Holomorphic discrete series representations on Siegel-Jacobi domains

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$\mathfrak{AGGRHS}$ – 2010

22-26 November, 2010, Lorentz Center, Leiden

Outline

• **For mathematicians**: *Jacobi group*– $G^J_n = H_n \rtimes \mathrm{Sp}(n,\mathbb{R})$
$(2n+1)(n+1)$- dim. G^J_n - group of *Harish-Chandra* type (Satake)

Generalized Jacobi groups (Takase, Yang, Lee...)

• **In Physics**: Schrödinger (Hagen, or conformal Galilean...) group.

• Denomination *Jacobi group* G_n^J introduced by Eichler and Zagier, *Theory of Jacobi forms* (1985), inspired by Pyatetskii-Shapiro

• The denomination *Jacobi group* was adopted also in the monograph Berndt & Schmidt *Elements of the Representation Theory of the Jacobi group* (1998).

• Kirillov; K. B. Wolf: *Weyl-symplectic group*-1975;

• R. Berndt (1984), E. Kähler (1983); *Poincaré group* or *The New Poincaré group* investigated by Erich Kähler

• **Jacobi & Physics** The Jacobi group is an important object in connection with Quantum Mechanics, Geometric Quantization, Optics-*squeezed states* .

The Jacobi group G^J (cf Berndt, Takase, Lee)

$\mathrm{Sp}(n,\mathbb{R})$: $\sigma \in M_{2n}(\mathbb{R})$, ${}^t\sigma J_n \sigma = J_n$,

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \tag{2.1}$$

The Siegel upper half space $\mathfrak{H}_n$: $\Omega \in M_n(\mathbb{C})$, $\Omega =^t \Omega$, $\mathrm{Im}\,\Omega > 0$.
$\mathrm{Sp}(n,\mathbb{R})$ acts transitively on $\mathfrak{H}_n$, $\sigma\Omega = (a\Omega + b)(c\Omega + d)^{-1}$.

Siegel-Jacobi domain

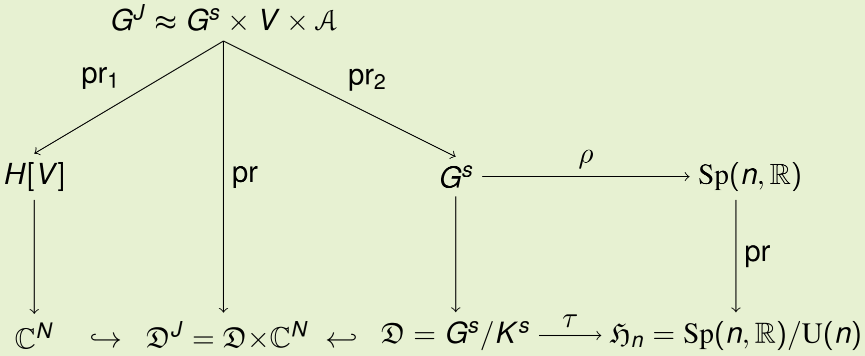
G^s - a Zariski connected ss real algebraic group of Hermitian type.
K^s - a maximal compact subgroup of G.
$\mathfrak{D} = G^s/K^s$ - the associated Hermitian symmetric domain.
Suppose $\exists$ homomorphism $\rho : G^s \rightarrow \mathrm{Sp}(n, \mathbb{R})$ & holomorphic map $\tau : \mathfrak{D} \rightarrow \mathfrak{H}_n$, $\tau(gz) = \rho(g)\tau(z)$, $g \in G^s$, $z \in \mathfrak{D}$.
Jacobi group $G^J = G^s \ltimes H[V]$, V - symplectic $\mathbb{R}$-space, D : - nondegenerate alternating bilinear form $V \times V \rightarrow \mathcal{A}$, $\mathcal{A}$ - center of $H[V]$, $G^J \approx G^s \times V \times \mathcal{A}$:

$$gg' = (\sigma\sigma', \rho(\sigma)v' + v, \varkappa + \varkappa' + \frac{1}{2}D(v, \rho(\sigma)v')), \; g = (\sigma, v, \varkappa) \in G^J \quad (2.2)$$

Siegel-Jacobi domain associated to the Jacobi group G^J:

$$\mathfrak{D}^J = \mathfrak{D} \times \mathbb{C}^N \cong G^J/(K^s \times \mathcal{A}), \dim \; V = 2N.$$

Jacobi group G^J & Siegel-Jacobi domain



Group of Harish-Chandra type

$w_0 \in \mathfrak{D}$, $I_{\tau(w_0)}$ - the complex structure on V corresponding to $\tau(w_0) \in \mathfrak{H}_n$.
$V_{\mathbb{C}} = V_+ \oplus V_-$ - complexification of V; $V_{\pm} = \{v \in V_{\mathbb{C}} | I_{\tau(w_0)} v = \pm \mathrm{i} v\}$.
$w \in \mathfrak{D}$, $v \in V_{\mathbb{C}} \rightarrow v_w = v_+ - \tau(w) v_- \in V_+$, $(v = v_+ + v_-, v_{\pm} \in V_{\pm})$.
G^J - algebraic group of Harish-Chandra type:
G - Zariski connected $\mathbb{R}$-group.
Suppose there are given a Zariski connected $\mathbb{R}$-subgroup K of G, $\mathfrak{k}$, connected unipotent $\mathbb{C}$ -subgroups $P_{\pm}$ of $G_{\mathbb{C}}$, $\mathfrak{p}_{\pm}$.
G - of Harish-Chandra type:
(HC 1) $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}_+ + \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_-$, $\left[\mathfrak{k}_{\mathbb{C}}, \mathfrak{p}_{\pm}\right] \subset \mathfrak{p}_{\pm}$, $\overline{\mathfrak{p}_+} = \mathfrak{p}_-$; (HC 2) the map $P_+ \times K_{\mathbb{C}} \times P_- \rightarrow G_{\mathbb{C}}$ - holomorphic injection of $P_+ \times K_{\mathbb{C}} \times P_-$ onto its open image $P_+ K_{\mathbb{C}} P_-$; (HC 3) $G \subset P_+ K_{\mathbb{C}} P_-$ and $G \cap K_{\mathbb{C}} P_- = K$.
Notation: $g \in P_+ K_{\mathbb{C}} P_- \subset G_{\mathbb{C}}$, $(g)_+ \in P_+$, $(g)_0 \in K_{\mathbb{C}}$, $(g)_- \in P_-$ – components of g, $g = (g)_+ (g)_0 (g)_-$.

Generalized Harish-Chandra embedding

H- linear algebraic group, $\mathring{H}$ - identity connected component (in the usual topology).

Generalized Harish-Chandra embedding $\mathcal{D} = \mathring{G}/\mathring{K} \hookrightarrow \mathfrak{p}_+$: $g\mathring{K} \longmapsto z$, $g \in \mathring{G}$, $z \in \mathfrak{p}_+$, $\exp z = (g)_+$.

$(G_{\mathbb{C}} \times \mathfrak{p}_+)' = \{(g, z) \in G_{\mathbb{C}} \times \mathfrak{p}_+ | \, g \exp z \in P_+ K_{\mathbb{C}} P_-\}$;

$(\mathfrak{p}_+ \times \mathfrak{p}_+)' = \{(z_1, z_2) \in \mathfrak{p}_+ \times \mathfrak{p}_+ | (\exp \bar{z}_2)^{-1} \exp z_1 \in P_+ K_{\mathbb{C}} P_-\}$.

Canonical automorphy factor $J : (G_{\mathbb{C}} \times \mathfrak{p}_+)' \rightarrow K_{\mathbb{C}}$, *canonical kernel function* $K : (\mathfrak{p}_+ \times \mathfrak{p}_+)' \rightarrow K_{\mathbb{C}}$ for G,

$(g, z) \in (G_{\mathbb{C}} \times \mathfrak{p}_+)', (z', z) \in (\mathfrak{p}_+ \times \mathfrak{p}_+)'$.

$$J(g, z) = (g \exp z)_0 \, , \quad K(z', z) = (((\exp \bar{z})^{-1} \exp z')_0)^{-1} . \qquad (2.3)$$

Theorem

a) G^J *acts transitively on* $\mathfrak{D}^J$, $(g=(\sigma, v, \varkappa)$, $x=(w,z))$,

$$gx = (\sigma w, v_{\sigma w} + {}^t(c\tau(w)+d)^{-1}z),\ \rho(\sigma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \qquad (2.4)$$

b) The canonical automorphy factor J *for* G^J

$$J(g,x) = (J_1(\sigma, w), 0, J_2(g,x)), g = (\sigma, v, \varkappa) \in G^J, x = (w,z) \in \mathfrak{D}^J, \qquad (2.5)$$

J_1 *is the canonical automorphy factor for* G^s,

$$J_2(g,x) = \varkappa + \frac{1}{2}D(v, v_{\sigma w}) + \frac{1}{2}D(2v + \rho(\sigma)z, J_1(\sigma, w)z). \qquad (2.6)$$

Continuation

c) The canonical kernel function K for the Jacobi group G^J

$$K(x,x') = (K_1(w,w'), 0, K_2(x,x')), x = (w,z) \in \mathfrak{D}^J, \qquad (2.7)$$

K_1 *- the canonical kernel function for* G^s,

$$K_2(x,x') = D(2\bar{z}' + \frac{1}{2}\overline{\tau(w')}z, qz) + \frac{1}{2}D(\bar{z}', q\tau(w)\bar{z}'),\ q = \rho(K_1(w,w'))^{-1}. \qquad (2.8)$$

Jacobi group G_n^J

Heisenberg group $\mathrm{H}_n(\mathbb{R})$: (λ, μ, κ), $\lambda, \mu \in M_{1\,n}(\mathbb{R})$, $\kappa \in \mathbb{R}$,

$$(\lambda, \mu, \kappa) \circ (\lambda', \mu', \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda'). \quad (2.9)$$

$G_n^J = \mathrm{Sp}(n, \mathbb{R}) \ltimes \mathrm{H}_n(\mathbb{R})$:

$$(\sigma, (\lambda, \mu, \kappa)) \cdot (\sigma', (\lambda', \mu', \kappa')) = (\sigma\sigma', (\lambda\sigma', \mu\sigma', \kappa) \circ (\lambda', \mu', \kappa')), \quad (2.10)$$

The *Jacobi group* G_n^J acts transitively on *Jacobi-Siegel space* $\mathfrak{H}_n^J = \mathfrak{H}_n \times \mathbb{C}^n$, $g(\Omega, \zeta) = (\Omega_g, \zeta_g)$,

$$\Omega_g = (a\Omega + b)(c\Omega + d)^{-1},\ \zeta_g = \nu(c\Omega + d)^{-1},\ \nu = \zeta + \lambda\Omega + \mu. \quad (2.11)$$

Proposition

The canonical automorphy factor J_1, the canonical kernel function K_1 for $\mathrm{Sp}(n,\mathbb{R})$:

$$J_1(\sigma,\Omega) = \begin{pmatrix} {}^t(c\Omega+d)^{-1} & 0 \\ 0 & c\Omega+d \end{pmatrix}, \ \sigma \in \mathrm{Sp}(n,\mathbb{R}), \tag{2.12}$$

$$K_1(\Omega',\Omega) = \begin{pmatrix} 0 & \overline{\Omega}-\Omega' \\ (\Omega'-\overline{\Omega})^{-1} & 0 \end{pmatrix}, \ \Omega,\Omega' \in \mathfrak{H}_n. \tag{2.13}$$

The canonical automorphy factor $\theta = J_2(g,(\Omega,\zeta))$ for G^J_n

$$\theta = \kappa + \lambda\, {}^t\zeta + \nu\, {}^t\lambda - \nu(c\Omega+d)^{-1}c\, {}^t\nu, \ \nu = \zeta + \lambda\Omega + \mu. \tag{2.14}$$

The canonical automorphy kernel K_2 for G^J_n

$$K_2\left((\zeta',\Omega'),(\zeta,\Omega)\right) = -\frac{1}{2}(\zeta'-\bar{\zeta})(\Omega'-\bar{\Omega}')^{-1}({}^t\zeta' -{}^t\bar{\zeta}). \tag{2.15}$$

The Siegel disk

$\mathfrak{D}_n = \{W \in M_n(\mathbb{C}) | W =^t W, I_n - W\bar{W} > 0\}$.
$\mathrm{Sp}(n,\mathbb{R})_* = \mathrm{Sp}(n,\mathbb{C}) \cap \mathrm{U}(n,n)$: $\omega \in M_{2n}(\mathbb{C})$

$$\omega = \begin{pmatrix} p & q \\ \bar{q} & \bar{p} \end{pmatrix}, \quad {}^t p\bar{p} - {}^t\bar{q}q = I_n, \; {}^t p\bar{q} = {}^t\bar{q}p, \quad p, q \in M_n(\mathbb{C}). \qquad (2.16)$$

$\mathrm{Sp}(n,\mathbb{R})_*$ - transitively on $\mathfrak{D}_n \cong \mathrm{Sp}(n,\mathbb{R})_*/\mathrm{U}(n)$, $K_{n*} \cong \mathrm{U}(n)$ - maximal compact subgroup of $\mathrm{Sp}(n,\mathbb{R})_*$ $p \in \mathrm{U}(n)$, $q = 0$.
$\omega W = (pW + q)(\bar{q}W + \bar{p})^{-1}$.

G^J_{n*} - the Jacobi group - $(\omega, (\alpha, \varkappa))$, $\omega \in \mathrm{Sp}(n,\mathbb{R})_*$, $\alpha \in \mathbb{C}^n$, $\varkappa \in \mathrm{i}\mathbb{R}$,

$$(\omega', (\alpha', \varkappa'))(\omega, (\alpha, \varkappa)) = \left(\omega'\omega, (\alpha + \beta, \varkappa + \varkappa' + \beta^t\bar{\alpha} - \bar{\beta}^t\alpha)\right),$$

$\beta = \alpha' p + \bar{\alpha}'\bar{q}$.

The Siegel-Jacobi disk

The Heisenberg group $\mathrm{H}_n(\mathbb{R})_* = (I_n, (\alpha, \varkappa)) \in G^J_{n*}$, $\alpha \in \mathbb{C}^n$, $\varkappa \in \mathrm{i}\mathbb{R}$.
$\mathcal{A}_* \cong \mathbb{R}$ of $\mathrm{H}_n(\mathbb{R})_*$: $(I_n, (0, \varkappa)) \in G^J_{n*}$, $\varkappa \in \mathrm{i}\mathbb{R}$.
$\exists$ an isomorphism $\Theta : G^J_n \rightarrow G^J_{n*}$, $\Theta(g) = g_* = (\omega, (\alpha, \varkappa))$,
$g = (\sigma, (\lambda, \mu, \kappa))$,

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \omega = \begin{pmatrix} p_+ & p_- \\ \bar{p}_- & \bar{p}_+ \end{pmatrix}, \tag{2.17}$$

$$p_\pm = \frac{1}{2}(a \pm d) \pm \frac{\mathrm{i}}{2}(b \mp c), \quad \alpha = \frac{1}{2}(\lambda + \mathrm{i}\mu), \quad \varkappa = -\mathrm{i}\frac{\kappa}{2}. \tag{2.18}$$

$\mathfrak{D}^J_n = \mathfrak{D}_n \times \mathbb{C}^n \cong G^J_{n*}/(\mathrm{U}(n) \times \mathbb{R})$ – *the Siegel-Jacobi disk.*
G^J_{n*}, transitively on $\mathfrak{D}^J_n$, $g_*(W, z) = (W_{g_*}, z_{g_*})$, $g_* = (\omega, (\alpha, \varkappa)) \in G^J_{n*}$:

$$W_{g_*} = (pW + q)(\bar{q}W + \bar{p})^{-1}, \ z_{g_*} = (z + \alpha W + \bar{\alpha})(\bar{q}W + \bar{p})^{-1}. \tag{2.19}$$

Partial Cayley transform

$\phi : \mathfrak{D}^J_n \rightarrow \mathfrak{H}^J_n$:

$$\Omega = \mathrm{i}(I_n + W)(I_n - W)^{-1},\ \zeta = 2\,\mathrm{i}\,z(I_n - W)^{-1}, (W, z) \in \mathfrak{D}^J_n. \quad (2.20)$$

ϕ - biholomorphic map, $g\phi = \phi g_*$, $g \in G^J_n$, $g_* = \Theta(g)$.

The inverse partial Cayley transform $\phi^{-1} : \mathfrak{H}^J_n \rightarrow \mathfrak{D}^J_n$

$$W = (\Omega - \mathrm{i}I_n)(\Omega + \mathrm{i}I_n)^{-1},\ z = \zeta(\Omega + \mathrm{i}I_n)^{-1}, (W, z) = \in \mathfrak{D}^J_n. \quad (2.21)$$

Proposition

The canonical automorphy factor J_{1*}, the canonical kernel function K_{1*} for $\mathrm{Sp}(n,\mathbb{R})_*$

$$J_{1*}(\omega, W) = \begin{pmatrix} {}^t(\bar{q}W+\bar{p})^{-1} & 0 \\ 0 & \bar{q}W+\bar{p} \end{pmatrix}, \quad (2.22)$$

$$K_{1*}(W', W) = \begin{pmatrix} I_n - W'\bar{W} & 0 \\ 0 & {}^t(I_n - W'\bar{W})^{-1} \end{pmatrix}. \quad (2.23)$$

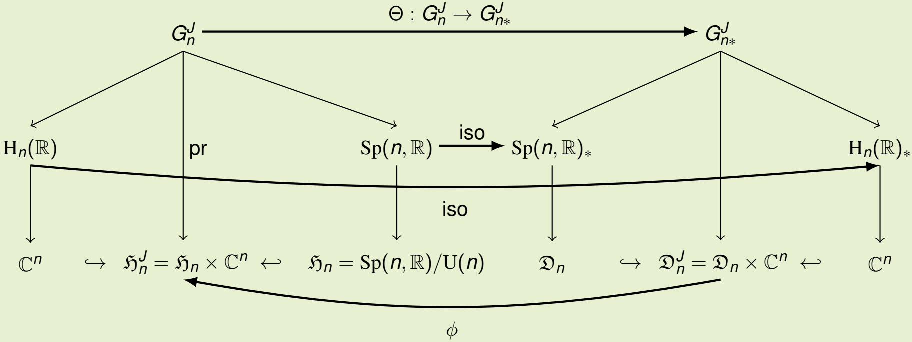
Canonical automorphy factor $\theta_* = J_2(g_*, (W,z))$ for G^J_{n*} ($g_* = (\omega, (\alpha, \varkappa)) \in G^J_{n*}$)

$$\theta_* = \kappa_* + z\,{}^t\alpha + \nu_*\,{}^t\alpha - \nu_*(\bar{q}W+\bar{p})^{-1}\bar{q}\,{}^t\nu_*, \quad \nu_* = z + \alpha W + \bar{\alpha}. \quad (2.24)$$

Can. autom. kernel for G^J_{n*} $K_{2*}((W',z'),(W,z)) = A(W',z';W,z)$

$$A(W',z';W,z) = (\bar{z} + \frac{1}{2}z'\bar{W})(I_n - W'\bar{W})^{-1}\,{}^tz' + \frac{1}{2}\bar{z}(I_n - W'\bar{W})^{-1}W'\,{}^t\bar{z}$$

Summary of notation



Fock space of functions

$\mathcal{F}_{mW}$, $W \in \mathfrak{D}_n, m > 0$ - Fock space of functions $\Phi \in \mathcal{O}(\mathbb{C}^n)$, $\|\Phi\|_{mW} < \infty$,

$$(\Phi, \Psi)_{mW} = (2\pi m)^n \left(\det(1 - W\bar{W})\right)^{-1/2} \times \int_{\mathbb{C}^n} \Phi(z)\overline{\Psi(z)} \exp(-8\pi m A(W, z)) d\nu(z), \quad (3.1)$$

$$d\nu(\zeta) = \pi^{-n} \prod_{i=1}^{n} d\mathrm{Re}\zeta_i \, d\mathrm{Im}\zeta_i, \zeta \in \mathbb{C}^n \quad (3.2)$$

$$A(W, z) = (\bar{z} + \frac{1}{2} z\bar{W})(I_n - W\bar{W})^{-1}\, {}^t z + \frac{1}{2} \bar{z}(I_n - W\bar{W})^{-1} W\, {}^t\bar{z}. \quad (3.3)$$

$\mathcal{F}_{m0}$ – the usual Bargmann space.

Gaussian functions

$G_U : \mathfrak{D}^J_n \to \mathbb{C}$, $U \in \mathbb{C}^n$, $G_U(W,Z) = G(U,Z,W)$,

$$G(U,Z,W) = \exp(U\,{}^tZ + \frac{1}{2}UW\,{}^tU) = \sum_{s\in\mathbb{N}^n} \frac{U^s}{s!} P_s(Z,W), (W,Z) \in \mathcal{D}^J_n \tag{3.4}$$

$$U^s = \prod_{i=1}^{n} U_i^{s_i}, \quad s! = \prod_{i=1}^{n} s_i!, \quad |s| = \sum_{i=1}^{n} s_i, \quad \delta_{sr} = \prod_{i=1}^{n} \delta_{s_i r_i}, \tag{3.5}$$

$U = (U_1, ..., U_n) \in M_{1n}(\mathbb{C}) \cong \mathbb{C}^n$, $s = (s_1, ..., s_n) \in \mathbb{N}^n$,
$r = (r_1, ..., r_n) \in \mathbb{N}^n$.
$P_s : \mathcal{D}^J_n \to \mathbb{C}$, $s \in \mathbb{N}^n$, - the matching functions of Neretin. Here

$$P_s(Z,W) = \sum_{a\in A_n,\, \tilde{a}\le s} \frac{s!}{2^{\hat{a}} a!(s-\tilde{a})!} Z^{s-\tilde{a}} W^a, \tag{3.6}$$

Notation & Calculation

A_n – symmetric matrices $a = (a_{ij})_{1\le i,j\le n}$, $a_{ij} \in \mathbb{N}$,

$$W^a = \prod_{1\le i\le j\le n} W_{ij}^{a_{ij}},\ a! = \prod_{1\le i\le j\le n} a_{ij},\ \tilde{a}_k = \sum_{i=1}^{n} a_{ik},\ \hat{a} = \sum_{i=1}^{n} a_{ii}$$

, $\tilde{a} \le s$: $\tilde{a}_i \le s_i$ for $1 \le i \le n$. Using the equations

$$\int_{\mathbb{C}^n} U^s \bar{U}^r d\nu(U) = \delta_{sr} s!, \tag{3.7}$$

$$\int_{\mathbb{C}^n} G(U,Z',W')G(\bar{U},\bar{Z},\bar{W})d\nu(U) = \det(1-W'\bar{W})^{-1/2}\exp A(W',z';W,z),$$

we obtain

$$\left(\det(1-W'\bar{W})\right)^{-1/2}\exp A(W',z';W,z) = \sum_{s\in\mathbb{N}^n}\frac{1}{s!}P_s(Z',W')\overline{P_s(Z,W)}.$$

$$\Phi_{Ws}(z) = \frac{1}{\sqrt{s!}}P_s(2\sqrt{2\pi m}z,W),\ s\in\mathbb{N}^n, \Phi_{Ws}\in\mathfrak{F}_{mW}. \tag{3.8}$$

Proposition

Given $W \in \mathfrak{D}_n$, the set of polynomials $\{\Phi_{Ws} | s \in \mathbb{N}^n\}$ forms an orthonormal basis of the Fock space $\mathfrak{F}_{mW}$. The kernel function of $\mathfrak{F}_{mW}$ admits the expansion

$$\left(\det(1 - W\bar{W})\right)^{-1/2} \exp\left(2\pi m A(W, z'; W, z)\right) = \sum_{s \in \mathbb{N}^n} \Phi_{Ws}(z')\overline{\Phi_{Ws}(z)}. \tag{3.9}$$

Proof. Given $U \in \mathbb{C}^n$, $W \in \mathcal{D}_n$, define the function $\Psi_{UW} : \mathbb{C}^n \to \mathbb{C}$, that $\Psi_{UW}(z) = G(U, 2\sqrt{2\pi m}z, W)$. Change of variables: $Z = 2\sqrt{2\pi m}z \Rightarrow$

$$\|\Psi_{UW}\|^2_{mW} = \pi^{-n} \det(1 - W\bar{W})^{-1/2} \int_{\mathbb{C}^n} \exp(B(U, Z, W) - A(Z, W)) d\nu(Z), \tag{3.10}$$

$$B(U, Z, W) = U^t Z + \bar{U} Z^\dagger - \frac{1}{2} U W^t U - \frac{1}{2} \bar{U} \bar{W} U^+. \tag{3.11}$$

Change of variables: $Z' = (1 - W\bar{W})^{-1/2}\left(Z - \bar{U} - WU\right)$, the relation $d\nu(Z) = \det(1 - W\bar{W}) d\nu(Z')$,&

Proof - continuation

$$\int_{\mathbb{C}^n} \exp\left(-\overline{Z'}^t Z' - \frac{1}{2}(Z'\bar{W}^t Z' + \overline{Z}' W Z'^{\dagger})\right) d\nu(Z') = \pi^n \left(\det(1 - W\bar{W})\right)^{-1/2}$$

$\Rightarrow \|\Psi_{UW}\|^2_{mW} = \exp(UU^{\dagger})$. Then

$$\sum_{s,r\in\mathbb{N}^n} \frac{U^s\bar{U}^r}{\sqrt{s!r!}} (\Phi_{Ws}, \Phi_{Wr})_{mW} = \sum_{s\in\mathbb{N}^n} \frac{1}{s!} U^s\bar{U}^s. \tag{3.12}$$

Comparing the coefficients of $U^s\bar{U}^r$ in the series of both sides of (3.12):

$$(\Phi_{Ws}, \Phi_{Wr})_{mW} = \delta_{sr} s!, \quad s, r \in \mathbb{N}^n. \tag{3.13}$$

■

Notation

$$f_s(W,z) = \frac{1}{\sqrt{s!}} P_s(2\sqrt{2\pi m}z, W), f_s : \mathcal{D}^J_n \to \mathbb{C},\; s \in \mathbb{N}^n. \qquad (3.14)$$

$\mathcal{H}_0(\mathcal{D}^J_n)$ – complex linear subspace of holomorphic functions $f \in \mathcal{O}(\mathcal{D}^J_n)$, basis $\{f_s | s \in \mathbb{N}^n\}$.

$\mathfrak{F}_m(\mathcal{D}^J_n)$ – the Hilbert space of all functions $f \in \mathcal{O}(\mathcal{D}^J_n)$, $\langle f, f \rangle_m < \infty$. The inner product $\langle ., . \rangle_m$ such that the set $\{f_s | s \in \mathbb{N}^n\}$ is an orthonormal basis.

Proposition

a) The generating function of the basis $\{f_s | s \in \mathbb{N}^n\}$ *can be expressed as*

$$\exp(8\pi m U\,^t z + \frac{1}{2} U W\,^t U) = \sum_{s \in \mathbb{N}^n} \frac{U^s}{\sqrt{s!}} f_s(W, z). \qquad (3.15)$$

The kernel function of $\mathfrak{F}_m(\mathcal{D}^J_n)$ *admits the expansion*

$$\left(\det(1 - W'\bar{W})\right)^{-1/2} \exp A(W', z'; W, z) = \sum_{s \in \mathbb{N}^n} f_s(W', z') \overline{f_s(W, z)}. \qquad (3.16)$$

b) $f \in \mathcal{O}(\mathcal{D}^J_n)$ *is a solution of the system of differential equations*

$$\frac{\partial^2 f}{\partial z_j \partial z_k} = 8\pi m (1 + \delta_{jk}) \frac{\partial f}{\partial W_{jk}}, \; 1 \leq j \leq k \leq n, \qquad (3.17)$$

if and only if $f \in \mathcal{H}_0(\mathcal{D}^J_n)$.

$\mathcal{H}^{mk}$

G^J_n, δ - rational representation of $GL(n,\mathbb{C})$, $\delta|_{\mathrm{U}(n)}$ – scalar irreducible representation of $\mathrm{U}(n)$ with highest weight k, $k\in\mathbb{Z}$, $\delta(A)=(\det A)^k$. $\chi=\delta\otimes\bar{\chi}^m$, $m\in\mathbb{R}$, central character χ^m of $\mathcal{A}\cong\mathbb{R}$: $\chi^m(\kappa)=\exp(2\pi\mathrm{i}m\kappa)$, $\kappa\in\mathcal{A}$. $\forall$ scalar holomorphic irreducible representation of G^J_n — index m, weight k. $m>0$ and $k>n+1/2$. $\mathcal{H}^{mk}$ – the Hilbert space functions $\varphi\in\mathcal{O}(\mathfrak{H}^J_n)$, $\|\varphi\|_{\mathfrak{H}^J_n}<\infty$,

$$(\varphi,\psi)_{\mathfrak{H}^J_n}=C\int_{\mathfrak{H}^J_n}\varphi(\Omega,\zeta)\overline{\psi(\Omega,\zeta)}\,\mathcal{K}^{mk}(\Omega,\zeta)^{-1}d\mu(\Omega,\zeta), \tag{4.1}$$

$$d\mu(\Omega,\zeta)=(\det Y)^{-n-2}\prod_{1\le i\le n}d\xi_i\,d\eta_i\prod_{1\le j\le k\le n}dX_{jk}\,dY_{jk}, \tag{4.2}$$

$$\mathcal{K}^{mk}(\Omega,\zeta)=\mathrm{K}^{mk}((\Omega,\zeta),(\Omega,\zeta))=\exp\left(4\pi m\eta Y^{-1}\,{}^t\eta\right)(\det Y)^k, \tag{4.3}$$

$$\mathrm{K}^{mk}((\zeta',\Omega'),(\zeta,\Omega))=\left(\det(\frac{\mathrm{i}}{2}\bar{\Omega}-\frac{\mathrm{i}}{2}\Omega')\right)^{-k}\exp\left(2\pi\mathrm{i}mK\left((\zeta',\Omega'),(\zeta,\Omega)\right)\right),$$

Theorem - Takase

π^{mk} - unitary representation of G_n^J on $\mathcal{H}^{mk}$, the automorphic factor $\mathcal{J}^{mk}$

$$\left(\pi^{mk}(g^{-1})\varphi\right)(\Omega,\zeta) = \mathcal{J}^{mk}(g,(\Omega,\zeta))\varphi(\Omega_g,\zeta_g), \varphi \in \mathcal{H}^{mk}, g \in G_n^J, \quad (4.4)$$

$$\mathcal{J}^{mk}(g,(\zeta,\Omega)) = (\det(c\Omega + d))^{-k}\exp(2\pi i m\theta), \quad (4.5)$$

Theorem

Suppose $k > n + 1/2$. Then $\mathcal{H}^{mk} \neq \{0\}$ and π^{mk} is an irreducible unitary representation of G_n^J, square integrable modulo center.

$\mathcal{H}_*^{mk}$

$\mathcal{H}_*^{mk}$ – complex pre-Hilbert space $\psi \in \mathcal{O}(\mathfrak{D}_n^J)$, $\|\psi\|_{\mathfrak{D}_n^J} < \infty$

$$(\psi_1, \psi_2)_{\mathfrak{D}_n^J} = C_* \int_{\mathfrak{D}_n^J} \psi_1(W,z)\overline{\psi_2(W,z)}\left(\mathcal{K}_*^{mk}(W,z)\right)^{-1} d\,\nu(W,z),$$

$$\mathcal{K}_*^{mk}(W,z) = \left(\det(I_n - W\bar{W})\right)^{-k} \exp(8\pi mA(W,z)), \qquad (4.6)$$

$$d\nu(W,z) = (\det(1 - W\bar{W}))^{-n-2}\prod_{i=1}^{n} d\,\mathrm{Re} z_i\, d\,\mathrm{Im} z_i \prod_{1\leq j\leq k\leq n} d\,\mathrm{Re}\, W_{jk}\, d\,\mathrm{Im}\, W_{jk}.$$

$$\mathcal{K}_*^{mk}(W,z) = \mathrm{K}_*^{mk}((W,z),(W,z)),$$

$$\mathrm{K}_*^{mk}((z,W),(z',W')) = \left(\det(I_n - W'\bar{W})\right)^{-k}\exp\left(8\pi mA(W',z';W,z)\right).$$

The map $g_* \longmapsto \pi_*^{mk}(g_*)$, $\pi_*^{mk}(g_*)$: $\mathcal{H}_*^{mk} \to \mathcal{H}_*^{mk}$

$$\left(\pi_*^{mk}(g_*^{-1})\psi\right)(z, W) = J_*^{mk}(g_*, (z, W))\psi(z_{g_*}, W_{g_*}), \tag{4.7}$$

$\psi \in \mathcal{H}_*^{mk},\ g_* = (\omega, (\alpha, \varkappa)) \in G_{n*}^J$, $(z, W) \in \mathfrak{D}_n^J$,

$$J_*^{mk}(g_*, (z, W)) = \exp(2\pi\mathrm{i}m\theta_*)\left(\det(\bar{q}W + \bar{p})\right)^{-k}. \tag{4.8}$$

Proposition

Suppose $m > 0$, $k > n + 1/2$, *and* $C = 2^{n(n+3)} C_*$. *Then*
a) $\mathcal{H}_*^{mk} \neq \{0\}$ *and* π_*^{mk} *is an irreducible unitary representation of* G_{n*}^J *on the Hilbert space* $\mathcal{H}_*^{mk}$, *square integrable modulo center.*
b) $\exists$ *the unitary isomorphism* $T_*^{mk} : \mathcal{H}_*^{mk} \rightarrow \mathcal{H}^{mk}$

$$\varphi(\Omega, \zeta) = \psi\,(W, z)\,(\det(I_n - W))^k \exp(4\pi m z (I_n - W)^{-1}\, {}^t z), \tag{4.9}$$

$\psi \in \mathcal{H}_*^{mk}$, $\varphi = T_*^{mk}(\psi)$, $(W, z) \in \mathfrak{D}_n^J$, $(\Omega, \zeta) = \phi((-W, z)) \in \mathfrak{H}_n^J$.
The inverse isomorphism $T^{mk} : \mathcal{H}^{mk} \rightarrow \mathcal{H}_*^{mk}$*:*

$$\psi\,(W, z) = \varphi(\Omega, \zeta)\,(\det(I_n - \mathrm{i}\Omega))^k \exp\left(2\pi m \zeta (I_n - \mathrm{i}\Omega)^{-1}\, {}^t\zeta\right), \tag{4.10}$$

$\psi \in \mathcal{H}_*^{mk}$, $\psi = T^{mk}(\varphi)$, $(\Omega, \zeta) \in \mathfrak{H}_n^J$, $(-W, z) = \phi^{-1}((\Omega, \zeta)) \in \mathfrak{D}_n^J$.
c) The representations π^{mk} *and* π_*^{mk} *are unitarily equivalent.*

$\mathcal{H}^k$

$\mathcal{H}^k$ - $\Phi \in \mathcal{O}(\mathfrak{D}_n)$, $\|\Phi\|_{\mathfrak{D}_n} < \infty$,

$$(\Psi_1, \Psi_2)_k = \int_{\mathcal{D}_n} \Psi_1(W)\overline{\Psi_2(W)} \left(\det(1 - W\bar{W})\right)^{k-1/2} d\mu_{\mathcal{D}_n}(W),$$

$$d\mu_{\mathcal{D}_n}(W) = \left(\det(1 - W\bar{W})\right)^{-n-1} \prod_{1\leq j\leq k\leq n} \mathrm{d}\,\mathrm{Re}\, W_{jk}\, \mathrm{d}\,\mathrm{Im}\, W_{jk}.$$

We have $\mathcal{H}^k \neq \{0\}$ for $k > n + 1/2$, cf Berezin, Takase.

Proposition- THE LAST ONE!

$\{Q_a | a \in A_n\}$ – an orthonormal polynomial basis of $\mathcal{H}^k$. We introduce the polynomials

$$F_{sa}(W,z) = \sqrt{\frac{(8\pi m)^n}{C_* s!}} P_s(\sqrt{8\pi m} z, W) Q_a(W), \; s \in \mathbb{N}^n, \; a \in A_n. \quad (4.11)$$

Proposition

The set of polynomials $\{F_{sa} | s \in \mathbb{N}^n, \; a \in A_n\}$ forms an orthonormal basis of $\mathcal{H}^{mk}_*$. The kernel function of $\mathcal{H}^{mk}_*$ satisfies the expansion

$$(\det(1 - W'\bar{W}))^{-k} \exp A(W', z', W, z) = \sum_{s \in \mathbb{N}^n, a \in A_n} F_{sa}(W', z') \overline{F_{sa}(W, z)}.$$

The base of functions & Reproducing Kernel, $n = 1$

Proposition

$$f_{|n>;e_{k',k'+m}}(z,w) = \sqrt{\frac{\Gamma(n+2k)}{n!\Gamma(2k)}}\, w^n \frac{P_n(z,w)}{\sqrt{n!}}$$

$$P_n(z,w) = n! \sum_{k=0}^{[\frac{n}{2}]} (\frac{w}{2})^k \frac{z^{n-2k}}{k!(n-2k)!}, z, w \in \mathcal{D}^J_1. \tag{4.12}$$

$$K(z,w;\bar{z}',\bar{w}') = (1 - w\bar{w}')^{-2k} \exp \frac{2\bar{z}'z + z^2\bar{w}' + \bar{z}'^2 w}{2(1 - w\bar{w}')}$$

Berezin's quantization

$$\Lambda_n = \frac{k-3}{2\pi^{\frac{n(n+3)}{2}}} \prod_{i=1}^{n-1} \frac{(\frac{k-3}{2} - n + i)\Gamma(k+i-2)}{\Gamma[k+2(i-n-1)]}.$$

Compare with the case of the symplectic group: *a shift of* p to $p - 1/2$ *in the normalization constant* $\Lambda_n = \pi^{-n} J^{-1}(p)$.

Kähler two-form on $\mathfrak{D}^J_n$

$$
\begin{aligned}
-\mathrm{i}\omega_n &= \frac{k}{2}\mathrm{Tr}(C\wedge\bar{C})+\mathrm{Tr}(A^t\bar{M}\wedge\bar{A}),\\
A &= dz+dW\bar{x},\\
C &= MdW,\ \ M=(1-W\bar{W})^{-1}\\
x &= (1-W\bar{W})^{-1}(z+W\bar{z}), W\in\mathfrak{D}_n, z\in\mathbb{C}^n,
\end{aligned}
\tag{5.1}
$$

The Kähler two-form on $\mathfrak{H}^J_n$

Under the partial Cayley transform the two-form ω_n on $\mathfrak{D}^J_n$ becomes on $\mathfrak{H}^J_n$

$$-\mathrm{i}\omega'_n = \frac{k}{2}\mathrm{Tr}(p \wedge \bar{p}) + \frac{2}{\mathrm{i}}\mathrm{Tr}(B^t \wedge \bar{B}) \quad (5.2)$$

$$p = (\bar{v} - v)^{-1} dv; \; B = du - dv(v - \bar{v})^{-1}(u - \bar{u}).$$

“n”-dimensional generalization of Berndt-Kähler two-form ω'_1.

Remark $\mathfrak{D}^J_n$ and $\mathfrak{H}^J_n$ are called by Jae-Hyun Yang *Siegel-Jacobi domains* . Kähler calls $\mathfrak{H}^J_1$ *Phasenraum der Materie*, v is *Pneuama*, u is *Soma*.

Comparation with Yang's results

$$(v, u) \leftrightarrow (\Omega, \frac{Z^t}{2\mathrm{i}}). \tag{5.3}$$

Remark

The case $m = 1$ in Theorem 1 in Yang is our (5.2), while our relation (5.1) is theorem 5 in Yang; Our factor Q_1

$$Q_1 = 2^{-n(n+3)}[\det(2i(\bar{v} - v))]^{-(n+2)} = 2^{-n(n+1)}[\det(\operatorname{Im} v)]^{-(n+2)} \tag{5.4}$$

corresponds to Yang's result expressed in Lemma A in the same situation $m = 1$.

Comparation with Yang's result on Jacobi-Sigel half plane

Yang's notation $\Omega = X + \mathrm{i}Y; Z = U + \mathrm{i}V$. Our (5.2) expressed in Yang's notation

$$
\begin{aligned}
-\mathrm{i}w'_n &= \frac{k}{8}\mathrm{Tr}(Y^{-1}d\Omega \wedge Y^{-1}d\bar{\Omega}) \\
&+ \frac{1}{8}\mathrm{Tr}[(dZ - VY^{-1}d\Omega)Y^{-1} \wedge (d\bar{Z}^t - d\bar{\Omega}Y^{-1}V^t)]
\end{aligned}
$$

$$
\begin{aligned}
-\mathrm{i}w'_n &= \frac{k}{8}\mathrm{Tr}(Y^{-1}\,d\Omega \wedge Y^{-1}\,d\bar{\Omega}) \\
&+ \frac{1}{8}\mathrm{Tr}(dZY^{-1} \wedge d\bar{Z}^t) + \frac{1}{8}\mathrm{Tr}(VY^{-1}d\Omega Y^{-1} \wedge \bar{\Omega}y^{-1}V^t) \\
&- \frac{1}{8}\mathrm{Tr}(dZY^{-1} \wedge \bar{\Omega}Y^{-1}V^t) - \frac{1}{8}\mathrm{Tr}(VY^{-1}d\Omega Y^{-1} \wedge d\bar{Z}^t)
\end{aligned}
$$

Yang's Kähler metric on Jacobi-Siegel disc

$$
\begin{aligned}
& \frac{1}{4}\, d\tilde{s}^2_{n,m;A,B} = \\
& A\,\mathrm{tr}\big((I_n - W\overline{W})^{-1} dW (I_n - \overline{W}W)^{-1} d\overline{W}\big) \\
+ \quad & B\Big\{\mathrm{tr}\big((I_n - W\overline{W})^{-1}\, {}^t(d\eta)\, d\overline{\eta}\,\big) \\
+ \quad & \mathrm{tr}\big((\eta\overline{W} - \overline{\eta})(I_n - W\overline{W})^{-1} dW \\
& \quad (I_n - \overline{W}W)^{-1}\, {}^t(d\overline{\eta})\big) \\
+ \quad & \mathrm{tr}\big((\overline{\eta}W - \eta)(I_n - \overline{W}W)^{-1} d\overline{W} \\
& \quad (I_n - W\overline{W})^{-1}\, {}^t(d\eta)\,\big) \\
- \quad & \mathrm{tr}\big((I_n - W\overline{W})^{-1}\, {}^t\eta\,\eta\,(I_n - \overline{W}W)^{-1} \\
& \quad \overline{W} dW (I_n - \overline{W}W)^{-1} d\overline{W}\,\big) \\
- \quad & \mathrm{tr}\big(W(I_n - \overline{W}W)^{-1}\, {}^t\overline{\eta}\,\overline{\eta}\,(I_n - W\overline{W})^{-1} \\
& \quad dW (I_n - \overline{W}W)^{-1} d\overline{W}\,\big)
\end{aligned}
$$

Yang's- result- continuation

$$
\begin{aligned}
+ \quad & \operatorname{tr}\Big((I_n - W\overline{W})^{-1}\, {}^t\eta\,\overline{\eta}\,(I_n - W\overline{W})^{-1} \\
& \quad dW(I_n - \overline{W}W)^{-1} d\overline{W}\,\Big) \\
+ \quad & \operatorname{tr}\Big((I_n - \overline{W})^{-1}\, {}^t\overline{\eta}\,\eta\,\overline{W}\,(I_n - W\overline{W})^{-1} \\
& \quad dW(I_n - \overline{W}W)^{-1} d\overline{W}\,\Big) \\
+ \quad & \operatorname{tr}\Big((I_n - \overline{W})^{-1}(I_n - W)(I_n - \overline{W}W)^{-1} \\
& \quad {}^t\overline{\eta}\,\eta\,(I_n - \overline{W}W)^{-1}\,(I_n - \overline{W})(I_n - W)^{-1} \\
& \quad dW(I_n - \overline{W}W)^{-1} d\overline{W}\,\Big)\,\Big\}
\end{aligned}
$$

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