On the geometry of the Siegel-Jacobi manifolds

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Talk based on:

- 10 papers in 2005-2009, some are in ArXiv at DG,
- + talks at conferences
- + NEW RESULTS

Outline

• **For mathematicians**: *Jacobi group*– $G^J_n = H_n \rtimes \mathrm{Sp}(n, \mathbb{R})$
$(2n+1)(n+1)$- dim. Generalized Jacobi groups (Takase, Yang, Lee...)

G^J_n - group of *Harish-Chandra* type (Satake), and *Coherent State* type group (Moskovici & Verona, Lisiecki, Neeb,...)

• **In Physics**: Schrödinger (Hagen, or conformal Galilean...) group.

1972, U. Niederer: *the maximal kinematical invariance group of the free Schrödinger equation– 12-parameter group*

Barut & Raczka; Dobrev & all: Schrödinger group in $(n+1)$-space-time dimensions

In the case $n = 1$ with $t \in \mathbb{C}, \Im(t) > 0$, $x \in \mathbb{C}$, corresponds to $G^J_1(\mathbb{R})$ of Eichler & Zagier, Kähler, Berndt.

• Denomination *Jacobi group* G_n^J was introduced by Eichler and Zagier, *Theory of Jacobi forms* (1985), inspired by Pyatetskii-Shapiro, who referred to *Fourier-Jacobi* expansion, and to some coefficients as *Jacobi forms*

• The denomination *Jacobi group* was adopted also in the monograph Berndt & Schmidt *Elements of the Representation Theory of the Jacobi group* (1998).

• Kirillov: $\mathfrak{st}(n, \mathbb{R})$ – 1974; $\mathfrak{tsp}(2n+2, \mathbb{R})$ – 2004

• K. B. Wolf: *Weyl-symplectic group*-1975; Integral transforms in science and engineering, Plenum (1979); Geometric Optics on Phase Space, Springer (2004)

• R. Berndt (1984), E. Kähler (1983); *Poincaré group* or *The New Poincaré group* investigated by Erich Kähler as the 10-dimensional group G^K (a double cover of the de Sitter group $SO_0(4,1)$) which invariates the metric $ds^2 = t^{-2}(dx^2 + dy^2 + dz^2 + dt^2)$. Quaternionic 2×2 matrices.

Jacobi & Physics

The Jacobi group is an important object in connection with Quantum Mechanics, Geometric Quantization, Optics.

Jacobi group describes *squeezed states* in Quantum Optics (see Stoler).

Applications of squeezed states: detection of gravitational waves, quantum communications, entanglement, quantum cryptography, teleportation, ...

For the harmonic oscillator: $\Delta x = \Delta p = 1/\sqrt{2}$ (in units of $\hbar$). "The squeezed states": $\Delta x < 1/\sqrt{2}$. The squeezed states are a particular class of *minimum uncertainty states* — states which saturates the Heisenberg uncertainty relation = CS states bazed on the Siegel upper half plane $\mathcal{H}_n$.

Appendix- general definitions

Coherent state (à la Perelomov): $(G, \pi, \mathcal{H})$ G = Lie group, π = unitary irreducible representation of G on the complex separable Hibert space $\mathcal{H}$. A common realization of coherent states as space of holomorphic functions defined on the homogeneous manifold $M = G/H$, square integrable with respect to a scalar product determined by the reproducing kernel K. Usually M is a Kähler manifold.
$\boldsymbol{X} := d\pi(X), X \in \mathfrak{g}$

The Jacobi algebra $\mathfrak{g}_1^J$

The (6-dim) Jacobi algebra

$$\mathfrak{g}_1^J := \mathfrak{h}_1 \rtimes \mathfrak{su}(1,1), \tag{2.1}$$

$\mathfrak{h}_1$ is an ideal in $\mathfrak{g}_1^J$

$$\begin{aligned}
&[a, a^\dagger] = 1,\ [K_0, K_\pm] = \pm K_\pm\ ,\ [K_-, K_+] = 2K_0, \\
&[a, K_+] = a^\dagger\ ,\ \left[K_-, a^\dagger\right] = a, \\
&2\left[K_0,\ a^\dagger\right] = a^\dagger,\ 2\,[K_0, a] = -a.
\end{aligned}$$

Perelomov's CS vectors

$$ae_0 = 0, \;\; \boldsymbol{K}_- e_0 = 0, \; \boldsymbol{K}_0 e_0 = k e_0; \; k > 0, 2k = 2, 3, ...$$

For SU(1, 1), D_k^+ the positive discrete series representations (Bargmann).
To G_1^J we associate Perelomov's CS vectors

$$e_{z,w} := \mathrm{e}^{za^\dagger + w\boldsymbol{K}_+} e_0, \; z, w \in \mathbb{C}, \; |w| < 1. \tag{2.3}$$

on (4-dim) manifold

$$M := H_1/\mathbb{R} \times \mathrm{SU}(1,1)/\mathrm{U}(1) = \mathbb{C} \times \mathcal{D}_1 (:= \mathcal{D}_1^J)$$

The *displacement operator*

$$D(\alpha) := \exp(\alpha a^\dagger - \bar{\alpha} a) \tag{2.4}$$

S – the unitary squeezed operator – the D^k_+ representation of the group SU(1, 1), $\underline{S}(z) = S(w)$, z, $w \in \mathbb{C}$, $|w| < 1$:

$$\begin{aligned} \underline{S}(z) &:= \exp(z\boldsymbol{K}_+ - \bar{z}\boldsymbol{K}_-),\ w = \frac{z}{|z|}\tanh(|z|); \\ S(w) &= \exp(w\boldsymbol{K}_+)\exp(\eta\boldsymbol{K}_0)\exp(-\bar{w}\boldsymbol{K}_-); \end{aligned}$$

The normalized (squeezed) CS vector

$$\Psi_{\alpha,w} := D(\alpha)S(w)e_0. \tag{2.6}$$

The base of functions & Reproducing Kernel

Proposition

$$f_{|n>;e_{k',k'+m}}(z,w) = \sqrt{\frac{\Gamma(n+2k)}{n!\Gamma(2k)}}w^n\frac{P_n(z,w)}{\sqrt{n!}}$$

$$P_n(z,w) = n!\sum_{k=0}^{[\frac{n}{2}]}(\frac{w}{2})^k\frac{z^{n-2k}}{k!(n-2k)!}, z,w \in \mathcal{D}_1^J. \quad (2.7)$$

$$K(z,w;\bar{z}',\bar{w}') := (e_{\bar{z},\bar{w}}, e_{\bar{z}',\bar{w}'}) = (1-w\bar{w}')^{-2k}\exp\frac{2\bar{z}'z + z^2\bar{w}' + \bar{z}'^2 w}{2(1-w\bar{w}')}$$

$$\Psi_{\alpha,w} = (1-w\bar{w})^k\exp(-\frac{\bar{\alpha}}{2}z)e_{z,w}, \; z = \alpha - w\bar{\alpha}. \quad (2.8)$$

The composition law & the action

Proposition

$$G_1^J := HW \rtimes SU(1,1)$$

$$(g_1,\alpha_1,t_1)\circ(g_2,\alpha_2,t_2) = (g_1\circ g_2, g_2^{-1}\cdot\alpha_1+\alpha_2, t_1+t_2+\Im(g_2^{-1}\cdot\alpha_1\bar{\alpha}_2)),$$

$$g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, |a|^2-|b|^2 = 1,$$

$$\alpha_g = a\,\alpha + b\,\bar{\alpha},$$

$$h := (g,\alpha)\in G_1^J, \pi(h) := S(g)D(\alpha), g\in SU(1,1), \alpha\in\mathbb{C},$$

$$z_1 = \frac{\alpha-\bar{\alpha}w+z}{\bar{b}w+\bar{a}};\ w_1 = g\cdot w = \frac{aw+b}{\bar{b}w+\bar{a}}, (z,w)\in\mathcal{D}_1^J := \mathbb{C}\times\mathcal{D}_1. \quad (2.9)$$

Scalar product ; Kähler two-form

$$
\begin{aligned}
(\phi,\psi) & = \Lambda_1 \int_{z\in\mathbb{C};|w|<1} \bar{f}_\phi(z,w) f_\psi(z,w)(1-w\bar{w})^{2k} \times \\
& \exp -\frac{|z|^2}{1-w\bar{w}} \exp -\frac{z^2\bar{w}+\bar{z}^2 w}{2(1-w\bar{w})} d\nu_1,
\end{aligned}
$$

$$
d\nu_1 = \frac{d\Re w d\Im w}{(1-w\bar{w})^3} d\Re z d\Im z, \text{ the } G_1^J\text{-invariant measure}
$$

$$
\Lambda_1 = \frac{4k-3}{2\pi^2}.
$$

$$
-\mathrm{i}\,\omega_1 = \frac{2k}{(1-w\bar{w})^2} dw\wedge d\bar{w} + \frac{A\wedge\bar{A}}{1-w\bar{w}}, \; A = dz + \bar{\alpha}_0 dw,
$$

$$
\alpha_0 = \frac{z+\bar{z}w}{1-w\bar{w}}.
$$

Action

Rolf Berndt – the real Jacobi group $G^J(\mathbb{R})$. Kähler and Berndt – Jacobi group $G^J_0(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ acting on the manifold $\mathcal{X}^J_1 := \mathcal{H}_1 \times \mathbb{C}$. $\mathcal{H}_1$ – upper half plane $\mathcal{H}_1 := \{v \in \mathbb{C} | \Im(v) > 0\}$.

Remark

The action of $G^J_0(\mathbb{R})$ on $\mathcal{X}^J_1$ is given by $(g,(v,z)) \rightarrow (v_1, z_1)$, $g = (M, l)$, where

$$v_1 = \frac{av+b}{cv+d}, z_1 = \frac{z + l_1 v + l_2}{cv+d}; \ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), (l_1, l_2) \in \mathbb{R}^2.$$

Partial Cayley transform & Kähler 2-form of Kähler

Remark

The action $C^{-1}G_0^J(\mathbb{R})C$, descends on the basis as the biholomorphic map: $\check{C}^{-1}:\mathcal{X}_1^J:=\mathcal{H}_1\times\mathbb{C}\rightarrow\mathcal{D}_1^J:=\mathcal{D}_1\times\mathbb{C}$:

$$w=\frac{v-i}{v+i};\ z=\frac{2iu}{v+i}, w\in\mathcal{D}_1,\ v\in\mathcal{H}_1, z\in\mathbb{C}. \qquad (3.1)$$

Remark

When expressed in the coordinates $(v,u)\in\mathcal{X}_1^J$ which are related to the coordinates $(w,z)\in\mathcal{D}_1^J$ by the map (3.1), our Kähler two-form ω_1

$$-i\omega_1=\frac{2k}{(1-w\bar{w})^2}dw\wedge d\bar{w}+\frac{A\wedge\bar{A}}{1-w\bar{w}},\ A=dz+\bar{\alpha}_0dw,\ \alpha_0=\frac{z+\bar{z}w}{1-w\bar{w}}$$

is identical with the one considered by Kähler-Berndt.

Jacobi algebra

$$\mathfrak{g}_n^J := \mathfrak{h}_n \rtimes \mathfrak{sp}(n,\mathbb{R})$$

$$[a_i, a_j^+] = \delta_{ij};\ [a_i, a_j] = [a_i^+, a_j^+] = 0.$$

$$\begin{aligned} [a_k^+, K_{ij}^+] &= [a_k, K_{ij}^-] = 0,\ \ 2[a_i, K_{kj}^+] = \delta_{ik} a_j^+ + \delta_{ij} a_k^+, \\ 2[K_{ij}^0, a_k^+] &= \delta_{jk} a_i^+,\ \ 2[a_k, K_{ij}^0] = \delta_{ik} a_j,\ \ 2[K_{kj}^-, a_i^+] = \delta_{ik} a_j + \delta_{ij} a_k. \end{aligned}$$

Composition law

$$G_n^J := H_n \rtimes \mathrm{Sp}(n,\mathbb{R}), g = \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{R})_{\mathbb{C}},$$

$$(\alpha_2, t_2) \circ (\alpha_1, t_1) = (\alpha_2 + \alpha_1, t_2 + t_1 + \mathrm{Im}(\alpha_2 \bar{\alpha}_1)).$$

$$(g_1, \alpha_1, t_1) \circ (g_2, \alpha_2, t_2) = (g_1 \circ g_2, g_2^{-1} \cdot \alpha_1 + \alpha_2, t_1 + t_2 + \mathrm{Im}(g_2^{-1} \cdot \alpha_1 \bar{\alpha}_2)).$$

$$g \cdot \tilde{\alpha} : A\alpha + B\bar{\alpha}$$

The action of Jacobi group on Siegel-Jacobi disc

$M = \mathcal{D}_n^J = H_n/\mathbb{R} \times \mathrm{Sp}(n,\mathbb{R})/\mathrm{U}(n) = \mathbb{C}^n \times \mathcal{D}_n$, ($M$ – $\frac{n(n+3)}{2}$-dimensional manifold)

$$\mathcal{D}_n := \{Z \in M(n,\mathbb{C}) | Z = Z^t,\ 1 - Z\bar{Z} > 0\}$$

The action of the group G_n^J on $\mathcal{D}_n^J$: $(g,\alpha) \times (W,z) \rightarrow (W_1, z_1)$:

$$W_1 = (AW + B)(\bar{B}W + \bar{A})^{-1}; \quad (4.2a)$$

$$z_1 = (WB^* + A^*)^{-1}(z + \alpha - W\bar{\alpha}). \quad (4.2b)$$

Coherent states

$$e_{z,W} = \exp(\boldsymbol{X})e_0,\ \boldsymbol{X} := \sum_i z_i a_i^+ + \sum_{ij} w_{ij}\boldsymbol{K}_{ij}^+,\ z = (z_i), \tag{4.3}$$

$$a_i e_o = 0,\ i = 1, \cdots, n. \tag{4.4}$$

$$\boldsymbol{K}_{ij}^+ e_0 \neq 0,\ \ \boldsymbol{K}_{ij}^- e_0 = 0,\ \ \boldsymbol{K}_{ij}^0 e_0 = \frac{k_i}{4}\delta_{ij}e_0,\ i,j = 1, \ldots, n \tag{4.5}$$

$e_0 = e_0^H \otimes e_0^K$: e_0^H – the minimum weight vector (vaccum) for the Heisenberg-Weyl group H_n; e_0^K – extremal weight vector for $\mathrm{Sp}(n, \mathbb{R})$ corresponding to the weight $k = \{k_i\}$ in (4.5).

Differential action of the generators

$$
\begin{aligned}
\boldsymbol{a} &= \frac{\partial}{\partial z}\\
\boldsymbol{a}^{+} &= z + W\frac{\partial}{\partial z}\\
\mathbb{K}^{-} &= \frac{\partial}{\partial W}\\
\mathbb{K}^{0} &= \frac{k}{4} + \frac{1}{2}\frac{\partial}{\partial z}\otimes z + \frac{\partial}{\partial W}W\\
\mathbb{K}^{+} &= \frac{W'}{4} + \frac{1}{2}z\otimes z + \frac{1}{2}\left(W\frac{\partial}{\partial z}\otimes z + z\otimes\frac{\partial}{\partial z}W\right) + W\frac{\partial}{\partial W}W.
\end{aligned}
$$

$(A\otimes B)_{kl} = a_k b_l$, $A = (a_k)$, $B = (b_l)$, $k = \mathrm{diag}(k_1, \ldots, k_n)$,
$w'_{kl} = (k_k + k_l)w_{kl}$, $k, l = 1, \ldots, n$, $w_{kl} = w_{lk}$, $\frac{\partial}{\partial w_{kl}} = \frac{\partial}{\partial w_{lk}}$.

Scalar product on $\mathcal{D}_n^J$

$$(\phi,\psi)=\Lambda_n\int_{z\in\mathbb{C}^n;1-W\bar{W}>0}\bar{f}_\phi(z,W)f_\psi(z,W)QK^{-1}dzdW \qquad (4.7)$$

$K=(e_{z,W},e_{z,W})$ - the reproducing kernel, Q - density of the volume form,

$$K=\det(M)^{\frac{k}{2}}\exp\frac{1}{2}[2<z,Mz>+<W\bar{z},Mz>+<z,MW\bar{z}>]$$

$$Q=\det(1-W\bar{W})^{-(n+2)},\ M=(1-W\bar{W})^{-1}$$

$$dz=\prod_{i=1}^{n}\Re z_i\Im z_i;\ \ dW=\prod_{1\le i\le j\le n}\Re w_{ij}\Im w_{ij}.$$

Berezin's quantization

$$\Lambda_n = \frac{k-3}{2\pi^{\frac{n(n+3)}{2}}}\prod_{i=1}^{n-1}\frac{(\frac{k-3}{2}-n+i)\Gamma(k+i-2)}{\Gamma[k+2(i-n-1)]}.$$

Compare with the case of the symplectic group: *a shift of p to $p-1/2$ in the normalization constant $\Lambda_n=\pi^{-n}J^{-1}(p)$.*

Kähler two-form on $\mathcal{D}_n^J$

$$-\mathrm{i}\omega_n = \frac{k}{2}\mathrm{Tr}(C\wedge\bar{C}) + \mathrm{Tr}(A^t\bar{M}\wedge\bar{A}), \tag{4.8}$$

$$A = dz + dW\bar{x},$$

$$C = MdW,\ \ M = (1 - W\bar{W})^{-1}$$

$$x = (1 - W\bar{W})^{-1}(z + W\bar{z}), W \in \mathcal{D}_n, z \in \mathbb{C}^n,$$

The Siegel-Jacobi manifold $\mathcal{X}_n^J = \mathcal{H}_n \times \mathbb{R}^{2n}$

$\mathcal{H}_n$ – Siegel upper half plane

$$\mathcal{H}_n := \{Z \in M(n,\mathbb{C}) | Z = U + \mathrm{i}V, U, V \in M(n,\mathbb{R}), (V) > 0,$$
$$U^t = U; V^t = V\}$$

Let $g = (M, l) \in G_n^J(\mathbb{R})_0$, i.e.

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{R}),\ \ l = (l_1, l_2) \in \mathbb{R}^{2n}, \qquad (5.1)$$

and $v \in \mathcal{H}_n,\ z \in \mathbb{C}^n \equiv \mathbb{R}^{2n}$.

The action of the group $G_n^J(\mathbb{R})_0$ on $\mathcal{X}_n^J$: $(M, l) \times (v, z) \rightarrow (v_1, z_1)$,

$$v_1 = (Av + B)(Cv + D)^{-1}; \qquad (5.2a)$$
$$z_1 = (z + vl_1^t + l_2^t)(Cv + D)^{-1}. \qquad (5.2b)$$

The Kähler two-form on $\mathcal{X}_n^J$

Under the partial Cayley transform

$$w = (v - i)(v + i)^{-1};\ z = (v + i)^{-1} 2iu$$

of $\mathcal{X}_n^J \rightarrow \mathcal{D}_n^J$, *the two-form* ω_n *on* $\mathcal{D}_n^J$ *becomes on* $\mathcal{X}_n^J$

$$-\mathrm{i}\omega'_n = \frac{k}{2}\mathrm{Tr}(p \wedge \bar{p}) + \frac{2}{\mathrm{i}}\mathrm{Tr}(B^t \wedge \bar{B}) \tag{5.3}$$

$$p = (\bar{v} - v)^{-1} dv;\ B = du - dv(v - \bar{v})^{-1}(u - \bar{u}).$$

“n”-dimensional generalization of Berndt-Kähler two-form ω'_1.

Remark $\mathcal{D}_n^J$ and $\mathcal{X}_n^J$ are called by Jae-Hyun Yang *Siegel-Jacobi spaces*. Kähler calls $\mathcal{X}_1^J$ *Phasenraum der Materie*, v is *Pneuama*, u is *Soma*.

Comparation with Yang's results

$\mathcal{H}_{n,m} = \mathcal{H}_n \times \mathbb{C}^{(m,n)}$- the Siegel-Jacobi space. Sp$(n,\mathbb{R})$ acts on $\mathcal{H}_n$ transitively by (5.2a). Sp$(n,\mathbb{R})/U(n) \cong \mathcal{H}_n$ - Hermitean symmetric space, Sigel upper half plane.

$$H_{\mathbb{R}}^{(n,m)} = \left\{(\lambda,\mu,\kappa)|\lambda,\mu\in\mathbb{R}^{(m.n)},\kappa\in\mathbb{R}^{(m,n)}\right\}$$

– (generalized) Heisenberg group.

$$G^J = \mathrm{Sp}(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

– Jacobi group with the multiplication law

$$(M_0,(\lambda_0,\mu_0,\kappa_0))\cdot(M,(\lambda,\mu,\kappa)) = (M_0M,(\tilde{\lambda}_0+\lambda,\tilde{\mu}_0+\mu,\kappa_0+\kappa+\tilde{\lambda}_0\mu^t-\tilde{\mu}_0\lambda^t))$$

$(\tilde{\lambda}_0,\tilde{\mu}_0) = (\lambda_0,\mu_0)M$. G^J acts on the Siegel-Jacobi space $\mathcal{H}_{n,m}$ transitively by

$$(M,(\lambda,\mu,\kappa))(\Omega,Z) = (M\circ\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1})$$

$G^J/K^J \cong \mathcal{H}_{n,m}$ is a non-reductive complex manifold, where $K^J = \mathrm{U}(n)\times\mathrm{Sym}(n,\mathbb{R})$.

Comparation with Yang's results

Identifications: (v, u) in our partial Cayley transform (5.2) with Yang's partial Cayley transform (11) in Yang

$$\Omega = \mathrm{i}(I + W)(I - W)^{-1}; Z = 2\mathrm{i}\eta(I - W)^{-1}, \tag{5.4}$$

$$(v, u) \leftrightarrow (\Omega, \frac{Z^t}{2\mathrm{i}}). \tag{5.5}$$

Remark

The case $m = 1$ in Theorem 1 in Yang is our (5.3), while our relation (4.8) is theorem 5 in Yang; Our factor Q_1

$$Q_1 = 2^{-n(n+3)}[\det(2i(\bar{v} - v))]^{-(n+2)} = 2^{-n(n+1)}[\det(\operatorname{Im} v)]^{-(n+2)} \tag{5.6}$$

corresponds to Yang's result expressed in Lemma A in the same situation $m = 1$.

Comparation with Yang's result on Jacobi-Sigel half plane

Yang's notation $\Omega = X + \mathrm{i}Y; Z = U + \mathrm{i}V$. Our (5.3) expressed in Yang's notation

$$
\begin{aligned}
-\mathrm{i}w_n' &= \frac{k}{8}\mathrm{Tr}(Y^{-1}d\Omega \wedge Y^{-1}d\bar{\Omega}) \\
&+ \frac{1}{8}\mathrm{Tr}[(dZ - VY^{-1}d\Omega)Y^{-1} \wedge (d\bar{Z}^t - d\bar{\Omega}Y^{-1}V^t)]
\end{aligned}
$$

$$
\begin{aligned}
-\mathrm{i}w_n' &= \frac{k}{8}\mathrm{Tr}(Y^{-1}\,d\Omega \wedge Y^{-1}\,d\bar{\Omega}) \\
&+ \frac{1}{8}\mathrm{Tr}(dZY^{-1} \wedge d\bar{Z}^t) + \frac{1}{8}\mathrm{Tr}(VY^{-1}d\Omega Y^{-1} \wedge \bar{\Omega}y^{-1}V^t) \\
&- \frac{1}{8}\mathrm{Tr}(dZY^{-1} \wedge \bar{\Omega}Y^{-1}V^t) - \frac{1}{8}\mathrm{Tr}(VY^{-1}d\Omega Y^{-1} \wedge d\bar{Z}^t)
\end{aligned}
$$

Yang's Kähler two-form on Jacobi-Siegel disc

$$
\begin{aligned}
&\frac{1}{4}\, d\tilde{s}^2_{n,m;A,B} = \\
& A\,\mathrm{tr}\big((I_n - W\overline{W})^{-1} dW (I_n - \overline{W}W)^{-1} d\overline{W}\big) \\
+ \quad & B\Big\{\mathrm{tr}\big((I_n - W\overline{W})^{-1}\, {}^t(d\eta)\, d\overline{\eta}\,\big) \\
+ \quad & \mathrm{tr}\big((\eta\overline{W} - \overline{\eta})(I_n - W\overline{W})^{-1} dW \\
& \quad (I_n - \overline{W}W)^{-1}\, {}^t(d\overline{\eta})\big) \\
+ \quad & \mathrm{tr}\big((\overline{\eta}W - \eta)(I_n - \overline{W}W)^{-1} d\overline{W} \\
& \quad (I_n - W\overline{W})^{-1}\, {}^t(d\eta)\,\big) \\
- \quad & \mathrm{tr}\big((I_n - W\overline{W})^{-1}\, {}^t\eta\, \eta\, (I_n - \overline{W}W)^{-1} \\
& \quad \overline{W} dW (I_n - \overline{W}W)^{-1} d\overline{W}\,\big) \\
- \quad & \mathrm{tr}\big(W(I_n - \overline{W}W)^{-1}\, {}^t\overline{\eta}\, \overline{\eta}\, (I_n - W\overline{W})^{-1} \\
& \quad dW (I_n - \overline{W}W)^{-1} d\overline{W}\,\big)
\end{aligned}
$$

Yang's- result- continuation

$$
\begin{aligned}
+ \quad & \mathrm{tr}\Big((I_n - W\overline{W})^{-1}\, {}^t\eta\,\overline{\eta}\,(I_n - W\overline{W})^{-1} \\
& \quad dW(I_n - \overline{W}W)^{-1} d\overline{W}\Big) \\
+ \quad & \mathrm{tr}\Big((I_n - \overline{W})^{-1}\, {}^t\overline{\eta}\,\eta\,\overline{W}\,(I_n - W\overline{W})^{-1} \\
& \quad dW(I_n - \overline{W}W)^{-1} d\overline{W}\Big) \\
+ \quad & \mathrm{tr}\Big((I_n - \overline{W})^{-1}(I_n - W)(I_n - \overline{W}W)^{-1} \\
& \quad {}^t\overline{\eta}\,\eta\,(I_n - \overline{W}W)^{-1}\,(I_n - \overline{W})(I_n - W)^{-1} \\
& \quad dW(I_n - \overline{W}W)^{-1} d\overline{W}\Big)\ \Big\}
\end{aligned}
$$

The Schrödinger-Weil representation

Squzeed states: $e_0 \equiv e_0^H$. Realization of the squeezed states-

$$\mathbf{K}_{ij}^+ = \frac{1}{2}a_i^+ a_j^+, \ \mathbf{K}_{ij}^- = \frac{1}{2}a_i a_j, \ \mathbf{K}_{ij}^0 = \frac{1}{4}(a_i^+ a_j + a_j a_i^+), \ i,j = 1,\ldots,n \quad (6.1)$$

coresponding to the eigenvalues $k_i = k = 2$ in (4.5). *The fundamental principle* in representation theory of the Jacobi group G^J: any representation π of the Jacobi G^J is obtained in a unique way as $\pi = \pi_{SW}^m \otimes \tilde{\pi}$; *the Schrödinger-Weil representation* π_{SW}^m is a certain projective representation of G^J of index m and $\tilde{\pi}$ is a representation of the metaplectic group $\mathrm{Mp}(n,\mathbb{R})$, considered as a projective representation of $\mathrm{Sp}(n,\mathbb{R})$.

G^J - group of Harish-Chandra type- Satake

$\hat{\pi}(\mathfrak{g}^J_{\mathbb{C}}) = \mathfrak{p}_+ + \mathfrak{k} + \mathfrak{p}_-$, $\mathfrak{p}_- = \mathfrak{p}_+^\dagger$, $\mathfrak{p}_+$ – raising operators $a^+_i, \mathbf{K}^+_{ij}$, $\mathfrak{k}$ is generated by I, $\mathbf{K}^0_{ij}$. If $X_1, \cdots, X_n \in \mathfrak{U}(\mathfrak{p}_+)$, we introduce a holomporphic family of elements $E_z = \exp(z_1X_1 + \cdots + z_nX_n)$, Ω is an open subset of $\mathbb{C}^n$. $\Phi_z = E_z\Phi_0$, Φ_0 is an cyclic vector in the domain $\mathcal{D}$ and $\mathfrak{p}_-\Phi_0 = 0$, $\mathfrak{k}\Phi_0 = \langle\Phi_0\rangle_{\mathbb{C}}$.
$T : \mathcal{H} \rightarrow \mathcal{H}_{\rm hol}(\Omega)$ $T(\phi)(z) = \langle\Phi_{\bar{z}}, \phi\rangle, \phi \in \mathcal{H}$, $z \in \Omega$.
We take $\mathcal{H} = \mathcal{H}_0 = L^2(\mathbb{R}^n)$, $\mathcal{D} = \mathcal{S}(\mathbb{R}^n)$. CS-vectors (4.3) in biboson operators $\Phi_{\alpha,W} = \exp(\sum\alpha_ia^+_i + \frac{1}{2}\sum w_{ij}a^+_ia^+_j)e^H_0$. The vector numbers form a complete orthonormal base of analytic vectors of the Hilbert space $\mathcal{H}_0$, with vacuum vector ϕ_0.
$T_0 : \mathcal{H}_0 \rightarrow \mathcal{D}^J$, $T_0(\phi)(\alpha, W) = \langle\Phi_{\bar{\alpha},\bar{W}}, \phi\rangle$, $\Phi \in \mathcal{H}_0$, $\Phi_0 = \phi_0$.

Notation

$\mathcal{H}(\mathcal{D}_n^J)$ - holomorphic functions on $\mathcal{D}_n^J$,

$$T^{[s]} = \prod_{i=1}^{n} t_i^{s_i}, \quad [s!] = \prod_{i=1}^{n} s_i!, \quad |s| = \sum_{i=1}^{n} s_i, \quad \delta_{sr} = \prod_{i=1}^{n} \delta_{s_i r_i},$$

$T^t = (t_1, \dots, t_n) \in \mathbb{C}^n$. $[s] = (s_1, \dots, s_n)$, $[s],\ [r] \in S_n = \mathbb{N}^n$.

$$G_T(Z,W) = \exp(T^t Z + \frac{1}{2} T^t W T) = \sum_{s \in S_n} \frac{T^{[s]}}{[s!]} P_{[s]}(Z,W), \tag{7.1}$$

$$P_{[s]}(Z,W) = \sum_{q,\, s-q \in S_n} \frac{[s!]}{[(s-q)!][q!]} Z^{[s-q]} S_{[q]}(W), \tag{7.2}$$

$$\exp(\frac{1}{2} T^t W T) = \sum_{r \in \mathbb{N}^n} \frac{T^{[r]}}{[r!]} S_{[r]}(W),\ S_{[r]} = 0 \textit{ for odd} \tag{7.3}$$

Scalar product $(,)$ on $\mathcal{H}(\mathcal{D}^J_n)$ of the form (4.7)

$$(f,g) = \Lambda \int_{\mathcal{D}^J_n} \bar{f}(Z,W) g(Z,W) \rho(W,\bar{W}) \exp[-A(Z,W)] \mathbf{d}^{2n} Z \mathbf{d}^{n(n+1)} W,$$

$$A(Z,W) = Z^+ MZ + \frac{1}{2} Z^t \bar{W} MZ + \frac{1}{2} Z^+ MW\bar{Z},\ M = (1 - W\bar{W})^{-1},\ W \in \mathcal{D}_n.$$

$$(P_{[r]}, P_{[s]}) = \delta_{sr}[s!]\lambda, \; s, r \in S_n, \; \lambda = \pi^n \int_{\mathcal{J}_n} \rho(W, \bar{W})(1 - W\bar{W})^{1/2} \, \mathbf{d}^{n(n+1)} W.$$

$\lambda = 1$ for $\rho(W) = \det(1 - W\bar{W})^{k/2-n-2}$.
The family of polynomials $\{P_{[s]}\}_{s \in S_n}$ is orthogonal.

$$f_{[s]} = (\lambda[s!])^{-1/2} P_{[s]}. \tag{7.4}$$

Proposition

a) The family of polynomials $\{f_{[s]}\}_{s\in S_n}$ *form a complete orthonormal basis of analytic vectors for the Hilbert space* $T_0(\mathcal{H}_0)$*. The generating function of* $\{f_{[s]}\}_{s\in S_n}$ *can be written as*

$$\exp(T^t Z + \frac{1}{2}T^t WT) = \sqrt{\lambda}\sum_{s\in S_n}\frac{T^{[s]}}{\sqrt{[s!]}}f_{[s]}(Z,W)\ . \qquad (7.5)$$

Propositon- continuation

b) $f \in \mathcal{H}(\mathcal{D}^J_n)$ is a solution of the system of differential equations ("heat equation")

$$\left(\frac{\partial^2}{\partial z_j \partial z_k} - 2\frac{\partial}{\partial w_{jk}}\right) f = 0, \quad 1 \leq j \leq k \leq n, \tag{7.6}$$

if and only if $f \in \mathcal{H}_0(\mathcal{D}^J_n)$.
c)Let $\pi_o = T_0 \pi^m_{SW} T_0^{-1}$, as in §6. The base of differential operators can be expressed as in (4.6) with $k = 2$.
d) The reproducing kernel on $\mathcal{H}_0(\mathcal{D}^J_n)$ admits the expansion:

$$\det(1 - W\bar{W})^{-1/2} \exp A(Z, W) = \sum_{s \in S_n} f_{[s]}(\bar{Z}, \bar{W}) f_{[s]}(Z, W). \tag{7.7}$$

Proposition

Let $Q_{[a]}$ be the polynomials

$$\det(1 - W\bar{W})^{(1-k)/2} = \sum_{a\in A_n} Q_{[a]}(\bar{W})Q_{[a]}(W).$$

Then $f_{[s]}(Z,W)Q_{[a]}(W)$ is an orthogonal base for the reproducing kernel K which admits the series expansion

$$\begin{aligned} K(Z,W) &= \det(1 - W\bar{W})^{-k/2}\exp A(Z,W) = \\ &= \sum_{s\in S_n,[a]\in A_n} f_{[s]}(\bar{Z},\bar{W})Q_{[a]}(\bar{W})f_{[s]}(Z,W)Q_{[a]}(W) \end{aligned}$$

Equations of motion determined by linear Hamiltonians

$$H = \epsilon_i a_i + \bar{\epsilon}_i a_i^+ + \epsilon_{ij}^0 K_{ij}^0 + \epsilon_{ij}^- K_{ij}^- + \epsilon_{ij}^+ K_{ij}^+$$

Classical evolution (in the meaning of Berezin) associated to the (linear) Jacobi oscillators: a *Matrix Riccati equation*.

$$\begin{aligned} \mathrm{i}\dot{z} &= \epsilon + W\bar{\epsilon} + \epsilon^+ zW + \frac{1}{2} z\epsilon^0, \\ \mathrm{i}\dot{W} &= W\epsilon^+ W + W\epsilon^0 + \epsilon^-. \end{aligned}$$

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