is a 1-coboundary if and only if there exists $h \in \Gamma$ such that $\gamma = \tau(h)h^{-1}$.

Proof. The proof follows immediately form the definition. □

To each such a 1-cocycle f_γ we associate the γ-twisted involution $\tau\gamma : \mathbf{D} \longrightarrow \mathbf{D}$ and $\tau\gamma : \Gamma \longrightarrow \Gamma$. Indeed the involution $\tau\gamma : \mathbf{D} \longrightarrow \mathbf{D}$ is defined by

$$\tau\gamma(xK) = \tau(\gamma x K) = \tau(\gamma)\tau(x)K, \qquad x \in G \tag{2}$$

and the involution $\tau\gamma : \Gamma \longrightarrow \Gamma$ is defined by

$$\tau\gamma(\gamma_1) = \tau(\gamma\gamma_1\gamma^{-1}), \qquad \gamma_1 \in \Gamma. \tag{3}$$

Let

$$\mathbf{D}^{\tau\gamma} := \{ x \in \mathbf{D} \,|\, (\tau\gamma)(x) = x \,\}$$

and

$$\Gamma^{\tau\gamma} := \{ \gamma_1 \in \Gamma \,|\, (\tau\gamma)(\gamma_1) = \gamma_1 \,\}$$

be the fixed point sets.

Lemma 2. *Let $x \in \mathbf{D}$. Then $\pi_\Gamma(x) \in X_\Gamma^\tau$ if and only if there exists an element $\gamma \in \Gamma$ such that $x \in \mathbf{D}^{\tau\gamma}$.*

Proof. It is easy to prove this lemma. We leave the proof to the reader. □

Theorem A. *Assume Γ is torsion free. Let $\mathscr{C}_\Gamma$ be the set of all connected components of the fixed point set X_Γ^τ. Then the map $\Phi_\Gamma : H^1(S_\tau, \Gamma) \longrightarrow \mathscr{C}_\Gamma$ defined by*

$$\Phi_\Gamma([f_\gamma]) := \pi_\Gamma\big(\mathbf{D}^{\tau\gamma}\big) = \Gamma^{\tau\gamma}\backslash\mathbf{D}^{\tau\gamma}$$

determines a one-to-one correspondence between $H^1(S_\tau, \Gamma)$ and $\mathscr{C}_\Gamma$.

Proof. The proof can be found in [9, pp. 3-4]. □

Theorem B. *Let $S_\tau = \{1, \tau\}$ be a group of order 2. Then $Sp(g, \mathbb{R})$ has a S_τ-group structure via the action (2.7) and hence $U(g)$ also admits a S_τ-group structure through the restriction of the action (2.7) to $U(g)$. And $H^1(S_\tau, U(g))$ and $H^1(S_\tau, Sp(g, \mathbb{R}))$ are trivial.*

Proof. The proof can be found in [9, pp. 8-9]. However I will give a sketchy proof for the reader. Assume f_k is a 1-cocycle in $Z^1(S_\tau, U(g))$ with $k = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in U(g)$. Using the fact $\tau(k)k = I_g$, we see that

$$A = {}^tA, \quad B = {}^tB, \quad AB = BA \quad \text{and} \quad A^2 + B^2 = I_g.$$

Therefore we can find $h \in O(g)$ such that $h(A + iB)h^{-1} = \mathfrak{D} \in U(g)$ is diagonal. We take $\mu = \sqrt{\mathfrak{D}} \in U(g)$ by choosing a square root of each diagonal entry. We set $\delta = h^{-1}\mu h$. Then $k = \tau(\delta)\,\delta^{-1}$. By Lemma 1, f_k is a 1-coboundary. Hence $H^1(S_\tau, U(g))$ is trivial.

Let $G = Sp(g, \mathbb{R})$ as before. Suppose $f_M \in Z^1(S_\tau, G)$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$. Then we see that f_M is a 1-coboundary with values in G if and only if $\mathbb{H}_g^{M\tau} \neq \varnothing$. We can find $M_1 \in G$ such that $\mathbb{H}_g^{M_1\tau} \neq \varnothing$ and $f_M \sim f_{M_1} \sim f_\flat$. Therefore $f_M \sim f_\flat$, that is, f_M is a 1-coboundary with values in G. Hence $H^1(S_\tau, G)$ is trivial. □

Theorem C. *For all $m \geq 1$, the mapping $H^1(S_\tau, \Gamma_g(4m)) \longrightarrow H^1(S_\tau, \Gamma_g(2, 2m))$ is trivial.*

Proof. The proof can be found in [9, pp. 7-10]. We will give a sketchy proof for the reader. In order to prove this theorem, we need the following lemma.

Lemma 3. *If $\tau(\gamma)\gamma \in \Gamma_g(4m)$ with $\gamma \in \Gamma_g$, then $\gamma = \beta u$ for some $\beta \in \Gamma_g(2, 2m)$ and for some $u \in GL(g, \mathbb{Z})$.*

Lemma 4. *Let $\gamma \in \Gamma_g(2)$ and suppose $\Omega \in \mathbb{H}_g$ is not fixed by any element of Γ_g other than $\pm I_g$. Suppose $\tau(\Omega) = \gamma \cdot \Omega$. Then there exists an element $h \in \Gamma_g$ such that $\gamma = \tau(h)\, h^{-1}$.*

Lemma 4 is a consequence of the theorem of Silhol [26, Theorem 1.4] and Comessatti. Suppose f_γ is a cocycle in $Z^1(S_\tau, \Gamma_g(4m))$ with $\gamma \in \Gamma_g(4m)$. According to Theorem B, its image in $H^1(S_\tau, G))$ is a coboundary and so there exists $h \in G$ with $\gamma = \tau(h)\, h^{-1}$. Thus $\mathbb{H}^{\tau\gamma} = h \cdot (i\,\mathcal{P}_g)$. By Lemma 2.2, there exists $\Omega \in \mathbb{H}^{\tau\gamma}$ which are not fixed by any element of Γ_g other than $\pm I_{2g}$ and the set of such points is the complement of a countable union of proper real algebraic subvarieties of $\mathbb{H}^{\tau\gamma}$. According to Lemma 4, $\gamma = \tau(h)\, h^{-1}$ for some $h \in \Gamma_g$. By Lemma 3, we may write $h = \beta\, u$ for some $\beta \in \Gamma_g(2, 2m)$ and for some $u \in GL(g, \mathbb{Z})$. Then $\gamma = \tau(h)\, h^{-1} = \tau(\beta)\, \beta^{-1}$. By Lemma 1, the cohomology class $[f_\gamma]$ is trivial in $H^1(S_\tau, \Gamma_g(2, 2m))$. □

Theorem D. *Let $\Gamma_0 = \Gamma_g(2, 2m)$ and $\Gamma = \Gamma_g(4m)$. Then we have the following results:*

(a) $\mathbf{D}^\tau = G^\tau / K^\tau$;

(b) For each cohomology class $[f_\gamma] \in H^1(S_\tau, \Gamma)$, there exists $h \in \Gamma_0$ such that $\gamma = \tau(h)h^{-1}$, in which case

$$\mathbf{D}^{\tau\gamma} = h\, \mathbf{D}^\tau \qquad \text{and} \qquad \Gamma^{\tau\gamma} = h\, \Gamma^\tau h^{-1}.$$

(c) The association $f_\gamma \longrightarrow h$ (cf. see (2)) determines a one-to-one correspondence between $H^1(S_\tau, \Gamma)$ and $\Gamma \backslash \Gamma_0 / \Gamma_0^\tau$.

(d)

$$X_\Gamma^\tau := \coprod_{h \in \Gamma \backslash \Gamma_0 / \Gamma_0^\tau} h \Gamma^\tau h^{-1} \backslash h \mathbf{D}^\tau.$$

Proof. (a) follows from the fact that $H^1(S_\tau, U(g))$ is trivial (cf. Theorem B). (b) follows from Theorem C. (c) follows from Theorem C, and the facts that Γ is a normal subgroup of Γ_0 and that τ acts on $\Gamma \backslash \Gamma_0$ trivially. (d) follows from Theorem C. (c) follows from Theorem C, and the facts that Γ is a normal subgroup of Γ_0 and that τ acts on $\Gamma \backslash \Gamma_0$ trivially together with the fact that Γ s torsion free. □

Corollary. *Let m be a positive integer with $m \geq 1$. Let S_τ be as in Theorem A. Let $\Gamma = \Gamma_g(4m)$ and $X = \Gamma \backslash \mathbb{H}_g$. The set $X_\mathbb{R}$ of real points of X is given by*

$$X_\mathbb{R} = \coprod_h \Gamma_{[h]} \backslash h \cdot (i\,\mathcal{P}_g) = \Gamma \backslash \mathbb{H}^{\tau\Gamma},$$

where h is indexed by elements

$$h \in \Gamma_g(4m) \backslash \Gamma_g(2, 2m) / \Gamma_g^{[L]}(2) = H^1(S_\tau, \Gamma_g(4m))$$

and $\Gamma_{[h]} := h\, \Gamma_g^{[L]}(4m)\, h^{-1}$.

Proof. The proof follows from (c) and (d) in Theorem D. □

References

[1] A. A. Albert, *Symmetric and alternate matrices in arbitrary field, I.,* Trans. Amer. Math. Soc. **43** (1938), 386–436.

[2] A. Ash, D. Mumford, M. Rapoport and Y. Tai, *Smooth compactification of locally symmetric varieties*, Math. Sci. Press (53 Jordan Rd., Brookline, MA. 02146, USA), 1975.

[3] W. Baily, *Satake's compactification of* V_n^*, Amer. J. Math. **80** (1958), 348–364.

[4] W. Baily and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. Math. **84** (1966), 442–528.

[5] C. Birkenhake and H. Lange, *Complex Tori*, Progress in Math. vol. 177, Birkhäuser, Boston (1999).

[6] H. Comessati, *Sulle varietà abeliane reali* I, II, Ann. Mat. Pura. Appl. **2** (1924), 67-106 and **4** (1926), 27–72.

[7] G. Faltings and C.-L. Chai, *Degeneration of abelian varieties*, Ergebnisse der Math. **22**, Springer-Verlag, Berlin-Heidelberg-New York (1990).

[8] S. Helgason, *Groups and geometric analysis,* Academic Press, New York (1984).

[9] M. Goresky and Y. S. Tai, *The Moduli Space of Real Abelian Varieties with Level Structure*, Compositio Math. **139** (2003), 1–27.

[10] J. Igusa, *Theta Functions,* Springer-Verlag, Berlin-Heidelberg-New York (1971).

[11] M. Itoh, *On the Yang Problem (SFT),* preprint, Max-Planck Institut für Mathematik, Bonn (2011).

[12] A. W. Knapp, *Representation Theory of Semisimple Groups,* Princeton University Press, Princeton, New Jersey (1986).

[13] H. Lange and C. Birkenhake, *Complex Abelian Varieties*, Grundlehren der mathematischen Wissenschaften, Springer-Verlag (1992).

[14] H. Maass, *Siegel modular forms and Dirichlet series,* Lecture Notes in Math. **216**, Springer-Verlag, Berlin-Heidelberg-New York (1971).

[15] Y. Matsushima, *On the intermediate cohomology group of a holomorphic line bundle over a complex torus,* Osaka J. Math. **16** (1979), 617–631.

[16] H. Minkowski, *Gesammelte Abhandlungen,* Chelsea, New York (1967).

[17] D. Mumford, *Abelian Varieties*, Oxford University Press (1970): Reprinted (1985).

[18] I. Satake, *On the compactification of the Siegel space*, J. Indian Math. Soc. **20** (1956), 259–281.

[19] I. Satake, *Algebraic Structures of Symmetric Domains,* Kano Memorial Lectures 4, Iwanami Shoton, Publishers and Princeton University Press (1980).

[20] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series,* J. Indian Math. Soc. B. **20** (1956), 47-87.

[21] M. Seppälä and R. Silhol, *Moduli Spaces for Real Algebraic Curves and Real Abelian varieties*, Math. Z. **201** (1989), 151–165.

[22] G. Shimura. *On the Fourier coefficients of modular forms of several variables*, Nachr. Akad. Wiss. Göttingen Math-Phys. **17** (1975), 261-268.

[23] C. L. Siegel, *Symplectic Geometry,* Amer. J. Math. **65** (1943), 1-86; Academic Press, New York and London (1964); Gesammelte Abhandlungen, no. 41, vol. II, Springer-Verlag (1966), 274-359.

[24] R. Silhol, *Real Abelian varieties and the theory of Comessatti*, Math. Z. **181** (1982), 345–362.

[25] R. Silhol, *Real Algebraic Surfaces,* Lecture Notes in Math. **1392**, Springer-Verlag, Berlin-Heidelberg-New York (1989).

[26] R. Silhol, *Compactifications of moduli spaces in real algebraic geometry*, Invent. Math. **107** (1992), 151–202.

[27] J.-H. Yang, *Invariant differential operators on the Minkowski-Euclidean space*, arXiv:math/0611389v5 [math.DG], 2011.

[28] J.-H. Yang, *Invariant Differential Operators on the Siegel-Jacobi Space*, arXiv:math/0611388v4 [math.NT], 2011.

[29] J.-H. Yang, *Remark on harmonic analysis on the Siegel-Jacobi space*, arXiv:1107.0509v1 [math.NT], 2009.

[30] J.-H. Yang, *A note on a fundamental domain for Siegel-Jacobi space,* Houston J. Math. **32 (3)** (2006), 701–712 *or* arXiv:math.NT/0507218.

[31] J.-H. Yang, *Invariant metrics and Laplacians on Siegel-Jacobi space,* Journal of Number Theory **127** (2007), 83–102 *or* arXiv:math.NT/0507215.

[32] J.-H. Yang, *A partial Cayley transform for Siegel-Jacobi disk,* J. Korean Math. Soc. **45**, No. 3 (2008), 781-794 *or* arXiv:math.NT/0507216.
[33] J.-H. Yang, *Invariant metrics and Laplacians on Siegel-Jacobi disk,* Chinese Annals of Mathematics, Vol. 31B(1), 2010, 85-100 *or* arXiv:math.NT/0507217.

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[K] Papers related to Algebraic Geometry and Other Topics

POSTECH (포항공대)에서

COLLECTION OF PAPERS
DEDICATED TO PROFESSOR OH
ON HIS SIXTIETH BIRTHDAY

RECENT PROGRESS ON THE SCHOTTKY PROBLEM

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1. Introduction

Let H_g be the Siegel upper half plane of degree g. Let $A_g = H_g/Sp(2g,Z)$ be the *Siegel's* modular variety of degree g, i.e., the moduli space of principally polarized abelian varieties of dimension g. Let M_g be the moduli space of compact Riemann surfaces of genus g. Then we have the Jacobi mapping

$$w : M_g \to A_g$$

given by

$$M \mapsto J(M) := \textit{the Jacobian of } M$$

is injective by *Toreilli's* theorem and the Jacobian locus $w(M_g) := J_g$ forms a subvariety of dimension $3g-3$ in A_g. The Schottky problem is to find equations or characterizations of the Jacobian locus J_g or its closure $\bar{J}_g$ *in* A_g.

At first this problem had been investigated from the analytic point of view: to find explicit equations of J_g(or $\bar{J}_g$) *in* A_g defined by Siegel modular forms on H_g, for example, polynomials in the theta constants $\theta[m|n](0,\Omega)$ and their derivatives. The first result in this direction is due to F. Schottky [S] who gave a simple equation satisfied by the theta constants of Jacobians of genus 4. The fact that this equation characterizes the Jacobian locus J_4 was recently proved by Igusa [**I1**](see also Freitag [**F**]). Recently there has been much progress on the characterization of Jacobians by differential equations. In [**A-D1**], Arbarello and De Concini gave a set of such equations defining $\bar{J}_g$. Moreover the Novikov conjecture which states that a theta function satisfying the Kadomtsev-Petviasvili(briefly, **K-P**) differential equation is the theta function of a Jacobian was recently proved by

T. Shiota [**Sh**](Harvard thesis). Later the proof of the above Novikov conjecture was simplified by Arbarello and De Concini [**A-D2**]. Bert van Geeman showed that $\bar{J}_g$ is an irreducible component of the subvariety $\overline{S}_g$ in $\overline{A}_g$ defined by certain equations([**vG**]). Here $\overline{A}_g$ is the Satake compactification of A_g.

In this short article we review recent results on the Schottky problem and also consider the geometrical approach of this problem.

2. Reducibility of $\Theta \cap \Theta_{\mathbf{a}}$ and trisecany

All the varieties we consider in this article is defined over the complex number field **C**.

Let A be a g-dimensional abelian variety equipped with an irreducible principal polarization Θ, and we let Ω be a point of H_g corresponding to A. For a subvariety Z of A and an element a of A we denote the subvariety $Z+a$ by Z_a. The elementary *Riemann's theta series* associated to Ω is given by

$$(1)\quad \theta[\nu|\tau](z,\Omega)=\sum_{n}\exp 2\pi i\left\{\frac{1}{2}{}^t(n+\nu)\Omega(n+\nu)+{}^t(n+\nu)(z+\tau)\right\},$$

where $z\in\mathbf{C}^g$, ν, $\tau\in\mathbf{C}^g$(parameters called the characteristics) and n runs over Z^g. Then $\theta[0|0](z,\Omega)$ can be considered as a holomorphic section of the line bundle $\mathcal{O}(\Theta)$. The second-order theta functions are defined by

$$(2)\qquad \theta_2[\nu|\tau](z,\Omega)=\theta[\nu|\tau](2z,2\Omega).$$

Then the functions $\theta_2[\nu|\tau](z,\Omega)$ form a basis for the vector space of holomorphic sections of $\mathcal{O}(2\Theta)$. Following *Dubrovin's* notation, we write

$$(3)\qquad \hat{\theta}[\nu](z)=\hat{\theta}[\nu](z,\Omega)=\theta_2[\nu|0](z,\Omega).$$

And we have, for any $\nu\in(Z/2Z)^g$ and any $\lambda\in\Lambda=(I,\Omega)Z^{2g}$,

$$(4)\qquad \hat{\theta}[\nu](z+\lambda)=\eta(\lambda,z)^2\hat{\theta}[\nu](z),$$

where $\eta(\lambda,z)$ is the nowhere-vanishing entire function of z.

We set

$$\vec{\theta}(z) = \vec{\theta}(z, \Omega) = \left\{\hat{\theta}[\nu](z) : \nu \in (Z/2Z)^g\right\} \in \mathbf{C}^{N+1}, \tag{5}$$

where $N + 1 = 2^g$. *Riemann's* quadratic relation is given by

$$\theta(z+\alpha)\theta(z-\alpha) = {}^t\vec{\theta}(\alpha) \cdot \vec{\theta}(z). \tag{6}$$

From Riemann's quadratic relation it is easily seen that the vector $\vec{\theta}(z)$ is never the zero vector. Therefore, according to (4), we have a well-defined holomorphic mapping

$$\psi : A \to P^N, \qquad N = 2^g - 1, \tag{7}$$

given by

$$z \mapsto \vec{\theta}(z)$$

associated to the linear system $|2\Theta|$. The image $\psi(A)$ is called the Kummer variety of A or the Wirtinger variety of A.

A. Weil observed that for the Jacobian (JC, Θ) of a curve C the intersection $\Theta \cap \Theta_a$ is reducible when a is of the form $p - q$, with p, $q \in C$. More precisely, for any four distinct p, q, r, $s \in C$, we have $\Theta \cup \Theta_{p-q} \subset \Theta_{p-r} \cap \Theta_{s-q}$. This leads to consider, for a principally polarized abelian variety (A, Θ), a certain number of conditions which are satisfied when (A, Θ) is a Jacobian:

- (i) There exists a nonzero element a in A such that $\Theta \cap \Theta_a$ is reducible.
- (ii) There exists three distinct nonzero elements a, x, y in A such that $\Theta \cap \Theta_a \subset \Theta_x \cap \Theta_y$.
- (iii) The Kummer variety of (A, Θ) admits a trisecant line.
- (iv) The theta function associated to (A, Θ) satisfies the Kadomtsev-Petviashvili equation, simply the K-P equation(see section 3 for detail).

A. Beauville and O. Debarre **[B-D]** showed that each of the above mentioned conditions (ii), (iii) and (iv) implies the Andreotti-Mayer condition $\dim \operatorname{Sing} \Theta \geq g-4$. Moreover they showed that a principally polarized abelian variety (A, Θ) satifying the condition (i) satisfies the Andreotti-Mayer condition or contains an elliptic curve E with

$(\Theta \cdot E) = 2$. And then we can deduce immediately from this result and the Andreotti-Mayer theorem that $\bar{J}_g$ is an irreducible component of the subvariety of A_g consisting of (A, Θ) satisfying one of the above-mentioned conditions (i), (ii), (iii) and (iv).

Concerning the trisecancy of the Kummer variety, Gunning [G] obtained the following very beautiful characterization of Jacobians:

Theorem(Gunning[**G**]). *An irreducible principally polarized abelian variety A is a Jacobian if and only if there exists an irreducible curve $C \subset A$, satisfying the property that, for general $\alpha, \beta, \gamma \in C$ and for any point $\zeta \in A$ such that $2\zeta + \alpha + \beta + \gamma \in C$, the three points $\psi(\zeta + \alpha)$, $\psi(\zeta + \beta)$, $\psi(\zeta + \gamma)$ are collinear in P^N. Here $N = 2^g - 1$, and ψ denotes the morphism given by (7).*

Remark. Welters generalized Gunning's criterion by including an infinitesimal version of it([**W2**]). This version is obtained by letting the three points, α, β, γ come together.

3. Novikov's conjecture

Let C be a compact Riemann surface of genus g and let (I, Ω) be its canonical Riemann matrix, where $\Omega \in H_g$. Then $\Lambda = (I, \Omega)Z^{2g}$ is a lattice subgroup in R^{2g} and hence the compact complex Lie group $J(C) = \mathbf{C}^g/\Lambda$ is a complex torus of dimension g, called the Jacobi variety of C or the Jacobian of C. We recall that the elementary Riemann's theta function $\theta(z)$ is given by

$$\theta(z) = \theta[0|0](z, \Omega) = \sum_n \exp 2\pi i \left\{ \frac{1}{2} {}^t n \Omega n + {}^t n z \right\},$$

where $z \in \mathbf{C}^g$ and n runs over Z^g.

In his study of solutions of nonlinear equations of Korteveg de Vrie type, Krichever proved the following fact:

Theorem(Krichever[**K**]). *Let C be a compact Riemann surface of genus g and let $\theta(z)$ its elementary Riemann's theta function. Then there exist three vectors W_1, W_2, W_3 in $\mathbf{C}^g$ with $W_1 \neq 0$ such that, for every $z \in \mathbf{C}^g$, the function*

$$(8) \qquad u(x, y, t; z) = \frac{\partial^2}{\partial x^2} \log \theta(xW_1 + yW_2 + tW_3 + z)$$

satisfies the so-called Kadomtsev-Petviashvili equation (simply the K-P equation)

$$3u_{yy} = \frac{\partial}{\partial x}(u_t - 3uu_x - 2u_{xxx}). \tag{9}$$

S. P. Novikov[N] conjectured that a matrix $\Omega \in H_g$ is a Riemann matrix of a curve if and only if the corresponding theta function satisfies the K-P equation, in the sense we just explained. Following work of Mumford[**M1**] and Mulase[**M**], Novikov's conjecture has been recently proved by T. Shiota[**Sh**]. Approaching more geometrcally to the same problem, Arbarello and De Concini quite recently gave another proof of Novikov's conjecture[**A-D2**].

Using the trisecant formula, a beautiful result descoverd by J. D. Fay [**FY**], Mumford noticed [**M3**] that, when α, β, γ tend to 0 on a curve C, one then get the equation

$$\begin{aligned} D_1^4\theta \cdot \theta - 4D_1^3\theta \cdot \theta + 3(D_1^2\theta)^2 - 3(D_2\theta)^2 \\ + 3D_2^2\theta \cdot \theta + 3D_1\theta \cdot D_3\theta - 3D_1D_3\theta \cdot \theta + d_4\theta \cdot \theta = 0. \end{aligned} \tag{10}$$

Consequently we can then describe Novikov's conjecture(now Shiota's theorem) as follows:

Theorem([Sh] and [A-D2]). *Let A be an irreducible principally polarized abelian variety of dimension $g \geq 1$. Then A is the Jacobian of a compact Riemann surface of genus g if and only if there exist a constant d_4 and constant vector fields $D_1 \neq 0$, D_2, D_3 on A such that the equation (10) is satisfied.*

4. $\bar{J}_g$ is an irreducible component of $\overline{S}_g$

Let H_g be the Siegel upper half plane of degree g, and let $\Gamma_g = Sp(2g, Z)$ be the Siegel modular group of degree g. Then Γ_g acts on H_g transitively as follows:

$$M\langle\Omega\rangle = (A\Omega + B)(C\Omega + D)^{-1},$$

where

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \quad \text{and} \quad \Omega \in H_g.$$

We set congruence subgroups of Γ_g:

$$\Gamma_g(n) := \{M \in \Gamma_g : M \equiv I_g \pmod{n}\},$$
$$\Gamma_g(n, 2n) := \{M \in \Gamma_g(n) : \mathrm{diag}(A^tB) \equiv \mathrm{diag}(C^tD) \equiv 0 \pmod{2n}\}.$$

We denote by $\overline{A}_{g,n}$ resp. $\overline{A}_{g,(n,2n)}$ the Satake compactification of $A_{g,n} = H_g/\Gamma_{g(n)}$ resp. $A_{g,(n,2n)} = H_g/\Gamma_g(n,2n)$. The boundary $\overline{A}_{g,n} - A_{g,n}$ of $\overline{A}_{g,n}$ is a disjoint union of finite number of copies of $A_{k,n}$ with $0 \leq k \leq g-1$, each of these is called a boundary component of $\overline{A}_{g,n}$.

We define the theta functions with half integral characteristics:

$$\theta \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (z, \Omega)$$
$$= \sum_m \exp 2\pi i \left\{ \frac{1}{2}{}^t\left(m + \frac{\epsilon}{2}\right) \Omega \left(m + \frac{\epsilon}{2}\right) + {}^t\left(m + \frac{\epsilon}{2}\right)\left(z + \frac{\delta}{2}\right) \right\},$$

where m runs over Z^g, $z \in \mathbf{C}^g$, $\Omega \in H_g$ and ϵ, $\delta \in (Z/2Z)^g$. Note that

$$\theta \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (-z, \Omega) = (-1)^{{}^t\epsilon\delta} \theta \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (z, \Omega).$$

Therefore there are $2^{g-1}(2^g + 1)$ even and $2^{g-1}(2^g - 1)$ odd theta functions.

Let C be a curve of genus $g \geq 1$ and $\Omega \in H_g$ the Riemann matrix of the Jacobian $J(C)$. Then there exists an etale covering of degree 2 of C such that the Prym variety has a Riemann matrix $\tau \in H_{g-1}$ which satisfies the following equations for all ϵ, $\delta \in (Z/2Z)^{g-1}$:

$$\lambda \cdot \theta^2 \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (0, \tau) = \theta \begin{bmatrix} \epsilon & 0 \\ \delta & 0 \end{bmatrix} (0, \Omega) \cdot \theta \begin{bmatrix} \epsilon & 0 \\ \delta & 1 \end{bmatrix} (0, \Omega),$$

where $\lambda \in C$, $\lambda \neq 0$, is a constant independent of ϵ, $\delta \in (Z/2Z)^{g-1}$. The theta constants $\theta^2 \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (0, \tau)$, $\tau \in H_{g-1}$ are Siegel modular forms on $\Gamma_{g-1}(2,4)$ and they define a map

$$\theta_2 : A_{g-1,(2,4)} = H_{g-1}/\Gamma_{g-1}(2,4) \to P^l, \qquad l + 1 = 2^{g-2}(2^{g-1} + 1),$$

given by

$$\theta_2(\tau) = \left\{ \theta^2 \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (0, \tau) : {}^t\epsilon\delta = 0 \right\}, \qquad \tau \in H_{g-1}.$$

The maps θ_2 extends to a map of the Satake compactification $\overline{A}_{g-1,(2,4)}$ which we also denote by θ_2.

Let $I_{g-1} \subset C[X_0, \ldots, X_N]$ be the ideal defining the projective vartiety $\theta_2(\overline{A}_{g-1,(2,4)})$. For homogeneous $F \in I_{g-1}$ we define $\sigma(F)$: $H_g \mapsto C$ by

$$\sigma(F)(\Omega) = F\left(\ldots, \theta \begin{bmatrix} \epsilon & 0 \\ \delta & 0 \end{bmatrix} (0, \Omega) \cdot \theta \begin{bmatrix} \epsilon & 0 \\ \delta & 1 \end{bmatrix} (0, \Omega), \ldots \right),$$

where $\Omega \in H_g$. We observe that $\sigma(F) = 0$ if Ω is the Riemann matrix of a curve and that $\sigma(F)$ are Siegel modular forms on $\Gamma_g(4, 8)$.

Let $A(\Gamma_g(4, 8))$ be the graded ring generated by the Siegel modular forms on $\Gamma_g(4, 8)$. We define $S(\Gamma_g(4, 8)) \subset A(\Gamma_g(4, 8))$ to be the ideal generated by the $\sigma(F)$, $F \in I_{g-1}$ and F homogeneous, and the conjugates of $\sigma(F)$ under the action of $\Gamma_g/\Gamma_g(4, 8)$ on $A(\Gamma_g(4, 8))$. Let $\overline{S}_{g,(4,8)}$ be the subset of $\overline{A}_{g,(4,8)}$ defined by $S(\Gamma_g(4, 8))$. The canonical map $\overline{A}_{g,(4,8)} \to \overline{A}_g$ maps $\overline{S}_{g,(4,8)}$ onto a set $\overline{S}_g$. Obviously $\bar{J}_g \subset \overline{S}_g$. B. van Geeman showed the following fact:

Theorem(van Geeman, [**vG**]). *$\bar{J}_g$ is an irreducible component of $\overline{S}_g$.*

Inspired by van Geeman's work, R. Donagi[**Do**] defined the big Schottky locus defined by Schottky's relations and then showed that J_g is a component of the big Schottky locus.

5. Conclusion

The Prym varieties $\mathcal{P}_g$ *in* A_g forms an irreducible analytic subvariety of dimenson $3g$ for $g \geq 5$ containing J_g. As in the Schottky problem for Jacobians, it is natural to try to characterize the Prym varieties. Fortunately the geometrical approach introduced in section 2 and 3 extends to the Prym varieties. For example, we may obtain the method of quadrisecant planes. Recently Debarre[**D**] showed the following result analogous to the Andreotti-Mayer theorem: For $g \geq 7$,

$\mathcal{P}_g$ is an irreducible component of N_{g-6}, where N_{g-6} is the set of Prym varieties (P, Θ) such that $\dim \operatorname{Sing} \Theta \geq g-6$. Beauville, Debarre and Welters observed that the theta function of Prym variety associated to a singular stable curve satisfies a certain nonlinear partial differential equation, called the BKP equation. Novikov conjectured that this equation characterizes the Prym varieties of singular stable curves. Recently T. Shiota announced that he proved this conjecture.

References

[A-D1] E. Arbarello and C. De Concini, *On a set of equations characterizing Riemann matrices*, Ann. of Math. **120** (1984), 119–140.

[A-D2] ——————, *Another proof of a conjecture of S. P. Novikov on periods of abelian integrals on Riemann surfaces*, Duke Math. Jour. **54** (1987), 163–178.

[B-D] A. Beauville and O. Debarre, *Une relation entre deux approches du Problem de Schottky*, Inv. Math. **86** (1986), 195–207.

[D] O. Debarre, *Variétes de Prym et ensembles d'Andreotti-Mayer*, (to appear).

[D0] R. Donagi, *Big Schottky*, Inv. Math. **89** (1987), 567–599.

[F] E. Freitag, *Die Irreduzibillität der Schottky-Relation (Bemerkung zu einem Satz von J. Igusa)*, Arch. der Math. **48** (1983), 255–259.

[FY] J. D. Fay, "Theta functions on Riemann surfaces," Lecture Notes in Mathematics vol. **352**, Springer-Verlag, 1973.

[G] R. Gunning, *Some curves in abelian varieties*, Inv. Math. **78** (1982), 377–389.

[vG] B. van Geeman, *Siegel modular forms vanishing on the moduli space of curves*, Inv. Math. **78** (1984), 329–349.

[I1] J. Igusa, *On the irreducibillity of Schottky divisor*, J. Fac. Sci. Tokyo **28** (1981), 531–545.

[I2] ————, "Theta functions," Springer-Verlag, 1972.

[K] I. M. Krichever, *Methods of algebraic geometry in the theory of nonlinear equations*, Russia Math. Survey **32** (1977), 185–213.

[M] M. Mulase, *Cohomological structure in soliton equations and Jacobian varieties*, J. Diff. Geom. **19** (1984), 403–430.

[M1] D. Mumford, *An algebro-geometric construction of commuting operators and of solution to the Toda lattice equation*, in "Korteweg de Vries equation and related nonlinear equations," Proc. Int. Sym. Alg. Geo., Kyoto, 1977, Kinokuniya Book Store, Tokyo, 1978, pp. 115–153.

[M2] ——————, "Tata Lecture on theta II," Birchäuser, 1983.

[M3] D. Mumford and Forgarty, "Geometric invariant theory," Springer-Verlag, 1982.

[S] F. Schottky, *Zur Theorie der Abelschen Funktionen von vier Variabeln*, J. Riene Angew. Math. **102** (1888), 304–352.

[Sh] T. Shiota, *Characterization of Jacobian varieties in term of soliton equations*, Inv. Math. **83** (1986), 333–382.

[W1] G. Welters, *A characterization of non-hyperelliptic Jacobi varieties*, Inv. Math. **71** (1983), 473–440.

[W2] ________, *A criterion for Jacobi varieties*, Ann. of Math. **120** (1984), 497–504.

1992 년 제 2 호

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보형형식과 제타함수

양 재현 (인하대)

보형형식이란 대략 말하면, 어떤 변수변환에 관해 거의 변하지 않는 함수이며 이중에서 변수변환에 관해 완전히 불변인 함수를 보형함수라고 한다.

이제, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ 를 각각 정수환, 유리수체, 실수체, 복소수체라 하고, F 가 이들 중의 하나일 때 $F^{(k,l)}$ 는 F 의 원을 원소로 갖는 $k\times l$ 행렬들의 집합을 나타낸다. $A\in F^{(k,l)}$ 일 때 ${}^tA$ 는 A 의 전치행렬을 나타낸다.

보형형식의 예로서 지겔 보형형식이 있다. g 가 자연수일 때

$$H_g := \{ Z \in \mathbb{C}^{(g,g)} \mid Z = {}^tZ,\ Im\,Z > 0 \}$$

을 차수가 g 인 지겔 상반평면이라 하고

$$\Gamma_g := \{ \gamma \in \mathbb{Z}^{(2g,2g)} \mid {}^t\gamma\, J \gamma = J \}$$

를 지겔 모듈러군이라 하자. 여기서

$$J := \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}, \quad I_g := g\times g \text{ 단위행렬이다.}$$

$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$ 는 H_g 의 원 Z 에 $\gamma(Z) := (AZ+B)(CZ+D)^{-1}$과 같이 작용한다. $A_g := \Gamma_g \backslash H_g$ 는 H_g 안의 기본영역이며 g 차원의 복소아벨다양체 (abelian variety) 들의 모듈라이 공간이 된다. 종수가 g 인 긴밀 리만곡면들의 모듈라이 공간 M_g 는 Torelli의 정리에 의해 A_g 안에서 야코비 다양체

(Jacobi variety) 라고 불리우는 특수한 아벨다양체의 모듈라이 공간과 동일시되어 A_g의 해석적 부분공간이 된다. g = 1, 2, 3 인 경우는 $M_g = A_g$ 이다. 모듈라이 공간 M_g 를 A_g 안에서 구체적으로 표현하는 문제 (소위, Riemann-Schottky 문제) 는 아주 흥미로운 문제로 아직까지 만족스러울 정도로 해결되어 있지 않은 실정이다. 그리고, A_g 는 긴밀하지 않을 뿐 만 아니라 아주 특이 (singular) 하므로 A_g 상의 함수를 연구 한다는 것은 아주 까다롭다. 그래서, A_g 의 긴밀화(compactification) 의 문제가 Siegel 에 의해 제시되어 Satake, Baily, Borel 과 Mumford 학파에 의해 많이 연구되어 왔다. 이 연구의 결과는 대수기하학에서 특히, 특이공간 (singularity) 이론에 지대한 영향을 끼쳤다.

H_g 상의 함수 f (Z) 가 무게 k 의 시겔 보형형식이라는 것은

$$f((AZ+B)(CZ+D)^{-1}) = \det(CZ+D)^k f(Z), \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$$

와 같은 변환식을 성립시키는 함수를 뜻한다. $g=1$ 인 경우, 이 조건외에 cusp 조건도 성립해야 한다. 무게 0 인 보형형식 (즉, A_g 상의 함수) 을 보형함수라고 한다. 지겔 보형형식의 연구는 1930년 초반에 독일의 수학자 지겔 (Carl Ludwig Siegel) 에 의해 시작되어 지금까지 활발하게 진행되고 있다.

$g=1$ 인 경우에 좀 더 구체적으로 설명하겠다. $q := e^{2\pi i z}$, $(z \in H_1)$ 이라 두면,

$$\Delta(z) := \sum_{i=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24}$$

는 수렴하며, $ad-bc=1$ 의 조건을 만족하는 정수 a, b, c, d 에 관해

$$\Delta((az+b)(cz+d)^{-1}) = (cz+d)^{12} \Delta(z), \quad z \in H_1$$

의 변환식을 성립시키는 무게 12 인 보형형식이 된다. 여기서, $\tau(n)$ 은 1916년경에 Ramanujan 에 의해 연구되었던 Ramanujan의 tau 함수로 $\tau(1)=1$, $\tau(2)=-24$, $\tau(3)=252$, $\tau(4)=-1472$, $\tau(5)=4830$, $\tau(6)=-6048$, ... 는 정수이다. tau 함수는 분할함수 $p(n)$ 과 아주 밀접하게 관련되어 있다. Ramanujan 은 임의의 소수 p 에 대해 $|\tau(p)| < 2p^{\frac{11}{2}}$ 의 관계가 성립한다는 것을 예상하였다. 이 예상은 1960년대 후반에 벨기에의 수학자 P. Deligne 에 의해 증명되었으며 나아가 그는 무게 k 인 보형형식에 대해서도 이와 유사한 가설을 증명하였다. 이것은 대수기하학에서 Weil 예상의 증명으로부터 얻어진다.

보형형식을 구성하는 방법에는 아이젠쉬타인 급수 (Eisenstein series) 와 세타급수 (theta series) 의 두 가지 방법이 있다. 아이젠쉬타인 급수의 간단한 기본적인 예는 아래의 두 종류가 있다.

$$E_4(z) := 1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n = \frac{1}{2\zeta(4)}\sum_{(c,d)\neq(0,0)}\frac{1}{(cz+d)^4},$$

$$E_6(z) := 1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n = \frac{1}{2\zeta(6)}\sum_{(c,d)\neq(0,0)}\frac{1}{(cz+d)^6}$$

여기서, $\sigma_k(n) := \sum_{0<d|n} d^k$ 는 약수함수 (the divisor function) 이며 $\zeta(s)$는 나중에 언급하겠지만 리만 제타함수를 나타내고 $(c,d)\in Z^2$이다. $E_4(z)$ 와 $E_6(z)$는 각각 무게 4 , 6 의 보형형식이 되며, $\Delta(z)$ 와의 관계는

$$1728\ \Delta(z) = E_4(z)^3 - E_6(z)^2$$

와 같이 주어진다. q 의 멱수 k 가 작은 경우는 양변의 q^k 의 계수가 일치한다는 것을 쉽게 보일수 있지만 양변의 함수가 서로 일치한다는 사실을 보이는 것은 쉽지 않다. 그런데, 양변의 함수가 모두 무게 12 인 보형형식이라는 사실은 쉽게 보일수 있다. 무게 12 인 보형형식들의 집합이 1 차원 벡터공간이라는 사실로 부터 위의 등식이 성립함을 증명할 수 있다.

세타급수의 기본적인 예로는 아래의 3 가지가 있다. 이것은 Jacobi 세타급수이다 (실은, $\theta_1(z)=0$).

$$\theta_2(z) := \sum_{n=-\infty}^{\infty} q^{\frac{(n-1/2)^2}{2}} = 2q^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-q^n)(1+q^n)^2,$$

$$\theta_3(z) := \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} = \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-1/2})^2,$$

$$\theta_4(z) := \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n^2}{2}} = \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-1/2})^2.$$

상기의 세타급수들은 $\Gamma_1 := SL(2, Z)$ 보다 작은 이산군 Γ 에 대해 무게 $\frac{1}{2}$ 인 보형형식들이며 Jacobi 등식 $\theta_2(z)^4 = \theta_3(z)^4 - \theta_4(z)^4$ 을 만족한다. 그리고, 아이젠쉬타인 급수와 밀접한 관계가 있다. 예를 들면,

$$E_4(z) = \frac{1}{2}\{\theta_2(z)^8 + \theta_3(z)^8 + \theta_4(z)^8\}.$$

S 가 $r\times r$ unimodular 이고 양 (positive) 인 대칭행렬이면 아래의 세타급수

$$\theta_g(Z, S) := \sum_{G\in Z^{(r,g)}} e^{i\pi\, \mathrm{trace}({}^tGSGZ)},\quad Z\in H_g$$

는 무게 $\frac{r}{2}$ 인 지겔 보형형식이다. 예를 들면,

$$\theta_g(-Z^{-1}, S) = \det(Z)^{\frac{r}{2}} \cdot \theta_g(Z, S), \quad Z \in H_g$$

의 변수 변환식이 성립한다. E_8 와 E_{16} 이 각각 8×8, 16×16 제외 행렬 (exceptional matrix) 이면

$$\theta_1(z, E_8)^2 = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n = \theta_1(z, E_{16}), \quad z \in H_1$$

의 관계식을 얻는다. 상기의 함수는 $E_4(z)^2$ 와 일치한다. $g \geq 2$ 인 경우에 대해서도 $\theta_g(Z, E_8)^2$ 과 $\theta_g(Z, E_{16})$ 는 $g=2, 3$ 인 경우에는 일치하지만 $g=4$ 인 경우에는 완전히 일치하지는 않지만 모듈라이 공간 M_4 상에서는 일치한다.

제타 (zeta) 함수에는 여러 종류 (cf. 참고문헌 [9]) 가 있는데 그 중에서도 가장기본이 되는 제타함수는

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}, \quad \sigma := \mathrm{Re}\ s > 1$$

으로 정의된 소위 리만 제타함수이다. $\zeta(s)$ 는 이미 18세기 중엽에 Euler 에 의해

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}, \quad p = 2, 3, 5, \ldots \text{ 소수}$$

의 Euler 곱이 존재함이 증명되었으며 연구되었다. Euler는 s 가 양의 짝수인 경우

$$\zeta(2m) = \frac{2^{2m-1} \pi^{2m} B_{2m}}{(2m)!}, \quad m = 1, 2, 3, \ldots$$

와 같이 아주 간단한 형태로 $\zeta(s)$ 의 값을 구하였지만 양의 홀수인 경우는 아직도 $\zeta(s)$ 의 값을 간단한 형태로 나타내지 못하고 있다. 음의 정수인 경우는

$$\zeta(0) = -\frac{1}{2}, \quad \zeta(1-n) = -\frac{B_n(0)}{n}, \quad n = 2, 3, 4, \ldots$$

임을 쉽게 보일수 있다. 여기서, $B_m(x)$ 는 Bernoulli 다항식이다. 그 후, Riemann 에 의해 $\zeta(s)$ 는 복소평면 C 상으로의 해석적 접속 (analytic continuation) 을 지니고 $s=1$ 에서만 단순 극점을 가지며 아래와 같은 함수방정식

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

이 성립함이 증명되었다. 여기서, $\Gamma(s)$ 는 Gamma 함수를 나타내고 있다. 그리

고, $\zeta(s)$ 는 $\sigma>1$ 인 영역에서는 영점 (zero) 을 갖지 않으며 $\sigma \leq 0$ 인 영역에서는 $s=-2, -4, \ldots, -2n, \ldots$ 에서만 단순 영점만을 갖는다. 그러나, $0<\sigma<1$ 인 영역에서는 무한이 많은 영점을 가짐이 알려져 있다. 이러한 영점을 $\zeta(s)$ 의 자명하지 않는 영점 (nontrivial zero) 라고 부른다. " $\zeta(s)$ 의 자명하지 않는 영점은 $\sigma=\dfrac{1}{2}$ 인 선상에 있다 " 라는 리만 예상 (1859) 은 130여년 후인 아직까지도 해결되지 않고 있는 실정이다.

$\zeta(s)$ 의 연구 조사를 통해 소수의 성질을 알 수 있다. 예를 들면, $\zeta(1) = \prod_p (1 - p^{-1})^{-1} = \infty$ 이어서 소수가 무한히 많이 존재함을 알 수 있다. 또, 양의 실수 x 이하의 모든 소수의 갯수를 $\pi(x)$ 이라 할때, 소위, 소수정리, 즉

$$\pi(x) \sim \frac{x}{\log x} \qquad (x \to \infty)$$

의 점근공식 (asymptotic formula) 이 얻어졌다. 이의 결과를 얻는데 $\zeta(s)$ 의 성질이 필요하다. 상기의 점근공식으로 부터 소수들은 아주 불규칙적으로 분포되어 있음을 알 수 있다.

소수의 성질을 조사하기 위해서는 리만 제타함수 $\zeta(s)$ 이외에도 여러 종류의 제타함수의 연구가 되어 있다. 예를 들면, 앞에서 언급한 보형형식 $\Delta(z)$ 의 제타함수 $L_\Delta(s)$ 는

$$L_\Delta(s) := \sum_{n=1}^{\infty}\tau(n)n^{-s} = \prod_p (1-\tau(p)p^{-s}+p^{11-2s})^{-1}$$

와 같은 Euler 곱을 갖는다. $L_\Delta(s)$ 는 $\zeta(s)$ 와 다른 함수이다.

또,

$$L_k(s) := \sum_{n=1}^{\infty}\sigma_k(n)n^{-s} = \prod_p (1-\sigma_k(p)p^{-s}+p^{k-2s})^{-1}$$

$$= \zeta(s)\zeta(s-k)$$

인 관계식이 성립되어 $k=3, 5$ 인 경우는 $L_k(s)$ 를 보형형식 $E_4(z)$, $E_6(z)$ 의 제타함수라고 간주할 수 있다. 이 사실로 부터 보형함수의 중요성이 제타함수의 연구에서 드러난다.

소수 이론의 많은 결과는 여러 종류의 제타함수의 성질로 부터 얻어진다. 예를 들면, 유체론 (class field theory) 의 연구는 "Artin 형의 제타함수는 Hecke-Langlands 형의 제타함수로 쓸 수 있다" 라는 주장이라 할 수 있다. 여

기서, Hecke–Langlands 형의 제타함수는 보형형식의 제타함수를 뜻하며 $\sum_{n=1}^{\infty}\tau(n)n^{-s}$ 는 이의 아주 기본적인 예이다. 리만 제타함수는 Artin 형의 제타함수 또는 Hecke–Langlands 형의 제타함수의 특수한 예이다. 소수 이론적인 제타함수에는 상기의 두 제타함수이외에도 고차원 대수다양체의 Hasse–Weil 형의 제타함수가 있다. 지금까지 소개한 제타함수들의 연구는 1967년 경에 Langlands 에 의해 소개된 Langlands Program 에 포함된다. 이 Program 을 쉽게 서술한 참고문헌으로는 [3] 과 [4] 를 추천한다.

30 여 년 전에 Selberg (cf. [5], [7]) 는 trace 공식에 관해 연구하는 중에 리만 예상의 유사성이 증명될 수 있는 제타함수를 발견 하였다. 이것은 앞에서 언급한 3 종류의 소수론적인 제타함수와 다르며 소수를 닫힌 측지선 (closed geodesic) 이라 간주하고 소수론적인 제타함수와 유사하게 Euler 곱의 형태로 정의하였다. Selberg 제타함수의 영점 또는 극점은 Laplace 작용소의 고유치를 이용하여 설명되며 Laplace 작용소의 고유치가 양수라는 사실로 부터 리만 예상의 유사성이 증명된다. 따라서, 닫힌 폐곡선의 분포를 잘 알 수 있다.

이것을 구체적으로 기술하여 보면, 종수 $g\geq 2$ 인 리만 곡면 C 에 대해 Selberg 제타함수를

$$Z_C(s) := \prod_{\gamma}\prod_{n=0}^{\infty}(1 - N(\gamma)^{-s-n}), \quad \gamma : \text{닫힌 측지선}$$

이라 정의한다. 여기서, $l(\gamma)$ 를 닫힌 폐곡선 γ 의 길이라고 하면 $N(\gamma) := e^{-l(p)}$ 이다. Selberg에 의해 $Z_C(s)$ 는 차수가 2인 해석적(entire) 함수이고 $s=-k$ $(k=1,2,\ldots)$ 에서 차수가 $(2g-2)(2k+1)$, $s=0$에서 차수가 $2g-1$, $s=1$ 에서 차수가 1 인 자명한 영점을 지니고, $Z_C(s)$ 의 자명하지 않는 영점은 모두 $\mathrm{Re}\ s = \frac{1}{2}$ 인 선상에 놓여 있다는 사실을 보였다. 게다가 $Z_C(s)$ 는

$$Z_C(s) = Z_C(1-s)\cdot \exp\left\{A(F)\cdot \int_0^{s-\frac{1}{2}} r\tan(\pi r)\,dr\right\}$$

와 같은 함수적 방정식을 지님을 보였다. 여기서, $A(F)$는 어떤 기본 영역 F의 면적을 나타내고 있다. Δ_C 를 Laplace 작용소라고 하면

$$\det(\Delta_C) = c^{g-1}\cdot Z_C'(1)$$

인 관계식이 성립한다. 여기서, $c=1.96\cdots$ 는 정수이며 $\det(\Delta_C)$ 는 소위 물리학자들이 일컫는 Δ_C 의 함수적 행렬식 (the functional determinant of Δ_C) 이며 제타 청칙화 (zeta regularization) 의 과정에 의해 정의된다 (cf. [2], [6], [8]). 가

령, 수열 $\{ \lambda_n \mid 0<\lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots ; \lambda_k \uparrow +\infty \}$가 어떤 작용소 L의 고유값들의 집합이라 하자. 그리고, $\{\lambda_n\}$ 이 아래의 성질 (a)와 (b)를 갖는다고 가정하자.

(a) 급수 $Z(s,L)=\sum_n \lambda_n^{-s}$ 는 Re s 가 넉넉하게 큰 값에 대해 수렴한다.

(b) 급수 $Z(s,L)=\sum_n \lambda_n^{-s}$ 는 해석적 접속을 지니며 $s=0$에서 해석적(*regular*)이다.

이 때, 작용소 L 의 함수적 행렬식 det(L) 를

$$\det(L) := \exp\{-Z'(0,L)\}$$

와 같이 정의한다. 그러면, Selberg 제타함수 $Z_C(s)$ 는

$$Z_C(s)=\left\{ e^{(s-\frac{1}{2})^2} \cdot \det\left((-\Delta_{S^2}+\frac{1}{4})^{\frac{1}{2}}-\frac{1}{2}+s\right)\right\}^{2g-2} \cdot \det((-\Delta_C+s^2-s))$$

와 같은 관계식을 갖는다는 것을 보일 수 있다 ([1], [6], [8]). 여기서, Δ_{S^2} 는 구면 S^2 의 Laplace 작용소 이다. 그리고,

$$\Gamma^*(s) = \exp\left\{\left((s-\frac{1}{2})^2-P_1(s)-2P_2(s)\right)\Gamma_1(s)\,\Gamma_2(s)\right\}^{2-2g}$$

이라 놓자. 그러면, 상기의 사실로 부터,

$$\Gamma^*(s)\,Z_C(s) = \det((-\Delta_C+s^2-s))$$

는 차수가 2인 entire 함수이며 변환 $s \mapsto 1-s$ 에 관해 불변임을 보일 수 있다. 여기서, P_1, P_2 는 어떤 다항식이고 Γ_k ($k=1,2,\cdots$) 는 중복 Gamma 함수이다. $g=1$ 인 경우는 타원 곡선 $C := \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\cdot\tau)$, $(\tau\in M_1=A_1)$ 에 관해서는 $\det(\Delta_C) = y^2 \mid \Delta(\tau) \mid^{\frac{1}{6}}$ 이 된다. 여기서, $\tau=x+iy$ (x, y는 실수) 이다.

제타함수의 특수값은 여러 측면에서 연구되어 왔지만 원래는 Leibniz 에 의해 얻어진

$$\sum_{n:\text{홀수}}^{\infty}(-1)^{\frac{n-1}{2}}\cdot n^{-1} = \frac{\pi}{4}$$

또는 Euler 에 의해 얻어진

$$\zeta(2) = \sum_{n=1}^{\infty} n^{-2} = \frac{\pi^2}{6}$$

에서 시작되었다. 또, $\sum_{n-1}^{\infty}(-1)^{n-1}\cdot n^{-1} = \log 2$ 인 관계식도 이와 유사한 예이

다. 이 사실은 현 이론 (string theory) 에서 적용된다고 한다. 현 이론을 연구하는 학자들은 " $\zeta(s)$ 가 $s=1$ 에서 발산하지만 스핀 (spin) 된 $\sum_{n=1}^{\infty}(-1)^{n-1}\cdot n^{-s}$ 이라든가 $\sum_{n:홀수}^{\infty}(-1)^{\frac{n-1}{2}}\cdot n^{-s}$ 가 $s=1$ 에서 유한값을 갖는다는 사실은 현 이론에서는 발산이 일어났지만 이것을 스핀한 초현 이론 (superstring theory) 에서는 유한이 된다" 는 표현으로서 설명하고 있다 (필자는 이 의미를 이해하지 못하고 있지만).

이미 언급하였듯이 Euler 에 의해

$$\prod_p (1 - p^{-2})^{-1} = \frac{\pi^2}{6}\ ,\ \ p=2,3,\ 5,\ \ldots\ 소수$$

와 같은 아름다운 관계식이 얻어 졌다. 좌변은 완전히 수론에 해당되는 문제인 반면에 우변은 물리학적인 상수이다. 이 사실로 부터 수론과 물리학 사이에 아주 밀접한 관계가 있을 것이라는 것을 유추할 수 있다. 1980년 중반에 많은 수학자와 물리학자들에 의해 현 이론이 전개되어 왔다. 가령, M_g 상의 뽈리야코프 측도 (Polyakov measure) 를 계산하기 위해 높이함수 (height function) 를 사용한다든가 ... . Y. Manin 은 우리를 둘러 싸고 있는 우주는 adelic (수론에서 쓰는 용어로서, C. Chevalley 에 의해 창안되었음) 하다고 예상하고 현 이론을 adelic 하게 연구할 수 있다고 예상하고 있다.

끝으로, 보형형식과 제타함수의 연구는 수론, 대수기하학, 군표현론, 다변수 복소해석학, 물리학 등의 여러 분야와 밀접하게 연관되어 있을 뿐 만 아니라 이 연구의 역사가 길다. 많은 국내의 젊고 유능한 수학도들이 이 심오한 분야에 관심을 가지고 연구하여 주길 바라며 이만 줄인다.

참 고 문 헌

1. P. Cartier and A. Voros, Une nouvelle interpretation de la formule des traces del Selberg, Grothendieck Festschrift, vol. 2, Birkhäuser (1991), 1-67.
2. D. Fried, Analytic torsion and closed geodesics on hyperbolic manifold, Inv. Math. 84 (1986), 523-540.
3. S. Gelbart, An elementary introduction to the Langlands Program, Bull. AMS, vol. 10, No. 2 (1984), 177-219.
4. S. Gelbart and F. Shahidi, Analytic properties of automorphic L-functions, preprint.

5. D. Hejhal, The Selberg Trace Formula for PSL(2, R), Springer Lecture Notes, No. 548 (1976), vol. 1 & No. 1001 (1983), vol. 2.
6. P. Sarnak, Determinants of Laplacians, Commu. Math. Phys. 110 (1987), 113-120.
7. A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, J. Indian Math. Soc. 20 (1956), 47-87.
8. A. Voros, Spectral functions, special functions and the Selberg zeta functions, Commu. Math. Phys. 110 (1987), 439-465.
9. Encyclopedic Dictionary of Mathematics, MIT Press, pp. 1694-1720.

최근의 preprint 소개

1. 저 자 : P. Buser and P. Sarnak (Princeton)

제 목 : On the period matrix of a Riemann Surface of Large Genus

Abstract : Riemann showed that a period matrix of a compact Riemann surface of genus $g \geq 1$ satisfies certain relations. We give a further simple combinatorial property, related to the length of the shortes : non-zero lattice vertor, satisfied by such a period matrix, see (1. 13). In particular, it is shown that for large genus the entire locus of Jacobians lies in a very small neighborhood of the boundary of the space of principally polarized abelian varieties.

We apply this to the problem of congruence subgroups of arithmetic lattices in $SL_2(\mathbb{R})$. We show that, with the exception of a finite number of arithmetic lattices in $SL_2(\mathbb{R})$, every such lattice has a subgroup of index at most 2 which is noncongruence. A notable exception is the modular group $SL_2(\mathbb{Z})$.

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On Theta Functions

Dedicated to Professor U-Hang Ki on his sixtieth birthday

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In this paper, we identitfy theta functions with smooth functions on the Heisenberg group with certain conditions and give a connection of theta functions with lattice representations.

1. Introduction

For any positive integers g and h, we consider the Heisenberg group

$$H_{\mathbf{R}}^{(g,h)} := \left\{ (\lambda, \mu, \kappa) \mid \lambda, \mu \in \mathbf{R}^{(h,g)},\ \kappa \in \mathbf{R}^{(h,h)},\ \kappa + \mu\, {}^t\lambda \ symmetric \right\}$$

endowed with the following multiplication law

$$(\lambda, \mu, \kappa) \circ (\lambda', \mu', \kappa') := (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda').$$

The Heisenberg group $H_{\mathbf{R}}^{(g,h)}$ is embedded in the symplectic group $Sp(g+h, \mathbf{R})$ via the mapping

$$H_{\mathbf{R}}^{(g,h)} \ni (\lambda, \mu, \kappa) \longmapsto \begin{pmatrix} E_g & 0 & 0 & {}^t\mu \\ \lambda & E_h & \mu & \kappa \\ 0 & 0 & E_g & -{}^t\lambda \\ 0 & 0 & 0 & E_h \end{pmatrix} \in Sp(g+h, \mathbf{R}).$$

This Heisenberg group is a 2-step nilpotent Lie group and is important in the study of toroidal compactification of Siegel moduli spaces. In fact, $H_{\mathbf{R}}^{(g,h)}$ is obtained as the unipotent radical of the parabolic subgroup of $Sp(g+h, \mathbf{R})$ associated with the rational boundary component F_g (cf. [N] p. 21).

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In this paper, we identify theta functions with smooth functions on the Heisenberg group $H_{\mathbf{R}}^{(g,h)}$ with some conditions and give a connection between theta functions and lattice representations. In section two, we investigate the Heisenberg group $H_{\mathbf{R}}^{(g,h)}$ and obtain Heisenberg commutation relations. In section three, we give an explicit description of theta functions due to J. Igusa (cf. [I] or [M1]) and identify theta functions with smooth functions on $H_{\mathbf{R}}^{(g,h)}$ with certain conditions. In the final section, we give a connection of theta functions with lattice representations.

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NOTATIONS: We denote by $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ the ring of integers, the field of real numbers, and the field of complex numbers respectively. $\mathbf{C}_1^{\times}$ denotes the multiplicative group consisting of all complex numbers z with $|z| = 1$. $Sp(g, \mathbf{R})$ denotes the symplectic group of degree g. H_g denotes the Siegel upper half plane of degree g. The symbol ":=" means that the expression on the right is the definition of that on the left. We denote by $\mathbf{Z}^+$ the set of all positive integers. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. For $A \in F^{(k,k)}$, $\sigma(A)$ denotes the trace of A. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. E_k denotes the identity matrix of degree k. For a positive integer n, $Sym\,(n, K)$ denotes the vector space consisting of all symmetric $n \times n$ matrices with entries in a field K.

2. Heisenberg Groups

First of all, we observe that $H_{\mathbf{R}}^{(g,h)}$ is a 2-step nilpotent Lie group. It is easy to see that the inverse of an element $(\lambda, \mu, \kappa) \in H_{\mathbf{R}}^{(g,h)}$ is given by

$$(\lambda, \mu, \kappa)^{-1} = (-\lambda, -\mu, -\kappa + \lambda\, {}^t\mu - \mu\, {}^t\lambda).$$

Now we put

(2.1) $$[\lambda, \mu, \kappa] := (0, \mu, \kappa) \circ (\lambda, 0, 0) = (\lambda, \mu, \kappa - \mu\, {}^t\lambda).$$

Then $H_{\mathbf{R}}^{(g,h)}$ may be regarded as a group equipped with the following multiplication

(2.2) $$[\lambda, \mu, \kappa] \diamond [\lambda_0, \mu_0, \kappa_0] := [\lambda + \lambda_0, \mu + \mu_0, \kappa + \kappa_0 + \lambda\, {}^t\mu_0 + \mu_0\, {}^t\lambda].$$

The inverse of $[\lambda,\mu,\kappa]\in H_{\mathbf{R}}^{(g,h)}$ is given by

$$[\lambda,\mu,\kappa]^{-1}=[-\lambda,-\mu,-\kappa+\lambda\,{}^t\mu+\mu\,{}^t\lambda].$$

We set

$$K:=\left\{\,[0,\mu,\kappa]\in H_{\mathbf{R}}^{(g,h)}\;\middle|\;\mu\in\mathbf{R}^{(h,g)},\ \kappa={}^t\kappa\in\mathbf{R}^{(h,h)}\,\right\}. \tag{2.3}$$

Then K is a commutative normal subgroup of $H_{\mathbf{R}}^{(g,h)}$. Let $\hat{K}$ be the Pontrajagin dual of K, i.e., the commutative group consisting of all unitary characters of K. Then $\hat{K}$ is isomorphic to the additive group $\mathbf{R}^{(h,g)}\times Sym\,(h,\mathbf{R})$ via

$$<a,\hat{a}>:=e^{2\pi i\sigma(\hat{\mu}\,{}^t\mu+\hat{\kappa}\kappa)},\quad a=[0,\mu,\kappa]\in K,\ \hat{a}=(\hat{\mu},\hat{\kappa})\in\hat{K}. \tag{2.4}$$

We put

$$S:=\left\{\,[\lambda,0,0]\in H_{\mathbf{R}}^{(g,h)}\;\middle|\;\lambda\in\mathbf{R}^{(h,g)}\,\right\}\cong\mathbf{R}^{(h,g)}. \tag{2.5}$$

Then S acts on K as follows:

$$\alpha_\lambda([0,\mu,\kappa]):=[0,\mu,\kappa+\lambda\,{}^t\mu+\mu\,{}^t\lambda],\quad [\lambda,0,0]\in S. \tag{2.6}$$

It is easy to see that the Heisenberg group $\left(H_{\mathbf{R}}^{(g,h)},\diamond\right)$ is isomorphic to the semidirect product $S\propto K$ of S and K whose multiplication is given by

$$(\lambda,a)\cdot(\lambda_0,a_0):=(\lambda+\lambda_0,a+\alpha_\lambda(a_0)),\quad \lambda,\lambda_0\in S,\ a,a_0\in K.$$

On the other hand, S acts on $\hat{K}$ by

$$\alpha_\lambda^*(\hat{a}):=(\hat{\mu}+2\hat{\kappa}\lambda,\hat{\kappa}),\quad [\lambda,0,0]\in S,\quad a=(\hat{\mu},\hat{\kappa})\in\hat{K}. \tag{2.7}$$

Then we have the relation $<\alpha_\lambda(a),\hat{a}>=<a,\alpha_\lambda^*(\hat{a})>$ for all $a\in K$ and $\hat{a}\in\hat{K}$.

We have two types of S-orbits in $\hat{K}$.

Type I. Let $\hat{\kappa}\in Sym\,(h,\mathbf{R})$ with $\hat{\kappa}\neq 0$. The S-orbit of $\hat{a}(\hat{\kappa}):=(0,\hat{\kappa})\in\hat{K}$ is given by

$$\hat{\mathcal{O}}_{\hat{\kappa}}:=\left\{\,(2\hat{\kappa}\lambda,\hat{\kappa})\in\hat{K}\;\middle|\;\lambda\in\mathbf{R}^{(h,g)}\,\right\}\cong\mathbf{R}^{(h,g)}. \tag{2.8}$$

Type II. Let $\hat{y}\in\mathbf{R}^{(h,g)}$. The S-orbit $\hat{\mathcal{O}}_{\hat{y}}$ of $\hat{a}(\hat{y}):=(\hat{y},0)$ is given by

$$\hat{\mathcal{O}}_{\hat{y}}:=\{\,(\hat{y},0)\,\}=\hat{a}(\hat{y}). \tag{2.9}$$

We have

$$\hat{K} = \left(\bigcup_{\hat{\kappa} \in Sym(h,\mathbf{R})} \hat{O}_{\hat{\kappa}} \right) \bigcup \left(\bigcup_{\hat{y} \in \mathbf{R}^{(h,g)}} \hat{O}_{\hat{y}} \right)$$

as a set. The stabilizer $S_{\hat{\kappa}}$ of S at $\hat{a}(\hat{\kappa}) = (0, \hat{\kappa})$ is given by

$$S_{\hat{\kappa}} = \{0\}. \tag{2.10}$$

And the stabilizer $S_{\hat{y}}$ of S at $\hat{a}(\hat{y}) = (\hat{y}, 0)$ is given by

$$S_{\hat{y}} = \left\{ [\lambda, 0, 0] \,\Big|\, \lambda \in \mathbf{R}^{(h,g)} \right\} = S \cong \mathbf{R}^{(h,g)}. \tag{2.11}$$

The following matrices

$$X^0_{kl} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(E_{kl} + E_{lk}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad 1 \le k \le l \le h,$$

$$X_{ka} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ E_{ka} & 0 & 0 & 0 \\ 0 & 0 & 0 & -{}^tE_{ka} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad 1 \le k \le h, \ 1 \le a \le g,$$

$$\hat{X}_{lb} := \begin{pmatrix} 0 & 0 & 0 & {}^tE_{lb} \\ 0 & 0 & E_{lb} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad 1 \le l \le h, \ 1 \le b \le g$$

form a basis of the Lie algebra $\mathcal{H}^{(g,h)}_{\mathbf{R}}$ of the real Heisenberg group $H^{(g,h)}_{\mathbf{R}}$. Here E_{kl} denotes the $h \times h$ matrix with entry 1 where the k-th row and the l-th column meet, all other entries 0 and E_{ka} (resp. E_{lb}) denotes the $h \times g$ matrix with entry 1 where the k-th (resp. the l-th) row and the a-th (resp. the b-th) column meet, all other entires 0. By an easy calculation, we see that the following vector fields

$$D^0_{kl} := \frac{\partial}{\partial \kappa_{kl}}, \quad 1 \le k \le h,$$

$$D_{ka} := \frac{\partial}{\partial \lambda_{ka}} - \left(\sum_{p=1}^{k} \mu_{pa} \frac{\partial}{\partial \kappa_{pk}} + \sum_{p=k+1}^{h} \mu_{pa} \frac{\partial}{\partial \kappa_{kp}} \right), \quad 1 \le k \le h, \ 1 \le a \le g,$$

$$\hat{D}_{lb} := \frac{\partial}{\partial \mu_{lb}} + \left(\sum_{p=1}^{l} \lambda_{pb} \frac{\partial}{\partial \kappa_{pl}} + \sum_{p=l+1}^{h} \lambda_{pb} \frac{\partial}{\partial \kappa_{lp}} \right), \quad 1 \le k \le h, \ 1 \le a \le g$$

form a basis for the Lie algebra of left-invariant vector fields on the Lie group $H_{\mathbf{R}}^{(g,h)}$.

Lemma 2.1. *We have the following Heisenberg commutation relations*

$$\begin{aligned}
&[D^0_{kl}, D^0_{mn}] = [D^0_{kl}, D_{ma}] = [D^0_{kl}, \hat{D}_{ma}] = 0,\\
&[D_{ka}, D_{lb}] = [\hat{D}_{ka}, \hat{D}_{lb}] = 0,\\
&[D_{ka}, \hat{D}_{lb}] = 2\delta_{ab} D^0_{kl},
\end{aligned}$$

where $1 \leq k,l,m,n \leq h$, $1 \leq a,b \leq g$ *and* δ_{ab} *denotes the Kronecker delta symbol.*

Proof. The proof follows from a straightforward calculation.

We put

$$\begin{aligned}
Z^0_{kl} &:= -\sqrt{-1}D^0_{kl}, \quad 1 \leq k \leq l \leq h,\\
Y^+_{ka} &:= \frac{1}{2}(D_{ka} + \sqrt{-1}\hat{D}_{ka}), \quad 1 \leq k \leq h,\ 1 \leq a \leq g,\\
Y^-_{lb} &:= \frac{1}{2}(D_{lb} - \sqrt{-1}\hat{D}_{lb}), \quad 1 \leq l \leq h,\ 1 \leq b \leq g.
\end{aligned}$$

Then it is easy to see that the vector fields Z^0_{kl}, Y^+_{ka}, Y^-_{lb} form a basis of the complexification of the real Lie algebra $\mathcal{H}_{\mathbf{R}}^{(g,h)}$.

Lemma 2.2. *We have the following commutation relations*

$$\begin{aligned}
&[Z^0_{kl}, Z^0_{mn}] = [Z^0_{kl}, Y^+_{ma}] = [Z^0_{kl}, Y^-_{ma}] = 0,\\
&[Y^+_{ka}, Y^+_{lb}] = [Y^-_{ka}, Y^-_{lb}] = 0,\\
&[Y^+_{ka}, Y^-_{lb}] = \delta_{ab}\, Z^0_{kl},
\end{aligned}$$

where $1 \leq k,l,m,n \leq h$ *and* $1 \leq a,b \leq g$.

Proof. It follows immediately from Lemma 2.1.

We let $E^{\bullet}_{kl} := E_{kl} + E_{lk}$ for $1 \leq k \leq l \leq h$. We put

$$\begin{aligned}
R_{kl}(r) &:= \exp 2rX^0_{kl} = (0,0,rE^{\bullet}_{kl}), \quad r \in \mathbf{R},\\
P_{ma}(s) &:= \exp sX_{ma} = (sE_{ma},0,0), \quad s \in \mathbf{R},\\
Q_{nb}(t) &:= \exp t\hat{X}_{nb} = (0,tE_{nb},0), \quad t \in \mathbf{R},
\end{aligned}$$

where $1 \leq k \leq l \leq h$, $1 \leq m,n \leq h$ and $1 \leq a,b \leq g$. Then these one-parameter subgroups generate the Heisenberg group $H_{\mathbf{R}}^{(g,h)}$. They satisfy the *Weyl commutation relations*:

$$P_{ma}(s) \circ Q_{ma}(t) = Q_{ma}(t) \circ P_{ma}(s) \circ R_{mm}(st) \quad \text{(all others commute)},$$

where $1 \leq m \leq h$ and $1 \leq a \leq g$.

3. Theta Functions

We fix an element $\Omega \in H_g$ once and for all. From now on, we put $i = \sqrt{-1}$. Let $\mathcal{M}$ be a positive definite, symmetric even integral matrix of degree h. A holomorphic function $f : \mathbf{C}^{(h,g)} \longrightarrow \mathbf{C}$ satisfying the following equation

$$(3.1) \qquad f(W + \xi\Omega + \eta) = e^{-\pi i \sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi + 2\xi\,{}^tW)\}} f(W), \quad W \in \mathbf{C}^{(h,g)}$$

for all $\xi, \eta \in \mathbf{Z}^{(h,g)}$ is called a *theta function of level $\mathcal{M}$ with respect to Ω*. The set $R^{\Omega}_{\mathcal{M}}$ of all theta functions of level $\mathcal{M}$ with respect to Ω is a complex vector space of dimension $(\det \mathcal{M})^g$ with a basis consisting of theta functions

$$(3.2) \quad \vartheta^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega, W) := \sum_{N \in \mathbf{Z}^{(h,g)}} e^{\pi i \sigma\{\mathcal{M}((N+A)\Omega\,{}^t(N+A) + 2W\,{}^t(N+A))\}},$$

where A runs over a complete system of the cosets $\mathcal{M}^{-1}\mathbf{Z}^{(h,g)}/\mathbf{Z}^{(h,g)}$.

Definition 3.1. Let S be a positive definite, symmetric real matrix of degree h and let $A, B \in \mathbf{R}^{(h,g)}$. We define the theta function

$$(3.3) \quad \vartheta^{(S)}\begin{bmatrix} A \\ B \end{bmatrix}(\Omega, W) := \sum_{N \in \mathbf{Z}^{(h,g)}} e^{\pi i \sigma\{S((N+A)\Omega\,{}^t(N+A) + 2(W+B)\,{}^t(N+A))\}}$$

with characteristic (A, B) converging normally on $H_g \times \mathbf{C}^{(h,g)}$.

We have a general definition of theta functions.

Definition 3.2. Let V be a finite dimensional complex vector space and let $L \subset V$ be a lattice of V. A *theta function* on V *relative to* L is a nonzero holomorphic function ϑ on V satisfying the following condition

$$\vartheta(W + \xi) = e^{2\pi i (Q_\xi(W) + c_\xi)} \vartheta(W),$$

where Q_ξ is a $\mathbf{C}$-linear form on V and c_ξ is an element of $\mathbf{C}$, for every $W \in V$ and $\xi \in L$.

The mapping $J : L \times V \longrightarrow \mathbf{C}^\times$ defined by

$$J(\xi, W) := e^{2\pi i (Q_\xi(W) + c_\xi)}, \quad \xi \in L, \quad W \in V$$

is easily seen to be an automorphic factor. We observe that for all $\xi_1, \xi_2 \in L$ and $W \in V$,

$$Q_{\xi_1 + \xi_2}(W) + c_{\xi_1 + \xi_2} \equiv Q_{\xi_1}(W + \xi_2) + Q_{\xi_2}(W) + c_{\xi_1} + c_{\xi_2} \mod \mathbf{Z}.$$

J is called the automorphic factor of the theta function ϑ on V relative to L.

Theorem 3.3 (Igusa [I], p.67). *Let $J : L \times V \longrightarrow \mathbf{C}^\times$ be the automorphic factor of a theta function ϑ on V relative to L. Then there exists a unique triple (Q, ℓ, ψ) such that*

$$J(\xi, W) = e^{\pi\{Q(W,\xi)+\frac{1}{2}Q(\xi,\xi)+2i\ell(\xi)\}}\psi(\xi), \quad \xi \in L, \ W \in V, \tag{3.4}$$

where

(1) *Q is a quasi-hermitian form on $V \times V$,*

(2) *the hermitian form $H := Her\,(Q)$ defined by*

$$H(W_1, W_2) := \frac{1}{2i}\{Q(iW_1, W_2) - Q(W_1, iW_2)\}, \quad W_1, \ W_2 \in V$$

is a Riemann form with respect to L, that is, $H = {}^t\bar{H} > 0$ and $(\,\mathrm{Im}\, H)(L \times L) \subset \mathbf{Z}$,

(3) *$\ell : V \longrightarrow \mathbf{C}$ is a $\mathbf{C}$-linear form on V,*

(4) *ψ is a second degree character of L which is associated with $A := \mathrm{Im}\, H$,*

(5) *ψ is strongly associated with A.*

Proof. The proof can be found in Igusa [I], p.67.

Remark 3.4. $\psi : L \longrightarrow \mathbf{C}_1^\times$ is a semi-character of L satisfying the functional equation

$$\psi(\xi_1 + \xi_2) = e^{\pi i A(\xi_1,\xi_2)}\psi(\xi_1)\psi(\xi_2), \quad \xi_1, \xi_2 \in L. \tag{$*$}$$

Definition 3.5. A theta function with the automorphic factor of the form (3.4) is called a *theta function of type* (Q, ℓ, ψ). We denote by $L(Q, \ell, \psi)$ the union of theta functions of type (Q, ℓ, ψ) and the constant 0. A theta function of type (Q, ℓ, ψ) is said to be *normalized* if $Sym\,Q = 0$ and $\ell = 0$. Here $Sym\,Q : V \times V \longrightarrow \mathbf{C}$ is a symmetric $\mathbf{C}$-linear form on $V \times V$ defined by

$$(Sym\,Q)(z, w) := \frac{1}{2i}\{Q(iz, w) + Q(z, iw)\}, \quad z, w \in V.$$

We observe that $Q = Her\,Q + Sym\,Q$. We note that $Sym\,Q = 0$ if and only if $Q = Her\,Q = H$. We denote by $Th(H, \psi, L)$ the union of the set of all normalized theta functions of type $(H, 0, \psi)$ and the constant 0. It is easily seen that if $\vartheta \in Th(H, \psi, L)$, for all $W \in V$, $\xi \in L$, we have

$$\vartheta(W + \xi) = e^{\pi H(W+\frac{1}{2}\xi,\xi)}\psi(\xi)\,\vartheta(W). \tag{3.5}$$

Theorem 3.6.. *Let S be a positive definite, symmetric real matrix of degree h and let A, B be two $h \times g$ real matrices. Then for $\Omega \in H_g$ and $W \in \mathbb{C}^{(h,g)}$, we have*

$(\theta.1)$ $\vartheta^{(S)} \begin{bmatrix} A \\ B \end{bmatrix} (\Omega, -W) = \vartheta^{(S)} \begin{bmatrix} -A \\ -B \end{bmatrix} (\Omega, W),$

$(\theta.2)$ $$\vartheta^{(S)} \begin{bmatrix} A \\ B \end{bmatrix} (\Omega, W + \lambda\Omega + \mu)$$
$$= e^{-\pi i \sigma\{S(\lambda\Omega\, {}^t\lambda + 2(W+\mu)\, {}^t\lambda)\}}\, e^{-2\pi i\sigma(SB\, {}^t\lambda)} \cdot \vartheta^{(S)} \begin{bmatrix} A+\lambda \\ B+\mu \end{bmatrix} (\Omega, W)$$
for all $\lambda, \mu \in \mathbb{R}^{(h,g)}$.

$(\theta.3)$ $\vartheta^{(S)} \begin{bmatrix} A \\ B \end{bmatrix} (\Omega, W) = e^{\pi i \sigma\{S(A\Omega\, {}^tA) + 2(W+B)\, {}^tA)\}}\, \vartheta^{(S)} \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\Omega, W + A\Omega + B).$

Moreover, if S is a positive definite, symmetric integral matrix of degree h, we have

$(\theta.4)$ $\vartheta^{(S)} \begin{bmatrix} A+\xi \\ B+\eta \end{bmatrix} (\Omega, W) = e^{2\pi i\sigma(SA^t\eta)}\, \vartheta^{(S)} \begin{bmatrix} A \\ B \end{bmatrix} (\Omega, W)$
for all $\xi, \eta \in \mathbb{Z}^{(h,g)}$.

$(\theta.5)$ $$\vartheta^{(S)} \begin{bmatrix} A \\ B \end{bmatrix} (\Omega, W + \xi\Omega + \eta)$$
$$= e^{-\pi i\sigma\{S(\xi\Omega\, {}^t\xi + 2W\, {}^t\xi)\}} \cdot e^{2\pi i\sigma\{S(A\, {}^t\eta - B\, {}^t\xi)\}} \cdot \vartheta^{(S)} \begin{bmatrix} A \\ B \end{bmatrix} (\Omega, W)$$

for all $\xi, \eta \in \mathbb{Z}^{(h,g)}$.

Proof. $(\theta.1)$ follows immediately from the definition (3.3). $(\theta.2)$ follows immediately from the relation

$$\begin{aligned} & (N+A)\Omega\, {}^t(N+A) + 2(W + \lambda\Omega + \mu + B)\, {}^t(N+A) \\ = \;& (N+A+\lambda)\,\Omega\, {}^t(N+A+\lambda) + 2(W+\mu+B)\, {}^t(N+A+\lambda) - (N+A)\Omega\, {}^t\lambda \\ & + \lambda\Omega\, {}^t(N+A) - \lambda\Omega\, {}^t\lambda - 2(W+\mu+B)\, {}^t\lambda. \end{aligned}$$

If we put $A = B = 0$ and replace λ, μ by A, B in $(\theta.2)$, then we obtain $(\theta.3)$. For $\xi, \eta \in \mathbb{Z}^{(h,g)}$, we have

$$\begin{aligned} & \vartheta^{(S)} \begin{bmatrix} A+\xi \\ B+\eta \end{bmatrix} (\Omega, W) \\ = \;& \sum_{N \in \mathbb{Z}^{(h,g)}} e^{\pi i\sigma\{S((A+N+\xi)\Omega\, {}^t(A+N+\xi) + 2(W+B)\, {}^t(A+N+\xi))\}} \\ & \times e^{2\pi i\sigma\{S\eta\, {}^t(N+\xi)\}} \cdot e^{2\pi i\sigma(S\, {}^t\eta A)} \end{aligned}$$

$$= \quad e^{2\pi i\sigma(SA\,{}^t\eta)}\cdot\vartheta^{(S)}\begin{bmatrix}A\\B\end{bmatrix}(\Omega,W).$$

Here in the last equality we used the fact that $\sigma(S\eta\,{}^t(N+\xi))\in\mathbf{Z}$ because S is integral. $(\theta.5)$ follows from $(\theta.2)$, $(\theta.4)$ and the fact that $\sigma(S\eta\,{}^t\xi)$ is integral.

For a positve definite, symmetric real matrix S of degree h, $\Omega\in H_g$ and $A,B\in\mathbf{R}^{(h,g)}$, we put

(3.6) $\chi_{S,\Omega,A,B}(\xi\Omega+\eta):=\chi_{S,\Omega,A,B}(\xi,\eta):=e^{2\pi i\sigma\{S(A\,{}^t\eta-B\,{}^t\xi)\}},\quad \xi,\eta\in\mathbf{Z}^{(h,g)}.$

We define

$$(3.7)\qquad q_{S,\Omega}(W):=\frac{1}{2}\sigma\left(SW(\Omega-\bar\Omega)^{-1}\,{}^tW\right),\quad W\in\mathbf{C}^{(h,g)}$$

and also define

$$(3.8)\qquad H_{S,\Omega}(W_1,W_2):=2i\sigma\left(SW_1(\Omega-\bar\Omega)^{-1}\,{}^t\overline{W}_2\right),\quad W_1,W_2\in\mathbf{C}^{(h,g)}.$$

It is easy to check that $H_{S,\Omega}$ is a positive hermitian form on $\mathbf{C}^{(h,g)}$.

Lemma 3.7. *For $W\in\mathbf{C}^{(h,g)}$ and $l\in\mathbf{Z}^{(h,g)}\Omega+\mathbf{Z}^{(h,g)}$, we have*

$$(3.9)\qquad q_{S,\Omega}(W+l)=q_{S,\Omega}(W)+q_{S,\Omega}(l)+\sigma\left(Sl(\Omega-\bar\Omega)^{-1}\,{}^tW\right).$$

and

$$(3.10)\ H_{S,\Omega}(W+\frac{l}{2},l)=\sigma\left(S(W+\frac{l}{2})(\operatorname{Im}\Omega)^{-1}\,{}^tl\right)-2i\sigma\left(S(W+\frac{l}{2})\,{}^t\xi\right),$$

where $l=\xi\Omega+\eta,\ \xi,\eta\in\mathbf{Z}^{(h,g)}$.

Proof. It follows immediately from a straightforward computation.

Lemma 3.8. *Let S be a positive definite, symmetric integral matrix of degree h. For $\Omega\in H_g$, we let $L_\Omega:=\mathbf{Z}^{(h,g)}\Omega+\mathbf{Z}^{(h,g)}$ be the lattice in $\mathbf{C}^{(h,g)}$. We define the mapping $\psi_{S,\Omega}:L_\Omega\longrightarrow\mathbf{C}_1$ by*

$$(3.11)\qquad \psi_{S,\Omega}(\xi\Omega+\eta):=e^{\pi i\sigma(S\eta\,{}^t\xi)},\quad \xi,\eta\in\mathbf{Z}^{(h,g)}.$$

Then

(A) *$\psi_{S,\Omega}$ is a second-degree character of L_Ω associated with* $\operatorname{Im}H_{S,\Omega}$.

(B) *$\psi_{S,\Omega}\cdot\chi_{S,\Omega,A,B}$ is a second-degree character of L_Ω associated with* $\operatorname{Im}H_{S,\Omega}$.

Proof. (A) We fix $l = \xi\Omega + \eta \in L_\Omega$ with $\xi, \eta \in \mathbf{Z}^{(h,g)}$. We define $f_l : L_\Omega \longrightarrow \mathbf{C}_1^\times$ by

$$f_l(l_1) := \frac{\psi_{S,\Omega}(l_1 + l)}{\psi_{S,\Omega}(l_1)\,\psi_{S,\Omega}(l)}\,, \quad l_1 \in L_\Omega.$$

It is easy to see that f_l is a character of L_Ω and hence to see that the map from $L_\Omega \times L_\Omega$ to $\mathbf{C}_1^\times$ defined by

$$(l_1, l_2) \longmapsto \frac{\psi_{S,\Omega}(l_1 + l_2)}{\psi_{S,\Omega}(l_1)\,\psi_{S,\Omega}(l_2)}$$

is a bicharacter of L_Ω, i.e., a character of L_Ω in l_1 and l_2. Hence $\psi_{S,\Omega}$ is a second degree character of L_Ω. In order to show that $\psi_{S,\Omega}$ is associated with $H_{S,\Omega}$, it is enough to prove that

$$\psi_{S,\Omega}(l_1 + l_2) = e^{\pi i A_{S,\Omega}(l_1,l_2)}\,\psi_{S,\Omega}(l_1)\,\psi_{S,\Omega}(l_2) \tag{3.12}$$

for all $l_1, l_2 \in L_\Omega$. Here $A_{S,\Omega}$ denotes the imaginary part of the positive hermitian form $H_{S,\Omega}$. By an easy computation, we have

$$A_{S,\Omega}(l_1, l_2) = \sigma\{S(\xi_1\,{}^t\eta_2 - \eta_1\,{}^t\xi_2)\}, \tag{3.13}$$

where $l_i = \xi_i\Omega + \eta_i \in L_\Omega\,(1 \leq i \leq 2)$. Hence (3.12) follows immediately from (3.13).

(B) We fix $l = \xi\Omega + \eta \in L_\Omega$ with $\xi, \eta \in \mathbf{Z}^{(h,g)}$. We put $\tilde{\psi}_{S,\Omega,A,B} := \psi_{S,\Omega}\cdot\chi_{S,\Omega,A,B}$. Then the map $\tilde{f}_l : L_\Omega \longrightarrow \mathbf{C}_1^\times$ defined by

$$\tilde{f}_l(l_1) := \frac{\tilde{\psi}_{S,\Omega,A,B}(l_1 + l)}{\tilde{\psi}_{S,\Omega,A,B}(l_1)\,\tilde{\psi}_{S,\Omega,A,B}(l_2)}\,, \quad l_1 \in L_\Omega$$

is a character of L_Ω. So $\tilde{\psi}_{S,\Omega,A,B}$ is a second degree character of L_Ω. In order to show that $\tilde{\psi}_{S,\Omega}$ is associated with $A_{S,\Omega}$, it suffices to prove that

$$\tilde{\psi}_{S,\Omega,A,B}(l_1 + l_2) = e^{\pi i A_{S,\Omega}(l_1,l_2)}\,\tilde{\psi}_{S,\Omega,A,B}(l_1)\,\tilde{\psi}_{S,\Omega,A,B}(l_2) \tag{3.14}$$

for all $l_1, l_2 \in L_\Omega$. An easy calculation yields (3.14).

Theorem 3.9. *We assume that S is a positive definite, symmetric integral matrix of degree h. Let $\Omega \in H_g$. We denote by R_S^Ω the vector space of all holomorphic functions $f : \mathbf{C}^{(h,g)} \longrightarrow \mathbf{C}$ satisfying the transformation behaviour*

$$f(W + \xi\Omega + \eta) = e^{-\pi i\sigma\{S(\xi\Omega\,{}^t\xi + 2\xi\,{}^tW)\}}\,f(W), \quad W \in \mathbf{C}^{(h,g)}$$

for all $\xi, \eta \in \mathbf{Z}^{(h,g)}$. *Then the mapping*

$$\Theta : R_S^{\Omega} \longrightarrow Th(H_{S,\Omega}, \psi_{S,\Omega}, L_\Omega)$$

defined by

$$(\Theta(F))(W) := e^{2\pi i q_{S,\Omega}(W)} F(W), \quad F \in R_S^{\Omega},\ W \in \mathbf{C}^{(h,g)}$$

is an isomorphism of vector spaces, where L_Ω *and* $\psi_{S,\Omega}$ *are the same as in lemma 3.8.*

Proof. First of all, we will show the image $\Theta(R_S^{\Omega})$ is contained in $Th(H_{S,\Omega}, \psi_{S,\Omega}, L_\Omega$ If $F \in R_S^{\Omega}$, $W \in \mathbf{C}^{(h,g)}$ and $l = \xi\Omega + \eta \in L_\Omega$, then we have

$$\begin{aligned}
\Theta(F)(W+l) &= e^{2\pi i q_{S,\Omega}(W+l)}\, F(W+l) \\
&= e^{2\pi i \{q_{S,\Omega}(W) + q_{S,\Omega}(l) + \sigma(Sl(\Omega-\bar{\Omega})^{-1}\,{}^tW)\}} \\
&\quad \times e^{-\pi i \sigma\{S(\xi\Omega\,{}^t\xi + 2W\,{}^t\xi)\}}\, F(W) \qquad (\textit{by lemma } 3.7) \\
&= e^{2\pi i \sigma\{S(W+\frac{l}{2})(\Omega-\bar{\Omega})^{-1}\,{}^tl\}} \\
&\quad \times e^{-\pi i \sigma\{S(\xi\Omega\,{}^t\xi + 2W\,{}^t\xi + 2W\,{}^t\xi)\}} \cdot \Theta(F)(W) \\
&= e^{\pi H_{S,\Omega}(W+\frac{l}{2}, l)} \cdot e^{-\pi i \sigma(S\eta\,{}^t\xi)}\, \Theta(F)(W) \\
&= e^{\pi H_{S,\Omega}(W+\frac{l}{2}, l)}\, \psi_{S,\Omega}(l)\, \Theta(F)(W).
\end{aligned}$$

Thus $\Theta(F)$ is contained in the set $Th(H_{S,\Omega}, \psi_{S,\Omega}, L_\Omega)$. It is easy to see that the mapping Θ is an isomorphism.

Proposition 3.10. *Let* S *be as above in Theorem 3.9 and* $A, B \in \mathbf{R}^{(h,g)}$. *We denote by* $R_{S,A,B}^{\Omega}$ *the union of the set of all theta functions with characteristic* (A, B) *with respect to* S *and* Ω *and the constant* 0. *Then we have an isomorphism*

$$R_{S,A,B}^{\Omega} \cong Th(H_{S,\Omega}, \psi_{S,\Omega} \cdot \chi_{S,\Omega,A,B}, L_\Omega).$$

Proof. First we observe that $\psi_{S,\Omega} \cdot \chi_{S,\Omega,A,B}$ is a second degree character of L_Ω associated with $A_{S,\Omega}$ (cf. lemma 3.8 (B)). In a similiar way in the proof of Theorem 3.9, using $(\theta.5)$, we can show that the mapping

$$\Theta_{A,B}(f)(W) := e^{2\pi i q_{S,\Omega}(W)}\, f(W), \quad f \in R_{S,A,B}^{\Omega},\ W \in \mathbf{C}^{(h,g)}$$

has its image in $Th(H_{S,\Omega}, \psi_{S,\Omega} \cdot \chi_{S,\Omega,A,B}, L_\Omega)$.

Proposition 3.11. *Let* S *be as above in Theorem 3.9 and let* $A, B \in \mathbf{R}^{(h,g)}$. *Then we have an isomorphism*

$$Th(H_{S,\Omega}, \psi_{S,\Omega}, L_\Omega) \cong TH(H_{S,\Omega}, \psi_{S,\Omega} \cdot \chi_{S,\Omega,A,B}, L_\Omega).$$

Proof. The proof follows from the fact that the dimension of the complex vector space $Th(H_{S,\Omega}, \psi_{S,\Omega}, L_\Omega)$ is equal to that of $Th(H_{S,\Omega}, \psi_{S,\Omega} \cdot \chi_{S,\Omega,A,B}, L_\Omega)$. It is well known that the dimension of $Th(H_{S,\Omega}, \psi_{S,\Omega}, L_\Omega)$ is equal to the Pfaffian of $A_{S,\Omega}$ relative to L_Ω (cf. [I], p.72).

Remark 3.12. From theorem 3.9, proposition 3.10 and proposition 3.11, R_S^Ω is isomorphic to $R_{S,A,B}^\Omega$ for any $A, B \in \mathbf{R}^{(h,g)}$.

Now as before, we fix an element $\Omega \in H_g$ and let S be a positive symmetric *integral* matrix of degree h. Then the lattice $L := \mathbf{Z}^{(h,g)} \times \mathbf{Z}^{(h,g)}$ acts on $\mathbf{C}^{(h,g)}$ freely by

$$(\xi, \eta) \cdot W = W + \xi\Omega + \eta, \quad \xi, \eta \in \mathbf{Z}^{(h,g)}, \quad W \in \mathbf{C}^{(h,g)}.$$

Lemma 3.13. *Let $A, B \in \mathbf{R}^{(h,g)}$. We let $J_{S,\Omega,A,B} : L \times \mathbf{C}^{(h,g)} \longrightarrow \mathbf{C}^\times$ the mapping defined by*

$$(3.15) \qquad J_{S,\Omega,A,B}(l, W) := e^{\pi i \sigma\{S(\xi\Omega\, {}^t\xi + 2W\, {}^t\xi)\}} \cdot e^{-2\pi i \sigma\{S(A\, {}^t\eta - B\, {}^t\xi)\}},$$

where $l = (\xi, \eta) \in L$ and $W \in \mathbf{C}^{(h,g)}$. Then $J_{S,\Omega,A,B}$ is an automorphic factor for the lattice L.

Proof. For brevity, we write $J := J_{S,\Omega,A,B}$. For any two elements $l_i = (\xi_i, \eta_i)$ ($i = 1, 2$) of L and $W \in \mathbf{C}^{(h,g)}$, we must show that

$$(3.16) \qquad J(l_1 + l_2, W) = J(l_1, l_2 + W)\, J(l_2, W).$$

Using the fact that $\sigma(2S\eta_2\, {}^t\xi_1)$ is an even integer, an easy computation yields (3.16)

The Heisenberg group $H_{\mathbf{R}}^{(g,h)}$ with multiplication $\diamond$ acts on $\mathbf{C}^{(h,g)}$

$$[\lambda_0, \mu_0, \kappa_0] \cdot (\lambda\Omega + \mu) := (\lambda_0 + \lambda)\Omega + (\mu_0 + \mu), \quad \lambda, \mu \in \mathbf{R}^{(h,g)}.$$

Since the center $\mathcal{Z} = \{ [0, 0, \kappa] \mid \kappa = {}^t\kappa \in \mathbf{R}^{(h,h)} \}$ of $H_{\mathbf{R}}^{(g,h)}$ is the stabilizer of $H_{\mathbf{R}}^{(g,h)}$ at the identity element $[0, 0, 0]$, the homogeneous space $H_{\mathbf{R}}^{(g,h)}/\mathcal{Z}$ is identified with $\mathbf{C}^{(h,g)}$ via

$$[\lambda, \mu, \kappa] \cdot \mathcal{Z} \longmapsto [\lambda, \mu, \kappa] \cdot 0 = \lambda\Omega + \mu.$$

Thus the automorphic factor $J_{S,\Omega,A,B}$ for the lattice L may be lifted to the automorphic factor $\tilde{J}_{S,\Omega,A,B} : H_{\mathbf{R}}^{(g,h)} \times \mathbf{C}^{(h,g)} \longrightarrow \mathbf{C}^\times$ defined by

$$(3.17) \quad \tilde{J}_{S,\Omega,A,B}(g_0, W) = e^{\pi i \sigma\{S(\lambda\Omega\, {}^t\lambda + 2W\, {}^t\lambda + \kappa)\}} \cdot e^{-\pi i \sigma\{S(A\, {}^t\mu - B\, {}^t\lambda)\}},$$

where $g_0=[\lambda,\mu,\kappa]\in H_{\mathbf{R}}^{(g,h)}$.

We denote by $\mathcal{A}_{S,\Omega}$ be the complex vector space consisting of $\mathbf{C}$-valued smooth functions φ on $H_{\mathbf{R}}^{(g,h)}$ satisfying the following conditions

(a) $\varphi([\xi,\eta,0]\diamond g_0)=\varphi(g_0)$ for all $\xi,\eta\in\mathbf{Z}^{(h,g)}$ and $g_0\in H_{\mathbf{R}}^{(g,h)}$,

(b) $\varphi(g_0\diamond[0,0,\kappa])=e^{\pi i\sigma(S\kappa)}\,\varphi(g_0)$ for all $\kappa={}^t\kappa\in\mathbf{R}^{(h,h)}$ and $g_0\in H_{\mathbf{R}}^{(g,h)}$,

(c) $\left(\mathcal{L}_{X_{ka}}-\sum_{b=1}^{g}\Omega\mathcal{L}_{\hat{X}_{kb}}\right)=0$ for all $1\le k\le h$ and $1\le a\le g$.

Here if X is an element of the Lie algebra of $H_{\mathbf{R}}^{(g,h)}$,

$$\left(\mathcal{L}_X\varphi\right)(g_0)=\frac{d}{dt}\bigg|_{t=0}\varphi(g_0\diamond\exp tX),\quad g_0\in H_{\mathbf{R}}^{(g,h)}.$$

Theorem 3.14. *Let S and Ω be as before. Then the vector space R_S^Ω is isomorphic to the vector space $\mathcal{A}_{S,\Omega}$ via the mapping*

$$f\longmapsto\varphi_f(g_0):=\tilde{J}_{S,\Omega,0,0}(g_0,\,0)\,f(g_0\cdot 0),$$

where $g_0\in H_{\mathbf{R}}^{(g,h)}$ and $f\in R_S^\Omega$.
The inverse of the above isomorphism is given by

$$\varphi\longmapsto f_\varphi(W):=\tilde{J}_{S,\Omega,0,0}(g_0,\,0)^{-1}\,\varphi(g_0),\quad\varphi\in\mathcal{A}_{S,\Omega},$$

where $W=g_0\cdot 0$. This definition does not depend on the choice of g_0 with $W=g_0\cdot 0$.

Proof. For brevity, we write $\tilde{J}:=\tilde{J}_{S,\Omega,0,0}$. If $\gamma=[\xi,\eta,0]\in H_{\mathbf{R}}^{(g,h)}$ with $\xi,\eta\in\mathbf{Z}^{(h,g)}$, we have for all $g_0\in H_{\mathbf{R}}^{(g,h)}$

$$\begin{aligned}\varphi_f(\gamma\diamond g_0)&=\tilde{J}(\gamma\diamond g_0,\,0)\,f((\gamma\diamond g_0)\cdot 0)\\&=\tilde{J}(\gamma,\,g_0\cdot 0)\,\tilde{J}(g_0,\,0)\,f(g_0\cdot 0+\xi\Omega+\eta)\\&=\tilde{J}(\gamma,\,g_0\cdot 0)\,\tilde{J}(g_0,0)\,J((\xi,\eta),\,g_0\cdot 0)^{-1}\,f(g_0\cdot 0)\\&=\tilde{J}(\gamma,0)\,f(g_0\cdot 0)\\&=\varphi_f(g_0).\end{aligned}$$

And if $\kappa={}^t\kappa\in\mathbf{R}^{(h,h)}$, we have

$$\begin{aligned}\varphi_f(g_0\diamond[0,0,\kappa])&=\tilde{J}(g_0\diamond[0,0,\kappa],\,0)\,f((g_0\diamond[0,0,\kappa])\cdot 0)\\&=\tilde{J}(g_0,\,[0,0,\kappa]\cdot 0)\,\tilde{J}([0,0,\kappa],\,0)\,f(g_0\cdot 0)\\&=e^{\pi i\sigma(S\kappa)}\,\tilde{J}(g_0,0)\,f(g_0\cdot 0)\\&=e^{\pi\sigma(S\kappa)}\,\varphi_f(g_0).\end{aligned}$$

Finally, we introduce a system of complex coordinates on $\mathbb{C}^{(h,g)}$ with respect to Ω:

$$W = \lambda\Omega + \mu, \quad \overline{W} = \lambda\overline{\Omega} + \mu, \quad \lambda, \mu \text{ real.}$$

We set

$$dW = \begin{pmatrix} dW_{11} & dW_{12} & \dots & dW_{1g} \\ dW_{21} & dW_{22} & \dots & dW_{2g} \\ \vdots & \vdots & \ddots & \vdots \\ dW_{h1} & dW_{h2} & \dots & dW_{hg} \end{pmatrix}, \quad \frac{\partial}{\partial W} = \begin{pmatrix} \frac{\partial}{\partial W_{11}} & \frac{\partial}{\partial W_{21}} & \dots & \frac{\partial}{\partial W_{h1}} \\ \frac{\partial}{\partial W_{12}} & \frac{\partial}{\partial W_{22}} & \dots & \frac{\partial}{\partial W_{h2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial}{\partial W_{1g}} & \frac{\partial}{\partial W_{2g}} & \dots & \frac{\partial}{\partial W_{hg}} \end{pmatrix}.$$

Then an easy computation yields

$$\begin{aligned} \frac{\partial}{\partial \lambda} &= \Omega\frac{\partial}{\partial W} + \overline{\Omega}\frac{\partial}{\partial \overline{W}}, \\ \frac{\partial}{\partial \mu} &= \frac{\partial}{\partial W} + \frac{\partial}{\partial \overline{W}}. \end{aligned}$$

Thus we obtain the following

$$\frac{\partial}{\partial \overline{W}} = \frac{i}{2}\left(\operatorname{Im}\Omega\right)^{-1}\left(\frac{\partial}{\partial \lambda} - \Omega\frac{\partial}{\partial \mu}\right). \tag{3.18}$$

Since f is holomorphic, according to (3.18), f satisfies the conditions

$$\left(\frac{\partial}{\partial \lambda_{ka}} - \sum_{b=1}^{g} \Omega_{ab}\frac{\partial}{\partial \mu_{kb}}\right) f(W) = 0, \quad 1 \le k \le h, \ 1 \le a \le g. \tag{3.19}$$

Conversely, if a smooth function on $\mathbb{C}^{(h,g)}$ satisfies the condition (3.19), it is holomorphic.

In order to prove that φ_f satisfies the condition (c), we first compute $\mathcal{L}_{X_{ka}}\varphi_f$ and $\mathcal{L}_{\hat{X}_{lb}}\varphi_f$ for $1 \le k, l \le h$ and $1 \le a, b \le g$. If $g = [\lambda, \mu, \kappa] \in H_{\mathbf{R}}^{(g,h)}$ and $S = (S_{kl})$,

$$\begin{aligned} \left(\mathcal{L}_{X_{ka}}\varphi_f\right)(g) &= \frac{d}{dt}\bigg|_{t=0} \varphi_f(g \diamond \exp tX_{ka}) \\ &= \frac{d}{dt}\bigg|_{t=0} \varphi_f([\lambda, \mu, \kappa] \diamond [tE_{ka}, 0, 0]) \\ &= \frac{d}{dt}\bigg|_{t=0} \tilde{J}([\lambda + tE_{ka}, \mu, \kappa], 0)\, f((\lambda + tE_{ka})\Omega + \mu) \end{aligned}$$

$$= \left.\frac{d}{dt}\right|_{t=0} e^{\pi i\sigma\{S(\lambda+tE_{ka})\Omega\,{}^t(\lambda+tE_{ka})\}}\, e^{\pi i\sigma(S\kappa)}\, f((\lambda+tE_{ka})\Omega+\mu)$$
$$= e^{\pi i\sigma\{S(\kappa+\lambda\Omega\,{}^t\lambda)\}}\left\{2\pi i\left(\sum_{b=1}^{g}\sum_{l=1}^{h} S_{kl}\Omega_{ab}\lambda_{ab}\right)+\frac{\partial}{\partial\lambda_{ka}}\right\} f(W).$$

On the other hand,

$$\begin{aligned}
(\mathcal{L}_{\hat{X}_{lb}}\varphi_f)(g) &= \left.\frac{d}{dt}\right|_{t=0}\varphi_f(g\diamond\exp t\hat{X}_{lb})\\
&= \left.\frac{d}{dt}\right|_{t=0}\varphi_f([\lambda,\mu,\kappa]\diamond[0,tE_{lb},0])\\
&= \left.\frac{d}{dt}\right|_{t=0}\varphi_f([\lambda,\mu+tE_{lb},\kappa+t\lambda\,{}^tE_{kb}+tE_{kb}\,{}^t\lambda])\\
&= e^{\pi i\sigma\{S(\kappa+\lambda\Omega\,{}^t\lambda)\}}\left.\frac{d}{dt}\right|_{t=0} e^{2\pi it\sigma(S\lambda\,{}^tE_{lb})} f(\lambda\Omega+(\mu+tE_{lb}))\\
&= e^{\pi i\sigma\{S(\kappa+\lambda\Omega\,{}^t\lambda)\}}\left\{2\pi i\left(\sum_{m=1} S_{lm}\lambda_{mb}\right)+\frac{\partial}{\partial\mu_{lb}}\right\} f(W).
\end{aligned}$$

Thus

$$\Big(\mathcal{L}_{X_{ka}}-\sum_{b=1}^{g}\Omega_{ab}\mathcal{L}_{\hat{X}_{kb}}\Big)\varphi_f(g)$$
$$= e^{\pi i\sigma\{S(\kappa+\lambda\Omega\,{}^t\lambda)\}}\left\{\frac{\partial}{\partial\lambda_{ka}}-\sum_{b=1}^{g}\Omega_{ab}\frac{\partial}{\partial\mu_{kb}}\right\} f(W)=0.$$

This completes the proof.

4. Relation of lattice representations to theta functions

In this section, we state the connection between the lattice representations and theta functions. As before, we write $V=\mathbb{R}^{(h,g)}\times\mathbb{R}^{(h,g)}\cong\mathbb{C}^{(h,g)}$, $L=\mathbb{Z}^{(h,g)}\times\mathbb{Z}^{(h,g)}$ and $\mathcal{M}$ is a positive symmetric half-integral matrix of degree h. The function $q_{\mathcal{M}}: L\longrightarrow\mathbb{R}/2\mathbb{Z}=[0,2)$ defined by

(4.1) $$q_{\mathcal{M}}((\xi,\eta)) := 2\,\sigma(\mathcal{M}\xi\,{}^t\eta),\quad (\xi,\eta)\in L$$

satisfies the condition

$$q_{\mathcal{M}}(\ell_0+\ell_1)\equiv q_{\mathcal{M}}(\ell_0)+q_{\mathcal{M}}(\ell_1)-2\sigma\{\mathcal{M}(\lambda_0\,{}^t\mu_1-\mu_0\,{}^t\lambda_1)\}\pmod 2$$

for all $\ell_0 = (\lambda_0, \mu_0) \in L$ and $\ell_1 = (\lambda_1, \mu_1) \in L$.

We consider the following normal subgroup Γ_L of $H_{\mathbf{R}}^{(g,h)}$:

$$\Gamma_L := \left\{ (\lambda, \mu, \kappa) \in H_{\mathbf{R}}^{(g,h)} \mid (\lambda, \mu) \in L,\ \kappa \in \mathbf{R}^{(h,h)} \right\}.$$

We let $\varphi_{\mathcal{M},q_{\mathcal{M}}} : \Gamma_L \longrightarrow \mathbf{C}_1^\times$ be the character of Γ_L defined by

$$\varphi_{\mathcal{M},q_{\mathcal{M}}}((\ell,\kappa)) = e^{2\pi i\,\sigma(\mathcal{M}\kappa)}\, e^{\pi i\, q_{\mathcal{M}}(\ell)}, \quad (\ell,\kappa) \in \Gamma_L.$$

For brevity, we write $G := H_{\mathbf{R}}^{(g,h)}$. We denote by $\mathcal{H}_{\mathcal{M},q_{\mathcal{M}}}$ the Hilbert space consisting of measurable functions $\phi : G \longrightarrow \mathbf{C}$ which satisfy the conditions (4.2) and (4.3):

(4.2) $\quad \phi((\ell,\kappa)\circ g) = \varphi_{\mathcal{M},q_{\mathcal{M}}}((\ell,\kappa))\,\phi(g)$ for all $(\ell,\kappa) \in \Gamma_L$ and $g \in G$.

$$\int_{\Gamma_L \backslash G} \|\phi(\dot g)\|^2\, d\dot g < \infty, \quad \dot g = \Gamma_L \circ g. \tag{4.3}$$

Then the *lattice representation*

$$\pi_{\mathcal{M},q_{\mathcal{M}}} := \mathrm{Ind}_{\Gamma_L}^{G}\, \varphi_{\mathcal{M},q_{\mathcal{M}}}$$

of G induced from the character $\varphi_{\mathcal{M},q_{\mathcal{M}}}$ is realized in $\mathcal{H}_{\mathcal{M},q_{\mathcal{M}}}$ as

$$(\pi_{\mathcal{M},q_{\mathcal{M}}}(g_0)\,\phi)(g) = \phi(gg_0), \quad g_0, g \in G,\ \phi \in \mathcal{H}_{\mathcal{M},q_{\mathcal{M}}}.$$

Let $\mathbf{H}_{\mathcal{M},q_{\mathcal{M}}}$ be the vector space consisting of measurable functions $F : V \longrightarrow \mathbf{C}$ satisfying the conditions (4.4) and (4.5):

$$F(\lambda+\xi, \mu+\eta) = e^{2\pi i\,\sigma\{\mathcal{M}(\xi\, {}^t\eta + \lambda\, {}^t\eta - \mu\, {}^t\xi)\}}\, F(\lambda,\mu) \tag{4.4}$$

for all $(\lambda,\mu) \in V$ and $(\xi,\eta) \in L$.

$$\int_{L\backslash V} \|F(\dot v)\|^2\, d\dot v = \int_{I_\lambda \times I_\mu} \|F(\lambda,\mu)\|^2\, d\lambda d\mu < \infty, \tag{4.5}$$

where

$$I_\lambda := \overbrace{[0,1]\times[0,1]\times\cdots\times[0,1]}^{(h\times g)-\text{times}} \subset \left\{ [\lambda,0,0] \mid \lambda \in \mathbf{R}^{(h,g)} \right\}$$

and

$$I_\mu := \overbrace{[0,1]\times[0,1]\times\cdots\times[0,1]}^{(h\times g)-\text{times}} \subset \left\{ [0,\mu,0] \mid \mu \in \mathbf{R}^{(h,g)} \right\}$$

Given $\phi \in \mathcal{H}_{\mathcal{M},q_{\mathcal{M}}}$ and a fixed element $\Omega \in H_g$, we put

$$
\begin{aligned}
(4.6)\qquad E_\phi(\lambda,\mu) &:= \phi((\lambda,\mu,0)), \quad \lambda,\mu \in \mathbf{R}^{(h,g)},\\
(4.7)\qquad F_\phi(\lambda,\mu) &:= \phi([\lambda,\mu,0]), \quad \lambda,\mu \in \mathbf{R}^{(h,g)},\\
(4.8)\qquad F_{\Omega,\phi}(\lambda,\mu) &:= e^{-2\pi i\,\sigma(\mathcal{M}\lambda\Omega\,{}^t\lambda)}\,F_\phi(\lambda,\mu), \quad \lambda,\mu \in \mathbf{R}^{(h,g)}.
\end{aligned}
$$

In addition, we put for $W = \lambda\Omega + \mu \in \mathbf{C}^{(h,g)}$,

$$
(4.9)\qquad \vartheta_{\Omega,\phi}(W) := \vartheta_{\Omega,\phi}(\lambda\Omega+\mu) := F_{\Omega,\phi}(\lambda,\mu).
$$

We observe that E_ϕ, F_ϕ, $F_{\Omega,\phi}$ are functions defined on V and $\vartheta_{\Omega,\phi}$ is a function defined on $\mathbf{C}^{(h,g)}$.

Proposition 4.1. *If $\phi \in \mathcal{H}_{\mathcal{M},q_{\mathcal{M}}}$, $(\xi,\eta) \in L$ and $(\lambda,\mu) \in V$, then we have the formulas*

$$
\begin{aligned}
(4.10)\quad E_\phi(\lambda+\xi,\mu+\eta) &= e^{2\pi i\,\sigma\{\mathcal{M}(\xi\,{}^t\eta+\lambda\,{}^t\eta-\mu\,{}^t\xi)\}}\,E_\phi(\lambda,\mu).\\
(4.11)\quad F_\phi(\lambda+\xi,\mu+\eta) &= e^{-4\pi i\,\sigma(\mathcal{M}\xi\,{}^t\mu)}\,F_\phi(\lambda,\mu).\\
(4.12)\quad F_{\Omega,\phi}(\lambda+\xi,\mu+\eta) &= e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi+2\lambda\Omega\,{}^t\xi+2\mu\,{}^t\xi)\}}\,F_{\Omega,\phi}(\lambda,\mu).
\end{aligned}
$$

If $W = \lambda\Omega + \eta \in \mathbf{C}^{(h,g)}$, then we have

$$
(4.13)\qquad \vartheta_{\Omega,\phi}(W+\xi\Omega+\eta) = e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi+2W\,{}^t\xi)\}}\,\vartheta_{\Omega,\phi}(W).
$$

Moreover, F_ϕ is an element of $\mathbf{H}_{\mathcal{M},q_{\mathcal{M}}}$.

Proof. We note that

$$
(\lambda+\xi,\mu+\eta,0) = (\xi,\eta,-\xi\,{}^t\mu+\eta\,{}^t\lambda)\circ(\lambda,\mu,0).
$$

Thus we have

$$
\begin{aligned}
E_\phi(\lambda+\xi,\mu+\eta) &= \phi((\lambda+\xi,\mu+\eta,0))\\
&= \phi((\xi,\eta,\ -\xi\,{}^t\mu+\eta\,{}^t\lambda)\circ(\lambda,\mu,0))\\
&= e^{2\pi i\sigma\{\mathcal{M}(\xi\,{}^t\eta+\lambda\,{}^t\eta-\mu\,{}^t\xi)\}}\,\phi((\lambda,\mu,0))\\
&= e^{2\pi i\sigma\{\mathcal{M}(\xi\,{}^t\eta+\lambda\,{}^t\eta-\mu\,{}^t\xi)\}}\,E_\phi(\lambda,\mu).
\end{aligned}
$$

This proves the formula (4.10). We observe that

$$
[\lambda+\xi,\mu+\eta,0] = (\xi,\eta,\ -\xi\,{}^t\mu-\mu\,{}^t\xi-\eta\,{}^t\xi)\circ[\lambda,\mu,0].
$$

Thus we have

$$
\begin{aligned}
F_\phi(\lambda+\xi,\mu+\eta) &= \phi([\lambda+\xi,\mu+\eta,0]) \\
&= e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\,{}^t\mu+\mu\,{}^t\xi+\eta\,{}^t\xi)\}} \\
&\quad \times e^{2\pi i\,\sigma(\mathcal{M}\xi\,{}^t\eta)}\,\phi([\lambda,\mu,0]) \\
&= e^{-4\pi i\,\sigma(\mathcal{M}\xi\,{}^t\mu)}\,\phi([\lambda,\mu,0]) \\
&= e^{-4\pi i\,\sigma(\mathcal{M}\xi\,{}^t\mu)}\,F_\phi(\lambda,\mu).
\end{aligned}
$$

This proves the formula (4.11). According to (4.11), we have

$$
\begin{aligned}
F_{\Omega,\phi}(\lambda+\xi,\mu+\eta) &= e^{-2\pi i\,\sigma\{\mathcal{M}(\lambda+\xi)\Omega\,{}^t(\lambda+\xi)\}}\,F_\phi(\lambda+\xi,\mu+\eta) \\
&= e^{-2\pi i\,\sigma\{\mathcal{M}(\lambda+\xi)\Omega\,{}^t(\lambda+\xi)\}} \\
&\quad \times e^{-4\pi i\,\sigma(\mathcal{M}\xi\,{}^t\mu)}\,F_\phi(\lambda,\mu) \\
&= e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi+2\lambda\Omega\,{}^t\xi+2\mu\,{}^t\xi)\}} \\
&\quad \times e^{-2\pi i\,\sigma(\mathcal{M}\lambda\Omega\,{}^t\lambda)}\,F_\phi(\lambda,\mu) \\
&= e^{-2\pi i\,\sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi+2\lambda\Omega\,{}^t\xi+2\mu\,{}^t\xi)\}}\,F_{\Omega,\phi}(\lambda,\mu).
\end{aligned}
$$

This proves the formula (4.12). The formula (4.13) follows immediately from the formula (4.12). Indeed, if $W=\lambda\Omega+\mu$ with $\lambda,\mu\in\mathbf{R}^{(h,g)}$, we have

$$
\begin{aligned}
\vartheta_{\Omega,\phi}(W+\xi\Omega+\eta) &= F_{\Omega,\phi}(\lambda+\xi,\mu+\eta) \\
&= e^{-2\pi i\sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi+2(\lambda\Omega+\mu)\,{}^t\xi)\}}\,F_{\Omega,\phi}(\lambda,\mu) \\
&= e^{-2\pi i\sigma\{\mathcal{M}(\xi\Omega\,{}^t\xi+2W\,{}^t\xi)\}}\,\vartheta_{\Omega,\phi}(W).
\end{aligned}
$$

Remark 4.2. The function $\vartheta_{\Omega,\phi}(W)$ is a theta function of level $2\mathcal{M}$ with respect to Ω if $\vartheta_{\Omega,\phi}$ is holomorphic. For any $\phi\in\mathcal{H}_{\mathcal{M},q_\mathcal{M}}$, the function $\vartheta_{\Omega,\phi}$ satisfies the well known transformation law of a theta function. In this sense, the lattice representation $(\pi_{\mathcal{M},q_\mathcal{M}},\mathcal{H}_{\mathcal{M},q_\mathcal{M}})$ is closely related to theta functions.

References

[C] P. Cartier, *Quantum Mechanical Commutation Relations and Theta Functions*, Proc. of Symposia in Pure Mathematics, **9**, A.M.S.(1966), 361-383.

[I] J. Igusa, *Theta Finctions*, Springer-Verlag, New York/Berlin , 1972.

[M1] D. Mumford, *Tata Lectures on Theta I*, Birkhäuser, Boston/Basel/Stuttgart, 1983.

[M2] ———, *Tata Lectures on Theta III*, Birkhäuser, Boston/Basel/Stuttgart, 1991.

[N] Y. Namikawa, *Toroidal Compactification of Siegel Spaces*, Lecture Notes in Math.,812, Springer-Verlag, New York/Berlin, 1980.

[Y1] J.-H. Yang, *Harmonic Analysis on the Quotient Spaces of Heisenberg Groups*, Nagoya Math. J. 123(1991), 103-117.

[Y2] ———, *Harmonic Analysis on the Quotient Spaces of Heisenberg Groups II* , J. Number Theory 49, No. 1(1994), 63-72.

[Y3] ———, *A decomposition theorem on differential polynomials of theta functions of high level*, to appear in Japanese J. of Mathematics, The Mathematical Society of Japan, New Series 22, No. 1 (1996).

[Y4] ———, *Lattice Representations of Heisenberg groups*, preprint

[Y5] ———, *Unitary Representations of the Heisenberg group* $H_R^{(g,h)}$, preprint.

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STABLE AUTOMORPHIC FORMS

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1. INTRODUCTION

Originally the notion of stable automorphic forms was at first introduced in the symplectic group by E. Freitag [F]. Those automorphic forms were called *stable modular forms* by Freitag. Thereafter R. Weissauer investigated stable modular forms in the sense of Freitag intensively for the study of Eisenstein series [W]. In this paper, we generalize the concept of stable modular forms to that of stable automorphic forms on a semisimple real Lie group, a reductive real Lie group and a Jacobi group. The motivation of introducing the notion of stable automorphic forms is to investigate the geometric properties of finite or infinite dimensional arithmetic varieties associated with those automorphic forms.

This paper is organized as follows. In section 2, we review the notion of infinite dimensional algebraic varieties due to I. R. Shafarevich [Sh1-2]. We introduce the notion of stable functions. In section 3, we introduce the notion of stable automorphic forms on a semisimple real Lie group. As an example, we consider stable automorphic forms on an infinite dimensional symplectic group $Sp(\infty, \mathbb{R})$ introduced at first by E. Freitag [F]. Using his result (cf. Theorem 3.9), we characterize the so-called *universal Satake compactification.* In section 4, we formulate the notion of stable automorphic forms on a reductive real Lie group. Similiarly you may formulate the notion of stable automorphic forms on $G_\infty(\mathbb{A}_F)$ and stable automorphic representations, where $\mathbb{A}_F$ is the adéle ring of a number field F. In section 5, we introduce the notion of *stable Jacobi forms* on the Jacobi group. We investigate the geometric properties of the universal families of abelian varieties.

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Notations. We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers and the field of complex numbers respectively. $\mathbb{Z}^+$ and $\mathbb{Z}_+$ denote the set of all positive integers and the set of all nonnegative integers respectively. The symbol ":=" means that the expression on the right is the definition of that on the left. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix A, $\sigma(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose of M. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. E_n denotes the identity matrix of degree n.

2. Preliminaries

First we review the notion of infinite dimensional algebraic groups due to I. R. Shafarevich (cf. [K] and [Sh1-2]).

Definition 2.1. By an infinite dimensional algebraic variety over a field k we mean the inductive limit X of a directed system (X_i, f_{ij}) of algebraic varieties over the field k, where $f_{ij} : X_i \longrightarrow X_j \,(i < j)$ are closed embeddings. We write $X := \overset{\lim}{\longrightarrow} X_i$.

Throughout this paper, we shall consider only the case where the set of indices is the set $\mathbb{Z}^+$ of all positive integers. Each of the X_i will be considered to be equipped with its Zariski topology and we endow X with the topology of the inductive limit where a set $Z \subset X$ is closed if and only if its preimage in each X_i is closed. In particular, each X_i is closed in X.

Definition 2.2. A continous mapping $f : X \longrightarrow Y$ of two infinite dimensional algebraic varieties is called a *morphism* if for any X_i in the system (X_i) defining X, there exist at least one Y_j in the system (Y_j) defining Y such that $f(X_i) \subset Y_j$ and the restriction $f : X_i \longrightarrow Y_j$ is a morphism of finite dimensional algebraic varieties. Irreducibility and connectedness of an infinite dimensional algebraic variety are defined as irreducibilty and connectedness of the corresponding topological space.

Definition 2.3. An infinite dimensional algebraic variety G with a group structure is called an *infinite dimensional algebraic group* if the inverse mapping $x \longmapsto x^{-1}$ and the multiplication $(x, y) \longmapsto xy$ are morphisms.

In a similiar way, we may define the notions of infinite dimensional smooth manifolds, infinite dimensional complex manifolds, infinite dimensional real or complex Lie groups and so on with a usual topology and suitable morphisms.

Let X be an infinite dimensional space with its directed system (X_i, f_{ij}). Let V be a fixed finite dimensional complex vector space. We assume that

(I) to each X_i there is given the vector space C_i of functions on X_i with values in V

and that

(II) there is given an inverse system (C_i, Φ_{ij}) of linear maps $\Phi_{ij} : C_j \longrightarrow C_i \,(i < j)$ such that

$$\Phi_{ik} = \Phi_{ij} \circ \Phi_{jk} \quad \text{for all } i < j < k.$$

Now we let

$$\begin{aligned} C := & \varprojlim (C_i, \Phi_{ij}) \\ = & \left\{ (f_k) \in \prod_{i \in \mathbb{Z}+} C_i \mid \Phi_{ij}(f_j) = f_i \text{ for all } i < j \right\} \end{aligned}$$

be the inverse limit of the system (C_i, Φ_{ij}). Elements of C are called *stable functions.*

3. Stable automorphic forms on a semisimple real Lie group

3.1. Stable automorphic forms

Let G be an infinite dimensional semisimple real Lie group with its inductive system (G_i, ϕ_{ij}) of semisimple real Lie groups G_i and the group monomorphisms $\phi_{ij} : G_i \longrightarrow G_j \,(i < j)$. We fix a finite dimensional complex vector space V.

We now assume that

(I) there is given a sequence (K_i) of comapct subgroups such that each K_i is a maximal compact subgroup of G_i and $\phi_{ij}(K_i) \subset K_j$ for all $i < j$.

(II) there is given a sequence (Γ_i) such that each Γ_i is a discrete subgroup of G_i and $\phi_{ij}(\Gamma_i) \subset \Gamma_j$ for all $i < j$.

(III) there is given a sequence (ρ_i) such that each ρ_i is a representation of K_i on V compatible with the morphisms ϕ_{ij}, that is, if $i < j$, then $\rho_j(\phi_{ij}(k)) = \rho_i(k)$ for all $k \in K_i$.

For each $i \in \mathbb{Z}^+$, we let $A(\Gamma_i, \rho_i)$ be the complex vector space of all automorphic forms of type (ρ_i, Γ_i). We recall [Bo] that if $f \in A(\Gamma_i, \rho_i)$, then f satisfies the following conditions (1)-(3) :

(1) $f(\gamma g k) = \rho_i(k)^{-1} f(g)$ for all $k \in K_i$, $g \in G_i$ and $\gamma \in \Gamma_i$.

(2) f is $Z(\mathfrak{g}_i)$-finite.

(3) f satisfies a suitable growth condition.

Here $Z(\mathfrak{g}_i)$ denotes the center of the universal enveloping algebra $U(\mathfrak{g}_i)$ of the Lie algebra $\mathfrak{g}_i$ of G_i.

We also assume that

(IV) there is a sequence (L_{ij}) of linear maps

$$L_{ij} : A(\Gamma_j, \rho_j) \longrightarrow A(\Gamma_i, \rho_i), \quad i < j$$

satisfying the conditions

$$L_{ik} = L_{ij} \circ L_{jk} \quad \text{for all } i < j < k.$$

Elements of the inverse limit

(3.1) $$A := \varprojlim A(\Gamma_i, \rho_i)$$

are called *stable automorphic forms* on an infinite dimensional semisimple real Lie group G. If there is no confusion, we briefly say stable automorphic forms.

We put

$$K := \varinjlim K_i, \quad \Gamma := \varinjlim \Gamma_i \quad \text{and} \quad \rho := \varinjlim \rho_i.$$

We call ρ a *stable representation* of K or simply a *stable representation*. It is easy to see that a stable automorphic form f in A satisfies the following conditions (S1)-(S3):

(S1) $f(\gamma g k) = \rho(k)^{-1} f(g)$ for all $k \in K$, $g \in G$ and $\gamma \in \Gamma$.

(S2) f is $Z(\mathfrak{g})$-finite.

(S3) f satisfies a suitable growth condition.

$Z(\mathfrak{g}) = \xrightarrow{\lim} Z(\mathfrak{g}_i)$ denotes the center of the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of an infinite dimensional semisimple Lie group G.

Theorem 3.1. The dimension of A is finite.

Proof. The proof follows from the definition of A and the fact that the dimension of $A(\Gamma_i, \rho_i)$ for each i is finite due to Harish-Chandra [H-C]. □

3.2. Example

For each $n \in \mathbb{Z}^+$, we let

$$G_n := Sp(n, \mathbb{R}), \quad K_n := U(n), \quad \Gamma_n := Sp(n, \mathbb{Z})$$

be the symplectic group of degree n, the unitary group of degree n and the Siegel modular group of degree n respectively. And we put $G_0 = K_0 = \Gamma_0 = \{\text{identity}\}$.

Let $\rho := (\rho_n)$ be a stable representation of $K := U(\infty)$. For each $n \in \mathbb{Z}^+$, we let $A(\rho_n)$ be the vector space of automorphic forms of type (ρ_n, Γ_n). For each $n \in \mathbb{Z}^+$, we extend ρ_n to the complexification $GL(n, \mathbb{C})$ of K_n and also denote by ρ_n the extension of ρ_n to $GL(n, \mathbb{C})$. We note that each coset space $G_n/K_n\,(n \in \mathbb{Z}^+)$ is a hermitian symmetric space of noncompact type and is biholomorphic to the Siegel upper half plane

$$H_n := \left\{ Z \in \mathbb{C}^{(n,n)} \mid Z = {}^tZ, \ \ \mathrm{Im}\, Z > 0 \right\}. \tag{3.2}$$

We recall that G_n acts on H_n transitively by

$$g < Z > := (AZ + B)(CZ + D)^{-1}, \tag{3.3}$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_n$ and $Z \in H_n$. Thus G_n/K_n is identified with H_n via

$$G_n/K_n \ni gK_n \longmapsto g < iE_n > \in H_n.$$

Now for each $n \in \mathbb{Z}^+$, we define the automorphic factor $J_n : G_n \times H_n \longrightarrow GL(V)$ by

$$J_n(g, Z) := \rho_n(CZ + D), \tag{3.4}$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_n$ and $Z \in H_n$.

For all $m, n \in \mathbb{Z}^+$ with $m < n$, we define the *Siegel operator* $L_{m,n}$ on $A(\rho_n)$ by

$$(L_{m,n} f)(g) := J_m(g, iE_m)^{-1} \lim_{t \longrightarrow \infty} J_n(g_t, iE_n) f(g_t), \tag{3.5}$$

where $f \in A(\rho_n)$ and $g_t \in G_n$ is defined by

$$g_t := \begin{pmatrix} A & 0 & B & 0 \\ 0 & t^{1/2}E_{n-m} & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & t^{-1/2}E_{n-m} \end{pmatrix}, \qquad t > 0 \tag{3.6}$$

for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_m$.

Proposition 3.2. The limit in (3.5) exists and $L_{m,n}$ is a linear mapping of $A(\rho_n)$ into $A(\rho_m)$.

Proof. □

For each $n \in \mathbb{Z}^+$, we denote by $[\Gamma_n, \rho_n]$ the vector space of Siegel modular forms on H_n of type ρ_n. We recall that a Siegel modular form f in $[\Gamma_n, \rho_n]$ is a holomorphic function $f : H_n \longrightarrow V$ satisfying the condition

$$f(\gamma < Z >) = \rho_n(CZ + D)f(Z), \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n. \tag{3.7}$$

For $n = 1$, f requires a cuspidal condition. We have the well-known classical *Siegel operator* $\Phi_{m,n} : [\Gamma_n, \rho_n] \longrightarrow [\Gamma_m, \rho_m]$ defined by

$$(\Phi_{m,n} f)(Z) := \lim_{t \longrightarrow \infty} f \begin{pmatrix} Z & 0 \\ 0 & itE_{n-m} \end{pmatrix}, \quad Z \in H_m. \tag{3.8}$$

We observe that (3.8) is well defined and is a linear mapping.

For an element $F \in A(\rho_n)$, we define the function $P_n F$ on H_n by

$$(P_n F)(g < iE_n >) := J_n(g, iE_n)F(g), \tag{3.9}$$

where $g \in G_n$. Then $P_n F$ satisfies the condition (3.7). For an element $f \in [\Gamma_n, \rho_n]$, we define the function $Q_n f$ on G_n by

$$(Q_n f)(g) := J_n(g, iE_n)^{-1} f(g < iE_n >), \quad g \in G_n. \tag{3.10}$$

It is easy to see that the image of $[\Gamma_n, \rho_n]$ under Q_n is contained in $A(\rho_n)$ and that Q_n is a linear mapping of $[\Gamma_n, \rho_n]$ into $A(\rho_n)$. From now on, we denote by $A_h(\rho_n)$ the image of $[\Gamma_n, \rho_n]$ under Q_n.

Let $\mathfrak{g}_n$ be the Lie algebra of G_n and $\mathfrak{g}_n^{\mathbb{C}}$ its complexification. Then

$$\mathfrak{g}_n^{\mathbb{C}} = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \;\middle|\; B = {}^tB, \;\; C = {}^tC \right\}.$$

We let $\hat{J}_n := iJ_n$ with $J_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$. We define an involution σ_n of G_n by

$$\sigma_n(g) := \hat{J}_n g \hat{J}_n^{-1}, \quad g \in G_n.$$

The differential map $d\sigma_n = \mathrm{Ad}(\hat{J}_n)$ of σ_n extends complex linearly to the complexification $\mathfrak{g}_n^{\mathbb{C}}$ of $\mathfrak{g}_n$. $\mathrm{Ad}(\hat{J}_n)$ has 1 and -1 as eigenvalues. The (+1)-eigenspace of $\mathrm{Ad}(\hat{J}_n)$ is given by

$$\mathfrak{k}_n^{\mathbb{C}} := \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \;\middle|\; {}^tA + A = 0, \; B = {}^tB \right\}.$$

We note that $\mathfrak{k}_n^{\mathbb{C}}$ is the complexification of the Lie algebra $\mathfrak{k}_n$ of a maximal compact subgroup $K = G_n \cap SO(2n, \mathbb{R}) \cong U(n)$ of G_n. The (-1)-eigenspace of $\mathrm{Ad}(\hat{J}_n)$ is given by

$$\mathfrak{p}_n^{\mathbb{C}} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \;\middle|\; A = {}^tA,\ B = {}^tB \right\}.$$

We observe that $\mathfrak{p}_n^{\mathbb{C}}$ is not a Lie algebra. But $\mathfrak{p}_n^{\mathbb{C}}$ has the following decomposition

$$\mathfrak{p}_n^{\mathbb{C}} = \mathfrak{p}_{n,+} \oplus \mathfrak{p}_{n,-},$$

where

$$\mathfrak{p}_{n,+} = \left\{ \begin{pmatrix} X & iX \\ iX & -X \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \;\middle|\; X = {}^tX \right\}$$

and

$$\mathfrak{p}_{n,-} = \left\{ \begin{pmatrix} Y & -iY \\ -iY & -Y \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \;\middle|\; Y = {}^tY \right\}.$$

We observe that $\mathfrak{p}_{n,+}$ and $\mathfrak{p}_{n,-}$ are abelian subalgebras of $\mathfrak{g}_n^{\mathbb{C}}$.

Proposition 3.3. A function F in $A_h(\rho_n)$ is characterized by the following conditions:

(3.11) $F(\gamma g k) = \rho_n(k)^{-1} F(g)$ for all $\gamma \in \Gamma_n$, $g \in G_n$ and $k \in K_n$.

(3.12) $X^- F = 0$ for all $X^- \in \mathfrak{p}_{n,-}$.

(3.13) For any $M \in G_n$, the function $\psi : G_n \longrightarrow V$ defined by

$$\psi(g) := \rho_n(Y^{-\frac{1}{2}}) F(Mg), \quad g \in G_n, \quad g < iE_n >:= X + iY$$

is bounded in the domain $Y \geq Y_0 > 0$ for some $Y_0 = {}^tY_0 > 0$.

Proposition 3.4. The mapping P_n and Q_n are compatible with the Siegel operators $L_{m,n}$ and $\Phi_{m,n}$ ($m < n$). That is, for any $m, n \in \mathbb{Z}^+$ with $m < n$, we have

$$L_{m,n} \circ Q_n = Q_m \circ \Phi_{m,n} \quad \text{on } [\Gamma_n, \rho_n] \tag{3.14}$$

and

$$P_m \circ L_{m,n} = \Phi_{m,n} \circ P_n \quad \text{on } A_h(\rho_n). \tag{3.15}$$

Proof. □

Using the Siegel operator $\Phi_{m,n}$, we define the inverse limit

$$[\Gamma,\, \rho] := \varprojlim [\Gamma_n,\, \rho_n]. \tag{3.16}$$

For $n \in \mathbb{Z}^+$, we put

$$M_n := \oplus_\tau [\Gamma_n,\, \tau],$$

where τ runs over all isomorphism classes of irreducible rational finite dimensional representations of the general linear group $GL(n, \mathbb{C})$ of degree n. For $n = 0$, we set $M_0 := \mathbb{C}$. For an irreducible finite dimensional representation $\tau = (\lambda_1, \cdots, \lambda_n)$ of $GL(n, \mathbb{C})$ with $\lambda_1 \geq \cdots \geq \lambda_n$, $\lambda_i \in \mathbb{Z}\,(1 \leq i \leq n)$, the integer $k(\tau) := \lambda_n$ is called the *weight* of τ.

For $n \in \mathbb{Z}^+$, we define

$$M_n^* := \oplus_{\tau:\mathrm{even}} [\Gamma_n, \tau],$$

where τ runs over all isomorphism classes of irreducible rational finite dimensional even representations of $GL(n, \mathbb{C})$ such that the highest weight $\lambda(\tau)$ of τ is even, i.e., $\lambda(\tau) \in (2\mathbb{Z})^n$. For $n = 0$, we also set $M_0^* := \mathbb{C}$. Clearly the Siegel operator $\Phi_{m,n}$ maps M_n (resp. M_n^*) into M_m (resp. M_m^*).

We let

$$M := \varprojlim M_n \quad \text{and} \quad M^* := \varprojlim M_n^*.$$

It is easy to see that both M and M^* have commutative ring structures compatible with the Siegel operators $\Phi_{*,*}$. Obviously M^* is a subring of M.

Proposition 3.5.

$$M = \oplus_\rho [\Gamma,\, \rho],$$

where ρ runs over all stable irreducible representations.

Proof. □

Definition 3.6. Elements of M are called *stable modular forms* and elements of M^* are called *even stable modular forms.*

The concept of stable modular forms was first introduced by E. Freitag [F]. Thereafter the study of stable modular forms was intensively investigated by R. Weissauer [W].

3.3. Application to algebraic geometry

In this subsection, we give an application of stable automorphic forms to algebraic geometry.

First of all, for any two nonnegative integers $k, l \in \mathbb{Z}_+$ with $k < l$, we define the mapping $\varphi_{kl} : H_k \longrightarrow H_l$ by

$$(3.17) \qquad \varphi_{kl}(Z) := \begin{pmatrix} Z & 0 \\ 0 & iE_{l-k} \end{pmatrix}, \quad Z \in H_k.$$

We let

$$(3.18) \qquad H := \varinjlim H_k$$

be the inductive limit of the direct system (H_k, φ_{kl}). For the convenience of the reader, we write down G, Γ and K explicitly. For any $k, l \in \mathbb{Z}_+$ with $k < l$, we define the mapping $\pi_{kl} : G_k \longrightarrow G_l$ by

$$(3.19) \qquad \pi_{kl}\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) := \begin{pmatrix} A & 0 & B & 0 \\ 0 & E_{l-k} & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & E_{l-k} \end{pmatrix}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_k$$

and also the mapping $\rho_{kl} : \Gamma_k \longrightarrow \Gamma_l$ by (3.19) with $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_k$. We recall that for $k, l \in \mathbb{Z}_+$ with $k < l$, the mapping $u_{kl} : U(k) \longrightarrow U(l)$ defined by

$$(3.20) \qquad u_{kl}(A + iB) := \begin{pmatrix} A & 0 \\ 0 & E_{l-k} \end{pmatrix} + i \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \quad A + iB \in U(k)$$

yields the inductive limit $K := U(\infty)$.

Lemma 3.7. Let k and l be two nonnegative integers with k, l. Then for any $\gamma \in \Gamma_k$ and $Z \in H_k$, we have

$$(3.21) \qquad \varphi_{kl}(\gamma < Z >) = \rho_{kl}(\gamma) < \varphi_{kl}(Z) > .$$

Proof. (3.21) follows from an easy computation. □

For $k \in \mathbb{Z}_+$, we let $\mathbb{A}_k := \Gamma_k \backslash H_k$ be the Siegel modular variety of degree k. For any $k, l \in \mathbb{Z}_+$ with $k < l$, we define the mapping $s_{kl} : \mathbb{A}_k \longrightarrow \mathbb{A}_l$ by

$$(3.22) \qquad s_{kl}([Z]) := [\varphi_{kl}(Z)] = \left[\begin{pmatrix} Z & 0 \\ 0 & iE_{l-k} \end{pmatrix}\right],$$

where $[Z] \in \mathbb{A}_k$ with $Z \in H_k$ and $[Z]$ denotes the equivalence class of Z. According to Lemma 3.7, (3.22) is well defined. We let

$$(3.23) \qquad \mathbb{A} := \varinjlim \mathbb{A}_k$$

be the inductive limit of the system $(\mathbb{A}_k, s_{kl})$.

Proposition 3.8. G acts on H transitively and Γ acts on H properly discontinously. H is isomorphic to G/K. And we have

$$\mathbb{A} = \Gamma \backslash G/K.$$

For $d \in \mathbb{Z}_+$, we let $[\Gamma_n, d]$ be the vector space of all Siegel modular forms on H_n of weight d. We review some properties of the Siegel operator $\Phi_{n-1,n} : [\Gamma_n, d] \longrightarrow [\Gamma_{n-1}, d]$ (cf. (3.8)). According to the theory of singular modular forms in [F] and [R], $\Phi_{n-1,n}$ is injective if $n > 2d$ and $\Phi_{n-1,n}$ is an isomorphism if $n > 2d+1$. H. Maass [M] proved that $\Phi_{n-1,n}$ is an isomorphism if d is even and $d > 2n$.

For $n \in \mathbb{Z}_+$, we put

$$A_n := \oplus_{d=0}^{\infty}[\Gamma_n, d] \quad \text{for } n \geq 1 \quad \text{and} \quad A_0 := \mathbb{C}. \tag{3.24}$$

Then A_n is a $\mathbb{Z}_+$-graded ring which is integrally closed and of finite type over $\mathbb{C} := [\Gamma_n, 0]$. We observe that for $m < n$, the Siegel operator $\Phi_{m,n}$ maps M_n into M_m preserving the weights and that $\Phi_{m,n}$ is a ring homomorphism of A_n into A_m. Thus (A_n, $\Phi_{m,n}$) forms an inverse system of rings over $\mathbb{Z}_+$. We let

$$A := \varprojlim A_n \tag{3.25}$$

be the inverse limit of the system $(A_n, \Phi_{m,n})$. That is,

$$A = \left\{ (f_k) \in \prod_{l \in \mathbb{Z}_+} A_l \mid \Phi_{k,l}(f_l) = f_k \quad \text{for any } k < l \right\}.$$

If $f = (f_n) \in A$, then for each $n \in \mathbb{Z}_+$, we write

$$f_n = \sum_{d=0}^{\infty} f_{n,d}, \qquad f_{n,d} \in [\Gamma_n, d]. \tag{3.26}$$

We note that $\Phi_{m,n}(f_{n,d}) = f_{m,d}$ for all $m, n \in \mathbb{Z}_+$ with $m < n$. For each $d \in \mathbb{Z}_+$, the sequence $\{ (f_{k,d}) \mid k \in \mathbb{Z}_+ \}$ is called a *stable sequence* of weight d. We denote by S_d the complex vector space consisting of all stable sequences of weight d. Then it is easy to see that

$$A = \oplus_{d=0}^{\infty} S_d. \tag{3.27}$$

Then A is a $\mathbb{Z}_+$-graded ring. It is known that $\dim_{\mathbb{C}} S_d = \dim_{\mathbb{C}} [\Gamma_n, d]$ if $k > 2d$ (cf. [F], p, 203).

Let S be a positive definite symmetric, unimodular even integral matrix of degree m. Then we define the theta series $\theta_S^{(n)}(Z),\ Z \in H_n$ by

$$\theta_S^{(n)}(Z) := \sum_{G \in \mathbb{Z}^{(m,n)}} e^{\pi i \sigma(S[G]Z)}, \quad Z \in H_n. \tag{3.28}$$

Then we can show that $\theta_S^{(n)}(Z)$ is a Siegel modular form of weight $m/2$ on Γ_n.

We state the result obtained by E. Freitag [F].

Theorem 3.9 ([F], Theorem 2.5). A is a polynomial ring in a countably infinite set of indeterminates over $\mathbb{C}$ given by

$$A = \mathbb{C}[\theta_S^{(n)} \mid n \in \mathbb{Z}_+],$$

where S runs over the set of all equivalence classes of irreducible positive definite symmetric, unimodular even integral matrices.

Remark 3.10. The homogeneous quotient field $Q(A)$ of A is a rational function field with countably infinitely many variables. But in general A_n is not a polynomial ring. It is well known that the homogeneous function field $Q(A_n)$ of A_n is an algebraic function field with the transcendence degree $\frac{1}{2}n(n+1)$.

For any $m, n \in \mathbb{Z}_+$ with $k < l$, the Siegel operator $\Phi_{m,n} : A_n \longrightarrow A_m$ induces the morphism $\Phi_{m,n}^* : \mathrm{Proj}\, A_m \longrightarrow \mathrm{Proj}\, A_n$ of projective schemes. The Satake compactification $\mathbb{A}_n^* = \mathrm{Proj}\, \mathbb{A}_n$ of $\mathbb{A}_n$ contains $\mathbb{A}_n$ as a Zariski open dense subset. As a set, $\mathbb{A}_n^*$ is the disjoint union of $\mathbb{A}_n$ and its rational boundary components, i.e.,

$$\mathbb{A}_n^* = \mathbb{A}_n \cup \mathbb{A}_{n-1} \cup \cdots \mathbb{A}_1 \cup \mathbb{A}_0, \quad \mathbb{A}_0 = \{\infty\}.$$

Obviously ($\mathbb{A}_n$, $\Phi_{m,n}^*$) forms an inductive system of schemes over $\mathbb{Z}_+$. We let

$$\mathbb{A}^* := \varinjlim \mathbb{A}_n^* \tag{3.29}$$

be the inductive limit of ($\mathbb{A}_n^*$, $\Phi_{m,n}^*$). We call the infinite dimensional variety $\mathbb{A}^*$ the *universal Satake compactification*.

Theorem 3.11. The universal Satake compactification $\mathbb{A}$ has the following properties:

(1) $\mathbb{A}^* = \mathrm{Proj}\, A$.

(2) $\mathbb{A}^*$ is an infinite dimensional projective variety which contains $\mathbb{A}$ as a Zariski open dense subset. So $\mathbb{A}^*$ is also called the *Satake compactification* of $\mathbb{A}$.

Now we shall describe the analytic local ring of the image of the boundary point in $\mathbb{A}_n^*$ under f_n^*, where $f_n^* : \mathbb{A}_n^* \longrightarrow \mathbb{A}^* \,(\, n \in \mathbb{Z}^+ \,)$ is the canonical morphism. Let $[Z_k] \in \mathbb{A}_k \,(\, 0 \leq k \leq n-1,\; Z_k \in H_k \,)$ be a boundary point in $\mathbb{A}_n^* - \mathbb{A}_n$. We set $Z_{k,\infty}^* := f_k^*([Z_k])$.

Theorem 3.12. The analytic local ring at $Z_{k,\infty}^*$ in $\mathbb{A}^*$ consists of all sequences $(f_m)_{m=0}^{\infty}$ with $\Phi_{m,m+1} f_{m+1} = f_m$ such that each $f_{k+m}\,(m \geq 1)$ is a convergent series of type

$$f_{k+m}(Z, W_m, \Omega_m) = \sum_{T_m} \phi_{T_m}(Z, W_m) e^{2\pi i \sigma(T_m \Omega_m)},$$

where Z is an element in a sufficiently small open neighborhood V of Z_k in H_k invariant under the action of Γ_k, $W_m \in \mathbb{C}^{(k,m)}$, $\Omega_m \in H_m$ and T_m runs over the set of all semi-positive symmetric half-integral matrices of degree m. In addition, each $\phi_{T_m}(Z, W_m)(m \geq 1)$ is a Jacobi form of weight 0 and index T_m defined on $V \times \mathbb{C}^{(k,m)}$. The notion of Jacobi forms will be discussed in section 5.

4. Stable automorphic forms on a reductive real Lie group

4.1. Automorphic forms on a reductive real group

In this subsection, we review the notion of automorphic forms on a reductive real group due to A. Borel and H. Jacquet [B-J].

Let G be a connected reductive group over $\mathbb{Q}$, $\mathcal{Z}$ the greatest $\mathbb{Q}$-split torus of the center of G and K a maximal compact subgroup of $G(\mathbb{R})$. Let $\mathfrak{g}$ be the Lie algebra of $G(\mathbb{R})$, $U(\mathfrak{g})$ its universal enveloping algebra over $\mathbb{C}$ and $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. Let $\mathcal{H} := \mathcal{H}(G(\mathbb{R}), K)$ be the Hecke algebra of $G(\mathbb{R})$ and K.

Definition 4.1. Let Γ be an arithmetic subgroup of $G(\mathbb{Q})$. A smooth function $f : G(\mathbb{R}) \longrightarrow \mathbb{C}$ is called an *automorphic form* for (Γ, K) or simply for Γ if it satisfies the following conditions:

(a) $f(\gamma x) = f(x)$ for all $\gamma \in \Gamma$ and $x \in G(\mathbb{R})$.

(b) There exists an idempotent $\xi \in \mathcal{H}$ such that $f * \xi = f$.

(c) There exists an ideal J of finite codimension of $Z(\mathfrak{g})$ which annihilates f.

(d) f is slowly increasing.

An automorphic form satisfying these conditions (a)-(d) is said to be of type (ξ, J). We denote by $A(\Gamma, \xi, J, K)$ the space of all automorphic forms for (Γ, K) of type (ξ, J).

Theorem 4.2 ([B-J], [HC]). The space $A(\Gamma, \xi, J, K)$ is finite dimensional.

4.2. Stable automorphic forms on a reductive group

We consider a sequence $\{ G_i \mid i \in \mathbb{Z}^+ \}$ of connected reductive groups over $\mathbb{Q}$ such that $G_\infty(\mathbb{R})$ is an infinite dimensional reductive real group with its inductive system $(G_i(\mathbb{R}), \phi_{ij})$ of reductive real groups $G_i(\mathbb{R})$ and the group monomorphisms $\phi_{ij} : G_i(\mathbb{R}) \longrightarrow G_j(\mathbb{R})$ for $i < j$.

We now assume that the following:

(I) There is given a sequence (K_i) of compact subgroups such that each K_i is a maximal compact subgroup of $G_i(\mathbb{R})$ and $\phi_{ij}(K_i) \subset K_j$ for all $i < j$.

(II) There is given an inductive system (Γ_i, ψ_{ij}) of arithmetic subgroups such that each Γ_i is an arithmetic subgroup of $G_i(\mathbb{Q})$ and $\psi_{ij} : \Gamma_i \longrightarrow \Gamma_j$ is a closed embedding for all $i < j$.

(III) There is given an inductive system $(\mathcal{H}_i, \alpha_{ij})$ of the Hecke algebras $\mathcal{H}_i$, a sequence (ξ_i) of idempotents ξ_i in $\mathcal{H}_i$ and a sequence (J_i) of ideals of finite codimension in $\mathcal{H}_i$ such that each $\alpha_{ij} : \mathcal{H}_i \longrightarrow \mathcal{H}_j$ is an algebra monomorphism, $\alpha_{ij}(\xi_i) = \xi_j$ and $\alpha_{ij}(J_i) \subset J_j$ for all $i < j$.

Moreover we assume that

(IV) there is a sequence (Φ_{ij}) of linear maps

$$\Phi_{ij} : A(\Gamma_j, \xi_j, J_j, K_j) \longrightarrow A(\Gamma_i, \xi_i, J_i, K_i), \quad i < j$$

satisfying the conditions

$$\Phi_{ik} = \Phi_{ij} \circ \Phi_{jk} \quad \text{for all } i < j < k.$$

Elements of the inverse limit

$$A_\infty := \varprojlim A(\Gamma_i, \xi_i, J_i, K_i)$$

are called *stable automorphic forms* on an infinite dimensional reductive group $G_\infty(\mathbb{R})$ or simply *stable automorphic forms.* In a similiar way, we may define the notion of stable automorphic forms on $G_\infty(\mathbb{A}_F)$ and stable automorphic representations, where $\mathbb{A}_F$ denotes the ring of adéles of a number field F.

Remark 4.3. An example of stable automorphic forms on an infinite dimensional general linear group $GL(\infty, \mathbb{R})$ was given by D. Grenier [G] using the analogues of the Siegel operators (cf. [G]).

5. Stable Jacobi forms

In this section, we introduce the notion of stable Jacobi forms. Since the so-called Jacobi group is associated to such forms and is a nonreductive Lie group, we need a new approach different from the previous ones.

5.1. Jacobi forms

First of all, we fix two positive integers m and n. We let

$$H_{\mathbb{R}}^{(n,m)} := \left\{ (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\, {}^t\lambda \text{ symmetric} \right\}$$

be the Heisenberg group endowed with the following multiplication law

$$(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') := (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda').$$

We define the semidirect product of $Sp(n,\mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$

$$G_{n,m}^J := Sp(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)} \tag{5.1}$$

endowed with the following multiplication law

$$\begin{aligned}&(M,(\lambda,\mu;\kappa)) \cdot (M',(\lambda',\mu';\kappa'))\\ &:= (MM', (\tilde{\lambda} + \lambda', \tilde{\mu} + \mu',; \kappa + \kappa' + \tilde{\lambda}\, {}^t\mu' - \tilde{\mu}\, {}^t\lambda')),\end{aligned}$$

where $M, M' \in Sp(n,\mathbb{R})$ and $(\tilde{\lambda}, \tilde{\mu}) := (\lambda,\mu)M'$. The group $G_{n,m}^J$ is called the *Jacobi group* of degree (n,m). It is easy to see that the Jacobi group $G_{n,m}^J$ acts on the homogeneous space $H_n \times \mathbb{C}^{(m,n)}$ transitively by

$$(M,(\lambda,\mu;\kappa)) \cdot (Z,W) := (M < Z >, (W + \lambda Z + \mu)(CZ+D)^{-1}), \tag{5.2}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(Z,W) \in H_n \times \mathbb{C}^{(m,n)}$. We recall that $Sp(n,\mathbb{R})$ acts on H_n transitively by

$$M < Z > := (AZ+B)(CZ+D)^{-1}, \quad Z \in H_n.$$

From now on, for brevity, we write $G_n := Sp(n,\mathbb{R})$ and $H_{n,m} := H_n \times \mathbb{C}^{(m,n)}$. Let ρ be a rational representation of $GL(n,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbb{R}^{(m,m)}$ be a symmetric half-integral semi-positive definite matrix of degree m. Let $C^\infty(H_{n,m}, V_\rho)$ be the algebra of all C^∞ functions on $H_{n,m}$ with values in V_ρ. For $f \in C^\infty(H_{n,m}, V_\rho)$, we define

$$\begin{aligned}&(f|_{\rho,\mathcal{M}}[(M,(\lambda,\mu;\kappa))])(Z,W)\\ &:= e^{-2\pi i \sigma(\mathcal{M}[W+\lambda Z+\mu](CZ+D)^{-1}C)} \times e^{2\pi i\sigma(\mathcal{M}(\lambda Z^t\lambda + 2\lambda^t W + (\kappa+\mu^t\lambda)))}\\ &\quad \times \rho(CZ+D)^{-1} f(M < Z >, (W+\lambda Z+\mu)(CZ+D)^{-1}),\end{aligned} \tag{5.3}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R}),\ (\lambda,\mu,\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(Z,W) \in H_{n,m}$.

Definition 5.1. Let ρ and $\mathcal{M}$ be as above. We let $\Gamma_n := Sp(n,\mathbb{Z})$ be the Siegel modular group of degree n. Let

$$H_{\mathbb{Z}}^{(n,m)} := \{\, (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)} \mid \lambda,\mu \in \mathbb{Z}^{(m,n)},\ \kappa \in \mathbb{Z}^{(m,m)}\,\}.$$

A *Jacobi form* of index $\mathcal{M}$ with respect to ρ on Γ_n is a holomorphic function $f \in C^\infty(H_{n,m}, V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho,\mathcal{M}}[\tilde{\gamma}] = f$ for all $\tilde{\gamma} \in \Gamma_{n,m}^J := \Gamma_n \ltimes H_{\mathbb{Z}}^{(n,m)}$.

(B) f has a Fourier expansion of the following form :

$$f(Z,W) = \sum_{\substack{T\geq 0 \\ \text{half-integral}}} \sum_{R\in\mathbb{Z}^{(n,m)}} c(T,R)\cdot e^{2\pi i\,\sigma(TZ)}\cdot e^{2\pi i\sigma(RW)}$$

with $c(T,R) \neq 0$ only if $\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}\,{}^tR & \mathcal{M} \end{pmatrix} \geq 0.$

If $n \geq 2$, the condition (B) is superfluous by Köcher principle (cf. [Z] Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma_n)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ_n. Ziegler (cf. [Z] Theorem 1.8 or [E-Z] Theorem 1.1) proved that the vector space $J_{\rho,\mathcal{M}}(\Gamma_n)$ is finite dimensional. For more results on Jacobi forms with $n > 1$ and $m > 1$, we refer to [D], [Kr], [Y1-6] and [Z].

Definition 5.2. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ is said to be a *cusp* (or *cuspidal*) form if $\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}\,{}^tR & \mathcal{M} \end{pmatrix} > 0$ for any T, R with $c(T,R) \neq 0$. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ is said to be *singular* if it admits a Fourier expansion such that a Fourier coefficient $c(T,R)$ vanishes unless $\det \begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}\,{}^tR & \mathcal{M} \end{pmatrix} = 0$.

Example 5.3. Let $S \in \mathbb{Z}^{(2k,2k)}$ be a symmetric, positive definite unimodular even integral matrix and $c \in \mathbb{Z}^{(2k,m)}$. We define the theta series

$$\vartheta_{S,c}^{(n)}(Z,W) := \sum_{\lambda\in\mathbb{Z}^{(2k,n)}} e^{\pi i\{\sigma(S\lambda Z\,{}^t\lambda)+2\sigma({}^tcS\lambda\,{}^tW)\}},\quad Z\in H_n,\ W\in\mathbb{C}^{(m,n)}.$$

We put $\mathcal{M} := \frac{1}{2}\,{}^tcSc$. We assume that $2k < n + \operatorname{rank}(\mathcal{M})$. Then it is easy to see that $\vartheta_{S,c}^{(n)}$ is a singular form in $J_{k,\mathcal{M}}(\Gamma_n)$ (cf. [Z], p. 212).

5.2. The Lie algebra of the Jacobi group $G_{n,m}^J$

We first note that the stabilizer $K^J_{n,m}$ of $G^J_{n,m}$ at $(iE_n, 0)$ (cf. (5.2)) is given by

$$K^J_{n,m} := \left\{ (k, (0,0;\kappa)) \in G^J_{n,m} \mid k \in U(n), \quad \kappa \in \mathbb{R}^{(m,m)} \right\}.$$

So the homogeneous space $G^J_{n,m}/K^J_{n,m}$ is identified with $H_{n,m}$ via

$$G^J_{n,m}/K^J_{n,m} \ni g\, K^J_{n,m} \longmapsto g \cdot (iE_n, 0) \in H_{n,m}, \qquad g \in G^J_{n,m}.$$

The Lie algebra $\mathfrak{g}_n$ of G_n has a Cartan decomposition

$$\mathfrak{g}_n = \mathfrak{k}_n \oplus \mathfrak{p}_n, \tag{5.4}$$

where

$$\mathfrak{k}_n := \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \;\middle|\; A + {}^tA = 0, \quad B = {}^tB \right\},$$
$$\mathfrak{p}_n := \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \;\middle|\; A = {}^tA, \quad B = {}^tB \right\}.$$

Then $\theta_n := \mathrm{Ad}\,(\hat{J}_n)$ is a Cartan involution because

$$-B(W, \theta_n(W)) = -B(X,X) + B(Y,Y) > 0$$

for all $W = X + Y, \ X \in \mathfrak{k}_n, \ Y \in \mathfrak{p}_n$. Here B denotes the Cartan-Killing form for $\mathfrak{g}_n$. Indeed,

$$B(X,Y) = 2(n+1)\,\sigma(XY), \qquad X, Y \in \mathfrak{g}_n.$$

The vector space $\mathfrak{p}_n$ is identified with the tangent space of H_n at iE_n. The correspondence

$$\frac{1}{2}\begin{pmatrix} B & A \\ A & -B \end{pmatrix} \longmapsto A + iB$$

yields an isomorphism of $\mathfrak{p}_n$ onto $\mathrm{Symm}^2(\mathbb{C}^n)$. The Lie algebra $\mathfrak{g}^J_{n,m}$ of the Jacobi group $G^J_{n,m}$ has a decomposition

$$\mathfrak{g}^J_{n,m} = \mathfrak{k}^J_{n,m} + \mathfrak{p}^J_{n,m}, \tag{5.5}$$

where

$$\mathfrak{k}^J_{n,m} = \left\{ (X, (0,0;\kappa) \mid X \in \mathfrak{k}_n, \ \kappa = {}^t\kappa \in \mathbb{R}^{(m,m)} \right\},$$
$$\mathfrak{p}^J_{n,m} = \left\{ (Y, (P,Q;0) \mid Y \in \mathfrak{p}_n, \ P, Q \in \mathbb{R}^{(m,n)} \right\}.$$

We observe that $\mathfrak{k}^J_{n,m}$ is the Lie algebra of $K^J_{n,m}$. Thus the tangent space of the homogeneous space $H_{n,m} \cong G^J_{n,m}/K^J_{n,m}$ at $(iE_n, 0)$ is given by

$$\mathfrak{p}^J_{n,m} := \mathfrak{p}_n \oplus (\mathbb{R}^{(m,n)} \times \mathbb{R}^{(m,n)}) \cong \mathfrak{p}_n \oplus \mathbb{C}^{(m,n)}.$$

We define a complex structure I^J on the tangent space $\mathfrak{p}^J_{n,m}$ of $H_{n,m}$ at iE_n by

$$\text{(5.6)} \qquad I^J\left(\begin{pmatrix} Y & X \\ X & -Y \end{pmatrix}, (P,Q)\right) := \left(\begin{pmatrix} X & -Y \\ -Y & -X \end{pmatrix}, (Q,-P)\right).$$

Identifying $\mathbb{R}^{(m,n)} \times \mathbb{R}^{(m,n)}$ with $\mathbb{C}^{(m,n)}$ via

$$(P,Q) \longmapsto iP + Q, \quad P, Q \in \mathbb{R}^{(m,n)},$$

we may regard the complex structure I^J as a real linear map

$$I^J(X+iY, Q+iP) = (-Y+iX, -P+iQ),$$

where $X+iY \in \mathrm{Symm}^2(\mathbb{C}^n)$, $Q+iP \in \mathbb{C}^{(m,n)}$. I^J extends complex linearly on the complexification $\mathfrak{p}^J_{n,m,\mathbb{C}} = \mathfrak{p}^J_{n,m} \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathfrak{p}^J_{n,m}$. $\mathfrak{p}^J_{n,m,\mathbb{C}}$ has a decomposition

$$\text{(5.7)} \qquad \mathfrak{p}^J_{n,m,\mathbb{C}} = \mathfrak{p}^J_{n,m,+} \oplus \mathfrak{p}^J_{n,m,-},$$

where $\mathfrak{p}^J_{n,m,+}$ (resp. $\mathfrak{p}^J_{n,m,-}$) denotes the $(+i)$-eigenspace (resp. $(-i)$-eigenspace) of I^J. Precisely, both $\mathfrak{p}^J_{n,m,+}$ and $\mathfrak{p}^J_{n,m,-}$ are given by

$$\mathfrak{p}^J_{n,m,+} = \left\{ \left(\begin{pmatrix} X & iX \\ iX & -X \end{pmatrix}, (P, iP)\right) \,\middle|\, X \in \mathrm{Symm}^2(\mathbb{C}^n), \quad P \in \mathbb{C}^{(m,n)} \right\}$$

and

$$\mathfrak{p}^J_{n,m,-} = \left\{ \left(\begin{pmatrix} X & -iX \\ -iX & -X \end{pmatrix}, (P, -iP)\right) \,\middle|\, X \in \mathrm{Symm}^2(\mathbb{C}^n), \quad P \in \mathbb{C}^{(m,n)} \right\}.$$

With respect to this complex structure I^J, we may say that f is *holomorphic* if and only if $\xi f = 0$ for all $\xi \in \mathfrak{p}^J_{n,m,-}$.

5.3. Characterization of Jacobi Forms as Functions on the Jacobi Group $G^J_{n,m}$

In this subsection, we lift a Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ to a smooth function Φ_f on the Jacobi group $G^J_{n,m}$ and characterize the lifted function Φ_f on $G^J_{n,m}$.

We recall that for given ρ and $\mathcal{M}$, the canonical automorphic factor $J_{\mathcal{M},\rho} : G^J_{n,m} \times H_{n,m} \longrightarrow GL(V_\rho)$ is given by

$$J_{\mathcal{M},\rho}(g,(Z,W)) = e^{-2\pi i \sigma(\mathcal{M}[W+\lambda Z+\mu](CZ+D)^{-1}C)} \times e^{2\pi i \sigma(\mathcal{M}(\lambda Z\,{}^t\lambda + 2\lambda\,{}^tW + \kappa + \mu\,{}^t\lambda))} \rho(CZ+D)^{-1},$$

where $g = (M,(\lambda,\mu;\kappa)) \in G^J_{n,m}$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_n$. It is easy to see that the automorphic factor $J_{\mathcal{M},\rho}$ satisfies the cocycle condition:

$$(5.8) \qquad J_{\mathcal{M},\rho}(g_1g_2,(Z,W)) = J_{\mathcal{M},\rho}(g_2,(Z,W))\, J_{\mathcal{M},\rho}(g_1, g_2 \cdot (Z,W))$$

for all $g_1,\ g_2 \in G^J_{n,m}$ and $(Z,W) \in H_{n,m}$.

Since the space $H_{n,m}$ is diffeomorphic to the homogeneous space $G^J_{n,m}/K^J_{n,m}$, we may lift a function f on $H_{n,m}$ with values in V_ρ to a function Φ_f on $G^J_{n,m}$ with values in V_ρ in the following way. We define the lifting

$$(5.9) \qquad L_{\rho,\mathcal{M}} : \mathcal{F}(H_{n,m},V_\rho) \longrightarrow \mathcal{F}(G^J_{n,m},V_\rho), \quad L_{\rho,\mathcal{M}}(f) := \Phi_f$$

by

$$\begin{aligned}\Phi_f(g) :&= (f|_{\rho,\mathcal{M}}[g])(iE_n,0) \\ &= J_{\mathcal{M},\rho}(g,(iE_n,0))\, f(g\cdot(iE_n,0)),\end{aligned}$$

where $g \in G^J_{n,m}$ and $\mathcal{F}(H_{n,m},V_\rho)$ (resp. $\mathcal{F}(G^J_{n,m},V_\rho)$) denotes the vector space consisting of functions on $H_{n,m}$ (resp. $G^J_{n,m}$) with values in V_ρ.

For brevity, we set $\Gamma := \Gamma_n = Sp(n,\mathbb{Z})$ and $\Gamma^J_{n,m} = \Gamma \ltimes H^{(n,m)}_{\mathbb{Z}}$. We let $\mathcal{F}^\Gamma_{\rho,\mathcal{M}}$ be the space of all functions f on $H_{n,m}$ with values in V_ρ satisfying the transformation formula

$$(5.10) \qquad f|_{\rho,\mathcal{M}}[\gamma] = f \quad \text{for all } \gamma \in \Gamma^J_{n,m}.$$

And we let $\mathcal{F}^\Gamma_{\rho,\mathcal{M}}(G^J_{n,m})$ be the space of functions $\Phi : G^J_{n,m} \longrightarrow V_\rho$ on $G^J_{n,m}$ with values in V_ρ satisfying the following conditions (5.11) and (5.12):

$$(5.11) \quad \Phi(\gamma g) = \Phi(g) \quad \text{for all } \gamma \in \Gamma^J_{n,m} \text{ and } g \in G^J_{n,m}.$$

(5.12) $\Phi(g\,r(k,\kappa)) = e^{2\pi i\sigma(\mathcal{M}\kappa)}\rho(k)^{-1}\Phi(g), \quad \forall\ r(k,\kappa) := [k,(0,0;\kappa)] \in K^J_{n,m}.$

Lemma 5.4. The space $\mathcal{F}^{\Gamma}_{\rho,\mathcal{M}}$ is isomorphic to the space $\mathcal{F}^{\Gamma}_{\rho,\mathcal{M}}(G^J_{n,m})$ via the lifting $L_{\rho,\mathcal{M}}$.

Proof. Let $f \in \mathcal{F}^{\Gamma}_{\rho,\mathcal{M}}$. If $\gamma \in \Gamma^J_{n,m}$, $g \in G^J_{n,m}$ and $r(k,\kappa) = [k,(0,0;\kappa)] \in K^J_{n,m}$, then we have

$$\begin{aligned}\Phi_f(\gamma g) &= (f|_{\rho,\mathcal{M}}[\gamma g])(iE_n,0)\\ &= ((f|_{\rho,\mathcal{M}}[\gamma])|_{\rho,\mathcal{M}}[g])(iE_n,0)\\ &= (f|_{\rho,\mathcal{M}}[g])(iE_n,0) \qquad (\text{since } f \in \mathcal{F}^{\Gamma}_{\rho,\mathcal{M}})\\ &= \Phi_f(g)\end{aligned}$$

and

$$\begin{aligned}\Phi_f(g\,r(k,\kappa)) &= J_{\mathcal{M},\rho}(g\,r(k,\kappa),(iE_n,0))\,f(g\,r(k,\kappa)\cdot(iE_n,0))\\ &= J_{\mathcal{M},\rho}(r(k,\kappa),(iE_n,0))\,J_{\mathcal{M},\rho}(g,(iE_n,0))\,f(g\cdot(iE_n,0))\\ &= e^{2\pi i\sigma(\mathcal{M}\kappa)}\rho(k)^{-1}\Phi_f(g).\end{aligned}$$

Here we identified $k = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in K_n$ with $A + iB \in U(n)$.

Conversely, if $\Phi \in \mathcal{F}^{\Gamma}_{\rho,\mathcal{M}}(G^J_{n,m})$, $G^J_{n,m}$ acting on $H_{n,m}$ transitively, we may define a function f_Φ on $H_{n,m}$ by

$$f_\Phi(g\cdot(iE_n,0)) := J_{\mathcal{M},\rho}(g,(iE_n,0))^{-1}\Phi(g). \tag{5.13}$$

Let $\gamma \in \Gamma^J_{n,m}$ and $(Z,W) = g\cdot(iE_n,0)$ for some $g \in G^J_{n,m}$. Then using the cocycle condition (5.8), we have

$$\begin{aligned}(f_\Phi|_{\rho,\mathcal{M}}[\gamma])(Z,W) &= J_{\mathcal{M},\rho}(\gamma,(Z,W))f_\Phi(\gamma\cdot(Z,W))\\ &= J_{\mathcal{M},\rho}(\gamma,g\cdot(iE_n,0))\,f_\Phi(\gamma g\cdot(iE_n,0))\\ &= J_{\mathcal{M},\rho}(\gamma,g\cdot(iE_n,0))J_{\mathcal{M},\rho}(\gamma g,(iE_n,0))^{-1}\Phi(\gamma g)\\ &= J_{\mathcal{M},\rho}(\gamma,g\cdot(iE_n,0))\,J_{\mathcal{M},\rho}(\gamma,g\cdot(iE_n,0))^{-1}\\ &\quad J_{\mathcal{M},\rho}(g,(iE_n,0))^{-1}\Phi(g)\\ &= J_{\mathcal{M},\rho}(g,(iE_n,0))^{-1}\Phi(g)\\ &= f_\Phi(g\cdot(iE_n,0)) = f_\Phi(Z,W).\end{aligned}$$

This completes the proof. □

Now we have the following two algebraic representations $T_{\rho,\mathcal{M}}$ and $\dot{T}_{\rho,\mathcal{M}}$ of $G^J_{n,m}$ defined by

$$T_{\rho,\mathcal{M}}(g)f := f|_{\rho,\mathcal{M}}[g^{-1}], \quad g \in G^J_{n,m}, \ f \in \mathcal{F}^\Gamma_{\rho,\mathcal{M}} \tag{5.14}$$

and

$$\dot{T}_{\rho,\mathcal{M}}(g)\Phi(g') := \Phi(g^{-1}g'), \quad g, g' \in G^J_{n,m}, \quad \Phi \in \mathcal{F}^\Gamma_{\rho,\mathcal{M}}(G^J_{n,m}). \tag{5.15}$$

Then it is easy to see that these two models $T_{\rho,\mathcal{M}}$ and $\dot{T}_{\rho,\mathcal{M}}$ are intertwined by the lifting $\mathrm{L}_{\rho,\mathcal{M}}$.

Proposition 5.5. The vector space $J_{\rho,\mathcal{M}}(\Gamma_n)$ is isomorphic to the space $\tilde{A}_{\rho,\mathcal{M}}(\Gamma^J_{n,m})$ of smooth functions Φ on $G^J_{n,m}$ with values in V_ρ satisfying the following conditions:

(1a) $\Phi(\gamma g) = \Phi(g)$ for all $\gamma \in \Gamma^J_{n,m}$.
(1b) $\Phi(g\, r(k,\kappa)) = e^{2\pi i\sigma(\mathcal{M}\kappa)}\rho(k)^{-1}\Phi(g)$ for all $g \in G^J_{n,m}$, $r(k,\kappa) \in K^J_{n,m}$.
(2) $Y^-\Phi = 0$ for all $Y^- \in \mathfrak{p}^J_{n,m,-}$.
(3) For all $M \in Sp(n,\mathbb{R})$, the function $\psi : G^J_{n,m} \longrightarrow V_\rho$ defined by

$$\psi(g) := \rho(Y^{-\frac{1}{2}})\, \Phi(Mg), \quad g \in G^J_{n,m}$$

is bounded in the domain $Y \geq Y_0$. Here $g\cdot(iE_n, 0) = (Z, W)$ with $Z = X+iY$, $Y > 0$.

Corollary 5.6. $J^{\text{cusp}}_{\rho,\mathcal{M}}(\Gamma_n)$ is isomorphic to the subspace $\tilde{A}^0_{\rho,\mathcal{M}}(\Gamma^J_{n,m})$ of $\tilde{A}_{\rho,\mathcal{M}}(\Gamma^J_{n,m})$ with the condition (3') the function $g \longmapsto \Phi(g)$ is bounded.

5.4. Stable Jacobi forms

Let $\mathcal{M}$ be a fixed positive definite symmetric half-integral matrix of degree m. Let $\rho_\infty := (\rho_n)$ be a stable representation of $GL(\infty,\mathbb{C})$. That is, for each $n \in \mathbb{Z}^+$, ρ_n is a finite dimensional rational representation of $GL(n,\mathbb{C})$ and ρ_∞ is compatible with the embeddings $\alpha_{kl} : GL(k,\mathbb{C}) \longrightarrow GL(l,\mathbb{C})\,(\,k < l\,)$ defined by

$$\alpha_{kl}(A) := \begin{pmatrix} A & 0 \\ 0 & E_{l-k} \end{pmatrix}, \quad A \in GL(k,\mathbb{C}), \quad k < l.$$

For $k, l \in \mathbb{Z}^+$ with $k < l$, we define the mapping $\Phi_{l,k,\mathcal{M}}$ of $\tilde{A}_{\rho_l,\mathcal{M}}(\Gamma^J_{l,m})$ into the functions on $G^J_{k,m}$ by

$$(\,\Phi_{l,k,\mathcal{M}} F\,)(g) := J_{\mathcal{M},\rho_k}(g,\, (iE_k, 0)) \lim_{t \longrightarrow \infty} J_{\mathcal{M},\rho_l}(g_t,\, (iE_l, 0))^{-1} F(g_t), \tag{5.16}$$

where $F \in \tilde{A}_{\rho_l,\mathcal{M}}(\Gamma^J_{l,m})$, $g = (M, (\lambda, \mu; \kappa)) \in G^J_{k,m}$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_k$ and

$$g_t := \left(\begin{pmatrix} A & 0 & B & 0 \\ 0 & t^{1/2}E_{l-k} & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & t^{-1/2}E_{l-k} \end{pmatrix}, ((\lambda, 0), (\mu, 0); \kappa) \right) \in G^J_{l,m}$$

Proposition 5.7. The limit (5.16) always exists and the image of $\tilde{A}_{\rho_l,\mathcal{M}}(\Gamma^J_{l,m})$ under $\Phi_{l,k,\mathcal{M}}$ is contained in $\tilde{A}_{\rho_k,\mathcal{M}}(\Gamma^J_{k,m})$. Obviously the mapping

$$\Phi_{l,k,\mathcal{M}} : \tilde{A}_{\rho_l,\mathcal{M}}(\Gamma^J_{l,m}) \longrightarrow \tilde{A}_{\rho_k,\mathcal{M}}(\Gamma^J_{k,m})$$

is a linear mapping.

Proof. □

The mapping $\Phi_{l,k,\mathcal{M}}$ is called the *Siegel-Jacobi operator*. For any $n \in \mathbb{Z}^+$, we put

(5.17) $$\tilde{A}_{n,\mathcal{M}} := \oplus_\rho \tilde{A}_{\rho,\mathcal{M}}(\Gamma^J_{n,m}),$$

where ρ runs over all isomorphism classes of irreducible rational representations of $GL(n, \mathbb{C})$. For $n = 0$, we set $\tilde{A}_{0,\mathcal{M}} := \mathbb{C}$.

For each $n \in \mathbb{Z}^+$, we put

(5.18) $$\tilde{A}^*_{n,\mathcal{M}} := \oplus_{\rho_*} \tilde{A}(\rho_*, \mathcal{M}),$$

where ρ_* runs over all isomorphism classes of irreducible rational representations of $GL(n, \mathbb{C})$ with highest weight $\lambda(\rho_*) \in (2\mathbb{Z})^n$. It is obvious that if $k < l$, then the Siegel-Jacobi operator $\Phi_{l,k,\mathcal{M}}$ maps $\tilde{A}_{l,\mathcal{M}}$ (resp. $\tilde{A}^*_{l,\mathcal{M}}$) into $\tilde{A}_{k,\mathcal{M}}$ (resp. $\tilde{A}^*_{k,\mathcal{M}}$).

We let

(5.19) $$\tilde{A}_{\infty,\mathcal{M}} := \varprojlim \tilde{A}_{k,\mathcal{M}} \quad \text{and} \quad \tilde{A}^*_{\infty,\mathcal{M}} := \varprojlim \tilde{A}^*_{k,\mathcal{M}}$$

be the inverse limits of ($\tilde{A}_{k,\mathcal{M}}$, $\Phi_{l,k,\mathcal{M}}$) and ($\tilde{A}^*_{k,\mathcal{M}}$, $\Phi_{l,k,\mathcal{M}}$) respectively.

Proposition 5.8. $\tilde{A}_{\infty,\mathcal{M}}$ has a commutative ring structure compatible with the Siegel-Jacobi operators. Obviously $A^*_{\infty,\mathcal{M}}$ is a subring of $A_{\infty,\mathcal{M}}$.

Proof. □

For a stable irreducible representation $\rho_\infty = (\rho_n)$ of $GL(\infty, \mathbb{C})$, we define

$$\tilde{A}_{\rho_\infty, \mathcal{M}} := \varprojlim \tilde{A}_{\rho_n, \mathcal{M}}(\Gamma^J_{n,m}). \tag{5.20}$$

Proposition 5.9. We have

$$\tilde{A}_{\infty, \mathcal{M}} = \oplus_{\rho_\infty} \tilde{A}_{\rho_\infty, \mathcal{M}},$$

where ρ_∞ runs over all isomorphism classes of stable irreducible representations of $GL(\infty, \mathbb{C})$.

Proof. □

Definition 5.10. Elements in $\tilde{A}_{\infty, \mathcal{M}}$ are called *stable automorphic forms* on $G^J_{\infty,m}$ of index $\mathcal{M}$ and elements of $\tilde{A}^*_{\infty, \mathcal{M}}$ are called *even stable automorphic forms* on $G^J_{\infty,m}$ of index $\mathcal{M}$.

For $n \geq 1$, we define

$$\tilde{A}_n := \oplus_\rho \oplus_{\mathcal{M}} \tilde{A}_{\rho, \mathcal{M}}(\Gamma^J_{n,m}), \tag{5.21}$$

where ρ runs over all isomorphism classes of irreducible rational representations of $GL(n, \mathbb{C})$ and $\mathcal{M}$ runs over all equivalence classes of positive definite symmetric, half-integral matrices of any degree ≥ 1. We set $\tilde{A}_0 := \mathbb{C}$.

For $n \geq 1$, we also define

$$\tilde{A}^*_n := \oplus_{\rho_*} \oplus_{\mathcal{M}} \tilde{A}_{\rho_*, \mathcal{M}}(\Gamma^J_{n,m}), \tag{5.22}$$

where ρ_* runs over all isomorphism classes of irreducible rational representations of $GL(n, \mathbb{C})$ with highest weight $\lambda(\rho_*) \in (2\mathbb{Z})^n$ and $\mathcal{M}$ runs over all equivalence classes of positive definite symmetric half-integral matrices of any degree ≥ 1.

Let $\rho_\infty = (\rho_n)$ be a stable irreducible rational representation of $GL(\infty, \mathbb{C})$. For each irreducible rational representation ρ_n of $GL(n, \mathbb{C})$ appearing in ρ_∞, we put

$$\tilde{A}(\rho_n; \rho_\infty) := \oplus_{\mathcal{M}} \tilde{A}_{\rho_n, \mathcal{M}}(\Gamma^J_{n,m}), \tag{5.23}$$

where $\mathcal{M}$ runs over all equivalence classes of positive definite symmetric half-integral matrices of any degree ≥ 1. Clearly the Siegel-Jacobi operator $\Phi_{l,k} := \oplus_{\mathcal{M}} \Phi_{l,k,\mathcal{M}}\,(\,k < l\,)$ maps $\tilde{A}(\rho_l; \rho_\infty)$ into $\tilde{A}(\rho_k; \rho_\infty)$.

Using the Siegel-Jacobi operators, we can define the inverse limits

$$\tilde{A}(\rho_\infty) := \varprojlim \tilde{A}(\rho_n;\rho_\infty), \quad \tilde{A}_\infty := \varprojlim \tilde{A}_n \quad \text{and} \quad \tilde{A}^*_\infty := \varprojlim \tilde{A}^*_n.$$

Theorem 5.11.

$$\tilde{A}_\infty = \oplus_{\rho_\infty} \tilde{A}(\rho_\infty),$$

where ρ_∞ runs over all equivalence classes of stable irreducible representations of $GL(\infty,\mathbb{C})$.

Proof. □

Let $\rho_\infty = (\rho_n)$ be a stable representation of $GL(\infty,\mathbb{C})$. For $k,l \in \mathbb{Z}^+$ with $k<l$, we define the Siegel-Jacobi operator $\Psi_{l,k,\mathcal{M}} : J_{\rho_l,\mathcal{M}}(\Gamma_l) \longrightarrow J_{\rho_k,\mathcal{M}}(\Gamma_k)$ by

$$(5.24) \qquad (\Psi_{l,k,\mathcal{M}} f)(Z,W) := \lim_{t\to\infty} f\left(\begin{pmatrix} Z & 0 \\ 0 & itE_{l-k}\end{pmatrix}, (W,0)\right),$$

where $f \in J_{\rho_l,\mathcal{M}}(\Gamma_l)$ and $(Z,W) \in H_{k,m}$. For a more detail on $\Psi_{l,k,\mathcal{M}}$, we refer to [Y2] and [Z].

Proposition 5.12. Let $\rho_\infty = (\rho_n)$ be a stable representation of $GL(\infty,\mathbb{C})$. For a fixed positive definite symmetric half-integral matrix $\mathcal{M}$ of degree m, the liftings $L_{\rho_k,\mathcal{M}}\,(\,k\in\mathbb{Z}^+\,)$ are compatible with the action of the Siegel operators $\Phi_{l,k,\mathcal{M}}$ and $\Psi_{l,k,\mathcal{M}}$. That is, the following relation holds:

$$(5.25) \qquad \Phi_{l,k,\mathcal{M}} \circ L_{\rho_l,\mathcal{M}} = L_{\rho_k,\mathcal{M}} \circ \Psi_{l,k,\mathcal{M}}$$

for all $k,l \in \mathbb{Z}^+$ with $k<l$.

Proof. □

For a positive integer r with $r<n$, we let $V_\rho^{(r)}$ be the subspace of V_ρ spanned by the values $\{\,(\Psi_{n,r}f)(Z,W)\mid f\in J_{\rho,\mathcal{M}}(\Gamma_n),\ (Z,W)\in H_r\times\mathbb{C}^{(h,r)}\,\}$. Then $V_\rho^{(r)}$ is invariant under the action of the group

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & E_{n-r}\end{pmatrix} \,:\, a\in GL(r,\mathbb{C}) \right\} \cong GL(r,\mathbb{C}).$$

We can show that if $V_\rho^{(r)} \neq 0$ and (ρ,V_ρ) is irreducible, then $(\rho^{(r)},V_\rho^{(r)})$ is also irreducible.

Theorem 5.13. The action of the Siegel-Jacobi operator is compatible with that of the Hecke operator.

We refer to [Y14] for a precise detail on the Hecke operators and the proof of the above theorem.

Problem 5.14. Discuss the injectivity, surjectivity and bijectivity of the Siegel-Jacobi operator.

This problem was partially discussed by Yang [Y14] and Kramer [Kr2] in the special cases. For instance, Kramer [Kr2] showed that if n is arbitrary, $m = 1$ and $\rho : GL(n,\mathbb{C}) \longrightarrow \mathbb{C}^\times$ is a one-dimensional representation of $GL(n,\mathbb{C})$ defined by $\rho(a) := (\det(a))^k$ for some $k \in \mathbb{Z}^+$, then the Siegel-Jacobi operator

$$\Psi_{n,n-1} : J_{k,m}(\Gamma_n) \longrightarrow J_{k,m}(\Gamma_{n-1})$$

is surjective for $k \gg m \gg 0$.

Theorem 5.15. Let $1 \leq r \leq n-1$ and let ρ be an irreducible finite dimensional representation of $GL(n,\mathbb{C})$. Assume that $k(\rho) > n + r + \mathrm{rank}(\mathcal{M}) + 1$ and that k is even. Then

$$J^{\mathrm{cusp}}_{\rho^{(r)},\mathcal{M}}(\Gamma_r) \subset \Psi_{n,r}(J_{\rho,\mathcal{M}}(\Gamma_n)).$$

Here $J^{\mathrm{cusp}}_{\rho^{(r)},\mathcal{M}}(\Gamma_r)$ denotes the subspace consisting of all cuspidal Jacobi forms in $J_{\rho^{(r)},\mathcal{M}}(\Gamma_r)$.

Idea of Proof. For each $f \in J^{\mathrm{cusp}}_{\rho^{(r)},\mathcal{M}}(\Gamma_r)$, we can show by a direct computation that

$$\Psi_{n,r}(E^{(n)}_{\rho,\mathcal{M}}(Z,W;f)) = f,$$

where $E^{(n)}_{\rho,\mathcal{M}}(Z,W;f)$ is the Eisenstein series of Klingen's type associated with a cusp form f. For a precise detail, we refer to [Z]. □

Remark 5.16. Dulinski [Du] decomposed the vector space $J_{k,\mathcal{M}}(\Gamma_n)\,(k \in \mathbb{Z}^+)$ into a direct sum of certain subspaces by calculating the action of the Siegel-Jacobi operator on Eisenstein series of Klingen's type explicitly.

For two positive integers r and n with $r \leq n-1$, we consider the bigraded ring

$$J^{(r)}_{*,*}(\ell) := \oplus_{k=0}^{\infty} \oplus_{\mathcal{M}} J_{k,\mathcal{M}}(\Gamma_r(\ell))$$

and

$$M^{(r)}_{*}(\ell) := \oplus_{k=0}^{\infty} J_{k,0}(\Gamma_r(\ell)) = \oplus_{k=0}^{\infty}[\Gamma_r(\ell),k],$$

where $\Gamma_r(\ell)$ denotes the principal congruence subgroup of Γ_r of level ℓ and $\mathcal{M}$ runs over the set of all symmetric semi-positive half-integral matrices of degree m. Let

$$\Psi_{r,r-1,\mathcal{M},\ell} : J_{k,\mathcal{M}}(\Gamma_r(\ell)) \longrightarrow J_{k,\mathcal{M}}(\Gamma_{r-1}(\ell))$$

be the Siegel-Jacobi operator defined by (5.24). This induces the linear mapping

$$\Psi_{r,r-1,\ell} : J_{*,*}^{(r)}(\ell) \longrightarrow J_{*,*}^{(r-1)}(\ell).$$

Problem 5.17. Let $m = 1$. Investigate $\mathrm{Proj}\, J_{*,*}^{(r)}(\Gamma_r(\ell))$ over $M_*^{(r)}(\ell)$ and the quotient space

$$Y_r(\ell) := (\Gamma_r(\ell) \ltimes (\ell\mathbb{Z})^2)\backslash(H_r \times \mathbb{C}^r)$$

for $1 \leq r \leq n-1$.

The difficulty to this problem comes from the following facts (A) and (B):

(A) $J_{*,*}^{(r)}(\ell)$ is not finitely generated over $M_*^{(r)}(\ell)$.

(B) $J_{k,\mathcal{M}}^{\mathrm{cusp}}(\Gamma_r(\ell)) \neq \ker \Psi_{r,r-1,\mathcal{M},\ell}$ in general.

These are the facts different from the theory of Siegel modular forms. We remark that Runge ([Ru], pp. 190-194) discussed some parts about the above problem.

References

[B] W.L. Baily, *Satake's compactification of V_n*, Amer. J. Math. **80** (1958), 348-364.

[Bo] A. Borel, *Introduction to automorphic forms*, Proc. Sympos. Pure Math., vol. 9, Amer. Math. Soc., Providence, R.I. (1966), 199-210.

[B-J] A. Borel and H. Jacquet, *Automorphic forms and automorphic representations*, Proc. Symposia in Pure Math. **XXXIII, Part 1** (1979), 189-202.

[C] C.-L. Chai, *Compactification of Siegel moduli schemes*, Cambridge Univ. Press, 1985.

[D] J. Dulinski, *A decomposition theorem for Jacobi forms*, Math. Ann. **303** (1995), 473-498.

[E-Z] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progress in Math. vol. 55, Birkhäuser, 1985.

[F-C] G. Faltings and C.-L. Chai, *Degeneration of abelian varieties*, Springer-Verlag, Berlin-Heidelberg-New York, 1990.

[F] E. Freitag, *Stabile Modulformen*, Math. Ann. **230** (1977), 197-211.

[G] D. Grenier, *An analogue of Siegel's ϕ-operator for automorphic forms for $GL(n,\mathbb{Z})$*, Trans. Amer. Math. Soc. **331, No. 1** (1992), 463-477.

[H] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, 1977.

[HC] Harish-Chandra, *Automorphic forms on semi-simple Lie groups*, Notes by J.G.M. Mars, Lecture Notes in Math., vol. 62, Springer-Verlag, Berlin-Heidelberg-New York, 1968.

[I] J. Igusa, *A desingularization problem in the theory of Siegel modular functions*, Math. Ann. **168** (1967), 228-260.

[K] T. Kambayashi, *Reviews of [Sh1]*, Math. Reviews. # 5697 **58** (1979).

[Kr] J. Kramer, *An arithmetic theory of Jacobi forms in higher dimension*, J. reine angew. Math. **458** (1995), 157-182.

[M] H. Maass, *Über die Darstellung der Modulformen n-th Grades durch Poincarésche Reihnen*, Math. Ann. **123** (1951), 125-151.

[R] H. Resnikoff, *Automorphic forms of singular weight are singular forms*, Math. Ann. **215** (1975), 173-193.

[Ru] B. Runge, *Theta functions and Siegel-Jacobi functions*, Acta Math. **175** (1995), 165-196.

[S] I. Satake, *On the compactification of the Siegel spcae*, J. Indian Math. Soc. **20** (1956), 259-281.

[Sh1] I. R. Shafarevich, *On some infinite dimensional groups*, Rend. Mat. e. Appl.(5) **25** (1966), 208-212.

[Sh2] ———, *On some infinite dimensional groups. II*, Math. USSR. Izvestija **18, No. 1** (1982), 185-194.

[T] A. Terras, *Harmonic analysis on symmetric spaces and applications II*, Springer-Verlag, 1988.

[W] R. Weissauer, *Stabile Modulformen und Eisensteinreihen*, vol. 1219, Springer-Verlag, 1986.

[Y1] J.-H. Yang, *Some results on Jacobi forms of higher degree*, Proc. RIMS Symposium on Automorphic Forms and Associated Zeta Functions, Kyoto (1992), 35-50.

[Y2] ———, *The Siegel-Jacobi operator*, Abh. Math. Sem. Univ. Hamburg **63** (1993), 135-146.

[Y3] ———, *Remarks on Jacobi forms of higher degree*, Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, the Pyungsan Institute for Mathematical Sciences, Seoul (1993), 33-58.

[Y4] ———, *Singular Jacobi forms*, Trans. of American Math. Soc. **347, No. 6** (1995), 2041-2049.

[Y5] ———, *Construction of vector valued modular forms from Jacobi forms*, Canadian J. of Math. **47 (6)** (1995), 1329-1339.

[Y6] ———, *A geometrical theory of Jacobi forms of higher degree*, Proceedings of Symposium on Hodge Theory and Algebraic Geometry (edited by Tadao Oda), Sendai, Japan (1996), 125-147.

[Z] C. Ziegler, *Jacobi Forms of Higher Degree*, Abh. Math. Sem. Univ. Hamburg **59** (1989), 191-224.

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리만 가설에 관하여

양재현 / 평산 수학연구소 및 인하대

1. 머리말

소수는 수 중에서 가장 기본이 되는 수이다. 소수로써 거의 모든 수를 설명할 수 있기 때문이다. 오래 전부터 위대한 수학자들은 소수의 신비와 분포에 관하여 연구하여 왔다.
1859년에 리만[1]은 베를린 학술원의 회원으로 선정되었다. 베를린 학술원의 헌장에 의하면, 새로이 선출된 회원은 반드시 최근의 연구업적을 보고하게 되어 있었다. 그래서 리만은 『주어진 수보다 작은 소수의 개수에 관하여 (On the number of primes less than a given magnitude)』의 제목으로 보고서를 학술원에 제출하였다.(참고문헌 [12] 참조) 그는 이 보고서에서 리만 제타함수의 성질들을 열거하고 소위, "리만 가설 (the Riemann Hypothesis)" 을 제시하였다.

이미 이 전에 소수의 분포에 관하여 오일러[2], 르장드르[3], 가우스[4] 등의 위대한 수학자에 의하여 연구되었다. 오일러는 소수의 분포를 연구하기 위하여 아래의 제타함수

(1) $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ (단, s는 실수)

를 공부하였다. 그는

(2) $\zeta(s) = \prod_p (1-p^{-s})^{-1}$

의 관계식을 보였다. 여기서 $\prod_p$ 는 모든 소수 p 들의 곱을 나타낸다. 관계식 (2)는 「오일러 곱(Euler product)」이라고 불린다. 이 사실로부터 소수의 개수가 무한임을 알 수 있다.

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1) Georg Friedrich Bernhard Riemann (1826~1866)
2) Leonhard Euler (1707~1783)
3) Adrien Marie Legendre (1752~1833)
4) Carl Friedrich Gauss (1777~1855)

x를 주어진 양의 실수라고 하고

$$\pi(x) := |\{p \in \mathbb{Z}^+ \mid 2 \le p < x, \quad p\text{는 소수}\}|$$

라고 하자. 여기서 $\mathbb{Z}^+$는 모든 자연수들의 집합을 나타내고 $|S|$는 집합 S의 개수를 나타낸다. 오일러는

(3) $$\lim_{x \to \infty} \frac{\pi(x)}{\frac{x}{\log x}} = 1$$

이라는 것을 가설로 제시하였다. 오일러, 르장드르, 가우스와 같은 위대한 수학자들이 (3)을 증명하려고 시도하였지만 실패하였다. 1854년에 체비쉐프[5]는 논문집 『Memoires de l'Academie des Sciences de Saint Petersburg』에서

(4) $$A_1 < \frac{\pi(x)}{\frac{x}{\log x}} < A_2$$

의 등식을 증명하였다. (단, $0.992 < A_1 < 1$ 이고 $1 < A_2 < 1.105$ 임.) 그러나 체비쉐프는 (3)의 극한값이 존재한다는 사실은 증명하지 않았다.

1850년경에 리만은 (1)에서 실수 변수 s뿐만 아니라 복소수 변수 s까지 생각하였다. 그는 $\mathrm{Re}\, s > 1$을 만족하는 영역에서 $\zeta(s)$는 해석적 함수이고 해석적 접속(analytic continu-ation)을 지님을 증명하였다. 게다가 $\zeta(s)$의 함수방정식을 발견하였다. 끝으로 그는

$$0 = \zeta(-2) = \zeta(-4) = \zeta(-6) = \cdots = \zeta(-2n) = \cdots \quad (\text{단, } n\text{은 자연수})$$

임을 증명하고

(RH) "$\zeta(s)$의 다른 영점(zero)은 모두 $\mathrm{Re}\, s = \frac{1}{2}$의 선상에 놓여 있다."

라는 사실을 주장하였다. 그러나 리만은 이 주장을 증명하지 않았다. 그의 사후에 제타함수 $\zeta(s)$는 「리만 제타함수(the Riemann zeta function)」라고 불렸고 주장 (RH)는 『리만 가설』이라고 불렸다. 그 후 프랑스 수학자 Jacques Hadamard (1865~1963)와 Charles de la Vallée-Poussin (1866~1962) 등과 같은 유명한 수학자들이 리만 가설을 해결하려고 하였지만 실패하였다. 아직까지도 이 가설은 풀리지 않고 있다. 1941년에 프랑스 수학자

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5) Pafnuti L'vovich Chebyshev (1821~1894)

베이유[6]는 함수체(function field)인 경우에 (RH)를 증명하였고, 1949년에 유한체(finite field) 상에서 정의되는 대수다양체의 제타함수에 대하여 (RH)와 유사한 소위, 『베이유 가설(Weil conjecture)』을 제시하였다.(참고문헌 [16]과 [17] 참조) 그 후, 1974년에 벨기에 수학자 데리네[7]가 매끄러운 사영다양체(nosingular projective variety)인 경우에 베이유 가설이 옳다는 것을 증명하였다.(참고문헌 [1] 참조) 이 업적과 하지 이론의 업적으로 데리네는 1978년에 수학의 노벨상인 필즈상을 수상하였다. 1980년에 일반적인 다양체(complete variety)인 경우에 베이유 가설이 진실이라는 사실을 증명하였다.(참고문헌 [2] 참조)

리만 가설은 정수론 분야에서 중요한 『소수 정리 (the Prime Number Theorem)』와 아주 밀접한 관계가 있다. 가령, 주장 (3)은 $\zeta(1+it)\neq 0$ (단, $t\neq 0$ 인 실수) 이라는 주장과 동치이다.

본인은 이 강연에서 리만 가설의 내용을 쉽게 설명하고 소수 정리와의 연관성에 관하여 가능하면 쉽게 다루려고 한다. 또, 소수에 관한 여러 문제(가령, 골드바하[8] 가설, 쌍둥이 소수 짝의 문제, Bertrand[9]의 주장)들을 소개하겠다.

2. 리만 제타함수 $\zeta(s)$

리만 가설의 내용을 어느 정도 이해하기 위해서는 우선,

(ㄱ) 복소수(complex number)의 개념
(ㄴ) 해석적(解析的; analytic or holomorphic) 함수의 개념
(ㄷ) 유리형(meromorphic) 함수의 개념
(ㄹ) 해석적 접속(analytic continuation)의 개념

등의 기본적인 여러 개념을 알아야 한다.

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6) André Weil (1906~1998)
7) Pierre Deligne (1944~)
8) Christian Goldbach (1690~1764)
9) Joseph Louis François Bertrand (1822~1900)

상기의 개념을 간략하게 설명하겠습니다. 복소수의 개념은 여러분 모두가 잘 알고 있기 때문에 설명은 생략하겠습니다. 복소함수 $f(z)$ 가 z_0 의 근방에서 극한값

$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ 을 가질 때 함수 $f(z)$ 는 z_0 에서 해석적이다라고 한다. 영역(a region) D 의 모든 점에서 복소 함수 $f(z)$ 가 해석적일 때 $f(z)$ 는 D 상에서 해석적이다라고 한다. 그리고 $\frac{f(z)}{g(z)}$ (단, $f(z)$ 와 $g(z)$ 는 해석적 함수이고 $g(z) \neq 0$ 임)의 형태의 함수를 유리형 함수라고 한다. 복소 함수 $f(z)$ 가 z_0 의 근방에서

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots\cdots + \frac{a_1}{z-z_0} + \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

(단, $a_{-m} \neq 0$ 임)의 형태일 때 함수 $f(z)$ 는 z_0 에서 m 계의 극점 (*a pole of order m*)을 갖는다고 한다. 영역 $D(\subset C)$ 에서 정의되는 해석적 함수 $f(z)$ 가 주어져 있다고 하자. D 를 포함하는 영역 $E(\neq D)$ 상에 유리형 함수 $F(z)$ 가 존재하여 D 상에서는 $f(z) = F(z)$ 일 때 함수 $F(z)$ 를 $f(z)$ 의 해석적 접속이라고 한다. 예를 들면, 기하급수로 주어지는 함수

$$f(z) = \sum_{k=0}^{\infty} z^k$$

는 중심이 원점인 단위원 내부 $D = \{ z \in C \mid |z| < 1 \}$ 에서 정의되는 해석적 함수이다. 그런데 함수 $F(z) = \frac{1}{1-z}$ 는 $E = \{ z \in C \mid z \neq 1 \}$ 상에서 정의되는 해석적 함수이며 D 상에서는 $f(z) = F(z)$ 이다. 그러므로 $F(z) = \frac{1}{1-z}$ 를 $f(z)$ 의 해석적 접속이라 말할 수 있다.

도움말 : (1) 자연대수

$$e := \sum_{n=0}^{\infty} \frac{1}{n!} \qquad (\text{단, } 0! = 1)$$

는 무리수이다.

(2) $z = x + iy$ (단, x, y 는실수)가 복소수이고 $a > 0$ 일 때

$\mathrm{Re}\, z := x$ (즉, z의 실수부분), $\mathrm{Im}\, z := y$ (즉, z의 허수부분)

$$e^z := e^x \cdot e^{iy} = e^x(\cos y + i \sin y), \quad a^z := e^{z \ln a}, \quad |z| := (x^2+y^2)^{1/2}$$

와 같이 정의한다. 가령, n이 자연수일 때 $|n^z| = e^{x \ln n} = n^x$ 이다.

(3) $x > 1$일 때 무한급수 $\sum_{n=1}^{\infty} \frac{1}{n^x}$ 는 수렴한다. 그리고 무한급수

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

는 발산한다.

(4) $D := \{ z \in \mathbb{C} \mid \operatorname{Re} z > 1 \}$이라 놓으면 D는 domain(open and connected set)이다. 여기서, $\mathbb{C}$ 는 복소수 전체의 집합을 나타내는 복소수 체이다. 제타함수 $\zeta(s)$는 D상에서 절대수렴하며 Weierstrass-M 테스트에 의하여 $\zeta(z)$는 해석적 함수이다.

리만은 다음의 정리를 증명하였다.

정리 1. (1) $\zeta(s)$는 전 복소평면 상으로 해석적으로 접속이 가능하며 $s=1$에서만 단순극점(a simple pole)을 지니며 이의 residue는 1 이다.

(2) 리만 제타함수는

$$\text{(F.E.)} \qquad \pi^{-s/2}\Gamma(s/2)\,\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

와 같은 함수방정식을 만족한다. 여기서

$$\Gamma(s) := \int_0^{\infty} t^{s-1} e^{-t} dt \quad (\text{단}, \ \operatorname{Re} s > 0)$$

으로 정의되며 $\Gamma(s)$는 해석적 접속을 지닌다.

(3) $0 = \zeta(-2) = \zeta(-4) = \cdots = \zeta(-2n) = \cdots$ (단, n은 자연수).

머리말에서 언급한 보고서에서 리만은 리만 가설을 제시하였다.

이제, 리만 제타함수의 성질을 열거하겠다.

(R1) $\operatorname{Re} s > 1$이면 $\zeta(s) \neq 0$이다.

(R2) $k \in \mathbb{Z}^+$가 자연수일 때

$$\zeta(2k) = (-1)^k \pi^{2k} \frac{2^{2k-1}}{(2k-1)!} \left(-\frac{B_{2k}}{2k}\right)$$

이다. 여기서 B_k는

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k$$

으로 정의되는 베르누이(Bernoulli) 수이다. 가령,

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(8) = \frac{\pi^8}{9450},$$

$$\zeta(10) = \frac{\pi^{10}}{93555}, \quad \zeta(12) = \frac{691}{638512875} \pi^{12}, \cdots.$$

(R3) $k \in \mathbb{Z}^+$가 자연수일 때

$$\zeta(-k) = -\frac{1}{k+1} \sum_{r=0}^{k+1} \binom{k+1}{r} B_r$$

이다. 단, $\binom{k+1}{r}$ 는 $k+1$ 개중에서 r개를 뽑는 경우의 수이다. 즉,

$$\zeta(-2n) = 0, \qquad \zeta(1-2n) = -\frac{B_{2n}}{2n}, \qquad n = 1, 2, 3, \cdots.$$

(R4) $\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1) \qquad (s \to 1)$

이다. 여기서

$$\gamma := \lim_{n \to \infty} \left(1 + \cdots + \frac{1}{n} - \log n\right)$$

은 오일러 상수이다.

(R5)

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \frac{s}{12} - \frac{s(s+1)(s+2)}{720} + \frac{s(s+1)(s+2)(s+3)(s+4)}{30240}$$
$$- \cdots\cdots + \frac{B_n}{n!} s(s+1)\cdots(s+n-2).$$

(단, $s = 0, -1, -2, \cdots, -n+1$._

리만 가설 (RH)는 아직까지도 증명되지 않았다. 리만 가설을 해결하기 위해 노르웨이 수학자 셀버그[10]는 1950년경에 소위, 셀버그 트레이스 공식(trace formula)을 창안해내었다. 이 트레이스 공식은 매우 심오하고 아름다운 이론으로 Lie 군의 표현론, 보형형식론, 수리물리, 미분기하학 등의 분야에 응용되었다. 지난 10여 년 전에는 독일 수학자 데닝어[11]는 코호모로지 접근 방법으로 motivic L-함수의 여러 성질들을 유도하였으며 이의 리만 가설을 해결하려고 시도하였다. 물론, motivic L-함수 또는 motivic 제타함수는 리만 제타함수의 경우를 일반화한 함수이다.

함수 $\mu: \mathbb{Z}^+ \to \mathbb{Z}$ 를 아래와 같이 정의하자. 임의의 $x \in \mathbb{Z}^+$ 에 대하여 $x = p_1 p_2 \cdots p_k$ (단, p_i들은 소수이며 같을 수가 있다.) 이면

$$\mu(x) := \begin{cases} 0 & \text{if } x \textit{ is square} \\ 1 & \text{if } \textit{all } p_i \textit{ are distinct} \text{ and } k \textit{ is even} \\ -1 & \text{if } \textit{all } p_i \textit{ are distinct} \text{ and } k \textit{ is odd.} \end{cases}$$

가령, $\mu(12) = \mu(25) = 0$, $\mu(6) = 1$, $\mu(70) = -1$ 임을 쉽게 알 수 있다.
함수 $M: \mathbb{Z}^+ \to \mathbb{Z}$ 를

$$M(N) := \sum_{n=1}^{N} \mu(n)$$

으로 정의한다.

정리 2. (RH) is true iff $M(N)$ grows no faster than a constant multiple of $N^{\frac{1}{2}+\varepsilon}$ as $N \to \infty$ for any $\varepsilon > 0$.

이 정리는 오래 전에 증명되었다.

T가 양수일 때 $N(T)$를 사각형 $0 < \text{Re}\, s < 1$, $0 < \text{Im}\, s < T$ 안에 있는 $\zeta(s)$의 영점들의 개수라고 하자. 1905년에 H. von Mangoldt (1854~1925)는

10) Alte Selberg (1917~)
11) Christopher Deninger (1959~)

$$N(T) = \frac{1}{2\pi} T \log T - \frac{1+\log 2\pi}{2\pi} T + O(\log T)$$

와 같은 점근 공식을 얻었다. 그리고 리만 제타함수는

$$(s-1)\zeta(s) = \frac{1}{2} e^{bs} \frac{1}{\Gamma(\frac{s}{2}+1)} \prod_{\rho} (1-\frac{s}{\rho}) e^{s/\rho}$$

의 무한곱 관계식을 만족한다. 여기서, b는 상수이고 $\prod_{\rho}$는 $\zeta(s)$의 자명하지 않은 모든 영점들의 곱을 나타낸다.

3. 소수 정리 (the Prime Number Theorem)

소수는 수 중에서 가장 기본이 되는 수이다. 소수의 개수가 무한임을 앞에서 보였다. 이제 소수가 어떻게 분포되어 있는가를 살펴보자.
가령, 9,999,900 과 10,000,000 사이에는

9,999,901 ; 9,999,907 ; 9,999,929 ; 9,999,931 ; 9,999,937 ;
9,999,943 ; 9,999,971 ; 9,999,973 ; 9,999,991

와 같이 9개의 소수가 있다. 그런데 10,000,000 과 10,000,100 사이에 있는 소수는

10,000,019 ; 10,000,079

밖에 없다. 이 예에서 보듯이 소수의 분포에 관하여 무엇이라 말할 수 없는 입장이다. $2^{21,701}-1$ 은 1979년 전까지 알려진 가장 큰 소수이다. 몇 년 전에 이보다 더 큰 소수가 알려졌다는 소식을 들었다.

연습문제 1. $2^{21,701}-1$ 이 소수임을 보여라.

소수 정리 3.

(3)' $\lim_{n\to\infty} \frac{\pi(n)}{n} \log n = 1.$

J. Hadamard, de la Vallée-Poussin, A. Selberg 등의 수학자들에 의하여 상기의 소수 정리가 증명되었다. 이의 증명과정에서 리만 제타함수 $\zeta(s)$의 자명하지 않는 영점(zero)들이 모두 y 축과 $x=1$ 직선 사이에 있다는 사실을 사용하고 있다. 그래서 소수 정리는 리만 가설 (RH)와 매우 밀접한 관계가 있음을 알 수 있다.

또, 우리는

$$(5)\quad \pi(n) \sim 1+\cdots+\sum_{k=1}^{\infty}\frac{1}{k\cdot\zeta(k+1)}\cdot\frac{(\log n)^k}{k!}$$

임을 증명할 수 있다. (5)로부터 소수 정리와 리만 제타함수의 이론과 어느 정도 연관되어 있음을 어렴풋이 알 수 있다.

정리 4. 소수 정리는 $\zeta(1+it)\neq 0$ (단, $t\neq 0$인 실수)이라는 주장과 동치이다.

정리 4로부터 소수 정리와 리만 제타함수와는 아주 밀접한 관계가 있음을 재확인할 수 있다. 만약에 리만 가설 (RH)가 진실이라면, 우리는 소수의 분포에 관한 보다 자세하고 구체적인 정보와 지식을 얻을 수 있다.

소수에 관한 흥미로운 여러 문제를 소개하겠다.

문제 1. $4n+1$ (단, n은 자연수)의 형태의 소수가 무한히 많으냐? 예를 들면,

5, 13, 17, 29, … , 10006721, …

디리클레[12]의 정리. k와 l이 서로 소인 자연수라고 하자. 그러면, $kn+l$(단, n은 자연수)의 형태의 소수의 개수는 무한이다.

그래서 문제 1은 디리클레의 정리에 의하여 풀린다.

문제 2 (골드바하 가설). "4 보다 큰 임의의 짝수는 홀수인 두 소수의 합으로 쓸 수 있다." 라고 골드바하는 주장하였다. 이 주장은 아직까지 해결되지 않았다. 그래서 이 주장

12) Peter Gustav Lejeune Dirichlet (1805~1859)

은 「골드바하」 가설이라고 불리고 있다. 가령,

$6=3+3$, $8=3+5$, $10=3+7$, $12=5+7$, $14=7+7$, $16=3+13, \cdots$.

이 가설이 옳다고 하면,

"7보다 큰 임의의 홀수는 홀수인 세 소수의 합이다."

이란 주장이 옳다는 사실을 쉽게 알 수 있다.

문제 3 (쌍둥이 소수 문제).

$\{3,5\}$; $\{5,7\}$; $\cdots$ $\{10016957, 10016959\}$; $\cdots$; $\{10^9+7,\ 10^9+9\}$; $\cdots$

와 같이 차가 2 인 소수 짝을 「소수 쌍둥이(prime twin)」 이라고 한다. 100,000 보다 작은 소수 쌍둥이의 개수는 1224 개이고 1,000,000 보다 작은 소수 쌍둥이의 수는 8164 개이다. 1957년 전 까지 알려진 소수 쌍둥이 중에서 가장 큰 것은 {1000000009649, 1000000009651} 이었다.

(PTP) "소수 쌍둥이의 개수는 무한인가 ?"

(PTP) 문제는 아직까지도 해결되지 않았다. 보다 나아가 아래의 문제를 제기할 수 있다.

(PTP*) "$\{\ p\mid$ 단, p, $p+2$, $p+6$ 은 모두 소수 $\}$**의 개수는 무한인가?"**

물론 이 문제도 풀리지 않았다

문제 4. N을 주어져 있는 자연수라고 하자. 아래 형태의 수

$$n^2-n+p,\qquad 0\le n\le N,\quad \text{단, } n \text{ 은 자연수}$$

가 모두 소수가 되게 하는 소수 p가 있느냐? 이 문제도 역시 아직까지도 풀리지 않았다.

예. (1) $N=16$, $p=17$.

(2) $N=40$, $p=41$.

문제 5. n^2+1 (단, n은 자연수)의 형태의 소수의 개수는 무한인가? 가령,

$$2,\ 5,\ 17,\ 37,\ \cdots,\ 65537,\ \cdots.$$

이 문제의 해답은 아직까지도 모르고 있다.

문제 6. p_n을 n번째 소수라고 하자. 집합 $\{p_n - p_{n-1} \mid n$은 자연수$\}$의 원소 중에서 가장 큰 값은? 또, 아래의 극한값

$$\lim_{n\to\infty}(p_n - p_{n-1}) = ?$$

은 무엇일까?

문제 7. n이 자연수라고 하자. "n과 $2n$사이에 소수가 존재하느냐?" 이 질문은 Bertrand의 문제로 알려져 왔는데 체비쉐프에 의하여 이 질문이 옳다는 사실이 밝혀졌다.

참 고 문 헌

P. Deligne, La conjecture de Weil, I, Publ. Math. IHES, Vol. 43 (1974), 273-307.

P. Deligne, La conjecture de Weil, II, Publ. Math. IHES, Vol. 52 (1980), 137-252.

C. Deninger, On the Γ-factors attached to motives, Invent. Math. vol. 104 (1991), 245-261.

C. Deninger, Local L-factors of motives and regularized determinants, Invent. Math. Vol. 107 (1992), 135-150.

C. Deninger, Evidence for a cohomological approach to analytic number theory, Proc. of European Congress of Mathematicians, Birkhäuser (1994), 491-510.

D. A. Hejhal, The Selberg trace formula and the Riemann zeta function, Duke Math. J. Vol. 43, No. 3 (1976), 441-482.

L. K. Hua, Introduction to Number Theory, Springer-Verlag (1982).

N. Koblitz, Introduction to Elliptic Curves and Modular Forms, Springer-Verlag

(1984).

Y. Manin, Lectures on Zeta Functions and Motives, Lecture Notes (Columbia Univ.), 1991.

M. Monastyrsky, Riemann, topology and physics, Birkhäuser (1987).

Y. Motohashi, Spectral theory of the Riemann zeta function, Cambridge Univ. Press (1997).

B. Riemann, Über die Anzahl der Primzahlen unter einen gegebenen Grösse, Gesammelte mathematischen Werke (1859), 145-153 (Dover, 1953).

A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, J. Indian Math. Soc., Vol. 20 (1956), 47-87.

C. L. Siegel, Über Riemanns Nachlass zur analytischen Zahlentheorie, Quellen und Studien zur Geschichte der Mathematik, Astromie und Physik 2 (1932), 45-80.

A. Voros, Spectral Functions, Special Functions and the Selberg Trace Formula, Communications Math. Physics, Vol. 111 (1987), 439-465.

A. Weil, On the Riemann Hypotheses in function-fields, Proc. Nat. Acad. Sci., Vol. 27 (1941), 345-347.

A. Weil, Numbers of solutions of equations in finite fields, Bull. Amer. Math. Soc., Vol. 55 (1949), 497-508.

D. Zagier, Eisenstein series and the Riemann zeta function, Tata Institute for Fundamental Research, Bombay (1979), 275-301.

D. Zagier, Zetafunktionen und quadratische Körper, Hochschultext, Springer-Verlag (1981).

리만의 생애

리만(Bernhard Riemann)은 1826년 9월 17일 하노버 왕국의 브레제렌쯔(Breselenz)라는 마을에서 목사의 아들로 태어났다. 1846년에 그의 부친의 권유로 괴팅겐 대학의 신학과에 입학하였다가 수학에 관한 열정이 매우 강렬하여 얼마 후 철학과로 전과하였다. 그 당시에 괴팅겐 대학의 철학과에는 천문학자 골드슈미트(Carl Wolfgang Benjamin Goldschmidt, 1807~1851), 수학자 스턴(Moritz Stern, 1807~1894)과 위대한 수학자 가우스(Carl Friedrich Gauss, 1777~1855)가 재직하고 있었다. 괴팅겐에서 일년을 보낸 후 베를린 대학에 가서 2년간 거기서 연구하였다. 이 당시에 베를린 대학에는 야코비(Carl Gustav Jacob Jacobi, 1804~1851), 스타인너(Jakob Steiner, 1796~1863), 디리클레(Peter Gustav Lejeune Dirichlet, 1805~1859), 아이젠쉬타인(Ferdinand Gotthold Max Eisenstein, 1823~1852)등의 수학자가 있었다.

리만은 『Grundlagen für eine allgemeine Theorie der Functionen einen veränderlichen complexen Grösse (복소 함수의 일반적인 이론에 관한 기초)』의 논문으로 박사학위를 취득하였고 1853년에는 『On the Representability of a Function by a Trigonometric Series』의 논문을 괴팅겐 대학에 제출하여 Habilitation을 받았다. 1857년에 그의 유명한 논문 『Theorie der Abelschen Functionen (아벨 함수의 이론에 관하여)』이 Crelle 수학저널의 제54호에 게재되었다. 이 논문의 대부분의 내용은 리만이 1851년과 1856년 사이에 연구하여 얻었던 결과들이다. 1859년에 이미 1쪽에서 소개되었던 『On the numbers of primes less than a given magnitude』을 베를린 학술원에 제출하여 베를린 학술원의 정식 회원으로 선정되었다.

1855년에 가우스가 죽자 디리클레가 가우스의 교수직을 승계하였다. 1859년에 숙환으로 디리클레가 죽자 이 교수직을 리만이 계승하여 정교수가 되었다. 리만은 리만 기하학을 창시하였고 복소함수론, 아벨함수론, 소수의 분포 이론, 수리물리학 등의 분야에 뛰어난 업적을 남겼다.

1862년 가을에 심한 독감을 앓은 후 결핵에 걸려 그 후 투병생활을 하였다. 정부의 보조금을 받아 병을 치료하기 위하여 날씨가 좋은 이탈리아에 가서 요양하기도 하였다. 1866년 7

월 20일에 Selasca 라는 조그만 마을에서 그의 짧은 생을 마감하였다. 그의 시신은 이 근처에 있는 Biganzola 라는 마을에 매장되었다. 그의 비석에는 아래와 같이 새겨져 있다.

"Here lies in God Georg Friedrich Bernhard Riemann-Göttingen professor, born in Brezelenz, September 17, 1826, died in Selasca, July 20, 1866. Denen die Gott lieben müssen alle Dinge zum Besten dienen" (To those whom God loves ought all to be successful).

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충남중등수학교육연구회 전문성 신장 세미나

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• 장소 : 공주대학교 산학연구관 강당

차 례

소수의 아름다움

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자연수가 인간의 실생활에서 중요하다는 사실을 누구나 인정할 것이다. 그러면 우리 인간은 자연수의 성질을 알아야 할 필요가 있다. 자연수는 소수들의 곱으로 나타낼 수 있다는 사실을 쉽게 알 수 있다. 여기서 **소수란 약수가 1과 자기 자신뿐인 수**이다. 예를 들면, 2, 3, 5, 7, 11, 13, 17, 19, 23, … 은 소수이며 2는 짝수인 유일한 소수이다. 자연수를 이해하기 위해서는 소수를 철저하게 이해해야 한다. 따라서 소수가 수의 기본이 된다는 사실을 인정할 것이다.

오늘의 강연은 소수의 아름다움, 심오함과 신비성을 잘 설명하고 있는 **리만가설**(Riemann Hypothesis)**과 골드바하 가설**(Goldbach Conjecture)에 관한 이야기와 이와 관련된 여러 흥미로운 문제들을 소개한다.

먼저 리만가설에 관한 역사적인 배경과 내용에 관하여 간략하게 설명하겠다. 오래 전부터 위대한 수학자들은 소수의 신비와 분포에 관하여 연구하여 왔다. 1859년에 리만[1)]은 베를린 학술원의 회원으로 선정되었다. 베를린 학술원의 헌장에 의하면, 새로이 선출된 회원은 반드시 최근의 연구업적을 보고하게 되어 있었다. 그래서 리만은 『주어진 수보다 작은 소수의 개수에 관하여 (On the number of primes less than a given magnitude)』의 제목으로 보고서를 학술원에 제출하였다.(참고문헌 [16] 참조) 그는 이 보고서에서 리만 제타함수의 성질들을 열거하고 소위, "리만 가설"을 제시하였다.

1) Georg Friedrich Bernhard Riemann (1826~1866) : 리만의 일생과 업적에 관해 참고문헌 [14]을 참고할 것.

이미 이 전에 소수의 분포에 관하여 오일러[2], 르장드르[3], 가우스[4] 등의 위대한 수학자에 의하여 연구되었다. 오일러는 소수의 분포를 연구하기 위하여 아래의 제타함수

(1) $$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (단, s 는 실수)$$

를 공부하였다. 그는

(2) $$\zeta(s) = \prod_p (1-p^{-s})^{-1}$$

의 관계식을 보였다. 여기서 $\prod_p$ 는 모든 소수 p 들의 무한곱을 나타낸다. 관계식

(2)는 「오일러 곱(Euler product)」이라고 불린다. 이 사실로부터 소수의 개수가 무한임을 알 수 있다. x 를 주어진 양의 실수라고 하고

$$\pi(x) := |\{ p \in \mathbb{Z}^+ \mid 2 \le p < x, \ p는 소수 \}|$$

라고 하자. 여기서 $\mathbb{Z}^+$는 모든 자연수들의 집합을 나타내고 $|S|$는 집합 S의 개수를 나타낸다. 가우스는

(3) $$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\log x}} = 1$$

일 것이라는 추측을 제시하였다. 르장드르, 가우스와 같은 위대한 수학자들이 (3)을 증명하려고 시도하였지만 실패하였다. 1854년에 체비쉐프[5]는 논문집 『Memoires de l'Academie des Sciences de Saint Petersburg』에서

(4) $$A_1 < \frac{\pi(x)}{\frac{x}{\log x}} < A_2$$

의 등식을 증명하였다. (단, $0.992 < A_1 < 1$ 이고 $1 < A_2 < 1.105$ 임.)

1850년경에 리만은 (1)에서 실수 변수 s 뿐만 아니라 복소수 변수 s 까지 생각하였다. 그는 $\mathrm{Re}\, s > 1$을 만족하는 영역에서 $\zeta(s)$는 해석적 함수이고 해석적 접속(analytic continuation)을 지님을 증명하였다. 게다가 논문 [16]에서 그는 $\zeta(s)$의 함수방정식을 발견하였을 뿐만 아니라

$$0 = \zeta(-2) = \zeta(-4) = \zeta(-6) = \cdots = \zeta(-2n) = \cdots \qquad (단, n은 자연수)$$

임을 증명하고

2) Leonhard Euler (1707~1783) : 스위스 수학자. 금년에 탄생 300주년을 맞이하여 스위스와 러시아에서 많은 국제학술회의 및 여러 행사가 거행되었음.
3) Adrien Marie Legendre (1752~1833) : 프랑스 수학자.
4) Carl Friedrich Gauss (1777~1855) : 위대한 독일 수학자.
5) Pafnuti L'vovich Chebyshev (1821~1894) : 러시아 수학자.

(RH) "$\zeta(s)$의 다른 영점(zero)은 모두 $\mathrm{Re}\,s=\frac{1}{2}$의 선상에 놓여 있다."

라는 사실을 추측하였다. 그러나 리만은 이 추측을 증명하지 않았다. 그의 사후에 제타함수 $\zeta(s)$는 「리만 제타함수(the Riemann zeta function)」라고 불렸고 주장 (RH)는 『리만 가설』이라고 불렸다. 그 후 프랑스 수학자 Jacques Hadamard (1865~1963)와 벨기에 수학자 Charles de la Vallée-Poussin (1866~1962) 등과 같은 유명한 수학자들이 리만 가설을 해결하려고 하였지만 실패하였다. 아직까지도 이 가설은 풀리지 않고 있다. 1941년에 프랑스 수학자 베이유[6]는 함수체(function field)인 경우에 (RH)를 증명하였고, 1949년에 유한체(finite field) 상에서 정의되는 대수다양체의 제타함수에 대하여 (RH)와 유사한 소위, 『베이유 가설(Weil conjecture)』을 제시하였다.(참고문헌 [20]과 [21] 참조) 그 후, 1974년에 벨기에 수학자 델리네[7]가 매끄러운 사영다양체(nosingular projective variety)인 경우에 베이유 가설이 옳다는 것을 증명하였다.(참고문헌 [3] 참조) 이 업적과 하지 이론의 업적으로 델리네는 1978년에 수학의 노벨상인 필즈상을 수상하였다. 1980년에 일반적인 다양체(complete variety)인 경우에 베이유 가설이 진실이라는 사실을 증명하였다.(참고문헌 [4] 참조) 리만 가설은 정수론 분야에서 중요한 『소수 정리 (the Prime Number Theorem)』와 아주 밀접한 관계가 있다. 가령, 주장 (3)은 $\zeta(1+it)\neq 0$(단, $t\neq 0$ 인 실수) 이라는 주장과 동치이다.

리만 가설의 내용을 어느 정도 이해하기 위해서는 우선,

(ㄱ) 복소수(complex number)의 개념
(ㄴ) 해석적(解析的; analytic or holomorphic) 함수의 개념
(ㄷ) 유리형(meromorphic) 함수의 개념
(ㄹ) 해석적 접속(analytic continuation)의 개념

등의 기본적인 여러 개념을 알아야 한다. 복소수의 개념은 여러분 모두가 잘 알고 있기 때문에 설명은 생략하겠다. 복소함수 $f(z)$가 z_0의 근방에서 극한값 $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$을 가질 때 함수 $f(z)$는 z_0에서 해석적이다라고 한다. 영역(a region) D의 모든 점에서 복소 함수 $f(z)$가 해석적일 때 $f(z)$는 D상에서 해석

6) André Weil (1906~1998) : Wolf 상을 수상하였음. 시카고 대학과 고등연구소 교수를 역임하였음.
7) Pierre Deligne (1944~) : 1978년에 필즈상을 수상한 벨기에 수학자. 현재 고등연구소 교수로 재임.

적이다라고 한다. 그리고 $\frac{f(z)}{g(z)}$ (단, $f(z)$ 와 $g(z)$ 는 해석적 함수이고 $g(z)\neq 0$ 임)의 형태의 함수를 유리형 함수라고 한다. 복소 함수 $f(z)$ 가 z_0 의 근방에서

$$f(z)=\frac{a_{-m}}{(z-z_0)^m}+\frac{a_{-m+1}}{(z-z_0)^{m-1}}+\cdots\cdots+\frac{a_1}{z-z_0}+\sum_{k=0}^{\infty}a_k(z-z_0)^k$$

(단, $a_{-m}\neq 0$ 임)의 형태로 나타낼 수 있을 때 함수 $f(z)$ 는 z_0 에서 m 계의 극점 (a *pole of order* m)을 갖는다고 한다. 이제부터 $\mathbb{C}$ 는 복소평면 즉, 복소수 전체의 집합을 나타내기로 한다. 영역 $D(\subset \mathbb{C})$ 에서 정의되는 해석적 함수 $f(z)$ 가 주어져 있다고 하자. D 를 포함하는 영역 $E(\neq D)$ 상에 유리형 함수 $F(z)$ 가 존재하여 D 상에서는 $f(z)=F(z)$ 일 때 함수 $F(z)$ 를 $f(z)$ 의 **해석적 접속**이라고 한다. 예를 들면, 기하급수로 주어지는 함수

$$f(z)=\sum_{k=0}^{\infty}z^k$$

는 중심이 원점인 단위원 내부 $D=\{z\in\mathbb{C} \mid |z|<1\}$ 에서 정의되는 해석적 함수이다. 그런데 함수 $F(z)=\frac{1}{1-z}$ 는 $E=\{z\in\mathbb{C} \mid z\neq 1\}$ 상에서 정의되는 해석적 함수이며 D 상에서는 $f(z)=F(z)$ 이다. 그러므로 $F(z)=\frac{1}{1-z}$ 를 $f(z)$ 의 해석적 접속이라 말할 수 있다.

도움말 : (a) 1737년에 오일러는 자연대수

$$e:=\sum_{n=0}^{\infty}\frac{1}{n!} \qquad (\text{단, } 0!=1)$$

가 무리수임을 증명하였고, 1873년에 프랑스 수학자 Charles Hermite (1822~1901)는 자연대수 e 가 초월수임을 증명하였다.

(b) $z=x+iy$ (단, x, y 는실수)가 복소수이고 $a>0$ 일 때

$\mathrm{Re}\,z:=x$ (즉, z의 실수부분), $\mathrm{Im}\,z:=y$ (즉, z의 허수부분)

$e^z:=e^x\cdot e^{iy}=e^x(\cos y+i\sin y)$, $a^z:=e^{z\ln a}$, $|z|:=(x^2+y^2)^{1/2}$

와 같이 정의한다. 가령, n 이 자연수일 때 $|n^z|=e^{x\ln n}=n^x$ 이다.

(c) $x>1$일 때 무한급수 $\sum_{n=1}^{\infty}\frac{1}{n^x}$ 는 수렴한다. 그리고 무한급수

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots$$

는 발산한다.

(d) $D:=\{\, z\in\mathbb{C} \mid \mathrm{Re}\, z>1 \,\}$이라 놓으면 D는 domain(open and connected set)이다. 제타함수 $\zeta(s)$는 D상에서 절대수렴하며 Weierstrass-M 테스트에 의하여 $\zeta(z)$는 해석적 함수이다.

리만은 다음의 정리를 증명하였다.

정리 1. (1) $\zeta(s)$는 전 복소평면 상으로 해석적으로 접속이 가능하며 $s=1$에서만 단순 극점(a simple pole)을 지니며 이의 residue는 1 이다.

(2) 리만 제타함수는

(FE) $$\pi^{-s/2}\Gamma(s/2)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

와 같은 함수방정식을 만족한다. 여기서

$$\Gamma(s):=\int_0^{\infty}t^{s-1}e^{-t}dt \quad (\text{단, } \mathrm{Re}\, s>0)$$

으로 정의되며 $\Gamma(s)$는 해석적 접속을 지닌다.

(3) $0=\zeta(-2)=\zeta(-4)=\cdots=\zeta(-2n)=\cdots$ (단, n은 자연수).

이제, 리만 제타함수의 성질을 열거하겠다. $\zeta(s)$에 관한 참고문헌으로 [10,11, 12, 15]을 소개한다.

(R1) $\mathrm{Re}\, s>1$이면 $\zeta(s)\neq 0$이다.

(R2) $k\in\mathbb{Z}^+$가 자연수일 때

$$\zeta(2k)=(-1)^k\pi^{2k}\frac{2^{2k-1}}{(2k-1)!}\left(-\frac{\mathrm{B}_{2k}}{2k}\right)$$

이다. 여기서 B_k는

$$\frac{t}{e^t-1}=\sum_{k=0}^{\infty}\frac{B_k}{k!}t^k$$

으로 정의되는 베르누이(Bernoulli) 수이다. 가령,

$$\zeta(2)=\frac{\pi^2}{6},\quad \zeta(4)=\frac{\pi^4}{90},\quad \zeta(6)=\frac{\pi^6}{945},\quad \zeta(8)=\frac{\pi^8}{9450},$$

$$\zeta(10)=\frac{\pi^{10}}{93555},\quad \zeta(12)=\frac{691}{638512875}\pi^{12},\cdots.$$

(R3) $k\in\mathbb{Z}^+$가 자연수일 때

$$\zeta(-k)=-\frac{1}{k+1}\sum_{r=0}^{k+1}\binom{k+1}{r}B_r$$

이다. 단, $\binom{k+1}{r}$ 는 $k+1$ 개중에서 r개를 뽑는 경우의 수이다. 즉,

$$\zeta(-2n)=0,\qquad \zeta(1-2n)=-\frac{B_{2n}}{2n},\qquad n=1,2,3,\cdots.$$

(R4) $\zeta(s)=\frac{1}{s-1}+\gamma+O(s-1)\qquad (s\to 1)$

이다. 여기서

$$\gamma:=\lim_{n\to\infty}\left(1+\cdots+\frac{1}{n}-\log n\right)$$

은 오일러 상수이다. 아직까지도 오일러 상수가 무리수인지 초월수인지를 모르고 있다.

(R5)

$$\zeta(s)=\frac{1}{s-1}+\frac{1}{2}+\frac{s}{12}-\frac{s(s+1)(s+2)}{720}+\frac{s(s+1)(s+2)(s+3)(s+4)}{30240}$$
$$-\cdots\cdots+\frac{B_n}{n!}s(s+1)\cdots(s+n-2).$$

단, $s=0,-1,-2,\cdots,-n+1$.

리만 가설 (RH)는 아직까지도 증명되지 않았다. 리만 가설을 해결하기 위해 노르웨이 수학자 셀버그8)는 1950년경에 소위, 셀버그 트레이스 공식(trace formula)을 창안해내었다. 이 트레이스 공식은 매우 심오하고 아름다운 이론으로 Lie 군의 표현론, 보형형식론, 수리물리, 미분기하학 등의 분야에 응용되었다 (참고문헌 [10, 17, 19, 22, 23]). 지난 10여 년 전에는 독일 수학자 데닝어9)는 코호모로지 접근 방법으로 motivic L-함수의 여러 성질들을 유도하였으며 이의 리만 가설을 해결하려고 시도하였다 (참고문헌 [5, 6, 7]). 물론, motivic L-함수 또는 motivic 제타함수는 리만 제타함수의 경우를 일반화한 함수이다 (참고문헌 [13]).

8) Alte Selberg (1917~2007) : 보다 자세한 것은 부록을 참조할 것.
9) Christopher Deninger (1959~) : 독일 수학자. 1998년 ICM에서 기조강연을 하였음.

함수 $\mu: \mathbb{Z}^+ \to \mathbb{Z}$ 를 아래와 같이 정의하자. 임의의 $x \in \mathbb{Z}^+$ 에 대하여 $x = p_1 p_2 \cdots p_k$ (단, p_i들은 소수이며 같을 수가 있다.) 이면

$$\mu(x) := \begin{cases} 0 & \text{if } x \text{ is divisible by the square of a prime,} \\ 1 & \text{if all } p_i \text{ are distinct and } k \text{ is even,} \\ -1 & \text{if all } p_i \text{ are distinct and } k \text{ is odd.} \end{cases}$$

가령, $\mu(12) = \mu(25) = 0$, $\mu(6) = 1$, $\mu(70) = -1$ 임을 쉽게 알 수 있다.
함수 $M: \mathbb{Z}^+ \to \mathbb{Z}$ 를

$$M(N) := \sum_{n=1}^{N} \mu(n)$$

으로 정의한다.

정리 2. (RH) is true iff $M(N)$ grows no faster than a constant multiple of $N^{\frac{1}{2}+\epsilon}$ as $N \to \infty$ for any $\epsilon > 0$.

이 정리는 오래 전에 증명되었다.

2004년에 Xavier Gourdon과 Patrick Demichel 은 Odlyzko-Schönhage 계산법을 이용하여 1조개이상의 리만제타함수의 영점을 발견하였다. 리만의 유고집에 의하면 리만이 첫 여섯 번째까지의 영점을 계산하였다고 전해지고 있다. 리만의 유고집을 검토한 후 지겔[10]은 소위 리만-지겔 공식(the Riemann-Siegel Formula)을 발견하였다 (참고문헌 18]). 이 공식의 중요성이 40여년이 지나서야 밝혀졌다. 이 공식에 기반을 두고 많은 사람들이 $\zeta(s)$의 영점을 발견하는 계산법을 개발하여 왔다. 아래에 재미있는 말을 인용한다.

I don't believe in God, but I believe in the Riemann Hypothesis. For the latter there are more than 400,000,000,000 reasons to believe.

-Manoj Verma-

T 가 양수일 때 $N(T)$를 사각형 $0 < \mathrm{Re}\, s < 1$, $0 < \mathrm{Im}\, s < T$ 안에 있는 $\zeta(s)$의 영점들의 개수라고 하자. 1905년에 H. von Mangoldt (1854~1925)는

10) Carl Ludwig Siegel (1896~1981) : 독일의 위대한 수학자. 첫 번째 Wolf 상을 수상하였음. 괴팅겐 대학과 프린스턴 고등연구소 교수를 역임하였음.

$$N(T)=\frac{1}{2\pi}T\log T-\frac{1+\log 2\pi}{2\pi}T+O(\log T)$$

와 같은 점근 공식을 얻었다. 그리고 리만 제타함수는

$$(s-1)\zeta(s)=\frac{1}{2}e^{bs}\frac{1}{\Gamma(\frac{s}{2}+1)}\prod_{\rho}(1-\frac{s}{\rho})e^{s/\rho}$$

의 무한곱 관계식을 만족한다. 여기서, b는 상수이고 $\prod_{\rho}$는 $\zeta(s)$의 자명하지 않는 모든 영점들의 곱을 나타낸다.

이제 소수가 어떻게 분포되어 있는가를 살펴보자.
가령, 9,999,900 과 10,000,000 사이에는

9,999,901 ; 9,999,907 ; 9,999,929 ; 9,999,931 ; 9,999,937 ;
9,999,943 ; 9,999,971 ; 9,999,973 ; 9,999,991

와 같이 9개의 소수가 있다. 그런데 10,000,000 과 10,000,100 사이에 있는 소수는

10,000,019 ; 10,000,079

밖에 없다. 이 예에서 보듯이 소수의 분포에 관하여 무엇이라 말할 수 없는 입장이다.

$2^{32,582,657}-1$ (9,808,358 자리수)

이 지금까지 알려진 가장 큰 소수이다. 이 소수는 2006년 9월 4일에 Curtis Cooper와 Steven Boone에 의해 발견되었으며, 44번째 Mersenne 소수이다. 그리고 Mersenne 소수가 아니며 알려진 가장 큰 소수는

$19,249\times 2^{13,018,586}+1$ (3,918,990 자리수)

이다.

소수정리 3.

(3)' $$\lim_{n\to\infty}\frac{\pi(n)}{n}\log n=1.$$

J. Hadamard, de la Vallée-Poussin, A. Selberg 등의 수학자들에 의하여 상기의 소수 정리가 증명되었다. 이의 증명과정에서 리만 제타함수 $\zeta(s)$의 자명하지 않는 영점(zero)들이 모두 y 축과 $x=1$ 직선 사이에 있다는 사실을 사용하고 있다. 그래서 소수 정리는 리만 가설 (RH)와 매우 밀접한 관계가 있음을 알 수 있다. 또, 우리는

(5) $$\pi(n)\sim 1+\cdots+\sum_{k=1}^{\infty}\frac{1}{k\cdot\zeta(k+1)}\cdot\frac{(\log n)^k}{k!}$$

임을 증명할 수 있다. (5)로부터 소수 정리와 리만 제타함수의 이론과 어느 정도 연관되어 있음을 어렴풋이 알 수 있다.

정리 4. 소수 정리는 $\zeta(1+it)\neq 0$ (단, $t\neq 0$ 인 실수)이라는 주장과 동치이다.

정리 4로부터 소수 정리와 리만 제타함수와는 아주 밀접한 관계가 있음을 재확인할 수 있다. 만약에 리만 가설 (RH)가 진실이라면, 우리는 소수의 분포에 관한 보다 자세하고 구체적인 정보와 지식을 얻을 수 있다.

예를 들어, 다음의 흥미로운 문제를 생각하여 보자.

문제 A. $4n+1$ (단, n 은 자연수) 의 형태의 소수의 개수가 무한개 인가 ?
예를 들면,

5, 13, 17, 29, ⋯ , 10006721, ⋯ 등은 $4n+1$ 형태의 소수이다.

디리클레[11]의 정리. k와 l이 서로 소인 자연수라고 하자. 그러면, $kn+l$ (단, n 은 자연수)의 형태의 소수의 개수는 무한이다.

그래서 문제 A는 디리클레의 정리에 의하여 해결된다.

이제 골드바하 가설에 관한 역사적인 배경과 이 가설과 관련된 여러 최근의 놀랄만한 결과들을 소개하겠다.

11) Peter Gustav Lejeune Dirichlet (1805~1859) : 수론을 연구한 독일 수학자.

1742년 6월 7일에 프러시아 수학자 골드바흐[12)]는

(6) 『5 보다 큰 자연수는 세 개의 소수들의 합으로 나타낼 수 있다』

라고 추측하는 내용이 담긴 편지를 스위스 수학자 오일러에게 보냈다. 예를 들면,

6=2+2+2, 7=2+2+3, 8=2+3+3, 9=3+3+3, 10=2+3+5,

11=3+3+5, 12=2+5+5, 13=3+5+5, 14=2+5+7, 15=3+5+7,

16=2+3+11=2+7+7, 17=3+3+11=3+7+7, 18=2+5+11,

19=3+5+11=5+7+7, 20=2+7+11, 21=3+7+11=5+5+11=7+7+7,

22=2+3+17=2+7+13, ··············

등이다. 오일러는 이 편지를 받은 후 이 문제에 흥미를 가지며

(7) 『2 보다 큰 짝수는 두 개의 소수의 합으로 나타낼 수 있다』

라는 가설을 내놓았다. 가령,

4=2+2, 6=3+3, 8=3+5, 10=3+7=5+5, 12=5+7,

14=3+11=7+7, 16=3+13=5+11, 18=5+13=7+11,

20=3+17, 22=5+17=11+11, 24=5+19=7+17=11+13,

26=13+13=7+19, 28=11+17, ··············

등이다. 주장 (6)을 삼변수(ternary) 골드바흐 가설이라 하고 주장 (7)을 이변수(binary) 골드바흐 가설이라고 부른다. 주장 (7)이 우리가 지금까지 익히 알고 있는 골드바흐 가설이다. 또한 우리는

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12) 크리스천 골드바흐 [Christian Goldbach (1690~1764)] : 프러시아 (지금은 러시아) 수학자. 수론 연구에
많은 업적을 남겼음.

(8) 『 9보다 큰 홀수는 세 개의 홀수인 소수의 합으로 나타낼 수 있다』

라는 추측을 약한(weak) 골드바흐 가설이라 한다. 예를 들면,

11=3+3+5, 13=3+3+7=3+5+5, 15=3+5+7=5+5+5,

15=3+5+7=5+5+5, 17=3+7+7=5+5+7, ……………

등이다.

약 260년이 지난 지금까지도 골드바흐 가설은 풀리지 않았다! 유명한 전문 수학자들뿐만 아니라 아마추어 수학자들이 이 가설을 풀려고 시도하였지만 실패하였다. 1973년에 중국 수학자 첸징륜[13](Jing-Run Chen)는 그의 논문 [1], [2]에서 체(sieve) 방법을 사용하여

(9) 『 p 가 소수로 $p+2$ 가 소수이든가 $p+2$ 가 두 소수의 곱이 되는 p 가 무수히 많이 존재한다 』

는 사실을 증명하였다. 게다가 첸징륜은 상기의 논문에서

(10) 『 충분히 큰 임의의 짝수는 모두 한 소수와 두 소수의 곱의 형태로 쓸 수 있다 』

라는 사실을 증명하였다. 예를 들면,

$14=3+11=5+3\times 3$, $32=13+19=7+5\times 5$,

$102=11+91=11+7\times 13$, ……………

등이다. 그래서 보통 골드바흐 가설을 <1+1> 로 표시하고 첸징륜의 결과는 <1+2> 로 표시한다고 한다. 처음의 1 은 하나의 소수를 나타내고 다음 숫자 2는 두 개의 소수의 곱을 의미한다. 첸칭륜의 놀랄만한 결과 (10)은 1966년에 발견되

13) 진경윤 [陳景潤 (1933~1996)] : 1953년에 중국 Xiamen 대학을 졸업하고, 중국 과학 학술원에서 유명한 중국 수학자 Hua Luogeng (1910~1985)의 지도아래 해석적 수론을 공부하였음.

어 증명되었다는 사실이 전 세계에 알려졌다. 그러나 그 당시에 중국 문화의 역사를 후퇴시켰던 문화혁명이란 소용돌이로 인하여 7년이 지나서야 (10)의 완벽한 증명이 수학저널 [1]에 게재되어 출판되었다. 첸칭륜은 이 연구결과로 인하여 하루아침에 전 세계에서 유명한 인사가 되었다. 보다 자세한 것은 부록을 참고하길 바란다.

1975년에 휴 몽고메리(Hugh Montgomery)와 로버트 보간(Robert Vaughan)은 **『대부분의 짝수들은 두 개의 소수의 합으로 나타낼 수 있다』**는 사실을 증명하였고, 2002년에는 영국 수학자 로저 히드-브라운(Roger Heath-Brown)은 그의 동료와 함께 **『상당히 큰 짝수는 두 개의 소수와 2^{13} 의 합으로 쓸 수 있다』**라는 사실을 증명하였다.

잘 알려진 쌍둥이 소수(twin primes) 문제에 관하여 이야기할까한다.

$$3,5\ ;\ 5,7\ ;\ \cdots\ 10016957, 10016959\ ;\ \cdots\ ;\ 10^9+7, 10^9+9\ ;\ \cdots$$

와 같이 차가 2 인 소수 짝을 「소수 쌍둥이(prime twin)」이라고 한다. 100,000 보다 작은 소수 쌍둥이의 개수는 1224 개이고 1,000,000 보다 작은 소수 쌍둥이의 수는 8164 개이다. 지금까지 알려진 소수 쌍둥이 중에서 가장 큰 것은

$$2{,}003{,}663{,}613 \cdot 2^{195{,}000} - 1,\ \ 2{,}003{,}663{,}613 \cdot 2^{195{,}000} + 1$$

이다. 이것은 지난 2007년 1월 15일에 발견되었다. 이제 흥미롭고 자연스런 문제를 제기할 수 있다.

(PTP) "소수 쌍둥이의 개수는 무한인가 ?"

(PTP) 문제는 아직까지도 해결되지 않았다. 많은 전문가들은 소수 쌍둥이의 개수가 무한이라고 추측하고 있다. 그래서 이 추측을 **소수 쌍둥이 추측**(Twin Prime Conjecture)라고 한다. 보다 나아가 아래의 문제를 제기할 수 있다.

(PTP*) "{ p | 단, $p, p+2, p+6$ 은 모두 소수 }의 개수는 무한인가?"

물론 이 문제도 풀리지 않았다.

소수 쌍둥이 추측과 관련된 놀랄만한 결과가 약 2년 전에 얻어졌다. 2005년에 미국 수학자 골드스톤(Dan Goldston), 항가리 수학자 핀츠(Janos Pintz) 와 터키 수학자 일디림(Cem Yildirim)은 함께 쓴 논문 [8]에서

$$\text{f}-\lim_{n\to\infty}\frac{p_{n+1}-p_n}{\log p_n}=0$$

임을 증명하였다. 여기서, f−lim[14] 는 limit inferior 을 나타내고, p_n 는 n 번째 소수이다. 게다가 그들은 최근에 논문 [9]에서

$$\text{f}-\lim_{n\to\infty}\frac{p_{n+1}-p_n}{\sqrt{\log p_n(\log\log p_n)^2}}<\infty$$

임을 증명하였다. 터키 수학자 일디림은 2002년 7월에 전북대학에서 개최된 국제학술회의에서 「리만제타함수와 소수의 분포에 관하여」란 제목으로 초청강연을 하였다. 그 후 연세대학에서 비슷한 토픽으로 집중강연을 하기도 하였다. 그는 아직 결혼하지 않고 독신으로 지내고 있다.

다음은 자연수가 아닌 수인 원주율 π 에 대하여 간략하게 언급하겠다.
원주율 π 는

(원둘레) ÷ (지름)

으로 정의되는 수이다. 이 정의에서 원주율 π 는 기하학적이며 아름다운 수임을 알 수 있다. 원주율 π 는 자연수도 아니고 분수도 아님이 알려져 있다. 실제로

$$\pi = 3.14159265358979323846264338327950288 4197\cdots\cdots$$

로 주어지며 끝이 없는 무한 소수이다. 또한 π 는

$$\pi=4\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\cdots\right)$$

14) 한글(hwp)에서 limit inferior의 기호를 나타낼 수 없어 상기와 같은 기호를 사용하였음.

의 무한 합으로 나타낼 수 있다. 최근에 들리는 소문에 의하면 뉴욕에 거주하고 있는 러시아 출신의 처드노프스키(Chudnovsky) 형제는 소수점 이하 80억 자리의 계산을 이미 끝내고 소수점 이하 1조 자리까지의 계산을 진행 중이라고 한다. 그리고 18세기에 스위스 수학자 오일러는

$$\frac{6}{\pi^2}=\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{25}\right)\left(1-\frac{1}{49}\right)\left(1-\frac{1}{121}\right)\left(1-\frac{1}{169}\right)\cdots$$

즉,

$$\frac{6}{\pi^2}=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{11^2}\right)\left(1-\frac{1}{13^2}\right)\cdots$$

와 같은 아름다운 공식이 성립한다는 사실을 증명하였다. 이 공식에서 원주율 π 는 모든 소수와 밀접한 관계가 있다는 것을 추측할 수 있다. 1882년에 독일 수학자 Ferdinand von Lindemann (1852~1939)는 π 가 초월수임을 증명하였다. 즉, 원주율 π 가 유리수 계수를 갖는 다항식의 영점(zero)이 될 수 없다. 그래서 많은 수학자들뿐만 아니라 일반 사람들이 원주율 π 에 매료되어 이에 관한 저서도 많이 발간하였으며 연구하였다. 참으로 원주율 π 는 기하학적이고 정수론적인 성질을 지닌 아주 신비롭고, 아름다운 수일뿐만 아니라 우주적인 수라고 할 수 있다.

참고로 지금 특히 중국 수학자들을 포함하여 여러 수학자들이 다음의 문제를 연구하고 있다는 사실을 언급하고 싶다.

문제 : 홀수인 자연수를 항상

$$2^n+p, \quad 단, \quad p\text{는 소수임}$$

의 형태로 쓸 수 있을까 ?

가령, $5=2^1+3,\quad 7=2^2+3,\quad 9=2^2+7,\quad 13=2^1+11,\ \cdots,$

$29=2^4+13,\quad 31=2^1+29,\ \cdots,\quad 37=2^3+29,\cdots,$

끝으로 소수에 관한 흥미로운 문제들을 소개하겠다.

문제 I. N을 주어져 있는 자연수라고 하자. 아래 형태의 수

$$n^2-n+p, \quad 0 \le n \le N, \quad \text{단, } n \text{은 자연수}$$

가 모두 소수가 되게 하는 소수 p가 있느냐 ? 이 문제도 역시 아직까지도 풀리지 않았다.

예. (1) $N=16,\ p=17.$ (2) $N=40,\ p=41.$

문제 II. n^2+1 (단, n은 자연수)의 형태의 소수의 개수는 무한인가 ? 가령,

$$2,\ 5,\ 17, 37,\ \cdots,\ 65537,\ \cdots .$$

이 문제의 해답은 아직까지도 모르고 있다.

문제 III. p_n을 n번째 소수라고 하자. 집합 $\{p_n-p_{n-1} | \, n \text{은 자연수}\}$의 원소 중에서 가장 큰 값은 ? 또, 아래의 극한값

$$\lim_{n\to\infty}(p_n-p_{n-1}) = \ ?$$

은 무엇일까 ?

문제 IV. n이 자연수라고 하자.

n과 $2n$사이에 소수가 존재하느냐?

이 질문은 Bertrand의 문제로 알려져 왔는데 체비쉐프에 의하여 이 질문이 옳다는 사실이 밝혀졌다.

[참고문헌]

[1] J.-R. Chen, On the representation of a large even integer as the sum of a prime and a product of at most two primes, Sci. Sinica, 16 (1973), 157-176.

[2] J.-R. Chen, On the representation of a large even integer as the sum of a prime and the product of at most two primes [Chinese], J. Kexuse Tongbao, 17 (1966), 385-386.

[3] P. Deligne, La conjecture de Weil, I, Publ. Math. IHES, Vol. 43 (1974), 273-307.

[4] P. Deligne, La conjecture de Weil II, Publ. Math. IHES, Vol. 52(1980), 137-252.

[5] C. Deninger, On the Γ-factors attached to motives, Invent. Math. vol. 104 (1991), 245-261.

[6] C. Deninger, Local L-factors of motives and regularized determinants, Invent. Math. Vol. 107 (1992), 135-150.

[7] C. Deninger, Evidence for a cohomological approach to analytic number theory, Proc. of European Congress of Mathematicians, Birkhäuser (1994), 491-510.

[8] D. Goldston, J. Pintz and C. Yildirim, Primes in tuples I, to appear in Annals of Math. or math.NT/0508185.

[9] D. Goldston, J. Pintz and C. Yildirim, Primes in tuples II, arXiv:math. NT/0710.2728v1.

[10] D. A. Hejhal, The Selberg trace formula and the Riemann zeta function, Duke Math. J. Vol. 43, No. 3 (1976), 441-482.

[11] L. K. Hua, Introduction to Number Theory, Springer-Verlag (1982).

[12] N. Koblitz, Introduction to Elliptic Curves and Modular Forms, Springer-Verlag (1984).

[13] Y. Manin, Lectures on Zeta Functions and Motives, Lecture Notes (Columbia Univ.), 1991.

[14] M. Monastyrsky, Riemann, topology and physics, Birkhäuser (1987).

[15] Y. Motohashi, Spectral theory of the Riemann zeta function, Cambridge Univ. Press (1997).

[16] B. Riemann, Über die Anzahl der Primzahlen unter einen gegebenen Grösse, Gesammelte mathematischen Werke (1859), 145-153 (Dover, 1953).

[17] A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, J. Indian Math. Soc., Vol. 20 (1956), 47-87.

[18] C. L. Siegel, Über Riemanns Nachlass zur analytischen Zahlentheorie, Quellen und Studien zur Geschichte der Mathematik, Astromie und Physik 2 (1932), 45-80.

[19] A. Voros, Spectral Functions, Special Functions and the Selberg Trace Formula, Communications Math. Physics, Vol. 111 (1987), 439-465.

[20] A. Weil, On the Riemann Hypotheses in function-fields, Proc. Nat. Acad. Sci., Vol. 27 (1941), 345-347.

[21] A. Weil, Numbers of solutions of equations in finite fields, Bull. Amer. Math. Soc., Vol. 55 (1949), 497-508.

[22] D. Zagier, Eisenstein series and the Riemann zeta function, Tata Institute for Fundamental Research, Bombay (1979), 275-301.

[23] D. Zagier, Zetafunktionen und quadratische Körper, Hochschultext, Springer-Verlag (1981).

참고로 리만가설, 골드바흐 가설, 소수 쌍둥이 추측과 관련된 흥미로운 논문과 책을 아래에 소개하겠다.

[a] J. B. Conrey, The Riemann Hypothesis, Notices of the AMS, 50 (2003), no. 3, 341-353.

[b] H. M. Edwards, Riemann's Zeta Function, Academic Press, New York, 1974.

[c] D. A. Goldston, Are there infinitely many twin primes ?, arXiv:0710.2123v1 [math.NT] (2007).

[d] A. Odlyzko, The 10^{22}-nd zero of the Riemann zeta function, Contemp. Math. Series, AMS. Providence, RI (2001), 139-144.

[e] K. Soundararajan, Small gaps between prime numbers: the work of Goldston-Pintz-Yildirim, Bulletin of the AMS, vol. 44, No. 1 (2007), 1-18.

[f] J. Wu, Chen's double sieve, Goldbach's conjecture and the twin prime problem 2, arXiv:0709.3764v1 [math.NT] (2007).

[부록]

셀버그(Atle Selberg)의 명복을 빌며

지난 8월 6일에 노르웨이 수학자 애틀레 셀버그가 90세의 나이로 세상을 떠났다는 소식을 최근에 여러 매체로 통하여 접하게 되었다. 뉴욕 타임즈, LA 타임즈, 워싱턴 포스트 등의 주요 일간지가 그의 사망 소식을 전하며 그의 일생과 뛰어난 연구 업적을 다루었다. 셀버그는 필자와 학문적으로 인연이 많은 수학자 중의 한 사람이다. 필자가 U. C. Berkeley에서 박사과정을 하고 있을 때 인도 수학회의 수학저널에 게재된 그의 유명한 논문[15]을 이해하려고 지도교수와 함께 몇 달 동안 세미나를 하면서 고군분투한 기억이 지금도 생생하게 떠오른다. 유학 첫해에 버클리에서 필자는 이치로 사타케 교수의 강의를 수강하면서 셀버그의 논문과 그의 업적을 알게 되었다. 물론 위대한 수학자 지겔(Carl Ludwig Siegel, 1896~1981)과 베이유(André Weil, 1906~1998)의 위대한 업적도 알게 되었다. 그때 수학이란 학문이 어떠한 학문인가를 진지하게 생각하였던 기억이 떠오른다.

셀버그는 1917년 6월 14일에 노르웨이의 랑게순드(Langesund)에서 태어났다. 17세가 되던 해에 우연히 노르웨이 수학저널에 게재된 인도 수학자 라마누잔에 관한 논문 『The Indian Srinivasa Ramanujan, a remarkable mathematical genius[16]』을 읽게 되었다. 이 논문은 주로 라마누잔의 흥미로운 일생과 그의 신비로운 여러 공식을 다루고 있었다. 이를 계기로 그는 수학에 매료되어 수학에 관심을 갖게 되었고, 1936년 노르웨이의 수도인 오슬로에서 개최되었던 ICM에서 독일의 수학자인 헥케(Erick Hecke, 1887~1947)의 강연을 듣고 큰 감명을 받은 후 수학자의 길을 걷겠다는 일생에 중대한 결심을 하게 되었다고 먼 훗날에 회고를 하고 있다[17]. 그는 1943년에 오슬로 대학에서 리만 제타 함수의 영점에 관한 연구로 박사학위를 취득하였다. 1942년부터 1947년까지 오슬로 대학에서 특별연구원으로 근무하다가 1947년 지겔의 초청으로 프린스턴 고등연구소(이하 IAS로 약칭)에 초청되었다. IAS에서 1년을 보낸 후 시라쿠스(Syracus) 대학의 수학과에 부교수로 임명되었다. 1948년 여름에 복소 함수론의 이론을 전혀 사용하지 않

15) Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, J. Indian Math. Soc. B., vol. 20 (1956), 47-87.

16) 이 논문의 저자는 오슬로 대학의 수학과의 교수인 스퇴르머 (Carl Störmer)이다.

17) 『Reflections around the Ramanujan centenary』: Alte Selberg, Collected Papers, Vol. I, Springer-Verlag (1989), 695-709.

고 아주 초등적인 방법으로 소수정리[18]를 증명하였다. 이 증명은 기술적으로는 초보적이지만 계산 자체는 상당히 복잡하다. 그는 이 증명에 관한 논문[19]을 작성하여 다음 해에 프린스턴 대학의 수학연보에 발표하였다. 그는 이 업적으로 바일(Hermann Weyl, 1885~1955)과 지겔(1896~1981)의 추천으로 1949년에 다시 IAS로 초청된 후 1950년에 영구직 연구원으로 임명되었다. 1950년에 미국의 캠브리지의 하버드 대학에서 개최되었던 ICM에서 필즈상을 수상하였고 1986년에는 울프상(Wolf Prize)을 수상하였다. 1951년에 IAS의 영구직 교수가 되었으며 1987년에 정년퇴임을 한 후 세상을 떠나기 전까지 IAS의 명예교수로 있었다.

그의 큰 연구 업적 중의 하나가 소위 셀버그 대각합 공식(the Selberg Trace Formula)이다. 이 공식은 Poisson 합 공식의 일반화라고 할 수 있는데 이 심오하고 아름다운 연구 내용은 앞에서 소개한 인도 수학저널(주석 [1]을 참고)에서 소개되어 그 후 보형형식의 이론, 군 표현론과 이론 물리학 분야에 지대한 영향을 끼쳤다. 예를 들면, 이 대각합 공식의 이론은 후에 랑그랭즈에게 큰 영감을 주어 1960년대 중반에 소위 『랑그랭즈 프로그램』의 탄생에 큰 영향을 끼쳤다. 현재 캐나다 토론토 대학의 아더(James Arthur) 교수가 대각합 공식의 이론을 일반화하며 수십 년 간 계속 연구하여 오고 있다. 셀버그는 리만가설[20]의 해결을 위해 노력하여 왔으며 이 가설과 관련된 여러 연구를 하였다. 즉, 리만 제타 함수의 영점에 관한 연구결과를 발표하였고, 셀버그 제타함수와 체 이론(sieve theory)[21]을 창안하여 연구하였다. 그의 이름을 딴 셀버그 적분, 란킨-셀버그 L-함수, 셀버그 고유값 가설 등을 보더라도 그가 해석적 정수론 분야에서 위대한 업적을 남겼다는 사실을 알 수 있다. 특히 유명한 중국의 수학자 첸징륜(陳景潤, Jingrun Chen, 1933~1996)은 그의 논문[22]에서 셀버그의 체 방법(sieve method)을 사용하여 『p 가 소수로 $p+2$ 가 소수이든가 $p+2$ 가 두 소수의 곱이 되는 소수 p 가 무수히 많이 존재한다』는 사실과 『충분히 큰 임의의 짝수는 모두 한 소수와 두 소수의 곱의 합의 형태로 나타낼 수 있다』라는 탁월한 결과를 증명하였다. 예를 들면,

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18) Prime Number Theorem : 『20세기 수학자들과의 만남 [저자: 양재현, 경문사 (2005년 2쇄)』의 280~285쪽을 참고하길 바람.

19) An elementary proof of prime number theorem, Annals of Mathematics, vol. 50 (1949), No. 2, 305~313.

20) the Riemann Hypothesis : 이에 관하여 일반인들이 쉽게 이해할 수 있도록 쓴 논문으로 『리만 가설에 관하여 : 저자 양재현』[[20세기 수학자들과의 만남]; 경문사 2005년 2쇄; 268~290쪽과 298~302쪽)을 추천함.

21) Alte Selberg, 『Lectures on sieves』, Collected Papers, Vol. II, Springer-Verlag (1991), 65-251.

22) J. R. Chen, On the representation of a large even integer as the sum of a prime and the product of at most two primes, Sci. Sinica 16 (1973), 157-176 또는 J. Kexue Tongbao 17 (1966), 385-386.

$$32 = 7 + 5 \times 5, \qquad 100 = 23 + 7 \times 11, \qquad 102 = 11 + 13 \times 7.$$

첸칭룬의 상기의 두 번째 결과는 소위 『골드바흐 가설』을 해결하려고 시도하며 연구하는 가운데서 얻어 졌다. 1999년에 중국 정부는 이 뛰어난 업적을 기념하기 위하여 『골드바흐 가설에 관한 최고의 업적』이란 제목으로 첸칭룬의 실루에트를 실은 우표를 발행하였다. 또한 그의 유명한 부등식

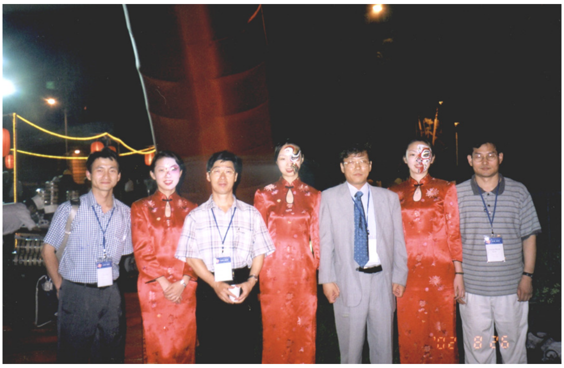
$$P_x(1,2) \geq \frac{0.67 x C_x}{(\log x)^2}$$

도 이 우표에 인쇄되어 있다. 그의 박사학위 논문 지도교수가 다름아닌 저명한 루오겡 후아(Luogeng Hua, 華羅庚[화라경]; 1910~1985)이다.

필자는 2001년 2월에 랑그랭즈 교수의 초청으로 IAS에 단기간 초청된 적이 있었다. 매주 월요일에서 금요일까지 매일 3시에 풀드 홀(Fuld Hall)의 1층에서 다과회가 있는데 한번은 이 자리에서 셀버그를 본 적이 있었다. 그 당시 84세의 나이에도 불구하고 건강하여 보였으며 미남형의 얼굴이었다. 필자는 그가 아주 우아하고 멋있게 살아오며 아름답고 심오한 진리를 탐구 하며 곱게 늙었다는 느낌을 강렬하게 받았던 기억이 난다. 이전에 우리나라에 셀버그 같은 뛰어난 수학자가 있었더라면 대한민국의 수학의 위상이 국제적으로 높은 위상에 있고 적지 않는 뛰어난 수학자가 배출되었으리라는 생각을 가끔 하여 보았다. 이제 국내에도 셀버그처럼 심오하고 아름다운 수학 분야의 연구를 하며 세계 수학계에 지대한 공헌을 할 뿐만 아니라 우아하고 고상하게 늙어가는 수학자를 보기를 기대하여 본다.

끝으로 그의 별세에 심심한 애도의 뜻을 표함과 동시에 그의 명복을 빌며 이 글을 끝맺는다.

[L] Essays



북경(Beijing)에서 중국 미인들과 손잡고 (2002년 8월)

John Nash와 중국 학생들 (ICM, August 2002, Beijing)

추상 수학의 어머니 에미 뇌데르(Emmy Noether, 1882~1935)의 생애

양재현 (인하대)

편집자주: 양재현 교수 지음 "20세기 수학자들과의 만남" (1998, 도서출판 애플기획 발행. 8,000원)의 60~67쪽에서 저자의 양해를 얻어 옮긴 것이다. 표지의 그림은 호주 수학재단(Australian Mathemaics Trust)에서 파는 T셔츠의 도안으로 뇌데르의 이론을 예시한 것이다. "18**Z**(18수 배수의 집합)으로 시작하는 모든 체인(Chain)은 유한하다."는 것을 그림으로 나타내었다.

상대성 이론의 창시자인 물리학자 알버트 아인슈타인이 한 여성 수학자의 죽음을 애도한다는 글이 1935년 5월 5일에 발간된 뉴욕타임즈에 실려 많은 사람의 관심을 끈 적이 있었다. 아인슈타인이 언급한 여성은 다름아닌 현대 추상 대수학의 창시자 에미 뇌데르(Amalie Emmy Noether)이다. 뇌데르는 20세기 초반, 남성들의 독무대이다시피 한 독일 수학계에서 많은 시련과 난관을 극복하여 20세기의 위대한 수학자로서의 지위를 확보하였다. 그녀는 괴팅겐대학에서 강의하며 연구하는 중에 1930년대 초반 나치의 탄압에 견디다 못해 1933년 괴팅겐대학에서 쫓겨나 미국에 정착한지 2년이 채 안된 1935년 4월 14일에 53세의 나이로 한많은 생을 마감하였다.

에미 뇌데르는 독일 남부 지방의 조그만 도시 에르랑겐(Erlangen)에서 1882년 3월 23일에 유태인 부모인 막스 뇌데르(Max Noether, 1844~1921)와 아이다 아말리아 카우프만(Ida Amalia Kauffmann, 1852~1915) 사이의 맏딸로 태어났다. 둘째 남동생 프리츠(Fritz Noether, 1884~?)는 1922년에 브레스라우(Breslau)대학에서 응용수학 분야의 교수로 역임하다가 1934년 나치의 탄압에 못이겨 러시아로 망명하여 시베리아에 있는 톰스크대학의 교수로 임용되었다. 그녀의 아버지는 1844년에 만하임에서 태어나 1875년에 에르랑겐대학의 수학교수가 되었으며 독일 수학계에서 대수함수의 이론으로 잘 알려진 수학자였다.

그 당시에는 에르랑겐대학은 수학 분야에서 명성이 나 있었다. 왜냐하면, 슈타우트(Christian von Staudt, 1798~1867), 크라인(Felix Klein, 1849~1925), 고르단(Paul Gordan, 1837~1912), 막스 뇌데르 등의 유명한 수학자들이 수학교수로 역임하였기 때문이었다. 특히, 클라인의 1872년의 교수 취임 연설은 매우 유명하여 「에르랑겐 프로그램」으로 지금까

지 잘 알려져 있다.

뇌데르는 어렸을 때에는 그다지 뛰어난 학생이 아니었고, 18세가 되던 해부터 수학에 깊은 관심을 가지기 시작하였다. 그 당시에는 여성들에게 대학의 입학이 공식적으로 허용되지 않았기 때문에 교수들의 특별한 허락을 얻어 1900~1902년에는 에르랑겐대학에서, 1903~1904년에는 괴팅겐대학에서 청강하였다. 1904년에는 에르랑겐대학의 입학 허가를 받아 고르단과 함께 연구하였으며, 1907년에는 그의 지도를 받으며 쓴 논문「Über die Bildung des Formensystems der ternären biquadratischen Form : Crelle 134 (1908), pp. 23-90 (영역: On the construction of the systems of forms for the ternary biquadratic forms)으로 박사 학위를 받았다. 고르단이 1910년에 은퇴한 후에는 피셔(Ernst Fischer, 1875~1954)와 슈미트(Erhard Schmidt, 1876~1959)와 함께 연구하기도 하였다. 그녀는 그 후 힐버트의 불변론의 연구에 큰 영향을 받아 여러 편의 불변론에 관한 논문을 발표하여 수학자로서의 명성을 얻기 시작하였다.

1916년, 힐버트와 클라인은 그녀에게 능력을 발휘할 기회를 주기 위하여 괴팅겐에 초청하였다. 그 당시 힐버트는 일반 상대성이론에 몰두하고 있었고 클라인은 상대성이론과 이와「에르랑겐 프로그램」사이의 연관성에 관하여 연구하고 있었다. 그래서 그들은 그녀의 불변론에 관한 해박한 지식이 그들의 연구에 큰 도움이 되리라고 믿고 있었다. 실제로, 그녀는 곧 미분 불변량(differential invariants)에 관한 여러 결과를 내어 그들의 연구에 도움을 주었다.

남녀 성별 차별이 심하던 당시, 그녀는 사강사(Privatdozent) 자격증이 없었기 때문에 생계유지가 쉽지 않았다. 제1차 세계 대전에 패한 후, 독일 공화국이 여러 규제를 완화하는 바람에 1919년이 되어서야 사강사 자격을 취득할 수가 있었다. 그리고 1922년에는 괴팅겐대학에서 영구직이 아닌 교수직(nichtbeamteter ausserordentlicher Professer mit Lehrauftrag)을 얻었다. 이 교수직은 봉급이 없는 형식적인 직함이었다. 교수직은 1933년까지 역임하였다.

1919년 이후로는, 불변론에서 환의 아이디얼 이론으로 관심을 바꾸어, 1921년 논문「Idealtheorie in Ringbereichen : Math. Ann. 83 (1921), pp. 24-66」을 발표하였다. 이 논문은 현대 대수학의 기본이 되었다. 이 논문에서, 서양장기 세계 챔피언인 라스커(Emanuel Lasker, 1868~1941)가 다항식 환에서 아이디얼이 준소 아이디얼들의 교집합으로 분해된다는 결과를 일반화하여 오름 연쇄조건을 만족하는 가환 환에도 같은 결과가 성립함을 보임으로써 오늘날의 뇌데르 환이 탄생하게 되었다.

그녀는 지도하고 있는 학생들을 아주 사랑하였을 뿐만 아니라 그들의 사생활까지도 깊은 관심을 보일 정도로 학생들에게 자상하였다. 학생들 중에 듀링(Max Deuring, 1907~), 비트(Ernst Witt, 1911~), 베버(Werner Weber, 1906~), 크룰(Wolfgang Krull, 1899~1971) 등은 후에 유명한 수학자가 되었다. 또한 학생들과 함께 소위,「The Noether boys」라고 불리는 다소 시끄럽고 격렬한 그룹을 형성하였을 정도로 학생들 간의 친분이 돈독하였다.

1924년경 네델란드에서 온 반 데르 바에르덴(B. L. van der Waerden, 1903~)는 그녀에게서 학문적으로 많이 배우고, 그녀와 함께 연구를 하기도 하였다. 그 후, 그는 암스테르담으로 돌아가 두 권으로 된「현대 대수학」이라는 책을 저술하였는데, 그 중 제2권에서는 뇌데르의 주요 연구 내용을 다루었다. 1927년부터는 하세(Helmut Hasse, 1898~1979)와 부라우어(Richard Brauer, 1901~1977) 등의 수학자들과 함께 비가환대수에 관하여 공

동 연구를 하였다. 이들의 공동 연구는 상당히 성공적이었다. 1920년대에 괴팅겐대학을 자주 방문하였던 러시아 수학자 알렉산드로프(Pavel Sergeevich Alexandrov, 1896~1982)와 두터운 친분을 가졌고 그의 위상학적인 연구에 큰 영향을 끼쳤다. 1930년에는 한 학기 동안을 모스크바대학에서 보내었으며 거기서 폰트라야진(Lev Semenovich Pontrjagin, 1908~)을 만나 친분을 돈독히 하였다. 1928~1929년 사이 한 학기를 프랑크푸르트대학에서 강의를 하였으고 이 기간 동안 지겔은 괴팅겐대학의 방문교수 자격으로 강의를 하였다. 강의에는 소질이 없었다고 전해지고 있다. 연구와 강의 이외에도 독일의 수학저널「Mathematische Annalen」의 편집인으로 여러 해 동안 일했고, 많은 논문들이 그녀의 이름보다도 동료들과 제자들의 이름으로 저널에 게재되었다고 한다.

1930년대 초반에 와서는 나치의 등장으로 인하여 유태인에 대한 탄압 정책이 시행되었다. 힐버트는 68세의 고령으로 은퇴하여 후임으로 취리히에서 온 봐일이 괴팅겐대학 수학과의 전반적인 행정을 책임지고 있었다. 란다우(Edmund Landau, 1877~1938), 쿠랑(Richard Courant, 1888~1972), 뇌데르 등의 유태인들은 학생들의 수업 거부와 학교 당국의 부당한 처사로 괴팅겐에서 연구하기가 어렵게 되었을 뿐만 아니라 살기 조차도 힘들게 되었다. 1933년 여름에 와서는 모든 유태인 교수들이 괴팅겐대학을 사직하여야 했으며, 대부분의 유태인들은 미국과 러시아 등지로 망명을 해야만 하였다. 부자였던 란다우는 망명을 가지 않고 있다가 1938년에 비참한 생을 마쳤다. 이전에, 란다우는 지겔의 지도교수였다. 봐일은 나치의 이러한 처사를 도저히 참을 수가 없어 그동안 망설여 왔던 프린스턴 고등연구소장인 아브라험 플렉스너의 초청을 수락하여 미국으로 망명하였다.

뇌데르는 1933년 여름에 미국의 브린 모르(Bryn Mawr)여자대학교의 객원교수로 초청받아 미국으로 망명하였다. 미국에 온 후에는 프린스턴 고등연구소에 초청되어 강의를 하기도 하였다. 또한 고등연구소의 여러 교수들과 친분을 다지기도 하였다. 그녀는 1935년에 받은 종양 수술의 휴유증의 악화로 얼마 되지 않아 갑작스럽게 세상을 떠났다. 그의 사망 소식이 전세계에 알려지자 많은 수학자들이 애도하였다.

생전에, 그녀의 업적이 인정을 받아 1928년의 볼로냐(Bologna)와 1932년의 취리히에서의 국제수학자총회에서 기조연설을 두차례 하였으며, 1932년에는 아르틴과 함께「Alfred Ackermann Teubner memorial award」라고 불리는 상을 공동 수상하였다. 그 당시 받은 상금은 500 마르크였다.

그녀는 땅딸막한데다가 뚱뚱하여 외모면에서는 여자로서의 매력은 없었지만, 마음은 곱고, 솔직담백하여 내면적으로는 상당히 매력이 있어 많은 남성 수학자들과 가깝게 지낼 수 있었다. 그녀는 평생동안 독신으로 지냈다. 이전에, 유명한 여성 수학자인 소냐 코발래프스키(Sonya Kovalevsky, 1850~1891)가 있다. 코발래프스키는 미모가 뛰어난데다가 여성다운 매력을 지녔다고 한다. 모스크바 수학가의 집안에서 자라난 그녀는 수학으로 인하여 인생이 불행하였던 반면에 뇌데르는 수학의 연구 과정에서 큰 즐거움을 얻었다.

참고문헌

August Dick (1981) *Emmy Noether, 1882~1935*. translated by H. I. Blocher, Birkhäuser .

고형준 (1997). The Mother of Abstract Algebra, Emmy Amalie Noether (1882~1935). 대한수학회 뉴스레터 **54**, pp. 45-47.

해외연구소 소개

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독일의 전 수도인 Bonn에는 세계적으로 널리 알려진 막스-플랑크 수학연구소(이하 MPIfM으로 약칭)가 있다. 필자는 1994년에 2개월, 1997년에 4개월 기간동안 MPI에 초청교수로 초빙되어 연구한 적이 있다. MPIfM은 1982년에 정식으로 설립되어 초대 소장인 F. Hirzebruch (1927-)에 의하여 그가 1996년에 은퇴하기까지 15년 동안 성공적으로 운영되어 독일 수학계뿐만 아니라 세계 수학계에 여러 측면에서 큰 공헌을 하였다. 그래서 1996년에는 옛 동독에 있는 Leipzig에 또 다른 막스-플랑크 수리과학연구소 (Max-Planck Institute for Mathematics in the Sciences ; 이하 MPIMS 약칭)가 설립되었다. 필자는 참고문헌 [6]에서 MPIfM에 관하여 간략하게 소개하였지만 여기서는 MPIfM의 설립배경, 활동내용, 인적구성 등에 관하여 보다 자세하게 살펴보기로 하겠다. 그리고 MPIMS에 대하여 아주 간략하게 언급하고 보다 자세한 것은 참고문헌 [3]과 MPIMS 홈페이지 [8]을 참조하길 바란다.

통일이 된 독일에는 현재 약 80개의 막스-플랑크 연구소가 있다. 이미 언급하였다시피 수학분야에서는 2개의 막스-플랑크 연구소가 있다. 막스-플랑크 연구소는 뮌헨에 본부를 두고 있는 막스-플랑크 협회 (Max-Planck Gesellschaft ; 이하 MPG 약칭)의 행정적인 지원과 재정적인 지원을 받으며 운영되며 수시로 MPG에 의하여 엄격한 평가를 받고 있다. MPG는 독립적이고 비영리 연구기관으로 1911년에 창립된 카이저-빌헬름 협회(Kaiser-Wilhelm-Gesellschaft ; 이하 KWG 약칭)의 유업을 이어 받아 1948년 2월에 설립되었다. 제2차 세계대전이 끝난 1945년에 KWG의 측근들이 노벨 물리학상의 수상자인 Max-Planck (1858-1947)에게 다시 한번 더 KWG의 회장직을 맡으며 KWG의 사업을 수행하여 줄 것을 부탁하였다. 그는 87세라는 고령으로 건강이 좋지 않아 회장직을 사양하였지만 1-2년 밖에 하지 않는다는 조건부로 수락하였다. 1946년에 KWG의 이름이 MPG로 바뀌었다. MPG의 초대 회장은 1944년에 노벨 화학상을 수상한 Otto Hahn (1879 -1968)이었다. 괴팅겐 시를 위해 크게 이바지한 공로로 Max-Planck의 가족묘와 Otto Hahn의 가족묘가 괴팅겐의 시립묘지에 안치되어 있으며 수학자로는 David Hilbert (1862 -1943)의 부부묘와 C. L. Siegel의 묘가 이 시립묘지에 안치되어 있다. (참고문헌 [6], 235쪽) MPG의 설립 목적은 일반 대학에서 수행하기가 힘든

기초분야의 수준 높은 연구를 연구소를 통하여 할 수 있도록 지원하는데 있다. 일반적으로 어떤 기초 분야에서 뛰어난 한 사람의 학자에 의하여 막스-플랑크 연구소가 설립되는데 나중에 적당한 후계자가 나타나지 않으면 이 연구소는 문을 닫게 된다. 이런 이유로 1996년에 4개의 연구소가 사라졌다. Otto Hahn은 1960년에 회장직을 1939년에 노벨 화학상을 수상한 Adolf Butenandt에게 넘겨주었다. 그 후 4명의 회장을 거쳐 현재 MPG의 회장은 이전에 괴팅겐의 막스-플랑크 생물리 화학연구소장직을 지냈던 Peter Gruss 이다.

설립 배경

19세기 초반부터 1930년대 초반까지 세계 수학계를 주도적으로 이끌어 왔던 괴팅겐 학파가 나치의 등장으로 인하여 베를린 학파와 함께 1930년 대 중반부터 급속도로 붕괴되기 시작하면서 독일의 수학이 1940년에 와서는 상당한 부분이 무너졌다. (참고문헌 [4], [5]) D. Hilbert (1862-1943)가 1933년에 "No more Mathematics in Göttingen"이라고 탄식하기도 하였다. (참고문헌 [4], 205 쪽) 1930년대 초반까지만 하여도 수학분야에서 유럽에 비하여 상당히 뒤떨어져 있던 미국에서는 프린스턴 고등연구소(이하 IAS 약칭)의 설립으로 유럽에서 활동하던 적지 않은 저명한 수학자들이 미국으로 몰려와 자리를 잡으며 연구하였다. 예를 들면, H. Weyl, E. Noether, C. L. Siegel, R. Courant, Von Neumann, K. Gödel 등이다. 제2차 세계대전 후에는 미국의 수학 수준이 유럽의 수준을 앞지르고 있었다. 예전에는 미국의 유능한 젊은 수학자들이 독일로 유학을 가는 형편이었지만 1950년대에는 거꾸로 독일의 젊은 유능한 수학자들이 미국으로 유학을 가는 형편이었다. 뮌헨대학에서 박사학위를 받은 Friedrich Hirzebruch (1927-)는 1952-54년의 기간동안 IAS에 연구원으로 초청되어 자신의 연구영역을 넓히며 연구하였다. 그 후 그는 프린스턴대학의 조교수로 근무하다가 1956년에 Bonn 대학의 교수로 초빙되었다. 그는 귀국한 후부터 줄곧 독일에 IAS와 유사한 수학연구소가 설립되어야 한다는 주장을 하였다. 예전에 괴팅겐대학의 교수였던 C. L. Siegel과 R. Courant의 부정적인 의견과 주위의 여러 여건이 여의치 못하여 수학연구소의 설립이 실현되지는 못하였지만 그 대신에 Bonn Mathematische Arbeitstagung (이하 Tagung 이라 약칭)이 정부의 재정적인 지원을 받으며 매년 Bonn에서 개최되었다. 1957년에 첫 번째 Tagung이 1,000 DM의 재정적인 지원으로 Bonn에서 개최되었다. 이때 Atiyah, Grothendieck, Kuiper, Tits 등의 저명한 젊은 수학자들이 초청되어 강연하였다. 그 후로는 정부의 지원금이 계속 늘어나면서 매년 Bonn에서 Hirzebruch에 의하여 Tagung이 개최되었다.

MPIfM의 성립 배경을 보다 잘 알기 위해서는 Mathematisches Forschungsinstitut Oberwolfach (이하 Oberwolfach로 약칭)과 Sonderforschungsbereiche (이하 SFB 로 약칭)를 간략하게 소개할 필요가 있다.

Oberwolfach는 1944년에 Wilhelm Süss (1895-1958)에 의하여 설립된 수학연구소로 이 연구소의 주요 목적은 연구보다는 학술회의를 자주 개최함과 동시에 수학 관련의 참고서적을 가능하면 많이 소장하여 많은 수학자들에게 혜택을 제공하는 것이었다. Süss는 제2차 세계대전 중에 나치 정권 아래에서 독일 수학회장을 역임하였고 Freiburg 대학의 총장을 지냈던 인물이다. 현재 이 연구소는 수학 연구공동체 (Gesellschaft für Mathematische Forschung e.v. ; 이하 GMF로 약칭)에 의하여 운영되고 있다. GMF는 1959년에 설립되었으며 Baden-Württemberg 주로부터 재정적인 지원을 받고 있다. 매년 거의 매주 약 50명 정도의 수학자가 참여하는 조그만 학술회의가 개최되고 있다. 1958년에 Hirzebruch는 Oberwolfach를 IAS와 유사한 수학연구소로 탈바꿈해야 한다는 제안서를 작성하여 여러 곳에 배포하였다. 그러나 수학연구소의 설립의 시도는 실패로 돌아가고 말았다.

SFB는 1969년에 DFG(Deutsche Forschungsgemeinschaft : 독일 과학재단)의 막대한 재정적인 지원 아래에서 Bonn 대학에 설립되었다. 첫해인 1969년에 SFB는 DFG로부터 143,525 DM를 지원받았다. 그 후에 SFB의 활발한 학술활동으로 인하여 Bonn 대학의 수학과는 국제적인 명성을 얻게 되었다. SFB는 1985년 12월 31일에 막을 내렸다.

SFB가 얼마나 오래 동안 계속 DFG의 지원을 받을 것인가는 불투명하였다. 이러는 가운데 1977-78년의 기간에 Hirzebruch는 MPG의 회장인 Reimar Lüst와 몇 차례 만나 SFB가 끝난 후 Bonn 대학에서의 앞으로의 수학연구활동의 전망에 관하여 토의하며 의견을 나누었다. 이 기간에 Hirzebruch는 Lüst에게 SFB를 MPIfM으로 탈바꿈하는 과정을 상세히 설명하였다. 그는 1978년 11월 1일에 3페이지 반 분량의 MPIfM 설립에 관한 제안서를 작성하여 MPG에 제출하였다. Lüst의 협조를 얻어 마침내 1980년 3월 7일에 MPG는 Bonn에 Hirzebruch를 초대 소장으로 하는 MPI를 설립하기로 결정하였다.

Hirzebruch를 중심으로 그의 주위의 여러 수학자와 함께 20여년 기간의 줄기찬 노력 끝에 MPIfM이 설립되었다고 할 수 있다.

연구소의 인적구성

MPIfM의 초대 소장은 Hirzebruch 이었으며 1996년에 은퇴하였다. 그는 Bonn 대학의 수학과 교수직과 MPIfM의 소장직을 겸직하며 임무를 성공적으로 수행하였다. 겸직 때문에 강의부담은 줄었지만 Bonn 대학에서 강의하였다. 은퇴한 지금도 그는 연구실을 계속 지니고 있으며 명예 소장으로 재직하고 있다. 그는 1988년에 Wolf 상을 수상하여 MPIfM의 위상을 높이기도 하였다.

Hirzebruch가 은퇴한 후로는 MPIfM의 운영방식이 변경되었다. 현재 4명의 영구직 교수가 있으며, 이들 모두가 소장이다. Yuri Manin (1937-), Gerd Faltings (1954-), Don Zagier (1950-)과 G. Harder 등이 현재 영구직 교수이며 managing director는 Harder이다. Faltings는 1983년에 Mordell 가설을 해결하여 이 업적으로 1986년에 Fields 상을 수상하였으며 프린스턴 대학의 수학과 교수를 역임하였다. 현재 Manin은 Northwestern 대학의 교수직을 겸직하고 있고, Harder는 Bonn 대학의 교수직을 겸직하고 있으며 Zagier 는 College de France의 교수직을 겸직하고 있다.

현재 약 90명의 방문연구원들이 있으며 이들 중에는 단기 방문연구자와 장기 방문 연구자가 있다. 아마도 매년 수백 명의 수학자들이 방문하여 연구하든가 아니면 초청되어 강연하는 것으로 알고 있다. 도서관, 전산, 행정담당의 수십 명의 staff들이 있다.

연구소의 연구활동

MPIfM은 주로 대수기하학, 정수론, arithmetic geometry, 보형형식론, 표현론, 복소기하학, 대수 및 미분 위상수학, 미분기하, 수리물리,

singularity, algebraic군, arithmetic군 등의 이론을 중점적으로 연구한다.

MPIfM의 연구활동은 weekly activity, long−term activity, 세미나, arbeitstagung, pre-print 발간으로 이루어져 있다.

먼저 weekly activity에 관하여 간략하게 설명하겠다. 학기 기간 동안 매주 월요일에는 오후 1시 45분에 대수기하학 및 복소기하학 세미나가 있고 오후 3시에는 위상수학 세미나가 있다. 매주 화요일에는 오후 2시에 수리물리학과 관련된 분야의 세미나가 있으며 얼마 전까지만 해도 Y. Manin에 의하여 주관되었다. 필자가 1997년에 MPIfM에 있을 때는 Manin은 거울대칭(mirror symmetry)과 quantum cohomology에 관하여 집중강연을 하였다.

매주 수요일에는 12시 30분경부터 number theory lunch가 있다. 오후 2시경에 정식으로 정수론 세미나가 시작된다. 정수론을 연구하는 교수와 방문연구원들이 세미나의 초청연사와 함께 세미나가 시작하기 1시간 30분 전에 레스토랑에서 점심을 하며 분위기를 띄운다. 이것이 소위 number theory lunch이다. 이 lunch에는 Hirzebruch, Faltings, Zagier는 거의 항상 참석하고 가끔 Harder가 참석한다. 하여튼 흥미로운 모임이다. 매주 목요일에는 오후 3시경에 콜로키엄의 성격을 띤 세미나가 있다.

long−term activity는 국제학술회의, work-shop, activity 등의 형태를 통하여 이루어진다. 2002년 5월−8월의 기간에는 Activity on Frobenius manifolds, quantum cohomology and singularities 가 Manin의 주관으로 수행되었고 1월−6월의 기간에는 Special activity in analytic number theory가 D. R. Heath−Brown 의 주관으로 수행되었다.

2002년 6월 10−14일의 기간에는 Zagier의 주관으로 Maass wave forms, Selberg zeta functions and Spin chains에 관한 workshop이 개최되었다. 2003년 8월 18−22일의 기간에는 Manin의 주관으로 Workshop on Noncommutative Geometry and Number Theory가 개최될 예정이고 2003년 9월 15−19일의 기간에는 Harder의 주관으로 Inter-national Conference on Arithmetic and Algebraic Geometry가 개최될 예정이다.

2002년 1월부터 M. Marcolli의 주관으로 매주 목요일 2시경에 gauge theory seminar가 이루어지고 있다.

앞에서 언급하였듯이 1957년부터 시작된 Tagung은 첫 번째부터 30번째 Tagung까지는 Hirzebruch에 의하여 개최되었다. 이것을 First Series의 Tagung이라 하고 1993년부터는 매 2년마다 6월경에 Second Series의 Taugung이 MPIfM의 4명의 소장들에 의하여 개최되고 있다. 금년에는 Second Series의 여섯 번째의 Tagung이 지난 6월 13−19일 기간 동안 열렸다. 이 Tagung에 1998년에 Fields상을 수상한 Maxim Kontsevich가 초청되어 개회 강연(opening lecture)을 하였다.

그리고 연구소의 영구직 교수와 방문연구원들이 연구소에서 이루어진 결과를 preprint의 형태로 발간하고 있다.

특색

1998년 이전에는 MPIfM의 건물은 Bonn 기차역에서 라인강을 건너 걸어서 약 30분 걸리는 곳(Gottfried−Claren 26)에 있었다. 여기서 약 300−400미터 되는 곳에 라인강이 도도히 흐르고 있다. 이 곳의 수학자들은 아주 자유로운 분위기에서 연구에 몰두할 수 있고 함께 토론하고 의견을 교환할 수 있는 사람이 있어 MPIfM에서 많은 좋은 결과가 나오고 있

다. 연구소의 수학자들은 아이디어가 떠오르지 않는다든가 다루고 있는 문제가 해결되지 않고 답답할 때는 아름다운 라인강변을 걸으며 마음의 피곤을 풀기도 한다. 틈이 나면 주말에는 가족과 친지들과 유람선을 타고 라인강의 경치를 감상할 수도 있다.

이전에 Bonn이 독일의 수도였지만 인구가 약 20만 밖에 되지 않는 조그만 도시이기 때문에 생활비가 그렇게 많이 들지 않는다. 이 연구소에서 주는 급여로 부족함이 없이 생활할 수 있다. 방문 수학자들은 숙소를 구하는데 큰 어려움이 없는 걸로 알고 있다. 예를 들면 연구소에서 라인강 건너편에 우니 크럽(Universität Club : 주로 Bonn 대학의 방문교수들을 위해 약 15년 전에 세워진 아파트이며 Bonn 대학에서 이 아파트를 관리하고 있다.)이 있어 연구소의 일부 방문객들이 이용하고 있다. 만약에 음악, 오페라, 미술 등의 예술에 관심이 있으면 Köln 에 가서 심포니, 오페라 공연 등을 즐길 수 있다. 물론 Bonn에도 연주 홀이나 오페라 하우스가 있지만 Köln이 도시가 크고 많은 면에서 Bonn보다 낫기 때문이다. Bonn에서 Köln에 가는데 기차로 약 15-20 분 걸린다.

1998년에 MPIfM은 Bonn 역의 근처로 이사를 갔다. MPIfM의 새 건물은 예전에 궁(palace)이었다가 우체국으로 사용되었다. 이 우체국을 개조하여 연구소의 새 건물이 만들어졌다. 새 건물은 Bonn 기차역에서 걸어서 1-2분 걸리는 곳에 있고 Bonn의 중심지에 위치하고 있다. 예전의 연구소의 건물은 아주 콤팍트하고 예쁜 4층 건물로 부엌과 욕실이 딸려 있어 마치 일반 아파트 같은 인상을 풍기고 있었다. 이 건물의 1층에는 많은 수학 서적과 저널이 진열되어 있고 약 30-40명을 수용할 수 있는 세미나실이 하나 있었다. 지하에는 오래 된 수학 저널과 자료들이 보관되어 있고, 4층에는 휴게실(common room)이 있어, 매일 4시경에 여러 수학자들이 모여 잡담을 한다든가 토론을 하기도 하였다. 2-3층은 주로 연구원들의 연구실로 이루어져 있었다. 대충 알 수 있다시피 예전의 건물은 공간이 협소하여 많은 자료들을 보관할 수 없을 뿐만 아니라 Bonn 대학과 다소 떨어져 있기 때문에 불편한 점이 있었다. 그래서 오래 전부터 Hirzebruch는 보다 공간이 넓은 건물로 이전하기 위해 MPG와 계속 협상한 끝에 현재의 건물로 이전하게 되었다. 지난 1월에 MPIfM을 잠시 방문하였는데 100명 이상을 수용할 수 있는 대형 강의실과 몇 개의 조그만 세미나실이 있으며 도서관은 이전보다 훨씬 넓었다. 그리고 연구실도 이전보다 훨씬 넓고 여유로워 보였다. 무엇보다도 Bonn 대학과 가까워져서 연구 활동하기가 전보다 훨씬 좋아졌다. IAS나 IHES(Institut des Hautes Études Scientifiques)처럼 목가적이고 수도원적인 분위기는 아니지만 나름대로 외부와의 방음시설이 잘 되어 있어 건물내부는 조용하기 짝이 없다.

앞에서 언급하였다시피 연구소의 소장들이 겸직을 하고 있기 때문에 각자의 소속 대학에서 강의를 한다는 것이 다른 연구소와 다른 점이다. 그리고 소장들이 Bonn 대학의 대학원생들을 논문 지도한다는 점도 일반 다른 연구소와의 차이점이다. 예를 들면 1998년에 Fields 상을 수상한 M. Kontsevich는 1994년에 Bonn 대학에서 Ph.D.를 받았는데 공식적인 지도교수는 D. Zagier이다. 말할 것도 없이 Hirzebruch와 Harder는 Bonn 대학의 교수직도 가지고 있기 때문에 여러 Ph.D.를 배출하였다. Hirzebruch는 방문객들에게 공동연구를 권하고 있다.

막스-플랑크 수리과학연구소(MPIMS)

MPIMS는 1996년 3월 1일에 Leipzig에 설립되었다. MPIMS의 영구직 교수가 4명인데 모두가 소장이다. acting director는 Jürgen Jost이고 Wolfgang Hackbusch, Stefan Müller, Eberhard Zeidler가 소장이다.

MPIfM처럼 2년마다 acting director는 바뀐다. Leipzig 대학과 연계하여 운영되고 있다. MPIfM에서 다루지 않는 분야를 연구하고 있다. 이 연구소는 수리물리학과 관련된 리만 및 대수기하학, mathematical model, continuum mechanics, neutral networks, general relativity theory, quantum field theory, problems of mathematical biology, scientific computing 등의 분야를 주로 연구하고 있다.

이 연구소의 건물은 Leipzig 기차역에서 가까운 곳(Inselstrasse 22-26)에 있으며 Leipzig의 중심지역이라 할 수 있다. 필자는 1998년에 Berlin에서 개최된 ICM 기간 중에 잠시 틈을 내어 Leipzig를 방문한 적이 있었다. 관광버스를 타고 Leipzig를 대충 구경하였는데 기차역 주변만 개발되어 있고 나머지 지역은 통독되기 이전의 모습 그대로였다. 지금은 이 시의 상당 부분이 개발되었으리라 생각이 든다. 역 주변에는 위대한 작곡가 J. S. Bach (1685-1750)가 20여 년 동안 살던 집과 그의 묘가 안치되어 있는 교회가 있어, 시에서 이 주변을 아름답게 개발하여 관광수입을 올리고 있었다. 그리고 Faust를 쓴 J. W. Goethe (1749-1832)가 약 2년간 학창시절을 Leipzig에서 보냈다고 한다. 철학자 G. W. Leibniz (1646-1716)가 Leipzig 대학에서 학창시절을 보냈고, Felix Klein, Sophus Lie, E. Hopf, E. Kähler 등의 유명한 수학자들이 Leipzig 대학에서 근무하였다.

필자는 아직까지 MPIMS를 방문한 적이 없어 연구소의 홈페이지 [8]을 통하여 대충 이 연구소에 관하여 요약하였다.

재정적 지원 및 국내 수학연구소의 필요성

MPIfM과 MPIMS는 거의 전적으로 MPG를 통하여 정부로부터 재정적으로 지원을 받고 있다. 수학 수준이 우리보다 나은 독일에서도 국제적인 수학연구소를 설립하는데 많은 시간을 기다려야만 하였다. 미국에서는 수년전에 Clay Mathematics Institute (이하 CMI로 약칭)와 American Institute of Mathematics (이하 AIM으로 약칭)와 같은 수학연구소가 개인의 기부금으로 설립되었다. 이 연구소의 운영이 자연스럽게 성공적으로 잘 되고 있다고 생각한다. 인터넷 사이트 [9]와 [10]을 참조하길 바란다.

우리나라에서 CMI와 AIM과 같은 수학연구소를 설립하기 위해서는 우선 수학에 관한 정확한 인식을 일반 대중들이 할 수 있도록 대한수학회를 중심으로 체계적으로 노력하여야 한다는 것이 필자의 사견이다. 국내 대학에서는 수학에서 깊고 수준 높은 분야를 연구하기가 힘든 것이 현실이다. 국내의 뛰어난 수학자가 깊고 수준 높은 분야를 논문의 개수에 연연하지 않고 오랜 기간 연구할 수 있는 여건이 필요하다. 수학연구소가 이런 여건을 제공할 수 있어야 한다는 사실을 정부 당국(과기처, 교육부, 과학재단 등)뿐만 아니라 일반 대중에게 정확하고 체계적으로 알려야 한다고 생각한다. 예를 들면, 일본의 위대한 수학자 Kiyoshi Oka (1901-78)는 평생 약 10편의 논문 밖에 발표하지 않았다. 선배 수학자 Teiji Takagi (1875-1960)는 정부에 건의하여 Oka가 경제적 부담과 강의부담 없이 자유롭게 연구할 수 있도록

하여 주었다. 그래서 Oka는 산속으로 들어가 약 7년 동안 연구에 몰두할 수 있었다. 그 당시 세계에는 수학연구소가 거의 없던 시절이었다. 앞에서도 강조하였듯이 수학연구소가 이러한 역할을 하여야 한다. 이렇게 함으로써 국내 수학의 수준이 세계적인 수준으로 도약하리라 생각한다. 최근에 와서는 국내의 수학 수준이 상당히 높아 져서 이제는 국내에서 정부의 지원금과 개인의 기부금으로 운영되는 수학연구소의 설립이 절실하다고 생각한다.

참고문헌 및 인터넷 사이트

[1] Nobert Schappacher, Max-Planck Institut fűr Mathematik, the Mathematical Intelligencer, vol. 7, No. 2 (1985), 41-52.

[2] Max-Planck Gesellschaft (Berichte und Mitteilungen), Max-Planck-Institut für Mathematik, Bonn (1987).

[3] J. Jost, S. Müller and E. Zeidler, A New Max-Planck Institute for Mathematics in the Sciences in Leipzig, Notices of Ameri. Math. Soc., vol. 43, No. 2 (1996, October), 1125-1126.

[4] Constance Reid, Hilbert, Springer-Verlag (1996).

[5] Constance Reid, Courant, Springer-Verlag (1996).

[6] 양재현, 20세기 수학자들과의 만남, 도서출판 애플기획, 1999년 (중판).

[7] http://www.mpim-bonn.mpg.de (MPIfM의 홈페이지).

[8] http://www.mis.mpg.de (MPIMS의 홈페이지).

[9] http://www.claymath.org (Clay Mathematics Institute의 홈페이지).

[10] http://www.aimath.org (American Institute of Mathematics의 홈페이지).

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마침내 Poincaré Conjecture가 해결되다 !

양재현 (인하대학교)

지난 8월 22일 스페인의 수도 마드리드에서 개최된 국제수학자총회(간단히, ICM)의 개회식에서 4명의 수학자 Andrei Okounkov[1], Grigory (또는 Grisha) Perelman, Terrence Tao[2], Wendelin Werner[3]가 필즈상을 수상하였다. 9명의 저명한 수학자들로 구성된 필즈상 심사위원회가 약 6개월간 필즈상 수상자 선정 작업에 들어간다. 보통 시상식 3개월 전까지는 최종적으로 수상자가 선정된다. 그러면 국제수학연맹(간단히 IMU)의 회장은 전화로 수상자에게 이 선정사실을 알리고 시상식 전까지는 비밀로 할 것을 약속받는다. 필즈상 심사위원회의 명단도 시상식 전까지는 비밀이다. 올해는 IMU 회장인 John Ball은 6월경에 모든 수상자들에게 전화로 이 사실을 알렸다. 그러나 Perelman은 필즈상의 수상을 거부하여 과학 (특히, 수학)에 관심을 갖고 있는 많은 사람들에게 놀라움을 안겨주었다. 이 뉴스가 유명 일간지인 뉴욕타임즈, 워싱턴포스트 등의 세계 각국의 주요일간지에 대대적으로 보도되어 세계의 관심사가 되었다.(참고문헌 [9, 11, 12, 13, 16]) 지난 5월 말경에 Perelman이 필즈상 수상자로 확정되었다는 후문이 있다. Perelman이 수상을 거부하자 Ball은 6월 말경에 상트 페테스부르크(St. Petersburg) 근교에 살고 있는 그를 찾아가 마드리드에서 개최되는 ICM에서 기조 초청강연(plenary address)을 하여 줄 것을 간곡하게 부탁하였을 뿐만 아니라 수상 거부를 번복하기를 권하였다. 그러나 그는 한마디로 단호하

1) Andrei Okounkov (1969~) : 모스크바에서 출생하여 1995년에 모스크바 국립대학에서 박사학위를 취득하였음. 2004년에 유럽수학회 상을 수상하고 2006년에는 필즈상을 수상함. 현재 프린스턴 대학의 교수임.

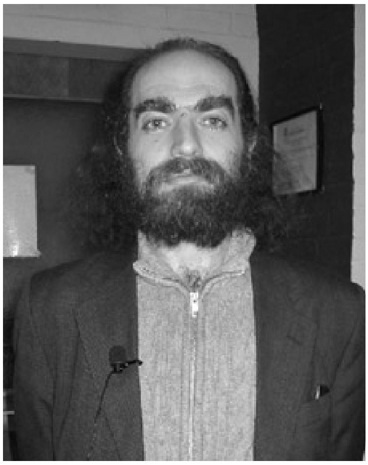
2) Terrence Tao (1975~) : 호주에서 출생하여 1996년에 프린스턴 대학에서 박사학위를 취득하였음. Salem 상, Bochner 상 등의 여러 상을 수상하고 2006년에는 필즈상을 수상함. 현재 UCLA의 교수임.

3) Wendelin Werner (1968~) : 독일에서 출생하였으며 현재 프랑스 국적을 가지고 있음. 1993년에 프랑스 파리 VI 대학에서 박사학위를 취득하였음. 유럽수학회 상, Polya 상 등 여러 상을 수상하고 2006년에는 필즈상을 수상함. 현재 파리 VI 대학의 교수임.

게 거절하였다. 그는 지난 12월에 오랫동안 몸을 닫고 있는 Steklov 수학연구소4)의 교수직을 사퇴하고 이전에 수학교사였던 모친과 단 둘이서 모친의 연금으로 가난하게 살고 있다는 뉴스가 최근에 영국의 모 일간지에 게재되었다.(참고문헌 [9]) 그의 동료들은 하나같이 그는 세속적이지 않은 순수한 사람이며 특히 물질적인 욕심이 없는 사람이라고 한다. 그는 장시간 산보하는 것을 좋아한다고 한다.

Perelman (1966~)은 지난 2003년에 Poincaré conjecture 를 해결하였다는 소문으로 널리 알려진 러시아의 젊은 수학자이다. 잘 아시다시피 이 가설은 1904년에 프랑스 수학자 Henri Poincaré (1854~1912)에 의하여 제기된 전설적인 문제이다. **"단순연결(simply connected)이며 닫힌(closed) n 차원의 다양체는 n 차원의 구와 위상동형 (homeomorphic)이다"**라는 주장이 이 가설의 내용이다. Poincaré 의 사후 2002년까지는 많은 저명한 수학자들이 이 가설을 해결하였다고 주장하였지만 그 후 이들의 주장이 모두 틀렸다는 것이 판명되었다. 1960년대 초반에 5차원 이상인 경우에는 이 가설이 옳다는 사실이 미국 수학자 Stephen Smale에 의하여 증명되었으며, 4 차원인 경우는 1983년에 미국 수학자 Michael Freedman에 의하여 해결되었다. 이 업적으로 Smale 과 Freedman 은 각각 1966년과 1986년에 필즈상을 수상하였다. 일차원인 경우는 간단명료하고 2차원인 경우는 19세기에 이미 해결되었다. 그래서 3차원 경우가 2002년 전까지는 위상수학 분야의 중요한 미해결 문제로 남아있었다. Perelman이 아래에 소개되는 논문 [Pel1]을 인터넷에 올리기 약 6개월 전에 영국 수학자 Martin Dunwoody가 본인이 Poincaré 가설을 해결하였다고 주장하였다. 뉴욕타임저는 2002년 4월 25일에 "UK Math Wiz May Have Solved Problem"이란 제목으로 이 사실을 보도하였다. 물론 그의 증명이 곧 틀렸다는 사실이 판명되었지만, 뉴욕타임즈가 이 가설을 기사로 취급할

4) Steklov Mathematical Institute : 1924년 레닌그라드(Leningrad)에 러시아 수학자 Vladimir Steklov (1864~1926)의 이름을 따서 설립됨.



정도이었으니 이 가설에 대한 관심이 널리 퍼져 있다는 사실을 알 수 있다. 2000년에 이 가설은 Clay 수학연구소5)(간단히 CMI)의 일곱 개의 밀레니엄 문제 중의 하나가 되었다.

Perelman은 2002년 11월 11일, 2003년 3월 10일과 2003년 7월 17일에 아래의 3편의 논문을 차례로 인터넷에 올렸다.

[Pel1] "The entropy formula for the Ricci flow and its geometric applications". arXiv: math.DG/0211159 (November 11, 2002) [39 pages].

[Pel2] "Ricci flow with surgery on three-manifolds". arXiv:math.DG/0303109 (March 10, 2003) [22 pages].

[Pel3] "Finite extinction time for the solutions to the Ricci flow on certain three-manifolds". arXiv:math.DG/0307245 (July 17, 2003) [7 pages].

Perelman은 2002년 11월에 논문 [Pel1]을 인

5) Clay Mathematics Institute : 1998년 미국 보스턴의 사업가 Landon T. Clay의 출연금으로 설립되었음. 2000년 5월 24일에 파리에서 이 연구소는 오랫동안 해결되지 않은 중요한 수학문제들 중에서 일곱 문제를 골라서 각 문제에 100만 달러의 상금을 내걸었음.

터넷에 올린 후 즉시 Richard Hamilton, Shing-Tung Yau, Gang Tian, John Morgan 등의 10여 명의 수학자 등에게 논문 [Pel1]의 초록을 전자우편으로 보냈다. 그러나 Yau와 Hamilton 은 회답을 하지 않았다. 그는 1994년 이후로 한 편의 논문도 발표하지 않아서 많은 동료들은 수학을 그만 두었다고 생각하고 있었다. 그런데 갑자기 2002년 11월에 Poincaré 가설의 해결의 실마리를 던지는 논문을 작성하자 많은 수학자들이 그의 연구결과에 지대한 관심을 가지게 되었다. 그래서 그는 MIT, 프린스턴 대학, 콜롬비아 대학, 뉴욕 주립대학에 초청되어 2003년 4월부터 한 달간 그의 연구결과를 강연하였다. 2003년 4월에 프린스턴 대학에서 강연을 하였는데, 이 강연에는 Andrew Wiles, John Nash, Jr., John Conway, John Ball 등의 저명한 수학자들이 참석하였다. 그는 Stony Brook 뉴욕 주립대학에서는 연구결과를 여러 번 강연을 하였는데 Hamilton 교수가 참석하지 않아 낙담하였다고 전해지고 있다. 실은 그는 누구보다도 그의 강연 내용을 잘 이해할 수 있는 Hamilton의 조언과 견해를 들으며 그와 토론하고 싶었기 때문이었다. Morgan 교수가 이 강연에 참석하여 강연을 듣고 크게 감명을 받아 콜롬비아 대학에 초청하였다. 이 초청강연은 토요일 아침에 있었는데 이 대학에 재직하고 있는 Hamilton은 강연이 끝날 무렵에 나타나 강연이 끝난 후 전혀 질문을 하지 않았을 뿐만 아니라 함께 점심을 하면서도 전혀 질문을 하지 않았다고 한다. Anderson과 Milnor 등의 수학자들은 Perelman의 강연을 듣고 그의 연구결과를 긍정적으로 받아 들였던 것 같다.(참고문헌 [1], [4]) Perelman은 미국에서 여러 차례의 강연을 한 후 고향인 상트 페터스부르크로 돌아가 이때부터 매우 친한 동료 수학자 몇 사람들과만 접촉하고 학문적으로 일체 외부와 연락을 단절하고 은둔생활을 하기 시작하였다.

상기에 언급한 Perelman의 3편의 논문은 Poincaré 가설을 포함하는 Geometrization Conjecture를 해결하였다는 것이다. Geometrization Conjecture는 1982년 필즈상 수학자인 William Thurston 에 의하여 제기된 문제이다.(참고문헌 [6]) 그래서 이 분야에 관심을 가지고 있었던 적지 않은 수학자들이 이 논문을 정독하며 검토하기 시작하였다. 크게 3 그룹이 이 논문의 검증 작업에 들어갔다는 사실이 알려지고 있다. 첫 번째가 Yau, Hamilton 과 Yau의 동료들로 이루어진 그룹이고, 둘째가 Bruce Kleiner와 John Lott 로 이루어진 그룹이며, 마지막으로 Tian과 Morgan 으로 이루어진 그룹이다. 첫 번째 그룹은 2003년 미국 과학재단으로부터 약 100만 달러의 지원을 받으며 연구가 수행되었으며, 나머지 두 그룹은 CMI의 재정적인 지원을 받으며 연구가 수행되었다. 이 그룹들은 Perelman의 연구업적을 기반으로 하여 아래의 논문과 책들을 완성하여 공개하였다.

[C-Z] A Complete Proof of the Poincaré and Geometrization Conjectures-application of the Hamilton-Perelman theory of the Ricci flow, by Huai-Dong Cao and Xi-Ping Zhu, Asian Journal of Mathematics, Vol. 10, No. 2, June 2006, 165-492 [327 pages].

[K-L] Notes on Perelman's Papers, by Bruce Kleiner and John Lott, arXiv.org, May 25, 2006 [192 pages].

[M-T] Ricci Flow and the Poincaré Conjecture, by John Morgan and Gang Tian, arXiv.org, July 25, 2006 [473 pages].

[C-Z]는 올해에 수학저널 Asian Journal of Mathematics (간단히 AJM)에 게재되었고 나머지 2편의 논문은 인터넷에 등재되었다. CMI의 홈페이지에서 상기의 3 편의 논문을 발견할 수 있다. 위에서 보시다시피 기하해석학(geometric analysis)의 분야에서는 적지 않은 중국 수학자들과 미국 수학자들이 선두적인 역할을 하고 있다는 사실을 알 수 있다. 적지만 러시아와 일본의 수학자들도 이 분야에서 중요한 역할을 수행하고 있다는 사실도 알 수 있다.(참고문헌 [5] 참조) Yau의 동료인 Cao와 Zhu 은 Hamilton과 Perelman의 연구결과를 기초로 하여 Yau와 Hamilton의 도움을 받아가며 새로운 방법으로 Poincaré 가설뿐만 아니라 Geometrization Conjecture 를 해결하였다고 주장

하는 논문 [C-Z]를 2005년 12월 12일에 수학저널 AJM 에 투고하였으며 금년 4월 16일에 게재승인을 받아 지난 6월에 출판되었다. 327 쪽이나 되는 긴 논문이 4~5 개월이란 짧은 기간 동안 심사되어 게재승인을 받은 사실에 대해 많은 수학자들이 의구심을 가지고 있다. AJM의 편집위원장인 Yau는 지난 2005년 9월부터 2006년 3월까지 공동저자인 Zhu가 Perelman의 연구결과와 그들의 연구결과에 관하여 하버드 대학에서 매주 3차례 세미나를 하여 왔기 때문에 이 논문을 빠른 시일에 용이하게 심사를 할 수 있었다고 해명하였다. Yau는 올해 4월 13일에 AJM의 31 명의 편집위원들에게 논문 [C-Z]의 사본과 심사위원들의 의견서도 첨부하지도 않고 이 논문의 게재승인에 대한 의견서를 3일 이내에 보내 줄 것을 요구하는 서신을 전자우편으로 보냈다고 한다.(참고문헌 [14]) Cao는 4월 16일에 논문의 게재승인을 전자우편을 통하여 통보받았다. 앞으로 이 일말의 사건이 큰 논쟁거리로 남을 것 같다. 논문 [C-Z]에 대하여 Perelman은 이 논문의 두 저자는 그의 연구에 대하여 제대로 이해하지 못했으며 이 논문에는 새로운 사실이 없다는 부정적인 의견을 피력하였다. Morgan 도 역시 이 논문에 대하여 이와 비슷한 부정적인 의견을 내비쳤다. Yau 그룹은 Kleiner와 Lott의 논문 [K-L]을 정확성이 결여되어 있는 강의록으로 무시하는 분위기이다. [M-T]에 대해서는 공동 저자인 Morgan은 Perelman 의 연구결과를 기반으로 하여 [M-T]에서 Poincaré 가설을 완벽하게 증명하였으며 Geometrization 가설은 다루지 않았다고 공식적으로 여러 기자들에게 말하였다. [M-T]는 심사 과정에 있으며 게재승인이 나면 CMI 에서 책으로 출판될 예정이다. Morgan과 Tian 은 Perelman 과 지난 3여 년 동안 전자우편으로 학문적 접촉을 통하여 [M-T]를 저술하였다.

Yau는 Cao와 Zhu의 논문 [C-Z]를 홍보하기 위해 2006년 6월 20일에 북경 소재의 그의 연구소(Morningside Center of Mathematics)[6]에서 「3차원 다양체의 구조에 관하여(Structures of Three-Manifolds)」의 제목으로 강연하였다.(참고문헌 [7, 8]) 이 강연에서 Hamilton의 업적을 높이 평가하며 강조하였다.

Perelman의 업적을 간략하게 다루겠다. Hamilton은 1982년에 Ricci flow를 창안하여 이를 이용하여 Geometrization Conjecture 를 증명하려고 시도하였다.(참고문헌 [2] 참조) Ricci flow는

$$\frac{\partial g(t)}{\partial t} = -2Ric(g(t))$$

으로 비선형 미분방정식이다. 여기서, t 는 시간 변수, $g(t)$ 는 3차원 다양체 상의 리만 계량(metric)이고 $Ric(g(t))$ 는 $g(t)$ 의 Ricci 곡률이다. 그러므로 Ricci flow는 비선형 열방정식(nonlinear heat equation)이라 간주할 수 있다. 그는 그 후 Ricci flow 에 관한 몇 편의 논문을 발표하면서 이 가설을 증명하기 위해서는 Ricci flow의 특이점들의 구조를 철저하게 분석하여 규명해야하고 Ricci flow의 정칙인 해들을 완전하게 분류해야 한다는 소위 Hamilton's Program을 제창하였다. 여기서, 구간 $[0, \infty)$에서 고르게 유계인(uniformly bounded) 단면곡률을 갖는 Ricci flow의 해를 정칙해(nonsingular solution)라고 하며, 그렇지 않는 단면곡률을 갖는 유한 구간 $[0, T)$에서의 해를 Ricci flow의 특이점(singularity)이라 한다. 이 특이점들을 어떻게 다루느냐가 어려운 문제로 남아 있었는데 Perelman이 심오한 통찰력으로 새로운 아이디어를 창안하여 이 난제를 극복하였다. Perelman은 그의 논문 [Pel1]의 3쪽의 상단 1-3 행에서

Thus, the implementation of Hamilton program would imply the geometrization conjecture for closed three-manifolds.

In this paper, we carry out some details of Hamilton program.

6) Morningside Center of Mathematics : 1996년에 중국 과학원과 홍콩 Morningside 주식회사의 기부금으로 Yau의 주도하에 북경에 설립된 수학연구소. 많은 젊은 수학자들이 이 연구소에 참여하고 있음. Yau의 수학연구소라고도 함.

이라 적고 있다. 앞에서 언급한 Perelman의 세 편의 논문은 이 분야의 전문가도 이해하기 힘들 정도로 많은 자세한 내용이 생략되고 아주 압축되어 완성되었다. Yau는 Perelman은 그의 세 편의 논문에서 Poincaré 가설을 상세하고 완벽하게 증명하지 않고 증명을 스케치하고 있다고 주장하고 있다. Yau는 Perelman이 Ricci flow의 특이점을 성공적으로 분류하고 이 점들에서의 Ricci flow의 해를 규명하는 새롭고 훌륭한 아이디어에 대해서는 높이 평가하였지만, 가설의 완벽한 증명은 논문 [C-Z]에서 그의 동료 Cao와 Zhu에 의하여 이루어졌다고 주장하고 있다.(참고문헌 [8, 10]) 그러나 다른 그룹의 수학자들은 이와 다른 의견을 내놓고 있다. 이들은 Perelman이 비록 증명은 자세하게 하지 않았지만 그의 논리과정에는 전혀 결함이 없기 때문에 근본적으로 Geometrization Conjecture 를 해결하였다고 주장하고 있다. Perelman은 이미 언급한 세 편의 논문을 권위있는 수학저널에 투고하지 않았다는 것이 수학계에선 이례적인 사건으로 받아지고 있다. 만약에 그가 전문적인 수학저널에 투고를 하였다면 틀림없이 익명의 논문 심사위원으로부터 상세하지 못한 부분과 결함을 설명하며 다시 논문을 작성하라는 서신을 받았을 것이다. 그러나 그는 센 자존심 때문에 많은 시간을 내어 이 논문을 다시 작성하지 않았을 것이다. 왜냐하면 심사위원의 지시대로 다시 쓴다면 이 논문은 지저분하게 길어 질 수 밖에 없고 그가 원하지 않는 논문의 형태가 될 것이기 때문이다. 그래서 그의 논문들은 게재가 거부되었을 것으로 생각한다.(참고문헌 [3]) 이 분야의 전문가는 아니지만 필자의 느낌은 마침내 Geometrization Conjecture (물론 Poincaré 가설)가 성공적으로 해결되었다는 것이다. 앞으로 2년 후에는 CMI가 상금 100만 달러의 주인을 가려낼 것이다. Perelman과 Hamilton 이 이 상금을 공동으로 나누어 가질 가능성이 크다고 필자는 생각한다. 지금의 사태로 미루어보아 아마도 Perelman이 이 상금의 수상도 거부할 것 같다. CMI의 소장인 James Carlson의 말에 의하면, 만약 그가 이 상금의 수상도 거부하면 이 상금은 러시아 수학발전기금 등의 다른 명목으로 사용될 수 있도록 노력하겠다고 기자회견에서 밝혔다.

Perelman은 1966년 6월 13일 레닌그라드(Leningrad; 상트 페터스부르크의 옛이름)에서 유태계 가정에서 출생하였다. 그의 부친은 전기공학자이고 모친은 수학교사로 근무한 적이 있었다. 그는 어려서부터 수학적 재능을 인정받아 1982년에 러시아 대표팀의 일원으로 국제수학 올림피아드에서 참가하여 만점을 받으며 금메달을 획득하였다. 그 후 레닌그라드 국립 대학교에 입학하여 1980년대 후반에 박사학위를 받았다. 그 후 Steklov 수학연구소의 연구원으로 취직하였으며, 1992년에 Courant 수학연구소에서 6개월, Stony Brook 뉴욕 주립대학에서 6개월 동안 초청되어 연구하였다. 그는 1993년에는 버클리 대학에 Miller Fellow로서 2년간 초청되어 연구하였으며 1994년에는 스위스 취리히에서 개최되었던 ICM에서 초청강연을 하기도 하였다. 1995년에는 스탠포드 대학 등 여러 대학에서 그를 채용하려고 좋은 조건을 제의하였지만 그는 모두 거절하고 상트 페터스부르크로 돌아가 버렸다. 한번은 스탠포드 대학에서 그를 채용하려고 그에게 이력서를 부탁하였는데, 그는 버럭 화를 내며 “If they know my work, they don't need my C.V. If they need my C.V., they don't know my work.”라고 반박하였다고 한다. 그는 약간 대머리인데다가 머리를 길게 기르고 손톱도 깍지 않고 기르고 있다고 한다. 그는 마치 라스푸틴(Rasputin)7) 처럼 생겼다고 한다. 그는 지난 12월에 Steklov 수학연구소의 교수직을 사퇴하고 상트 페터스부르크의 외곽에서 모친의 연금으로 모친과 함께 청빈하게 살고 있다고 한다.

이제 필자의 짧은 견해를 피력할까 한다. Perelman은 이미 언급하였다시피 1982년에 국제수학올림피아드에서 만점을 받으며 금메달을 수상하여 어려서부터 수학적 재능을 인정받았으며 1996년에는 유럽수학회에서 수여하는 젊은 수학자상을 거절하였다. 심사위원회들로 구성된 수학자들 중에서 그의 업적을 제대로 평가할 수 있는 사람이

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7) Grigori Yefimovich Rasputin (Russian: Григо́рий Ефи́мович Распу́тин; 그리고리 라스푸틴) (1869~1916) : 서부 시베리아 출신으로 로마노프 황실의 총애로 권력 휘두르던 러시아 괴짜 수도사. 1916년에 귀족들에게 피살

없었다는 것이 수상 거부의 이유 중의 하나이다. 이번의 필즈상 수상 거부는 전 세계에 센세이션을 불러 일으켰다. 이번의 수상거부로 그가 오만불손하다는 말도 나오고 있지만, 그의 순수한 마음과 고집적인 자존심에서 이런 행동이 나오지 않았나하는 생각이 든다. 그는 기자와의 인터뷰에서 아래와 같이 말하였다.(참고문헌 [12] 참조)

"I am not a politician ! I'm not going to decide whether to accept the prize of 100 million dollars offered by the Clay Mathematics Institute until it is offered."

필자는 Perelman의 학문을 이전에 접해 본 적도 그를 직접 만나본 적도 없지만 그를 접해 본 적이 있는 사람과 여러 기사를 통해서 그의 성격과 가치관을 어렴풋이 가늠할 수 있다. 그는 세속, 특히 물질적인 세속에 물들지 않은 매우 순수한 사람이다. 수학분야 뿐만 아니라 창의성을 요구하는 예술, 문학, 과학 등의 여러 분야에서도 순수한 마음을 지닌 사람들이 대체로 위대한 업적을 남기는 것 같다. 특히 명예, 재물을 쫓는 세속에 물든 사람들은 자기 분야에서 위대한 업적을 창출할 수 있는 경우가 아주 드물다. 가령, 수학 분야에서 위대한 업적을 내었던 독일의 위대한 수학자 Gauss와 Riemann 은 세속에 물들지 않은 경건하고 순수한 인물이었다. Perelman은 수학이란 학문 그 자체가 좋아 깊은 통찰력과 타의추종을 불허하는 탁월한 창의력으로 100년이라는 긴 역사를 가진 전설적인 난제, 즉, Poincaré 가설을 지난 3여 년 전에 해결하였다. 놀랄만한 사실은 그가 위상수학 분야에서 제기된 위상학적인 난제를 기하학과 해석학 분야에서 쓰이고 있는 도구(tool)와 테크닉을 이용하여 이 가설을 해결하였다는 것이다. 그의 Ricci flow의 특이점을 다루는 연구방법은 이와 유사한 다른 난제를 다루는 데에도 응용될 수 있다는 점에서 그의 최근의 연구업적은 높이 평가받고 있다. 많은 수학자들이 그의 업적이 깊고 아름답다는 찬사를 보낼 뿐만 아니라 그의 큰 업적을 이루어 낸 것은 21세기의 수학사뿐만 아니라 인류 문화사의 측면에서 대사건이라고 이구동성으로 말하고 있다. 또한 그의 업적을 **"인간정신의 승리"**(a triumph of human mind or thought)라고 하기도 한다. 필자는 이 표현이 아주 적절하다고 생각한다. 결론적으로 그는 세계 수학계에 큰 기쁨을 주었고 수학 발전에 지대한 기여를 하였다.

Perelman 은 지난 3년 동안 은둔생활을 하면서 소수의 수학자들과 전자우편을 통하여 학문적으로 교류를 하였던 것으로 짐작할 수 있다. 왜냐하면 일본 수학자 Shioya와 Yamaguchi 의 논문(참고문헌 [5])이 그의 도움으로 2005년 독일 수학저널에 게재되었다. 특히 그는 Morgan 과 Tian 등의 저명한 수학자들과도 전자우편을 통하여 학문적으로 자주 교류하였던 것으로 드러나고 있다. 왜냐하면 논문 [M-T]가 그의 도움으로 작성되었다는 사실을 Morgan 의 ICM 인터뷰에서 짐작할 수 있다. (참고문헌 [15]) 앞으로 적어도 2년 동안은 CMI와 여러 전문가들에 의하여 그의 세 편의 논문 [Pel1-3]이 더욱더 철저하게 검증을 받을 뿐만 아니라 Cao와 Zhu의 논문 [C-Z]도 역시 철저한 검증을 받을 것이다.

신문기사에 의하면 그의 주변에 명예, 재물을 밝히지 않는 비세속적인 사람(그의 가족, 지도교수, 동료 학자 등)들이 많다고 한다. 이로 미루어 보아 그가 유럽수학회 상과 필즈상의 수상을 거부한 사실을 이상하게 받아 드릴 일이 아니라고 생각한다. 가령 러시아의 시인이며 작가인 Boris Pasternak (1890~1960)와 프랑스의 실존주의 철학자인 Jean-Paul Sartre (1905~1980)는 각각 1958년과 1964년에 노벨 문학상의 수상을 거부하였고, 베트남의 외교관이며 정치가인 Le Duc Tho (1911~1990)는 1973년에 노벨 평화상의 수상을 거부하였다. 이들은 명예로운 상에 관심을 가지지 않았던 사람이다. 들리는 소문에 의하면 Perelman은 전에 비해 수학에 대한 열정이 식어 연구 활동을 하지 않고 있다고 한다. 지난 3 여 년 동안의 여러 정황으로 미루어 보아 그는 수학계에 환멸을 느끼고 영원히 수학의 연구 활동을 그만두고 수학분야와 무관한 일을 할 가능성도 배제할 수 없다. 세계 수학계를 위해서는 이런 일이 일어나지 않기를 바랄 뿐이다.

그가 수학계와 인연을 끊는다면 세계 수학계에 큰 손실이 될 것이다. 그와 관련된 여러 주변의 일들이 원만하게 처리되어 그가 즐거운 마음으로 수학계에 돌아와 타의 추종을 불허하는 탁월한 창의력을 발휘하여 다시 한 번 인간정신의 승리를 이루어 주었으면 하는 것이 필자의 간절한 바람이다.

[참고문헌]

[1] Michael Anderson, Geometrization of 3-manifolds via the Ricci flow, Notices of the American Math. Soc., Vol. 2 (2004), 184-93.

[2] Richard S. Hamilton, Three-manifolds with positive Ricci curvature, Journal of Differential Geometry, Vol. 17 (1982), 695-729.

[3] Ally Jackson, Conjectures No More ? Consensus Forming on the Proof of the Poincaré and geometrization Conjectures, Notices of the American Math. Soc., Vol. 53, No. 8 (2006), 897-901.

[4] John Milnor, Towards the Poincaré Conjecture and the classification of 3-manifolds, Notices of the American Math. Soc., Vol. 10 (2003), 1226-33.

[5] Takashi Shioya and Takao Yamaguchi, Volume-collapsed three-manifolds with a lower curvature bound, Math. Ann. Vol. 333 (2005), 131-55.

[6] William P. Thurston, Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, Bull. Amer. Math. Soc. (N. S.), Vol. 6 (1982), 357-81.

[7] Shing-Tung Yau, Perspectives on geome-tric analysis, ArXiv:math. DG/0602363.

[8] Shing-Tung Yau, Structures of Three-Manifolds, ArXiv:math. DG/0607821.

[9] Lobastova, Nadejda; Hirst, Michael, "World's top maths genius jobless and living with mother", The Daily Telegraph, August 20, 2006.

[10] "Chinese mathematicians solve global puzzle", China View (Xinhua), 3 June 2006.

[11] "Fields Medal – Grigory Perelman" (PDF), International Congress of Mathematicians 2006, 22 August 2006.

[12] Overbye, Dennis, "An Elusive Proof and Its Elusive Prover", New York Times, August 15th, 2006.

[13] "Maths genius declines top prize", BBC News, 22 August 2006.

[14] Sylvia Nasar and David Gruber, Manifold Destiny : A legendary problem and the battle over who solved it, the New Yorker, August 28th (2006).

[15] ICM Daily News (Madrid, Spain) : August 22th, 23th, 26th (2006).

[16] George Johnson, The Math Was Complex, the Intentions, Strikingly Simple, the New York Times, August 27th (2006).

Perelman의 업적에 관하여[1)]

-Poincaré 가설과 geometrization 가설-

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I. Poincaré 가설

Poincaré 가설에 관한 역사는 참고문헌 [17]을 참고하길 바란다. 이제부터는 3차원 다양체인 경우만 다루기로 한다. 3차원 다양체에서의 이 가설은 다음과 같이 기술된다.

Poincaré 가설 : 단순연결(simply connected)이며 닫힌(closed) 3차원 다양체는 3차원 구(sphere) S^3 와 위상동형(homeomorphic)이다. 이것은 또한, 단순연결이며 닫힌 3 차원 다양체는 S^3 와 미분동형(diffeomorphic)[2)]이다라고도 기술할 수 있다.

각주 2)에 의하여 앞으로 다양체는 매끄러운(smooth) 다양체이다. 상기의 가설은 1904년에 Henri Poincaré (1854~1912)에 의하여 제기된 어렵고 중요한 문제이다. 그 후부터 2002년 11월 11일 이전 까지는 이 가설이 미해결인 채로 있었는데, 위의 날짜에 Perelman에 의해 이 가설이 옳다는 사실이 증명되었다. 그는 미국 콜롬비아 교수인 Richard Hamilton(1943~)이 창안한 Ricci flow 방정식의 특이점들을 명확하게 규명함으로써 상기의 가설을 해결하였다. 놀랄만한 사실은 그가 위상수학 분야에서 제기된 위상학적인 난제를 기하학과 해석학 분야에서 쓰이고 있는 도구(tool)와 테크닉을 이용하여 이 가설을 해결하였다는 것이다. 이 가설의 해결은 위상수학이 기하학과 해석학의 분야와 매우 밀접한 관계가 있음을 보여주고 있다.

1976년 Shing-Tung Yau(1949~)는 Kähler 다양체상에서의 Kähler-Einstein 계량(metric)을 구성하여, 이 계량을 이용하여 소위, Severi 가설[3)]을

1) 저자는 9월 초쯤 대한수학회소식지 편집위원회로부터 올해의 Fields Medal 수상자인 러시아 수학자 Grisha Perelman(1966~)의 업적에 관한 글을 청탁받아 본 원고를 쓰게 되었다.

2) Every topological 3-manifold admits a differential structure and every homeomorphism between smooth 3-manifolds can be approximated by a diffeomorphism. Thus, classification results about topological 3-manifolds up to homeomorphism and about smooth 3-manifolds up to diffeomorphism are equivalent.

3) Every complex surface that can be deformed to the complex projective plane is itself the complex projective plane.

증명하였다. 이 가설은 복소기하학 분야에서의 Poincaré 가설이라 간주될 수 있다. 그래서 위상수학과 기하학 분야에서의 난제가 해석학의 어려운 테크닉의 힘으로 해결될 수 있다는 사실이 입증되고 있다.

II. Geometrization 프로그램

위상수학(topology)의 탄생 이후 다양체에 관한 관심이 고조되었고, 이 다양체를 분류하려는 시도가 초기부터 대두되었다. 2차원 다양체는 (방향을 줄 수 있는 다양체에서만 논하면) 2차원의 구(sphere), 토러스(torus)와 종수(genus)가 2 이상인 닫힌 곡면 밖에는 없다. 여기에서 특이할 만한 것은 구 S^2 에는 곡률이 1인 기하구조(spherical geometry)를 줄 수 있고, 토러스에는 곡률이 0 (Euclidean 구조), 그 밖에는 곡률이 -1 인 소위 쌍곡 구조(hyperbolic structure)를 줄 수 있다. 즉 위상구조와 기하구조 사이에는 아주 밀접한 관계가 있는 것이다. 이렇듯 위상다양체를 분류하는데 리만기하를 이용할 수 있다는 발상이 3차원 다양체에 적용된 것이 소위 W. P. Thurston(1946~)의 **geometrization 프로그램**이다. 그는 주어진 3차원 다양체를 조금 더 간단한 조각(또는 성분)들로 나누면, 각각의 조각들이 아주 대칭성이 많은 리만 구조를 가질 것이라 추측하였다. 간단한 조각들로 나눈다는 것은 다음을 의미한다. 우선 3차원 다양체가 **기약** (irreducible)이란 모든 매장된(embedded) 2차원 구가 3차원 ball을 bound 할 때 이다.

Helmut Kneser(1898~1973)는 1929년에 다음의 정리를 얻었다.

Kneser 정리 (1929): 주어진 3차원 다양체 M 을 3차원 ball을 bound 하지 않으며 매장된 2차원 구들을 따라 자르면 유한개의 조각들로 나누어지고 각각의 조각들은 기약이 된다.

도움말 : 3차원 다양체 M 의 상기의 분해의 유일성(uniqueness)은 John Milnor(1931~)에 의하여 증명되었다. 참고문헌 [16]을 참조하기 바란다.

이렇게 잘린 조각들을 그 안의 비압축적 토러스(incompressible torus)를 따라 자르면 최종적으로 잘린 조각들이 대칭성이 많은 아래의 8 개중의 한 가지 기하 구조를 갖는다는 것이 **geometrization 프로그램** 또는 **geometrization 가설** (간단히, GC)이다. 여기서 대칭성이 많은 구조란 국소적으로 등질인(locally homogeneous) 리만구조를 말한다. 즉, 이것은 등질인 계량(homogeneous metric)을 갖는 자연스런 모델(canonical model)공간을 적당한 이산군(discrete group)으로 나누어 만들어 지는 다양체이다. 등질 공간을 이산군으로 나누어 유한 체적의 다양체를 만들 수 있는 Lie 군은 8가지 밖에 없다는 것을 보이기는 그리 어려운 일은 아니 다. 이 8개의 기하 구조를 나열하면

$$S^3,\ E^3,\ H^3,\ S^2\times R,\ H^2\times R,$$
$$Sol,\ Nil,\ \widetilde{PSL(2,R)}$$

이다. 여기서 E^3 는 3차원 유크리드 공간, H^2 는 Poincaré 상반평면, H^3는 곡률이 -1인 Hadamard 3차원 다양체이고, Sol 은 $R^2\times R^*$ 인 가해군(solvable group)으로서 $t\in R^*$가 R^2 상에 (t, t^{-1})로 작용하는 공간 이고, Nil 은 3차원 Heisenberg 군, 즉 대각 성분이 전부 1인 3×3 상삼각(upper triangular) 실행렬들의 집합이다. 그리고 $\widetilde{PSL(2,R)}$ 는 H^2 의 단위 접다발(tangent bundle)의 보편적 덮개(universal cover)이다. Sol, Nil, $\widetilde{PSL(2,R)}$ 상에는 불변 계량(left invariant metric)이 주어져 있음을 유의하기 바란다. 특히 Thurston은 H^3-구조, 즉 쌍곡 구조에 관심을 많이 가지고 Haken hyperbolization 정리[4]를 증명하였다. 3차원 다양체 M 안에 들어 있는 곡면 F 가 **비압축적**(incompressible)이란 기본군 $\pi_1(F)$가 무한군으로 포함사상(inclusion map)에 의해 $\pi_1(M)$안으로 핵(kernel)이 없이 들어 갈 때를 의미한다. 3차원 다양체가 비압축적 곡면을 가지고 경계 이외에는 비압축적 토러스를 갖지 않으면 H^3-구조를 갖는다는 것이다. **그래프 다양체**(graph manifold)란 곡면상의 S^1-다발(circle bundle over surface)인 조각들을 토러스 경계를 따라 붙여서 만든 다양체를 말한다. 이 다양체는 1967년에 Friedhelm Waldhausen 에 의해

4) If M is an irreducible Haken 3-manifold, i.e., M contains an incompressible surface of genus greater than one, then the geometrization conjecture is true for M.

서 소개되어 연구되었다. 참고로 GC는 Poincaré 가설을 도출한다. 왜냐하면 $\pi_1(M)=\{1\}$ 이면 비압축적 토러스가 매장되어 있을 수 없으므로 M 자체가 더 이상 자를 수 없는 조각으로서 8가지 중에서 하나의 구조를 가져야 하는데, $\pi_1(M)$이 유한군이므로 S^3-구조를 가져야하고 또 $\pi_1(M)=\{1\}$ 이므로 S^3와 위상동형일수 밖에 없다. Thurston의 geometrization 프로그램에 관한 보다 상세한 내용은 참고문헌 [1], [17]과 [23]을 참고하길 바란다.

반면 Hamilton과 Perelman은 S^3의 구조에 많은 관심을 가지고 Ricci 곡률에 의해 만들어진 Ricci flow 를 연구하면서 Poincaré 가설을 해결하기 위해 노력하였다. Thurston의 geometrization 프로그램과 마찬가지로 어떤 다양체에 등질인 계량(homogeneous metric)을 줄 수 있는가를 찾는 것인데, heat flow 에서와 마찬가지로 주어진 다양체 상에 어떤 계량을 주고 Ricci flow를 따라 계량을 흐르게 하면 결국에는 계량이 균일하게 배분되어 등질 계량으로 수렴할 것이라는 착안이었다. 이 예상이 적중하여 마침내 100여 년 간의 미해결 문제였던 Poincaré 가설이 Perelman 에 의해 해결되었다.

III. Hamilton 의 프로그램

닫힌 2차원 공간 N 에서의 Gauss-Bonnet 공식

$$\int_N K\,dA = 2\pi\chi(N)$$

은 N의 가우스 곡률과 위상(topology)과의 깊고 아름다운 관계를 보여 주는 공식이라 할 수 있다. 여기서 K 는 리만 계량 g 의 가우스 곡률이며, dA는 g 의 면적요소이고 $\chi(N)$ 은 N 의 오일러 지표이다. 이 공식을 토대로 하여 미분기하학자들은 고차원의 리만 다양체 M 상에서 자연스런 리만 계량을 발견하여 이의 곡률을 연구함으로써 M 의 위상학적인 성질을 알아내려고 하였다. 다양체 상에 스칼라 곡률이 상수인 리만 계량을 찾는 문제가 소위 Yamabe 문제이다. Ricci 곡률이 상수인 Einstein 계량을 구하려면, 일반적으로 다루기 매우 힘든 Einstein 방정식을 해의 존재성에 관하여 연구해야 한다.

조화사상 heat flow에 관한 Eells 와 Sampson 의 연구(참고문헌 [7])에 영향을 받아서, 1982년 Hamilton 은 스칼라 곡률이 양의 함수이고 단면곡률(sectional curvature)이 양의 상수인 Einstein 계량을 구하기 위해 n 차원 다양체 M 상에 소위 Ricci flow

$$(1) \qquad \frac{\partial g(t)}{\partial t} = -2\,Ric(g(t))$$

를 소개하였다. 여기서, t 는 시간 변수, $g(t)$ 는 3차원 다양체 상의 리만 계량(metric)이고 $Ric(g(t))$ 는 $g(t)$ 의 Ricci 곡률이다. 그러므로 Ricci flow 는 비선형 열방정식(nonlinear heat equation)이라 간주할 수 있다. 리만 곡률 텐서 $\mathrm{R}m$ 을 국소적으로 R_{ijkl} 로 표시하면 Ricci 텐서 (R_{ik}) 는 $R_{ik}=\sum_{j,l=1}^{n} g^{jl}R_{ijkl}$ 이고 스칼라 곡률 $R(x)$ 는 $R=\sum_{i,k=1}^{n} g^{ik}R_{ik}$ 로 주어진다. 여기서 (g^{ij}) 는 계량 텐서 (g_{ij}) 의 역행렬이다. 그러므로 Ricci flow (간단히, RF) (1)은

$$(1)^* \qquad \frac{\partial}{\partial t} g_{ij} = -2\,R_{ij}$$

으로 나타낼 수 있다. 일반적으로 RF (1)의 해 $g(t)$ 에 대하여 변수 t 에 따라 다양체 $(M, g(t))$ 의 부피는 다르기 때문에 $r=\dfrac{\int R\,dV}{\int dV}$ 이라 놓으면 정규화된 (normalized) Ricci flow (간단히, NRF)

$$(2) \qquad \frac{\partial}{\partial t} g_{ij} = -2\,R_{ij} + \frac{2}{n}\,r\,g_{ij}$$

를 얻는다. NRF (2)의 해 $g(t)=(g_{ij}(t))$ 에 대하여 다양체 $(M, g(t))$의 부피는 t 에 무관하게 일정하다.(참고문헌 [2], [24])

Hamilton 은 다음의 정리들을 증명하였다.

정리 1 (Hamilton, [8]; Short-time Existence Theorem). M 이 닫힌 n 차원 다양체이고 초기 계량 조건 $g(0)$ 가 주어져 있다고 하자. 그러면 적당한 양의 실수 T 가 존재하여, 유한 구간 $[0, T)$ 에서 초기치 조건을 만족하는 NRF (2)의

해 $g(t)$ 가 유일하게 존재한다.

정리 2 (Hamilton, [8]; Positive curvature solutions). M 이 닫힌 3 차원 다양체이고 초기 계량 $g(0)$ 가 양의 Ricci 곡률을 가진다고 가정하자. 그러면 구간 $[0, \infty)$ 에서 초기치 조건을 만족하는 NRF (2)의 해 $g(t)$가 유일하게 존재하며, $t \to \infty$ 일 때 계량 $g(t)$는 단면곡률이 양의 상수인 계량 g_∞ 로 지수적으로(exponentially) 빠르게 수렴한다. 특히 M 은 S^3/Γ (구 공간 형; spherical space-form)와 미분동형(diffeomorphic)이다. 여기서 Γ 는 등장사상 $\gamma: S^3 \to S^3$ 으로 이루어진 유한군이다.

정리 2는 보통 Space-form Theorem 으로 알려져 있다.

정리 3 (Hamilton, [12]). M 이 닫힌 3 차원 다양체이라 하고, 모든 $t > 0$ 에 대하여 NRF (2)의 해 $g(t)$ 가 존재한다고 가정하자. 그리고 모든 $t > 0$ 에 대하여 리만 곡률 텐서 $\mathrm{R}m$ 이 $|\mathrm{R}m(t)| < C$ 의 조건을 만족한다고 가정하자. 여기서 C 는 시간 변수에 무관한 어떤 양의 실수이다. 그러면 M 에 대해 Geometrization Conjecture 가 성립한다. 게다가 $t \to \infty$ 이면 $(M, g(t))$ 는 비압축적 토러스(incompressible torus)를 따라서

$$M = M_{thick} \cup M_{thin}$$

으로 분해된다. $M_{thick} \cap M_{thin}$은 M_{thick}와 M_{thin} 의 공통 경계곡면이며 유한개의 토러스들의 disjoint 합집합이다. 여기서 $(M_{thick}, g(t))$는 부피가 유한인 완전 쌍곡 다양체들의 disjoint 합집합으로 수렴하고, $(M_{thin}, g(t))$는 붕괴한다(collapse). 다시 말하면 $(M_{thin}, g(t))$는 그래프 다양체(graph manifold)이다. 여기서 $|\mathrm{R}m(t)|$ 는 단면곡률의 최대 절대값을 나타낸다.

정리 4 (Hamilton [10]; Curvature pinching estimate). $g(t)$ 가 닫힌 3차원 다양체 M 상의 RF (1)의 해라고 하자. 그러면, $t \to \infty$ 일 때 $\phi(t) \to 0$ 의 성질을 갖는 비증가(non-increasing) 함수 $\phi: (-\infty, \infty) \to R$ 와 $g(0)$ 에만 의존하는 상수 C 가 존재하여

$$(3) \quad \mathrm{R}m(x,t) \geq -C - \phi(R(x,t)) \cdot |R(x,t)|$$

의 부등식을 얻는다.

정리 4로부터 (x, t) 에서의 $g(t)$ 의 모든 단면곡률 R_{ijkl} 은 부등식 (3)의 우변 값에 의하여 아래에서 유계(bounded below)임을 알 수 있다.

타원 미분방정식과 포물 미분방정식을 연구하는데 Harnack 부등식은 중요한 역할을 한다. 특히 Ricci flow 의 특이점을 분석하여 이해하기 위해서는 Harnack 부등식은 근본적으로 중요한 역할을 하는 도구로써 사용되고 있다. 1993년에 Hamilton은 n 차원 다양체 상의 Ricci flow 에 대한 Harnack 부등식을 증명하였다.

정리 5 (Hamilton, [9]; The Harnack estimate for the RF). M 을 음이 아닌 곡률 작용소를 지닌 n 차원 긴밀인 다양체라고 하자. 그리고 $(M, g(t))$를 M 상의 RF의 해라고 하자. 그러면, 임의의 벡터장 W와 2-form U 에 대하여

$$(4) \quad M_{ij}W^iW^j + 2P_{ijk}U^{ij}W^k + R_{ijkl}U^{ij}U^{kl} \geq 0$$

의 부등식이 성립한다. 여기서 $\nabla_i = \nabla_{\frac{\partial}{\partial x^i}}$ 는 $g(t)$의 Levi-Civita 접속의 공변 미분이고 $\Delta = g^{ij}\nabla_i\nabla_j$ 는 라플라스 연산자이라 하면

$$P_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik}$$

이고

$$M_{ij} = \Delta R_{ij} - \frac{1}{2}\nabla_i\nabla_j R + 2g^{kp}g^{lq}R_{ikjl}R_{pq} - g^{kl}R_{il}R_{jl} + \frac{1}{2t}R_{ij}$$

이다.

도움말 : B. Chow 와 S.-C. Chu 는 그들의 논문 [4, 5]에서 Harnack 부등식 (4)를 기하학적인 측면에서 분석하였다.

Hamilton 은 다음의 가설을 제기하였다.

가설 6. 임의의 초기 계량 g_0 을 가진 닫힌 3차

원 다양체 (M, g_0) 상의 Ricci flow 에 대하여 Harnack-타잎의 추정(estimate)이 성립한다.

이 가설은 Thurston의 GC 를 해결하는데 매우 중요한 역할을 한다.

일반적으로 닫힌 3 차원 다양체 상에서는 NRF (2)의 해 $g(t)$ 의 곡률 Rm 은 $t \to T$ 일 때 유계이지 않다. 여기서 $[0, T)$ 는 (2)의 해가 존재하기 위한 최대 시간 구간이다. 이러한 해를 (2)의 **특이점**(singularity)이라 한다. 반면에 무한 구간 $[0, \infty)$ 에서 고르게 유계인(uniformly bounded) 단면곡률을 갖는 RF (1)의 해를 **비특이해**(nonsingular solution)라고 한다.

Thurston 의 GC 를 해결하기 위해, Hamilton 은 RF (1) 을 철저하게 연구하여, 초기 계량 $g(0)$ 가 주어져 있다면, 적절한 기하학적인 surgery 를 수행하여, RF (1)이 기하학적인 구조(geometrical structure)로 발전(또는 진화; evolve)한다는 사실을 증명하려고 시도한 소위 Hamilton의 프로그램을 창안하였다. 보다 구체적으로 설명하면, Hamilton의 프로그램은 3차원 다양체 상의 Ricci flow 의 연구를 아래의 두 연구 [A]와 [B]로 나누어 수행하는 어려운 연구과정이다.

[A] RF (1)의 특이점들을 분석하고, 기하학적인 surgery 를 유한 번 시행하여 비특이해를 구하는 과정.

[B] 무한 구간 $[0, \infty)$ 에서 고르게 유계인 (uniformly bounded) 단면곡률을 갖는 RF (1)의 비특이해들을 분류하는 과정.

IV. Perelman의 업적

이제 간략하게 Perelman의 탁월한 업적을 기술하겠다. 우선 Kneser의 정리에 의하여 일반성을 잃지 않고 3차원 다양체 M 을 기약(irreducible)이라 가정해도 무방하다. 그의 연구는 아래와 같이 다섯 단계로 나누어진다.

단계 1. 특이점 (또는 surgery) 시간 T_i 에서의 NRF 의 특이점들의 형성 (formation)은 표준적(standard)이다: 특이점들이 "캡(cap)" 또는 "목(neck)"의 형태이다.

그림 1 : 표준 특이점(standard singularities)

단계 2. surgery 를 시행하라. 즉, 특이점 시간 T_i 의 근처에서 M 으로부터 목(neck) 과 캡(cap)을 잘라 내고, 이것을 유한 곡률을 갖는 구면캡 (spherical cap)으로 대치하여 얻어지는 다양

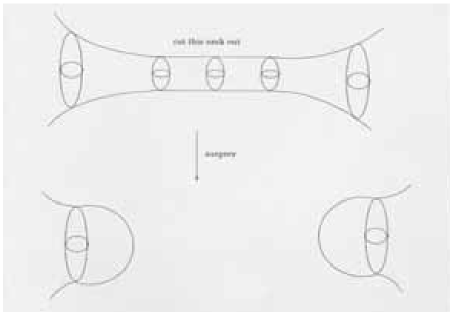

그림 2 : 목(neck; 또는 튜브(tube)) surgery

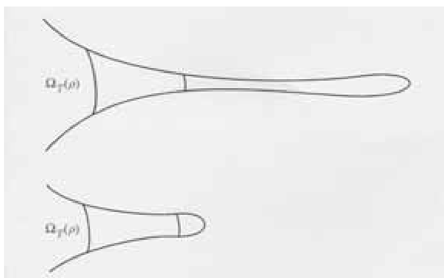

그림 3 : 캡(cap) surgery

체를 M_* 로 표시한다. 그런 후 M_* 에서 단면곡률이 양수인 성분(component)들을 제거하여 새로운 다양체 M^* 를 얻는다. 계속하여 M^* 상에 Ricci flow 를 시행한다.

도움말 : $M^* \neq \emptyset$ 인 경우에는 단면곡률이 양수인 성분을 제거하여도 M 의 위상은 변하지 않는다. $M^* = \emptyset$ 인 경우는 M 상에 단면곡률이 양수인 계량이 존재하므로 M 은 S^3/Γ 와 미분동형이다. $M^* = \emptyset$ 인 경우에는, 유한 번의 surgery를 한 Ricci flow가 유한 시간이 지난 후에 **사멸**(extinct)한다라고 말한다. M^* 가 GC를 만족하면 M 도 GC 를 만족한다.

단계 3. 기껏해야 국소적으로 유한개의 특이점 시간이 존재한다. 유한 시간 구간 $[0, T)$ 에서 많아야 유한 번의 surgery만 하여 Ricci flow를 계속하여 시행할 수 있다.

단계 4. 충분히 큰 시간 $t \gg 0$ 에 대해 비압축적 토러스(incompressible torus)들을 따라서 M^* 는

$$M^* = M_{thick} \cup M_{thin}$$

으로 분해된다. $t \to \infty$ 일 때 M_{thick} 의 성분들은 부피가 유한인 완전 쌍곡 다양체로 수렴하고 M_{thin} 의 성분들은 붕괴한다.

단계 5. M_{thin} 은 그래프 다양체이다.

Perelman은 M 의 특이점들의 구조가 아래와

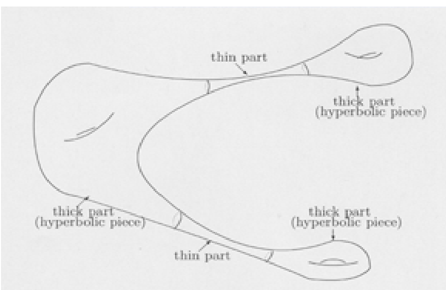

그림 4 : Thick-thin 분해

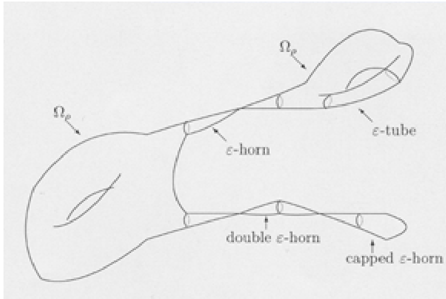

그림 5 : 특이점의 구조

같은 형태라는 사실을 보였다.

그림 4와 그림 5는 각각 Yau의 논문 [25]의 20쪽과 18쪽에 있는 그림을 가져왔다.

Perelman은 그의 논문 [20]에서 $\varepsilon-$neck, $\varepsilon-$cap, $\varepsilon-$tube, $\varepsilon-$circuit, $\varepsilon-$horn, double $\varepsilon-$horn, capped $\varepsilon-$horn 의 개념을 소개하면서 특이점들을 분석하며 연구하였다. 한정된 지면으로 인하여 이들의 개념들의 설명은 생략하겠다. 예를 들면, 그림 5에서 알 수 있듯이 double $\varepsilon-$horn 의 양쪽 끝 부분에서는 스칼라 곡률이 무한대로 접근하며, $\varepsilon-$tube 의 양쪽 끝 부분에서는 스칼라 곡률이 유계이다. 또한 그는 $\varepsilon-$cap과 $\varepsilon-$tube 안에 포함되어 있는 특이점들을 분석하며 연구하기 위해서 S^2를 따라서 행한 surgery 방법(surgery along S^2)을 창안하였다. 한마디로 요약하면 그는 surgery를 지닌 Ricci flow[5] 를 철저하고 심도있게 연구하여 GC 를 해결하였다.

Hamilton 은 단계 1을 해결하지 못했다. 반면에 Perelman은 그의 논문 [19]에서 단계 1을 해결하였고, [20]에서 단계 2, 3, 4 를 성공적으로 해결하였다. 그런데 Perelman 은 단계 5에 대해서는 자세하게 설명하지 않았다. 그래서 많은 수학자들이 네 번째 논문을 기다렸지만, 그는 지난 3년 동안 단계 5에 관한 논문을 내놓지 않았다. 그러나 단계 5는 위상학적으로 어렵지 않게 처리될

5) 참고문헌 [11]에서 Hamilton 은 어떤 특별한 곡률을 가지는 4차원 다양체의 위상을 연구하는 과정에서 처음으로 surgery 를 지닌 Ricci flow (즉, Ricci flow with surgery) 를 소개하였다.

수 있다고 위상학자들은 믿고 있다. 실제로 M 의 기본군 $\pi_1(M)$ 이 무한군인 경우는 Takashi Shioya와 Takao Yamaguchi 가 그들의 논문 [22]에서 Perelman 의 도움으로 단계 5를 증명하였다. M 의 기본군 $\pi_1(M)$ 이 유한군인 경우인 경우에, Perelman 은 그의 논문 [21]에서 특이점 시간 $T_1, \cdots, T_n$ 에서 surgery 를 한 후, $t \to T_{n+1}$ 일 때 M^*의 스칼라 곡률이 M^* 상의 모든 점에서 ∞ 으로 blow up 한다는 사실을 주장하였다. 이 경우에는 시간 T_{n+1} 의 근처에서 M^*의 단면곡률은 양수이므로 RF의 해는 $t = T_{n+1}$ 에서 사멸한다. 따라서 $M = S^3/\Gamma$ 이다. 결론적으로 3차원 다양체의 Poincaré 가설을 증명한 것이다. 한편 T. Colding과 W. Minicozzi II 는 그들의 논문 [6]에서 (M, g_0)이 유향이며 닫힌 3차원 다양체로서 aspherical6) 하지 않다고 하면, 초기치 조건을 만족하는 Ricci flow의 해 $g(t)$는 유한 시간 후에 사멸한다는 사실을 증명하였다.

예를 들어 다음과 같은 방정식을 보자. 반경이 r 인 E^{n+1} 안의 구 S^n 를 생각하면,

$$g_{ij} = r^2 \ \widehat{g}_{ij}$$

이고 Ricci 텐서는 $R_{ij} = (n-1) \ \widehat{g}_{ij}$ 으로 주어진다. 여기서 $\widehat{g}_{ij}$ 는 반지름이 1 인 구 S^n 상의 계량이고, Ricci flow 방정식이

$$\frac{\partial g_{ij}}{\partial t} = -2(n-1)R_{ij} \quad \text{즉,} \quad \frac{\partial r^2}{\partial t} = -2(n-1)$$

으로 주어진다. $n > 1$ 이면 $[0, T)$ (단, $T = \frac{r^2(0)}{2(n-1)}$)에서 이의 해는

$$g_{ij}(t) = r^2(t) \ \widehat{g}_{ij}(t),$$

$$(\text{단, } r^2(t) = r^2(0) - 2(n-1)t)$$

으로 주어진다. 다시 말하면, 구는 유한시간 내에 점으로 수렴하고 일반적으로 Ricci 텐서가 모든 점에서 양수인 계량에서 출발한 Ricci flow는 점으로 수렴하고, 체적이 1이 되도록 스케일(scale)하면 구로 수렴함을 보일 수 있다.

Perelman의 surgery를 지닌 Ricci flow 에서는 특이점이 발생하는 곳에서 surgery를 하면 (이것은 S^2 를 따라 M 을 조각으로 나누는 과정이다) Ricci flow가 무한대까지 갈 수 있고, 무한대에 접근하면 비압축적 토러스(incompressible torus)를 따라 다양체가 자연스럽게 나누어지는데 쌍곡기하를 갖는 조각 (또는 성분)과 그래프 다양체(graph manifold)로 나누어진다.

Perelman.의 업적에 관한 보다 상세한 내용은 참고문헌 [3, 6, 14, 15, 18, 22, 24, 25]를 참고하길 바란다.

6) $k \geq 2$ 이면 $\pi_k(M) = 0$

V. 맺음말

앞에서 알 수 있다시피 Perelman은 Hamilton 의 프로그램을 성공적으로 수행하기위해 심오하고 아름다운 기하학적 아이디어를 발견하고 강력한 해석학의 도구를 테크니컬하게 사용하며 GC 를 해결하였다. 한편 Cao와 Zhu는 그들의 논문 [3]에서 Perelman의 방법과 다른 방법으로 GC 를 해결하였다고 주장하고 있다. 앞으로 [3]은 철저하게 전문가들에 의해 검증되어야 한다. 지난 약 4년간 논문 [19, 20, 21]은 적지 않은 뛰어난 수학자들에 의해 오류가 없다는 사실이 엄밀하게 검증되었기 때문에 Perelman이 최종적으로 GC 를 해결하였다고 말할 수 있다. GC가 중요한 난제이기 때문에 적어도 앞으로 2년 동안 Clay 수학연구소와 여러 전문가들에 의해 검증을 받을 것이다.

Perelman의 탁월한 창의력은 세계 수학계의 수준을 한 단계 끌어 올리고 인간정신의 승리를 이루었다. 한정된 짧은 시간 내에 제한된 지면에다 Perelman의 깊고 아름다운 업적을 간략하고 명확하게 소개한다는 것은 저자들에게는 매우 힘든 작업이었음을 밝히고 싶다. 그의 논문을 제대로 이해하지 못한 가운데서 쓴 이 부끄러운 글이 그에게 커다란 누를 끼치지 않을까 두렵다. 흥미로운 문헌 [13]을 독자들에게 일독을 권하면서 이 부끄러운 글을 마무리 한다.

[참고문헌]

[1] Michael Anderson, Geometrization of 3-manifolds via the Ricci flow, Notices of the American Math. Soc., Vol. 2 (2004), 184-93.

[2] H.-D. Cao and B. Chow, Recent developments on the Ricci flow, Bull. Amer. Math. Soc. (N.S.), vol. 36 (1999), 59-74.

[3] Huai-Dong Cao and Xi-Ping Zhu, A Complete Proof of the Poincaré and Geometrization Conjectures - application of the Hamilton-Perelman theory of the Ricci flow, Asian Journal of Mathematics, Vol. 10, No. 2, June 2006, 165-492.

[4] B. Chow and S.-C. Chu, A geometric interpretation of Hamilton's Harnack inequality for the Ricci flow, Math. Res. Let. 2 (1995), 701-718.

[5] B. Chow and S.-C. Chu, A geometric approach to the linear trace Harnack inequality for the Ricci flow, Math. Res. Let. 3 (1996), 549-568.

[6] Tobias H. Colding and William P. Minicozzi II, Estimates for extinction time for the Ricci flow on certain 3-manifolds and a question of Perelman, J. Ameri. Math. Soc. 18, no. 3 (2005), 561-569.

[7] J. Eells and J. Sampson, Harmonic mappings of Riemannian manifolds, Amer. J. Math. 86 (1964), 109-160.

[8] Richard Hamilton, Three-manifolds with positive Ricci curvature, Journal of Differential Geometry, Vol. 17 (1982), 695-729.

[9] -----, The Harnack estimate for the Ricci flow, Journal of Differential Geometry, Vol. 37 (1993), 225-243.

[10] -----, Formation of singularities in the Ricci flow, Surveys in Differential Geometry, Vol. 2, International Press(1995), 7-136.

[11] -----, Four-manifolds with positive isotropic curvature, Comm. Anal. Geom. 5, No. 1 (1997), 1-92.

[12] -----, Non-singular solutions of the Ricci flow on three-manifolds, Comm. Anal. Geom. 7 (1999), 695-729.

[13] Ally Jackson, Conjectures No More ? Consensus Forming on the Proof of the Poincaré and geometrization Conjectures, Notices of the American Math. Soc., Vol. 53, No. 8 (2006), 897-901.

[14] Michael Kapovich, Geometrization Conjecture and the Ricci Flow, preprint, September 10, 2003.

[15] Bruce Kleiner and John Lott, Notes on Perelman's Papers, arXiv.org, May 25, 2006.

[16] John Milnor, A uniqueness decomposition theorem for 3-manifolds, Amer. J. Math. 84 (1962), 1-7.

[17] ----, Towards the Poincaré Conjecture and the classification of 3-manifolds, Notices of the American Math. Soc., Vol. 10 (2003), 1226-33.

[18] John Morgan and Gang Tian, Ricci Flow and the Poincaré Conjecture, arXiv.org, July 25, 2006.

[19] Grisha Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv:math.DG/0211159 (November 11, 2002).

[20] -----, Ricci flow with surgery on three-manifolds, arXiv:math.DG/0303109 (March 10, 2003).

[21] -----, Finite extinction time for the solutions to the Ricci flow on certain three-manifolds". arXiv:math.DG/0307245 (July 17, 2003).

[22] Takashi Shioya and Takao Yamaguchi, Volume-collapsed three-manifolds with a lower curvature bound, Math. Ann. Vol. 333 (2005), 131-55.

[23] William P. Thurston, Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, Bull. Amer. Math. Soc. (N. S.), Vol. 6 (1982), 357-81.

[24] S.-T. Yau, Perspectives on geometric analysis, ArXiv:math.DG/0602363 (February 16, 2006).

[25] -----, Structures of Three-Manifolds, ArXiv:math.DG/0607821(August 21, 2006).

인도의 Harish-Chandra 연구소에 다녀와서

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2006년 12월 1~5일의 기간 동안 인도의 성시(聖市)인 알라하바드(Allahabad)에 있는 Harish-Chandra 연구소(간단히, HRI)에서 개최되었던 정수론 국제학술회의에 초청연사로 초청되어 약 9일 동안 인도를 처음으로 여행하였다. 2003년에 HRI에 두 달간 정식으로 초청된 적이 있었지만 여러 사정으로 초청을 수락하지 않았다. 11월 28일 인디아 항공편으로 인천공항을 떠나 인도의 수도 뉴델리에 밤늦은 오후 11시경에 도착하였다. 우리나라와는 달리 택시를 타기 위해서는 공항 출구 바로 옆에 있는 창구에서 목적지를 알려주고 택시 요금을 미리 지불하여야만 한다. 그러면 유창한 영어를 구사하는 택시 기사의 친절한 안내를 받으며 목적지에 무사히 도착할 수 있다. 그렇지 않으면 밖에서 아우성치며 손짓하는 수십 명의 택시 기사 중의 한 사람과 흥정하여 보다 저렴한 가격으로 목적지로 갈 수 있지만, 처음 인도를 여행하는 외국인은 어느 정도의 위험을 감수하여야만 한다(더군다나 늦은 밤이면). 한국에 비하면 그 곳의 택시 요금은 매우 저렴하다. 주최 측에서 마련한 인도 기술 연구소(IIT)[1]의 게스트 하우스에서 하루 밤을 보낸 후 다음날 사우디 항공편(Air Sahara)으로 바라나시(Varanasi)에 낮 12시 경에 도착하였다. 마중 나온 HRI의 대학원생 Bahu의 환영을 받으며 역시 초청된 헝가리의 학술회원인 Katai와 함께 소형 밴(van)을 타고 목적지인 알라하바드로 향하였다. 바라나시와 알라하바드와의 거리는 70~80km 밖에 되지 않지만 3시간 넘게 걸려 HRI에 도착하였다. 도중에 나로서는 흥미롭고 진귀한 경험을 하게 되었다. 길가에서 많은 소, 여러 무리의 낙타들, 심지어는 코끼리 까지 보게 되어 신비감을 맛보았으며, 인도인들이 숟가락과 젓가락 등을 사용하지 않고 맨손으로 자연스럽게 식사를 하는 모습을 보고 놀라움을 금치 못하였다. 길가에는 교통 경찰관과 신호등이 없어서인지

1) the Indian Institute of Technology : 델리, 뭄바이, 첸나이, 칸푸르 등 7개의 도시에 있음.

사람, 소, 차들이 뒤엉켜 있어 운전기사들이 경적 소리를 크게 울리며 사람들, 특히 소들을 피해가며 운전하는 흥미로운 모습을 보았다.

HRI에 도착한 후에 큰 게스트 하우스에서 방을 배정받고 긴 여정의 피로를 몇 시간 풀었다. 서울에서 이곳에 도착하는데 약 30시간이 걸렸다. 이날 저녁에 국제학술회의의 주관자인 B. Ramakrishnan 교수의 초청으로 나를 포함하여 여러 초청연사들이 알라하바드에서 최고급인 호텔에 저녁 초대를 받았다. N.-P. Skoruppa, K. Miyake, I. Katai, S. Akiyama, Y. Chen 등의 초청연사들과 함께 즐거운 시간을 보냈다. 이 호텔은 이곳에서 하나밖에 없는 우리나라 기준으로는 4성급 호텔로서 최고급 호텔이다. 최근에서야 맥도널드 가게가 처음으로 이곳에 입성하였다고 한다. 이 도시의 인구는 20~30만 명 밖에 되지 않는 조그만 도시이지만 정치적으로 뿐만 아니라 문화・종교・교육적인 측면에서 인도사회에 지대한 영향을 끼쳤던 유서 깊은 도시라는 사실을 곧 알게 되었다. 뒤에 이 도시에 관하여 보다 자세히 설명하겠다.

HRI에 관하여 간략하게 소개하겠다. 1966년에 콜카타(Kolkata; 예전이름은 Calcutta 임)에 있는 메타 기업(Mehta Trust)의 출연기금으로 메타연구소의 이름으로 알라하바드에 설립되었다. 알라하바드 대학에 의해 운영되다가 1980년 대 중반에 와서 우타르 프라데쉬(Uttar Pradesh) 州 정부가 현 부지인 쥬시(Jhusi) 지역의 66 에이커의 부지를 무상으로 제공함과 동시에 정부기관인 원자에너지 부처에서 재정적인 지원을 하게 되면서 이 연구소는 급속도로 성장하게 되었다. 2001년 부터는 Harish-Chandra 연구소로 개명되어 현재까지 이르고 있다.

아시다시피 Harish-Chandra(1923~1983)는 브라만 계급으로 이 근처에 있는 칸푸르(Kanpur)에서 태어나 어린 시절을 보낸 후 알라하바드 대학에서 4년간 이론물리학을 전공하였다. 졸업 후 방갈로르(Bangalore)에 있는 인도 과학원(IIS)[2]에서 대학원 과정을 마치고 영국의 캠브리지 대학으로 유학을 가서 노벨 물리학상 수상자인 Paul A. M. Dirac (1902~1984)의 지도를 받으며 물리학을 공부하였다. 그의 박사학위 논문의 연구과제는 로렌츠 군의 무한 차원의 유니터리 표현에 관한 연구였다. 1947년에 물리학의 특성상 완벽한 증명의 마무리 없이 박사학위를 받았지만 이에 만족하지 못하고 완벽한 증명을 요구하는 수학이란 학문에 빨려 들어가게 되었다. 미국의 콜롬비아 대학 교수를 역임하였으며 1963년에 프린스턴 고등연구소에 영구직 교수로 초빙되어 1983년 심장마비로 별세할 때까지 연구하였다. 1954년에 Cole 상을 수상하였으며 1974년에 인도 과학 학술원으로부터 Ramanujan 메달을 수여 받았다. 그는 리군의 조화해석이론과 보형형식의 이론에 큰 업적을 남겼다. 인도 전역에서 뿐만 아니라 특히 이 지역에서는 Harish-Chandra 는 존경과 추앙을 받는 인물이다.

다시 HRI로 돌아가 이 연구소는 매우 훌륭한 도서관을 갖추고 있으며, 대학원 박사과정에 수십 명의 대학원생들이 수학과 이론 물리학을 전공으로 연구하고 있다. 이 연구소의 현관 복도에는 Harish-Chandra의 흉상이 서 있으며, 건너편의 대형 강의실의 입구에는 설립자인 Metha의 흉상이 서 있다. 본부 건물의 앞마당에는 저명한 사람(정치인, 수학자, 과학자)들이 이 연구소를 방문하여 식수한 후 자신의 이름을 남겨 놓았다. A. Borel[3] (1923~1983), R. Langlands[4] (1936~)와 같은 저명한 수학자와 이들의 부인들이 식수하였다. 그리고 Harish-Chandra의 부인인 Lily도 이 연구소를 방문하여 식수를 한

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2) the Indian Institute of Science : 1909년에 Banglore에 설립되었음. 수준이 높은 과학 연구소임.

3) Armand Borel (1923~2003) : 스위스 수학자. 프린스턴 고등연구소의 교수를 역임하였음. 리군과 보형형식의 이론을 연구하였음.

4) Robert P. Langlands (1936~) : 캐나다의 수학자. 1960년대에 Langlands Program 을 제창하였으며, 유체론과 보형형식 이론에 큰 공헌을 한 공로로 Wolf 상, Shaw 상 등을 수상하였음. 올해 프린스턴 고등연구소에서 정년퇴임하였으며, 현재 명예교수로 있음.

후 그녀의 이름을 남겨 놓았다. 앞마당에서 약 300 미터 정도 나아가면 성스러운 갠지스 강이 있다. 강가의 여러 곳에서 화장을 하는 모습을 볼 수 있었다. 이 연구소는 아주 넓으며 주위는 담으로 둘러싸여 있고 군인들이 곳곳에서 경비를 보고 있다. 그들 가까이 가면 경례를 하며 친절하게 대해 준다. 연구소 캠퍼스 내에는 교수 아파트, 게스트 하우스, 여러 시설이 잘 갖추어져 있어 외국 방문자들에게는 전혀 불편함이 없다. 나의 경우는 음식 때문에 약간 고생을 하여 장기간 이 곳에서 체류하기는 쉽지 않을 것 같았다. 캠퍼스 바깥은 거리가 지저분하고 먼지투성인 데다가 무질서하여 처음에는 난장판처럼 보였지만 며칠 지난 후에는 이런 모습이 자연스럽게 느껴졌다. 도시의 중심가에서는 많은 소들이 자유롭게 돌아다니고 차들과 사람들이 신성시되는 소들을 피하는 광경이 흥미로웠다. 체류하는 동안 틈틈이 도서관에서 인터넷을 통하여 인도의 철학, 종교, 역사, 문화를 차츰차츰 이해하게 되면서 인도에 관하여 관심을 가지게 되었다.

대형 강의실에서 국제 학술회의가 열렸으며 14명의 외국인 초청연사와 16명의 인도 수학자들이 초청강연을 하였다. 이 곳에서 독일의 N.-P. Skoruppa, 캐나다의 M. Ram Murty, 일본의 K. Miyake, S. Akiyama, Y. Tanigawa, 헝가리의 I. Katai, 중국의 Y. G. Chen, 인도의 V. Srinivas, J. Sengupta, R. Balasubramanian, T. N. Shorey, E. Ghate, 러시아의 A. Raigorodskii 등의 초청연사들을 알게 되면서 이들과 즐거운 시간을 보냈다. 특히 정수론 분야에서는 지난 수년 전에 미국의 저명대학 (Princeton, Yale 대학 등)에서 박사학위를 취득하여 Langlands 프로그램과 관련된 분야를 연구하고 있는 젊고 앞날이 밝은 유능한 인도 수학자들이 하는 강연을 즐겼다. 필자는 『Remark on Harmonic Analysis on Siegel-Jacobi Space』의 제목으로 강연하였다. 필자가 강연하는 모습의 사진과 더불어 학술회의를 소개하는 기사가 주요 중앙 일간지인 『힌두스탄 타임즈(Hindustan Times)』에 게재되었을 뿐만 아니라 지역 신문에는 이 학술회의에 관한 기사가 두 번이나 게재되었다. 필자는 강연 중간에 대칭공간상의 미분작용소 이론에 관한 Harish-Chandra 의 업적을 언급하였다. 강연이 끝난 후에 한 중년의 인도 여성 수학자가 다가와 초청강연자 중에서 필자만이 Harish-Chandra의 업적을 언급하였다고 말하면서 이 연구소의 이름과 결부시키며 강연 내용에 관하여 높이 평가하여 주었다. 필자도 이 연구소에서 Harish-Chandra의 업적을 강연 중에 언급하여서 기쁘고 나름대로의 의미도 있었다고 생각하고 싶다. 약 70여명의 수학자들이 이 학술회의에 참가하였다. 4년 후에는 남부에 있는 하이데라바드(Hyderabad)에서 ICM이 열리기 때문에 인도 수학자들이 매우 즐거운 마음으로 준비하고 있을 뿐만 아니라 자랑스러워하고 있었다. 하이데라바드는 남부의 안드라 프라데쉬(Andra Pradesh) 주의 수도이며 인구가 대략 600만이며 인도에서 IT, ITES와 BT의 중심지로 알려져 있다. 기후도 다른 대도시에 비하여 온화한 편인데다가 고급 호텔과 여러 현대식 시설(예를 들면 컨벤션센터)들이 많아 큰 국제 학술회의를 개최하기에 적합한 도시로 인정받고 있다. 그래서 뉴델리와 Tata 연구소가 있는 뭄바이(Mumbai)를 제치고 3년 후인 2010년 8월에 하이데라바드에서 ICM이 개최된다. 인도는 지난 1세기의 기간 동안에 스리니바사 라마누잔[5], Harish-Chandra, 올해의 아벨상 수상자인 스리니바사 바라단[6] (Varadhan) 등의 여러 저명한 수학자들을 배출하였을 뿐만 아니라 현재 미국에서 활약하고 있는 저명한 젊은 수학자 M. Bhargave, C. Khare, K. Soundararajan 등을 배출하였다.

5) Srinivasa Ramanujan (1887~1920) : 인도의 전설적인 수학자. 정규 교육을 제대로 받지 못하였지만 수학적인 재능을 맘껏 발휘한 수학 천재. 영국의 유명한 수학자 G. Hardy와 공동 연구를 하였음.

6) Ravi Srinivasa Varadhan (1940~) : 인도의 첸나이(Chennai)에서 태어나 인도 통계연구소(ISI)에서 박사학위를 받은 후 미국으로 건너가 쿠란트 수학연구소의 교수가 되었음. 1996년에 미국 수학회에서 수여하는 Steele 상을 수상하였음. 2007년에는 노르웨이 정부에서 수여하는 아벨상과 상금 100 만 달러를 받았음.

알라하바드에 관하여 언급하지 않고서는 지나갈 수 없다. 앞에서 잠시 이 도시는 정치적으로 뿐만 아니라 문화・종교・교육적인 측면에서 인도 사회에 지대한 영향을 끼쳤던 유서 깊은 도시라고 언급하였다. 이 도시의 이름은 1583년에 무굴제국의 위대한 황제인 Akbar에 의하여 명명되었다. 이 도시의 옛 이름은 프라야그(Prayag)인데 이 의미는 산스크리트로 『희생의 도시』(place of sacrifice)라고 한다. 이 도시의 상감(Sangam)[7]이란 성스러운 지역에서 갠지스강(the Ganga), 야무나강(the Yamuna)과 보이지 않는 전설속의 "지혜의 강"인 사라스와티 강(the Saraswati)이 합류한다. 그래서 매년 많은 사람들이 여기에 와서 몸을 씻으며 속세에서 지은 죄를 용서받고 또한 임종을 앞둔 사람들은 이 근처에서 몸을 씻은 후 죽음을 맞는다고 한다. 특히 인도 힌두교도들은 상감 지역에서 목욕을 하면 모든 죄가 용서되고, 윤회의 고통에서 벗어나며 모든 소원이 이루어진다고 믿고 있다. 그래서인지 연구소에서 장작을 피워 화장을 하는 여러 광경을 접할 수 있었다. 학술회의 주최 측에서 12월 3일(일요일) 오전 세션을 마치고 참석자들에게 상감지역과 무굴제국의 황제, 황제비의 무덤이 있는 공원과 여러 곳을 관광시켜 주었다. 배를 타고 누르스름한 색을 띤 성스러운 갠지스 강 위에서 사진을 찍으며 여러 생각에 잠기기도 하였다. 매년 1월에 이 지역에 수천만 명의 순례자들이 모여들어 거의 한 달 보름 동안 이곳에서 생활을 한다고 한다. 힌두교, 회교, 불교 신자들뿐만 아니라 일반 사람들도 이곳을 모여 든다고 한다. 성스러운 힌두교의 성지 순례 행사인 『쿰브 멜라』(Kumbh Mela)[8]가 12년에 4번씩 하리드와르(Haridwar), 우자인(Ujjain), 나시크(Nasik)와 알라하바드에서 번갈아 가며 대대적으로 거행된다. 그래서 한 곳에서는 12년 만에 한 번씩 쿰브 멜라가 열린다. 힌두교의 창조 신화에 의하면 신과 악마들이 불멸의 음료인 신주(神酒)가 담겨져 있는 쿰브를 차지하기 위해서 12일 동안 치열한 싸움을 치른 후에 힌두교의 3대 신의 하나인 비쉬누가 손을 뻗어 쿰브를 낚아 챈 뒤 하늘의 안전한 곳에 도착하기 전에 네 곳에 멈췄다고 한다. 그 와중에 술의 몇 방울을 강물에 떨어 뜨렸는데 그 곳이 상기에 언급한 네 도시 지역이었다고 전해지고 있다. 매일 상감 지역에 수백 만 명의 순례자들이 모여들어 한 두 달간 축제가 계속된다. 지난 2001년에 1~2월 두 달 동안 알라하바드에서 쿰브 멜라가 거행되었다고 하니 2013년에 대대적인 쿰브 멜라가 열리게 되어 있다. 금년에는 상감 지역에서 6년째 되는 "반(Ardh)" 쿰브 멜라가 열려 약 7천 만 명이 이 축제에 참가하였다고 외신은 전하고 있다. 그리고 영국의 식민지 지배 아래에서 독립운동을 하며 투옥되었던 인도의 초대 수상인 네루[9]가 이 도시에서 태어나 자랐던 곳이고, 그의 외동딸인 인디라 간디[10]도 여기서 태어났다. 인도의 2대 수상인 샤스트리(1904~1966)도 이 도시에서 태어나 자랐으며, 8대 수상인 싱(Singh, 1931 ~)과 인디라 간디의 아들이며 9대 수상인 레지브 간디[11]도 비록 이 곳에서 태어나지는 않았지만 정치적인 입지를 굳혔던 곳이다. 네루는 갠지스 강을 『인도의 강, 인도 문화와 문명의 상징, 끝없이 변

7) 산스크리트에서 유래된 말로 "강물이 합쳐지는 곳"이라는 뜻임.

8) 힌두어로 『쿰브(Kumbh)』는 "물병" 이며 『멜라(Mela)』는 "축제" 라는 뜻임.

9) Jawaharlal Nehru (1889~1964) : 변호사의 아들로 태어나 인도의 독립 운동을 하다가 여러 차례 투옥되었음. 감옥에서 1930-33년 사이에 그 당시에 13살이었던 외동딸인 인디라 간디에게 역사 교육을 위해 보낸 편지들을 엮어서 저서 『Glimpses of World History』를 1945년에 출판하였음. 1942-46년 기간 동안의 투옥 중에 감옥에서 『Discovery of India』를 저술하였음. 인도의 초대 총리이며 17년 동안 총리직을 수행하였음.

10) Indira Gandhi (1917~1984) : 네루의 외동딸로 4대, 7대의 2차례 인도 총리를 지냈으며 15년 동안 총리직을 수행하였음. 1984년에 그녀의 경호원에 의해 암살되었음. 간디 성을 얻은 건 남편이 페로제 간디이기 때문임.

11) Rajv Gandhi (1944~1991) : 인디라 간디의 큰 아들로 인도의 9대 총리를 지냈음. 역시 1991년에 정치적인 이유로 암살되었음. 그의 부인인 소냐 간디는 현 국민회의당 대표로 정치활동을 하고 있음. 그의 아들 라훌(Rahul) 간디는 현재 하원의원으로 차세대의 총리 후보감으로 인정받고 있음.

하고 흐르면서도 영원하고 한결같은 강의 여신』이라고 묘사하였다. 이외에도 알라하바드는 많은 유명한 시인, 과학자, 종교인과 예술인들을 배출한 곳으로 유명하다. 이것으로 미루어보아 비록 작은 도시이지만 이 도시는 정치적으로 뿐만 아니라 문화・종교・교육적인 측면에서 인도에서 매우 유서 깊은 도시라고 할 수 있을 것이다.

인도에는 크게 23개의 공식 언어가 있다고 한다. 실제로는 800 여 개의 방언이 사용되고 있으며 수많은 종교적, 철학적, 정치적, 사회적 가치관이 혼재되어 공존하고 있다. 그중에서도 영어와 힌디어가 인도 전 지역에서 공용어로 사용되고 있다. 남부지역에서 온 HRI의 교수들은 힌디어를 배운 적이 없어 타밀어(Tamil)를 사용하며 영어는 지식인들에게는 필수적이다. 힌디어를 사용하는 사람은 전체 인구의 40 퍼센트 정도이고 힌디어를 모르는 총리가 취임한 적도 있다. 지금은 거의 대부분의 젊은 사람들은 TV를 통하여 힌디어를 배워서 힌디어를 할 수 있다고 한다. 사회에서 출세하기 위해서는 영어는 물론이고 힌디어도 알아야 하기 때문이다. 모든 초, 중, 고등학교에서 영어를 가르치고 있으며 이들 학교의 교육 수준이 매우 높다고 한다. 그런데 각 지방의 액센트가 너무 달라 처음에는 자기끼리의 영어를 알아듣기가 힘들다고 한다. 인도 인구가 현재는 대략 11억이라고 하며 국민소득은 매우 낮아 인구의 99% 정도는 경제적으로는 빈곤한 생활을 하고 있다. 그러나 세계 갑부 100위 안에 5~6명의 인도인이 있다. 법적으로는 카스트 제도가 인정이 되고 있지 않지만 아직도 사회적으로는 완전히 없어지지 않은 것 같다. 총리나 국회의원 등의 권력이 있는 정치인이 되려면 브라만 계급이라야만 유리하다. 전 인구의 약 80 퍼센트가 힌두교도이고, 약 15 퍼센트가 회교도이며 나머지 약 5 퍼센트는 불교, 기독교, 여러 종교의 신자들이 차지하고 있다. 필자가 짧은 기간에 인도에서 보고 느낀 것을 무모하게 적으면 다음과 같다. 수 천 년 전에는 인도 대륙에 수많은 작은 나라가 존재하였으며 그동안 많은 전쟁과 외침을 받으면서 합병되든가 아니면 지배를 받아왔다. 근대에는 거의 300년 동안 영국의 식민지 지배를 받았으며 그전에는 몽골의 징기스칸의 후손이며 페르시아인인 무굴제국의 황제들에 의하여 수 백 년 동안 지배를 받았다. 이 지배를 받는 동안에 회교 문화와 서양문화가 인도 고유의 문화와 융화되었지만 이 지역의 많은 부분의 문화는 사라지지 않고 보존되고 있다고 생각한다. 대부분의 사람들이 힌두교를 믿으며 신앙심이 깊고, 철학자와 승려들을 존경하는 사회를 형성하였다. 그러나 저마다 각 지역의 전통적인 관습과 언어를 고집하기 때문에 11억이나 되는 사람과 거대한 대륙을 하나로 효율적으로 통솔하기가 힘들 것이란 생각이 든다. 적어도 전 인구의 10~20 퍼센트가 되는 1억 이상의 사람들은 교육을 잘 받은 지식인이라고 생각한다. 이들이 인도 내에서 뿐만 아니라 세계 각지에 진출하여 큰 활약을 하고 있다. 몇 십 년 후에는 많은 지역이 하이데라바드처럼 현대식으로 개발되리라 믿는다. 그때가 되면 인도는 세계에서 정치적, 군사적, 경제적 파워를 과시하는 문화적인 선진국이 되리라는 것을 기대하여 본다.

인도는 9명의 노벨상 수상자를 배출하였다. 3명의 문학 노벨상 수상자인 키프링[12], 타고르[13], 나이폴[14], 2명의 물리학 노벨상 수상자인 라만[15], 찬드라세크하르[16], 2명의 의학 노벨상 수상자인 로스[17], 코라나[18], 1명의 경제학 노벨상 수상자인

12) Rudyard Kipling (1865~1936) : 1907년에 노벨 문학상을 수상하였음.

13) Rabindranath Tagor (1861~1941) : 인도의 철학가이며 시인, 1913년에 노벨 문학상을 수상하였음.

14) Vidiadhar Surajprasad Naipaul (1932~) : 영국의 문학가이지만 인도인들은 그를 인도인이라고 인정함. 2001년에 노벨 문학상을 수상하였음.

15) Sir Chandrasekhare Venkata Raman (1888~1970) : 인도의 물리학자이며 1930년에 노벨 물리학상을 수상하였음.

16) Subrahmanyan Chandrasekhar (1910~1995) : 인도의 물리학자. 1983년에 노벨 물리학상을 수상하였음.

17) Ronald Ross (1857~1932) : 1902년에 노벨 의학상을 수상하였음.

센[19]과 1명의 노벨 평화상 수상자인 성녀 테레사[20]가 있다. 이것으로 미루어보아 인도는 철학, 문학, 물리학, 의학, 경제학 분야에서는 수준이 높은 연구가 진행되고 있는 국가라는 사실을 알 수 있다.

12월 6일 귀국하는 날에는 주최 측에서 바라나시 공항으로 안내하여 주면서 바라나시에서 북동쪽으로 13 킬로미터 정도 떨어져 있는 유서 깊은 사르나트(Saranath) 시를 관광시켜 주었다. 사르나트는 부처님이 알라하바드에서 남쪽으로 50 km 떨어진 콘단나(Kondanna)에서 6 여 년 동안의 고행을 통해 해탈한 후에 이곳에 와서 처음으로 설교를 하려고 하였지만 아무도 오지 않아 사슴 앞에서 처음으로 설법을 토하였다고 전해지고 있다. 사르나트는 산스크리트로 사슴이 뛰노는 공원이란 뜻이며 우리말로 녹야원(鹿野園)이라 번역이 되었다. 다메크 스투파, 아쇼카 석주, 네 마리의 사자상을 보고 감명을 받았다. 매년 세계 각지에서 많은 관광객들이 이곳으로 몰려오고 있다고 한다. 다메크 스투파는 부처님이 다섯 비구들에게 최초로 설법을 한 것을 기념하기 위해서 아쇼카 왕에 의하여 건립되었다. 인도에 여행할 기회가 있는 분들에게 사르나트를 추천하고 싶다. 성지 중의 하나인 바라나시는 시간적인 여유가 없어서 관광하지 못했다. 짧은 기간의 인도의 체류기간 동안 인도의 성지인 알라하바드 주변을 여행하면서 인도의 성스러운 모습을 보게 되어 매우 기쁘고 즐거웠다. 또한 틈이 나는 대로 인도의 철학, 종교, 문학, 시, 문화, 역사들을 공부하려고 한다. 프랑스의 위대한 수학자 앙드레 베이유[21]는 우타르 프라데쉬 주의 북부에 있는 알리가르 회교대학에서 1930~32년 동안 강의를 하였으며, 산스크리트를 배워 평생 동안 산스크리트에 관심을 가지고 공부하였다고 한다. 독일의 위대한 수학자 지겔[22]도 뭄바이의 Tata 연구소에 4차례 초청받아 장기간 체류하면서 산스크리트를 배웠다고 한다. 산스크리트로 수학논문을 쓰려고 시도하였다고도 전해진다.

18) Har Gobind Khorana (1922~) : 1968년에 노벨 의학상을 수상하였음. 현재 MIT 교수로 재직 중임.

19) Amartye Kumar Sen (1933~) : 1998년에 노벨 경제학상을 수상하였음.

20) Maria Teresa (1910~1997) : 루마니아에서 태어났지만 인도로 귀화하였음. 1979년에 노벨 평화상을 수상하였음.

21) André Weil (1906~1998) : 프랑스 수학자. 미국의 시카고 대학과 프린스턴 고등연구소에서 교수로 역임하였음. Wolf 상을 수상하였음. 정수론과 대수기하학 분야에 큰 공헌을 하였음. 시몬느 베이유는 그의 여동생임.

22) Carl Ludwig Siegel (1896~1981) : 20세기의 위대한 수학자. 정수론과 천체역학의 분야에 큰 공헌을 하였음. 제1회 Wolf 상을 수상하였음. 괴팅겐 대학과 프린스턴 고등연구소의 교수를 역임하였음.

원로수학자
와의 대화

프린스턴에서의 이임학 교수와의 만남

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1996년 10월 8~12일의 짧은 기간 동안 이임학 교수님과 함께 지냈던 추억을 회상할까 한다. 이 전(아마도 1980년 대 후반)에 이임학(1922~2005), 임덕상(1928~1983) 교수님에 관한 에피소드와 학문적 업적에 관하여 여러 수학과 선배(권경환, 이정림)들로부터 직접 들었던 기억이 난다. 그러나 그 분들은 부모님 연배가 되어 만나 본 적이 없어 크게 관심을 가지지 않았다. 필자가 서울대학교 대학원 석사과정(1976년 3월~1977년 2월, 1978년 9월~1979년 8월, 군복무: 1977년 2월~1978년 5월)에 있을 때 임덕상, 권경환, 한경택, 이동훈 교수님들이 AID 차관으로 수학과에 초청되어 강의하였다. 이 당시에 필자의 지도교수이셨던 김종식 교수님이 주도적으로 이들 교수님들과 접촉하여 초청하였던 것으로 알고 있다 (그 당시 필자는 학생 신분인지라 정확한 상황은 알 수가 없다). 그래서 1978년 2학기에 한경택 교수님(펜실베니아 주립대)의 다변수 복소함수론 강의와 이동훈 교수님의 리(Lie)군론 강의를 들을 기회를 가졌고, 1979년 1학기에는 권경환 교수님 (미시건 주립대학)의 대수적 위상수학 강의를 한 달 보름 정도 청강하였다. 당시 강의 노트 없이 자연스럽게 강의하시는 권경환 교수님의 모습에 감명을 받았다(가끔 강의 쪽지를 가지고 오셨지만). 1979년 7월 미국으로의 출국 준비(캘리포니아 주립대-버클리 유학)때문에 권경환 교수님의 강의를 계속 들을 수 없었다. 1979년 1학기 기간에는 권경환 교수님과 여러 번 대화를 나눈 적이 있다. 권 교수님의 가족(사모님, 어린 아들, 딸 둘)들을 학교에서 만났으며 1990년대 초반에는 포항공대에서 권 교수님 부부와 저녁 식사를 함께 한 적이 있다. 권 교수님은 스메일(Stephen Smale : 1966년 Fields 상 수상자)과 미시건 대학에서 함께 공부하였다는 이야기를 해 주셨고, 버클리에 가면 스메일 교수를 만나 보도록 조언을 하여 주셨다. 버클리에서 스메일 교수를 만나 권 교수님의 안부를 전하니 그가 매우 기뻐하며 나를 반겼다. 1978년 1학기에는 임덕상 교수님이 대수기하학을 대학원 과정에서 강의하였던 것으로 알고 있다. 이 기간 동안은 필자가 육군본부에서 군복무를 하였기 때문에 임덕상 교수님의 강의를 들을 수가 없었다. 임 교수님은 그의 변형이론에 관한 논문[1)]으로 그로텐딕(Alexander Grothendieck, 1928~ : 1966년 Fields 상 수상자)의 연구에 큰 기여를 하였던 대수기하학 분야에서

알려져 있는 유명한 수학자이다. 불행하게도 얼마 후에 간암으로 돌아가셨다는 비보를 접하게 되었다. 권경환, 이정림 등 살아계시는 선배 교수들을 언급하는 이유는 이 분들이 필자의 학문에 직접적인 영향을 끼쳤기 때문이다. 그리고 1950년대의 어려운 환경 속에서도 수학에 대한 열정만으로 연구하였다는 사실은 그 당시에는 참으로 감동적으로 받아들여졌다.

지난 10여 년 동안 이임학 교수님에 관한 여러 편의 글을 읽어보니 이 교수님은 1940년대에 필자가 1970년대의 대학 시절에 수학 공부하였던 것보다 더 많이 공부하셨다는 사실을 알게 되었다. 이 교수님은 1953년 출국하기 전에 이미 타카기의 해석개론, 레프쉐츠의 위상수학, 독일어 판의 Van der Waerden의 현대대수학과 같은 명저를 이미 독파하였다. 이 교수님은 수학적 재능을 타고 났을 뿐만 아니라 어려운 여건 가운데서도 훌륭한 몇 분의 교수의 지도와 조언을 받았음을 알 수 있다. 필자의 경우는 대학 시절에 반정부 데모로 인하여 학교가 휴교를 하여 많은 내용을 배울 수 없었고 공부도 하지 않았다. 필자는 유학 첫 해 3개월 동안 엄청난 양(대학 6년 동안 배우고 공부했던 것보다 더 많은 양)의 공부를 하였으며 그 후 몇 개월 동안 위대한 수학자 지겔, 웨이유, 셀버그, 하리쉬-찬드라의 업적을 배우게 되었다. 새로운 세계가 필자에게 열렸다.

그러나 필자는 이임학 교수님을 1996년 이전에 만나 본 적이 없어 이 교수님에 관해 큰 관심을 가지지 않았다. 필자가 안식년으로 1996년 9~11월에 하버드 대학을 방문하여 연구하고 있을 때의 일이다. 프린스턴 고등연구소 울펜손홀(Wolfensohn Hall)에서 랭글랜즈(Robert Langlands : 1936~)의 60세 기념 학술회의[2]가 1996년 10월 9~12일 기간 열렸는데, 이 기간 동안 이임학 교수님을 만나는 행운의 기회를 가졌다. (김대산, 김동균, 채희준, 김주리 교수도 만났고 R. Berndt, R. Weissauer 독일 동료교수도 만났다.) 랭글랜즈는 브리티시 콜럼비아 대학 재학 시절에 이 교수님의 강의를 여러 번 수강하였으며 이 교수님을 학문적으로 존경하였다. 그래서 랭글랜즈가 특별히 이 교수님을 개인적으로 이 학술회의에 초청하였다. 이 교수님이 학술회의에서 강연은 하지 않았지만 강의실에서 랭글랜즈와 함께 나란히 앉아 강연을 듣는 모습이 매우 인상적이었다. 당시 대부분의 참가자들이 Palmer Inn 이라는 호텔에 묵고 있었다. 이 교수님도 필자를 포함하여 한국에서 온 다른 동료들과 함께 이 호텔에 묵고 있어서 자연스럽게 저녁을 함께 하게 되었다. 정확하지는 않지만 아마도 6~7명의 한국 수학자가 이 학술회의에 참석하였던 것으로 기억하고 있다. 저녁 식사 후에 식당과 이 교수님의 방에서 한국에서 온 동료들과 많은 이야기를 나누었다. 이 교수님은 본인의 어려웠던 학창시절, 유학 시절, 브리티시 콜럼비아 대학에서의 교수 생활에 관하여 많은 이야기를 해주셨다. 김동균 교수는 1970년대부터 이 교수님은 랭글랜즈 프로그램에 관심을 가지고 연구하여 왔는데 불행하게도 이에 관한 논문을 한 편도 발표하지 않았다고 전해 주었다. 이 교수님은 1970년대 초반에 브리티시 콜럼비아 대학에 여러 번 랭글랜즈를 초청하여 이 프로그램의 강연을 들으며 많은 학문적인 의견을 교환하였다고 전해 주셨다.

그 당시 이 교수님의 나이가 74세이셨지만 건강하셨다. 이 교수님은 군살이 없이 말랐지만 날렵하

1) Dock Sang Rim, Formal deformation theory, Lecture Notes in Mathematics, Groupes de Monodromie en Géométrie Algébrique (SGA 7 I), Seminaire de Géométrie Algébrique du Bois-Marie (1967-1969), edited by A. Grothendieck with the collaboration with M. Raynaud and D. S. Rim, Springer-Verlag (1972), 32-132.

2) Conference on Automorphic Forms, Geometry and Analysis, October 9-12, 1996, Organizers: James Arthur, William Casselman, Robert Kottwitz, Institute for Advanced Study, School of Mathematics, Princeton, New Jersey.

시고 식사도 잘 하셨다. 건강의 비결이 무엇이냐고 물었더니 이 교수님은 거의 매일 수영을 하신다고 말하셨다. 이 학술회의에 여러 명의 젊은 한국 수학자들이 참가하여서 기뻐하시던 교수님의 모습이 떠오른다. 필자는 이 교수님과 학문적인 이견이 있어 논쟁을 하였던 기억이 난다. 가령 필자는 정수론 분야에서 지겔[3]의 업적을 높이 평가하였는데 이 교수님은 지겔을 높이 평가하지 않고, 와일즈[4]의 업적을 높이 평가하였다. 이 교수님은 지겔의 업적에 관하여 모르고 있는 느낌을 받았다. 참고로 지겔에 관한 타니야마의 글[5]을 참고하길 바란다. 이 교수님은 필자의 지도교수인 코바야시(Shoshichi Kobayashi, 캘리포니아 주립대-버클리)와 이 학술회의의 조직위원이고 초청연사인 카셀만(William Casselman, 브리티시 콜럼비아 대학)도 높이 평가하지 않았다. 예전에 코바야시 교수가 브리티시 콜럼비아 대학에서 몇 년간 근무한 적이 있었기 때문이다. 이것을 보더라도 이 교수님은 본인의 학문적인 업적에 큰 자부심을 가지고 있었고 웬만한 수학자는 거들떠보지도 않았던 것으로 짐작이 간다.

1960년 후반부터 1970년 초반은 랭글랜즈 프로그램의 태동이 시작되는 시기였다. 이 당시 이 교수님은 자신의 제자가 이 프로그램의 선두 주자가 되어 활동하는 모습을 보고 기뻐하였을 뿐만 아니라 이 프로그램의 중요성을 간파하고 이 프로그램의 한 주제인 아르틴 가설(Artin conjecture)에 관심을 가지고 도전하였던 것으로 알고 있다. 하지만 불행하게도 이임학 교수님은 그의 제자 랭글랜즈 그룹에 끼지 못했다. 그 이유는 알 수 없지만 14세라는 나이 차이도 있고 동양인이라는 핸디캡도 있지 않았나 하는 생각이 든다. 이 프로그램의 연구는 많은 기교적인(technical) 기법을 요구하는데다가 밀어붙이는 힘이 있어야 하고 주로 젊은 사람들이 연구에 참여하였기 때문에 아마도 이 그룹에 참여하기가 힘들었을 것으로 짐작이 간다. 한편, 보렐[6]은 이 교수님과 한 살 차이지만 이 그룹에 참여하여 연구하였다. 보렐은 이 프로그램에 관한 survey 논문[7]을 발표하였고 이와 관련된 학술회의에서 초청강연을 수차례 한 것으로 알고 있다. 그렇지만 보렐은 이 프로그램의 연구에 큰 공헌을 하지는 못했다.

헤어지기 전 날(1996년 10월 11일) 저녁에 이 교수님은 필자의 호텔 방에 직접 찾아와서 장래에 밴쿠버를 방문할 기회가 있으면 꼭 자기에게 연락을 달라고 하시면서 이 교수님이 저녁을 한 턱 내겠다고 필자에게 약속을 하셨다. 이어서 이 교수님은 다음 주에는 대한수학회의 초청으로 한국을 방문하게 된다는 말씀도 하셨다. 필자는 어제 인터넷을 검색하다가 우연히 1996년 10월 25일에 한국과학기술회관에서 선배 교수인 권경환, 이정림, 고영소, 주진구 교수님들이 이 교수님과 가졌던 인터뷰 내용을 읽게 되었다[8]. 이 대담 중에 이 교수님이 한국을 방문하기 직전에 참석한 프린스턴 학술회의에 관하여 간략하게 언급하였다. 필자는 이 인터뷰 글을 읽고 감회가 깊었다. 지난 2005년 3월에

3) Carl Ludwig Siegel (1896~1981) : 해석적 수론과 천체역학을 연구한 20세기의 위대한 독일 수학자. 1978년 제1회 울프상(Wolf Prize)을 수상함. 프린스턴 고등연구소와 독일 프랑크푸르트대학, 괴팅겐대학에서 교수를 역임하였음. 그의 이름이 있는 Siegel modular form, Siegel upper half plane, Siegel-Jacobi space, Siegel's lemma, Brauer-Siegel theorem, Thue-Siegel-Roth theorem, Siegel-Weil formula, Riemann- Siegel formula 등의 여러 수학전문용어가 있음

4) Andrew Wiles (1953~) : 수론을 연구하는 영국 수학자. 1995년에 350 여 년 동안 미해결 문제였던 페르마 마지막 정리를 해결하여 울프상, 특별필즈상, 쇼상 등 여러 상을 받았음. 영국 황실로부터 기사작위를 받았음. 소문에 의하면 프린스턴 대학에서 영국의 모대학(옥스포드 또는 캠브리지)으로 이직한다고 함

5) Yutaka Taniyama, On A. Weil, Bulletin of the AMS, Vol. 46, No. 4 (October 2009), 667-668.

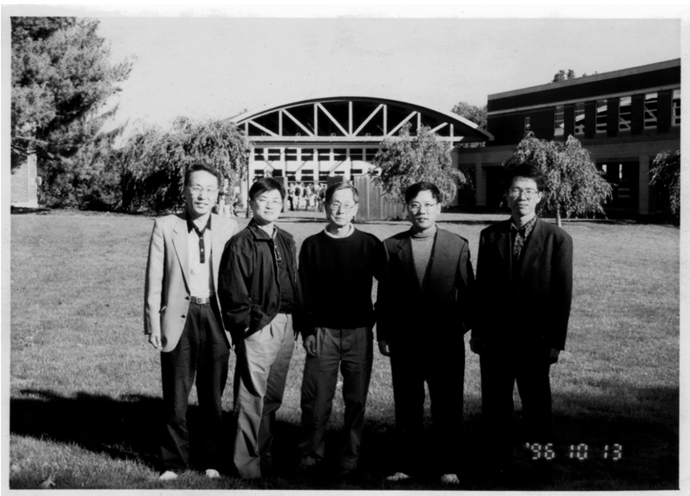
6) Armand Borel (1923~2003) : 프린스턴 고등연구소의 교수로 30여년 역임하였음

7) Automorphic L-functions, Proceedings of the Symposium in Pure Mathematics of the AMS [Automorphic Forms, Representations, and L-functions], edited by A. Borel and W. Casselman, Vol. XXXIII, Part 2, AMS (1979), 27-86.

발간된 『대한수학회소식』에서 이 교수님이 그해 1월 9일에 돌아가셨다는 소식을 접하게 되었다. 그 후 여러 후배 수학자들이 이 교수님을 기리는 글을『대한수학회소식』을 통하여 기고하였던 것으로 알고 있다. 이 교수님의 업적에 관해서는 장범식 브리티시 콜럼비아 대학 명예교수님의 글[9]을 참고하길 바라며, 지면에서는 포항공대 이정림 명예교수님의 글[10]을 인용하겠다.

"디유돈네(J. Dieudonne)가 쓴 '순수 수학의 파노라마'[11]라는 유명한 책이 있다. 이 책에서 이 교수님은 군론의 주도적인 공헌자의 하나로 기록되어 있다. 또한, 그의 이름은 매사추세츠 공과대학에서 발행되는 수학사전(일본 이와나미 수학사전의 영역판)에도 기록되어 있다. 그 수학 사전에 기록되어 있는 다른 한국 수학자는 임덕상 교수뿐이다."

1960년대 초반에 이미 이임학, 임덕상 교수님과 같은 세계적인 대한민국 수학자가 있었다는 사실은 지금 젊은 수학자들에게 큰 힘이 될 것이다. 아쉬운 것은 임덕상 교수님에 관한 자료는 거의 없다는 것이다. 수고스럽지만 어느 한 분이 임덕상 교수님에 관한 생애와 업적에 관하여 연구 조사하여 논문으로 발표하기를 바란다. 이젠 젊은 수학자들이 뛰어난 이 두 분의 선배 수학자들을 본받아 자부심을 가지고 수준 높은 연구를 하여 대한민국 수학을 빛내 주기를 바란다. 2014년 8월에는 서울에서 국제수학자대회(ICM)가 개최된다. 많은 국내 젊은 수학자들이 초청강연을 하기를 바란다. 그리고 가까운 장래에 위대한 대한민국 수학자가 나타나기를 기대한다. 물론 필즈상, 울프상, 아벨상, 쇼상의 대한민국 수상자가 탄생하기를 바라며 이만 줄인다.



프린스턴 고등연구소 울펜손홀 앞 잔디에서 이임학 선생님과 함께
<왼쪽부터> 김대산(서강대), 김동균(고려대), 이임학(UBC), 필자, 채희준(홍익대)
(김동균 교수 제공)

8) 이임학박사와의 대담(대한수학회사 1권: 1998년) : 대담자(권경환, 이정림, 고영소, 주진구)

9) 고 이임학 교수의 업적, [대한수학회소식 제 100호 (2005년 3월), 14~15쪽]

10) 나의 스승 고 이임학 선생님을 추모하며, [대한수학회소식 제 100호 (2005년 3월), 11~13쪽]

11) Jean Dieudonne, A Panorama of Pure Mathematics, Academic Press (1982), 196쪽.

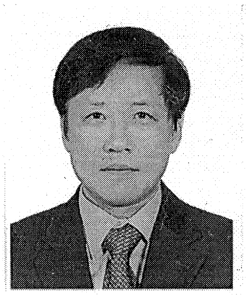
교수신문

인쇄하기 창닫기

극한의 긴장과 '수학의 진리'

수학이야기 ① 천재수학자의 심리 세계

2012년 05월 31일 (목) 10:50:31 양재현 인하대교수 editor@kyosu.net


양재현 인하대 교수

수학은 머리 아픈 학문일까. 수학이란 학문은 도대체 현실과 어떤 깊은 관련을 맺고 있을까. 수학자들은 영화에서처럼 괴짜들일까. 기초학문이 흔들리는 시대, 수학자들은 세상을 어떻게 바라보고 있을까. 우리 시대 수학자들의 고민을 양재현 인하대 교수가 '수학이야기'(격주 연재)로 풀어간다.

수학자는 수학적 진리를 발견하거나 창조하는 활동을 하는 사람이다. 여기서 수학적 진리란 시공을 초월해 영원히 변하지 않는 불멸의 우주적 진리를 의미한다. 수학자의 창조적 작품은 定理라는 형태로 표출된다. 정리는 수학적으로 의미가 있어야 할 뿐만 아니라 아름답고 심오해야 가치가 있는 것이다. 이런 관점에서 수학자의 창조적 활동은 시인, 화가, 음악가와 철학자들의 창조적 활동과 비슷하다. 그래서 수학을 예술인 동시에 철학이라고도 한다.

수학자의 탐구 활동은 새롭고 자연스런 좋은 문제를 찾는 데서부터 시작된다. 일단 좋은 문제를 찾으면, 이 문제를 체계적으로 풀어가며 새로운 수학분야를 개척해 간다. 실은 좋은 문제를 찾는 것은 어려운 일이다.

가령, 리만가설, 골드바흐가설과 버치-스위너톤-다이어가설은 아름답고 심오한 좋은 문제들이다. 설령 흥미롭고 좋은 문제를 찾았다하더라도 연구 도중에 큰 난관에 부딪쳐 진척이 되지 않는 경우가 허다하다.

학계의 혹독한 비판 … 목숨도 끊을 각오로

영국의 천재수학자 앤드류 와일즈는 8여 년동안 가족이외에는 거의 외부와의 접촉을 끊고 외롭게 자신과 싸워가며 연구한 끝에 370여 년동안 풀리지 않았던 페르마 마지막 정리를 1995년에 안정타원곡선에 대한 시무라-타니야마 추론을 증명함으로써 해결했다.

그는 진화론을 주창한 다윈처럼, 천성적으로 여럿이 어울리는 것보다 혼자 있는 것을 좋아하는 내성적인 사람이다. 그래서 아마도 1986년부터 철저한 비밀을 유지하며 모험심을 가지고 페르마의

이 어려운 문제에 도전했을 것이다. 이 문제를 해결할 수 없을지도 모른다는 두려움을 지닌 채 이 문제를 해결하려고 애썼다.

1993년 8월에는 증명에 부분적인 오류가 발견돼 그 후 14개월 동안은 마음이 초조하고 불안한 가운데 정신적으로 상당한 스트레스를 받았지만 결국 이 오류를 고쳤다. 만약에 이 오류를 고치지 못했더라면 수학계로부터 온갖 혹독한 비판을 받아 큰 압박감으로 인해 정신병이 났을지도 모른다.

드물지만 정신적인 질환이 있는 천재수학자들이 이러한 극심한 스트레스를 극복하지 못하고 스스로 목숨을 끊는 경우가 있다. 예를 들면, 버클리대 교수였던 안드레아스 프로어, 일본의 천재수학자들인 타니야마 유타카, 혼다 타이라와 신타니 타쿠로는 스트레스를 이겨내지 못하고 젊은 나이에 자살했다.

3차원 대수다양체의 극소모델이론의 업적으로 1990년에 필즈상을 수상한 일본 수학자 모리 시게후미는 수학이란 학문이 너무 힘들어 두 아들에게 수학과의 진학을 권하지 않았다는 사실을 수년 전에 필자에게 말했다. 두 아들은 교토대 화학과와 오사카대 의대에 진학했다.

내성적인데다 융통성 없는 '천재수학자들'

영화「뷰티풀 마인드」로 유명한 천재수학자 존 내쉬는 평생 정신분열증을 앓아왔으며, 한 때는 리만가설을 풀려고 시도하다가 그의 정신분열 증세가 더욱 악화돼 힘든 생활을 해야만 했다. 다행스럽게도 1950년에 쓴 박사학위논문의 「비협력 게임」으로 1994년에 노벨경제학상을 수상했고, 1999년에는 미국수학회에서 수여하는 스틸상을 수상했다.

수학은 특성상 완벽한 이론과 증명을 요구하는 정신적인 학문이다. 어려운 수학문제를 해결하려면 수학자는 뛰어난 독창력과 강한 집중력이 필요하다. 연구 도중에 어려운 장벽에 막혀 진척이 없을 때는 수학자들은 초조하고 불안한 마음을 갖는 경우가 많다.

이런 이유로 대부분의 저명한 수학자들은 어느 정도의 심리적인 스트레스를 받아가며 연구하고 있다. 대체로 천재수학자들은 내성적이고 비사교적일 뿐만 아니라 융통성이 결여돼 있다. 와일즈, 모리, 내쉬는 비사교적이고 내성적인 수학자들이다.

미국의 변호사 수전 케인은 그녀의 저서『침묵(Quiet)』에서 "세상은 외향적인 사람을 선호하지만 정작 세상을 바꾸는 것은 내성적인 사람이다. 다윈, 루스벨트, 간디, 스티브 잡스, 위즈니악 등의 인물들은 내성적인 사람들이었다. 이들은 옳다고 생각한 바를 끝까지 관철시켜 큰 업적을 이뤘다. 인간관계가 복잡한 외향적인 사람들과는 달리 이것저것 고려할 게 없었기 때문이다"라고 주장했다.

흥미로운 조사에 의하면 수학자들은 다른 분야의 과학자들에 비해 자폐증 또는 아스페르거 증세가 더 많으며, 과학자들은 인문학을 연구하는 학자들에 비해 이들 증세가 더 많다고 한다. 그러나 새롭고 아름다울 뿐만 아니라 심오한 수학적 진리를 발견 또는 창조하는 즐거움이 있기 때문에 수학자들은 어렵고 힘들지만 즐거운 마음으로 열심히 수학이란 학문에 정진하고 있다. 수학은 인간정신을 계발하는 고귀한 학문이다.

양재현 인하대·수학통계학부

캘리포니아대(버클리)에서 박사를 했다. 하버드대, 막스플랑크 수학연구소 등에서 초청교수를 지냈다. 「소수의 아름다움」, 『20세기 수학자들과의 만남』 등 다수의 논저가 있다.

ⓒ 교수신문(http://www.kyosu.net) 무단전재 및 재배포금지 | 저작권문의

인쇄하기 창닫기

교수신문

인쇄하기 창닫기

끼니 거르면서도 연구에 골몰 … 중국 젊은이들에게 '불굴의 의지' 선사한 수학자

수학이야기 ③ 골드바흐 추측과 첸징룬

2012년 07월 09일 (월) 11:11:12 양재현 인하대·수학통계학부 editor@kyosu.net

1742년 6월 7일, 프러시아 수학자 골드바흐(Christian Goldbach, 1690~1764)는 "5 보다 큰 자연수는 세 개의 소수들의 합으로 나타낼 수 있다"라고 추측하는 내용이 담긴 편지를 스위스 수학자 오일러에게 보냈다. 소수는 약수가 1과 자신 밖에 없는 1보다 큰 자연수이다. 예를 들면, 2, 3, 5, 7, 11, 13, 17과 79는 소수이다. 오일러는 이 편지를 받은 후 이 문제에 흥미를 가지며 '2 보다 큰 짝수는 두 개의 소수의 합으로 나타낼 수 있다'라는 가설을 내놓았다. 이것이 소위 '골드바흐 추측'이다. 가령 14=3+11=7+7, 32=13+19, 102=13+89 등이다. 270년이 지난 지금까지도 이 추측은 풀리지 않았다.

1960년대 초반에 젊은 중국 수학자 첸징룬(陳景潤, 1933~1996)이 독한 마음을 먹고 골드바흐 추측을 풀려고 여러 해 동안 시도했다. 불행하게도 이 문제를 풀지는 못했다. 그러나 그는 '충분히 큰 모든 짝수는 한 소수와 두 소수의 곱과의 합으로 나타낼 수 있다'라는 사실을 체(sieve)방법을 사용해 증명했다. 다음과 같다.

$$14 = 5 + 3 \times 3,\quad 32 = 7 + 5 \times 5,\quad 102 = 11 + 7 \times 13$$

이것을 '첸의 정리(Chen's Theorem)'라고 부른다.

'첸의 정리'로 일약 스타덤 올라

첸징룬은 1933년 5월 22일 푸젠성 푸조우에서 우체국 직원의 세 번째 아들로 태어났다. 그는 14세 되는 해에 어머니를 여의었다. 부친의 수입이 적은데다 가족이 많아 가난한 환경 속에서 자랐다. 1949년에 샤먼(厦門; Xiamen)대학에 입학해 1953년 졸업했다. 북경의 한 중학교 교사로 채용됐지만 적응하지 못해 해고를 당했다.

그의 딱한 사정을 알게 된 샤먼대학 총장은 그를 이 대학의 사무직원으로 채용했다. 당시 그는 해석적 정수론에 관심을 가지고 후아루오겅(華羅庚, 1910~1985)의 저서 『Additive Theory of Prime Numbers』를 탐독하며 연구했다. 그는 논문 「On Tarry's problem」을 작성해 후아 교수에게 보냈다. 후아 교수는 그의 수학적 재능을 발견하고 1956년 8월에 중국 수학회 연례 학술회의에서 그에게 연구결과를 발표할 기회를 주었을 뿐만 아니라 1957년에는 그를 중국 과학원 수학부에 조수

로 채용했다. 첸징룬은 여기서 원 문제(circle problem), 약수 문제(divisor problem), 구 문제(sphere problem)과 워링의 문제(Waring's problem)등을 활기차게 연구했다.

1960년대에 들어와서는 노르웨이 수학자 애틀레 셀버그(Atle Selberg, 1917~2007)가 창안한 '체 방법(sieve method)'과 이의 이용에 대해 연구하면서 골드바흐 추측의 해결에 도전했다. 이때가 연구의 전성기였다. 그는 1966년에 첸의 정리를 발표한 후 하루아침에 국제적인 명성을 얻게 됐다. 끼니를 거르며 무리하게 연구한 끝에 건강을 상당히 해쳤다.

1966년~1976년 이른바 문화혁명 기간에는 많은 비판과 비난을 받았다. 정신적 육체적인 고통을 받아 건강은 더욱더 악화됐다. 문화혁명이 끝난 후에는 그의 뛰어난 연구 업적으로 중국 정부로부터 융성한 대우를 받았다. 그는 1978년에 중국 과학원 수학부 교수로 임명됐고 1980년에는 중국 과학원의 회원으로 선정됐다. 게다가 그는 일등급 국가 기초과학상, 헤리앙-헤리 상, 후아루오겅 수학상 등의 유명한 상도 받았다.

1997년 봄, 4개월 동안 독일 본에 있는 막스-플랑크 수학연구소 초빙교수로 있을 때 경제학을 연구하는 젊은 중국인 교수로부터 첸징룬에 관한 흥미로운 에피소드를 우연히 들었다. 첸은 중국에서 1978년부터 지금까지 과학 영웅 대우를 받아 왔으며 어린 중국 학생들의 우상이었다고 한다.

1980년대에는 어린 애들에게 크면 어떤 사람이 되고 싶으냐고 물으면 하나같이 '과학자요'하고 대답했다. 첸의 정리를 증명하기 위해 소비한 종이 양이 트럭 5대 분이 됐다고 한다. 약간 과장된 것 같지만, 증명을 위한 계산이 복잡하고 어려워 그는 많은 양의 종이를 소비하며 계산을 했을 것이다.

끝내 풀어내지 못했지만 노력 자체가 '쾌거'

약 8년 전에 필자도 하나의 공식을 얻기 위해 엄청난 양의 계산을 한 경험이 있다. 연구에 너무 몰두해 건강을 해쳤을 뿐만 아니라 결혼 시기도 놓친 딱한 첸징룬을 보고 그의 동료 교수들이 일간지에 구혼광고를 냈다고 한다. 어느 한 병원의 간호사가 이 구혼광고를 보고 자청해 구혼을 수락했다. 이 흥미로운 에피소드를 중국의 한 경제학 교수가 들려줬다.

첸은 1980년, 그녀와 결혼해 아들 하나를 얻었다. 1984년에는 파킨슨병에 걸려 투병생활을 하면서도 연구는 계속했다. 1996년엔 폐렴 증세로 건강이 악화돼 그해 3월 19일, 첸은 세상을 떠났다. 1999년에 중국 정부는 그의 업적을 기리기 위해 그의 실루엣과 그가 얻은 유명한 부등식(아래)이 담긴 80펜(分) 가격의 우표를 발행했다.

이 우표의 타이틀은 哥德巴赫猜想的最佳結果(가덕파혁 시상적 최가결과: The Best Result of Goldbach Conjecture)이다. 그리고 그를 기리며 추모하기 위해 몇 개의 조각상이 세워졌다.

$$P_x(1,2) \geq \frac{0.67x\,C_x}{(\log x)^2}$$

2006년 4월 4일, 그의 모교 샤먼대학의 수학과 건물 앞에 그의 조각상과 그의 업적을 새긴 대리석 석판이 세워졌다. 첸의 상 뒤에 있는 대리석 석판에 그의 이름과 디리클레(Dirichlet), 유티라(Jutila), 린닉(Linnik), 판(Pan)이라는 4명의 수학자 이름이 새겨져 있다. 1996년에는 그의 이름을 따서 그해에 발견된 소행성에 애스터로이드 7681 첸징룬(the Asterroid 7681 Chenjingrun)이라 명명했다.

첸징룬의 피나는 탐구정신, 불굴의 도전정신, 독창력과 집중력은 높이 인정돼야 한다. 특히 그가 골드바흐 추측을 해결하기 위해 과감하게 도전하는 모험심은 대단하다. 그는 이 문제를 풀 수 없을지도 모른다는 두려움을 지닌 채 밤낮을 새워가며 연구한 끝에 이 추측을 해결하지 못했지만 이의 해답에 근접한 새롭고 아름다울 뿐만 아니라 심오한 진리를 발견하고 증명했다.

외롭게 자기 자신과 싸워가며 누추한 조그만 방에서 몽땅 연필을 들고 계산을 했던 모습을 상상해보라. 큰 난관에 부딪쳐 연구가 진척이 되지 않을 때는 분명히 엄청난 정신적 스트레스를 받았을 것이다. 그는 가난과 영양결핍에 시달린 가운데 심리적 스트레스를 받아 건강을 상당히 해쳤을 것으로 추정된다.

그의 순수하면서 진정한 탐구정신과 불굴의 도전 정신에 감복한 신이 그에게 수학적 진리, 즉 첸의 정리를 선사하지 않았나하는 생각을 해 본다. 그는 중국의 많은 젊은이들에게 탐구정신, 모험심과 인내심을 심어줬다. 그래서 지금 중국에는 세계적으로 저명한 수학자뿐만 아니라 과학자가 상당히 많다.

양재현 인하대·수학통계학부

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인쇄하기 창닫기

교수신문

150년 봉인해제 기다리는 우아한 난제

수학이야기 14. 리만가설과 베른하르트 리만

2013년 04월 15일 (월) 16:36:19

양재현 인하대·수학통계학부 editor@kyosu.net

지난 150여 년 동안 수학분야에서 지금까지 풀리지 않고 있는 상당히 중요한 난제인 '리만가설'이 있다. 1859년 8월에 리만(1826~1866)은 베를린 학술원의 회원으로 선정됐다. 베를린 학술원의 헌장에 의하면,새로이 선출된 회원은 반드시 최근의 연구업적을 보고하게 돼 있었다. 그래서 리만은 「주어진 수 보다 작은 소수의 개수에 관하여」의 제목으로 된 8페이지의 보고서를 학술원에 제출했다.

그의 사후 30여 년이 지나서야 이 보고서의 중요성이 인식되어 ζ(s)는 '리만 제타함수'라고 불렸고, 추측(RH)은 '리만가설'이라 불렸다. 불행히도 그의 사후 70여 년 동안 아무도 리만의 공식을 이해하지 못했다. 1932년에 지겔은 우연히 동료 수학자의 말을 듣고 괴팅겐 대학의 수학도서관에서 입수한『Nachlass』를 읽고 리만의 공식을 면밀히 검토한 후 일반 수학자들이 구체적으로 이해하기 쉽도록 새롭게 공식을 만들어 저널에 발표했다. 이 공식이 소위 유명한 리만-지겔 공식이다. 이 공식을 이용해 여러 수학자들이 2004년에 슈퍼컴퓨터의 도움으로 1013개 이상의 영점을 발견했다. 몽고메리와 오드리츠코는 연속하는 소수들의 간격 분포가 어떤 에르미트 작용소의 고유값 간격 분포와 동일하다는 사실을 보였다.

19세기 후반에 리만가설은 정수론과 해석학을 융합한 수학의 분야인 해석적 정수론을 탄생시켰다. 하디, 리틀우드, 아르틴, 베이유, 하세, 셀버그, 루이 드 브랑주, 콘네 등의 저명한 수학자들이 리만가설의 진위를 밝히려고 했지만 모두 실패했다. 지난 40여 년 동안 마이클 베리 등의 수리물리학자들이 양자물리학적인 방법으로 이 가설의 해결에 도전하고 있다.

아직까지도 이 가설은 풀리지 않고 있다. 현재 거의 모든 수학자들이 리만가설을 현존하고 있는 수학 문제 중에서 가장 심오하고 중요한 문제라고 믿고 있다. 리만 가설은 힐베르트의 여덟 번째 문제인 동시에, 지난 2000년에 미국의 클레이 수학연구소가 100만 달러 상금을 내건 일곱 개의 새천년 문제들 중 하나이기도 하다.

필자는 처음으로 독일어로 작성된 리만의 보고서 원문과 영문으로 번역된 것을 동시에 읽었을 때 제대로 이해하지 못했다. 나중에 상당부분 이해하게 됐을 때 놀랐고 충격을 받았다. 8페이지 밖에 되지 않는 보고서에 심오하고 아름다운 수학적 진리를 담고 있는 내용이 매우 우아하게 기술되어 있다는 것이다. 아마도 리만은 이 보고서를 쓸 당시에 그의 스승인 가우스와 디리클레가 예전에 수행했던 소수에 관한 연구를 한층 더 깊게 하려고 노력했던 것 같다.

그는 매일 자기 전에 하루 일과를 반성하고 기도를 했다고 한다. 그는 신앙심이 깊고 훌륭한 목사였던 부친과 자주 서신교환을 하면서 그의 애로점과 고충을 나누었다. 영적인 생활을 추구한 그는 항상 신과 대화를 하며 서로 교감을 나누었다.

리만은 1826년 9월 17일 하노버 왕국의 브레제렌츠라는 조그만 마을에서 목사의 아들로 태어났다. 1846년에 그의 부친의 권유로 괴팅겐 대학의 신학과에 입학했다가 수학에 관한 열정이 매우 강렬해 얼마 후 철학과로 전과했다.

그 당시에 철학과에는 위대한 수학자 가우스가 재직하고 있었다. 괴팅겐에서 1년을 보낸 후 베를린 대학에 가서 2년간 거기서 연구하면서 해석학 분야를 많이 배웠다. 베를린에서 야코비, 디리클레, 아이젠슈타인과 학문적으로 교류하며 많은 영향을 받았다. 1851년에 박사학위를 받았고, 1853년에 Habilitation을 취득했다. 그는 리만 기하학을 창시했고 복소함수론, 아벨함수론, 소수의 분포 이론, 수리물리학 등의 분야에 뛰어난 업적을 남겼다. 이런 업적을 인정받아 1859년에 정교수가 됐을 뿐만 아니라 여러 학술원의 회원으로 선정되기도 했다.

1859년 전까지는 가난 때문에 경제적으로 힘든 시절을 보내어 건강을 많이 해쳤다. 1862년 7월에 엘리제 코흐와 결혼해 딸을 낳았다. 1862년 가을에 결핵에 걸려 투병생활을 하다가 1866년 7월 20일에 이탈리아 셀라스카라는 조그만 휴양도시에서 요절했다. 그의 업적은 20세기의 수학과 물리학에 지대한 영향을 끼쳤다.

리만은 19세기 중반에 어떻게 위대한 여러 업적을 창출할 수 있었을까. 필자의 단견을 피력해 볼까 한다. 리만은 어린 시절에 수학적 재능을 발휘했고 수학에 대한 열정이 강렬했다. 그의 부친이 신앙심이 깊은 지성인이어서 리만은 영적인 삶을 영위하는 환경 속에서 자라났다.

그리고 그의 주변에 위대한 학자들이 다수 있어 그들과 돈독한 친분을 쌓으며 학문적으로 많은 영향을 받았다. 가우스로부터 기하학과 정수론을, 베버로부터 물리학을, 헤르바르트로부터는 철학을, 디리클레로부터 정수론을, 야코비와 아이젠슈타인으로부터는 함수론과 해석학을 배웠다. 리만의 순수함, 영적인 삶, 창의성, 깊은 사고력과 주위의 창의적인 연구 환경이 어우러져서 위대한 업적이 탄생되지 않았나하는 생각을 해본다.

양재현 인하대·수학통계학부
캘리포니아대(버클리)에서 박사를 했다. 하버드대, 막스플랑크 수학연구소등에서초청교수를지냈다. 『소수의아름다움』, 『20세기 수학자들과의 만남』 등의 저서가 있다.

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교수신문

인쇄하기 창닫기

형식주의와 논리주의 그리고직관주의의 대립

수학이야기 17. 수학의 진리는 발견되는가?(上)

2013년 05월 27일 (월) 16:10:21

양재현 인하대 교수 editor@kyosu.net

지난 100여 년 동안 수학의 진리가 발견되는가 아니면 인간에 의해 창조되느냐의 논쟁이 이어오고 있다. 많은 수학자들이 수학의 기초에 관한 불확실한 의문을 품어왔다. 이 의문을 파헤치려고 노력했던 대표적인 수학자들은 다비드 힐베르트(1862~1943), 버트런드 러셀(1872~1970), L.E.J. 브로우베르(1881~1966)이다.

1898년에 발간된 힐베르트의 저서『기하학의 기초』, 1910~1913년에 출판된 러셀과 알프레드 화이트헤드(1861-1947)의 공저『수학의 원리』의 3권,과 1908년에 발표된 브로우베르의 논문「논리원칙의 비신뢰성에 관해」를 통해 힐베르트는 형식주의, 러셀은 논리주의, 브로우베르는 직관주의를 표방했다. 독일어로 인쇄된『기하학의 기초』는 그 당시 유럽의 여러 주요 언어로 번역돼 학계에 엄청난 영향을 끼쳤고, 『수학의 원리』는 많이 읽히지는 않았지만 수학자들 사이에서 인정을 받았다. 힐베르트는 기하학의 공리화 방법으로 수학의 기초에 관한 의문에 만족스런 해답을 찾으려 했다.

그는 언어의 결함을 없애기 위해 수학의 명제를 형식적인 기호의 나열로 바꾸고, 그런 명제들 중에서 모순을 야기시키는 요소를 제거하며 수학의 공리를 체계화하고 설계한 후에 모순이 없는 명제를 유도하는 수학의 기초를 세우려고 했다. 체르멜로(1871~1953)는 힐베르트의 프로그램을 구현해 집합론의 체르멜로-프란켈의 공리계를 제안했다. 베르트는"칸토어(1845~1918)의 집합론은 인간의 활동을 순수 지성의 영역에서 가장 아름답게 실현시킨 수학적 사고의 경이로운 결실"이라고 극찬하면서 칸토어의 업적을 높이 평가했다. 반면에 칸토어의 집합론을 격렬하게 혹평했던 크로네커(1823~1891)의 주장은 수학발전에 걸림돌이 된다고 여겼다.

칸토어를 보는 상반된 평가

크로네커는 "신은 자연수를 창조했고, 그 외 나머지 수들은 인간에 의해 만들어졌다"라는 명언을 남긴 위대한 수론학자였다. 힐베르트는 유클리드 기하학의 구조가 연역적인 면에서는 장점이 있지만, 정교하지 못한 정의, 논리상의 결함으로 가득 차있다고 생각해 기하학에 관한 확고한 기초를 마련하기 위해 특정한 가정들을 제거하는 등 직관에 대한 의존을 완전히 배제하려고 했다. 이런 시도의 결과로 소위 형식주의 학파가 탄생됐다.

1870년대에 독일의 수리논리학자인 고트로프 프레게(1848~1925)는 대부분의 수학진리가 논리 명제들로 이루어진 훨씬 작은 집합으로 유도될 수 있음을 인식하고, 논리주의가 이론적으로 연구될 수 있다고 믿었다. 그는 이런 인식을 바탕으로 1884년에 저서『산술의 기초』를 출판했다.

그러나 러셀이 이 저서에 관심을 가졌던 1903년 이전까지는 프레게는 학계에서 전혀 알려지지 않은 학자였다. 러셀은 이 저서를 읽고 큰 감명을 받고선, 순수수학이 몇 개의 근본적인 개념의 조합으로 구성될 수 있으며, 모든 수학 명제들이 적은 개수의 기본적인 논리 원리에서 유도될 수 있다고 믿었다. 이것이 러셀의 논리주의의 태동이었다. 얼마 후에 러셀의 논리주의는 앙리 푸앵카레(1854~1912)와 브로우베르의 신랄한 비판을 받게 된다.

네덜란드 수학자 브로우베르는 힐베르트의 형식주의와 러셀의 논리주의를 모두 혹독하게 비판했다. 그리고 칸토어의 超限겍(transfinite number)의 이론에 대해서도 비판적이었다. 그는 排中律(the principle of excluded middle)을 거부하며 유한 개수의 추론 과정으로는 참도 거짓도 성립되지 않는 명제가 존재한다고 주장했다.

감정 싸움으로 번진 관점의 차이

더 나아가 그는 형식주의는 그 체계의 틀 안에서는 수학적 결실을 맺는 것이 없다고 주장했다. 이로써 그는 소위 직관주의를 창시했다. 이로 인해 그전까지만 해도 서로 우호적이었던 힐베르트와 브로우베르는 서로 감정적인 싸움으로 번져 결국에는 입에 담기 힘든 불미스런 일이 터지고 말았다.

앞서 부연한 형식주의, 논리주의, 직관주의는 수학의 기초를 정립하려는 시도의 관점에서 많은 수학자들의 관심을 끌었다. 힐베르트의 수제자인 헤르만 바일(1885~1955)은 브로우베르의 직관주의를 옹호했다. (계속)

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교수신문

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신의 계시인가, 인간의 창조인가

수학이야기 18. 수학의 진리는 발견되는가?(下)

2013년 06월 24일 (월) 16:21:04

양재현 인하대·수학통계학부 editor@kyosu.net

1930~1931년에 오스트리아 수학자인 쿠르트 괴델(1906~1978)이 증명한'불완전성 정리'에 의해 힐베르트의 형식주의에 한계가 있음이 입증됐다. 불완전성 정리에 의하면, 어떤 임의의 체계가 주어져 있을 때, 그 체계 내에서 증명될 수 없는 명제가 항상 존재한다고 주장하고 있다. 보다 구체적으로 설명하면, 산술체계를 포함해 모순이 없는 공리계에서는 참이지만 증명될 수 없는 명제가 존재하며, 또한 그 공리계는 자신의 무모순성을 증명할 수 없다는 것이다. 예를 들면'골드바흐 추측'이 증명될 수 없는 수학의 진리일 수도 있다는 것이다.

이것은 인간 인식에 근본적으로 한계가 있음을 보여주고 있다. 게다가 괴델은 1938년칸토어가 제기한'연속체 가설'의 무모순성을 증명했다. 그 후 1963년에 폴 코헨(1934~2007)이'일반 연속체 가설'의 독립성을 증명해 이 업적으로 1966년에 필즈상을 수상했다. 코헨은'강제법'이라는 기법을 사용해 일반 연속체 가설과 선택 공리는 체르멜로-프란켈(ZF) 공리계로부터 결정될 수 없음을 증명했다.

인간 인식의 근본적 한계

괴델과 코헨의 연속체 가설의 무모순성과 독립성을 종합하면, "ZFC 집합론에 있어서는 일반 연속체 가설의 참과 거짓도 증명할 수 없다. 즉 일반 연속체 가설은 ZFC 집합론의 공리계로부터 독립돼 있다"라고 표현할 수 있다. 불완전성 정리와 연속체 가설의 무모순성의 증명은 시사 주간지 타임지에 의해 높이 평가돼, 괴델은 타임지가 선정한 20세기에 가장 영향력을 끼쳤던 인물 100명 중의 한 사람이 됐다. 괴델의 불완전성 정리의 출현 이후로 대부분의 수학자들은 철학적인 문제인 수학의 기초에 관한 의문에 관심을 갖지 않기 시작했다.

상기의 형식주의, 논리주의, 직관주의를 근거해, 수학적 진리가 신이 창조한 우주의 심연에서 발견되는지 아니면 인간에 의해 창조되는지의 의문에 대해 논할까 한다. 칸토어, 힐베르트, 하디 등의 수학자들은수학 진리는 우주에 널려 있으며, 신의 계시를 받아 발견되는 것이라 믿었다. 18세기의 위대한 독일 철학자 임마누엘 칸트(1724~1804)는"수학진리는 인간에 의해 창조된다"라고 역설하고 있다. 푸앵카레와 샤를 에르미트(1822~1901) 등의 프랑스 수학자들은 수학 진리는 발견될 뿐만 아니라 인간에 의해 창조된다는 중용의 자세를 취하고 있다. 어떤 수학자들은 이 질문에 결정을 내리지 못하고 있다.

필자는 수학 진리가 창조된다는 주장보다는 신의 계시를 받아 발견된다는 주장에 더 무게를 두는 중용의 자세를 취하고 있다. 필자가 발표한 몇 편의 논문은 신의 계시를 받아 얻어진 공식을 지니고 있다. 필자는 이 공식이 세계 수학사에 영원히 남을 것으로 믿고 있다.

이 공식을 발견하고 이 공식이 참이라는 사실을 증명했을 때, 이루 형언할 수 없는 희열을 맛보았다. 예를 들면, 비유클리드 기하와 유클리드 기하의 밀접한 결합으로 이루어진 지겔-야코비 공간에서 아름답고 심오할 뿐만 아니라 자연스런 문제들을 발견하고, 이 문제의 해답을 얻기 위해 노력했던 경험이 있다. 오랫동안 고군분투하며 연구하는 중에 갑작스럽게 신의 계시를 받아 아름다운 공식을 발견하고, 이 공식의 참을 증명하는 과정에서 여러 차례의 오류를 범하며, 결국에는 기발한 아이디어와 기법을 창안해 증명했다.

독일·프랑스 수학자의 견해 차이

필자의 지론은 위대한 수학 진리는 우주의 깊숙한 곳에서 신의 계시를 받아 발견되며, 이 수학 진리를 이해 분석하며 증명하는 데 필요한 아이디어와 기법은 인간에 의해 창조된다는 것이다. 수학 진리의 증명은 개성이 다른 여러 수학자들의 창조 과정을 통해 얻어진 서로 다른 다양한 아이디어와 기법으로 완성된다. 그 중에서도 심오하고 아름다울 뿐만 아니라 간결한 아이디어와 기법이 살아남고, 이것들이 다른 연관된 문제들을 해결하는데 사용되기도 한다.

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(B-03) *A partial Cayley transform of Siegel-Jacobi disk,* Jour. of Korean Math. Soc. **45**, No. 3 (2008), 781-794 or arXiv:math.NT/0507216.

(B-04) *Invariant metrics and Laplacians on Siegel-Jacobi disk*, Chinese Annals of Mathematics, Vol. **31** B(1), 2010, pp. 85-100.

(B-05) (with Young-Hoon Yong, Su-Na Huh, Jung-Hee Shin and Gil-Hong Min) *Sectional Curvatures of the Siegel-Jacobi Space*, Bull. Korean Math. Soc. **50**, No. 3 (2013), 787-799.

(B-06) *Invariant differential operators on Siegel-Jacobi space and Maass-Jacobi forms*, Proceedings of the International Conference on Geometry, Number Theory and Representation Theory (October 10-12, 2012, Incheon, Korea), pp. 37-63 (2013).

[C] Theory of the Heisenberg Group $H_R^{(n,m)}$

(C-01) *Harmonic analysis on the quotient spaces of Heisenberg groups*, Nagoya Math. J. **123** (1991), 103-117.

(C-02) *Harmonic analysis on the quotient spaces of Heisenberg groups II*, Jour. of Number Theory **49** (1994), no. 1, 63-72.

(C-03) *A decomposition theorem on differential polynomials of theta functions of high level*, Japanese J. Math. (New Series). **22** (1996), no. 1, 37-49.

(C-04) *Fock representations of the Heisenberg group $H_R^{(g,h)}$*, J. Korean Math. Soc. **34** (1997), no. 2, 345-370.

(C-05) *Lattice representations of Heisenberg groups*, Math. Ann. **317** (2000), 309-323

[D] Theory of Maass-Jacobi Forms

(D-01) *A Note on Maass-Jacobi Forms*, Kyungpook Math. J., **43** (2003), no. 4, 547-566 or arXiv:math.NT/0612387.

(D-02) *마쓰-야코비 형식에 관한 연구 (On Maass-Jacobi Forms: Korean)*, Comm. Korean Math. Soc. (초청논문: invited paper), vol. **18** (2003), no. 3, 393-427.

(D-03) *A Note on Maass-Jacobi Forms II*, Kyungpook Math. J., **53** (2013), no. 1, 49-86.

[E] Theory of Vector Bundles

(E-01) *A note on holomorphic vector bundles over complex tori*, Bulletin of Korean Math. Soc., **23** (1986), no. 2, 149-154.

(E-02) *Holomorphic vector bundles*, Bull. Inst. Basic Sci., Inha Univ., **8** (1987), 1-14.

(E-03) *Holomorphic vector bundles over complex tori*, J. Korean Math. Soc. **26** (1989), no. 1, 117-142.

[F] Representation Theory of a Lie Group

(F-01) *The Method of Orbits for Real Lie Groups*, Kyungpook Math. J., **42** (2002), no. 2, 199-272 or arXiv:math.NT/0602056

(F-02) *The Weil representations of the Jacobi group*, Proceedings of the International Conference on Geometry, Number Theory and Representation Theory (October 10-12, 2012, Incheon, Korea), pp. 169-204 (2013).

[G] Theory of Elliptic Curves

(G-01) *Note on Taniyama-Shimura-Weil conjecture*, Proceedings of the Symposium on Number Theory and related topics (Masan, 1994; Pusan, 1994), Pyungsan Inst. Math. Sci., Seoul (1995), 29-46.

(G-02) *타원곡선에 관한 지난 20년 간의 연구 동향 (Past twenty years of the theory of elliptic curves: Korean)*, Comm. Korean Math. Soc. (초청논문 : Invited paper), **14** (1999), No. 3, 449-477.

(G-03) *The Birch-Swinnerton-Dyer Conjecture*, Proceedings of the 2002 International Conference on Related Subjects to Clay Problems, the Institute of Pure and Applied Mathematics, Chonbuk National Univ., Jeonju, Korea, Vol. **1** (2002), 134-151 or arXiv:math.HO (math:NT)/0611423.

[H] Theory of Siegel Modular Forms

(H-01) *Geometric theory of Siegel modular forms*, Proceedings of Topology and Geometry Research Center, **1** (December, 1990), 73-83.

(H-02) *Theory of the Siegel Modular Variety*, Proceedings of the International Conference on Number Theory and Cryptography, Edited by S. D. Adhikari and B. Ramakrishnan, Harish-Chandra Institute, Allahabad, India: A Publication of Hindustan Book Agency (2009), 219-278 [arXiv:math.NT (math.AG)/0706.4268]

[I] Topics on the Langlands Program

(I-01) *A note on the Artin conjecture*, Trends in Mathematics, Vol. **9**, No. 1 (2006), 99-107 or arXiv:math.HO(math.NT)/0611424.

(I-02) *Langlands Functoriality Conjecture*, Kyungpook Math. J., **49**, no. 2 (2009), 355-387.

[J] Harmonic Analysis on the Minkowski-Euclid Space

(J-01) *Harmonic Analysis on the Minkowski-Euclid Space $P_n \times R^{(m,n)}$, II : Unitary Representations of $GL(n,R) \ltimes R^{(m,n)}$*, Kyungpook Math. J. **45** (2005), 303-338.

(J-02) *Invariant Differential Opertors on the Minkowski-Euclid Space*, J. Korean Math. Soc., 50 (2013), No. 2, pp. 275-306.

(J-03) *Polarized Real Tori*, arXiv:0908.0252 [math.AG (math.NT)](2009).

[K] Papers related to Algebraic Geometry and Other Topics

(K-01) *Recent progress on the Schottky problem*, paper dedicated to Prof. Oh (Yoon Yong) for his 60th birthday, KAIST, Taejeon, Korea (1988), 131-140.

(K-02) *Automorphic forms and zeta functions (Korean)*, the Notices of the Pyungsan Institute for Mathematical Sciences, **2** (1992), 1-9.

(K-03) *On theta functions*, Kyungpook Math. J. **35** (1996), no. 3, 857-875.

(K-04) *Stable automorphic forms, Proceedings of Japan-Korea Joint Seminar on transcendental Number Theory and Related Topics*, edited by Iekata Shiokawa and Jin-Woo Son, Masan (1998), 101-126.

(K-05) *On the Riemann Hypothesis (Korean)*, Proceedings of Math. Festival, Math. Love, **2** (2000), no. 3, 314-327. [리만 가설에 관하여, 세계 수학의 해 기념 제2회 Math Festival 프로시딩 제2집 3권, 314-327.]

(K-06) *소수의 아름다움*, 뉴스레터 MATHCHUNGNAM 제10호, 충남 중등수학 교육 연구회 (2007), 1-20.

[L] Essays

(L-01) *추상수학의 어머니 에미 뇌테르 (Emmy Noether, 1882~1935)의 생애*, 한국 수학교육학회 뉴스레터 제15권 제6호 통권 제64호 (1999년 11월), 26-28.

(L-02) *막스-플랑크 수학연구소*, 대한수학회 소식 제90호 (2003년 7월), 17-23.

(L-03) *마침내 Poincaré Conjecture가 해결되다 !*, 대한수학회 소식 제 109호 (2006년 9월), 12-18.

(L-04) (공저: 김인강) *Perelman의 업적에 관하여 – Poincaré 가설과 geometrization 가설*, 대한수학회 소식 제 109호 (2006년 9월), 19-26.

(L-05) *인도의 Harish-Chandra 연구소에 다녀와서*, 대한수학회 소식 제 115호 (2007년 9월), 8-13.

(L-06) *프린스턴에서의 이임학 교수와의 만남*, 대한수학회지 제 132호 (2010년 7월), 24-27.

(L-07) *천재수학자의 심리 세계 : 극한의 긴장과 수학의 진리*, 교수신문 제646호 (2012년 5월 28일)

(L-08) *곧드바흐 추측과 췐징룬 : 끼니 거르면서도 연구에 골몰 — 중국 젊은이들에게 '불굴의 의지' 선사한 수학자*, 교수신문 제652호 (2012년 7월 9일)

(L-09) *리만 가설과 베른하르트 리만 : 150년 봉인해제 기다리는 우아한 난제*, 교수신문 제680호 (2013년 4월 15일)

(L-10) *수학의 진리는 발견되는가 ? (上)* : *형식주의와 논리주의 그리고 직관주의의 대립*, 교수신문 제686호 (2013년 5월 27일)

(L-11) *수학의 진리는 발견되는가 ? (下) : 신의 계시인가, 인간의 창조인가*, 교수신문 제690호 (2013년 6월 24일)

Bibliography
[Jae-Hyun Yang: 양재현 梁在賢]

[A] Theory of Jacobi Forms

(A-01) (with Jin-Woo Son) *A note on Jacobi forms of higher degree*, J. Korean Math. Soc., **28** (1991), no. 2, 341-358.

(A-02) *Some results on Jacobi forms of higher degree*, RIMS Symposium on Automorphic Forms and associated Zeta Functions, Kyoto, January 6-10, 1992: RIMS Kokyuroku, No**. 805**(1992), 36-50.

(A-03) *On Jacobi forms of higher degree*, the Notices of the Pyungsan Institute for Mathematical Sciences, **3** (1993), 3-29 [Korean].

(A-04) *The Siegel-Jacobi Operator*, Abhandlungen Math. Sem. Univ. Hamburg. **63** (1993), 135-146.

(A-05) *Vanishing theorems on Jacobi forms of higher degree*, J. Korean Math. Soc. **30** (1993), no. 1, 185-198.

(A-06) *Remarks on Jacobi forms of higher degree*, Proceedings of the 1993 Workshop on Automorphic Forms and Related Topics, Pyungsan Inst. Math. Sci., Seoul (1993), 33-58.

(A-07) *Stable Jacobi forms*, Proceedings of the workshop on Pure Mathematics, Topics in Number Theory and Algebra, **13**, the Daewoo Foundation, Seoul (1993), Part. I.

(A-08) *Singular Jacobi forms*, Transactions of American Math. Soc. **347** (1995), no. 6, 2041-2049.

(A-09) *Construction of Vector Valued Modular Forms from Jacobi Forms*, Canadian Jour. Math. **47** (1995), 1329-1339 or arXiv:math.NT/0612474.

(A-10) *Kac-Moody algebras, the Monstrous Moonshine, Jacobi forms and infinite products*, Proceedings of the Symposium on Number Theory, Geometry and Related Topics (Iksan City, 1995), 13-82, Pyungsan Inst. Math. Sci., Seoul, 1996 or arXiv:math.NT/0612474.

(A-11) *A geometrical theory of Jacobi forms of higher degree*, Kyungpook Math. J., **40** (2000), 209-234 or arXiv:math.NT/0602267.

[B] Harmonic Analysis on Siegel-Jacobi Space

(B-01) *Remarks on Harmonic Analysis on Siegel-Jacobi Space*, arXiv:math.NT/0612230.

(B-02) *A note on a fundamental domain for Siegel-Jacobi space*, Houston Journal of Mathematics, Vol. **32**, No. 3 (2006), 701-712 or arXiv:math.NT/0507218.

(B-03) *Invariant Metrics and Laplacians on Siegel-Jacobi Space*, Journal of Number Theory **127** (2007), 83-102 or arXiv:math.NT/0507215.

(B-04) *A partial Cayley transform of Siegel-Jacobi disk*, Jour. of Korean Math. Soc. **45**, No. 3 (2008), 781-794 or arXiv:math.NT/0507216.

(B-05) *Invariant metrics and Laplacians on Siegel-Jacobi disk*, Chinese Annals of Mathematics, Vol. **31** B (1), 2010, 85-100.

(B-06) (with Young-Hoon Yong, Su-Na Huh, Jung-Hee Shin and Gil-Hong Min) *Sectional Curvatures of the Siegel-Jacobi Space*, Bull. Korean Math. Soc. **50** (2013), No. 3, pp. 787-799.

(B-07) *Invariant differential operators on Siegel-Jacobi space and Maass-Jacobi forms*, Proceedings of the International Conference on Geometry, Number Theory and Representation Theory (October 10-12, 2012, Incheon, Korea), pp. 37-63 (2013).

(B-08) (with Minoru Itoh and Hiroyuki Ochiai) *Invariant differential operators on Siegel-Jacobi space*, preprint (2013).

[C] Theory of the Heisenberg Group $H_R^{(n,m)}$

(C-01) *Harmonic analysis on the quotient spaces of Heisenberg groups*, Nagoya Math. J. **123** (1991), 103-117.

(C-02) *Harmonic analysis on the quotient spaces of Heisenberg groups II*, Jour. of Number Theory **49** (1994), no. 1, 63-72.

(C-03) *A decomposition theorem on differential polynomials of theta functions of high level*, Japanese J. Math. (New Series). **22** (1996), no. 1, 37-49.

(C-04) *Fock representations of the Heisenberg group* $H_R^{(g,h)}$, J. Korean Math. Soc. **34** (1997), no. 2, 345-370.

(C-05) (with Jin-Woo Son) *Unitary representations of the Heisenberg group* $H_R^{(g,h)}$, Proceedings of Japan-Korea Joint Seminar on Transcendental Number Theory and Related Topics, edited by Iekata Shiokawa and Jin-Woo Son, Masan (1998), 143-157.

(C-06) *Lattice representations of Heisenberg groups*, Math. Ann. **317** (2000), 309-323

[D] Theory of Maass-Jacobi Forms

(D-01) *A Note on Maass-Jacobi Forms*, Kyungpook Math. J., **43** (2003), no. 4, 547-566 or arXiv:math.NT/0612387.

(D-02) *마쓰-야코비 형식에 관한 연구 (On Maass-Jacobi Forms: Korean)*, Comm. Korean Math. Soc. (초청논문: invited paper), vol. **18** (2003), no. 3, 393-427.

(D-03) *On the group* $SL(2,R)\ltimes R^{(m,2)}$, J. Korean Math. Soc., **40** (2003), no. 5, 831-867.

(D-04) *A Note on Maass-Jacobi Forms II*, Kyungpook Math. J., **53** (2013), no. 1, 49-86.

[E] Theory of Vector Bundles

(E-01) *Einstein-Hermitian vector bundle*, thesis, U.C. Berkeley (1984).

(E-02) *Stability of vector bundles*, Proceedings of the 4th workshop on Pure and Applied Math., **4**, the Daewoo Foundation, Seoul (1984), 263-285.

(E-03) *Einstein-Hermitian vector bundles over a complex torus*, Bulletin of the Institute for Basic Sciences, Inha University, **6** (1985), 47-51.

(E-04) *A note on holomorphic vector bundles over complex tori*, Bulletin of Korean Math. Soc., **23** (1986), no. 2, 149-154.

(E-05) *Vector bundles over an abelian varieties*, Proceedings of the 6th workshop on Pure and Applied Math., **6**, the Daewoo Foundation, Seoul (1986).

(E-06) (with Byung Moon Jun) *Vector bundles associated with an automorphic factor*, J. Korean Math. Soc., **24** (1987), no. 2, 267-285.

(E-07) *Holomorphic vector bundles*, Bull. Inst. Basic Sci., Inha Univ., **8** (1987), 1-14.

(E-08) *Holomorphic vector bundles over complex tori*, J. Korean Math. Soc. **26** (1989), no. 1, 117-142.

[F] Representation Theory of a Lie Group

(F-01) *The Method of Orbits for Real Lie Groups*, Kyungpook Math. J., **42** (2002), no. 2, 199-272 or arXiv:math.NT/0602056

(F-02) *Harmonic Analysis on Homogeneous Space*, arXiv:math.NT(math.RT)/0601655.

(F-03) *The Weil representations of the Jacobi group*, Proceedings of the International Conference on Geometry, Number Theory and Representation Theory (October 10-12, 2012, Incheon, Korea), pp. 169-204 (2013).

(F-04) *The orbit methods for the Jacobi group*, unpublished [http://www2.ims.nus.sg/preprints/2003-24.pdf]

[G] Theory of Elliptic Curves

(G-01) *Note on Taniyama-Shimura-Weil conjecture*, Proceedings of the Symposium on Number Theory and related topics (Masan, 1994; Pusan, 1994), Pyungsan Inst. Math. Sci., Seoul (1995), 29-46.

(G-02) *타원곡선에 관한 지난 20년 간의 연구 동향 (Past twenty years of the theory of elliptic curves: Korean)*, Commun. Korean Math. Soc. (초청논문: Invited paper), **14** (1999), No. 3, 449-477.

(G-03) *The Birch-Swinnerton Dyer Conjecture, Proceedings of the 2002 International Conference on Related Subjects to Clay Problems*, the Institute of Pure and Applied Mathematics, Chonbuk National Univ., Jeonju, Korea, Vol**.** **1** (2002), 134-151 or arXiv:math.HO (math:NT)/0611423.

[H] Theory of Siegel Modular Forms

(H-01) *Geometric theory of Siegel modular forms*, Proceedings of Topology and Geometry Research Center, **1** (December, 1990), 73-83.

(H-02) *Theory of the Siegel Modular Variety*, Proceedings of the International Conference on Number Theory and Cryptography, Edited by S. D. Adhikari and B. Ramakrishnan, Harish-Chandra Institute, Allahabad, India: A Publication of Hindustan Book Agency (2009), 219-278 [arXiv:math.NT (math.AG)/0706.4268]

[I] Topics on the Langlands Program

(I-01) *A note on the Artin conjecture*, Trends in Mathematics, Vol. **9**, No. 1 (2006), 99-107 or arXiv:math.HO(math.NT)/0611424.

(I-02) *Trace formula on finite groups*, Commun. Korean Math. **21** (2006), No. 1, pp. 17-25 or arXiv:math.GR/0612362.

(I-03) *Langlands Functoriality Conjecture*, Kyungpook Math. J., **49**, no. 2 (2009), 355-387.

[J] Harmonic Analysis on the Minkowski-Euclid Space

(J-01) *Harmonic Analysis on the Minkowski-Euclid Space* $P_n \times R^{(m,n)}$*, II : Unitary Representations of* $GL(n,R) \ltimes R^{(m,n)}$, Kyungpook Math. J. **45** (2005), 303-338.

(J-02) *Invariant Differential Opertors on the Minkowski-Euclid Space*, J. Korean Math. Soc., 50 (2013), No. 2, pp. 275-306.

(J-03) *Polarized Real Tori*, arXiv:0908.0252 [math.AG (math.NT)](2009).

[K] Papers related to Algebraic Geometry and Other Topics

(K-01) *Abelian varieties*, Proceedings of the 5th workshop on Pure and Applied Math., **5**, the Daewoo Foundation, Seoul (1985), 349-366.

(K-02) *Symplectic geometry*, Proceedings of Summer Symposium of Korean Scientists inside and outside, Seoul, Korea (1985), 343-351.

(K-03) (with J.-W. Son) *Survey on theta functions, identities for abelian integrals and subvarieties of Jacobi varieties*, Bull. Inst. Basic Sci., Inha Univ., **8** (1987), 15-21.

(K-04) *Satake compactification, Proceedings of the 7th workshop on Pure and Applied Math.*, **7**, the Daewoo Foundation, Seoul (1987), 74-87.

(K-05) *Recent progress on the Schottky problem*, paper dedicated to Prof. Oh (Yoon Yong) for his 60th birthday, KAIST, Taejeon, Korea (1988), 131-140.

(K-06) (with H. Yoon and B. Cho) *Siegel spaces as classifying spaces for Hodge structures*, Bull. Inst. Basic Sci., Inha Univ., **11** (1990), 17-25.

(K-07) *보형형식과 제타함수 (Automorphic forms and zeta functions, Korean)*, 평산연구회 소식 제2호 (1992), 1-9쪽 [the Notices of the Pyungsan Institute for Mathematical Sciences, **2** (1992)]

(K-08) (With Y.-S. Kim and J.-G. Nam) *Extendible Jordan algebras,* Jour. of the Natural Science, Wonkwang Univ., **13** (1994), no. 1, 1-9.

(K-09) *Supplementary note on the Freund-Witten adelic formula for Veneziano amplitudes*, Proceedings of the Symposium on Number Theory and related topics (Masan, 1994; Pusan, 1994), Pyungsan Inst. Math. Sci., Seoul (1995), 47-66.

(K-10) *On theta functions*, Kyungpook Math. J. **35** (1996), no. 3, 857-875.

(K-11) *A geometrical theory of Jacobi forms of higher degree*, Hodge 理論과 代數幾何學 講演記錄 (代表 小田忠雄) [1996年 6月 18日~21日: 東北大學], 125-147 (1996)

(K-12) *Stable automorphic forms*, Proceedings of Japan-Korea Joint Seminar on transcendental Number Theory and Related Topics, edited by Iekata Shiokawa and Jin-Woo Son, Masan (1998), 101-126.

(K-13) *On the Riemann Hypothesis (Korean)*, Proceedings of Math. Festival, Math. Love, **2** (2000), no. 3, 314-327. [리만 가설에 관하여, 세계 수학의 해 기념 제2회 Math Festival 프로시딩 제2집 3권, 314-327.]

(K-14) *소수의 아름다움*, 뉴스레터 MATHCHUNGNAM 제10호, 충남중등수학교육연구회 (2007), 1-20.

[L] Essays

(L-01) *추상수학의 어머니 에미 뇌테르 (Emmy Noether, 1882~1935)의 생애*, 한국수학교육학회 뉴스레터 제15권 제6호 통권 제64호 (1999년 11월), 26-28.

(L-02) *막스-플랑크 수학연구소*, 대한수학회 소식 제90호 (2003년 7월), 17-23.

(L-03) *마침내 Poincaré Conjecture 가 해결되다 !*, 대한수학회 소식 제109호 (2006년 9월), 12-18.

(L-04) (공저: 김인강) *Perelman의 업적에 관하여 − Poincaré 가설과 geometrization 가설*, 대한수학회 소식 제109호 (2006년 9월), 19-26.

(L-05) *인도의 Harish-Chandra 연구소에 다녀와서*, 대한수학회 소식 제115호 (2007년 9월), 8-13.

(L-06) *프린스턴에서의 이임학 교수와의 만남*, 대한수학회지 제132호 (2010년 7월), 24-27.

(L-07) *천재수학자의 심리 세계 : 극한의 긴장과 수학의 진리*, 교수신문 제 646호 (2012년 5월 28일)

(L-08) *골드바흐 추측과 첸징룬 : 끼니 거르면서도 연구에 골몰 — 중국 젊은이들에게 '불굴의 의지' 선사한 수학자*, 교수신문 제652호 (2012년 7월 9일)

(L-09) *괴팅겐 학파의 붕괴 : '메카'자리 되찾기 정말 어렵네*, 교수신문 제654호 (2012년 8월 27일)

(L-10) *헤르만 바일과 칼 루드비히 지겔 : 아리안인이었지만 유태인 도왔던 학문적 동반자*, 교수신문 제665호 (2012년 11월 19일)

(L-11) *은둔 수학자 그로텐딕 : 영성 찾아 피레네 골짜기로 숨어 버린 대수기하학의 거인*, 교수신문 제667호 (2012년 12월 3일)

(L-12) *아시아 수학 : 약진하고 있는 인도, 일본, 중국 — 갈 길 먼 한국 수학계*, 교수신문 제669호 (2012년 12월 17일)

(L-13) *리만 가설과 베른하르트 리만 : 150년 봉인해제 기다리는 우아한 난제*, 교수신문 제680호 (2013년 4월 15일)

(L-14) *국제수학자대회 (ICM) : 아시아 네 번째 대회 성공적으로 치루려면 ⋯*, 교수신문 제682호 (2013년 4월 29일)

(L-15) *프리드리히 히르체부르흐 (Friedrich Hirzebruch, 1927-2012) : 한글의 우수성과 아름다움을 알았던 수학자*, 교수신문 제684호 (2013년 5월 13일)

(L-16) *수학의 진리는 발견되는가 ? (上) : 형식주의와 논리주의 그리고 직관주의의 대립*, 교수신문 제686호 (2013년 5월 27일)

(L-17) *수학의 진리는 발견되는가 ? (下) : 신의 계신인가, 인간의 창조인가*, 교수신문 제690호 (2013년 6월 24일)

(L-18) *소수 쌍둥이 추측과 張益唐 : 세계 수학계를 놀라게 한 無名의 수학자들*, 교수신문 제696호 (2013년 8월 19일)

(L-19) *베트남 수학 고등연구소 (VIASM)*, 교수신문 제700호 (2013년 9월 23일)

(L-20) *필즈상과 노벨상 : 세계적인 과학 수준에 도달하기 위한 다섯 가지 제안,* 교수신문 제704호 (2013년, 10월 21일)

(L-21) *인재를 찾아 나서라*, 월간경영계 CEO Lounge 과학플러스 (2013년 4월호), 50-51.

(L-22) *현대수학의 동향*

(L-23) *위대한 피겨 스케이터 김연아 !*

(L-24) *고바야시 쇼시치 (小林昭七) 교수의 명복을 빌며*

(L-25) *수학과 수학자를 다룬 영화*

[M] Proceedings

(M-01) *Algebraic geometry and related topics*, Proceedings of the International Symposium held in Inchon, February 11-13, 1992, Edited by Jae-Hyun Yang, Yukihiko Namikawa and Kenji Ueno. Conference Proceedings and Lecture Notes in Algebraic Geometry, I. International Press, Cambridge, MA, 1993. vi+200 pp. ISBN: 1-57146-013-6 00B25 (14-06 32-06)

(M-02) *Automorphic forms and related topics*, Proceedings of the workshop held in Seoul, September 2-3, 1993. Edited by Jin-Woo Son and Jae-Hyun Yang. Pyungsan Inst. Math. Sci., Seoul, 1993, ii+92 pp.

(M-03) *Number theory and related topics*, Proceedings of the conferences held in Masan, October 11, 1994 and at Pusan National University, Pusan, Dec. 20-21, 1994. Edited by Jin-Woo Son and Jae-Hyun Yang. Pyungsan Institute for Mathematical Sciences, Seoul, 1995. vi+83 pp.

(M-04) *Number theory, geometry and related topics*, Proceedings of the Symposium held at Wonkwang University, Iksan City, September 23-24, 1995. Edited by Jin-Woo Son and Jae-Hyun Yang. Pyungsan Institute for Mathematical Sciences, Seoul, 1996. ii+122 pp.

(M-05) *Number Theory and Related Topics*, Proceedings of the International Conference held at Yonsei University, Seoul, October 20-22, 1998. Edited by Hyoung June Ko and Jae-Hyun Yang. Institute for Mathematical Sciences, Yonsei University, 1998. vi+141 pp.

(M-06) *Proceedings of the 2002 International Conference on Related Subjects to Clay Problems*, edited by Y. G. Kim, G. S. Seo, H. S. Park and J.-H.

Yang, the Institute of Pure and Applied Mathematics, Chonbuk National University, Jeonju, Korea, Vol. **1** (2002). ISBN 1598-7116.

(M-07) *Representation Theory and Automorphic Forms*, edited by Toshiyuki Kobayashi, Wilfried Schmid and Jae-Hyun Yang. Progress in Mathematics **255**, Birkhauser, Boston (2008). vi+210 pp. ISBN-13:978-0-8176-4505-2.

(M-08) *Automorphic Forms*, *L-Functions and Shimura Varieties,* Proceedings of the International Symposium held at Inha University, Incheon, November 14-17, 2008. Edited by Jae-Hyun Yang. National Institute for Mathematical Sciences, Daejeon, 2008 [Vol. 3, No. 10]. vi+152 pp. ISBN 1975-5384.

(M-09) *Invariant Theory and Related Topics*, Proceedings of NIMS Hot Topics Workshop held at Inha University, Incheon, February 17-19, 2010. Edited by Jae-Hyun Yang and Toru Umeda. National Institute for Mathematical Sciences, Daejeon, 2010 [Vol.5, No.2]. vi+162 pp.

(M-10) *Proceedings of the International Conference on Geometry*, Number Theory and Representation Theory held in Incheon, Korea (October 10-12, 2012), edited by Jae-Hyun Yang, Kyung Moon Sa (10101), Seoul (2013). vi+204 pp. ISBN 978-89-88615-35-5.

[N] Books

(N-01) *미분기하학 입문 (Introduction to Differential Geometry: Korean)*, 경문사 (Kyung Moon Sa, Seoul), 초판 (the first edition, 1987), 중판 (the second edition, 1988), 3판 (the third edition, 1997). vi+305 pp. ISBN 89-7282-013X 93410.

(N-02) *벡터속이론 (Theory of Vector Bundles : Korean)* [대우학술총서 자연과학 **60**], 민음사 (Minumsa, Seoul), 1988. vi+240 pp. ISBN 89-374-3560-8.

(N-03) *20세기 수학자들과의 만남 (Encounter with the great mathematicians of the 20th century : Korean)*, 애플기획 (Apple Press Company, Seoul), 초판 (the first edition, 1998. vi+226 pp.), 중판 (the 2nd edition, 1999. vi+267 pp.) ISBN 89-88772-00-8.

(N-04) *Lie 군의 표현론 (Representation Theory of Lie Groups : Korean)* [대우학술총서 자연과학 **127**] 민음사 (Minumsa, Seoul), 1998. vi+238 pp. ISBN 89-374-3627-2 : ISBN 89-374-3000-2 (세트).

(N-05) (공저: 방승진, 손진우 [with S.-J. Bang and J.-W. Son]) *선형대수학 (Linear Algebra : Korean)*, 경문사 (Kyung Moon Sa, Seoul), 1997. vi+325 pp. ISBN 89-7282-279-5.

(N-06) *20세기 수학자들과의 만남 (Encounter with the great mathematicians of the 20th century: Korean)*, 경문사 (Kyung Moon Sa, Seoul), 2005. vi+326 pp. ISBN 89-7282-660-X.

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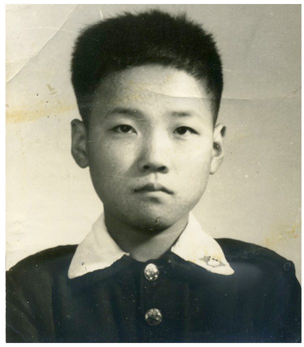

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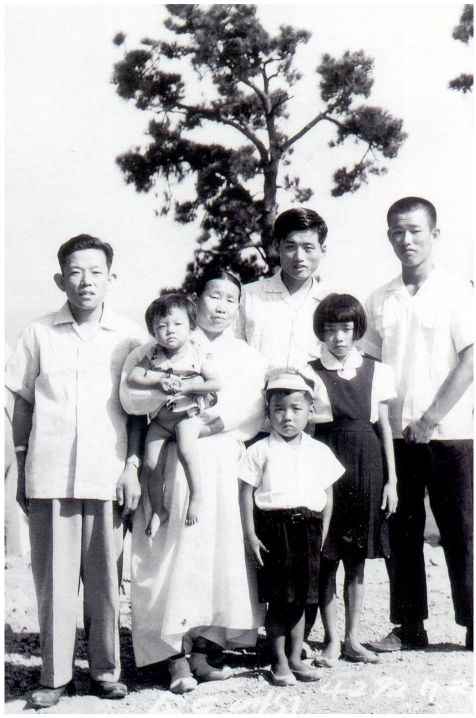

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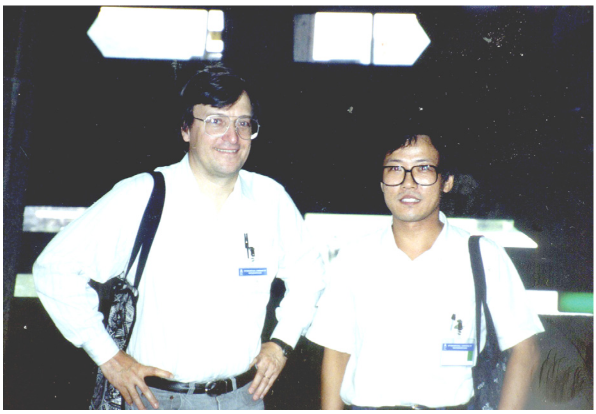

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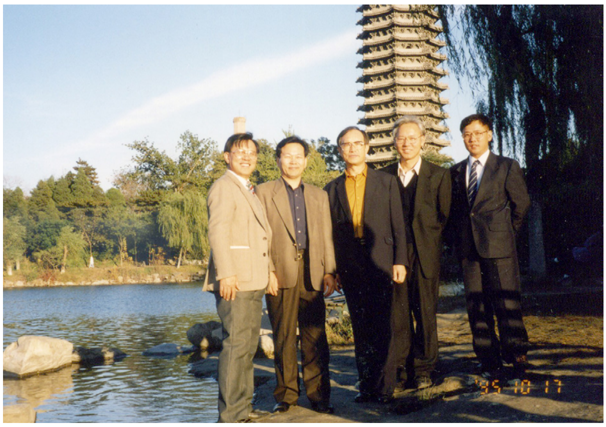

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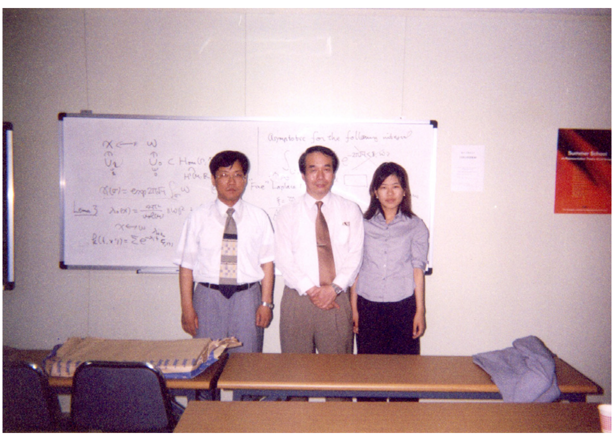

[25] 평산수학연구소에서 Sunada Toshikazu (砂田利一) 교수와 함께
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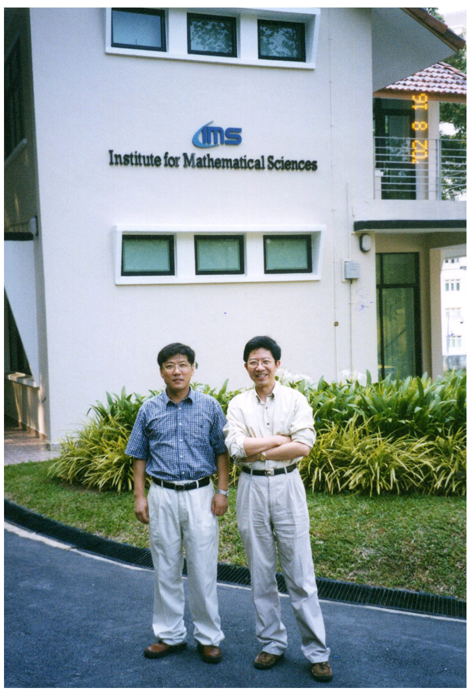

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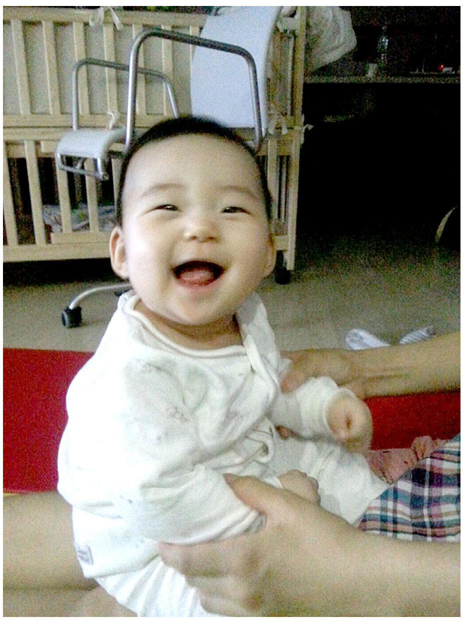

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