THE WEIL REPRESENTATIONS OF THE JACOBI GROUP

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Abstract. The Jacobi group is the semi-direct product of the symplectic group and the Heisenberg group. The Jacobi group is an important object in the framework of quantum mechanics, geometric quantization and optics. In this paper, we study the Weil representations of the Jacobi group and their properties. We also provide their applications to the theory of automorphic forms on the Jacobi group and representation theory of the Jacobi group.

1. Introduction

The Weil representation of the symplectic group was first introduced by A. Weil in his remarkable paper [42] to reformulate Siegel's analytic theory of quadratic forms [34] in group theoretical terms. The Weil representation plays a central role in the study of the transformation behaviors of theta series and has many applications to the theory of automorphic forms (cf. [9, 18, 19, 20, 21, 25, 32, 33]). The Jacobi group is defined to be the semi-direct product of the symplectic group and the Heisenberg group. The Jacobi group is an important object in the framework of quantum mechanics, geometric quantization and optics [1, 2, 3, 4, 10, 11, 12, 22, 26, 35, 43, 62]. The squeezed states in quantum optics represent a physical realization of the coherent states associated with the Jacobi group [12, 22, 35, 62]. In this paper, we show that we can construct several types of the Weil representations of the Jacobi group and present their applications to the theory of automorphic forms on the Jacobi group and representation theory of the Jacobi group.

For a given fixed positive integer n, we let

$$\mathbb{H}_n = \{\, \Omega \in \mathbb{C}^{(n,n)} \mid \Omega = {}^t\Omega, \quad \mathrm{Im}\,\Omega > 0 \,\}$$

be the Siegel upper half plane of degree n and let

$$Sp(n,\mathbb{R}) = \{ g \in \mathbb{R}^{(2n,2n)} \mid {}^tg J_n g = J_n \,\}$$

be the symplectic group of degree n, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l, ${}^tM$ denotes

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the transposed matrix of a matrix M, $\mathrm{Im}\,\Omega$ denotes the imaginary part of Ω and

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

We see that $Sp(n,\mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

$$g\cdot\Omega = (A\Omega + B)(C\Omega + D)^{-1},$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $\Omega \in \mathbb{H}_n$.

For two positive integers n and m, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \{\, (\lambda,\mu;\kappa) \mid \lambda,\mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \ \kappa + \mu\,{}^t\lambda \text{ symmetric }\}$$

endowed with the following multiplication law

$$(\lambda,\mu;\kappa)\circ(\lambda',\mu';\kappa') = (\lambda+\lambda',\mu+\mu';\kappa+\kappa'+\lambda\,{}^t\mu' - \mu\,{}^t\lambda').$$

We let

$$G^J = Sp(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)} \quad \text{(semi-direct product)}$$

be the Jacobi group endowed with the following multiplication law

$$\Big(g,(\lambda,\mu;\kappa)\Big)\cdot\Big(g',(\lambda',\mu';\kappa')\Big) = \Big(gg',(\widetilde{\lambda}+\lambda',\widetilde{\mu}+\mu';\kappa+\kappa'+\widetilde{\lambda}\,{}^t\mu' - \widetilde{\mu}\,{}^t\lambda')\Big)$$

with $g,g' \in Sp(n,\mathbb{R}), (\lambda,\mu;\kappa), (\lambda',\mu';\kappa') \in H_{\mathbb{R}}^{(n,m)}$ and $(\widetilde{\lambda},\widetilde{\mu}) = (\lambda,\mu)g'$.

Then we have the *natural action* of G^J on the Siegel-Jacobi space $\mathbb{H}_{n,m} := \mathbb{H}_n \times \mathbb{C}^{(m,n)}$ defined by

$$\Big(g,(\lambda,\mu;\kappa)\Big)\cdot(\Omega,Z) = \Big(g\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}\Big), \tag{1.1}$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega,Z) \in \mathbb{H}_{n,m}$. We refer to [49]-[53], [56]-[58] for more details on materials related to the Siegel-Jacobi space.

The aim of this article is to introduce three types of the Weil representations of the Jacobi group G^J and to study their applications to the theory of automorphic forms and representation theory. They are slightly different each other. They are essentially isomorphic. However each has its own advantage in applications to the theory of automorphic forms and representation theory.

This article is organized as follows. In Section 2, we review the Weil representation of the symplectic group and the Maslov index briefly. In Section 3, we define the Weil representation of the Jacobi group G^J using a cocycle class of G^J in $H^2(G^J,T)$. In Section 4, we define the Schrödinger-Weil representation of the Jacobi group that is used to study the transformation behaviors of certain theta series with toroidal variables. The the Schrödinger-Weil representation plays an important role in the construction of Jacobi forms, the theory of Maass-Jacobi forms and the study of Jacobi's theta sums. We deal with these applications in detail in Section 7. In Section 5, we recall the Weil-Satake representation of the Jacobi group formulated by Satake [31] on the Fock model of the Heisenberg group. In Section 6, we recall the concept of Jacobi forms of half integral weight to be used in a subsequent section. We review Siegel modular forms of half integral weight. In Section 7, we present the applications of the Schrödinger-Weil representation to constructing of Jacobi forms via covariant maps for the Schrödinger-Weil representation, the study of Maass-Jacobi forms and Jacobi's theta sums. We describe the works of the author [60], A. Piale [28] and J. Marklof [23]. In Section 8, we provides some applications of the Weil-Satake representation of G^J to the study of representations of G^J which were obtained by Takase [36, 37, 39]. Takase [36] showed that there is a bijective correspondence between the unitary equivalence classes of unitary representations of a two-fold covering group of the symplectic group and the unitary equivalence classes of unitary representations of the Jacobi group. Using this representation theoretical fact, Takase [39] established a bijective correspondence between the space of cuspidal Jacobi forms and the space of Siegel cusp forms of half integral weight which is compatible with the action of Hecke operators.

Notations : We denote by $\mathbb{R}$ and $\mathbb{C}$ the field of real numbers, and the field of complex numbers respectively. We denote by $\mathbb{R}^*_+$ the multiplicative group of positive real numbers. $\mathbb{C}^*$ (resp. $\mathbb{R}^*$) denotes the multiplicative group of nonzero complex (resp. real) numbers. We denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers respectively. $T = \{z \in \mathbb{C} \mid |z| = 1\}$ denotes the multiplicative group of complex numbers of modulus one. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transposed matrix of M. I_n denotes the identity matrix of degree n. For a positive integer m we denote by $S(m)$ the set of all $m \times m$ symmetric real matrices. We put $i = \sqrt{-1}$. For $z \in \mathbb{C}$, we define $z^{1/2} = \sqrt{z}$ so that $-\pi/2 < \arg(z^{1/2}) \leqq \pi/2$. Furthermore we put $z^{\kappa/2} = \left(z^{1/2}\right)^{\kappa}$ for every $\kappa \in \mathbb{Z}$. For a rational number field $\mathbb{Q}$, we denote by $\mathbb{A}$ and $\mathbb{A}^*$ the ring of adeles of $\mathbb{Q}$ and the multiplicative group of ideles of $\mathbb{Q}$ respectively.

2. The Weil Representation of the Symplectic Group

Let (V, B) be a symplectic real vector space of dimension $2n$ with a non-degenerate alternating bilinear form B. We consider the Lie algebra $\mathfrak{h} = V + \mathbb{R}E$ with the Lie bracket satisfying the following properties (2.1) and (2.2) :

$$[X, Y] = B(X, Y)E \quad \text{for all } X, Y \in V; \tag{2.1}$$

$$[Z, E] = 0 \quad \text{for all } Z \in \mathfrak{h}. \tag{2.2}$$

Let H be the Heisenberg group with its Lie algebra $\mathfrak{h}$. Via the exponential map $\exp : \mathfrak{h} \longrightarrow H$, H is identified with the $(2n + 1)$-dimensional vector space with following multiplication law :

$$\exp(v_1+t_1E)\cdot\exp(v_2+t_2E)=\exp\left(v_1+v_2+\left(t_1+t_2+\frac{B(v_1,v_2)}{2}\right)E\right),$$

where $v_1, v_2 \in V$ and $t_1, t_2 \in \mathbb{R}$. Let

$$Sp(B)=\big\{g\in GL(V)\mid B(gx,gy)=B(x,y)\quad\text{for all } x,y\in V\,\big\}$$

be the symplectic group of (V,B). Then $Sp(B)$ acts on H by

$$g\cdot\exp(v+tE)=\exp(gv+tE),\quad g\in Sp(B),\ v\in V,\ t\in\mathbb{R}.$$

For a fixed nonzero real number m, we let $\chi_m : H \longrightarrow T$ be the function defined by

$$\chi_m\big(\exp(v+tE)\big)=e^{2\pi imt},\quad v\in V,\ t\in\mathbb{R}.$$

Let $\mathfrak{l}$ be a Lagrangian subspace in (V,B). We put $L=\exp(\mathfrak{l}+\mathbb{R}E)$. Obviously the restriction of χ_m to L is a character of L. The induced representation

$$W_{\mathfrak{l},m}=\mathrm{Ind}_L^H\chi_m$$

is the so-called *Schrödinger representation* of the Heisenberg group H. The representation $H_{\mathfrak{l},m}$ of $W_{\mathfrak{l},m}$ is the completion of the space of continuous functions φ on H satisfying the following properties (2.3) and (2.4) :

$$\varphi(hl)=\chi_m(l)^{-1}\varphi(h),\quad h\in H,\ l\in L \tag{2.3}$$

and

(2.4) $h \mapsto |\varphi(h)|$ is square integrable with respect to an invariant measure on H/L.

We observe that $W_{\mathfrak{l},m}\big(\exp(tE)\big)=e^{2\pi imt}I_{H_{\mathfrak{l},m}}$, where $I_{H_{\mathfrak{l},m}}$ denotes the identity operator on $H_{\mathfrak{l},m}$. For brevity, we put $G=Sp(B)$. For a fixed element $g\in G$, we consider the representation $W^g_{\mathfrak{l},m}$ of H on $H_{\mathfrak{l},m}$ defined by

$$W^g_{\mathfrak{l},m}(h)=W_{\mathfrak{l},m}(g\cdot h),\quad h\in H. \tag{2.5}$$

Since $W_{\mathfrak{l},m}(\exp tE) = W^g_{\mathfrak{l},m}(\exp tE)$ for all $t\in\mathbb{R}$, according to Stone-von Neumann theorem, there exists a unitary operator $R_{\mathfrak{l},m}(g) : H_{\mathfrak{l},m} \longrightarrow H_{\mathfrak{l},m}$ such that $W^g_{\mathfrak{l},m}(h)R_{\mathfrak{l},m}(g)=R_{\mathfrak{l},m}(g)W_{\mathfrak{l},m}(h)$ for all $h\in H$. For convenience, we choose $R_{\mathfrak{l},m}(\mathbf{1}) = I_{H_{\mathfrak{l},m}}$, where $\mathbf{1}$ denotes the identity element of G. We note that $R_{\mathfrak{l},m}(g)$ is determined uniquely up to a scalar of modulus one. Since $R_{\mathfrak{l},m}(g_2)^{-1}R_{\mathfrak{l},m}(g_1)^{-1}R_{\mathfrak{l},m}(g_1g_2)$ is the unitary operator on $H_{\mathfrak{l},m}$ commuting with $W_{\mathfrak{l},m}$, according to Schur's lemma, we have a map $c_{\mathfrak{l},m} : G\times G \longrightarrow T$ satisfying the condition

$$R_{\mathfrak{l},m}(g_1g_2)=c_{\mathfrak{l},m}(g_1,g_2)R_{\mathfrak{l},m}(g_1)R_{\mathfrak{l},m}(g_2),\quad g_1,g_2\in G.$$

Therefore $R_{\mathfrak{l},m}$ is a projective representation of G with multiplier $c_{\mathfrak{l},m}$. It is easy to see that the map $c_{\mathfrak{l},m}$ satisfies the cocycle condition

$$c_{\mathfrak{l},m}(g_1g_2,g_3)\,c_{\mathfrak{l},m}(g_1,g_2)=c_{\mathfrak{l},m}(g_1,g_2g_3)\,c_{\mathfrak{l},m}(g_2,g_3)\quad\text{for all } g_1,g_2,g_3\in G.$$

The cocycle $c_{\mathfrak{l},m}$ produces the central extension $G_{\mathfrak{l},m}$ of G by T. The group $G_{\mathfrak{l},m}$ is the set $G \times T$ with the following group multiplication law:

$$(g_1,t_1)\cdot(g_2,t_2) := \big(g_1g_2,\ t_1t_2\, c_{\mathfrak{l},m}(g_1,g_2)^{-1}\big),\quad g_1,g_2\in G,\ t_1,t_2\in T. \tag{2.6}$$

We see that the map $\tilde{R}_{\mathfrak{l},m} : G_{\mathfrak{l},m} \longrightarrow GL(H_{\mathfrak{l},m})$ defined by

$$\tilde{R}_{\mathfrak{l},m}(g,t) := t\, R_{\mathfrak{l},m}(g),\quad g\in G,\ t\in\mathbb{R}$$

is a *true* representation of $G_{\mathfrak{l},m}$.

We now express the cocycle $c_{\mathfrak{l},m}$ in terms of the Maslov index. Let $\mathfrak{l}_1$, $\mathfrak{l}_2$, $\mathfrak{l}_3$ be three Lagrangian subspaces of (V,B). The *Maslov index* $\tau(\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_3)$ of $\mathfrak{l}_1$, $\mathfrak{l}_2$ and $\mathfrak{l}_3$ is defined to be the signature of the quadratic form Q on the $3n$ dimensional vector space $\mathfrak{l}_1\oplus\mathfrak{l}_2\oplus\mathfrak{l}_3$ given by

$$Q(x_1+x_2+x_3) = B(x_1,x_2)+B(x_2,x_3)+B(x_3,x_1),\quad x_i\in\mathfrak{l}_i,\ i=1,2,3.$$

For a sequence $\{\mathfrak{l}_1,\mathfrak{l}_2,\cdots,\mathfrak{l}_k\}$ of Lagrangian subspaces $\mathfrak{l}_1,\mathfrak{l}_2,\cdots,\mathfrak{l}_k\,(k\geq 4)$ in (V,B), we define the *Maslov index* $\tau(\mathfrak{l}_1,\mathfrak{l}_2,\cdots,\mathfrak{l}_k)$ by

$$\tau(\mathfrak{l}_1,\mathfrak{l}_2,\cdots,\mathfrak{l}_k) = \tau(\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_3)+\tau(\mathfrak{l}_1,\mathfrak{l}_3,\mathfrak{l}_4)+\cdots+\tau(\mathfrak{l}_1,\mathfrak{l}_{k-1},\mathfrak{l}_k).$$

For a Lagrangian subspace $\mathfrak{l}$ in (V,B), we put $\tau_{\mathfrak{l}}(g_1,g_2)=\tau(\mathfrak{l},g_1\mathfrak{l},g_1g_2\mathfrak{l})$ for $g_1,g_2\in G$.

Lemma 2.1. *Let $\mathfrak{l}_1,\mathfrak{l}_2,\cdots,\mathfrak{l}_k$ be Lagrangian subspaces in (V,B) with $k\geq 4$. Then we have*

(a) $\tau(\mathfrak{l}_1,\mathfrak{l}_2,\cdots,\mathfrak{l}_k)$ is invariant under the action of G and its value is unchanged under circular permutations.
(b) $\tau(\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_3) = -\tau(\mathfrak{l}_2,\mathfrak{l}_1,\mathfrak{l}_3) = -\tau(\mathfrak{l}_1,\mathfrak{l}_3,\mathfrak{l}_2)$.
(c) For any four Lagrangian subspaces $\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_3,\mathfrak{l}_4$ in (V,B),

$$\tau(\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_3)=\tau(\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_4)+\tau(\mathfrak{l}_2,\mathfrak{l}_3,\mathfrak{l}_4)+\tau(\mathfrak{l}_3,\mathfrak{l}_1,\mathfrak{l}_4).$$

(d) $\tau(\mathfrak{l}_1,\mathfrak{l}_2,\cdots,\mathfrak{l}_d)=\tau(\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l})+\tau(\mathfrak{l}_2,\mathfrak{l}_3,\mathfrak{l})+\cdots+\tau(\mathfrak{l}_{d-1},\mathfrak{l}_d,\mathfrak{l})+\tau(\mathfrak{l}_d,\mathfrak{l}_1,\mathfrak{l})$ for any Lagrangian subspace $\mathfrak{l}$ in (V,B) and $d\geq 3$.
(e) $\tau(\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_3,\mathfrak{l}_4) = -\tau(\mathfrak{l}_2,\mathfrak{l}_1,\mathfrak{l}_4,\mathfrak{l}_3)$.
(f) For any Lagrangian subspaces $\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_3,\mathfrak{l}_1',\mathfrak{l}_2',\mathfrak{l}_3'$ in (V,B), we have

$$\tau(\mathfrak{l}_1',\mathfrak{l}_2',\mathfrak{l}_3')=\tau(\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_3)+\tau(\mathfrak{l}_1',\mathfrak{l}_2',\mathfrak{l}_2,\mathfrak{l}_1)+\tau(\mathfrak{l}_2',\mathfrak{l}_3',\mathfrak{l}_3,\mathfrak{l}_2)+\tau(\mathfrak{l}_3',\mathfrak{l}_1',\mathfrak{l}_1,\mathfrak{l}_3).$$

(g) $\tau_{\mathfrak{l}}(g_1g_2,g_3)+\tau_{\mathfrak{l}}(g_1,g_2)=\tau_{\mathfrak{l}}(g_1,g_2g_3)+\tau_{\mathfrak{l}}(g_2,g_3)$ for all $g_1,g_2,g_3\in G$.

Proof. The proof can be found in [21]. □

Theorem 2.1. *For a Lagrangian subspace $\mathfrak{l}$ in (V,B) and a real number m, we have*

$$c_{\mathfrak{l},m}(g_1,g_2) = e^{-\frac{i\pi m}{4}\,\tau(\mathfrak{l},g_1\mathfrak{l},g_1g_2\mathfrak{l})}\quad for\ all\ g_1,g_2\in G.$$

Proof. The proof can be found in [21]. □

An *oriented vector space* of dimension n is defined to be a pair (U, e), where U is a real vector space of dimension n and e is an orientation of U, i.e., a connected component of $\bigwedge^n U - \{0\}$. For two oriented vector space $(\mathfrak{l}_1, e_1)$ and $(\mathfrak{l}_2, e_2)$ in a symplectic vector space (V, B), we define

$$s\big((\mathfrak{l}_1, e_1), (\mathfrak{l}_2, e_2)\big) := i^{n-\dim(\mathfrak{l}_1\cap\mathfrak{l}_2)}\, \varepsilon\big((\mathfrak{l}_1, e_1), (\mathfrak{l}_2, e_2)\big). \tag{2.7}$$

We refer to [21, pp. 64–66] for the precise definition of $\varepsilon\big((\mathfrak{l}_1, e_1), (\mathfrak{l}_2, e_2)\big)$. Let M be the space of all Lagrangian subspaces in (V, B) and $\widetilde{M}$ the manifold of all oriented Lagrangian subspaces in (V, B). Let $p : \widetilde{M} \longrightarrow M$ be the natural projection from $\widetilde{M}$ onto M. Now we will write $\tilde{\mathfrak{l}}$ for a Lagrangian oriented subspace $(\mathfrak{l}, e)$.

Theorem 2.2. *Let* $\tilde{\mathfrak{l}}_1,\, \tilde{\mathfrak{l}}_2,\, \tilde{\mathfrak{l}}_3 \in \widetilde{M}$. *Then*

$$e^{-\frac{i\pi}{2}\,\tau(p(\tilde{\mathfrak{l}}_1), p(\tilde{\mathfrak{l}}_1), p(\tilde{\mathfrak{l}}_1))} = s\big(\tilde{\mathfrak{l}}_1,\, \tilde{\mathfrak{l}}_2\big)\, s\big(\tilde{\mathfrak{l}}_2,\, \tilde{\mathfrak{l}}_3\big)\, s\big(\tilde{\mathfrak{l}}_3,\, \tilde{\mathfrak{l}}_1\big).$$

Proof. The proof can be found in [21, pp. 67–70]. □

Let $\mathfrak{l}$ be a Lagrangian subspace in (V, B). We choose an orientation $\mathfrak{l}^+$ on $\mathfrak{l}$. Then G acts on oriented Lagrangian subspace in (V, B). We define

$$s_{\mathfrak{l},m}(g) := s\big(\mathfrak{l}^+, g\mathfrak{l}^+\big)^m, \quad g \in G. \tag{2.8}$$

The above definition is well defined, i.e., does not depend on the choice of orientation on $\mathfrak{l}$. Since $s_{\mathfrak{l},m}(g^{-1}) = s_{\mathfrak{l},m}(g)^{-1}$, according to Theorem 2.1 and Theorem 2.2, we get

$$c_{\mathfrak{l},m}(g_1, g_2)^2 = s_{\mathfrak{l},m}(g_1)^{-1}\, s_{\mathfrak{l},m}(g_2)^{-1}\, s_{\mathfrak{l},m}(g_1 g_2) \quad \text{for all } g_1, g_2 \in G. \tag{2.9}$$

Hence we can see that

$$G_{2,\mathfrak{l},m} := \big\{\, (g, t) \in G_{\mathfrak{l},m} \mid t^2 = s_{\mathfrak{l},m}(g)^{-1} \,\big\} \tag{2.10}$$

is the subgroup of $G_{\mathfrak{l},m}$ (cf. Formula (2.6)) that is called the metaplectic group associated with a pair $(\mathfrak{l}, m)$. We know that $G_{2,\mathfrak{l},m}$ is a two-fold covering group of G. The restriction $R_{2,\mathfrak{l},m}$ of $\tilde{R}_{\mathfrak{l},m}$ to $G_{2,\mathfrak{l},m}$ is a true representation of $G_{2,\mathfrak{l},m}$ that is called the *Weil representation* of G associated with a pair $(\mathfrak{l}, m)$. We note that

$$R_{2,\mathfrak{l},m}(g, t) = t\, R_{\mathfrak{l},m}(g) = s_{\mathfrak{l},m}(g)^{-1/2} R_{\mathfrak{l},m}(g) \quad \text{for all } (g, t) \in G_{2,\mathfrak{l},m}. \tag{2.11}$$

We refer to [9, 15, 21] for more detail on the Weil representation.

3. The Weil Representation of the Jacobi Group G^J

Let $V = \mathbb{R}^{(m,n)} \times \mathbb{R}^{(m,n)}$ be the symplectic real veactor space with a nondegenerate alternating bilinear form on V given by

$$B((\lambda, \mu), (\lambda', \mu')) := \sigma(\lambda\, {}^t\mu' - \mu\, {}^t\lambda'), \quad (\lambda, \mu), (\lambda', \mu') \in \mathbb{R}^{(m,n)}.$$

We assume that $\mathcal{M}$ is a positive definite symmetric real matrix of degree m. We denote by $S(m)$ the set of all $m \times m$ symmetric real matrices. We let

$$\mathscr{W}_{\mathcal{M}} : H_{\mathbb{R}}^{(n,m)} \longrightarrow U(H_{\mathcal{M}}) \tag{3.1}$$

be the Schrödinger representation with central character $\mathscr{W}_{\mathcal{M}}((0,0;\kappa)) = e^{2\pi i\,\sigma(\mathcal{M}\kappa)} I_{H_{\mathcal{M}}}$, $\kappa \in S(m)$. Here $H_{\mathcal{M}}$ denotes the representation space of $\mathscr{W}_{\mathcal{M}}$. We note that $\mathscr{W}_{\mathcal{M}}$ is realized on $L^2\big(\mathbb{R}^{(m,n)}\big) \cong H_{\mathcal{M}}$ by

$$(3.2) \qquad \left(\mathscr{W}_{\mathcal{M}}(h)f\right)(x) = e^{2\pi i\,\sigma\left(\mathcal{M}(\kappa+\mu\,{}^t\lambda+2\,x\,{}^t\mu)\right)} f(x+\lambda), \quad x \in \mathbb{R}^{(m,n)},$$

where $h = (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $f \in L^2\big(\mathbb{R}^{(m,n)}\big)$. We refer to [44, 45, 47, 48] for more detail about $\mathscr{W}_{\mathcal{M}}$. The Jacobi group G^J acts on $H_{\mathbb{R}}^{(n,m)}$ by conjugation inside G^J. Fix an element $\widetilde{g} \in G^J$. The irreducible unitary representation $\mathscr{W}_{\mathcal{M}}^{\widetilde{g}}$ of $H_{\mathbb{R}}^{(n,m)}$ defined by

$$(3.3) \qquad \mathscr{W}_{\mathcal{M}}^{\widetilde{g}}(h) := \mathscr{W}_{\mathcal{M}}(\widetilde{g}\,h\,\widetilde{g}^{-1}), \quad h \in H_{\mathbb{R}}^{(n,m)}$$

has the property that $\mathscr{W}_{\mathcal{M}}^{\widetilde{g}}((0,0;\kappa)) = \mathscr{W}_{\mathcal{M}}((0,0;\kappa)) = e^{2\pi i\sigma(\mathcal{M}\kappa)}\cdot I_{H_{\mathcal{M}}}$ for all $\kappa \in S(m)$. According to Stone-von Neumann theorem, there exists a unitary operator $T_{\mathcal{M}}(\widetilde{g})$ on $H_{\mathcal{M}}$ such that $T_{\mathcal{M}}(\widetilde{g})\,\mathscr{W}_{\mathcal{M}}(h) = \mathscr{W}_{\mathcal{M}}^{\widetilde{g}}(h)\,T_{\mathcal{M}}(\widetilde{g})$ for all $h \in H_{\mathbb{R}}^{(n,m)}$. We observe that $T_{\mathcal{M}}(\widetilde{g})$ is determined uniquely up to a scalar of modulus one. According to Schur's lemma, we have a map $\widetilde{c}_{\mathcal{M}} : G^J \times G^J \longrightarrow T$ satisfying the relation

$$(3.4) \qquad T_{\mathcal{M}}(\widetilde{g}_1\widetilde{g}_2) = \widetilde{c}_{\mathcal{M}}(\widetilde{g}_1,\widetilde{g}_2)\,T_{\mathcal{M}}(\widetilde{g}_1)T_{\mathcal{M}}(\widetilde{g}_2), \quad \widetilde{g}_1,\widetilde{g}_2 \in G^J.$$

Therefore $T_{\mathcal{M}}$ is a projective representation of G^J and $\widetilde{c}_{\mathcal{M}}$ defines the cocycle class in $H^2(G^J,T)$. The cocycle $\widetilde{c}_{\mathcal{M}}$ satisfies the following properties

$$(3.5) \qquad \widetilde{c}_{\mathcal{M}}(h_1,h_2) = 1 \quad \text{for all } h_1,h_2 \in H_{\mathbb{R}}^{(n,m)};$$

$$(3.6) \qquad \widetilde{c}_{\mathcal{M}}(\tilde{g},e) = \widetilde{c}_{\mathcal{M}}(e,\tilde{g}) = \widetilde{c}_{\mathcal{M}}(e,e) = 1 \quad \text{for all } \tilde{g} \in G^J\,;$$

$$(3.7) \qquad \widetilde{c}_{\mathcal{M}}(\tilde{g},\tilde{g}^{-1}) = \widetilde{c}_{\mathcal{M}}(\tilde{g}^2,\tilde{g}^{-1})\,\widetilde{c}_{\mathcal{M}}(\tilde{g},\tilde{g}) \quad \text{for all } \tilde{g} \in G^J\,;$$

$$(3.8) \qquad T_{\mathcal{M}}(\tilde{g}^{-1}) = \widetilde{c}_{\mathcal{M}}(\tilde{g},\tilde{g}^{-1})^{-1}\,T_{\mathcal{M}}(\tilde{g})^{-1} \quad \text{for all } \tilde{g} \in G^J,$$

where e is the identity element of G^J. The cocycle $\widetilde{c}_{\mathcal{M}}$ yields the central extension $G^J_{\mathcal{M}}$ of G^J by T. The extension group $G^J_{\mathcal{M}}$ is the set $G^J \times T$ with the following group multiplication law:

$$(3.9) \qquad (\widetilde{g}_1,t_1)\cdot(\widetilde{g}_2,t_2) = \big(\widetilde{g}_1\widetilde{g}_2, t_1t_2\,\widetilde{c}_{\mathcal{M}}(\widetilde{g}_1,\widetilde{g}_2)^{-1}\big), \quad \widetilde{g}_1,\widetilde{g}_2 \in G^J,\ t_1,t_2 \in T.$$

It is obvious that $\big((I_{2n},(0,0;0)),1\big)$ is the identity element of $G^J_{\mathcal{M}}$ and

$$(\tilde{g},t)^{-1} = \big(\tilde{g}^{-1}, t^{-1}\,\widetilde{c}_{\mathcal{M}}(\tilde{g},\tilde{g}^{-1})\big)$$

if $(\tilde{g},t) \in G^J_{\mathcal{M}}$. We see easily that the map $\widetilde{T}_{\mathcal{M}} : G^J_{\mathcal{M}} \longrightarrow U(H_{\mathcal{M}})$ defined by

$$(3.10) \qquad \widetilde{T}_{\mathcal{M}}(\tilde{g},t) := t\,T_{\mathcal{M}}(\tilde{g}), \quad (\tilde{g},t) \in G^J_{\mathcal{M}}$$

is a true representation of $G^J_{\mathcal{M}}$. Here $U(H_{\mathcal{M}})$ denotes the group of unitary operators of $H_{\mathcal{M}}$. For the Lagrangian subspace $\mathfrak{l} = \big\{(0,\mu) \in V \mid \mu \in \mathbb{R}^{(m,n)}\big\}$, as (2.8) and (2.9) in Section 2, we can define the function $\widetilde{s}_{\mathcal{M}} : G^J \longrightarrow T$ satisfying the relation

$$(3.11) \qquad \widetilde{c}_{\mathcal{M}}(\widetilde{g}_1,\widetilde{g}_2)^2 = \widetilde{s}_{\mathcal{M}}(\widetilde{g}_1)^{-1}\,\widetilde{s}_{\mathcal{M}}(\widetilde{g}_2)^{-1}\,\widetilde{s}_{\mathcal{M}}(\widetilde{g}_1\widetilde{g}_2) \quad \text{for all } \widetilde{g}_1,\widetilde{g}_2 \in G^J.$$

Then it is easily seen that

$$G^J_{\mathcal{M},2} := \left\{ (\widetilde{g},t) \in G^J_{\mathcal{M}} \mid t^2 = \widetilde{s}_{\mathcal{M}}(\widetilde{g})^{-1} \right\} \tag{3.12}$$

is a two-fold covering group of G^J. The restriction $\widetilde{\omega}_{\mathcal{M}}$ of $\widetilde{T}_{\mathcal{M}}$ to $G^J_{\mathcal{M},2}$ is called the *Weil representation* of G^J associated with $\mathcal{M}$.

4. The Schrödinger-Weil Representation

Let $\mathscr{W}_{\mathcal{M}}$ be the Schrödinger representation of $H_{\mathbb{R}}^{(n,m)}$ defined by (3.1) in Section 3. The symplectic group $G = Sp(n,\mathbb{R})$ acts on $H_{\mathbb{R}}^{(n,m)}$ by conjugation inside G^J. We fix an element $g \in G$. We consider the unitary representation $\mathscr{W}^g_{\mathcal{M}}$ of $H_{\mathbb{R}}^{(n,m)}$ defined by

$$\mathscr{W}^g_{\mathcal{M}}(h) = \mathscr{W}_{\mathcal{M}}(ghg^{-1}), \qquad h \in H_{\mathbb{R}}^{(n,m)}. \tag{4.1}$$

Since $\mathscr{W}^g_{\mathcal{M}}\big((0,0;\kappa)\big) = \mathscr{W}_{\mathcal{M}}\big((0,0;\kappa)\big) = e^{2\pi i \sigma(\mathcal{M}\kappa)} I_{H_{\mathcal{M}}}$ for all $\kappa \in S(m)$, according to Stone-von Neumann theorem, $\mathscr{W}^g_{\mathcal{M}}$ is unitarily equivalent to $\mathscr{W}_{\mathcal{M}}$. Thus there exists a unitary operator $R_{\mathcal{M}}(g)$ of $H_{\mathcal{M}}$ satisfying the commutation relation $R_{\mathcal{M}}(g)\,\mathscr{W}_{\mathcal{M}}(h) = \mathscr{W}^g_{\mathcal{M}}(h)\,R_{\mathcal{M}}(g)$ for all $h \in H_{\mathbb{R}}^{(n,m)}$. We observe that $R_{\mathcal{M}}$ is determined uniquely up to a scalar of modulus one. According to Schur's lemma, we have a map $c_{\mathcal{M}} : G \times G \longrightarrow T$ satisfying the relation

$$R_{\mathcal{M}}(g_1g_2) = c_{\mathcal{M}}(g_1,g_2)\,R_{\mathcal{M}}(g_1)R_{\mathcal{M}}(g_2), \quad g_1,g_2 \in G. \tag{4.2}$$

Therefore $R_{\mathcal{M}}$ is a projective representation of G and $c_{\mathcal{M}}$ defines the cocycle class in $H^2(G,T)$. The cocycle $c_{\mathcal{M}}$ gives rise to the central extension $G_{\mathcal{M}}$ of G by T. The extension group $G_{\mathcal{M}}$ is the set $G \times T$ with the following group multiplication law:

$$(g_1,t_1)\cdot(g_2,t_2) = \big(g_1g_2, t_1t_2\,c_{\mathcal{M}}(g_1,g_2)^{-1}\big), \quad g_1,g_2 \in G,\ t_1,t_2 \in T. \tag{4.3}$$

We see that the map $\widetilde{R}_{\mathcal{M}} : G_{\mathcal{M}} \longrightarrow U(H_{\mathcal{M}})$ defined by

$$\widetilde{R}_{\mathcal{M}}(g,t) = t\,R_{\mathcal{M}}(g), \quad (g,t) \in G_{\mathcal{M}} \tag{4.4}$$

is a true representation of $G_{\mathcal{M}}$. For the Lagrangian subspace $\mathfrak{l} = \big\{(0,\mu) \in V \mid \mu \in \mathbb{R}^{(m,n)} \big\}$, as (2.8) and (2.9) in Section 2, we can define the function $s_{\mathcal{M}} : G \longrightarrow T$ satisfying the relation

$$c_{\mathcal{M}}(g_1,g_2)^2 = s_{\mathcal{M}}(g_1)^{-1}s_{\mathcal{M}}(g_2)^{-1}s_{\mathcal{M}}(g_1g_2) \quad \text{for all } g_1,g_2 \in G. \tag{4.5}$$

Hence we see that

$$G_{2,\mathcal{M}} = \left\{ (g,t) \in G_{\mathcal{M}} \mid t^2 = s_{\mathcal{M}}(g)^{-1} \right\} \tag{4.6}$$

is the metaplectic group associated with $\mathcal{M} \in S(m)$ that is a two-fold covering group of G. The restriction $R_{2,\mathcal{M}}$ of $\widetilde{R}_{\mathcal{M}}$ to $G_{2,\mathcal{M}}$ is the Weil representation of G associated with $\mathcal{M} \in S(m)$. Now we define the projective representation $\pi_{\mathcal{M}}$ of G^J by

$$\pi_{\mathcal{M}}(hg) := \mathscr{W}_{\mathcal{M}}(h)\,R_{\mathcal{M}}(g), \quad h \in H_{\mathbb{R}}^{(n,m)},\ g \in G. \tag{4.7}$$

We observe that any element $\widetilde{g}$ of G^J can be expressed in the form $\widetilde{g} = hg$ with $h \in H_{\mathbb{R}}^{(n,m)}$ and $g \in G$. Indeed, if $g, g_1 \in G$ and $h, h_1 \in H_{\mathbb{R}}^{(n,m)}$, then we have

$$\begin{aligned}
\pi_{\mathcal{M}}(hgh_1g_1) &= \pi_{\mathcal{M}}(hgh_1g^{-1}gg_1) \\
&= \mathscr{W}_{\mathcal{M}}(hgh_1g^{-1})\, R_{\mathcal{M}}(gg_1) \\
&= c_{\mathcal{M}}(g, g_1)\, \mathscr{W}_{\mathcal{M}}(h)\, \mathscr{W}_{\mathcal{M}}(gh_1g^{-1})\, R_{\mathcal{M}}(g) R_{\mathcal{M}}(g_1) \\
&= c_{\mathcal{M}}(g, g_1)\, \mathscr{W}_{\mathcal{M}}(h)\, \mathscr{W}_{\mathcal{M}}^{g}(h_1)\, R_{\mathcal{M}}(g) R_{\mathcal{M}}(g_1) \\
&= c_{\mathcal{M}}(g, g_1)\, \mathscr{W}_{\mathcal{M}}(h)\, R_{\mathcal{M}}(g)\, \mathscr{W}_{\mathcal{M}}(h_1)\, R_{\mathcal{M}}(g_1) \\
&= c_{\mathcal{M}}(g, g_1)\, \pi_{\mathcal{M}}(hg)\, \pi_{\mathcal{M}}(h_1g_1).
\end{aligned}$$

In the second equality, we used the fact that $H_{\mathbb{R}}^{(n,m)}$ is a normal subgroup of G^J. Therefore we get the relation

$$\pi_{\mathcal{M}}(hgh_1g_1) = c_{\mathcal{M}}(g, g_1)\, \pi_{\mathcal{M}}(hg)\, \pi_{\mathcal{M}}(h_1g_1) \tag{4.8}$$

for all $g, g_1 \in G$ and $h, h_1 \in H_{\mathbb{R}}^{(n,m)}$. From (4.8) we obtain the relation

$$T_{\mathcal{M}}(g) = R_{\mathcal{M}}(g), \quad \tilde{c}_{\mathcal{M}}(g, g') = c_{\mathcal{M}}(g, g') \qquad \text{for all } g, g' \in G. \tag{4.9}$$

Thus the representation $\pi_{\mathcal{M}}$ of G^J is naturally extended to the true representation $\omega_{\rm SW}^{\mathcal{M}}$ of $G_{2,\mathcal{M}}^J := G_{2,\mathcal{M}} \ltimes H_{\mathbb{R}}^{(n,m)}$. The representation $\omega_{\rm SW}^{\mathcal{M}}$ is called *Schrödinger-Weil representation* of the Jacobi group G^J associated with $\mathcal{M} \in S(m)$. Indeed we have

$$\omega_{\rm SW}^{\mathcal{M}}(h\cdot(g,t)) = t\, \pi_{\mathcal{M}}(hg), \quad h \in H_{\mathbb{R}}^{(n,m)},\ (g,t) \in G_{2,\mathcal{M}}. \tag{4.10}$$

We recall that the following matrices

$$\begin{aligned}
t(b) : &= \begin{pmatrix} I_n & b \\ 0 & I_n \end{pmatrix} \text{ with any } b = {}^t b \in \mathbb{R}^{(n,n)}, \\
g(\alpha) : &= \begin{pmatrix} {}^t\alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \text{ with any } \alpha \in GL(n, \mathbb{R}), \\
\sigma_n : &= \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}
\end{aligned}$$

generate the symplectic group $G = Sp(n, \mathbb{R})$ (cf. [8, p. 326], [24, p. 210]).

The Weil representation $R_{2,\mathcal{M}}$ is realized on the Hilbert space $L^2\big(\mathbb{R}^{(m,n)}\big)$ (cf. [42], [15]) :

$$\big(R_{\mathcal{M}}(t(b)f\big)(x) = e^{2\pi i \sigma(\mathcal{M}\, x\, b\, {}^t x)} f(x), \quad b = {}^t b \in \mathbb{R}^{(n,n)}\,; \tag{4.11}$$

$$\big(R_{\mathcal{M}}(g(\alpha)f\big)(x) = (\det \alpha)^{\frac{m}{2}} f(x\, {}^t\alpha), \quad \alpha \in GL(n, \mathbb{R}), \tag{4.12}$$

$$\big(R_{\mathcal{M}}(\sigma_n)f\big)(x) = \left(\frac{1}{i}\right)^{\frac{mn}{2}} (\det \mathcal{M})^{\frac{n}{2}} \int_{\mathbb{R}^{(m,n)}} e^{-4\pi i \sigma(\mathcal{M}\, y\, {}^t x)} f(y) dy. \tag{4.13}$$

According to Formulas (4.11)-(4.13), $R_{2,\mathcal{M}}$ is decomposed into two irreducible representations $R^{\pm}_{2,\mathcal{M}}$

$$R_{2,\mathcal{M}} = R^{+}_{2,\mathcal{M}} \oplus R^{-}_{2,\mathcal{M}}, \tag{4.14}$$

where $R^{+}_{2,\mathcal{M}}$ and $R^{-}_{2,\mathcal{M}}$ are the even Weil representation and the odd Weil representation respectively. Obviously the center $\mathscr{Z}^{J}_{2,\mathcal{M}}$ of $G^{J}_{2,\mathcal{M}}$ is given by

$$\mathscr{Z}^{J}_{2,\mathcal{M}} = \left\{ \big((I_{2n},1),(0,0;\kappa) \big) \in G^{J}_{2,\mathcal{M}} \right\} \cong S(m).$$

We note that $\omega^{\mathcal{M}}_{\rm SW}|_{G_{2,\mathcal{M}}} = R_{2,\mathcal{M}}$ and $\omega^{\mathcal{M}}_{\rm SW}(h) = \mathscr{W}_{\mathcal{M}}(h)$ for all $h \in H^{(n,m)}_{\mathbb{R}}$.

5. The Weil-Satake Representation

In this section we discuss the realization of the Weil representation on the Fock model and the Weil-Satake representation due to Satake (cf. [31]). We follow the notations in Section 3 and Section 4. For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$, we set

$$J(g,\Omega) = C\Omega + D, \quad \Omega \in \mathbb{H}_n. \tag{5.1}$$

Let $\mathcal{M}$ be an $m\times m$ symmetric real matrix. We define the map $J_{\mathcal{M}} : G^J \times \mathbb{H}_{n,m} \longrightarrow \mathbb{C}^*$ by

$$J_{\mathcal{M}}\big(\widetilde{g},(\Omega,Z)\big) := e^{2\pi i\,\sigma\left(\mathcal{M}[Z+\lambda\Omega+\mu](C\Omega+D)^{-1}C\right)} \cdot e^{-2\pi i\,\sigma\left(\mathcal{M}(\lambda\Omega\,{}^t\lambda+2\lambda\,{}^tZ+\kappa+\mu\,{}^t\lambda)\right)}, \tag{5.2}$$

where $\widetilde{g} = (g,(\lambda,\mu;\kappa)) \in G^J$ with $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ and $(\lambda,\mu;\kappa) \in H^{(n,m)}_{\mathbb{R}}$. Here $M[N] := {}^tNMN$ is a Siegel's notation for two matrices M and N. The $J_{\mathcal{M}}$ satisfies the cocycle condition

$$J_{\mathcal{M}}(\widetilde{g}_1\,\widetilde{g}_2,(\Omega,Z)) = J_{\mathcal{M}}(\widetilde{g}_1,\widetilde{g}_2\cdot(\Omega,Z))\,J_{\mathcal{M}}(\widetilde{g}_2,(\Omega,Z))$$

for all $\widetilde{g}_1,\widetilde{g}_2 \in G^J$ and $(\Omega,Z) \in \mathbb{H}_{n,m}$. We refer [31] and [52] for a construction of $J_{\mathcal{M}}$.

We introduce the coordinates (Ω,Z) on $\mathbb{H}_{n,m}$ and some notations.

$$\begin{aligned} \Omega &= X + iY, \quad X = (x_{\mu\nu}), \quad Y = (y_{\mu\nu}) \ \text{ real}, \\ Z &= U + iV, \quad U = (u_{kl}), \quad V = (v_{kl}) \ \text{ real}, \end{aligned}$$

$$[dX] = \bigwedge_{\mu\le\nu} dx_{\mu\nu}, \quad [dY] = \bigwedge_{\mu\le\nu} dy_{\mu\nu},$$

$$[dU] = \bigwedge_{k,l} du_{kl}, \quad [dV] = \bigwedge_{k,l} dv_{kl}.$$

Now we assume that $\mathcal{M}$ is *positive definite*. We define the function $\kappa_{\mathcal{M}} : \mathbb{H}_{n,m} \longrightarrow \mathbb{R}$ by

$$\kappa_{\mathcal{M}}(\Omega,Z) := e^{-4\pi\,\sigma({}^tV\mathcal{M}VY^{-1})}. \tag{5.3}$$

We fix an element Ω in $\mathbb{H}_n$. We let $H_{\mathcal{M},\Omega}$ be the complex Hilbert space consisting of all complex valued holomorphic functions f on $\mathbb{C}^{(m,n)}$ such that

$$\int_{\mathbb{C}^{(m,n)}} |f(Z)|^2\, d\nu_{\mathcal{M},\Omega}(Z) < \infty,$$

where

$$d\nu_{\mathcal{M},\Omega}(Z) = (\det 2\mathcal{M})^n (\det \mathrm{Im}\,\Omega)^{-m}\, \kappa_{\mathcal{M}}(\Omega, Z)\, [dU] \wedge [dV].$$

We define an irreducible unitary representation $\mathscr{U}_{\mathcal{M},\Omega}$ of $H_{\mathbb{R}}^{(n,m)}$ on $H_{\mathcal{M},\Omega}$ by

$$\big(\mathscr{U}_{\mathcal{M},\Omega}(h)f\big)(Z) := J_{\mathcal{M}}\big(h^{-1}, (\Omega, Z)\big)^{-1} f(Z - \lambda\Omega - \mu), \tag{5.4}$$

where $h = (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)}$, $f \in H_{\mathcal{M},\Omega}$ and $Z \in \mathbb{C}^{(m,n)}$. It is known that for any two elements Ω_1 and Ω_2 of $\mathbb{H}_n$, $\mathscr{U}_{\mathcal{M},\Omega_1}$ is equivalent to $\mathscr{U}_{\mathcal{M},\Omega_1}$ (cf. [31]). Therefore $\mathscr{U}_{\mathcal{M},\Omega}$ is called the *Fock representation* of $H_{\mathbb{R}}^{(n,m)}$ associated with $\mathcal{M}$. Clearly $\mathscr{U}_{\mathcal{M},\Omega}((0,0;\kappa)) = e^{-2\pi i\, \sigma(\mathcal{M}\kappa)}$. According to Stone-von Neumann theorem, $\mathscr{U}_{\mathcal{M},\Omega}$ is equivalent to $\mathscr{W}_{-\mathcal{M}}$ (cf. Formula (3.1)). Since the representation $\mathscr{U}_{\mathcal{M},\Omega}^g$ $(g \in G)$ of $H_{\mathbb{R}}^{(n,m)}$ defined by $\mathscr{U}_{\mathcal{M},\Omega}^g(h) = \mathscr{U}_{\mathcal{M},\Omega}(ghg^{-1})$ is equivalent to $\mathscr{U}_{\mathcal{M},\Omega}$, there exists a unitary operator $U_{\mathcal{M},\Omega}(g)$ of $H_{\mathcal{M},\Omega}$ such that $U_{\mathcal{M},\Omega}(g)\mathscr{U}_{\mathcal{M},\Omega}(h) = \mathscr{U}_{\mathcal{M},\Omega}^g(h) U_{\mathcal{M},\Omega}(g)$ for all $h \in H_{\mathbb{R}}^{(n,m)}$. Thus we obtain a projective representation $U_{\mathcal{M},\Omega}$ of G on $H_{\mathcal{M},\Omega}$ and a cocycle $\widehat{c}_{\mathcal{M},\Omega}$: $G \times G \longrightarrow T$ satisfying the condition

$$U_{\mathcal{M},\Omega}(g_1 g_2) = \widehat{c}_{\mathcal{M},\Omega}(g_1, g_2)\, U_{\mathcal{M},\Omega}(g_1)\, U_{\mathcal{M},\Omega}(g_2), \quad g_1, g_2 \in G.$$

Now $\widehat{c}_{\mathcal{M},\Omega}$ and $U_{\mathcal{M},\Omega}(g)$ will be determined explicitly (cf. [31], [36]). In fact,

$$\widehat{c}_{\mathcal{M},\Omega}(g_1, g_2) = \left(\frac{\gamma(g_2^{-1} g_1^{-1}\Omega, g_2^{-1}\Omega)}{\gamma(g_1^{-1}\Omega, \Omega)} \right)^m, \tag{5.5}$$

where

$$\gamma(\Omega_1, \Omega_2) := \left(\det \left(\frac{\Omega_1 - \overline{\Omega}_2}{2i} \right) \right)^{-\frac{1}{2}} (\det \mathrm{Im}\, \Omega_1)^{\frac{1}{4}}\, (\det \mathrm{Im}\, \Omega_2)^{\frac{1}{4}}, \quad \Omega_1, \Omega_2 \in \mathbb{H}_n.$$

We define the projective representation $\tau_{\mathcal{M},\Omega}$ of G^J by

$$\tau_{\mathcal{M},\Omega}(hg) := \mathscr{U}_{\mathcal{M},\Omega}(h)\, U_{\mathcal{M},\Omega}(g) \quad \text{for all } h \in H_{\mathbb{R}}^{(n,m)},\ g \in G. \tag{5.6}$$

Then $\tau_{\mathcal{M},\Omega}$ satisfies the following relation

$$\tau_{\mathcal{M},\Omega}(\widetilde{g}_1\, \widetilde{g}_2) = \widehat{c}_{\mathcal{M},\Omega}(g_1, g_2)\, \tau_{\mathcal{M},\Omega}(\widetilde{g}_1)\, \tau_{\mathcal{M},\Omega}(\widetilde{g}_2) \tag{5.7}$$

for all $\widetilde{g}_1 = (g_1, h_1)$, $\widetilde{g}_2 = (g_2, h_2) \in G^J$ with $g_1, g_2 \in G$ and $h_1, h_2 \in H_{\mathbb{R}}^{(n,m)}$.

We put

$$\beta_\Omega(g_1, g_2) := \widehat{c}_{\mathcal{M},\Omega}\big(g_1, g_2\big)^{\frac{1}{m}}, \quad g_1, g_2 \in G. \tag{5.8}$$

Then β_Ω satisfies the cocycle condition and the following relation

$$\beta_\Omega(g_1, g_2)^2 = \widehat{s}_\Omega(g_1)^{-1}\, \widehat{s}_\Omega(g_2)^{-1}\, \widehat{s}_\Omega(g_1 g_2), \quad g_1, g_2 \in G,$$

where

$$\widehat{s}_\Omega(g) = \,|\det J(g^{-1},\Omega)|^{-1}\,\left(\det J(g^{-1},\Omega)\right),\quad g\in G.$$

The cocycle class $[\beta_\Omega]$ in $H^2(G,T)$ defines the central extension $G_\Omega = G\times T$ of G by T with the following multiplication law

$$(g_1,t_1)\cdot(g_2,t_2) = \left(g_1g_2,t_1t_2\,\beta_\Omega(g_1,g_2)^{-1}\right).$$

We obtain a normal closed subgroup $G_{2,\Omega}$ of G_Ω given by

$$G_{2,\Omega} = \left\{\,(g,t)\in G_\Omega\,|\,t^2 = \,\widehat{s}_\Omega(g)^{-1}\,\right\}. \tag{5.9}$$

We can show that $G_{2,\Omega}$ is a two-fold covering group of G. We set for any $g\in G$ and $\Omega_1,\Omega_2\in\mathbb{H}_n$,

$$\varepsilon(g;\Omega_1,\Omega_2) := \frac{\gamma(g\cdot\Omega_1,g\cdot\Omega_2)}{\gamma(\Omega_1,\Omega_2)}. \tag{5.10}$$

We can see that for any element $g\in G$ and $\Omega\in\mathbb{H}_n$, the topological group $G_{2,\Omega}$ is isomorphic to $G_{2,g\cdot\Omega}$ via the correspondence

$$(g_0,t_0)\mapsto\left(g_0,t_0\,\varepsilon(g_0^{-1};g\cdot\Omega,\Omega)\right),\quad (g_0,t_0)\in G_{2,\Omega}.$$

Therefore it is enough to consider only the case $\Omega = iI_n$. We set $G_2 := G_{2,\,iI_n}$. We let

$$G_2^J := G_2\ltimes H_{\mathbb{R}}^{(n,m)}$$

be the two-fold covering group of G^J endowed with the multiplication law

$$\begin{aligned}&\Big((g,t),(\lambda,\mu;\kappa)\Big)\cdot\Big((g',t'),(\lambda',\mu';\kappa')\Big)\\ =\ &\Big((g,t)\cdot(g',t'),(\tilde\lambda+\lambda',\tilde\mu+\mu';\kappa+\kappa'+\tilde\lambda\,{}^t\mu'-\tilde\mu\,{}^t\lambda')\Big)\end{aligned}$$

with $(g,t),(g',t')\in G_2$, $(\lambda,\mu;\kappa),(\lambda',\mu';\kappa')\in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde\lambda,\tilde\mu) = (\lambda,\mu)g'$.

We note that any element $\widetilde\sigma$ of G_2^J can be written in the form $\widetilde\sigma = h(g,t)$ with $h\in H_{\mathbb{R}}^{(n,m)}$ and $(g,t)\in G_2$. We define a unitary representation $\widehat\omega_{\mathcal{M}} := \widehat\omega_{\mathcal{M},iI_n}$ of G_2^J by

$$\widehat\omega_{\mathcal{M}}(h(g,t)) := \,t^m\,\tau_{\mathcal{M},\,iI_n}(hg),\qquad h\in H_{\mathbb{R}}^{(n,m)},\ (g,t)\in G_2. \tag{5.11}$$

In fact, if $h, h_1 \in H_{\mathbb{R}}^{(n,m)}$ and $(g,t), (g_1,t_1) \in G_2$, then we obtain

$$\begin{aligned}
& \widehat{\omega}_{\mathcal{M}}\big(h(g,t)h_1(g_1,t_1)\big) \\
= \ & \widehat{\omega}_{\mathcal{M}}\big(h(g,t)h_1(g,t)^{-1}(g,t)(g_1,t_1)\big) \\
= \ & \widehat{\omega}_{\mathcal{M}}\big(h(g,t)h_1(g,t)^{-1}(gg_1, tt_1\beta_{iI_n}(g,g_1)^{-1})\big) \\
= \ & (tt_1)^m \beta_{iI_n}(g,g_1)^{-m} \tau_{\mathcal{M},iI_n}\big(h(g,t)h_1(g,t)^{-1}gg_1\big) \\
= \ & (tt_1)^m \beta_{iI_n}(g,g_1)^{-m} \mathscr{U}_{\mathcal{M},iI_n}\big(h(g,t)h_1(g,t)^{-1}\big) U_{\mathcal{M},iI_n}(gg_1) \\
= \ & (tt_1)^m \mathscr{U}_{\mathcal{M},iI_n}(h) \mathscr{U}^g_{\mathcal{M},iI_n}(h_1)\, U_{\mathcal{M},iI_n}(g)\, U_{\mathcal{M},iI_n}(g_1) \\
= \ & t^m t_1^m\, \mathscr{U}_{\mathcal{M},iI_n}(h)\, U_{\mathcal{M},iI_n}(g) \mathscr{U}_{\mathcal{M},iI_n}(h_1)\, U_{\mathcal{M},iI_n}(g_1) \\
= \ & t^m t_1^m\, \tau_{\mathcal{M},\,iI_n}(hg)\, \tau_{\mathcal{M},\,iI_n}(h_1 g_1) \\
= \ & \widehat{\omega}_{\mathcal{M}}\big(h(g,t)\big)\, \widehat{\omega}_{\mathcal{M}}\big(h_1(g_1,t_1)\big).
\end{aligned}$$

$\widehat{\omega}_{\mathcal{M}}$ is called the *Weil-Satake representation* of G^J associated with $\mathcal{M}$. In Section 8, we discuss some applications of the Weil-Satake representation $\widehat{\omega}_{\mathcal{M}}$ to the study of unitary representations of G^J.

6. **Jacobi Forms**

Let ρ be a rational representation of $GL(n,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbb{R}^{(m,m)}$ be a symmetric half-integral semi-positive definite matrix of degree m. Let $C^\infty(\mathbb{H}_{n,m}, V_\rho)$ be the algebra of all C^∞ functions on $\mathbb{H}_{n,m}$ with values in V_ρ. For $f \in C^\infty(\mathbb{H}_{n,m}, V_\rho)$, we define

$$\begin{aligned}
& (f|_{\rho,\mathcal{M}}[(g,(\lambda,\mu;\kappa))])(\Omega,Z) \\
& := e^{-2\pi i\sigma\left(\mathcal{M}(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}C\,{}^t(Z+\lambda\,\Omega+\,\mu)\right)} \times e^{2\pi i\sigma\left(\mathcal{M}(\lambda\,\Omega\,{}^t\lambda+\,2\,\lambda\,{}^tZ+\kappa+\mu\,{}^t\lambda)\right)} \\
& \quad \times \rho(C\Omega+D)^{-1} f(g\cdot\Omega, (Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}),
\end{aligned} \tag{6.1}$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$, $(\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$.

Definition 6.1. *Let ρ and $\mathcal{M}$ be as above. Let*

$$H_{\mathbb{Z}}^{(n,m)} := \left\{ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)} \,|\, \lambda,\mu \in \mathbb{Z}^{(m,n)},\ \kappa \in \mathbb{Z}^{(m,m)} \ \right\}.$$

A Jacobi form of index $\mathcal{M}$ with respect to ρ on a subgroup Γ of Γ_n of finite index is a holomorphic function $f \in C^\infty(\mathbb{H}_{n,m}, V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho,\mathcal{M}}[\tilde{\gamma}] = f$ *for all* $\tilde{\gamma} \in \widetilde{\Gamma} := \Gamma \ltimes H_{\mathbb{Z}}^{(n,m)}$.

(B) For each $M \in \Gamma_n$, $f|_{\rho,\mathcal{M}}[M]$ has a Fourier expansion of the following form :

$$\big(f|_{\rho,\mathcal{M}}[M]\big)(\Omega, Z) = \sum_{\substack{T={}^tT\geq 0\\ \textit{half-integral}}} \sum_{R\in\mathbb{Z}^{(n,m)}} c(T,R)\cdot e^{\frac{2\pi i}{\lambda_\Gamma}\sigma(T\Omega)}\cdot e^{2\pi i\sigma(RZ)}$$

with a suitable $\lambda_\Gamma \in \mathbb{Z}$ and $c(T,R) \neq 0$ only if $\begin{pmatrix} \frac{1}{\lambda_\Gamma}T & \frac12 R \\ \frac12 {}^tR & \mathcal{M}\end{pmatrix} \geqq 0.$

If $n \geq 2$, the condition (B) is superfluous by Köcher principle (cf. [63] Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ. Ziegler (cf. [63] Theorem 1.8 or [7] Theorem 1.1) proves that the vector space $J_{\rho,\mathcal{M}}(\Gamma)$ is finite dimensional. In the special case $\rho(A) = (\det(A))^k$ with $A \in GL(n,\mathbb{C})$ and a fixed $k \in \mathbb{Z}$, we write $J_{k,\mathcal{M}}(\Gamma)$ instead of $J_{\rho,\mathcal{M}}(\Gamma)$ and call k the *weight* of the corresponding Jacobi forms. For more results on Jacobi forms with $n > 1$ and $m > 1$, we refer to [49]-[53] and [63]. Jacobi forms play an important role in elliptic cusp forms to Siegel cusp forms of degree $2n$ (cf. [14]).

Definition 6.2. *A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma)$ is said to be a cusp (or cuspidal) form if* $\begin{pmatrix} \frac{1}{\lambda_\Gamma}T & \frac12 R \\ \frac12 {}^tR & \mathcal{M}\end{pmatrix} > 0$ *for any T, R with $c(T,R) \neq 0$. A Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma)$ is said to be singular if it admits a Fourier expansion such that a Fourier coefficient $c(T,R)$ vanishes unless det* $\begin{pmatrix} \frac{1}{\lambda_\Gamma}T & \frac12 R \\ \frac12 {}^tR & \mathcal{M}\end{pmatrix} = 0.$

Singular Jacobi forms were characterized by a certain differential operator and the weight by the author [51].

Without loss of generality we may assume that ρ is irreducible. Then we choose a hermitian inner product $\langle\ ,\ \rangle$ on V_ρ that is preserved under the unitary group $U(n) \subset GL(n,\mathbb{C})$. For two Jacobi forms f_1 and f_2 in $J_{\rho,\mathcal{M}}(\Gamma)$, we define the Petersson inner product formally by

$$\langle f_1, f_2\rangle := \int_{\Gamma_{n,m}\backslash\mathbb{H}_{n,m}} \langle\, \rho(Y^{\frac12})f_1(\Omega,Z), \rho(Y^{\frac12})f_2(\Omega,Z)\rangle\, \kappa_{\mathcal{M}}(\Omega,Z)\, dv, \tag{6.2}$$

where

$$dv = (\det Y)^{-(n+m+1)}[dX]\wedge[dY]\wedge[dU]\wedge[dV] \tag{6.3}$$

is a G^J-invariant volume element on $\mathbb{H}_{n,m}$. See (5.3) for the definition of $\kappa_{\mathcal{M}}(\Omega,Z)$. A Jacobi form f in $J_{\rho,\mathcal{M}}(\Gamma)$ is said to be square integrable if $\langle f,f\rangle < \infty$. We note that cusp Jacobi forms are square integrable and that $\langle f_1,f_2\rangle$ is finite if one of f_1 and f_2 is a cusp Jacobi form (cf. [63], p. 203).

We define the map $J_{\rho,\mathcal{M}} : G^J \times \mathbb{H}_{n,m} \longrightarrow GL(V_\rho)$ by

$$J_{\rho,\mathcal{M}}(\widetilde{g},(\Omega,Z)) = J_{\mathcal{M}}(\widetilde{g},(\Omega,Z))\,\rho(J(g,\Omega)) \qquad \text{(cf. (5.1) and (5.2))},$$

where $\widetilde{g} = (g, h) \in G^J$ with $g \in G$ and $h \in H_{\mathbb{R}}^{(n,m)}$. For a function f on $\mathbb{H}_n$ with values in V_ρ, we can lift f to a function Φ_f on G^J :

$$\begin{aligned}\Phi_f(\sigma): &= (f|_{\rho,\mathcal{M}}[\sigma])(iI_n, 0)\\ &= J_{\rho,\mathcal{M}}(\sigma, (iI_n, 0))^{-1} f(\sigma\cdot(iI_n, 0)), \quad \sigma \in G^J.\end{aligned}$$

A characterization of Φ_f for a cusp Jacobi form f in $J_{\rho,\mathcal{M}}(\Gamma)$ was given by Takase [36, pp. 162–164] and the author [54, pp. 252–254].

We allow a weight k to be half-integral. For brevity, we set $G = Sp(n, \mathbb{R})$. For any $g \in G$ and $\Omega, \Omega' \in \mathbb{H}_n$, we note that

$$\begin{aligned}(6.4)\quad \varepsilon(g; \Omega', \Omega) &= \det{}^{-\frac{1}{2}}\left(\frac{g\cdot\Omega' - \overline{g\cdot\Omega}}{2\,i}\right)\det{}^{\frac{1}{2}}\left(\frac{\Omega' - \overline{\Omega}}{2\,i}\right)\\ &\quad\times|\det J(g, \Omega')|^{-1/2}|\det J(g, \Omega)|^{-1/2}. \qquad \text{(cf. (5.10))}\end{aligned}$$

Here $J(g, \Omega) = C\Omega + D$ for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ (cf. (5.1)).

Let $\mathscr{S} = \left\{ S \in \mathbb{C}^{(n,n)} \mid S = {}^tS,\ \mathrm{Re}(S) > 0 \right\}$ be a connected simply connected complex manifold. Then there is a uniquely determined holomorphic function $\det^{\frac{1}{2}}$ on $\mathscr{S}$ such that

$$(6.5)\qquad \left(\det{}^{1/2} S\right)^2 = \det S \qquad \text{for all } S \in \mathscr{S}.$$

$$(6.6)\qquad \det{}^{1/2} S = (\det S)^{1/2} \qquad \text{for all } S \in \mathscr{S} \cap \mathbb{R}^{(n,n)}.$$

For each integer $k \in \mathbb{Z}$ and $S \in \mathscr{S}$, we put

$$\det{}^{k/2} S = \left(\det{}^{1/2} S\right)^k.$$

For each $\Omega \in \mathbb{H}_n$, we define the function $\beta_\Omega : G \times G \longrightarrow T$ by

$$(6.7)\qquad \beta_\Omega(g_1, g_2) = \epsilon(g_1; \Omega, g_2(\Omega)), \qquad g_1, g_2 \in G.$$

Then β_Ω satisfies the cocycle condition and the cohomology class of β_Ω of order two ;

$$(6.8)\qquad \beta_\Omega(g_1, g_2)^2 = \alpha_\Omega(g_2)\,\alpha_\Omega(g_1 g_2)^{-1}\alpha_\Omega(g_1),$$

where

$$(6.9)\qquad \alpha_\Omega(g) = \frac{\det J(g, \Omega)}{|\det J(g, \Omega)|}, \qquad g \in G,\ \Omega \in \mathbb{H}_n.$$

For any $\Omega \in \mathbb{H}_n$, we let

$$G_\Omega = \left\{ (g, \epsilon) \in G \times T \mid \epsilon^2 = \alpha_\Omega(g)^{-1} \right\}$$

be the two-fold covering group with multiplication law

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = \left(g_1 g_2, \epsilon_1\,\epsilon_2 \beta_\Omega(g_1, g_2)\right).$$

The covering group G_Ω depends on the choice of $\Omega \in \mathbb{H}_n$, i.e., the choice of a maximal compact subgroup of G. However for any two elements $\Omega_1, \Omega_2 \in \mathbb{H}_n$, G_{Ω_1} is isomorphic to G_{Ω_2} (cf. [38]). We put $G_* := G_{iI_n}$.

We define the automorphic factor $J_{1/2} : G_* \times \mathbb{H}_n \longrightarrow \mathbb{C}^*$ by

$$J_{1/2}(g_\epsilon, \Omega) := \epsilon^{-1}\, \varepsilon(g; \Omega, iI_n)\, |\det J(g,\Omega)|^{1/2}, \tag{6.10}$$

where $g_\epsilon = (g, \epsilon) \in G_\Omega$ with $g \in G$ and $\Omega \in \mathbb{H}_n$. It is easily checked that

$$J_{1/2}(g_* h_*, \Omega) = J_{1/2}(g_*, h\cdot\Omega) J_{1/2}(h_*, \Omega) \tag{6.11}$$

for all $g_* = (g, \epsilon)$, $h_* = (h, \eta) \in G_*$ and $\Omega \in \mathbb{H}_n$. and

$$J_{1/2}(g_*, \Omega)^2 = \det(C\Omega + D) \tag{6.12}$$

for all $g_* = (g, \epsilon) \in G$ with $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$.

Let $\pi_* : G_* \longrightarrow G$ be the projection defined by $\pi_*(g, \epsilon) = g$. Let Γ be a subgroup of the Siegel modular group Γ_n of finite index. Let $\Gamma_* = \pi_*^{-1}(\Gamma) \subset G_*$. Let χ be a finite order unitary character of Γ_*. Let $k \in \mathbb{Z}^+$ be a positive integer. We say that a holomorphic function $\phi : \mathbb{H}_n \longrightarrow \mathbb{C}^*$ is a Siegel modular form of a half-integral weight $k/2$ with level Γ if it satisfies the condition

$$\phi(\gamma_* \cdot \Omega) = \chi(\gamma_*)\, J_{1/2}(\gamma_*, \Omega)^k \phi(\Omega) \tag{6.13}$$

for all $\gamma_* \in \Gamma_*$ and $\Omega \in \mathbb{H}_n$. We denote by $M_{k/2}(\Gamma, \chi)$ be the vector space of all Siegel modular forms of weight $k/2$ with level Γ. Let $S_{k/2}(\Gamma, \chi)$ be the subspace of $M_{k/2}(\Gamma, \chi)$ consisting of $\phi \in M_{k/2}(\Gamma, \chi)$ such that

$$|\phi(\Omega)|\, \det(\mathrm{Im}\,\Omega)^{k/4} \ \text{ is bounded on } \mathbb{H}_n.$$

An element of $S_{k/2}(\Gamma, \chi)$ is called a Siegel cusp form of weight $k/2$.

Definition 6.3. *Let $\Gamma \subset \Gamma_n$ be a subgroup of finite index. We put $\Gamma_* = \pi_*^{-1}(\Gamma)$ and*

$$\widetilde{\Gamma}_* = \Gamma_* \ltimes H_{\mathbb{Z}}^{(n,m)}.$$

A holomorphic function $f : \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is said to be a Jacobi form of a weight $k/2 \in \frac{1}{2}\mathbb{Z}$ (k : odd) with level Γ and index $\mathcal{M}$ for the character χ of Γ_* of if it satisfies the following transformation formula

$$f(\widetilde{\gamma}_* \cdot (\Omega, Z)) = \chi(\gamma_*)\, J_{k,\mathcal{M}}(\widetilde{\gamma}_*, (\Omega, Z)) f(\Omega, Z) \quad \textit{for all } \widetilde{\gamma}_* \in \widetilde{\Gamma}_* \tag{6.14}$$

where $J_{k,\mathcal{M}} : \widetilde{\Gamma}_* \times \mathbb{H}_{n,m} \longrightarrow \mathbb{C}$ is an automorphic factor defined by

$$\begin{aligned} J_{k,\mathcal{M}}\big(\widetilde{\gamma}_*, (\Omega, Z)\big) := \ & e^{2\pi i \sigma\left(\mathcal{M}(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}C\,{}^t(Z+\lambda\Omega+\mu)\right)} \\ & \times e^{-2\pi i\sigma\left(\mathcal{M}(\lambda\Omega\,{}^t\lambda + 2\lambda\,{}^tZ + \kappa + \mu\,{}^t\lambda)\right)} J_{1/2}(\gamma_*, \Omega)^k, \end{aligned} \tag{6.15}$$

where $\widetilde{\gamma}_* = (\gamma_*, (\lambda, \mu; \kappa)) \in \widetilde{\Gamma}_*$ with $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$, $\gamma_* = (\gamma, \epsilon)$, $(\lambda, \mu, \kappa) \in H_{\mathbb{Z}}^{(n,m)}$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$.

7. Applications of the Schrödinger-Weil Representation

7.1. Construction of Jacobi Forms

We assume that $\mathcal{M}$ is a positive definite symmetric integral matrix of degree m. Let $\omega_{\mathcal{M}}$ be the Schrödinger-Weil representation of G^J constructed in Section 4. We recall that $\omega_{\mathcal{M}}$ is realized on the Hilbert space $L^2\big(\mathbb{R}^{(m,n)}\big)$ by Formulas (4.9)-(4.11). We define the mapping $\mathscr{F}^{(\mathcal{M})}:\mathbb{H}_{n,m}\longrightarrow L^2\big(\mathbb{R}^{(m,n)}\big)$ by

$$\mathscr{F}^{(\mathcal{M})}(\Omega,Z)(x)=\,e^{\pi i\,\sigma\{\mathcal{M}(x\,\Omega\,{}^tx+2\,x\,{}^tZ)\}},\quad (\Omega,Z)\in\mathbb{H}_{n,m},\ x\in\mathbb{R}^{(m,n)}. \tag{7.1}$$

For brevity we put $\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}:=\mathscr{F}^{(\mathcal{M})}(\Omega,Z)$ for $(\Omega,Z)\in\mathbb{H}_{n,m}$. We put

$$G^J_*:=G_*\ltimes H^{(n,m)}_{\mathbb{R}}.$$

We observe that G^J_* acts on $\mathbb{H}_{n,m}$ through the natural projection of G^J_* onto G^J. Let $J^*_{\mathcal{M}}:G^J_*\times\mathbb{H}_{n,m}\longrightarrow\mathbb{C}^\times$ be an automorphic factor for G^J_* on $\mathbb{H}_{n,m}$ defined by

$$\begin{aligned}J^*_{\mathcal{M}}(\widetilde{g},(\Omega,Z))\ &=e^{\pi i\,\sigma\left(\mathcal{M}(Z+\lambda\,\Omega+\mu)(C\Omega+D)^{-1}C\,{}^t(Z+\lambda\,\Omega+\mu)\right)}\\ &\qquad\times e^{-\pi i\,\sigma\left(\mathcal{M}(\lambda\,\Omega\,{}^t\lambda+2\,\lambda\,{}^tZ+\kappa+\mu\,{}^t\lambda)\right)}J_{1/2}((g,\epsilon),\Omega)^m,\end{aligned} \tag{7.2}$$

where $\widetilde{g}_*=((g,\epsilon),(\lambda,\mu;\kappa))\in G^J_*$ with $g=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa)\in H^{(n,m)}_{\mathbb{R}}$ and $(\Omega,Z)\in\mathbb{H}_{n,m}$.

Theorem 7.1. *Let m be an odd positive integer. The map $\mathscr{F}^{(\mathcal{M})}:\mathbb{H}_{n,m}\longrightarrow L^2\big(\mathbb{R}^{(m,n)}\big)$ defined by (7.1) is a covariant map for the Schrödinger-Weil representation $\omega_{\mathcal{M}}$ of G^J and the automorphic factor $J^*_{\mathcal{M}}$ for G^J_* on $\mathbb{H}_{n,m}$ defined by Formula (7.2). In other words, $\mathscr{F}^{(\mathcal{M})}$ satisfies the following covariance relation*

$$\omega_{\mathcal{M}}(\widetilde{g}_*)\mathscr{F}^{(\mathcal{M})}_{\Omega,Z}=J^*_{\mathcal{M}}\big(\widetilde{g}_*,(\Omega,Z)\big)^{-1}\mathscr{F}^{(\mathcal{M})}_{\widetilde{g}_*\cdot(\Omega,Z)} \tag{7.3}$$

for all $\widetilde{g}_*=((g,\epsilon),(\lambda,\mu;\kappa)\in G^J_*$ and $(\Omega,Z)\in\mathbb{H}_{n,m}$.

Proof. The proof can be found in [61] (cf. [60]). □

For a positive definite integral matrix $\mathcal{M}$ of degree m, we define the holomorphic function $\Theta_{\mathcal{M}}:\mathbb{H}_{n,m}\longrightarrow\mathbb{C}$ by

$$\Theta_{\mathcal{M}}(\Omega,Z)=\sum_{\xi\in\mathbb{Z}^{(m,n)}}e^{\pi\,i\,\sigma\left(\mathcal{M}(\xi\,\Omega\,{}^t\xi+2\,\xi\,{}^tZ)\right)},\quad (\Omega,Z)\in\mathbb{H}_{n,m}. \tag{7.4}$$

We can prove the following theorem.

Theorem 7.2. *The function $\Theta_{\mathcal{M}}$ is a Jacobi form of weight $\frac{m}{2}$ and index $\frac{\mathcal{M}}{2}$ with respect to a discrete subgroup $\Gamma^J_{\mathcal{M},*} := \Gamma_{\mathcal{M},*} \ltimes H^{(n,m)}_{\mathbb{Z}}$ of Γ^J_* for a suitable arithmetic subgroup $\Gamma_{\mathcal{M}}$ of Γ_n with $\Gamma_{\mathcal{M},*} = \pi_*^{-1}(\Gamma_{\mathcal{M}})$. That is, $\Theta_{\mathcal{M}}$ satisfies the functional equation*

$$\Theta_{\mathcal{M}}\big(\widetilde{\gamma}_* \cdot (\Omega, Z)\big) = \rho_{\mathcal{M}}(\gamma_*)\, J^*_{\mathcal{M}}\big(\widetilde{\gamma}_*, (\Omega, Z)\big)\Theta_{\mathcal{M}}(\Omega, Z), \quad (\Omega, Z) \in \mathbb{H}_{n,m}, \tag{7.5}$$

where $\rho_{\mathcal{M}}$ is a suitable character of $\Gamma_{\mathcal{M},*}$ and $\widetilde{\gamma}_* = (\gamma_*, (\lambda, \mu; \kappa)) \in \Gamma^J_{\mathcal{M},*}$.
Proof. The proof can be found in [60] when $\mathcal{M}$ is unimodular and even integral. In the case $\mathcal{M}$ is a symmetric positive integral matrix of odd degree m such that $\det(\mathcal{M}) = 1$ with a special arithmetic subgroup $\Gamma_{\mathcal{M},*}$, the proof can be found in [61]. In a similar way we can prove the above theorem. □

According to Theorem 1 and Theorem 2, we see that the theta series $\Theta_{\mathcal{M}}$ is closely related to the Schrödinger-Weil representation of the Jacobi group G^J. We note that the theta series

$$\Theta(\Omega) = \sum_{A \in \mathbb{Z}^n} e^{\pi i \sigma(A\,\Omega\,{}^tA)}, \qquad \Omega \in \mathbb{H}_n \tag{7.6}$$

is a Siegel modular form of weight $\frac{1}{2}$ with respect to the theta subgroup Γ_Θ of Γ_n, that is, Θ satisfies the following functional equation

$$\Theta(\gamma \cdot \Omega) = \zeta(\gamma)\, (\det(C\Omega + D))^{\frac{1}{2}}\, \Theta(\Omega), \qquad \Omega \in \mathbb{H}_n, \tag{7.7}$$

where $\zeta(\gamma)$ is a character of Γ_Θ with $|\zeta(\gamma)|^8 = 1$ and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_\Theta$. We refer to [24, pp. 189-201] for more detail. Indeed the function $\mathscr{F} : \mathbb{H}_n \longrightarrow L^2\big(\mathbb{R}^n\big)$ defined by

$$\mathscr{F}(\Omega)(x) = e^{\pi i \sigma(x\,\Omega\,{}^tx)}, \qquad \Omega \in \mathbb{H}_n \text{ and } x \in \mathbb{R}^n. \tag{7.8}$$

is a covariant map for the Weil representation ω of $Sp(n, \mathbb{R})$ and the automorphic form $\mathfrak{J}_{\frac{1}{2}} : Sp(n, \mathbb{R}) \times \mathbb{H}_n \longrightarrow \mathbb{C}^\times$ defined by

$$\mathfrak{J}_{\frac{1}{2}}(g, \Omega) = (\det(C\Omega + D))^{\frac{1}{2}}, \quad \Omega \in \mathbb{H}_n \tag{7.9}$$

with $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$. More precisely, if we put $\mathscr{F}_\Omega := \mathscr{F}(\Omega)$ for brevity, the vector valued map $\mathscr{F}$ satisfies the following covariance relation

$$\omega(g)\mathscr{F}_\Omega = (\det(C\Omega + D))^{-\frac{1}{2}}\, \mathscr{F}_{g\cdot\Omega} \tag{7.10}$$

for all $g \in Sp(n, \mathbb{R})$ and $\Omega \in \mathbb{H}_n$. We refer to [21] for more detail. This is a special case of Theorem 1 and Theorem 2.

7.2. Maass-Jacobi Forms

Recently in the case $n = m = 1$ A. Pitale [28] gave a new definition of nonholomorphic Maass-Jacobi forms of weight k and $m \in \mathbb{Z}^+$ as eigenfunctions of a certain differential operator $\mathcal{C}^{k,m}$, and constructed new examples of cuspidal Maass-Jacobi forms F_f of even weight k and index 1 from Maass forms f of weight half integral weight $k-1/2$ with respect to $\Gamma_0(4)$. Moreover he also showed that the map $f \mapsto F_f$ is Hecke equivariant and compatible with the representation theory of the Jacobi group G^J. We will describe his results in some detail.

For a positive integer N, we let

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \,\Big|\, c \equiv 0 \pmod{N} \right\}$$

be the congruence subgroup of $SL(2,\mathbb{Z})$ called the *Hecke subgroup* of level N. Let $\mathfrak{G}$ be the group which consists of all pairs $(\gamma, \phi(\tau))$, where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R})^+$ and $\phi(\tau)$ is a function on $\mathbb{H}$ such that

$$\phi(\tau) = t\,\det(\gamma) \left(\frac{(c\tau+d)}{|c\tau+d|} \right)^{1/2} \qquad \text{with } t \in \mathbb{C},\ |t| = 1.$$

The group law is given by

$$(\gamma_1, \phi_1(\tau)) \cdot (\gamma_2, \phi_2(\tau)) = \big(\gamma_1\gamma_2, \phi_1(\gamma_2 \cdot \tau)\phi_2(\tau)\big), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R})^+. \tag{7.11}$$

Then there is an injective homomorphism $\Gamma_0(4) \mapsto \mathfrak{G}$ given by

$$\gamma \mapsto \gamma^* := \big(\gamma, j(\gamma,\tau)\big), \tag{7.12}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ and

$$j(\gamma,\tau) := \left(\frac{c}{d}\right) \epsilon_d^{-1} \left(\frac{(c\tau+d)}{|c\tau+d|} \right)^{1/2} = \frac{\theta(\gamma\cdot\tau)}{\theta(\tau)}$$

with

$$\theta(\tau) := y^{1/4} \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 \tau}$$

and

$$\epsilon_d = \begin{cases} 1, & \text{if } d \equiv 1 \pmod{4}, \\ i, & \text{if } d \equiv 3 \pmod{4} \end{cases}.$$

And $\left(\frac{c}{d}\right)$ is defined as in [32, p. 442].

For an integer $k \in \mathbb{Z}$, we define the slash operator $||_{k-1/2}$ on functions on $\mathbb{H}$ as follows:

$$\big(f||_{k-1/2}(\gamma,\phi)\big)(\tau) := f(\gamma\cdot\tau)\,\phi(\tau)^{-(2k-1)}. \tag{7.13}$$

Definition 7.1. *A smooth function $f:\mathbb{H}\longrightarrow\mathbb{C}$ is called a Maass form of weight $k-1/2$ with respect to $\Gamma_0(4)$ if it satisfies the following properties (M1)-(M3):*

(M1) $f||_{k-1/2}\gamma^*=f$ *for all* $\gamma\in\Gamma_0(4)$.

(M2) $\Delta_{k-1/2}f=\Lambda f$ *for some* $\Lambda\in\mathbb{C}$, *where* $\Delta_{k-1/2}$ *is the Laplace-Beltrami operator given by*

$$\Delta_{k-1/2}=y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-i\,(k-1/2)\,y\frac{\partial}{\partial x}. \tag{7.14}$$

(M3) $f(\tau)=O(y^N)$ *as* $y\longrightarrow\infty$ *for some* $N>0$.

If, in addition, f vanishes at all the cusps of $\Gamma_0(4)$, then we say that f is a Maass cusp form.

We denote by $M_{k-1/2}(4)$ (resp. $S_{k-1/2}(4)$) be the vector space of all Maass forms (resp. Maass cusp forms) of weight $k-1/2$ with respect to Γ_0. As shown in [16] or [27], if $f\in M_{k-1/2}(4)$, then f has the following Fourier expansion

$$f(\tau)=\sum_{n\in\mathbb{Z}}c(n)W_{\mathrm{sgn}\frac{k-1/2}{2},\frac{il}{2}}(2\pi|n|y)\,e^{2\pi inx}, \tag{7.15}$$

where $\Lambda=-\left\{1/4+(l/2)^2\right\}$ and $W_{\mu,\nu}(y)$ is the classical Whittaker function which is normalized so that $W_{\mu,\nu}(y)\sim e^{-y/2}y^{\mu}$ as $y\longrightarrow\infty$. If $f\in S_{k-1/2}(4)$, then we have $c(0)=0$ in (7.15). We define the plus space by

$$M^+_{k-1/2}(4):=\left\{f\in M_{k-1/2}(4)\,|\,c(n)=0\text{ whenever }(-1)^{k-1}n\equiv 2,3\ \mathrm{mod}4)\,\right\}. \tag{7.16}$$

We set

$$S^+_{k-1/2}(4):=M^+_{k-1/2}(4)\cap S_{k-1/2}(4).$$

For a given integer $k\in\mathbb{Z}$ and $m\in\mathbb{Z}^+$, we let

$$j^{\rm nh}_{k,m}(\tilde g,(\tau,z)):=e^{2\pi i\,m\left\{\kappa-c(z+\lambda\tau+\mu)^2(c\tau+d)^{-1}+\lambda^2\tau+2\lambda z+\lambda\mu\right\}}\times\left(\frac{c\tau+d}{|c\tau+d|}\right)^{-k} \tag{7.17}$$

be the nonholomorphic automorphic factor for G^J on $\mathbb{H}\times\mathbb{C}$, where $\tilde g=(g,(\lambda,\mu;\kappa))$ with $g=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\mathbb{R}),\ \lambda,\mu,\kappa\in\mathbb{R}$ and $(\tau,z)\in\mathbb{H}\times\mathbb{C}$. For $\tilde g\in G^J(\mathbb{R}),\ (\tau,z)\in\mathbb{H}\times\mathbb{C}$ and a smooth function $F:\mathbb{H}\times\mathbb{C}\longrightarrow\mathbb{C}$, we set

$$(F|_{k,m}\tilde g)(\tau,z):=j^{\rm nh}_{k,m}(\tilde g,(\tau,z))F(\tilde g\cdot(\tau,z)). \tag{7.18}$$

Let $\Gamma^J:=SL(2,\mathbb{Z})\ltimes H^{(1,1)}_{\mathbb{Z}}$ be the discrete subgroup of $G^J(\mathbb{R}):=SL(2,\mathbb{R})\ltimes H^{(1,1)}_{\mathbb{R}}$.

Definition 7.2. *A smooth function $F:\mathbb{H}\times\mathbb{C}\longrightarrow\mathbb{C}$ is called a Maass-Jacobi form of weight k and index m with respect to Γ^J if it satisfies the following properties (MJ1)-(MJ3):*

(M1) $F(\tilde\gamma\cdot(\tau,z))=j^{nh}_{k,m}(\tilde\gamma,(\tau,z))^{-1}F(\tau,z)$ *for all* $\tilde\gamma\in\Gamma^J$ *and* $(\tau,z)\in\mathbb{H}\times\mathbb{C}$.

(M2) $\mathcal{C}^{k,m}F=\lambda_{k,m}(f)F$ *for some* $\lambda_{k,m}(f)\in\mathbb{C}$.

(M3) $F(\tau, z) = O(y^N)$ *as* $y \longrightarrow \infty$ *for some* $N > 0$.
If, in addition, f *satisfies the following cuspidal condition*

$$\int_0^1 \int_0^1 F\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}(0, \mu; 0)(\tau, z)\right) e^{-2\pi i(nx + r\mu)} dx d\mu = 0 \tag{7.19}$$

for all $n, r \in \mathbb{Z}$ *such that* $4mn - r^2 = 0$, *then we say that* f *is a Maass-Jacobi cusp form.*

In (M2), $\mathcal{C}^{k,m}$ *is the* $G^J(\mathbb{R})$*-invariant differential operator defined by*

$$\begin{aligned}
\mathcal{C}^{k,m} F \;=\;& \frac{5}{8} F - 2(\tau - \bar\tau)^2 F_{\tau\bar\tau} - (k-1)(\tau - \bar\tau) F_{\bar\tau} - k(\tau - \bar\tau) F_\tau \\
& + \frac{k(\tau - \bar\tau)}{8\pi i\, m} F_{zz} + \frac{(\tau - \bar\tau)^2}{4\pi i\, m} F_{\bar\tau zz} + \frac{k\,(\tau - \bar\tau)}{4\pi i\, m} F_{z\bar z} \\
& + \frac{(\tau - \bar\tau)(z - \bar z)}{4\,\pi i\, m} F_{zz\bar z} - 2\,(\tau - \bar\tau)(z - \bar z)\, F_{\tau\bar z} + \frac{(\tau - \bar\tau)^2}{4\,\pi i\, m} F_{\tau\bar z\bar z} \\
& + \left(\frac{(z - \bar z)^2}{2} + \frac{k(\tau - \bar\tau)}{8\,\pi i\, m}\right) F_{\bar z\bar z} + \frac{(\tau - \bar\tau)(z - \bar z)}{4\,\pi i\, m} F_{z\bar z\bar z}.
\end{aligned}$$

We denote by $J^{\rm nh}_{k,m}$ $\big($ resp. $J^{{\rm nh},cusp}_{k,m}$ $\big)$ the vector space of all Maass-Jacobi forms (resp. Maass-Jacobi cusp forms) of weight k and index m with respect to Γ^J.

For a Maass form $f \in M^+_{k-1/2}(4)$ with $k \in 2\mathbb{Z}$, he defined the function F_f on $\mathbb{H} \times \mathbb{C}$ by

$$F_f(\tau, z) := f^{(0)}(\tau)\, \widetilde{\Theta}^{(0)}(\tau, z) + f^{(1)}(\tau)\, \widetilde{\Theta}^{(1)}(\tau, z), \quad (\tau, z) \in \mathbb{H} \times \mathbb{C}. \tag{7.20}$$

We refer to [28, pp. 96-97] for the precise definition of $f^{(0)}$, $f^{(1)}$, $\widetilde{\Theta}^{(0)}$ and $\widetilde{\Theta}^{(1)}$. Pitale [28] showed that if $f \in M^+_{k-1/2}(4)$ with $k \in 2\mathbb{Z}$, then $F_f \in J^{\rm nh}_{k,1}$, and $F_f \in J^{{\rm nh},cusp}_{k,1}$ if and only if $f \in S^+_{k-1/2}(4)$. Furthermore he showed that if $\Delta_{k-1/2} f = \Lambda f$, then $\mathcal{C}^{k,1} F_f = 2\,\Lambda\, F_f$ under the assumption $f \in M^+_{k-1/2}(4)$ with $k \in 2\mathbb{Z}$.

For an odd prime p, the Jacobi Hecke operator T_p on $J^{\rm nh}_{k,1}$ (cf. [5, p. 168] or [7, p. 41] is defined by

$$T_p F := \sum_{\substack{M \in SL(2,\mathbb{Z})/\mathbb{Z}^{(2,2)} \\ \det(M) = p^2 \\ \gcd(M) = 1}} \; \sum_{(\lambda, \mu) \in (\mathbb{Z}/p\mathbb{Z})^2} F|_{k,1} \left(\det(M)^{-1/2} M(\lambda, \mu; 0)\right). \tag{7.21}$$

Theorem 7.3. *Let* $f \in S^+_{k-1/2}(4)\,(k \in 2\mathbb{Z})$ *be a Hecke eigenform with eigenvalue* λ_p *for every odd prime* p. *Then* $T_p = p^{k-3/2} \lambda_p\, F_f$ *for all odd prime* p. *Namely* F_f *is also an eigenfunction of all* T_p *for every odd prime* p.

Proof. The proof can be found in [28, pp. 104-106]. □

Let f be a Hecke eigenform in $S^+_{k-1/2}(4)\,(k\in 2\mathbb{Z})$ such that for every odd prime p we have $T_pf=\lambda_pf$ and $\Delta_{k-1/2}f=\Lambda f$ with $\Lambda=\frac{1}{4}(s^2-1)$. Let $\widetilde{\pi}_f=\otimes\widetilde{\pi}_{f,p}$ be the irreducible cuspidal genuine automorphic representation of a two-fold covering group $\widetilde{SL(2,\mathbb{A})}$ of $SL(2,\mathbb{A})$ corresponding to f (cf. [42, p. 386]). Now we let F_f be the Maass-Jacobi cusp form in $J^{\text{nh},cusp}_{k,1}$ constructed from an eigenform $f\in S^+_{k-1/2}(4)\,(k\in 2\mathbb{Z})$ by Formula (7.20). Then F_f is an eigenform of all T_p for every odd prime p and is an eigenfunction of the differential operator $\mathcal{C}^{k,1}$. We lift F_f to the function Φ_{F_f} on $G^J(\mathbb{A})$ as follows. By the strong approximation theorem for $G^J(\mathbb{A})$, we have the decomposition

$$G^J(\mathbb{A})=G^J(\mathbb{Z})\,G^J(\mathbb{R})\,\Pi_{p<\infty}G^J(\mathbb{Z}_p). \tag{7.22}$$

If $\tilde{g}=\gamma\tilde{g}_\infty k_0\in G^J(\mathbb{A})$ with $\gamma\in G^J(\mathbb{Z})$, $\tilde{g}_\infty\in G^J(\mathbb{R})$, $k_0\in\Pi_{p<\infty}G^J(\mathbb{Z}_p)$, we define

$$\Phi_{F_f}(\tilde{g}):=(F_f|_{k,m}\tilde{g}_\infty)(i,0). \tag{7.23}$$

Let Π_{F_f} be the space of all right translates of Φ_{F_f} on which $G^J(\mathbb{A})$ acts by right translation. Pitale [28] proved that

$$\Pi_{F_f}=\widetilde{\pi}_f\otimes\omega^1_{\text{SW}}, \tag{7.24}$$

where ω^1_{SW} is the Schrödinger-Weil representation of $G^J(\mathbb{A})$ (cf. [5]).

Remark 7.1. *For a Siegel cusp form of half integral weight, we have a result similar to Formula (7.24). See* [5] *for the case* $n=1$ *and* [38, 39] *for the case* $n\geq 1$.

Remark 7.2. *Berndt and Schmidt* [5] *gave a definition of Maass-Jacobi forms different from Definition 7.2. Yang* [55, 57] *gave a definition of Maass-Jacobi forms using the Laplacian of an invariant metric on the Siegel-Jacobi space* $\mathbb{H}_n\times\mathbb{C}^{(m,n)}$ *in the aspect of the spectral theory on* $L^2\big(\Gamma^J_n\backslash\mathbb{H}_n\times\mathbb{C}^{(m,n)}\big)$.

7.3. Theta Sums

We embed $SL(2,\mathbb{R})$ into $Sp(n,\mathbb{R})$ by

$$SL(2,\mathbb{R})\ni\begin{pmatrix}a&b\\c&d\end{pmatrix}\longmapsto\begin{pmatrix}aI_n&bI_n\\cI_n&dI_n\end{pmatrix}\in Sp(n,\mathbb{R}). \tag{7.25}$$

Every map $M=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\mathbb{R})$ admits the unique Iwasawa decomposition

$$M=\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}y^{1/2}&0\\0&y^{-1/2}\end{pmatrix}\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}=(\tau,\theta),$$

where $\tau=x+i\,y\in\mathbb{H}_1$ and $0\leq\theta<2\pi$. Then $SL(2,\mathbb{R})$ acts on $\mathbb{H}_1\times[0,2\pi)$ by

$$M\cdot(\tau,\theta):=\big(M\cdot\tau,\ \theta+\arg(c\tau+d)\ \text{mod}\ 2\pi\big), \tag{7.26}$$

where $M=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\mathbb{R}),\ \tau\in\mathbb{H}_1$ and $\theta\in[0,2\pi)$.

We put

$$G^J_{n,1} := Sp(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,1)}.$$

We take $\mathcal{M} = 1$ in Section 4. Then we let $\mathscr{W} = \mathscr{W}_{\mathcal{M}}$, $R = R_{\mathcal{M}}$ and $c = c_{\mathcal{M}}$ (see Section 4). If $M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in SL(2,\mathbb{R})$ for $i = 1,2,3$ with $M_3 = M_1 M_2$, then the cocycle c is given by

$$c(M_1, M_2) = e^{-i\pi n\, \mathrm{sign}(c_1 c_2 c_3)/4},$$

where

$$\mathrm{sign}(x) = \begin{cases} -1 & \text{if } x < 0, \\ \ \ 0 & \text{if } x = 0, \\ \ \ 1 & \text{if } x < 0. \end{cases}$$

For $(\tau, \theta) \in SL(2,\mathbb{R})$, we define

$$\widetilde{R}(\tau,\theta) := e^{-i\pi n\,\sigma_\theta/4}\, R(\tau,\theta), \tag{7.27}$$

where

$$\sigma_\theta = \begin{cases} 2\nu & \text{if } \theta = \nu\pi,\ \nu \in \mathbb{Z}, \\ 2\nu + 1 & \text{if } \nu\pi < \theta < (\nu+1)\pi,\ \nu \in \mathbb{Z}. \end{cases}$$

Then $\widetilde{R}$ is a unitary representation of the double covering group of $SL(2,\mathbb{R})$ (cf. [21]). Obviously $\widetilde{R}(i,\theta)\widetilde{R}(i,\theta') = \widetilde{R}(i,\theta+\theta')$.

We see that

$$\omega^1_{\mathrm{SW}}((\xi;t)(\tau,\theta)) = \mathscr{W}((\xi;t))\,\widetilde{R}(\tau,\theta), \tag{7.28}$$

where ω^1_{SW} denotes the Schrödinger-Weil representation of $G^J_{n,1}$ (see Formula (4.10)). Here $(\xi;t) \in H_{\mathbb{R}}^{(n,1)}$ and (τ,θ) is considered as an element of $Sp(n,\mathbb{R})$ by the embedding (7.25).

We denote by $\mathcal{S}(\mathbb{R}^n)$ the vector space of C^∞-functions on $\mathbb{R}^n$ that, as well as their derivatives, decrease rapidly at ∞. For any $f \in \mathcal{S}(\mathbb{R}^n)$, *Jacobi's theta sum* for f is defined to be the function

$$\Theta_f(\tau,\theta;\xi,t) := \sum_{\alpha \in \mathbb{Z}^n} \left[\omega^1_{\mathrm{SW}}\big((\xi;t)(\tau,\theta)\big) f\right](\alpha), \tag{7.29}$$

where $(\tau,\theta) \in SL(2,\mathbb{R}) \hookrightarrow Sp(n,\mathbb{R})$ and $(\xi;t) \in H_{\mathbb{R}}^{(n,1)}$ with $\xi = (\lambda,\mu)$, $\lambda,\mu \in \mathbb{R}^n$ and $t \in \mathbb{R}$. For $f, g \in \mathcal{S}(\mathbb{R}^n)$, the product of theta sums of the form

$$\Theta_f(\tau,\theta;\xi,t)\,\overline{\Theta_g(\tau,\theta;\xi,t)}$$

is independent of the t-variable.

Let us therefore define the semi-direct product group

$$G[n] := SL(2,\mathbb{R}) \ltimes \mathbb{R}^{2n}$$

with multiplication law

$$(M,\xi)(M',\xi') = (MM',\, \xi + M\xi'), \quad M, M' \in SL(2,\mathbb{R}),\ \xi,\xi' \in \mathbb{R}^{2n}.$$

The set

$$\Gamma[n] =: \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a\, b\, \mathfrak{s} \\ c\, d\, \mathfrak{s} \end{pmatrix} + \alpha \right) \;\Big|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}),\ \alpha \in \mathbb{Z}^{2n} \right\}$$

with $\mathfrak{s} = {}^t(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}) \in \mathbb{R}^n$ is a subgroup of $G[n]$. We can show that $\Gamma[n]$ is generated by

$$\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0 \right),\quad \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \mathfrak{s} \\ 0 \end{pmatrix} \right),\quad \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \alpha \right),\ \alpha \in \mathbb{Z}^{2n}.$$

We put, for brevity,

$$\Theta_f(\tau,\theta;\xi) :=\ \Theta_f(\tau,\theta;\xi,0).$$

J. Marklof [23] proved the following properties of Jacobi's theta sums.

Theorem 7.4. *Let f and g be two elements in $\mathcal{S}(\mathbb{R}^n)$. Then*

(1) $\Theta_f(\tau,\theta;\xi)\,\overline{\Theta_g(\tau,\theta;\xi)}$ is invariant under the action of the left action of $\Gamma[n]$.

(2) For any real number $R>1$, we have

$$\begin{aligned} &\Theta_f(\tau,\theta;\xi)\,\overline{\Theta_g(\tau,\theta;\xi)} \\ =\ & y^{n/2} \sum_{\alpha\in\mathbb{Z}^n} f_\theta\big((\alpha-\mu)\,y^{1/2}\big)\,\overline{g_\theta\big((\alpha-\mu)\,y^{1/2}\big)}\ +\ O_R\big(y^{-R}\big), \end{aligned}$$

where $\tau = x+\,i\,y\in\mathbb{H}_1,\ \xi=(\lambda,\mu)$ with $\lambda,\mu\in\mathbb{R}^n$ and

$$f_\theta =\ \widetilde{R}(i,\theta)f.$$

Proof. The proof can be found in [23, pp. 432-433]. □

The above properties of Jacobi's theta sums together with Ratner's classification of measures invariant under unipotent flows (cf. [29, 30]) are used to prove the important fact that under explicit diophantine conditions on $(\alpha,\beta)\in\mathbb{R}^2$, the local two-point correlations of the sequence given by the values $(m-\alpha)^2+(n-\beta)^2$ with $(m,n)\in\mathbb{Z}^2$, are those of a Poisson process (see [23] for more detail).

8. Applications of the Weil-Satake Representation

In this section we provide some applications of the Weil-Satake Representation $\widehat{\omega}_{\mathcal{M}} := \widehat{\omega}_{\mathcal{M},iI_n}$ to the theory of representations of the Jacobi group G^J. Throughout this section, for brevity, we put $G := Sp(n,\mathbb{R})$. We will keep the notations and the conventions in Section 5. We recall the notations $G_2 = G_{2,iI_n}$ and $G_2^J = G_2 \ltimes H_{\mathbb{R}}^{(n,m)}$ in Section 5. For a real Lie group $\mathfrak{G}$, we denote by $\widehat{\mathfrak{G}}$ the unitary dual of $\mathfrak{G}$. We define the following projections

$$\begin{aligned} p_2 : G_2 \longrightarrow G, & \qquad (g,t) \longmapsto g, \\ p^J : G^J \longrightarrow G, & \qquad (g,h) \longmapsto g, \\ p_2^J : G_2^J \longrightarrow G^J, & \qquad ((g,t),h) \longmapsto (g,h), \\ p_{2,J} : G_2^J \longrightarrow G_2, & \qquad \big((g,t),h\big) \longmapsto (g,t). \end{aligned}$$

Let $\mathscr{Z}$ be the center of G^J. Obviously $\mathscr{Z} \cong S(m)$.

Proposition 8.1. *Let $\chi_{\mathcal{M}}$ be the character of $\mathscr{Z}$ defined by $\chi_{\mathcal{M}}(\kappa) = e^{2\pi i \sigma(\mathcal{M}\kappa)}$ with $\kappa \in \mathscr{Z}$. We denote by $\widehat{G_2^J}\big(\overline{\chi}_{\mathcal{M}}\big)$ the set of all equivalence classes of irreducible representations η of G_2^J such that $\eta(\kappa) = \chi_{\mathcal{M}}(\kappa)^{-1}$ for all $\kappa \in \mathscr{Z}$. We put $\widetilde{\pi} = \pi \circ p_{2,J}$ for any $\pi \in \widehat{G_2^J}$. Then the correspondence*

$$\widehat{G_2} \longrightarrow \widehat{G_2^J}\big(\overline{\chi}_{\mathcal{M}}\big), \qquad \pi \longmapsto \widetilde{\pi} \otimes \widehat{\omega}_{\mathcal{M}}$$

is a bijection from $\widehat{G_2}$ to $\widehat{G_2^J}\big(\overline{\chi}_{\mathcal{M}}\big)$. Furthermore π is square integrable if and only if $\widetilde{\pi} \otimes \widehat{\omega}_{\mathcal{M}}$ is square square integrable modulo $\mathscr{Z}$.

Proof. The proof can be found in [36]. □

We now consider a holomorphic discrete series representation of G^J. Let K be the stabilizer of the action (1.1) at iI_n. Then

$$K = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in G \;\middle|\; A + i\,B \in U(n) \right\}.$$

Thus K can be identified with the unitary group $U(n)$. Let (ρ, V_ρ) be an irreducible representation of K with highest weight $\rho = (\rho_1, \cdots, \rho_n) \in \mathbb{Z}^n$ such that $\rho_1 \geq \cdots \geq \rho_n \geq 0$. Then ρ can be extended to a rational representation of $GL(n,\mathbb{C})$ that is also denoted by ρ. The representation space V_ρ of ρ has a Hermitian inner product $\langle\ ,\ \rangle$ on V_ρ such that $\langle \rho(g)u, v\rangle = \langle v, \rho(g^*)v\rangle$ for all $g \in GL(n,\mathbb{C}),\ u, v \in V_\rho$, where $g^* = {}^t\overline{g}$. We define the unitary representation τ_ρ of K by

$$\tau_\rho(k) := \rho\big(J(k, iI_n)\big), \quad k \in K. \tag{8.1}$$

For all $\widetilde{g} = (g,h) \in G^J$ with $g \in G$ and $(\Omega, Z) \in \mathbb{H}_{n,m}$, we define

$$J_{\rho,\mathcal{M}}\big(\widetilde{g}, (\Omega, Z)\big) := J_{\mathcal{M}}(\widetilde{g}, (\Omega, Z))\, \rho(J(g,\Omega)). \quad \text{(see (5.1) and (5.2))} \tag{8.2}$$

We note that for all $\widetilde{g} \in G^J,\ (\Omega, Z) \in \mathbb{H}_{n,m}$ and $u, v \in V_\rho$, we have the relation

$$\langle\, J_{\rho,\mathcal{M}}\big(\widetilde{g}, (\Omega, Z)\big)u, v\,\rangle = \langle\, u, J_{\rho,\mathcal{M}}\big(\widetilde{g}, (\Omega, Z)\big)^* v\,\rangle,$$

where

$$J_{\rho,\mathcal{M}}\big(\widetilde{g}, (\Omega, Z)\big)^* = \overline{J_{\mathcal{M}}\big(\widetilde{g}, (\Omega, Z)\big)}\, \rho\big({}^t\overline{J(g,\Omega)}\big).$$

We let $\mathbb{E}_{\rho,\mathcal{M}}$ be the Hilbert space consisting of V_ρ-valued measurable functions f on $\mathbb{H}_{n,m}$ satisfying the condtion

$$(f,f) = ||f||^2 = \int_{\mathbb{H}_{n,m}} \langle\, \rho(Y) f(\Omega, Z), f(\Omega, Z)\,\rangle\, \kappa_{\mathcal{M}}(\Omega, Z)\, dv,$$

where $\kappa_{\mathcal{M}}(\Omega, Z)$ and dv are defined in (5.3) and (6.15) respectively. We let $K^J := K \times S(m)$ be a subgroup of G^J. The representation $\Pi_{\rho,\mathcal{M}} := \mathrm{Ind}_{K^J}^{G^J}(\rho \otimes \overline{\chi}_{\mathcal{M}})$ induced from a representation $\rho \otimes \overline{\chi}_{\mathcal{M}}$ is realized on $\mathbb{E}_{\rho,\mathcal{M}}$ as follows: for any $\widetilde{g} \in G^J$ and $f \in \mathbb{E}_{\rho,\mathcal{M}}$, $\Pi_{\rho,\mathcal{M}}$ is given by

$$\big(\Pi_{\rho,\mathcal{M}}(\widetilde{g})f\big)(\Omega, Z) = J_{\rho,\mathcal{M}}(\widetilde{g}, (\Omega, Z))^{-1} f(\widetilde{g}^{-1} \cdot (\Omega, Z)). \tag{8.3}$$

Let $\mathbb{H}_{\rho,\mathcal{M}}$ be the subspace of $\mathbb{E}_{\rho,\mathcal{M}}$ consisting of holomorphic functions in $\mathbb{E}_{\rho,\mathcal{M}}$. It is easily seen that $\mathbb{H}_{\rho,\mathcal{M}}$ is a closed subspace of $\mathbb{E}_{\rho,\mathcal{M}}$ invariant under the action of $\Pi_{\rho,\mathcal{M}}$. We let $\pi_{\rho,\mathcal{M}}$ be the restriction of $\Pi_{\rho,\mathcal{M}}$ to $\mathbb{H}_{\rho,\mathcal{M}}$.

Takase [37] proved the following result.

Theorem 8.1. *Suppose $\rho_n > n + \frac{m}{2}$. Then $\mathbb{H}_{\rho,\mathcal{M}} \neq 0$ and $\pi_{\rho,\mathcal{M}}$ is an irreducible representation of G^J which is square integrable modulo $\mathscr{Z}$. Moreover the multiplicity of ρ in the restriction $\pi_{\rho,\mathcal{M}}|_K$ of $\pi_{\rho,\mathcal{M}}$ to K is equal to one.*

We let

$$K_2 = p_2^{-1}(K) = \left\{ (k,t) \in K \times T \mid \ t^2 = \det J(k, iI_n) \right\}.$$

The Lie algebra $\mathfrak{k}$ of K_2 and its Cartan subalgebra $\mathfrak{h}$ are given by

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \;\middle|\; A + {}^tA = 0,\ B = {}^tB \right\}$$

and

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \;\middle|\; C = \mathrm{diag}\,(c_1, c_2, \cdots, c_n) \right\}.$$

Here $\mathrm{diag}\,(c_1, c_2, \cdots, c_n)$ denotes the diagonal matrix of degree n. We define $\lambda_j \in \mathfrak{h}_{\mathbb{C}}^*$ by $\lambda_j\left(\begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix}\right) := \sqrt{-1}\, c_j$. We put

$$\mathbb{M}^+ = \left\{ \sum_{j=1}^n m_j \lambda_j \;\middle|\; m_j \in \frac{1}{2}\mathbb{Z},\ m_1 \geq \cdots \geq m_n,\ m_i - m_j \in \mathbb{Z} \text{ for all } i, j \right\}.$$

We take an element $\lambda = \sum_{j=1}^n m_j \lambda_j \in \mathbb{M}^+$. Let τ be an irreducible representation of K with highest weight $\tau = (\tau_1, \cdots, \tau_n) \in \mathbb{Z}^n$, where $\tau_j = m_j - m_n \,(1 \leq j \leq n-1)$. Let $\tau_{[\lambda]}$ be the irreducible representation of K_2 defined by

$$\tau_{[\lambda]}(k,t) := t^{2m_n} \cdot \tau(J(k, iI_n)), \quad (k,t) \in K_2. \tag{8.4}$$

Then $\tau_{[\lambda]}$ is the irreducible representation of K_2 with highest weight $\lambda = (m_1, \cdots, m_n)$ and $\lambda \longmapsto \tau_{[\lambda]}$ is a bijection from $\mathbb{M}^+$ to $\widehat{K}_2$, the unitary dual of K_2. According to [15, Theorem 7.2], we have a decomposition of the restriction $\widehat{\omega}_{\mathcal{M}}|_{K_2}$ into irreducible components:

$$\widehat{\omega}_{\mathcal{M}}|_{K_2} = \bigoplus_{\lambda} m_\lambda\, \tau_{[\lambda]},$$

where λ runs over

$$\lambda=\sum_{j=1}^{s}\tau_j\lambda_j+\frac{m}{2}\sum_{j=1}^{n}\lambda_j\in\mathbb{M}^+\quad(s=\mathrm{Min}\,\{m,n\}),$$
$$\tau_j\in\mathbb{Z}\text{ such that }\tau_1\geq\tau_2\geq\cdots\geq\tau_s\geq 0$$

and the multiplicity m_λ is given by

$$m_\lambda=\prod_{1\leq i<j\leq m}\left(1+\frac{\tau_i-\tau_j}{j-i}\right),$$

where $\tau_j=0$ if $j>s$. Let $\widehat{G}_{2,d}$ be the set of all equivalence classes of square integrable irreducible unitary representations of G_2. The correspondence

$$\pi\longmapsto\text{Harish-Chandra parameter of }\pi$$

is a bijection from $\widehat{G}_{2,d}$ to Λ^+, where

$$\Lambda^+=\left\{\sum_{j=1}^{n}m_j\lambda_j\in\mathbb{M}^+\;\middle|\;m_1>\cdots>m_n,\ m_i-m_j\neq 0\text{ for all }i,j,\ i\neq j\right\}.$$

See [41], Theorem 10.2.4.1 for the details.

We choose an element $\lambda=\sum_{j=1}^n m_j\,\lambda_j\in\mathbb{M}^+$. Let $\pi^\lambda\in\widehat{G}_{2,d}$ be the representation corresponding to the Harish-Chandra parameter

$$\sum_{j=1}^{n}(m_j-j)\,\lambda_j\in\Lambda^+.$$

The representation π^λ is realized as follows (see [17], Theorem 6.6): Let (τ,V_τ) be the irreducible representation of K with highest weight $\tau=(\tau_1,\cdots,\tau_n),\ \tau_i=m_i-m_n\,(\,1\leq j\leq n-1\,)$. Let $\mathscr{H}^\lambda$ be a Hilbert space consisting of V_τ-valued holomorphic functions φ on $\mathbb{H}_n$ such that

$$|\varphi|^2=\int_{\mathbb{H}_n}(\tau(Y)\,\varphi(\Omega),\varphi(\Omega))\,(\det Y)^{m_n}\,dv_\Omega<\infty,$$

where $dv_\Omega=(\det Y)^{-(n+1)}[dX]\wedge[dY]$ is a G-invariant volume element on $\mathbb{H}_n$. Then π^λ is realized on $\mathscr{H}^\lambda$ as follows: for any $\sigma=(g,t)\in G_2$ and $f\in\mathscr{H}^\lambda$,

$$\left(\pi^\lambda(\sigma)f\right)(\Omega)=J_{[\lambda]}(\sigma^{-1},\Omega)^{-1}f(\sigma^{-1}\Omega)\tag{8.5}$$

for all $\sigma=(g,t)\in G_2$ and $f\in\mathscr{H}^\lambda$. Here

$$J_{[\lambda]}(\sigma,\Omega)=\left\{t\,\beta_{iI_n}(g,g^{-1})\,|\det J(g,\Omega)|^{\frac{1}{2}}\,\frac{\gamma(g\Omega,g(iI_n))}{\gamma(\Omega,iI_n)}\right\}^{m_n}\tau(J(g,\Omega)).$$

Theorem 8.2. *Suppose $\tau_n>n+\frac{m}{2}$. We put $\lambda=\sum_{j=1}^n(\tau_j-\frac{m}{2})\lambda_j\in\mathbb{M}^+$. Then the unitary representation $\pi_{\tau,\mathcal{M}}\circ p_2^J$ of G_2^J is unitarily equivalent to the representation $(\pi^\lambda\circ p_{2,J})\otimes\widehat{\omega}_{\mathcal{M}}$.*

Proof. The proof can be found in [37]. □

Using Theorem 8.2, Takase [39] established a bijective correspondence between the space of cuspidal Jacobi forms and the space of Siegel cusp forms of half integral weight which is compatible with the action of Hecke operators. For example, the classical result (cf. [7] and [13])

$$J_{k,1}^{\mathrm{cusp}}(\Gamma_n) \cong S_{k-1/2}(\Gamma_0(4)) \tag{8.6}$$

can be obtained by the method of the representation theory. Here Γ_n denotes the Siegel modular group of degree n and $\Gamma_0(4)$ denotes the Hecke subgroup of Γ_n.

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