KYUNGPOOK Math J. 42(2002), 199-272

The Method of Orbits for Real Lie Groups

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(2000 Mathematics Subject Classification: Primary 22-XX, 20C35.)

In this paper, we outline a development of the theory of orbit method for representations of real Lie groups. In particular, we study the orbit method for representations of the Heisenberg group and the Jacobi group.

Contents

(Received: December 4, 2001. Revised: May 14, 2002.)
Key words and phrases: quantization, the Kirillov correspondence, nilpotent orbits, the Kostant-Sekiguchi correspondence, minimal representations, Heisenberg groups, the Jacobi group.
This work was supported by INHA UNIVERSITY Research Grant.(INHA-21382)

1. Introduction

Research into representations of Lie groups was motivated on the one hand by physics, and on the other hand by the theory of automorphic forms. The theory of unitary or admissible representations of noncompact reductive Lie groups has been developed systematically and intensively shortly after the end of World War II. In particular, Harish-Chandra, R. Langlands, Gelfand school and some other people made an enormous contribution to the theory of unitary representations of noncompact reductive Lie groups.

Early in the 1960s A.A. Kirillov [47] first initiated the orbit method for a nilpotent real Lie group attaching an irreducible unitary representation to a coadjoint orbit (which is a homogeneous symplectic manifold) in a perfect way. Thereafter Kirillov's work was generalized to solvable groups of type I by L. Auslander and B. Kostant [3] early in the 1970s in a nice way. Their proof was based on the existence of complex polarizations satisfying a positivity condition. Unfortunately Kirillov's work fails to be generalized in some ways to the case of compact Lie groups or semisimple Lie groups. Relatively simple groups like $SL(2,\mathbb{R})$ have irreducible unitary representations that do not correspond to any symplectic homogeneous space. Conversely, P. Torasso [85] found that the double cover of $SL(3,\mathbb{R})$ has a homogeneous symplectic manifold corresponding to no unitary representations. The orbit method for reductive Lie groups is a kind of a philosophy but not a theorem. Many large families of orbits correspond in comprehensible ways to unitary representations, and provide a clear geometric picture of these representations. The coadjoint orbits for a reductive Lie group are classified into three kinds of orbits, namely, hyperbolic, elliptic and nilpotent ones. The hyperbolic orbits are related to the unitary representations obtained by the parabolic induction and on the other hand, the elliptic ones are related to the unitary representations obtained by the cohomological induction. However, we still have no idea of attaching unitary representations to nilpotent orbits. It is known that there are only finitely many nilpotent orbits. In a certain case, some nilpotent orbits are corresponded to the so-called *unipotent representations*. For instance, a minimal nilpotent orbit is attached to a minimal representation. In fact, the notion of unipotent representations is not still well defined. The investigation of unipotent representations is now under way. Recently D. Vogan [93] presented a *new* method for studying the quantization of nilpotent orbits in terms of the restriction to a maximal compact subgroup even though it is not complete and is in a preliminary stage. J.-S. Huang and J.-S. Li [41] attached unitary representations to spherical nilpotent orbits for the real orthogonal and symplectic groups.

In this article, we describe a development of the orbit method for real Lie groups, and then in particular, we study the orbit method for the Heisenberg group and the Jacobi group in detail. This paper is organized as follows. In Section 2, we describe the notion of geometric quantization relating to the theory of unitary representations which led to the orbit method. The study of the geometric quantization was first made intensively by A.A. Kirillov [51]. In Section 3, we outline the beautiful Kirillov's work on the orbit method for a nilpotent real Lie group done early in the 1960s. In Section 4, we describe the work for a solvable Lie group of type I done by L. Auslander and B. Kostant [3] generalizing Kirillov's work. In Section 5, we roughly discuss the cases of compact or semisimple Lie groups where the orbit method does not work nicely. If G is compact or semisimple, the correspondence between G-orbits and irreducible unitary representations of G breaks down. In Section 6, for a real reductive Lie group G with Lie algebra $\mathfrak{g}$, we present some properties of nilpotent orbits for G and describe the Kostant-Sekiguchi correspondence between G-orbits in the cone of all nilpotent elements in $\mathfrak{g}$ and $K_{\mathbb{C}}$-orbits in the cone of nilpotent elements on $\mathfrak{p}_{\mathbb{C}}$, where $K_{\mathbb{C}}$ is the complexification of a fixed maximal compact subgroup K of G and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$ is the Cartan decomposition of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. We do not know yet how to quantize a nilpotent orbit in general. But for a maximal compact subgroup K of G, D. Vogan attaches a space with a representation of K to a nilpotent orbit. We explain this correspondence in a rough way. Most of the materials in this section come from the article [93]. In Section 7, we outline the notion of minimal orbits (that are nilpotent orbits), and the relation of the minimal representations to the theory of reductive dual pairs initiated first by R. Howe. We also discuss the recent works for a construction of minimal representations for various groups. For more detail, we refer to [65]. In Section 8, we study the orbit method for the Heisenberg group in some detail. In Section 9, we study the unitary representations of the Jacobi group and their related topics. The Jacobi group appears in the theory of Jacobi forms. That means that Jacobi forms are automorphic forms for the Jacobi group. We study the coadjoint orbits for the Jacobi group.

Notation. We denote by $\mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ the ring of integers, the field of real numbers, and the field of complex numbers respectively. The symbol $\mathbb{C}_1^\times$ denotes the multiplicative group consisting of all complex numbers z with $|z| = 1$, and the symbol $Sp(n, \mathbb{R})$ the symplectic group of degree n, H_n the Siegel upper half plane of degree n. The symbol ":=" means that the expression on the right hand side is the definition of that on the left. We denote by $\mathbb{Z}^+$ the set of all positive integers, by $F^{(k,l)}$ the set of all $k \times l$ matrices with entries in a commutative ring F. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. For $A \in F^{(k,k)}$, $\sigma(A)$ denotes the trace of A. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. We denote the identity matrix of degree k by E_k. For a positive integer n, $\mathrm{Symm}\,(n, K)$ denotes the vector space consisting of all symmetric $n \times n$ matrices with entries in a field K.

2. Quantization

The problem of quantization in mathematical physics is to attach a quantum mechanical model to a classical physical system. The notion of *geometric quantization* had emerged at the end of the 1960s relating to the theory of unitary group representations which led to the orbit method. The goal of geometric quantization is to construct quantum objects using the geometry of the corresponding classical objects as a point of departure. In this paper we are dealing with the group representations and hence the problem of quantization in representation theory is to attach a unitary group representation to a symplectic homogeneous space.

A classical mechanical system can be modelled by the phase space which is a symplectic manifold. On the other hand, a quantum mechanical system is modelled by a Hilbert space. Each state of the system corresponds to a line in the Hilbert space.

Definition 2.1. A pair (M,ω) is called a *symplectic* manifold with a nondegenerate closed differential 2-form ω. We say that a pair (M,c) is a *Poisson manifold* if M is a smooth manifold with a bivector $c=c^{ij}\partial_i\partial_j$ such that the Poisson brackets

$$\{f_1,f_2\}=c^{ij}\partial_i f_1\partial_j f_2 \tag{2.1}$$

define a Lie algebra structure on $C^\infty(M)$. We define a *Poisson* G-manifold as a pair $(M,\ f^M_{(\cdot)})$ where M is a Poisson manifold with an action of G and $f^M_{(\cdot)}:\mathfrak{g}\to C^\infty(M)\,(X\mapsto f^M_X)$ is a Lie algebra homomorphism such that the following relation holds:

$$\text{s}-\text{grad}(f^M_X)=L_X,\quad X\in\mathfrak{g},\quad X\in\mathfrak{g}. \tag{2.2}$$

Here L_X is the Lie vector field on M associated with $X\in\mathfrak{g}$, and s-grad(f) denotes the *skew gradient* of a function f, that is, the vector field on M such that

$$\text{s-grad}(f)g=\{f,g\}\quad\text{for all}\quad g\in C^\infty(M). \tag{2.3}$$

For a given Lie group G the collection of all Poisson G-manifolds forms the category $\mathcal{P}(G)$ where a morphism $\alpha:(M,f^M_{(\cdot)})\to(N,f^N_{(\cdot)})$ is a smooth map from M to N which preserves the Poisson brackets: $\{\alpha^*(\phi),\ \alpha^*(\psi)\}=\alpha^*(\{\phi,\psi\})$ and makes the following relation holds:

$$\alpha^*(f^N_X)=f^M_X,\quad X\in\mathfrak{g}. \tag{2.4}$$

Observe that the last condition implies that α commutes with the G-action.

First we explain the mathematical model of classical mechanics in the Hamiltonian formalism.

Let (M,ω) be a symplectic manifold of dimension $2n$. According to the Darboux theorem, the sympletic form ω can always be written in the form

$$\omega=\Sigma^n_{k=1}dp_k\wedge dq_k \tag{2.5}$$

in suitable canonical coordinates $p_1, \cdots, p_n, q_1, \cdots, q_n$. However, these canonical coordinates are not uniquely determined.

The symplectic from ω sets up an isomorphism between the tangent and cotangent spaces at each point of M. The inverse isomorphism is given by a bivector c, which has the form

$$c = \Sigma_{k=1}^{n} \frac{\partial}{\partial p_k}\frac{\partial}{\partial q_k} \tag{2.6}$$

in the same system of coordinates in which the equality (2.5) holds. In the general system of coordinates the form ω and the bivecter c are written in the form

$$\omega = \Sigma_{i<j}\omega_{ij}dx_i \wedge dx_j, \quad c = \Sigma_{i<j}c^{ij}\partial_i\partial_j \tag{2.7}$$

with mutually inverse skew-symmetric matrices (ω_{ij}) and (c^{ij}). The set $C^\infty(M)$ of all smooth functions on M forms a commutative associative algebra with respect to the usual multiplication. The Poisson bracket $\{\ ,\ \}$ defined by

$$\{F, G\} := \Sigma_{i,j}c^{ij}\partial_j F \cdot \partial_i G, \quad F, G \in C^\infty(M) \tag{2.8}$$

defines a Lie algebra structure on $C^\infty(M)$. The Jacobi identity for the Poisson bracket is equivalent to the condition $d\omega = 0$ and also to the vanishing of the Schouten bracket

$$[c, c]^{ijk} := \curvearrowleft_{ijk}\Sigma_m c^{im}\partial_m c^{jk} \tag{2.9}$$

where the sign $\curvearrowleft_{ijk}$ denotes the sum over the cyclic permutations of the indices i, j, k.

Physical quantities or *observables* are identified with the smooth functions on M. A state of the system is a linear functional on $C^\infty(M)$ which takes non-negative values on non-negative functions and equals 1 on the function which is identically equal to 1. The general form of such a functional is a probability measure μ on M. By a *pure state* is meant an extremal point of the set of states.

The dynamics of a system is determined by the choice of a Hamiltonian function or energy, whose role can be played by an arbitrary function $H \in C^\infty(M)$. The dynamics of the system is described as follows. The states do *not* depend on time, and the physical quantities are functions of the point of the phase space and of time. If F is any function on $M \times \mathbb{R}$, that is, any observable, the equations of the motion have the form

$$\dot{F} := \frac{\partial F}{\partial t} = \{H, F\}. \tag{2.10}$$

Here the dot denotes the derivative with respect to time. In particular, applying (2.10) to the canonical variables p_k, q_k, we obtain *Hamilton's equations*

$$\dot{q_k} = \frac{\partial H}{\partial p_k}, \quad \dot{p_k} = -\frac{\partial H}{\partial q_k}. \tag{2.11}$$

A set $\{F_1, \cdots, F_m\}$ of physical quantities is called *complete* if the conditions $\{F_i, G\} = 0\,(1 \le i \le m)$ imply that G is a constant.

Definition 2.2. Let (M, ω) be a symplectic manifold and $f \in C^\infty(M)$ a smooth function on M. The Hamiltonian vector field ξ_f of f is defined by

$$\xi_f(g) = \{f, g\}, \quad g \in C^\infty(M) \tag{2.12}$$

where $\{\ ,\ \}$ is the Poisson bracket on $C^\infty(M)$ defined by (2.8). Suppose G is a Lie group with a smooth action of G on M by symplectomorphisms. We say that M is a *Hamiltonian G-space* if there exist a linear map

$$\tilde{\mu} : \mathfrak{g} \longrightarrow C^\infty(M) \tag{2.13}$$

with the following properties (H1)-(H3) :
(H1) $\tilde{\mu}$ intertwines the adjoint action of G on $\mathfrak{g}$ with its action on $C^\infty(M)$;
(H2) For each $Y \in \mathfrak{g}$, the vector field by which Y acts on M is $\xi_{\tilde{\mu}(Y)}$;
(H3) $\tilde{\mu}$ is a Lie algebra homomorphism.

The above definition can be formulated in the category of Poisson manifolds, or even of possibly singular Poisson algebraic varieties. The definition is due to A. A. Kirillov [48] and B. Kostant [57].

The natural quantum analogue of a Hamiltonian G-space is simply a unitary representation of G.

Definition 2.3. Suppose G is a Lie group. A *unitary representation* of G is a pair $(\pi, \mathcal{H})$ with a Hilbert space $\mathcal{H}$, and

$$\pi : G \longrightarrow U(\mathcal{H})$$

a homomorphism from G to the group of unitary operations on $\mathcal{H}$.

We would like to have a notion of quantization passing from Definition 2.2 to Definition 2.3 : that is, from Hamiltonian G-space to unitary representations.

Next we explain the mathematical model of quantum mechanics. In quantum mechanics the physical quantities or observables are self-adjoint linear operators on some complex Hilbert space $\mathcal{H}$. They form a linear space on which two bilinear operations are defined :

$$A \circ B := \frac{1}{2}\,(AB + BA) \quad \text{(Jordan multiplication)} \tag{2.14}$$

$$[A, B]_\hbar := \frac{2\pi i}{\hbar}\,(AB - BA) \quad \text{(the commutator)} \tag{2.15}$$

where $\hbar$ is the Planck's constant.

With respect to (2.14) the set of observables forms a commutative but not associative algebra. With respect to (2.15) it forms a Lie algebra. These two operations (2.14) and (2.15) are the quantum analogues of the usual multiplication and the Poisson bracket in classical mechanics. The *phase space* in quantum mechanics consists of the non-negative definite operators A with the property that $\operatorname{tr} A = 1$. The *pure states* are the one-dimensional projection operators on $\mathcal{H}$. The *dynamics* of the system is defined by the *energy operator* $\hat{H}$. If the states do not depend on time but the quantities change, then we obtain the *Heisenberg picture.* The equations of motion are given by

$$\dot{\hat{A}} = [\hat{H}, \hat{A}]_\hbar, \quad \text{(Heisenberg's equation).} \tag{2.16}$$

The integrals of the system are all the operators which commute with $\hat{H}$. In particular, the energy operator itself does not change with time.

The other description of the system is the so-called Schrödinger picture. In this case, the operators corresponding to physical quantities do not change, but the states change. A pure state varies according to the law

$$\dot{\Psi} = \frac{2\pi i}{\hbar}\hat{H}\Psi \quad \text{(Schrödinger equation).} \tag{2.17}$$

The eigenfunctions of the Schrödinger operator give the stationary states of the system. We call a set of quantum physical quantities $\hat{A}_1, \cdots, \hat{A_m}$ *complete* if any operator $\hat{B}$ which commutes with $\hat{A}_i$ $(1 \le i \le m)$ is a multiple of the identity. One can show that this condition is equivalent to the irreducibility of the set $\hat{A}_1, \cdots, \hat{A_m}$.

Finally we describe the quantization problem relating to the orbit method in group representation theory. As I said earlier, the problem of geometric quantization is to construct a Hilbert space $\mathcal{H}$ and a set of operators on $\mathcal{H}$ which give the quantum analogue of this system from the geometry of a symplectic manifold which gives the model of classical mechanical system. If the initial classical system had a symmetry group G, it is natural to require that the corresponding quantum model should also have this symmetry. That means that on the Hilbert space $\mathcal{H}$ there should be a unitary representation of the group G.

We are interested in homogeneous symplectic manifolds on which a Lie group G acts transitively. If the thesis is true that every quantum system with a symmetry group G can be obtained by quantization of a classical system with the same symmetry group, then the irreducible representations of the group must be connected with homogeneous symplectic G-manifolds. The orbit method in representation theory intiated first by A. A. Kirillov early in the 1960s relates the unitary representations of a Lie group G to the coadjoint orbits of G. Later L. Auslauder, B. Kostaut, M. Duflo, D. Vogan, P. Torasso etc developed the theory of the orbit method to more general cases.

For more details, we refer to [47]- [51], [55], [57]-[58] and [90]-[91], [93].

3. The Kirillov Correspondence

In this section, we review the results of Kirillov on unitary representations of a nilponent real Lie group. We refer to [47]-[51], [55] for more detail.

Let G be a simply connected real Lie group with its Lie algebra $\mathfrak{g}$. Let $Ad_G : G \longrightarrow GL(\mathfrak{g})$ be the adjoint representation of G. That is, for each $g \in G$, $Ad_G(g)$ is the differential map of I_g at the identity e, where $I_g : G \longrightarrow G$ is the conjugation by g given by

$$I_g(x) := gxg^{-1} \quad \text{for} \ \ x \in G.$$

Let $\mathfrak{g}^*$ be the dual space of the vector space $\mathfrak{g}$. Let $Ad_G^* : G \longrightarrow GL(\mathfrak{g}^*)$ be the contragredient of the adjoint representation Ad_G. Ad_G^* is called the coadjoint representation of G. For each $\ell \in \mathfrak{g}^*$, we define the alternating bilinear form B_ℓ on $\mathfrak{g}$ by

$$B_\ell(X,Y) =< [X,Y], \ell >, \quad X, Y \in \mathfrak{g}. \tag{3.1}$$

Definition 3.1. (1) A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be *subordinate* to $\ell \in \mathfrak{g}^*$ if $\mathfrak{h}$ forms a totally isotropic vector space of $\mathfrak{g}$ relative to the alternating bilinear form B_ℓ define by (3.1), i.e., $B_\ell|_{\mathfrak{h}\times\mathfrak{h}} = 0$.

(2) A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ subordinate to $\ell \in \mathfrak{g}^*$ is called a *polarization* of $\mathfrak{g}$ for ℓ if $\mathfrak{h}$ is maximal among the totally isotropic vector subspaces of $\mathfrak{g}$ relative to B_ℓ. In other words, if P is a vector subspace of $\mathfrak{g}$ such that $\mathfrak{h} \subset P$ and $B_\ell|_{P\times P} = 0$, then we have $\mathfrak{h} = P$.

(3) Let $\ell \in \mathfrak{g}^*$ and let $\mathfrak{h}$ be a polarization of $\mathfrak{g}$ for ℓ. We let H the simply connected closed subgroup of G corresponding to the Lie subalgebra $\mathfrak{h}$. We define the unitary character $\chi_{\ell,\mathfrak{h}}$ of H by

$$\chi_{\ell,\mathfrak{h}}(\exp_H(X)) = e^{2\pi i <X,\ell>}, \quad X \in \mathfrak{h}, \tag{3.2}$$

where $\exp_H : \mathfrak{h} \longrightarrow H$ denotes the exponential mapping of $\mathfrak{h}$ to H. It is known that $\exp_H$ is surjective.

Using the Mackey machinary, Dixmier and Kirillov proved the following important theorem.

Theorem 3.1 (Dixmier-Kirillov, [19],[47]). *A simply connected real nilpotent Lie group G is monomial, that is, each irreducible unitary representation of G can be unitarily induced by a unitary character of some closed subgroup of G.*

Remark 3.2. More generally, it can be proved that a simply connected real Lie

group whose exponential mapping is a diffeomorphism is monomial.
Now we may state Theorem 3.1 explicitly.

Theorem 3.3 (Kirillov, [47]). *Let G be a simply connected nilpotent real Lie group with its Lie algebra $\mathfrak{g}$. Assume that there is given an irreducible unitary representation π of G. Then there exist an element $\ell \in \mathfrak{g}^*$ and a polarization $\mathfrak{h}$ of $\mathfrak{g}$ for ℓ such that $\pi \cong Ind_H^G \chi_{\ell,\mathfrak{h}}$, where $\chi_{\ell,\mathfrak{h}}$ is the unitary character of H defined by (3.2).*

Theorem 3.4 (Kirillov, [47]). *Let G be a simply connected nilpotent real Lie group with its Lie algebra $\mathfrak{g}$. If $\ell \in \mathfrak{g}^*$, there exists a polarization $\mathfrak{h}$ of $\mathfrak{g}$ for ℓ such that the monomial representation $Ind_H^G \chi_{\ell,\mathfrak{h}}$ is irreducible and of trace class. If ℓ' is an element of $\mathfrak{g}^*$ which belongs to the coadjoint orbit $Ad_G^*(G)\ell$ and $\mathfrak{h}'$ a polarization of $\mathfrak{g}$ for ℓ', then the monomial representations $Ind_H^G \chi_{\ell,\mathfrak{h}}$ and $Ind_{H'}^G \chi_{\ell',\mathfrak{h}'}$ are unitarily equivalent. Here H and H' are the simply connected closed subgroups corresponding to the Lie subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$ respectively. Conversely, if $\mathfrak{h}$ and $\mathfrak{h}'$ are polarizations of $\mathfrak{g}$ for $\ell \in \mathfrak{g}^*$ and $\ell' \in \mathfrak{g}^*$ respectively such that the monomial representations $Ind_H^G \chi_{\ell,\mathfrak{h}}$ and $Ind_{H'}^G \chi_{\ell',\mathfrak{h}'}$ of G are unitarily equivalent, then ℓ and ℓ' belong to the same coadjoint orbit of G in $\mathfrak{g}^*$. Finally, for each irreducible unitary representation τ of G, there exists a unique coadjoint orbit Ω of G in $\mathfrak{g}^*$ such that for any linear from $\ell \in \Omega$ and each polarization $\mathfrak{h}$ of $\mathfrak{g}$ for ℓ, the representations τ and $Ind_H^G \chi_{\ell,\mathfrak{h}}$ are unitarily equivalent. Any irreducible unitary representation of G is strongly trace class.*

Remark 3.5. (a) The bijection of the space $\mathfrak{g}^*/G$ of coadjoint orbit of G in $\mathfrak{g}^*$ onto the unitary dual $\hat{G}$ of G given by Theorem 3.4 is called the *Kirillov correspondence* of G. It provides a parametrization of $\hat{G}$ by means of the coadjoint orbit space.

(b) The above Kirillov's work was generalized immediately to the class known as exponential solvable groups, which are characterized as those solvable group G whose simply-connected cover $\tilde{G}$ is such that the exponential map $\exp : \tilde{\mathfrak{g}} \longrightarrow \tilde{G}$ is a diffeomorphism. For exponential solvable groups, the bijection between coadjoint orbits and representations holds, and can be realized using induced representations by an explicit construction using a polarization just as in the case of a nilpotent real Lie group. However, two difficulties arise: Firstly not all polarizations yield the same representation, or even an irreducible representation, and secondly not all representations are strongly trace class.

Theorem 3.6 (I.D. Brown). *Let G be a connected simply connected nilpotent Lie group with its Lie algebra $\mathfrak{g}$. The Kirillov correspondence*

$$\hat{G} \longrightarrow \mathcal{O}(G) = \mathfrak{g}^*/G$$

is a homeomorphism.

Theorem 3.7. *Let G be a connected simply connected nilpotent real Lie group with*

Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Let $p : \mathfrak{g}^* \longrightarrow \mathfrak{h}^*$ be the natural projection. Let H be the simply connected subgroup of G with its Lie algebra $\mathfrak{h}$. The following (a),(b) and (c) hold.

(a) Let π be an irreducible unitary representation of G corresponding to a coadjoint orbit $\Omega \subset \mathfrak{g}^*$ of G via the Kirillov correspondence. Then $Res_H^G \pi$ decomposes into the direct integral of irreducible representations of H corresponding to a coadjoint orbit $\omega (\subset \mathfrak{h}^*)$ of H such that $\omega \subset p(\Omega)$.

(b) Let τ be an irreducible unitary representation of H corresponding to a coadjoint orbit $\omega \subset \mathfrak{h}^*$ of H. Then the induced representation $Ind_H^G \tau$ decomposes into the direct integral of irreducible representations π_Ω of G corresponding to coadjoint orbits $\Omega \subset \mathfrak{g}^*$ such that $p(\Omega) \supset \omega$.

(c) Let π_1 and π_2 be the irreducible unitary representations of G corresponding to coadjoint orbits Ω_1 and Ω_2 respectively. Then the tensor product $\pi_1 \otimes \pi_2$ decomposes into the direct integral of irreducible representations of G corresponding to coadjoint orbits $\Omega \subset \mathfrak{g}^*$ such that $\Omega \subset \Omega_1 + \Omega_2$.

Theorem 3.8. *Let G be a connected simply connected nilpotent real Lie group with its Lie algebra $\mathfrak{g}$. Let π be an irreducible unitary representation of G corresponding to an orbit $\Omega \subset \mathfrak{g}^*$. Then the character χ_π is a distribution on $\mathcal{S}(G)$ and its Fourier transform coincides with the canonical measure on Ω given by the symplectic structure. Here $\mathcal{S}(G)$ denotes the Schwarz space of rapidly decreasing functions on G.*

A.A. Kirillov gave an explicit formula for the Plancherel measure on $\hat{G}$. We observe that for a nilpotent Lie group G, we may choose a subspace Q of $\mathfrak{g}^*$ such that generic coadjoint orbits intersect Q exactly in one point. We choose a basis $x_1, \cdots, x_l, y_1, \cdots, y_{n-l}$ in $\mathfrak{g}$ so that $y_1, \cdots, y_{n-l}$, considered as linear functionals on $\mathfrak{g}^*$, are constant on Q. Then $x_1, \cdots, x_l$ are coordinates on Q and hence on an open dense subset of $\mathcal{O}(G)$. For every $f \in Q$ with coordinates $x_1, \cdots, x_l$, we consider the skew-symmetric matrix $A = (a_{ij})$ with entries

$$a_{ij} =< [y_i, y_j], f >, \quad 1 \leq i, j \leq n - l.$$

We denote by $p(x_1, \cdots, x_l)$ the Pfaffian of the matrix A. We note that $p(x_1, \cdots, x_l)$ is a homogeneous polynomial of degree $\frac{n-l}{2}$.

Theorem 3.9. *Let G be a connected simply connected nilpotent Lie group. Then the Plancherel measure on $\hat{G} \cong \mathcal{O}(G)$ is concentrated on the set of generic orbits and it has the form*

$$\theta = p(x_1, \cdots, x_l) dx_1 \wedge \cdots \wedge dx_l$$

in the coordinates $x_1, \cdots, x_l$.

4. Auslander-Kostant's Theorem

In this section, we present the results obtained by L. Auslander and B. Kostant in [3] together with some complements suggested by I.M. Shchepochkina [80]-[81]. Early in the 1970s L. Auslander and B. Kostant described the unitary dual of all solvable Lie groups of type I.

Theorem 4.1. *A connected, simply connected solvable Lie group G belongs to type I if and only if the orbit space $\mathcal{O}(G) = \mathfrak{g}^*/G$ is a T_0-space and the canonical symplectic form σ is exact on each orbit.*

Remark 4.2. (a) Let G be a real Lie group with its Lie algebra $\mathfrak{g}$. Let $\Omega_\ell := Ad^*(G)\ell$ be the coadjoint orbit containing ℓ. From now on, we write Ad^* instead of Ad^*_G. Then Ω_ℓ is simply connected if a Lie group G is exponential. We recall that a Lie group G is said to be *exponential* if the exponential mapping $\exp : \mathfrak{g} \longrightarrow G$ is a diffeomorphism. But if G is solvable, Ω_l is not necessarily simply connected.

(b) Let G be a connected, simply connected solvable Lie group and for $\ell \in \mathfrak{g}^*$, we let G_ℓ be the stabilizer at ℓ. Then for any $\ell \in \mathfrak{g}^*$, we have $\pi_1(\Omega_l) \cong G_\ell/G^0_\ell$, where G^0_ℓ denotes the identity component of G_ℓ in G.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. A pair (ℓ, χ) is called a *rigged momentum* if $\ell \in \mathfrak{g}^*$, and χ is a unitary character of G_ℓ such that $d\chi_e = 2\pi i \ell|_{\mathfrak{g}_\ell}$, where $d\chi_e$ denotes the differential of χ at the identity element e of G_ℓ. We denote by $\mathfrak{g}^*_{\rm rigg}$ the set of all rigged momenta. Then G acts on $\mathfrak{g}^*_{\rm rigg}$ by

$$g \cdot (\ell, \chi) := (Ad^*(g)\ell, \chi \circ I_{g^{-1}}) = (\ell \circ Ad(g^{-1}), \chi \circ I_{g^{-1}}) \tag{4.1}$$

for all $g \in G$ and $(\ell, \chi) \in \mathfrak{g}^*_{\rm rigg}$. Here I_g denotes the inner automorphism of G defined by $I_g(x) = gxg^{-1}$ $(x \in G)$. We note that $\chi \circ I_{g^{-1}}$ is a unitary character of $G_{Ad^*(G)\ell} = gG_\ell g^{-1}$. We denote by $\mathcal{O}_{\rm rigg}(G)$ the set of all orbits in $\mathfrak{g}^*_{\rm rigg}$ under the action (4.1).

Proposition 4.3. *Let G be a connected, simply connected solvable Lie group. Then the following (a) and (b) hold.*

(a) The G-action commutes with the natural projection

$$\pi : \mathfrak{g}^*_{rigg} \longrightarrow \mathfrak{g}^* \qquad (\ell, \chi) \mapsto \ell. \tag{4.2}$$

(b) For a solvable Lie group G of type I, the projection π is surjective and the fiber over a point $\ell \in \mathfrak{g}^*$ is a torus of dimension equal to the first Betti number $b_1(\Omega_\ell)$ of Ω_ℓ.

Now we mention the main theorem obtained by L. Auslander and B. Kostant in [3].

Theorem 4.4 (Auslander-Kostant). *Let G be a connected, simply connected solvable Lie group of type I. Then there is a natural bijection between the unitary dual $\hat{G}$ and the orbit space $\mathcal{O}_{rigg}(G)$. The correspondence between $\hat{G}$ and $\mathcal{O}_{rigg}(G)$ is given as follows. Let $(\ell, \chi) \in \mathfrak{g}^*_{rigg}$. Then there always exists a complex subalgebra $\mathfrak{p}$ of $\mathfrak{g}_{\mathbb{C}}$ subordinate to ℓ. We let $L(G, \ell, \chi, \mathfrak{p})$ be the space of complex valued functions*

ϕ on G satisfying the following conditions

$$\phi(hg) = \chi(h)\phi(g), \quad h \in G_\ell \tag{4.3}$$

and

$$(L_X + 2\pi i < \ell, X >)\phi = 0, \quad X \in \mathfrak{p}, \tag{4.4}$$

where L_X is the right invariant complex vector field on G defined by $X \in \mathfrak{g}_{\mathbb{C}}$. Then we have the representation T of G defined by

$$(T(g_1)\phi)(g) := \phi(gg_1), \quad g, g_1 \in G. \tag{4.5}$$

We can show that under suitable conditions on $\mathfrak{p}$ including the Pukanszky condition

$$p^{-1}(p(\ell)) = \ell + \mathfrak{p}^\perp \subset \Omega_\ell \tag{4.6}$$

and the condition

$$codim_{\mathbb{C}}\, \mathfrak{p} = \frac{1}{2} rank\, B_\ell, \tag{4.7}$$

the representation T is irreducible and its equivalence class depends only on the rigged orbit Ω containing (ℓ, χ). Here $p : \mathfrak{g}_{\mathbb{C}}^* \longrightarrow \mathfrak{p}^*$ denotes the natural projection of $\mathfrak{g}_{\mathbb{C}}^*$ onto $\mathfrak{p}^*$ dual to the inclusion $\mathfrak{p} \hookrightarrow \mathfrak{g}_{\mathbb{C}}$. We denote by T_Ω the representation T of G obtained from $(\ell, \chi) \in \mathfrak{g}_{rigg}^*$ and $\mathfrak{p}$. The correspondence between $\mathcal{O}_{rigg}(G)$ and $\hat{G}$ is given by

$$(\ell, \chi) \in \Omega \mapsto T_\Omega.$$

Definition 4.5. Let H be a closed subgroup of a Lie group G. We say that a rigged orbit $\Omega' \in \mathcal{O}_{\rm rigg}(H)$ lies under a rigged orbit $\Omega \in \mathcal{O}_{\rm rigg}(G)$ (or equivalently, Ω lies over Ω') if there exist rigged momenta $(\ell, \chi) \in \Omega$ and $(\ell', \chi') \in \Omega'$ such that the following conditions are satisfied:

$$p(\ell) = \ell', \quad \chi = \chi' \quad \text{on } H \cap G_\ell. \tag{4.8}$$

We define the *sum of rigged orbits* Ω_1 and Ω_2 as the set of all $(\ell, \chi) \in \mathcal{O}_{\rm rigg}(G)$ for which there exist $(\ell_i, \chi_i) \in \Omega_i,\ i = 1, 2$, such that

$$\ell = \ell_1 + \ell_2, \quad \chi = \chi_1\chi_2 \ \text{ on } G_{\ell_1} \cap G_{\ell_2}. \tag{4.9}$$

I. M. Shchepochkina [80]-[81] proved the following.

Theorem 4.6. *Let G be a connected, simply connected solvable Lie group and H a closed subgroup of G. Then* *(a) The spectrum of* $Ind_H^G S_{\Omega'}$ *consists of those* T_Ω *for which* Ω *lies over* Ω'*, where* $S_{\Omega'}$ *is an irreducible unitary representation of* H *corresponding to a rigged orbit* Ω' *in* $\mathfrak{h}^*_{rigg}$ *by Theorem 4.4.*

(b) The spectrum of $Res_H^G T_\Omega$ *consists of those* $S_{\Omega'}$ *for which* Ω' *lies under* Ω*.*

(c) The spectrum of $T_{\Omega_1} \otimes T_{\Omega_2}$ *consists of those* T_Ω *for which* Ω *lies in* $\Omega_1 + \Omega_2$*.*

5. The Obstacle for the Orbit Method

In this section, we discuss the case where the correspondence between irreducible unitary representations and coadjoint orbits breaks down. If G is a compact Lie group or a semisimple Lie group, the correspondence breaks down.

First we collect some definitions. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\mathcal{S}(G)$ be the Schwarz space of rapidly decreasing functions on G. We define the Fourier transform $\mathcal{F}_f$ for $f \in \mathcal{S}(G)$ by

$$\mathcal{F}_f(\ell) = \int_{\mathfrak{g}} f(\exp X)\, e^{2\pi i \lambda(X)} dX, \quad \lambda \in \mathfrak{g}^*. \tag{5.1}$$

Then (5.1) is a well-defined function on $\mathfrak{g}^*$. As usual, we define the Fourier transform $\mathcal{F}_\chi$ of a distribution $\chi \in \mathcal{S}'(G)$ by

$$< \mathcal{F}_\chi, \mathcal{F}_f > = < \chi, f >, \quad f \in \mathcal{S}(G). \tag{5.2}$$

For $f \in \mathcal{S}(G)$ and an irreducible unitary representation T of G, we put

$$T(f) = \int_{\mathfrak{g}} f(\exp X)\, T(\exp X) dX.$$

Then we can see that for an irreducible unitary representation of a *nilpotent* Lie group G, we obtain the following formula

$$\operatorname{tr} T(f) = \int_\Omega \mathcal{F}_f(\lambda) d_\Omega \lambda, \tag{5.3}$$

where Ω is the coadjoint orbit in $\mathfrak{g}^*$ attached to T under the Kirillov correspondence and $d_\Omega\lambda$ is the measure on Ω with dimension $2k$ given by the form $\frac{1}{k!} B_\Omega \wedge \cdots \wedge B_\Omega$ (k factors) with the canonical symplectic form B_Ω on Ω.

Definition 5.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$. A coadjoint orbit Ω in $\mathfrak{g}^*$ is called *integral* if the two dimensional cohomology class defined by the canonical two form B_Ω belongs to $H^2(\Omega, \mathbb{Z})$, namely, the integral of B_Ω over a two dimensional cycle in Ω is an integer.

5.1. Compact Lie Groups

Let G be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Then the G-orbits in $\mathfrak{g}^*$ are simply connected and have Kähler structures(not unique). These Kähler manifolds are called *flag manifolds* because their elements are realized in terms of flags. Let T be a maximal abelian subgroup of G. Then $X = G/T$ is called the *full flag manifold* and other flag manifolds are called *degenerate* ones. From the exact sequence

$$\cdots \longrightarrow \pi_k(G) \longrightarrow \pi_k(X) \longrightarrow \pi_{k-1}(T) \longrightarrow \pi_{k-1}(G) \longrightarrow \cdots$$

and the fact that $\pi_1(G) = \pi_2(G)$ under the assumption that G is simply connected, we obtain

$$H_2(X, \mathbb{Z}) \cong \pi_2(X) \cong \pi_1(T) \cong \mathbb{Z}^{\dim T}. \tag{5.4}$$

Let Ω be a coadjoint orbit in $\mathfrak{g}^*$. We identify $\mathfrak{g}^*$ with $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}^*$ with $\mathfrak{g}_{\mathbb{C}}$ via the Killing form so that $\mathfrak{t}_{\mathbb{C}}^*$ goes to $\mathfrak{t}_{\mathbb{C}}$ and the weight $P \subset \mathfrak{t}_{\mathbb{C}}^*$ corresponds to a lattice in $i\mathfrak{t}^* \subset i\mathfrak{g}^* \cong i\mathfrak{g}$. Then $\Omega \cap \mathfrak{t}^*$ is a finite set which forms a single W-orbit, where W is the Weyl group defined as $W = N_G(T)/Z_G(T)$.

Proposition 5.2. *Let Ω_λ be the orbit passing through the point $i\lambda \in \mathfrak{t}^*$. Then*

(1) the orbit Ω_λ is integral if and only if $\lambda \in P$.

(2) $\dim \Omega_\lambda$ is equal to the number of roots non-orthogonal to λ.

Let Ω be an integral orbit of maximal dimension in $\mathfrak{g}^*$. Let $\lambda \in \Omega$ and let $\mathfrak{h}$ be a positive admissible polarization for λ. Here the admissibility for λ means that $\mathfrak{h}$ satisfies the following conditions:

(A1) $\mathfrak{h}$ is invariant under the action of G_λ,

(A2) $\mathfrak{h} + \bar{\mathfrak{h}}$ is a subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

Here G_λ denotes the stabilizer of G at λ.

Let χ_λ be the unitary character of G_λ defined by

$$\chi_\lambda(\exp X) = e^{2\pi i \lambda(X)}, \quad X \in \mathfrak{g}_\lambda, \tag{5.5}$$

where $\mathfrak{g}_\lambda$ is the Lie algebra of G_λ. We note that G_λ is connected. Let L_λ be the hermitian line bundle over $\Omega = G/G_\lambda$ defined by the unitary character χ_λ of G_λ. Then G acts on the space $\Gamma(L_\lambda)$ of holomorphic sections of L_λ as a representation of G. A. Borel and A. Weil proved that $\Gamma(L_\lambda)$ is non-zero and is an irreducible unitary representation of G with highest weight λ. This is the so-called *Borel-Weil Theorem.* Thereafter this theorem was generalized by R. Bott in the late 1950s as follows.

Theorem 5.3 (R. Bott). *Let ρ be the half sum of positive roots of the root system for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Then the cohomology space $H^k(X, L_\lambda)$ is non-zero precisely when*

$$\rho - i\lambda = w(\mu + \rho) \tag{5.6}$$

for some $\mu \in P_+,\ w \in W$ *and* $k = l(w)$*, the length of* w*. In this case the representation of* G *in* $H^k(\Omega, L_\lambda)$ *is equivalent to* $\pi_{-i\mu}$.

We note that the Borel-Weil Theorem strongly suggests relating π_λ to Ω_λ and, on the other hand, the Bott's Theorem suggests the correspondence $\pi_\lambda \leftrightarrow \Omega_{\lambda+\rho}$ which is a bijection between the unitary dual $\hat{G}$ of G and the set of all integral orbits of maximal dimension. It is known that

$$\dim \pi_\lambda = \mathrm{vol}(\Omega_{\lambda+\rho}). \tag{5.7}$$

The character formula (5.3) is valid for a compact Lie group G and provides an integral representation of the character:

$$\chi_\lambda(\exp X) = \frac{1}{p(X)} \int_{\Omega_\lambda} e^{2\pi i \lambda(X)} d_\Omega \lambda. \tag{5.8}$$

In 1990 N.J. Wildberger [96] proved the following.

Theorem 5.4. *Let* $\Phi : C^\infty(\mathfrak{g})' \longrightarrow C^\infty(G)'$ *be the transform defined by*

$$< \Phi(\nu), f >=< \nu, p \cdot (f \circ exp) >, \quad \nu \in C^\infty(\mathfrak{g})',\ f \in C^\infty(G)'. \tag{5.9}$$

Then for $Ad(G)$*-invariant distributions the convolution operators on* G *and* $\mathfrak{g}$ *are related by the transform above:*

$$\Phi(\mu) *_G \Phi(\nu) = \Phi(\mu *_{\mathfrak{g}} \nu). \tag{5.10}$$

The above theorem says that Φ straightens the group convolution, turning it into the abelian convolution on $\mathfrak{g}$. This implies the following geometric fact.

Corollary 5.5. *For any two coadjoint orbits* $\Omega_1, \Omega_2 \subset \mathfrak{g}$*, we let* $C_1 = exp\, \Omega_1$ *and* $C_2 = exp\, \Omega_2$*. Then the following holds.*

$$C_1 \cdot C_2 \subset \exp(\Omega_1 + \Omega_2). \tag{5.11}$$

5.2. Semisimple Lie Groups

The unitary dual $\hat{G}_u$ of a semisimple Lie group G splits into different series, namely, the principal series, degenerate series, complimentary series, discrete series and so on. These series may be attached to different types of coadjoint orbits. The principal series were defined first for complex semisimple Lie groups and for the real semisimple Lie group G which admit the *split* Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. These series are induced from characters of the Borel subgroup $B \supset H = \exp \mathfrak{h}$. The degenerate series are obtained by replacing B by a parabolic subgroup $P \supset B$. All these series are in a perfect correspondence with the family of coadjoint orbits which have a non-empty intersection with $\mathfrak{h}$. An irreducible unitary representation

π of G is said to be a *discrete series* if it occurs as a direct summand in the regular representation R of G on $L^2(G, dg)$. According to Harish-Chandra, if G is a real semisimple Lie group, $\hat{G}_d \neq 0$ if and only if G has a compact Cartan subgroup. Here $\hat{G}_d$ denotes the set of equivalent classes of discrete series of G. There is an interesting complimentary series of representations which are *not weakly contained* in the regular representation R of G. These can be obtained from the principal series and degenerate series by analytic continuation.

The principal series are related to the semisimple orbits. On the other hand, the nilpotent orbits are related to the so-called *unipotent* representations if they exist. In fact, the hyperbolic orbits are related to the representations obtained by the *parabolic induction* and the elliptic orbits are connected to the representations obtained by the *cohomological parabolic induction*. The notion of unipotent representations are not still well defined and hence not understood well. Recently J.-S. Huang and J.-S. Li [41] attached unitary representations to spherical nilpotent orbits for the real orthogonal and symplectic groups. The study of unipotent representations is under way. For some results and conjectures on unipotent representations, we refer to [1], [16], [41] and [90]-[91].

6. Nilpotent Orbits and the Kostant-Sekiguchi Correspondence

In this section, we present some properties of nilpotent orbits for a reductive Lie group G and describe the Kostant-Sekiguchi correspondence. We also explain the work of D. Vogan that for a maximal compact subgroup K of G, he attaches a space with a K-action to a nilpotent orbit. Most of the materials in this section are based on the article [93].

6.1. Jordan Decomposition

Definition 6.1.1. Let $GL(n)$ be the group of nonsingular real or complex $n \times n$ matrices. The *Cartan involution* of $GL(n)$ is the automorphism conjugate transpose inverse:

$$\theta(g) = {}^t\bar{g}^{-1}, \quad g \in GL(n). \tag{6.1}$$

A *linear reductive group* is a closed subgroup G of some $GL(n)$ preserved by θ and having finitely many connected components. A *reductive* Lie group is a Lie group $\tilde{G}$ endowed with a homomorphism $\pi : \tilde{G} \longrightarrow G$ onto a linear reductive group G so that the kernel of π is finite.

Theorem 6.1.2 (Cartan Decomposition). *Let $\tilde{G}$ be a reductive Lie group with $\pi : \tilde{G} \longrightarrow G$ as in Definition 6.1.1. Let*

$$K = G^\theta = \{ g \in G \mid \theta(g) = g \}$$

be a maximal compact subgroup of G. We write $\tilde{K} = \pi^{-1}(K)$, a compact subgroup of $\tilde{G}$, and use $d\pi$ to identify the Lie algebras of $\tilde{G}$ and G. Let $\mathfrak{p}$ be the (-1)-eigenspace *of $d\theta$ on the Lie algebra $\mathfrak{g}$ of G. Then the map*

$$\tilde{K} \times \mathfrak{p} \longrightarrow \tilde{G}, \quad (\tilde{k}, X) \mapsto \tilde{k} \cdot exp\, X, \quad \tilde{k} \in \tilde{K},\ X \in \mathfrak{p} \tag{6.2}$$

is a diffeomorphism from $\tilde{K} \times \mathfrak{p}$ onto $\tilde{G}$. In particular, $\tilde{K}$ is maximal among the compact subgroups of $\tilde{G}$.

Suppose $\tilde{G}$ is a reductive Lie group. We define a map $\theta : \tilde{G} \longrightarrow \tilde{G}$ by

$$\theta(\tilde{k} \cdot \exp X) = \tilde{k} \cdot \exp(-X), \quad \tilde{k} \in \tilde{K},\ X \in \mathfrak{p}. \tag{6.3}$$

Then θ is an involution, that is, the Cartan involution of $\tilde{G}$. The group of fixed points of θ is $\tilde{K}$.

The following proposition makes us identify the Lie algebra of a reductive Lie group with its dual space.

Proposition 6.1.3. *Let G be a reductive Lie group. Identify $\mathfrak{g}$ with a Lie algebra of $n \times n$ matrices (cf. Definition 6.1.1). We define a real valued symmetric bilinear form on $\mathfrak{g}$ by*

$$< X, Y >= Re\, tr(XY), \quad X, Y \in \mathfrak{g}. \tag{6.4}$$

Then the following (a),(b) and (c) hold:

(a) The form $<,>$ is invariant under $Ad(G)$ and the Cartan involution θ.

(b) The Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is orthogonal with respect to the form $<,>$, where $\mathfrak{k}$ is the Lie algebra of K (the group of fixed points of θ) which is the $(+1)$-eigenspace of $\theta := d\theta$ on $\mathfrak{g}$ and $\mathfrak{p}$ is the (-1)-eigenspace of θ on $\mathfrak{g}$. The form $<,>$ is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. And hence the form $<,>$ is nondegenerate on $\mathfrak{g}$.

(c) There is a G-equivariant linear isomorphism

$$\mathfrak{g}^* \cong \mathfrak{g}, \quad \lambda \mapsto X_\lambda$$

characterized by

$$\lambda(Y) =< X_\lambda, Y >, \quad Y \in \mathfrak{g}. \tag{6.5}$$

Definition 6.1.4. Let G be a reductive Lie group with Lie algebra $\mathfrak{g}$ consisting of $n \times n$ matrices. An element $X \in \mathfrak{g}$ is called *nilpotent* if it is nilpotent as a matrix. An element $X \in \mathfrak{g}$ is called *semisimple* if the corresponding complex matrix is diagonalizable. An element $X \in \mathfrak{g}$ is called *hyperbolic* if it is semisimple and its eigenvalues are real. An element $X \in \mathfrak{g}$ is called *elliptic* if it is semisimple and its eigenvalues are purely imaginary.

Proposition 6.1.5 (Jordan Decomposition). *Let G be a reductive Lie group* *with its Lie algebra $\mathfrak{g}$ and let $G = K \cdot exp\,\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ (see Theorem 6.1.2). Then the following (1)-(5) hold:*

(1) Any element $X \in \mathfrak{g}$ has a unique decomposition

$$X = X_h + X_e + X_n$$

characterized by the conditions that X_h is hyperbolic, X_e is elliptic, X_n is nilpotent and X_h, X_e, X_n commute with each other.

(2) After replacing X by a conjugate under $Ad(G)$, we may assume that $X_h \in \mathfrak{p}$, $X_e \in \mathfrak{k}$ and that $X_n = E$ belongs to a standard $\mathfrak{sl}(2)$ triple. We recall that a triple $\{H, E, F\} \subset \mathfrak{g}$ is called a standard $\mathfrak{sl}(2)$ triple, if they satisfy the following conditions

$$\theta(E) = -F, \quad \theta(H) = -H, \quad [H, E] = 2E, \quad [E, F] = H. \tag{6.6}$$

(3) The $Ad(G)$ orbits of hyperbolic elements in $\mathfrak{g}$ are in one-to-one correspondence with the $Ad(K)$ orbits in $\mathfrak{p}$.

(4) The $Ad(G)$ orbits of elliptic elements in $\mathfrak{g}$ are in one-to-one correspondence with the $Ad(K)$ orbits in $\mathfrak{k}$.

(5) The $Ad(G)$ orbits of nilpotent orbits are in one-to-one correspondence with the $Ad(K)$ orbits of standard $\mathfrak{sl}(2)$ triples in $\mathfrak{g}$.

6.2. Nilpotent Orbits

Let G be a real reductive Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$. We consider the complex special linear group $SL(2, \mathbb{C})$ which is the complexification of $SL(2, \mathbb{R})$. We define the involution $\theta_0 : SL(2, \mathbb{C}) \longrightarrow SL(2, \mathbb{C})$ by

$$\theta_0(g) = {}^t g^{-1}, \quad g \in SL(2, \mathbb{C}). \tag{6.7}$$

We denote its differential by the same letter

$$\theta_0(Z) = -\,{}^t Z, \quad Z \in \mathfrak{sl}(2, \mathbb{C}). \tag{6.8}$$

The complex conjugation σ_0 defining the real form $SL(2, \mathbb{R})$ is just the complex conjugation of matrices.

We say that a triple $\{H, X, Y\}$ in a real or complex Lie algebra is a *standard triple* if it satisfies the following conditions:

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H. \tag{6.9}$$

We call the element H (resp. X, Y) a *neutral* (resp. *nilpositive, nilnegative*) element of a standard triple $\{H, X, Y\}$.

We consider the standard basis $\{H_0, E_0, F_0\}$ of $\mathfrak{sl}(2,\mathbb{R})$ given by

$$H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{6.10}$$

Then they satisfy

$$[H_0, E_0] = 2E_0, \quad [H_0, F_0] = -2F_0, \quad [E_0, F_0] = H_0 \tag{6.11}$$

and

$$\theta_0(H_0) = -H_0, \quad \theta_0(E_0) = -F_0, \quad \theta_0(F_0) = -E_0. \tag{6.12}$$

We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and let θ be the corresponding Cartan involution. We say that a standard triple $\{H, X, Y\}$ is a *Cayley triple* in $\mathfrak{g}$ if it satisfies the conditions:

$$\theta(H) = -H, \quad \theta(X) = -Y, \quad \theta(Y) = -X. \tag{6.13}$$

According to (6.11) and (6.12), the triple $\{H_0, E_0, F_0\}$ is a Cayley triple in $\mathfrak{sl}(2,\mathbb{R})$.

For a real Lie algebra $\mathfrak{g}$, we have the following theorems.

Theorem 6.2.1. *Given a Cartan decomposition θ on $\mathfrak{g}$, any triple $\{H, X, Y\}$ in $\mathfrak{g}$ is conjugate under the adjoint group $Ad(G)$ to a Cayley triple $\{H', X', Y'\}$ in $\mathfrak{g}$.*

Theorem 6.2.2 (Jacobson-Morozov). *Let X be a nonzero nilpotent element in $\mathfrak{g}$. Then there exists a standard triple $\{H, X, Y\}$ in $\mathfrak{g}$ such that X is nilpositive.*

Theorem 6.2.3 (Kostant). *Any two standard triples $\{H, X, Y\}$, $\{H', X, Y'\}$ in $\mathfrak{g}$ with the same nilpositive element X are conjugate under G^X, the centralizer of X in the adjoint group of G.*

Let $\{H, X, Y\}$ be a Cayley triple in $\mathfrak{g}$. We are going to look for a semisimple element in $\mathfrak{g}$. For this, we need to introduce an auxiliary standard triple attached to a Cayley triple. We put

$$H' = i(X - Y), \quad X' = \frac{1}{2}(X + Y + iH), \quad Y' = \frac{1}{2}(X + Y - iH). \tag{6.14}$$

Then the triple $\{H', X', Y'\}$ in $\mathfrak{g}_{\mathbb{C}}$ is a standard triple, called the *Cayley transform* of $\{H, X, Y\}$.

Since

$$\theta(H') = H', \quad \theta(X') = -X', \quad \theta(Y') = -Y',$$

we have

$$H' \in \mathfrak{k}_{\mathbb{C}} \quad \text{and} \quad X', Y' \in \mathfrak{p}_{\mathbb{C}}. \tag{6.15}$$

Therefore the subalgebra $\mathbb{C} < H', X', Y' >$ of $\mathfrak{g}c$ spanned by H', X', Y' is stable under the action of θ. A standard triple in $\mathfrak{g}c$ with the property (6.15) is called *normal.*

Theorem 6.2.4. *Any nonzero nilpotent element $X \in \mathfrak{p}_{\mathbb{C}}$ is the nilpositive element of a normal triple (see Theorem 6.2.2).*

Theorem 6.2.5. *Any two normal triples $\{H, X, Y\}$, $\{H', X, Y'\}$ with the same nilpositive element X is $K_{\mathbb{C}}^X$-conjugate, where $K_{\mathbb{C}}^X$ denotes the centralizer of X in the complexification $K_{\mathbb{C}}$ of a maximal compact subgroup K corresponding to the Lie algebra $\mathfrak{k}$.*

Theorem 6.2.6. *Any two normal triples $\{H, X, Y\}$, $\{H, X', Y'\}$ with the same neutral element H are $K_{\mathbb{C}}^H$-conjugate.*

Theorem 6.2.7 (Rao). *Any two standard triples $\{H, X, Y\}$, $\{H', X', Y'\}$ in $\mathfrak{g}$ with $X - Y = X' - Y'$ are conjugate under G^{X-Y}, the centralizer of $X - Y$ in G. In fact, $X - Y$ is a semisimple element which we are looking for.*

Let $\mathcal{A}_{triple}$ be the set of all $Ad(G)$-conjugacy classes of standard triples in $\mathfrak{g}$. Let $\mathcal{O}_{\mathcal{N}}$ be the set of all nilpotent orbits in $\mathfrak{g}$. We define the map

$$\Omega : \mathcal{A}_{triple} \longrightarrow \mathcal{O}_{\mathcal{N}}^{\times} := \mathcal{O}_{\mathcal{N}} - \{0\} \tag{6.16}$$

by

$$\Omega([\{H, X, Y\}]) := \mathcal{O}_X, \quad \mathcal{O}_X := Ad(G) \cdot X, \tag{6.17}$$

where $[\{H, X, Y\}]$ denotes the G-conjugacy class of a standard triple $\{H, X, Y\}$. According to Theorem 6.2.2 (Jacobson-Morozov Theorem) and Theorem 6.2.3 (Kostant's Theorem), the map Ω is bijective.

We put

$$h_0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad x_0 = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad y_0 = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}. \tag{6.18}$$

It is easy to see that the triple $\{h_0, x_0, y_0\}$ in $\mathfrak{sl}(2, \mathbb{C})$ is a normal triple. The complex conjugation σ_0 acts on the triple $\{h_0, x_0, y_0\}$ as follows:

$$\sigma_0(h_0) = -h_0, \quad \sigma_0(x_0) = y_0, \quad \sigma_0(y_0) = x_0. \tag{6.19}$$

We introduce some notations. We denote by $\mathrm{Mor}(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{g}_{\mathbb{C}})$ the set of all nonzero Lie algebra homomorphisms from $\mathfrak{sl}(2, \mathbb{C})$ to $\mathfrak{g}_{\mathbb{C}}$. We define

$$\mathrm{Mor}^{\mathbb{R}}(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{g}_{\mathbb{C}}) = \{\, \phi \in \mathrm{Mor}(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{g}_{\mathbb{C}}) \mid \phi \text{ is defined over } \mathbb{R} \,\},$$

$$\mathrm{Mor}^{\theta}(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{g}_{\mathbb{C}}) = \{\, \phi \in \mathrm{Mor}(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{g}_{\mathbb{C}}) \mid \theta \circ \phi = \phi \circ \theta_0 \,\},$$

$$\mathrm{Mor}^{\mathbb{R},\theta}(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{g}_{\mathbb{C}}) = \mathrm{Mor}^{\mathbb{R}}(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{g}_{\mathbb{C}}) \cap \mathrm{Mor}^{\theta}(\mathfrak{sl}(2, \mathbb{C}), \mathfrak{g}_{\mathbb{C}}), \tag{6.20}$$

$$\mathrm{Mor}^{\sigma}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}}) = \{\, \phi \in \mathrm{Mor}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}}) \mid \sigma \circ \phi = \phi \circ \sigma_0 \,\},$$
$$\mathrm{Mor}^{\sigma,\theta}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}}) = \mathrm{Mor}^{\sigma}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}}) \cap \mathrm{Mor}^{\theta}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}}).$$

We observe that $\mathrm{Mor}^{\mathbb{R}}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$ is naturally isomorphic to $\mathrm{Mor}(\mathfrak{sl}(2,\mathbb{R}),\mathfrak{g})$, the set of all nonzero Lie algebra real homomorphisms from $\mathfrak{sl}(2,\mathbb{R})$ to $\mathfrak{g}$.

Proposition 6.2.8. *Suppose ϕ be a nonzero Lie algebra homomorphism from $\mathfrak{sl}(2,\mathbb{C})$ to a complex reductive Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Write*

$$H = \phi(H_0), \quad E = \phi(E_0), \quad F = \phi(F_0) \quad (\textit{see } (6.2.10)).$$

Then the following hold.

(1)

$$\mathfrak{g}_{\mathbb{C}} = \sum_{k\in\mathbb{Z}} \mathfrak{g}_{\mathbb{C}}(k),$$

where

$$\mathfrak{g}_{\mathbb{C}}(k) = \{X \in \mathfrak{g}_{\mathbb{C}} |\ [H,X] = kX\,\}, \quad k \in \mathbb{Z}.$$

(2) If we write

$$\mathfrak{l} = \mathfrak{g}_{\mathbb{C}}(0) \quad \textit{and} \quad \mathfrak{u} = \sum_{k>0} \mathfrak{g}_{\mathbb{C}}(k),$$

then $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a Levi decomposition of a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

(3) The centralizer of E is graded by the decomposition in (1). More precisely,

$$\mathfrak{g}_{\mathbb{C}}^{E} = \mathfrak{l}^{E} + \sum_{k>0} \mathfrak{g}_{\mathbb{C}}(k)^{E} = \mathfrak{l}^{E} + \mathfrak{u}^{E}.$$

(4) The subalgebra $\mathfrak{l}^E = \mathfrak{g}_{\mathbb{C}}^{H,E}$ is equal to $\mathfrak{g}_{\mathbb{C}}^{\phi}$, the centralizer in $\mathfrak{g}_{\mathbb{C}}$ of the image of ϕ. It is a reductive subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Consequently the decomposition in (3) is a Levi decomposition of $\mathfrak{g}_{\mathbb{C}}^{E}$.

Parallel results hold if $\{H,E,F\}$ are replaced by

$$h = \phi(h_0), \quad x = \phi(x_0), \quad y = \phi(y_0).$$

Proposition 6.2.9. *Suppose G is a real reductive Lie group, and let $\phi_{\mathbb{R}}$ be an element of $Mor^{\sigma}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$. Define $E_{\mathbb{R}}, H_{\mathbb{R}}, F_{\mathbb{R}}$ by*

$$H_{\mathbb{R}} = \phi_{\mathbb{R}}(H_0), \quad E_{\mathbb{R}} = \phi_{\mathbb{R}}(E_0), \quad F_{\mathbb{R}} = \phi_{\mathbb{R}}(F_0).$$

Then the following hold.

(1) $E_{\mathbb{R}},\ F_{\mathbb{R}}$ are nilpotent, and $H_{\mathbb{R}}$ is hyperbolic.

(2) If we define $L = G^{H_{\mathbb{R}}}$ to be the isotrophy group of the adjoint action at $H_{\mathbb{R}}$ and $U = exp(\mathfrak{u} \cap \mathfrak{g})$, then $Q = LU$ is the parabolic subgroup of G associated to $H_{\mathbb{R}}$.

(3) The isotrophy group $G^{E_{\mathbb{R}}}$ of the adjoint action at $E_{\mathbb{R}}$ is contained in Q, and respects the Levi decomposition :

$$G^{E_{\mathbb{R}}} = \left(L^{E_{\mathbb{R}}}\right)\left(U^{E_{\mathbb{R}}}\right).$$

(4) The subgroup $L^{E_{\mathbb{R}}} = G^{H,E_{\mathbb{R}}}$ is equal to $G^{\phi_{\mathbb{R}}}$, the centralizer in G of the image image of $\phi_{\mathbb{R}}$. It is a reductive subgroup of G. The The subgroup $U^{E_{\mathbb{R}}}$ is simply connected unipotent.

(5) Suppose that $\phi_{\mathbb{R},\theta}$ is an element of $Mor^{\sigma,\theta}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$. Then $G^{\phi_{\mathbb{R},\theta}}$ is stable under the action of θ, and we may take θ as a Cartan involution on this reductive group. In particular, $G^{E_{\mathbb{R}}}$ and $G^{\phi_{\mathbb{R},\theta}}$ have a common maximal compact subgroup

$$K^{\phi_{\mathbb{R},\theta}} = (L\cap K)^{E_{\mathbb{R}}}.$$

The above proposition provides good information about the action of G on the cone $\mathcal{N}_{\mathbb{R}}$ of all nilpotent orbits in $\mathfrak{g}$.

The following proposition gives information about the action of $K_{\mathbb{C}}$ on the cone $\mathcal{N}_\theta$ of all nilpotent elements in $\mathfrak{p}_{\mathbb{C}}$.

Proposition 6.2.10. *Suppose G is a real reductive Lie group with Cartan decomposition $G = K\cdot exp\,\mathfrak{p}$. Let $\phi_\theta \in Mor^\theta(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$. We define*

$$h_\theta = \phi_\theta(h_0),\quad x_\theta = \phi_\theta(x_0),\quad y_\theta = \phi_\theta(y_0).$$

Then we have the following results.

(1) x_θ and y_θ are nilpotent elements in $\mathfrak{p}c$, $h_\theta \in \mathfrak{k}c$ is hyperbolic and $ih_\theta \in \mathfrak{k}c$ is elliptic.

(2) The parabolic subalgebra $\mathfrak{q} = \mathfrak{l}+\mathfrak{u}$ constructed as in Proposition 6.2.8 using h_θ is stable under θ.

(3) If we define $L_K := (K_{\mathbb{C}})^{h_\theta}$ and $U_K = exp(\mathfrak{u}\cap\mathfrak{k}c)$, then $Q_K = L_K U_K$ is the parabolic subgroup of $K_{\mathbb{C}}$ associated to h_θ.

(4) $K_{\mathbb{C}}^{x_\theta} \subset Q_K$ and $K_{\mathbb{C}}^{x_\theta}$ respects the Levi decomposition

$$K_{\mathbb{C}}^{x_\theta} = (L_K^{x_\theta})\,(U_K^{x_\theta}).$$

(5) $L_K^{x_\theta} = K_{\mathbb{C}}^{h_\theta,x_\theta}$ is equal to $K_{\mathbb{C}}^{\phi_\theta}$, the centralizer in $K_{\mathbb{C}}$ of the image of ϕ_θ. It is a reductive algebraic subgroup of $K_{\mathbb{C}}$. The subgroup U_K^x is simply connected unipotent. In particular, the decomposition of (4) is a Levi decomposition of $K_{\mathbb{C}}^x$.

(6) Let $\phi_{\mathbb{R},\theta}$ be an element of $Mor^{\sigma,\theta}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$. Then $K_{\mathbb{C}}^{\phi_{\mathbb{R},\theta}}$ is stable under σ, and we may take σ as complex conjugation for a compact real form of this reductive algebraic group. In particular, $K_{\mathbb{C}}^x$ and $K_{\mathbb{C}}^{\phi_{\mathbb{R},\theta}}$ have a common maximal compact subgroup

$$K_{\mathbb{C}}^{\phi_{\mathbb{R},\theta}} = L_K^{x_\theta}\cap K.$$

6.3. The Kostant-Sekiguchi Correspondence

J. Sekiguchi [79] and B. Kostant (unpublished) established a bijection between the set of all nilpotent G-orbits in $\mathfrak{g}$ on the one hand and, on the other hand, the set of all nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathbb{C}}$. The detail is as follows.

Theorem 6.3.1. *Let G be a real reductive Lie group with Cartan involution θ and its corresponding maximal compact subgroup K. Let σ be the complex conjugation on the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. Then the following sets are in one-to-one correspondence.*

(a) G-orbits on the cone $\mathcal{N}_{\mathbb{R}}$ of nilpotent elements in $\mathfrak{g}$.

(b) G-conjugacy classes of Lie algebra homomorphisms $\phi_{\mathbb{R}}$ in $Mor^{\sigma}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$.

(c) K-conjugacy classes of Lie algebra homomorphisms $\phi_{\mathbb{R},\theta}$ in $Mor^{\sigma,\theta}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$.

(d) $K_{\mathbb{C}}$-conjugacy classes of Lie algebra homomorphisms ϕ_{θ} in $Mor^{\theta}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$.

(e) $K_{\mathbb{C}}$-orbits on the cone $\mathcal{N}_{\theta}$ of nilpotent elements in $\mathfrak{p}_{\mathbb{C}}$.

Here G acts on $\mathrm{Mor}^{\sigma}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$ via the adjoint action of G in $\mathfrak{g}$:

$$(g\cdot\phi_{\mathbb{R}})(\zeta)=Ad(g)(\phi_{\mathbb{R}}(\zeta)),\quad g\in G \text{ and } \quad \phi_{\mathbb{R}}\in \mathrm{Mor}^{\sigma}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}}). \tag{6.21}$$

Similarly K and $K_{\mathbb{C}}$ act on $\mathrm{Mor}^{\sigma,\theta}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$ and $\mathrm{Mor}^{\theta}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$ like (6.21) respectively. The correspondence between (a) and (e) is called the *Kostant-Sekiguchi correspondence* between the G-orbits in $\mathcal{N}_{\mathbb{R}}$ and the $K_{\mathbb{C}}$-orbits in $\mathcal{N}_{\theta}$. If $\phi_{\mathbb{R},\theta}$ is an element in $\mathrm{Mor}^{\sigma,\theta}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$ as in (c) , then the correspondence is given by

$$E=\phi_{\mathbb{R},\theta}(E_0)\leftrightsquigarrow x=\phi_{\mathbb{R},\theta}(x_0). \quad \text{(see (6.18))} \tag{6.22}$$

The proof of the above theorem can be found in [90], pp. 348-350.

M. Vergne [87] showed that the orbits $G\cdot E=Ad(G)E$ and $K_{\mathbb{C}}\cdot x=Ad(K_{\mathbb{C}})x$ are diffeomorphic as manifolds with K-action under the assumption that they are in the Kostant-Sekiguchi correspondence. Here E and x are given by (6.22).

Theorem 6.3.2 (M. Vergne). *Suppose $G=K\cdot exp(\mathfrak{p})$ is a Cartan decomposition of a real reductive Lie group G and $E\in\mathfrak{g}$, $x\in\mathfrak{p}c$ are nilpotent elements. Assume that the orbits $G\cdot E$ and $K_{\mathbb{C}}\cdot x$ correspond under the Kostant-Sekiguchi correspondence. Then there is a K-equivariant diffeomorphism from $G\cdot E$ onto $K_{\mathbb{C}}\cdot x$.*

Remark 6.3.3. The Kostant-Sekiguchi correspondence sends the zero orbit to the zero orbit, and the nilpotent orbit through the nilpositive element of a Cayley triple in $\mathfrak{g}$ to the orbit through the nilpositive element of its Cayley transform.

Remark 6.3.4. Let $G\cdot E$ and $K_{\mathbb{C}}\cdot x$ be in the Kostant-Sekiguchi correspondence, where $E\in\mathcal{N}_{\mathbb{R}}\subset\mathfrak{g}$ and $x\in\mathcal{N}_{\theta}\subset\mathfrak{p}_{\mathbb{C}}$. Then the following hold.

(1) $G_{\mathbb{C}}\cdot E=G_{\mathbb{C}}\cdot x$, where $G_{\mathbb{C}}$ denotes the complexification of G.

(2) $\dim_{\mathbb{C}}(K_{\mathbb{C}}\cdot x)=\frac{1}{2}\dim_{\mathbb{R}}(G\cdot E)=\frac{1}{2}\dim_{\mathbb{C}}(G_{\mathbb{C}}\cdot x)$.

(3) The centralizers $G^E,\ K_{\mathbb{C}}^x$ have a common maximal compact subgroup $K^{E,x}$ which is the centralizer of the span of E and x in K.

Remark 6.3.5. Let π be an irreducible, admissible representation of a reductive Lie group G. Recently Schmid and Vilonen gave a new geometric description of the Kostant-Sekiguchi correspondence (cf. [76], Theorem 7.22) and then using this fact proved that the associated cycle $\mathrm{Ass}(\pi)$ of π coincides with the wave front cycle $\mathrm{WF}(\pi)$ via the Kostant-Sekiguchi correspondence (cf. [77], Theorem 1.4).

6.4. The Ouantization of the K-action

It is known that a hyperbolic orbit could be quantized by the method of a parabolic induction, and on the other hand an elliptic orbit may also be quantized by the method of cohomological induction. However, we do not know yet how to quantize a nilpotent orbit. But D. Vogan attached a space with a representation of K to a nilpotent orbit.

We first fix a nonzero nilpotent element $\lambda_n \in \mathfrak{g}^*$. Let E be the unique element in $\mathfrak{g}$ given from λ_n via (6.5). According to Jacobson-Morosov Theorem or Theorem 6.3.1, there is a non-zero Lie algebra homomorphism $\phi_{\mathbb{R}}$ from $\mathfrak{sl}(2,\mathbb{R})$ to $\mathfrak{g}$ with $\phi_{\mathbb{R}}(E_0) = E$. We recall that E_0 is given by (6.10). $\phi_{\mathbb{R}}$ extends to an element $\phi_{\mathbb{R}} \in \mathrm{Mor}^{\sigma}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$. After replacing λ_n by a conjugate under G, we may assume that $\phi_{\mathbb{R}} = \phi_{\mathbb{R},\theta}$ intertwines θ_0 and θ, that is, $\phi_{\mathbb{R},\theta} \in \mathrm{Mor}^{\sigma,\theta}(\mathfrak{sl}(2,\mathbb{C}),\mathfrak{g}_{\mathbb{C}})$.

We define

$$x = \phi_{\mathbb{R},\theta}(x_0) \in \mathfrak{p}_{\mathbb{C}}, \quad x_0 = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}. \tag{6.23}$$

The isomorphism in (c), Proposition 6.1.3 associates to x a linear functional

$$\lambda_\theta \in \mathfrak{p}_{\mathbb{C}}^*, \quad \lambda_\theta(Y) =< x, Y >, \quad Y \in \mathfrak{g}. \tag{6.24}$$

We note that the element λ_θ is not uniquely determined by λ_n, but the orbit $K_{\mathbb{C}} \cdot \lambda_\theta$ is determined by $G \cdot \lambda_n$. According to Theorem 6.3.2, we get a K-equivariant diffeomorphism

$$G \cdot \lambda_n \cong K_{\mathbb{C}} \cdot \lambda_\theta. \tag{6.25}$$

Definition 6.4.1. (1) Let $\mathfrak{g}_{im}^*$ be the space of purely imaginary-valued linear functionals on $\mathfrak{g}$. We fix an element $\lambda_{im} \in \mathfrak{g}_{im}^*$. We denote the G-orbit of λ_{im} by

$$\mathcal{O}_{im} := G \cdot \lambda_{im} = Ad^*(G) \cdot \lambda_{im}. \tag{6.26}$$

We may define an imaginary-valued symplectic form ω_{im} on the tangent space

$$T_{\lambda_{im}}(\mathcal{O}_{im}) \cong \mathfrak{g}/\mathfrak{g}^{\lambda_{im}}, \tag{6.27}$$

where $\mathfrak{g}^{\lambda_{im}}$ is the Lie algebra of the isotropy subgroup $G^{\lambda_{im}}$ of G at λ_{im}. We denote by $Sp(\omega_{im})$ the group of symplectic real linear transformations of the tangent space (6.27). Then the isotropy action gives a natural homomorphism

$$j : G^{\lambda_{im}} \longrightarrow Sp(\omega_{im}). \tag{6.28}$$

On the other hand, we let $Mp(\omega_{im})$ be the metaplectic group of $Sp(\omega_{im})$. That is, we have the following exact sequence

$$1 \longrightarrow \{1, \epsilon\} \longrightarrow Mp(\omega_{im}) \longrightarrow Sp(\omega_{im}) \longrightarrow 0. \tag{6.29}$$

Pulling back (6.29) via (6.28), we have the so-called *metaplectic double cover* of the isotropy group $G^{\lambda_{im}}$:

$$1 \longrightarrow \{1, \epsilon\} \longrightarrow \tilde{G}^{\lambda_{im}} \longrightarrow G^{\lambda_{im}}. \tag{6.30}$$

That is, $\tilde{G}^{\lambda_{im}}$ is defined by

$$\tilde{G}^{\lambda_{im}} = \{(g, m) \in G^{\lambda_{im}} \times Mp(\omega_{im}) |\ j(g) = p(m)\,\}. \tag{6.31}$$

A representation χ of $\tilde{G}^{\lambda_{im}}$ is called *genuine* if $\chi(\epsilon) = -I$. We say that χ is *admissible* if it is genuine, and the differential of χ is a multiple of λ_{im}: namely, if

$$\chi(\exp x) = \exp(\lambda_{im}(x)) \cdot I, \quad x \in \mathfrak{g}^{\lambda_{im}}. \tag{6.32}$$

If admissible representations exist, we say that λ_{im} (or the orbit $\mathcal{O}_{im}$) is *admissible*. A pair (λ_{im}, χ) consisting of an element $\lambda_{im} \in \mathfrak{g}_{im}^*$ and an irreducible admissible representation χ of $\tilde{G}^{\lambda_{im}}$ is called an *admissible G-orbit datum*. Two such are called *equivalent* if they are conjugate by G.

We observe that if $G^{\lambda_{im}}$ has a finite number of connected components, an irreducible admissible representation of $\tilde{G}^{\lambda_{im}}$ is unitarizable. The notion of admissible G-orbit data was introduced by M. Duflo [25].

(2) Suppose $\lambda_\theta \in \mathfrak{p}_{\mathbb{C}}^*$ is a non-zero nilpotent element. Let $K_{\mathbb{C}}^{\lambda_\theta}$ be the isotropy subgroup of $K_{\mathbb{C}}$ at λ_θ. Define 2ρ to be the algebraic character of $K_{\mathbb{C}}^{\lambda_\theta}$ by which it acts on the top exterior power of the cotangent space at λ_θ to the orbit:

$$2\rho(k) := \det\left(Ad^*(k)|_{(\mathfrak{k}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}}^{\lambda_\theta})^*}\right), \quad k \in K_{\mathbb{C}}^{\lambda_\theta}. \tag{6.33}$$

The differential of 2ρ is a one-dimensional representation of $\mathfrak{k}_{\mathbb{C}}^{\lambda_\theta}$, which we denote also by 2ρ. We define $\rho \in (\mathfrak{k}_{\mathbb{C}}^{\lambda_\theta})^*$ to be the half of 2ρ. More precisely,

$$\rho(Z) = \frac{1}{2} \operatorname{tr}\left(ad^*(Z)|_{(\mathfrak{k}_{\mathbb{C}}/\mathfrak{k}_{\mathbb{C}}^{\lambda_\theta})^*}\right), \quad Z \in \mathfrak{k}_{\mathbb{C}}^{\lambda_\theta}. \tag{6.34}$$

A *nilpotent admissible $K_{\mathbb{C}}$-orbit datum* at λ_θ is an irreducible algebraic representation (τ, V_τ) of $K_{\mathbb{C}}^{\lambda_\theta}$ whose differential is equal to $\rho \cdot I_\tau$, where I_τ denotes the identity map on V_τ. The nilpotent element $\lambda_\theta \in \mathfrak{p}_{\mathbb{C}}^*$ is called *admissible* if a nilpotent admissible $K_{\mathbb{C}}$-orbit datum at λ_θ exists. Two such data are called *equivalent* if they are conjugate.

Theorem 6.4.2 (J. Schwarz). *Suppose G is a real reductive Lie group, K is a maximal compact subgroup, and $K_{\mathbb{C}}$ is its complexification. Then there is a natural bijection between equivalent classes of nilpotent admissible G-orbit data and equivalent classes of nilpotent admissible $K_{\mathbb{C}}$-orbit data.*

Suppose $\lambda_n \in \mathfrak{g}^*$ is a non-zero nilpotent element. Let $\lambda_\theta \in \mathfrak{p}_{\mathbb{C}}^*$ be a nilpotent element which corresponds under the Kostant-Sekiguchi correspondence. We fix a nilpotent admissible $K_{\mathbb{C}}$-orbit datum (τ, V_τ) at λ_θ. We let

$$\mathcal{V}_\tau := K_{\mathbb{C}} \times_{K_{\mathbb{C}}^{\lambda_\theta}} V_\tau \tag{6.35}$$

be the corresponding algebraic vector bundle over the nilpotent orbit $K_{\mathbb{C}} \cdot \lambda_\theta \cong K_{\mathbb{C}}/K_{\mathbb{C}}^{\lambda_\theta}$. We note that a complex structure on $\mathcal{V}_\tau$ is preserved by K but not preserved by G. We now *assume* that the boundary of $\overline{K_{\mathbb{C}} \cdot \lambda_\theta}$ (that is, $\overline{K_{\mathbb{C}} \cdot \lambda_\theta} - K_{\mathbb{C}} \cdot \lambda_\theta$) has a complex codimension at least two. We denote by denote by $X_K(\lambda_n, \tau)$ the space of algebraic sections of $\mathcal{V}_\tau$. Then $X_K(\lambda_n, \tau)$ is an algebraic representation of $K_{\mathbb{C}}$. That is, if $k_1 \in K_{\mathbb{C}}$ and $s \in X_K(\lambda_n, \tau)$, then $(k_1 \cdot s)(k\lambda_\theta) = s((k_1^{-1}k)\lambda_\theta)$. We call the representation $(K_{\mathbb{C}},\, X_K(\lambda_n, \tau))$ of $K_{\mathbb{C}}$ the *quantization of the K-action* on $G \cdot \lambda_n$ for the admissible orbit datum (τ, V_τ).

What this definition amounts to is a desideratum for the quantization of the G-action on $G \cdot \lambda_\theta$. That is, whatever a unitary representation $\pi_G(\lambda_n, \tau)$ we associate to these data, we hope that we have

$$K-\text{finite part of } \pi_G(\lambda_n, \tau) \cong X_K(\lambda_n, \tau). \tag{6.36}$$

When G is a complex Lie group, the coadjoint orbit is a complex symplectic manifold and hence of real dimension $4m$. Consequently the codimension condition is automatically satisfied in this case.

Remark 6.4.3. Nilpotent admissible orbit data may or may not exist. When they exist, there is a one-dimensional admissible datum (τ_0, V_{τ_0}). In this case, all admissible data are in one-to-one correspondence with irreducible representations of the group of connected components of $K_{\mathbb{C}}^{\lambda_\theta}$; the correspondence is obtained by tensoring with τ_0. If G is connected and simply connected, then this component group is just the fundamental group of the nilpotent orbit $K_{\mathbb{C}} \cdot \lambda_\theta$.

7. Minimal Representations

Let G be a real reductive Lie group. Let π be an admissible representation of G. Let $\mathfrak{g}$ be the Lie algebra of G. Three closely related invariants $WF(\pi)$, $AS(\pi)$ and $Ass(\pi)$ in $\mathfrak{g}^*$ which are called the *wave front set* of π, the *asymptotic support of the character* of π and the *associated variety* of π respectively, are attached to a given admissible representation π. The subsets $WF(\pi)$, $AS(\pi)$, and $Ass(\pi)$ are contained in the cone $\mathcal{N}^*$ consisting of nilpotent elements in $\mathfrak{g}^*$. They are all invariant under the coadjoint action of G. Each of them is a closed subvariety of $\mathfrak{g}^*$, and is the union of finitely many nilpotent orbits. It is known that $WF(\pi) = AS(\pi)$. W. Schmid and K. Vilonen [77] proved that $Ass(\pi)$ coincides with $WF(\pi)$ via the Kostant-Sekuguchi correspondence. The dimensions of all three invariants are the same, and is always even. We define the *Gelfand-Kirillov dimension* of π by

$$\dim_{G-K} \pi := \frac{1}{2} \dim WF(\pi) \tag{7.1}$$

If $\mathfrak{g}_{\mathbb{C}}$ is simple, there exist a unique nonzero nilpotent $G_{\mathbb{C}}$-orbit $\mathcal{O}_{min} \subset \mathfrak{g}_{\mathbb{C}}^*$ of minimal dimension, which is contained in the closure of any nonzero nilpotent $G_{\mathbb{C}}$-orbit. In this case, we have $\mathcal{O}_{min} = \mathcal{O}_{X_\alpha}$, where $\mathcal{O}_{X_\alpha}$ is the $G_{\mathbb{C}}$-orbit of a nonzero highest root vector X_α.

A nilpotent G-orbit $\mathcal{O} \subset \mathfrak{g}^*$ is said to be *minimal* if

$$\dim_{\mathbb{R}} \mathcal{O} = \dim_{\mathbb{C}} \mathcal{O}_{min}, \tag{7.2}$$

equivalently, $\mathcal{O}$ is nonzero and contained in $\mathcal{O}_{min} \cap \mathfrak{g}^*$. An irreducible unitary representation π of G is called *minimal* if

$$\dim_{G-K} \pi = \frac{1}{2} \dim_{\mathbb{C}} \mathcal{O}_{min}. \tag{7.3}$$

Remark 7.1. (1) If G is not of type A_n, there are at most finity many minimal representations. These are the unipotent representations attached to the minimal orbit $\mathcal{O}_{min}$.

(2) In many cases, the minimal representations are isolated in the unitary dual $\hat{G}_u$ of G.

(3) A minimal representation π is almost always *automorphic*, namely, π occurs in $L^2(\Gamma \backslash G)$ for some lattice Γ in G. The theory of minimal representation is the basis for the construction of large families of other interesting automorphic representations. For example, it is known that the end of complementary series of $Sp(n,1)$ and $F_{4,1}$ are both automorphic.

Remark 7.2. Let (π, V) be an irreducible admissible representation of π. We denote by V^K the space of K-finite vectors for π, where K is a maximal compact subgroup of G. Then V^K is a $U(\mathfrak{g}_{\mathbb{C}})$-module. We fix any vector $0 \neq x_\pi \in V^K$. Let $U_n(\mathfrak{g}_{\mathbb{C}})$ be the subspace of $U(\mathfrak{g}_{\mathbb{C}})$ spanned by products of at most n elements of $\mathfrak{g}_{\mathbb{C}}$. Put

$$X_n(\pi) := U_n(\mathfrak{g}_{\mathbb{C}}) x_\pi.$$

D. Vogan [88] proved that $\dim X_n(\pi)$ is asymptotic to $\frac{c(\pi)}{d!} \cdot n^d$ as $n \to \infty$. Here $c(\pi)$ and d are positive integers independent of the choice of x_π. In fact, d is the Gelfand-Kirillov dimension of π. We may say that $d = \dim_{G-K} \pi$ is a good measurement of the size of π.

Suppose $F = \mathbb{R}$ or $\mathbb{C}$. Let G be a connected simple Lie group over F and K a maximal compact subgroup of G. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition of $\mathfrak{g}$. If G/K is hermitian symmetric, all the minimal representations are known to be either holomorphic or antiholmorphic. They can be found in the list of unitary highest weight modules given in [27]. D. Vogan [89] proved the existence and unitarity of the minimal representations for a family of split simple group including E_8, F_4 and all classical groups except for the B_n-case ($n \geq 4$), which no minimal representation seems to exist. The construction of the minimal representations for G_2 was given by M. Duflo [24] in the complex case and by D. Vogan [93] in the real case. D. Kazhdan and G. Savin [46] constructed the spherical minimal representation for every simple, split, simply laced group. B. Gross and N. Wallach [32] constructed minimal representations of all exceptional groups of real rank 4. R. Brylinski and B. Kostant [13]-[14] gave a construction of minimal representations for any simple real Lie group G under the assumption that G/K is not hermitian symmetric and minimal representations exist.

For a complex group G not of type A_n, the Harish-Chandra module of the spherical minimal representation can be realized on $U(\mathfrak{g})/J$, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ and $J \subset U(\mathfrak{g})$ is the Joseph ideal of $\mathfrak{g}$ [43]. It is known that $Sp(2n, \mathbb{C})$ has a non-spherical minimal representation, that is, the odd piece of the Weil representation. The following natural question arises.

Question. Are there non-spherical minimal representations for complex Lie groups other than $Sp(2n, \mathbb{C})$?

Recently P. Torasso [86] gave a uniform construction of minimal representations for a simple group over any local field of characteristic 0 with split rank ≥ 3. He constructs a minimal representation for each set of admissible datum associated to the minimal orbit defined over F.

Let π_{min} be a minimal representation of G. It is known that the annihilator of the Harish-Chandra module of π_{min} in $U(\mathfrak{g})$ is the Joseph ideal. D. Vogan [89] proved that the restriction of π_{min} to K is given by

$$\pi_{min}|_K = \oplus_{n=0}^{\infty} V(\mu_0 + n\beta),$$

where β is a highest weight for the action of K on $\mathfrak{p}$, μ_0 is a fixed highest weight depending on π_{min} and $V(\mu_0 + n\beta)$ denotes the highest weight module with highest weight $\mu_0 + n\beta$. Indeed, there are two or one possibilities for β depending on whether G/K is hermitian symmetric or not.

Definition 7.3. (1) A *reductive dual pair* in a reductive Lie group G is a pair (A, B) of closed subgroups of G, which are both reductive and are centralizers of each other.

(2) A reductive dual pair (A, B) in G is said to be *compact* if at least one of A and B is compact.

Duality Conjecture. Let (A, B) be a reductive dual pair in a reductive Lie group G. Let π_{min} be a minimal representation of G. Can you find a Howe type correspondence between suitable subsets of the admissible duals of A and B by restricting π_{min} to $A \times B$?

In the 1970s R. Howe [39] first formulated the duality conjecture for the Weil representation (which is a minimal representation) of the symplectic group $Sp(n, F)$ over any local field F. He [40] proved the duality conjecture for $Sp(n, F)$ when F is archimedean and J. L. Waldspurger [94] proved the conjecture when F is non-archimedean with odd residue characteristic.

Example 7.4. Let G be the simply connected split real group E_8. Then $K = Spin(16)$ is a maximal compact subgroup of G. We take $A = B = Spin(8)$. It is easy to see that the pair (A, B) is not only a reductive dual pair in $Spin(16)$, but also a reductive dual pair in G. Let π_{min} be a minimal representation of G. J.-S. Li [64] showed that the restriction of π_{min} to $A \times B \subset G$ is decomposed as follows:

$$\pi_{min}|_{A\times B} = \oplus_\pi m(\pi) \cdot (\pi \otimes \pi),$$

where π runs over all irreducible representations of $Spin(8)$ and $m(\pi)$ is the multiplicity with which $\pi \otimes \pi$ occurs. It turns out that $m(\pi) = +\infty$ for all π.

Example 7.5. Let G be the simply connected quaternionic E_8 with split rank 4. We let $A = Spin(8)$ and $B = Spin(4, 4)$. Then the pair (A, B) is a reductive dual pair in G. Let π_{min} be the minimal representation of G. H. Y. Loke [62] proved that the restriction of π_{min} to $A \times B$ is decomposed as follows.

$$\pi_{min}|_{A\times B} = \oplus_\pi m(\pi) \cdot (\pi \otimes \pi'),$$

where π runs over all irreducible representations of A and π' is the discrete series representation of B which is uniquely determined by π. All the multiplicities are *finite*. But they are *unbounded*.

The following interesting problem is proposed by Li.

Problem 7.6. Let (A, B) be a *compact* reductive dual pair in G. Describe the explicit decomposition of the restriction $\pi_{min}|_{A\times B}$ of a minimal representation π_{min} of G to $A \times B$.

Remark 7.7. In [42], J. Huang, P. Paudzic and G. Savin dealt with the family of dual pairs (A, B), where A is the split exceptional group of type G_2 and B is compact.

8. The Heisenberg Group $H_{\mathbb{R}}^{(g,h)}$

For any positive integers g and h, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(g,h)} = \left\{ (\lambda, \mu, \kappa) \mid \lambda, \mu \in \mathbb{R}^{(h,g)},\ \kappa \in \mathbb{R}^{(h,h)},\ \kappa + \mu\, {}^t\lambda \text{ symmetric} \right\}$$

with the multiplication law

$$(\lambda, \mu, \kappa) \circ (\lambda', \mu', \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda').$$

Here $\mathbb{R}^{(h,g)}$ (resp. $\mathbb{R}^{(h,h)}$) denotes the all $h \times g$ (resp. $h \times h$) real matrices.

The Heisenberg group $H_{\mathbb{R}}^{(g,h)}$ is embedded to the symplectic group $Sp(g+h, \mathbb{R})$ via the mapping

$$H_{\mathbb{R}}^{(g,h)} \ni (\lambda, \mu, \kappa) \longmapsto \begin{pmatrix} E_g & 0 & 0 & {}^t\mu \\ \lambda & E_h & \mu & \kappa \\ 0 & 0 & E_g & -{}^t\lambda \\ 0 & 0 & 0 & E_h \end{pmatrix} \in Sp(g+h, \mathbb{R}).$$

This Heisenberg group is a 2-step nilpotent Lie group and is important in the study of toroidal compactifications of Siegel moduli spaces. In fact, $H_{\mathbb{R}}^{(g,h)}$ is obtained as the unipotent radical of the parabolic subgroup of $Sp(g+h, \mathbb{R})$ associated with the rational boundary component F_g (cf. [28] p. 123 or [69] p. 21). For the motivation of the study of this Heisenberg group we refer to [103]-[107] and [110]. We refer to [98]-[102] for more results on $H_{\mathbb{R}}^{(g,h)}$.

In this section, we describe the Schrödinger representations of $H_{\mathbb{R}}^{(g,h)}$ and the coadjoint orbits of $H_{\mathbb{R}}^{(g,h)}$. The results in this section are based on the article [108] with some corrections.

8.1. Schrödinger Representations

First of all, we observe that $H_{\mathbb{R}}^{(g,h)}$ is a 2-step nilpotent Lie group. It is easy to see that the inverse of an element $(\lambda, \mu, \kappa) \in H_{\mathbb{R}}^{(g,h)}$ is given by

$$(\lambda, \mu, \kappa)^{-1} = (-\lambda, -\mu, -\kappa + \lambda\, {}^t\mu - \mu\, {}^t\lambda).$$

Now we set

$$[\lambda, \mu, \kappa] := (0, \mu, \kappa) \circ (\lambda, 0, 0) = (\lambda, \mu, \kappa - \mu\, {}^t\lambda). \tag{8.1}$$

Then $H_{\mathbb{R}}^{(g,h)}$ may be regarded as a group equipped with the following multiplication

$$[\lambda, \mu, \kappa] \diamond [\lambda_0, \mu_0, \kappa_0] = [\lambda + \lambda_0, \mu + \mu_0, \kappa + \kappa_0 + \lambda\, {}^t\mu_0 + \mu_0\, {}^t\lambda]. \tag{8.2}$$

The inverse of $[\lambda, \mu, \kappa] \in H_{\mathbb{R}}^{(g,h)}$ is given by

$$[\lambda, \mu, \kappa]^{-1} = [-\lambda, -\mu, -\kappa + \lambda\, {}^t\mu + \mu\, {}^t\lambda].$$

We set

$$K = \left\{ [0, \mu, \kappa] \in H_{\mathbb{R}}^{(g,h)} \,\middle|\, \mu \in \mathbb{R}^{(h,g)},\ \kappa = {}^t\kappa \in \mathbb{R}^{(h,h)} \right\}. \tag{8.3}$$

Then K is a commutative normal subgroup of $H_{\mathbb{R}}^{(g,h)}$. Let $\hat{K}$ be the Pontrajagin dual of K, i.e., the commutative group consisting of all unitary characters of K. Then $\hat{K}$ is isomorphic to the additive group $\mathbb{R}^{(h,g)} \times \mathrm{Symm}\,(h,\mathbb{R})$ via

$$(8.4) \qquad < a, \hat{a} >= e^{2\pi i \sigma(\hat{\mu}\,{}^t\mu + \hat{\kappa}\kappa)}, \quad a = [0,\mu,\kappa] \in K,\ \hat{a} = (\hat{\mu},\hat{\kappa}) \in \hat{K}.$$

We put

$$(8.5) \qquad S = \left\{ [\lambda,0,0] \in H_{\mathbb{R}}^{(g,h)} \,\middle|\, \lambda \in \mathbb{R}^{(h,g)} \right\} \cong \mathbb{R}^{(h,g)}.$$

Then S acts on K as follows:

$$(8.6) \qquad \alpha_\lambda([0,\mu,\kappa]) = [0,\mu,\kappa + \lambda\,{}^t\mu + \mu\,{}^t\lambda], \quad [\lambda,0,0] \in S.$$

It is easy to see that the Heisenberg group $\left(H_{\mathbb{R}}^{(g,h)}, \diamond\right)$ is isomorphic to the semi-direct product $S \ltimes K$ of S and K whose multiplication is given by

$$(\lambda, a) \cdot (\lambda_0, a_0) = (\lambda + \lambda_0, a + \alpha_\lambda(a_0)), \quad \lambda, \lambda_0 \in S,\ a, a_0 \in K.$$

On the other hand, S acts on $\hat{K}$ by

$$(8.7) \qquad \alpha_\lambda^*(\hat{a}) = (\hat{\mu} + 2\hat{\kappa}\lambda, \hat{\kappa}), \quad [\lambda,0,0] \in S, \quad a = (\hat{\mu},\hat{\kappa}) \in \hat{K}.$$

Then, we have the relation $< \alpha_\lambda(a), \hat{a} >=< a, \alpha_\lambda^*(\hat{a}) >$ for all $a \in K$ and $\hat{a} \in \hat{K}$.

We have three types of S-orbits in $\hat{K}$.

Type I. Let $\hat{\kappa} \in \mathrm{Sym}\,(h,\mathbb{R})$ be nondegenerate. The S-orbit of $\hat{a}(\hat{\kappa}) := (0,\hat{\kappa}) \in \hat{K}$ is given by

$$(8.8) \qquad \hat{\mathcal{O}}_{\hat{\kappa}} = \left\{ (2\hat{\kappa}\lambda, \hat{\kappa}) \in \hat{K} \,\middle|\, \lambda \in \mathbb{R}^{(h,g)} \right\} \cong \mathbb{R}^{(h,g)}.$$

Type II. Let $(\hat{\mu},\hat{\kappa}) \in \mathbb{R}^{(h,g)} \times \mathrm{Sym}\,(h,\mathbb{R})$ with degenerate $\hat{\kappa} \neq 0$. Then

$$(8.9) \qquad \hat{\mathcal{O}}_{(\hat{\mu},\hat{\kappa})} = \left\{ \hat{\mu} + 2\hat{\kappa}\lambda, \hat{\kappa}) \,\middle|\, \lambda \in \mathbb{R}^{(h,g)} \right\} \subsetneqq \mathbb{R}^{(h,g)} \times \{\hat{\kappa}\}.$$

Type III. Let $\hat{y} \in \mathbb{R}^{(h,g)}$. The S-orbit $\hat{\mathcal{O}}_{\hat{y}}$ of $\hat{a}(\hat{y}) := (\hat{y}, 0)$ is given by

$$(8.10) \qquad \hat{\mathcal{O}}_{\hat{y}} = \{\, (\hat{y}, 0)\,\} = \hat{a}(\hat{y}).$$

We have

$$\hat{K} = \left(\bigcup_{\substack{\hat{\kappa} \in \mathrm{Sym}(h,\mathbb{R}) \\ \hat{\kappa}\ \text{nondegenerate}}} \hat{\mathcal{O}}_{\hat{\kappa}} \right) \bigcup \left(\bigcup_{\hat{y} \in \mathbb{R}^{(h,g)}} \hat{\mathcal{O}}_{\hat{y}} \right) \bigcup \left(\bigcup_{\substack{(\hat{\mu},\hat{\kappa}) \in \mathbb{R}^{(h,g)} \times \mathrm{Sym}(h,\mathbb{R}) \\ \hat{\kappa} \neq 0\ \text{degenerate}}} \hat{\mathcal{O}}_{(}\hat{\mu},\hat{\kappa}) \right)$$

as a set. The stabilizer $S_{\hat{\kappa}}$ of S at $\hat{a}(\hat{\kappa}) = (0, \hat{\kappa})$ is given by

$$S_{\hat{\kappa}} = \{0\}. \tag{8.11}$$

And the stabilizer $S_{\hat{y}}$ of S at $\hat{a}(\hat{y}) = (\hat{y}, 0)$ is given by

$$S_{\hat{y}} = \left\{ [\lambda, 0, 0] \,\middle|\, \lambda \in \mathbb{R}^{(h,g)} \right\} = S \cong \mathbb{R}^{(h,g)}. \tag{8.12}$$

From now on, we set $G = H_{\mathbb{R}}^{(g,h)}$ for brevity. It is known that K is a closed, commutative normal subgroup of G. Since $(\lambda, \mu, \kappa) = (0, \mu, \kappa + \mu\, {}^t\lambda) \circ (\lambda, 0, 0)$ for $(\lambda, \mu, \kappa) \in G$, the homogeneous space $X = K\backslash G$ can be identified with $\mathbb{R}^{(h,g)}$ via

$$Kg = K \circ (\lambda, 0, 0) \longmapsto \lambda, \quad g = (\lambda, \mu, \kappa) \in G.$$

We observe that G acts on X by

$$(Kg) \cdot g_0 = K\,(\lambda + \lambda_0, 0, 0) = \lambda + \lambda_0, \tag{8.13}$$

where $g = (\lambda, \mu, \kappa) \in G$ and $g_0 = (\lambda_0, \mu_0, \kappa_0) \in G$.

If $g = (\lambda, \mu, \kappa) \in G$, we have

$$k_g = (0, \mu, \kappa + \mu\, {}^t\lambda), \quad s_g = (\lambda, 0, 0) \tag{8.14}$$

in the Mackey decomposition of $g = k_g \circ s_g$ (cf.[67]). Thus if $g_0 = (\lambda_0, \mu_0, \kappa_0) \in G$, then we have

$$s_g \circ g_0 = (\lambda, 0, 0) \circ (\lambda_0, \mu_0, \kappa_0) = (\lambda + \lambda_0, \mu_0, \kappa_0 + \lambda\, {}^t\mu_0) \tag{8.15}$$

and so

$$k_{s_g \circ g_0} = (0, \mu_0, \kappa_0 + \mu_0\, {}^t\lambda_0 + \lambda\, {}^t\mu_0 + \mu_0\, {}^t\lambda). \tag{8.16}$$

For a real symmetric matrix $c = {}^tc \in \mathbb{R}^{(h,h)}$ with $c \neq 0$, we consider the one-dimensional unitary representation σ_c of K defined by

$$\sigma_c\left((0, \mu, \kappa)\right) = e^{2\pi i \sigma(c\kappa)}\, I, \quad (0, \mu, \kappa) \in K, \tag{8.17}$$

where I denotes the identity mapping. Then the induced representation $U(\sigma_c) := \mathrm{Ind}_K^G\, \sigma_c$ of G induced from σ_c is realized in the Hilbert space $\mathcal{H}_{\sigma_c} = L^2(X, d\dot{g}, \mathbb{C}) \cong L^2\left(\mathbb{R}^{(h,g)}, d\xi\right)$ as follows. If $g_0 = (\lambda_0, \mu_0, \kappa_0) \in G$ and $x = Kg \in X$ with $g = (\lambda, \mu, \kappa) \in G$, we have

$$\left(U_{g_0}(\sigma_c) f\right)(x) = \sigma_c\left(k_{s_g \circ g_0}\right)\left(f(xg_0)\right), \quad f \in \mathcal{H}_{\sigma_c}. \tag{8.18}$$

It follows from (8.16) that

$$\left(U_{g_0}(\sigma_c) f\right)(\lambda) = e^{2\pi i \sigma\{c(\kappa_0 + \mu_0\, {}^t\lambda_0 + 2\lambda\, {}^t\mu_0)\}}\, f(\lambda + \lambda_0). \tag{8.19}$$

Here, we identified $x = Kg$ (resp. $xg_0 = Kgg_0$) with λ (resp. $\lambda + \lambda_0$). The induced representation $U(\sigma_c)$ is called the *Schrödinger representation* of G associated with σ_c. Thus $U(\sigma_c)$ is a monomial representation.

Now, we denote by $mathcal H^{\sigma_c}$ the Hilbert space consisting of all functions $\phi : G \longrightarrow \mathbb{C}$ which satisfy the following conditions:

(1) $\phi(g)$ is measurable measurable with respect to dg,
(2) $\phi\left((0, \mu, \kappa) \circ g)\right) = e^{2\pi i\sigma(c\kappa)}\phi(g)$ for all $g \in G$,
(3) $\| \phi \|^2 := \int_X |\phi(g)|^2 \, d\dot{g} < \infty, \quad \dot{g} = Kg$,

where dg (resp. $d\dot{g}$) is a G-invariant measure on G (resp. $X = K\backslash G$). The inner product (,) on $\mathcal{H}^{\sigma_c}$ is given by

$$(\phi_1, \phi_2) = \int_G \phi_1(g)\,\overline{\phi_2(g)}\,dg \quad \text{for } \phi_1,\ \phi_2 \in \mathcal{H}^{\sigma_c}.$$

We observe that the mapping $\Phi_c : \mathcal{H}_{\sigma_c} \longrightarrow \mathcal{H}^{\sigma_c}$ defined by

$$(8.20) \qquad (\Phi_c(f))\,(g) = e^{2\pi i\sigma\{c(\kappa + \mu\,{}^t\lambda)\}}\, f(\lambda), \quad f \in \mathcal{H}_{\sigma_c},\ g = (\lambda, \mu, \kappa) \in G$$

is an isomorphism of Hilbert spaces. The inverse $\Psi_c : \mathcal{H}^{\sigma_c} \longrightarrow \mathcal{H}_{\sigma_c}$ of Φ_c is given by

$$(8.21) \qquad (\Psi_c(\phi))\,(\lambda) = \phi((\lambda, 0, 0)), \quad \phi \in \mathcal{H}^{\sigma_c},\ \lambda \in \mathbb{R}^{(h,g)}.$$

The Schrödinger representation $U(\sigma_c)$ of G on $\mathcal{H}^{\sigma_c}$ is given by

$$(8.22) \qquad (U_{g_0}(\sigma_c)\phi)\,(g) = e^{2\pi i\sigma\{c(\kappa_0 + \mu_0\,{}^t\lambda_0 + \lambda\,{}^t\mu_0 - \lambda_0\,{}^t\mu)\}}\, \phi\left((\lambda_0, 0, 0) \circ g\right),$$

where $g_0 = (\lambda_0, \mu_0, \kappa_0),\ g = (\lambda, \mu, \kappa) \in G$ and $\phi \in \mathcal{H}^{\sigma_c}$. (8.22) can be expressed as follows.

$$(8.23) \qquad (\,U_{g_0}(\sigma_c)\phi\,)\,(g) = e^{2\pi i\sigma\{c(\kappa_0 + \kappa + \mu_0\,{}^t\lambda_0 + \mu\,{}^t\lambda + 2\lambda\,{}^t\mu_0)\}}\, \phi((\lambda_0 + \lambda, 0, 0)).$$

Theorem 8.1. *Let c be a positive symmetric half-integral matrix of degree h. Then the Schrödinger representation $U(\sigma_c)$ of G is irreducible.*

Proof. The proof can be found in [99], Theorem 3.

8.2. The Coadjoint Orbits of Picture

In this subsection, we find the coadjoint orbits of the Heisenberg group $H_{\mathbb{R}}^{(g,h)}$ and describe the connection between the coadjoint orbits and the unitary dual of $H_{\mathbb{R}}^{(g,h)}$ explicitly.

For brevity, we let $G := H_{\mathbb{R}}^{(g,h)}$ as before. Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$. We observe that $\mathfrak{g}$ can be regarded as the subalgebra consisting of all $(g+h)\times(g+h)$ real matrices of the form

$$X(\alpha,\beta,\gamma)=\begin{pmatrix} 0 & 0 & 0 & {}^t\beta \\ \alpha & 0 & \beta & \gamma \\ 0 & 0 & 0 & -{}^t\alpha \\ 0 & 0 & 0 & 0 \end{pmatrix},\ \alpha,\beta\in\mathbb{R}^{(h,g)},\ \gamma={}^t\gamma\in\mathbb{R}^{(h,h)}$$

of the lie algebra $\mathfrak{sp}(g+h,\mathbb{R})$ of the symplectic group $Sp(g+h,\mathbb{R})$. An easy computation yields

$$[X(\alpha,\beta,\gamma),\ X(\delta,\epsilon,\xi)]=X(0,0,\alpha^t\epsilon+\epsilon^t\alpha-\beta^t\delta-\delta^t\beta).$$

The dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ can be identified with the vector space consisting of all $(g+h)\times(g+h)$ real matrices of the form

$$F(a,b,c)=\begin{pmatrix} 0 & {}^ta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & {}^tb & 0 & 0 \\ b & c & -a & 0 \end{pmatrix},\ a,b\in\mathbb{R}^{(h,g)},\ c={}^tc\in\mathbb{R}^{(h,h)}$$

so that

$$(8.24)\quad < F(a,b,c),X(\alpha,\beta,\gamma)>=\sigma(F(a,b,c)\,X(\alpha,\beta,\gamma))=2\sigma({}^t\alpha a+{}^tb\beta)+\sigma(c\gamma).$$

The adjoint representation Ad of G is given by $Ad_G(g)X=gXg^{-1}$ for $g\in G$ and $X\in\mathfrak{g}$. For $g\in G$ and $F\in\mathfrak{g}^*$, gFg^{-1} is not of the form $F(a,b,c)$. We denote by $(gFg^{-1})_*$ the

$$\begin{pmatrix} 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} - \text{part}$$

of the matrix gFg^{-1}. Then it is easy to see that the coadjoint representation $Ad_G^*: G\longrightarrow GL(\mathfrak{g}^*)$ is given by $Ad_G^*(g)F=(gFg^{-1})_*$, where $g\in G$ and $F\in\mathfrak{g}^*$. More precisely,

$$(8.25)\qquad Ad_G^*(g)F(a,b,c)=F(a+c\mu,b-c\lambda,c),$$

where $g=(\lambda,\mu,\kappa)\in G$. Thus the coadjoint orbit $\Omega_{a,b}$ of G at $F(a,b,0)\in\mathfrak{g}^*$ is given by

$$(8.26)\qquad \Omega_{a,b}=Ad_G^*(G)\,F(a,b,0)=\{F(a,b,0)\},\ \text{a single point}$$

and the coadjoint orbit Ω_c of G at $F(0,0,c)\in\mathfrak{g}^*$ with $c\neq 0$ is given by

$$(8.27)\qquad \Omega_c=Ad_G^*(G)\,F(0,0,c)=\{F(a,b,c)|a,b\in\mathbb{R}^{(h,g)}\}\cong\mathbb{R}^{(h,g)}\times\mathbb{R}^{(h,g)}.$$

Therefore the coadjoint orbits of G in $\mathfrak{g}^*$ fall into two classes :

(**I**) The single point $\{\Omega_{a,b} | a, b \in \mathbb{R}^{(h,g)}\}$ located in the plane $c = 0$.

(**II**) The affine planes $\{\Omega_c | c = {}^t c \in \mathbb{R}^{(h,h)},\ c \neq o\}$ parallel to the homogeneous plane $c = 0$.

In other words, the orbit space $\mathcal{O}(G)$ of coadjoint orbits is parametrized by

$$\begin{cases} c - \text{axis},\ c \neq 0,\ c = {}^t c \in \mathbb{R}^{(h,h)}; \\ (a,b) - \text{plane} \approx \mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)}. \end{cases}$$

The single point coadjoint orbits of the type $\Omega_{a,b}$ are said to be the *degenerate* orbits of G in $\mathfrak{g}^*$. On the other hand, the flat coadjoint orbits of the type Ω_c are said to be the *non-degenerate* orbits of G in $\mathfrak{g}^*$. Since G is connected and simply connected 2-step nilpotent Lie group, according to A. Kirillov (cf. [47] or [48] p.249, Theorem 1), the unitary dual $\widehat{G}$ of G is given by

$$\widehat{G} = \left(\mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)}\right) \coprod \left\{ z \in \mathbb{R}^{(h,h)} \mid z = {}^t z,\ z \neq 0 \right\}, \tag{8.28}$$

where $\coprod$ denotes the disjoint union. The topology of $\widehat{G}$ may be described as follows. The topology on $\{c - \text{axis} - (0)\}$ is the usual topology of the Euclidean space and the topology on $\{F(a,b,0) | a, b \in \mathbb{R}^{(h,g)}\}$ is the usual Euclidean topology. But a sequence on the c-axis which converges to 0 in the usual topology converges to the whole Euclidean space $\mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)}$ in the topology of $\hat{G}$. This is just the quotient topology on $\mathfrak{g}^*/G$ so that algebraically and topologically $\widehat{G} = \mathfrak{g}^*/G$.

It is well known that each coadjoint orbit is a symplectic manifold. We will state this fact in detail. For the present time being, we fix an element F of $\mathfrak{g}^*$ once and for all. We consider the alternating $\mathbb{R}$-bilinear form $\mathbf{B}_F$ on $\mathfrak{g}$ defined by

$$\mathbf{B}_F(X,Y) = < F, [X,Y] > = < ad^*_{\mathfrak{g}}(Y)F, X >,\ X, Y \in \mathfrak{g}, \tag{8.29}$$

where $ad^*_{\mathfrak{g}} : \mathfrak{g} \longrightarrow \text{End}(\mathfrak{g}^*)$ denotes the differential of the coadjoint representation $Ad^*_G : G \longrightarrow GL(\mathfrak{g}^*)$. More precisely, if $F = F(a,b,c),\ X = X(\alpha,\beta,\gamma)$, and $Y = X(\delta,\epsilon,\xi)$, then

$$\mathbf{B}_F(X,Y) = \sigma\{c(\alpha^t\epsilon + \epsilon^t\alpha - \beta^t\delta - \delta^t\beta)\}. \tag{8.30}$$

For $F \in \mathfrak{g}^*$, we let

$$G_F = \{g \subset G | Ad^*_G(g)F = F\}$$

be the stabilizer of the coadjoint action Ad^* of G on $\mathfrak{g}^*$ at F. Since G_F is a closed subgroup of G, G_F is a Lie subgroup of G. We denote by $\mathfrak{g}_F$ the Lie subalgebra of $\mathfrak{g}$ corresponding to G_F. Then it is easy to show that

$$\mathfrak{g}_F = rad\,\mathbf{B}_F = \{X \in \mathfrak{g} | ad^*_{\mathfrak{g}}(X)F = 0\}. \tag{8.31}$$

Here $rad\,\mathbf{B}_F$ denotes the radical of $\mathbf{B}_F$ in $\mathfrak{g}$. We let $\dot{\mathbf{B}_F}$ be the non-degenerate alternating $\mathbb{R}$-bilinear form on the quotient vector space $\mathfrak{g}/rad\,\mathbf{B}_F$ induced from $\mathbf{B}_F$.

Since we may identify the tangent space of the coadjoint orbit $\Omega_F \cong G/G_F$ with $\mathfrak{g}/\mathfrak{g}_F = \mathfrak{g}/rad\mathbf{B}_F$, we see that the tangent space of Ω_F at F is a symplectic vector space with respect to the symplectic form $\mathbf{B}_F$.

Now we are ready to prove that the coadjoint orbit $\Omega_F = Ad_G^*(G)F$ is a symplectic manifold. We denote by $\widetilde{X}$ the smooth vector field on $\mathfrak{g}^*$ associated to $X \in \mathfrak{g}$. That means that for each $\ell \in \mathfrak{g}^*$, we have

$$\widetilde{X}(\ell) = ad_{\mathfrak{g}}^*(X)\,\ell. \tag{8.32}$$

We define the differential 2-form B_{Ω_F} on Ω_F by

$$B_{\Omega_F}(\widetilde{X},\widetilde{Y}) = B_{\Omega_F}(ad_{\mathfrak{g}}^*(X)F, ad_{\mathfrak{g}}^*(Y)F) := \mathbf{B}_F(X,Y), \tag{8.33}$$

where $X, Y \in \mathfrak{g}$.

Lemma 8.2. *B_{Ω_F} is non-degenerate.*

Proof. Let $\widetilde{X}$ be the smooth vector field on $\mathfrak{g}^*$ associated to $X \in \mathfrak{g}$ such that $B_{\Omega_F}(\widetilde{X},\widetilde{Y}) = 0$ for all $\widetilde{Y}$ with $Y \in \mathfrak{g}$. Since $B_{\Omega_F}(\widetilde{X},\widetilde{Y}) = \mathbf{B}_F(X,Y) = 0$ for all $Y \in \mathfrak{g}$, $X \in \mathfrak{g}_F$. Thus $\widetilde{X} = 0$. Hence B_{Ω_F} is non-degenerate.

Lemma 8.3. *B_{Ω_F} is closed.*

Proof. If $\widetilde{X_1}, \widetilde{X_2}, \text{and} \widetilde{X_3}$ are three smooth vector fields on $\mathfrak{g}^*$ associated to $X_1, X_2, X_3 \in \mathfrak{g}$, then

$$\begin{aligned} dB_{\Omega_F}(\widetilde{X_1},\widetilde{X_2},\widetilde{X_3}) &= \widetilde{X_1}(B_{\Omega_F}(\widetilde{X_2},\widetilde{X_3})) - \widetilde{X_2}(B_{\Omega_F}(\widetilde{X_1},\widetilde{X_3})) + \widetilde{X_3}(B_{\Omega_F}(\widetilde{X_1},\widetilde{X_2})) \\ &\quad - B_{\Omega_F}([\widetilde{X_1},\widetilde{X_2}],\widetilde{X_3}) + B_{\Omega_F}([\widetilde{X_1},\widetilde{X_3}],\widetilde{X_2}) - B_{\Omega_F}([\widetilde{X_2},\widetilde{X_3}],\widetilde{X_1}) \\ &= - < F, [[X_1,X_2],X_3] + [[X_2,X_3],X_1] + [[X_3,X_1],X_2] > \\ &= 0 \qquad \text{(by the Jacobi identity)}. \end{aligned}$$

Therefore B_{Ω_F} is closed. □

In summary, (Ω_F, B_{Ω_F}) is a symplectic manifold of dimension $2hg$ or 0.

In order to describe the irreducible unitary representations of G corresponding to the coadjoint orbits under the Kirillov correspondence, we have to determine the polarizations of $\mathfrak{g}$ for the linear forms $F \in \mathfrak{g}^*$.

Case **I.** $F = F(a,b,0)$; the degenerate case.

According to (8.26), $\Omega_F = \Omega_{a,b} = \{F(a,b,0)\}$ is a single point. It follows from (8.30) that $\mathbf{B}_F(X,Y) = 0$ for all $X, Y \in \mathfrak{g}$. Thus $\mathfrak{g}$ is the unique polarization of $\mathfrak{g}$ for F. The Kirillov correspondence says that the irreducible unitary representation $\pi_{a,b}$ of G corresponding to the coadjoint orbit $\Omega_{a,b}$ is given by

$$\pi_{a,b}(\exp X(\alpha,\beta,\gamma)) = e^{2\pi i < F, X(\alpha,\beta,\gamma) >} = e^{4\pi i \sigma({}^t a\alpha + {}^t b\beta)}. \tag{8.34}$$

That is, $\pi_{a,b}$ is a one-dimensional degenerate representation of G.

Case **II.** $F = F(0,0,c),\ 0 \neq c = {}^t c \in \mathbb{R}^{(h,h)}$: the non-degenerate case.

According to (8.27), $\Omega_F = \Omega_c = \{F(a,b,c)|a,b \subset \mathbb{R}^{(h,g)}\}$. By (8.30), we see that

$$\mathfrak{k} = \{\ X(0,\beta,\gamma)|\beta \in \mathbb{R}^{(h,g)},\ \gamma = {}^t\gamma \in \mathbb{R}^{(h,h)}\} \tag{8.35}$$

is a polarization of $\mathfrak{g}$ for F, i.e.,$\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ subordinate to $F \in \mathfrak{g}^*$ which is maximal among the totally isotropic vector subspaces of $\mathfrak{g}$ relative to the alternating $\mathbb{R}$-bilinear form $\mathbf{B}_F$. Let K be the simply connected Lie subgroup of G corresponding to the Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$. We let

$$\chi_{c,\mathfrak{k}} : K \longrightarrow \mathbb{C}_1^\times$$

be the unitary character of K defined by

$$\chi_{c,\mathfrak{k}}(\exp X(0,\beta,\gamma)) = e^{2\pi i <F, X(0,\beta,\gamma)>} = e^{2\pi i\sigma(c\gamma)}. \tag{8.36}$$

The Kirillov correspondence says that the irreducible unitary representation $\pi_{c,\mathfrak{k}}$ of G corresponding to the coadjoint orbit $\Omega_F = \Omega_c$ is given by

$$\pi_{c,\mathfrak{k}} = \operatorname{Ind}_K^G \chi_{c,\mathfrak{k}}. \tag{8.37}$$

According to Kirillov's Theorem (cf. [47]), we know that the induced representation $\pi_{c,\mathfrak{k}}$ is, up to equivalence, independent of the choice of a polarization of $\mathfrak{g}$ for F. Thus we denote the equivalence class of $\pi_{c,\mathfrak{k}}$ by π_c. π_c is realized on the representation space $L^2(\mathbb{R}^{(h,g)}, d\xi)$ as follows:

$$(\pi_c(g)f)(\xi) = e^{2\pi i\sigma\{c(\kappa+\mu\,{}^t\lambda+2\xi\,{}^t\mu)\}} f(\xi+\lambda), \tag{8.38}$$

where $g = (\lambda,\mu,\kappa) \in G$ and $\xi \in \mathbb{R}^{(h,g)}$. Using the fact that

$$\exp X(\alpha,\beta,\gamma) = (\alpha,\beta,\gamma + \frac{1}{2}(\alpha\,{}^t\beta - \beta\,{}^t\alpha)),$$

we see that π_c is nothing but the Schrödinger representation $U(\sigma_c)$ of G induced from the one-dimensional unitary representation σ_c of K given by $\sigma_c((0,\mu,\kappa)) = e^{2\pi i\sigma(c\kappa)}I$. We note that π_c is the non-degenerate representation of G with central character $\chi_c : Z \longrightarrow \mathbb{C}_1^\times$ given by $\chi_c((0,0,\kappa)) = e^{2\pi i\sigma(c\kappa)}$. Here $Z = \{(0,0,\kappa)|\kappa = {}^t\kappa \in \mathbb{R}^{(h,h)}\}$ denotes the center of G.

It is well known that the monomial representation $(\pi_c, L^2(\mathbb{R}^{(h,g)}, d\xi))$ of G extends to an operator of operator of trace class

$$\pi_c(\phi) : L^2(\mathbb{R}^{(h,g)}, d\xi) \longrightarrow L^2(\mathbb{R}^{(h,g)}, d\xi) \tag{8.39}$$

for all $\phi \in C_c^\infty(G)$. Here $C_c^\infty(G)$ is the vector space of all smooth functions on G with compact support. We let $C_c^\infty(\mathfrak{g})$ and $C(\mathfrak{g}^*)$ the vector space of all smooth functions on $\mathfrak{g}$ with compact support and the vector space of all continuous functions on $\mathfrak{g}^*$ respectively. If $f \in \mathcal{C}_c^\infty(\mathfrak{g})$, we define the Fourier cotransform

$$\mathcal{C}F_{\mathfrak{g}} : C_c^\infty(\mathfrak{g}) \longrightarrow C(\mathfrak{g}^*)$$

by

$$(8.40) \qquad (\mathcal{C}F_{\mathfrak{g}}(f))\,(F') := \int_{\mathfrak{g}} f(X)\, e^{2\pi i <F',X>} dX,$$

where $F' \in \mathfrak{g}^*$ and dX denotes the usual Lebesgue measure on $\mathfrak{g}$. According to A. Kirillov (cf. [47]), there exists a measure β on the coadjoint orbit $\Omega_c \approx \mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)}$ which is invariant under the coadjoint action og G such that

$$(8.41) \qquad \mathrm{tr}\, \pi_c^1(\phi) = \int_{\Omega_c} \mathcal{C}F_{\mathfrak{g}}(\phi \circ \exp)(F')d\beta(F')$$

holds for all test functions $\phi \in C_c^\infty(G)$, where exp denotes the exponentional mapping of $\mathfrak{g}$ onto G. We recall that

$$\pi_c^1(\phi)(f) = \int_G \phi(x)\,(\pi_c(x)f)\, dx,$$

where $\phi \in C_c^\infty(G)$ and $f \in L^2(\mathbb{R}^{(h,g)}, d\xi)$. By the Plancherel theorem, the mapping

$$S(G/Z) \ni \varphi \longmapsto \pi_c^1(\varphi) \in TC(L^2(\mathbb{R}^{(h,g)}, d\xi))$$

extends to a unitary isometry

$$(8.42) \qquad \pi_c^2 : L^2(G/Z, \chi_c) \longrightarrow HS(L^2(\mathbb{R}^{(h,g)}, d\xi))$$

of the representation space $L^2(G/Z), \chi_c)$ of $\mathrm{Ind}_Z^G\, \chi_c$ onto the complex Hilbert space $HS(L^2(\mathbb{R}^{(h,g)}, d\xi))$ consisting of all Hilbert-Schmidt operators on $L^2(\mathbb{R}^{(h,g)}, d\xi)$, where $S(G/z)$ is the Schwartz space of all infinitely differentiable complex-valued functions on $G/Z \cong \mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)}$ that are rapidly decreasing at infinity and $TC(L^2(\mathbb{R}^{(h,g)}, d\xi))$ denotes the complex vector space of all continuous $\mathbb{C}$-linear mappings of $L^2(\mathbb{R}^{(h,g)}, d\xi)$ into itself which are of trace class.

In summary, we have the following result.

Theorem 8.4. *For $F = F(a, b, 0) \in \mathfrak{g}^*$, the irreducible unitary representation $\pi_{a,b}$ of G corresponding to the coadjoint orbit $\Omega_F = \Omega_c$ under the Kirillov correspondence is degenerate representation of G given by*

$$\pi_{a,b}(exp\, X(\alpha, \beta, \gamma)) = e^{4\pi i \sigma({}^t a\alpha - {}^t b\beta)}.$$

On the other hand, for $F = F(0, 0, c) \in \mathfrak{g}^*$ with $0 \neq c = {}^t c \in \mathbb{R}^{(h,h)}$, the irreducible unitary representation $(\pi_c, L^2(\mathbb{R}^{(h,g)}, d\xi))$ of G corresponding to the coadjoint orbit

Ω_c under the Kirillov correspondence is unitary equivalent to the Schrödinger representation $U(\sigma_c), L^2(\mathbb{R}^{(h,g)}, d\xi))$ and this non-degenerate representation π_c is square integrable module its center Z. For all test functions $\phi \in C_c^\infty(G)$, the character formula

$$tr\,\pi_c^2(\phi) = \mathcal{C}(\phi, c) \int_{\mathbb{R}^{(h,g)}} \phi(0,0,\kappa)\, e^{2\pi i \sigma(c\kappa)} d\kappa$$

holds for some constant $\mathcal{C}(\phi, c)$ depending on ϕ and c, where $d\kappa$ is the Lebesgue measure on the Euclidean space $\mathbb{R}^{(h,h)}$.

Now we consider the subgroup K of G given by

$$K = \{(0,0,\kappa) \in G \,|\, \mu \in \mathbb{R}^{(h,g)},\ \kappa = {}^t\kappa \in \mathbb{R}^{(h,h)}\}.$$

The Lie algebra $\mathfrak{k}$ of K is given by (8.35). The dual space $\mathfrak{k}^*$ of $\mathfrak{k}$ may be identified with the space

$$\{F(0,b,c) \,|\, b \in \mathbb{R}^{(h,g)},\ c = {}^tc \in \mathbb{R}^{(h,h)}\}.$$

We let $\mathrm{Ad}_K^* : K \longrightarrow GL(\mathfrak{k}^*)$ be the coadjoint representation of K on $\mathfrak{k}^*$. The coadjoint orbit $\omega_{b,c}$ of K at $F(0,b,c) \in \mathfrak{k}^*$ is given by

$$\omega_{b,c} = Ad_K^*(K)\, F(0,b,c) = \{F(0,b,c)\}, \text{ a single point.} \tag{8.43}$$

Since K is a commutative group, $[\mathfrak{k}, \mathfrak{k}] = 0$ and so the alternating $\mathbb{R}$-bilinear form $\mathbf{B}_f$ on $\mathfrak{k}$ associated to $F := F(0,b,c)$ identically vanishes on $\mathfrak{k} \times \mathfrak{k}$(cf. (8.29)). $\mathfrak{k}$ is the unique polarization of $\mathfrak{k}$ for $F = F(0,b,c)$. The Kirillov correspondence says that the irreducible unitary representation $\chi_{b,c}$ of K corresponding to the coadjoint orbit $\omega_{b,c}$ is given by

$$\chi_{b,c}(\exp X(0,\beta,\gamma)) = e^{2\pi i < F(0,b,c), X(0,\beta,\gamma) >} = e^{2\pi i \sigma(2\, {}^tb\beta + c\gamma)} \tag{8.44}$$

or

$$\chi_{b,c}((0,\mu,\kappa)) = e^{2\pi i \sigma(2\, {}^tb\mu + c\kappa)},\ \ (0,\mu,\kappa) \in K. \tag{8.45}$$

For $0 \neq c = {}^tc \in \mathbb{R}^{(h,h)}$, we let π_c be the Schrödinger representation of G given by (8.38). We know that the irreducible unitary representation of G corresponding to the coadjoint orbit

$$\Omega_c = \mathrm{Ad}_G^*(G)\, F(0,0,c) = \{F(a,b,c) \,|\, a, b \in \mathbb{R}^{(h,g)}\}.$$

Let $p : \mathfrak{g}^* \longrightarrow \mathfrak{k}^*$ be the natural projection defined by $p(F(a,b,c)) = F(o,b,c)$. Obviously we have

$$p(\Omega_c) = \left\{F(0,b,c) \,|\, b \in \mathbb{R}^{(h,g)}\right\} = \cup_{b \in \mathbb{R}^{(h,g)}}\, \omega_{b,c}.$$

According to Kirillov Theorem (cf. [48] p.249, Theorem1), The restriction $\pi_c|_K$ of π_c to K is the direct integral of all one-dimensional representations $\chi_{b,c}$ of K ($b \in$

$\mathbb{R}^{(h,g)}$). Conversely, we let $\chi_{b,c}$ be the element of $\hat{K}$ corresponding to the coadjoint orbit $\omega_{b,c}$ of K. The induced representation $\mathrm{Ind}_K^G \chi_{b,c}$ is nothing but the Schrödinger representation π_c. The coadjoint orbit Ω_c of G is the only coadjoint orbit such that $\Omega_c \cap p^{-1}(\omega_{b,c})$ is nonempty.

9. The Jacobi Group

In this section, we study the unitary representations of the Jacobi group which is a semi-product of a a symplectic group and a Heisenberg group, and their related topics. In the subsection 9.1, we present basic ingredients of the Jacobi group and the Iwasawa decomposition of the Jacobi group. In the subsection 9.2, we find the Lie algebra of the Jacobi group in some detail. In the subsection 9.3, we give a definition of Jacobi forms. In the subsection 9.4, we characterize Jacobi forms as functions on the Jacobi group satisfying certain conditions. In the subsection 9.5, we review some results on the unitary representations, in particular, the Weil representation of the Jacobi group. Most of the materials here are contained in [82]-[84]. In the subsection 9.6, we describe the duality theorem for the Jacobi group. In the final subsection, we study the coadjoint orbits for the Jacobi group and relate these orbits to the unitary representations of the Jacobi group.

9.1 The Jacobi Group G^J

In this section, we give the standard coordinates of the Jacobi group G^J and an Iwasawa decomposition of G^J.

9.1.1. The Standard Coordinates of the Jacobi Group G^J

Let m and n be two fixed positive integers. Let

$$Sp(n,\mathbb{R}) = \{ M \in \mathbb{R}^{(2n,2n)} \ \mid \ {}^tMJ_nM = J_n \}$$

be the symplectic group of degree n, where

$$J_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$$

is the symplectic matrix of degree n. We let

$$H_n = \{ \, Z \in \mathbb{C}^{(n,n)} \mid Z = {}^tZ, \ \ \mathrm{Im}\, Z \, > \, 0 \, \}$$

be the Siegel upper half plane of degree n. Then it is easy to see that $Sp(n,\mathbb{R})$ acts on H_n transitively by

$$M < Z >= (AZ + B)(CZ + D)^{-1}, \tag{9.1}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ and $Z \in H_n$.

We consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \left\{ (\lambda,\mu,\kappa) \middle| \lambda,\mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\,{}^t\lambda \text{ symmetric } \right\}$$

endowed with the following multiplication law

$$(9.2) \qquad (\lambda,\mu,\kappa)\circ(\lambda',\mu',\kappa') = (\lambda+\lambda',\mu+\mu',\kappa+\kappa'+\lambda\,{}^t\mu'-\mu\,{}^t\lambda').$$

We already studied this Heisenberg group in the previous section.

Now we let

$$G^J_{n,m} := Sp(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

the semidirect product of the symplectic group $Sp(n,\mathbb{R})$ and the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ endowed with the following multiplication law

$$(9.3)\ (M,(\lambda,\mu,\kappa))\cdot(M',(\lambda',\mu',\kappa')) = \left(MM',(\tilde{\lambda}+\lambda',\tilde{\mu}+\mu',\kappa+\kappa'+\tilde{\lambda}\,{}^t\mu'-\tilde{\mu}\,{}^t\lambda')\right)$$

with $M,M' \in Sp(n,\mathbb{R}), (\lambda,\mu,\kappa),\ (\lambda',\mu',\kappa') \in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde{\lambda},\tilde{\mu}) := (\lambda,\mu)M'$. We call $G^J_{n,m}$ the *Jacobi group* of *degree* (n,m). If there is no confusion about the degree (n,m), we write G^J briefly instead of $G^J_{n,m}$. It is easy to see that G^J acts on $H_{n,m} := H_n \times \mathbb{C}^{(m,n)}$ transitively by

$$(9.4) \qquad (M,(\lambda,\mu,\kappa))\cdot(Z,W) = (M < Z >, (W+\lambda Z+\mu)(CZ+D)^{-1}),$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R}),\ (\lambda,\mu,\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(Z,W) \in H_{n,m}$.

Now we define the linear mapping

$$Q\,:\,(\mathbb{R}^{(m,n)}\times\mathbb{R}^{(m,n)})\times(\mathbb{R}^{(m,n)}\times\mathbb{R}^{(m,n)}) \longrightarrow \mathbb{R}^{(m,m)}$$

by

$$Q((\lambda,\mu),(\lambda',\mu')) = \lambda{}^t\mu' - \mu{}^t\lambda', \quad \lambda,\mu,\lambda',\mu' \in \mathbb{R}^{(m,n)}.$$

Clearly we have

$$(9.5) \qquad {}^tQ(\xi,\eta) = -Q(\eta,\xi) \ \text{ for all}\xi,\eta \in \mathbb{R}^{(m,n)}\times\mathbb{R}^{(m,n)},$$

$$(9.6) \qquad Q(\xi M,\eta M) = Q(\xi,\eta) \ \text{ for all} M \in Sp(n,\mathbb{R}).$$

For a reason of the convenience, we write an element of G^J as

$$g = [M,(\lambda,\mu,\kappa)] := (E_{2n},(\lambda,\mu,\kappa))\cdot(M,(0,0,0)).$$

Then the multiplication becomes

$$[M,(\xi,\kappa)]\circ[M',(\xi',\kappa')]=[MM',(\xi+\xi'M^{-1},\kappa+\kappa'+Q(\xi,\xi'M^{-1}))].$$

For brevity, we set $G=Sp(n,\mathbb{R})$. We note that the stabilizer K of G at iE_n under the symplectic action is given by

$$K=\left\{\begin{pmatrix}A & -B\\ B & A\end{pmatrix}\in GL(2n,\mathbb{R})\,\middle|\, {}^tAB={}^tBA,\ \ {}^tAA+{}^tBB=E_n\right\}$$

and is a maximal compact subgroup of G. We also recall that the Jacobi group G^J acts on $H_{n,m}$ transitively via (9.4). Then it is easy to see that the stabilizer K^J of G^J at $(iE_n,0)$ under this action is given by

$$\begin{aligned}K^J&=\left\{[k,(0,0,\kappa)]\,\middle|\, k\in K,\ \ k={}^t k\in\mathbb{R}^{(m,m)}\right\}\\ &\cong K\times\{\ (0,0,\kappa)|\ \kappa={}^t\kappa\in\mathbb{R}^{(m,m)}\ \}\cong K\times \mathrm{Symm}^2(\mathbb{R}^m).\end{aligned}$$

Thus on $G^J/K^J\cong H_{n,m}$, we have the coordinate

$$g\cdot(iE_n,0):=(Z,W):=(X+iY,\lambda Z+\mu),\quad g\in G^J.$$

In fact, if $g=[M,(\lambda,\mu,\kappa)]\in G^J$ with $M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in G$,

$$\begin{aligned}Z&=M<iE_n>=(iA+B)(iC+D)^{-1}=X+iY,\\ W&=\{i(\lambda A+\mu C)+\lambda B+\lambda D\}(iC+D)^{-1}\\ &=\{\lambda(iA+B)+\mu(iC+D)\}(iC+D)^{-1}\\ &=\lambda Z+\mu.\end{aligned}$$

We set

$$dX=\begin{pmatrix}dX_{11} & \dots & dX_{1n}\\ \vdots & \ddots & \vdots\\ dX_{n1} & \dots & dX_{nn}\end{pmatrix},\quad dW=\begin{pmatrix}dW_{11} & \dots & dW_{1n}\\ \vdots & \ddots & \vdots\\ dW_{m1} & \dots & dW_{mn}\end{pmatrix}$$

and

$$\frac{\partial}{\partial X}=\begin{pmatrix}\frac{\partial}{\partial X_{11}} & \cdots & \frac{\partial}{\partial X_{1n}}\\ \vdots & \ddots & \vdots\\ \frac{\partial}{\partial X_{n1}} & \cdots & \frac{\partial}{\partial X_{nn}}\end{pmatrix},\quad \frac{\partial}{\partial W}=\begin{pmatrix}\frac{\partial}{\partial W_{11}} & \cdots & \frac{\partial}{\partial W_{m1}}\\ \vdots & \ddots & \vdots\\ \frac{\partial}{\partial W_{1n}} & \cdots & \frac{\partial}{\partial W_{mn}}\end{pmatrix}.$$

Similarly we set $dY=(dY_{ij}),\ d\lambda=(d\lambda_{pq}),\ d\mu=(d\mu_{pq}),\cdots$etc. By an easy calculation, we have

$$\begin{aligned}
\frac{\partial}{\partial W} &= \frac{1}{2i} Y^{-1} \left(\frac{\partial}{\partial \lambda} - \bar{Z} \frac{\partial}{\partial \mu} \right), \\
\frac{\partial}{\partial \overline{W}} &= \frac{i}{2} Y^{-1} \left(\frac{\partial}{\partial \lambda} - Z \frac{\partial}{\partial \mu} \right), \\
\frac{\partial}{\partial X} &= \frac{\partial}{\partial Z} + \frac{\partial}{\partial \bar{Z}} + \frac{\partial}{\partial W} \lambda + \frac{\partial}{\partial \overline{W}} \lambda, \\
\frac{\partial}{\partial Y} &= i \frac{\partial}{\partial Z} - i \frac{\partial}{\partial \bar{Z}} + i \frac{\partial}{\partial W} \lambda - i \frac{\partial}{\partial \overline{W}} \lambda.
\end{aligned}$$

We set

$$P_+ = \frac{1}{2} \left(\frac{\partial}{\partial X} - i \frac{\partial}{\partial Y} \right) = \frac{\partial}{\partial Z} + \frac{\partial}{\partial W} \, (\mathrm{Im}\, W) \, Y^{-1}$$

and

$$P_- = \frac{1}{2} \left(\frac{\partial}{\partial X} + i \frac{\partial}{\partial Y} \right) = \frac{\partial}{\partial \bar{Z}} + \frac{\partial}{\partial \overline{W}} \, (\mathrm{Im}\, W) \, Y^{-1}.$$

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Then

$$\mathfrak{g}_{\mathbb{C}} = \left\{ \begin{pmatrix} A & B \\ C & -{}^t A \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \;\middle|\; B = {}^t B, \;\; C = {}^t C \right\}.$$

We let $\hat{J} := iJ_n$ with $J_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$. We define an involution σ of G by

$$\sigma(g) := \hat{J} g \hat{J}^{-1}, \quad g \in G. \tag{9.7}$$

The differential map $d\sigma = \mathrm{Ad}\,(\hat{J})$ of σ extends complex linearly to the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. $\mathrm{Ad}\,(\hat{J})$ has 1 and -1 as eigenvalues. The (+1)-eigenspace of $\mathrm{Ad}\,(\hat{J})$ is given by

$$\mathfrak{k}_{\mathbb{C}} = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \;\middle|\; {}^t A + A = 0, \; B = {}^t B \right\}. \tag{9.8}$$

We note that $\mathfrak{k}_{\mathbb{C}}$ is the complexification of the Lie algebra $\mathfrak{k}$ of a maximal compact subgroup $K = G \cap SO(2n, \mathbb{R}) \cong U(n)$ of G. The (-1)-eigenspace of $\mathrm{Ad}\,(\hat{J})$ is given by

$$\mathfrak{p}_{\mathbb{C}} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \;\middle|\; A = {}^t A, \; B = {}^t B \right\}. \tag{9.9}$$

We observe that $\mathfrak{p}_{\mathbb{C}}$ is not a Lie algebra. But $\mathfrak{p}_{\mathbb{C}}$ has the following decomposition

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-,$$

where

$$\mathfrak{p}_+ = \left\{ \begin{pmatrix} X & iX \\ iX & -X \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \;\middle|\; X = {}^tX \right\} \tag{9.10}$$

and

$$\mathfrak{p}_- = \left\{ \begin{pmatrix} Y & -iY \\ -iY & -Y \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \;\middle|\; Y = {}^tY \right\}. \tag{9.11}$$

We observe that $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are abelian subalgebras of $\mathfrak{g}_\mathbb{C}$. Since $\mathrm{Ad}\,(\hat{J})[X,Y] = [\mathrm{Ad}\,(\hat{J})X,\, \mathrm{Ad}\,(\hat{J})Y]$ for all $X, Y \in \mathfrak{g}_\mathbb{C}$, we have

$$[\mathfrak{k}_\mathbb{C}, \mathfrak{k}_\mathbb{C}] \subset \mathfrak{k}_\mathbb{C}, \quad [\mathfrak{k}_\mathbb{C}, \mathfrak{p}_\mathbb{C}] \subset \mathfrak{p}_\mathbb{C}, \quad [\mathfrak{p}_\mathbb{C}, \mathfrak{p}_\mathbb{C}] \subset \mathfrak{k}_\mathbb{C}. \tag{9.12}$$

Since $\mathrm{Ad}(k)X = kXk^{-1}$ ($k \in K,\ X \in \mathfrak{g}_\mathbb{C}$), we obtain

$$\mathrm{Ad}(k)\mathfrak{p}_+ \subset \mathfrak{p}_+, \quad \mathrm{Ad}(k)\mathfrak{p}_- \subset \mathfrak{p}_-. \tag{9.13}$$

For instance, if $k = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K$, then

$$\mathrm{Ad}(k) \begin{pmatrix} X & \pm iX \\ \pm iX & -X \end{pmatrix} = \begin{pmatrix} X' & \pm iX' \\ \pm iX' & -X' \end{pmatrix}, \quad X = {}^tX, \tag{9.14}$$

where

$$X' = (A + iB)X\, {}^t(A + iB).$$

If we identify $\mathfrak{p}_-$ with $\mathrm{Symm}^2(\mathbb{C}^n)$ and K with $U(n)$ as a subgroup of $GL(n, \mathbb{C})$ via the mapping $K \ni \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \longrightarrow A + iB \in U(n)$, then the action of K on $\mathfrak{p}_-$ is compatible with the natural representation $\rho^{[1]}$ of $GL(n, \mathbb{C})$ on $\mathrm{Symm}^2(\mathbb{C}^n)$ given by

$$\rho^{[1]}(g)X = gX\, {}^tg, \quad g \in GL(n, \mathbb{C}), \quad X \in \mathrm{Symm}^2(\mathbb{C}^n).$$

The Lie algebra $\mathfrak{g}$ of G has a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{9.15}$$

where

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \;\middle|\; A + {}^tA = 0, \quad B = {}^tB \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \;\middle|\; A = {}^tA, \quad B = {}^tB \right\}.$$

Then $\theta := \mathrm{Ad}\,(\hat{J})$ is a Cartan involution because

$$-B(W, \theta(W)) = -B(X, X) + B(Y, Y) > 0$$

for all $W = X + Y,\ X \in \mathfrak{k},\ Y \in \mathfrak{p}$. Here B denotes the Cartan-Killing form for $\mathfrak{g}$. Indeed,

$$B(X,Y) = 2(n+1)\,\sigma(XY), \quad X, Y \in \mathfrak{g}. \tag{9.16}$$

The vector space $\mathfrak{p}$ is identified with the tangent space of H_n at iE_n. The correspondence

$$\frac{1}{2}\begin{pmatrix} B & A \\ A & -B \end{pmatrix} \longmapsto A + iB \tag{9.17}$$

yields an isomorphism of $\mathfrak{p}$ onto $\mathrm{Symm}^2(\mathbb{C}^n)$. The Lie algebra $\mathfrak{g}^J$ of the Jacobi group G^J has a decomposition

$$\mathfrak{g}^J = \mathfrak{k}^J + \mathfrak{p}^J, \tag{9.18}$$

where

$$\mathfrak{k}^J = \left\{ (X, (0,0,\kappa) \mid X \in \mathfrak{k},\ \ \kappa = {}^t\kappa \in \mathbb{R}^{(m,m)} \right\},$$
$$\mathfrak{p}^J = \left\{ (Y, (P,Q,0) \mid Y \in \mathfrak{p},\ \ P, Q \in \mathbb{R}^{(m,n)} \right\}.$$

Thus the tangent space of the homogeneous space $H_{n,m} \cong G^J/K^J$ at $(iE_n, 0)$ is given by

$$\mathfrak{p}^J \cong \mathfrak{p} \oplus (\mathbb{R}^{(n,m)} \times \mathbb{R}^{(n,m)}) \cong \mathfrak{p} \oplus \mathbb{C}^{(n,m)}.$$

We define a complex structure I^J on the tangent space $\mathfrak{p}^J$ of $H_{n,m}$ at iE_n by

$$I^J\left(\begin{pmatrix} Y & X \\ X & -Y \end{pmatrix}, (P,Q) \right) := \left(\begin{pmatrix} X & -Y \\ -Y & -X \end{pmatrix}, (Q,-P) \right). \tag{9.19}$$

Identifying $\mathbb{R}^{(m,n)} \times \mathbb{R}^{(m,n)}$ with $\mathbb{C}^{(m,n)}$ via

$$(P,Q) \longmapsto iP + Q, \quad P, Q \in \mathbb{R}^{(m,n)}, \tag{9.20}$$

we may regard the complex structure I^J as a real linear map

$$I^J(X + iY, Q + iP) = (-Y + iX, -P + iQ), \tag{9.21}$$

where $X + iY \in \mathrm{Symm}^2(\mathbb{C}^n),\ Q + iP \in \mathbb{C}^{(m,n)}$. I^J extends complex linearly on the complexification $\mathfrak{p}^J_{\mathbb{C}} = \mathfrak{p} \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathfrak{p}$. $\mathfrak{p}_{\mathbb{C}}$ has a decomposition

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}^J_+ \oplus \mathfrak{p}^J_-, \tag{9.22}$$

where $\mathfrak{p}^J_+$ (resp. $\mathfrak{p}^J_-$) denotes the $(+i)$-eigenspace (resp. $(-i)$-eigenspace) of I^J. Precisely, both $\mathfrak{p}^J_+$ and $\mathfrak{p}^J_-$ are given by

$$\mathfrak{p}^J_+ = \left\{ \left(\begin{pmatrix} X & iX \\ iX & -X \end{pmatrix}, (P, iP) \right) \;\middle|\; X \in \mathrm{Symm}^2(\mathbb{C}^n), \quad P \in \mathbb{C}^{(m,n)} \right\}$$

and

$$\mathfrak{p}_-^J = \left\{ \left(\begin{pmatrix} X & -iX \\ -iX & -X \end{pmatrix}, (P, -iP) \right) \;\middle|\; X \in \mathrm{Symm}^2(\mathbb{C}^n), \quad P \in \mathbb{C}^{(m,n)} \right\}.$$

With respect to this complex structure I^J, we may say that f is *holomorphic* if and only if $\xi f = 0$ for all $\xi \in \mathfrak{p}_-^J$.

We fix $g = [M, (l, \mu; \kappa)] \in G^J$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$. Let $T_g : H_n \longrightarrow H_n$ be the mapping defined by (9.4). We consider the behavior of the differential map dT_g of T_g at $(iE_n, 0)$

$$dT_g : T_{(iE_n,0)}(H_{n,m}) \longrightarrow T_{(Z,W)}(H_{n,m}), \quad (Z, W) := g \cdot (iE_n, 0).$$

Now we let $\alpha(t) = (Z(t), \xi(t))$ be a smooth curve in $H_{n,m}$ passing through $(iE_n, 0)$ with $\alpha'(0) = (V, iP + Q) \in T_{(iE_n,0)}(H_{n,m})$. Then

$$\begin{aligned} \gamma(t) : &= g \cdot \alpha(t) = (Z(g;t), \xi(g;t)) \\ &= (M < Z(t) >, (\xi(t) + \tilde{\lambda} Z(t) + \tilde{\mu})(CZ(t) + D)^{-1}) \end{aligned}$$

is a curve in $H_{n,m}$ passing through $\gamma(0) = (Z, W)$ with $(\tilde{\lambda}, \tilde{\mu}) = (\lambda, \mu)M$. Using the relation

$$\left.\frac{\partial}{\partial t}\right|_{t=0} (CZ(t) + D)^{-1} = -(iC + D)^{-1} C Z'(0)(iC + D)^{-1},$$

we have

$$\begin{aligned} \left.\frac{\partial}{\partial t}\right|_{t=0} Z(g;t) &= AZ'(0)(iC + D)^{-1} + (iA + B) \left.\frac{\partial}{\partial t}\right|_{t=0} (CZ(t) + D)^{-1} \\ &= AZ'(0)(iC + D)^{-1} - (iA + B)(iC + D)^{-1} CZ'(0)(iC + D)^{-1} \\ &= \{A\, {}^t(iC + D) - (iA + B)\, {}^tC\}\, {}^t(iC + D)^{-1} Z'(0)(iC + D)^{-1} \\ &= {}^t(iC + D)^{-1} Z'(0)(iC + D)^{-1}. \end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\bigg|_{t=0} \xi(g;t) &= (\xi'(0)+\tilde{\lambda}Z'(0))(iC+D)^{-1} \\ &\quad + (\xi(0)+i\tilde{\lambda}+\tilde{\mu})\,\frac{\partial}{\partial t}\bigg|_{t=0}(CZ(t)+D)^{-1} \\ &= (iP+Q+\tilde{\lambda}Z'(0))(iC+D)^{-1} \\ &\quad - (i\tilde{\lambda}+\tilde{\mu})(iC+D)^{-1}CZ'(0)(iC+D)^{-1} \\ &= (iP+Q)(iC+D)^{-1} \\ &\quad + \left\{\tilde{\lambda}\,{}^t(iC+D) - (i\tilde{\lambda}+\tilde{\mu})\,{}^tC\right\}{}^t(iC+D)^{-1}Z'(0)(iC+D)^{-1} \\ &= (iP+Q)(iC+D)^{-1} + \lambda\,{}^t(iC+D)^{-1}Z'(0)(iC+D)^{-1}.\end{aligned}$$

Here we used the fact that $(iC+D)^{-1}C$ is symmetric and the relation

$$\tilde{\lambda} = \lambda A + \mu C, \quad \tilde{\mu} = \lambda B + \mu D.$$

Therefore we obtain

$$\begin{aligned} Z'(g;0) &= {}^t(iC+D)^{-1}Z'(0)(iC+D)^{-1}, \\ \xi'(g;0) &= \xi'(0)(iC+D)^{-1} + \lambda\,{}^t(iC+D)^{-1}Z'(0)(iC+D)^{-1}.\end{aligned}$$

In summary, we have

Proposition 9.1. *Let* $g = [M,(\lambda,\mu;\kappa)] \in G^J$ *with* $M = \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in G$ *and let* $(Z,W) = g\cdot(iE_n,0)$. *Then the differential map* $dT_g : T_{(iE_n,0)}(H_{n,m}) \longrightarrow T_{(Z,W)}(H_{n,m})$ *is given by*

$$(v,w) \longmapsto (v(g),w(g)), \quad v \in \mathrm{Symm}^2(\mathbb{C}^n), \quad w \in \mathbb{C}^{(m,n)} \tag{9.23}$$

with

$$\begin{aligned} v(g) &= {}^t(iC+D)^{-1}v(iC+D)^{-1}, \\ w(g) &= w(iC+D)^{-1} + \lambda\,{}^t(iC+D)^{-1}v(iC+D)^{-1}.\end{aligned}$$

9.1.2. An Iwasawa Decomposition of the Jacobi Group G^J

First of all, we give the Iwasawa decomposition of $G = Sp(n,\mathbb{R})$. For a positive diagonal matrix H of degree n, we put

$$t(H) = \begin{pmatrix} H & 0 \\ 0 & H^{-1}\end{pmatrix}$$

and for an upper triangular matrix A with 1 in every diagonal entry and $B \in \mathbb{R}^{(m,n)}$, we write

$$n(A,B) = \begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix}.$$

We let A be the set of such all $t(H)$ and let N be the set of such all $n(A,B)$ such that $n(A,B) \in G$, namely, $A\,{}^tB = B\,{}^tA$. It is clear that A is an abelian subgroup of G and N is a nilpotent subgroup of G. Then we have the so-called *Iwasawa decomposition*

$$G = NAK = KAN. \tag{9.24}$$

Now we define the subgroups A^J, N^J and $\tilde{N}^J$ of G^J by

$$A^J = \left\{ t(H,\lambda) := [t(H),(\lambda,0,0)] \,\middle|\, t(H) \in A,\ \lambda \in \mathbb{R}^{(m,n)} \right\},$$
$$N^J = \left\{ n(A,B;\mu) := [n(A,B),(0,\mu,0)] \,\middle|\, n(A,B) \in N,\ \mu \in \mathbb{R}^{(m,n)} \right\}$$

and

$$\tilde{N}^J := \left\{ \tilde{n}(A,B;\mu,\kappa) = [n(A,B),(0,\mu,\kappa)] \,\middle|\, n(A,B) \in N,\ \mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)} \right\}.$$

For $t(H,\lambda),\ t(H',\lambda') \in A^J$, we have

$$t(H,\lambda) \circ t(H',\lambda') = t(HH', \lambda + \lambda' H^{-1}).$$

Thus A^J is the semidirect product of $\mathbb{R}^{(m,n)}$ and $\mathbb{D}^+$, where $\mathbb{D}^+$ denotes the subgroup of $GL(n,\mathbb{R})$ consisting of positive diagonal matrices of degree n. Furthermore we have for $t(H,\lambda) \in A^J$ and $\tilde{n}(A,B;\mu,\kappa) \in \tilde{N}^J$

$$\tilde{n}(A,B;\mu,\kappa) \circ t(H,\lambda) = [n(A,B)t(H), (\lambda A^{-1},\ \mu - \lambda A^{-1} B\,{}^tA,\ -\mu\,{}^tA^{-1}\,{}^t\lambda)]$$

and

$$\begin{aligned} t(H,\lambda) \circ \tilde{n}(A,B;\mu,\kappa) &= [t(H)n(A,B), (\lambda, \mu H,\ \kappa + \lambda H\,{}^t\mu)] \\ &= \left[\begin{pmatrix} HA & HB \\ 0 & H^{-1}\,{}^tA^{-1} \end{pmatrix}, (\lambda, \mu H, \kappa + \lambda H\,{}^t\mu) \right]. \end{aligned}$$

Therefore we have

$$\begin{aligned} & t(H,\lambda) \circ \tilde{n}(A,B;\mu,\kappa) \circ t(H,\lambda)^{-1} \\ =& [t(H)n(X)t(H^{-1}), (\lambda - \lambda HA^{-1}H^{-1},\ \mu H + \lambda HA^{-1}B\,{}^tAH, \\ & \quad \kappa + \lambda HA\,{}^tB\,{}^tA^{-1}H\,{}^t\lambda - \mu\,{}^tA^{-1}H\,{}^t\lambda)] \\ =& [n(HAH^{-1}, HBH),\ (\lambda - \lambda HA^{-1}H^{-1}, \mu H + \lambda HA^{-1}B\,{}^tAH, \\ & \quad \kappa + \lambda HA\,{}^tB\,{}^tA^{-1}H\,{}^t\lambda - \mu\,{}^tA^{-1}H\,{}^t\lambda)]. \end{aligned}$$

Thus there is a decomposition

$$G^J = \tilde{N}^J A^J K. \tag{9.25}$$

For $g \in G^J$, one has

$$\begin{aligned} g &= [n(A,B)t(H)\kappa, (\lambda,\mu,\kappa)], \quad \kappa \in K \\ &= \tilde{n}(A,B;\mu^*,\kappa^*) \circ t(H,\lambda^*) \circ k \end{aligned}$$

with

$$\lambda^* = \lambda H, \quad \mu^* = \mu + \lambda B^t A \ \text{ and } \ \kappa^* = \kappa + \mu^t\lambda + \lambda B\,{}^t(\lambda A).$$

Recalling the subgroup K^J of G^J defined by

$$K^J = \{\, [k,(0,0,\kappa)] \mid k \in K,\ \kappa = {}^t\kappa \in \mathbb{R}^{(m,m)} \,\},$$

we also have a decomposition

$$G^J = N^J A^J K^J. \tag{9.26}$$

For $g \in G^J$, one has

$$\begin{aligned} g &= [n(A,B)t(H)k, (\lambda,\mu,\kappa)] \\ &= n(A,B;\tilde{\mu}) \circ t(H,\tilde{\lambda}) \circ [k,(0,0,\tilde{\kappa})] \end{aligned}$$

with

$$\tilde{\lambda} = \lambda A, \quad \tilde{\mu} = \mu + \lambda A^{-1} B\,{}^t A, \quad \tilde{\kappa} = \kappa + (\mu + \lambda A^{-1} B\,{}^t A)\,{}^t\lambda.$$

We call the decomposition (9.25) or (9.26) an *Iwasawa decomposition* of G^J. Finally we note that the decomposition (9.25) or (9.26) may be understood as the product of the usual Iwasawa decomposition (9.24) of G with a decomposition

$$H_{\mathbb{R}}^{(n,m)} = \tilde{N}_0 A_0 \tag{9.27}$$

of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ into the group $A_0 = \{\, (\lambda,0,0) \mid \lambda \in \mathbb{R}^{(m,n)} \,\}$ which normalizes the maximal abelian subgroup $\tilde{N}_0 = \{\ (0,\mu,\kappa)\ \mid\ \mu \in \mathbb{R}^{(m,n)}, \kappa = \ {}^t\kappa \in \mathbb{R}^{(m,n)} \,\}$.

9.2. The Lie Algebra of the Jacobi Group G^J

In this section, we describe the Lie algebra $\mathfrak{g}^J$ of the Jacobi group G^J explicitly.

First of all, we observe that $\mathfrak{g}$ of G may be regarded as a subalgebra of $\mathfrak{g}^J$ by identifying $\mathfrak{g}$ with $\mathfrak{g} \times \{0\}$ and the Lie algebra $\mathfrak{h}$ of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ may be regarded as an ideal of $\mathfrak{g}^J$ by identifying $\mathfrak{h}$ with $\{0\} \times \mathfrak{h}$. We denote by E_{ij} the matrix with entry 1 where the i-th row and the j-th column meet, all other entries 0.

For $1 \le a,b,p \le m,\ 1 \le i,j,q \le n$, we set

$$A_{ij} := \begin{pmatrix} E_{ij}+E_{ji} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(E_{ij}+E_{ji}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_{ij} := \begin{pmatrix} 0 & 0 & E_{ij}+E_{ji} & 0 \\ 0 & 0 & 0 & 0 \\ E_{ij}+E_{ji} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_{ij} := \begin{pmatrix} E_{ij}-E_{ji} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E_{ij}-E_{ji} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$T_{ij} := \begin{pmatrix} 0 & 0 & E_{ij}+E_{ji} & 0 \\ 0 & 0 & 0 & 0 \\ -(E_{ij}+E_{ji}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D^0_{ab} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(E_{ab}+E_{ba}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D_{pq} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ E_{pq} & 0 & 0 & 0 \\ 0 & 0 & 0 & -E_{qp} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{D}_{pq} := \begin{pmatrix} 0 & 0 & 0 & E_{qp} \\ 0 & 0 & E_{pq} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We observe that the set

$$\left\{ S_{ij},\, T_{kl},\, D^0_{ab} \,\middle|\, 1 \le i < j \le n,\ 1 \le k \le l \le n,\ 1 \le a \le b \le m \right\}$$

form a basis of $\mathfrak{k}^J$ and the set

$$\left\{ A_{ij},\, B_{ij},\, D_{pq},\, \hat{D}_{rs} \,\middle|\, 1 \le i \le j \le n,\ 1 \le p,r \le m,\ 1 \le q,s \le n \right\}$$

form a basis of $\mathfrak{p}^J$ (cf. (9.15)). We note that

$$A_{ij} = A_{ji},\quad B_{ij} = B_{ji},\quad S_{ij} = -S_{ji},\quad T_{ij} = T_{ji},$$
$$D^0_{ab} = D^0_{ba},\quad D^2_{pq} = \hat{D}^2_{pq} = 0.$$

Lemma 9.2. *We have the following commutation relation :*

$$
\begin{aligned}
[A_{ij}, A_{kl}] &= \delta_{ik}S_{jl} + \delta_{il}S_{jk} + \delta_{jk}S_{il} + \delta_{jl}S_{ik},\\
[A_{ij}, B_{kl}] &= \delta_{ik}T_{jl} + \delta_{il}T_{jk} + \delta_{jk}T_{il} + \delta_{jl}T_{ik},\\
[A_{ij}, S_{kl}] &= \delta_{ik}A_{jl} - \delta_{il}A_{jk} + \delta_{jk}A_{il} - \delta_{jl}A_{ik},\\
[A_{ij}, T_{kl}] &= \delta_{ik}B_{jl} + \delta_{il}B_{jk} + \delta_{jk}B_{il} + \delta_{jl}B_{ik},\\
[B_{ij}, B_{kl}] &= \delta_{ik}S_{jl} + \delta_{il}S_{jk} + \delta_{jk}S_{il} + \delta_{jl}S_{ik},\\
[B_{ij}, S_{kl}] &= \delta_{ik}B_{jl} - \delta_{il}B_{jk} + \delta_{jk}B_{il} - \delta_{jl}B_{ik},\\
[B_{ij}, T_{kl}] &= -\delta_{ik}A_{jl} - \delta_{il}A_{jk} - \delta_{jk}A_{il} - \delta_{jl}A_{ik},\\
[S_{ij}, S_{kl}] &= -\delta_{ik}S_{jl} + \delta_{il}S_{jk} + \delta_{jk}S_{il} - \delta_{jl}S_{ik},\\
[S_{ij}, T_{kl}] &= -\delta_{ik}T_{jl} - \delta_{il}T_{jk} + \delta_{jk}T_{il} + \delta_{jl}T_{ik},\\
[T_{ij}, T_{kl}] &= -\delta_{ik}S_{jl} - \delta_{il}S_{jk} - \delta_{jk}S_{il} - \delta_{jl}S_{ik},\\
[D^0_{ab}, A_{ij}] &= [D^0_{ab}, B_{ij}] = [D^0_{ab}, S_{ij}] = [D^0_{ab}, T_{ij}] = 0,\\
[D^0_{ab}, D^0_{cd}] &= [D^0_{ab}, D_{pq}] = [D^0_{ab}, \hat{D}_{pq}] = 0,\\
[D_{pq}, A_{ij}] &= \delta_{qi}D_{pj} + \delta_{qj}D_{pi},\\
[D_{pq}, B_{ij}] &= [D_{pq}, T_{ij}] = \delta_{qi}\hat{D}_{pj} + \delta_{qj}\hat{D}_{pi},\\
[D_{pq}, S_{ij}] &= \delta_{qi}D_{pj} - \delta_{qj}D_{pi},\\
[D_{pq}, D_{rs}] &= 0, \quad [D_{pq}, \hat{D}_{rs}] = 2\delta_{qs}D^0_{pr},\\
[\hat{D}_{pq}, A_{ij}] &= -\delta_{qi}\hat{D}_{pj} - \delta_{qj}\hat{D}_{pi},\\
[\hat{D}_{pq}, B_{ij}] &= \delta_{qi}D_{pj} + \delta_{qj}D_{pi},\\
[\hat{D}_{pq}, S_{ij}] &= \delta_{qi}D_{pj} - \delta_{qj}\hat{D}_{pi},\\
[\hat{D}_{pq}, T_{ij}] &= -\delta_{qi}D_{pj} - \delta_{qj}D_{pi},\\
[\hat{D}_{pq}, \hat{D}_{rs}] &= 0.
\end{aligned}
$$

Here $1 \le a, b, c, d, p, r \le m,\ 1 \le i, j, k, l, q, s \le n$ and δ_{ij} denotes the Kronecker delta symbol.

Proof. The proof follows from a straightforward calculation. □

Corollary 9.3. *We have the following relation :*

$$
\begin{aligned}
&[\mathfrak{k}^J, \mathfrak{k}^J] \subset \mathfrak{k}^J, \quad [\mathfrak{k}^J, \mathfrak{p}^J] \subset \mathfrak{p}^J,\\
&[\mathfrak{p}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}.
\end{aligned}
$$

Proof. It follows immediately from Lemma 9.2. □

Remark 9.4. We remark that the relation

$$[\mathfrak{p}^J, \mathfrak{p}^J] \subset \mathfrak{k}^J$$

does not hold.

Now we set

$$\begin{aligned}
Z^0_{ab} &:= -\sqrt{-1}D^0_{ab},\\
Y^{\pm}_{pq} &:= \frac{1}{2}(D_{pq} \pm \sqrt{-1}\hat{D}_{pq}),\\
Z^{+}_{ij} &:= -S_{ij},\\
Z^{-}_{ij} &:= -\sqrt{-1}T_{ij},\\
X^{\pm}_{ij} &:= \frac{1}{2}(A_{ij} \pm \sqrt{-1}B_{ij}).
\end{aligned}$$

Lemma 9.5. *We have the following commutation relation :*

$$\begin{aligned}
&[Z^0_{ab}, Z^0_{cd}] = [Z^0_{ab}, Y^{\pm}_{pq}] = [Z^0_{ab}, Z^{\pm}_{ij}] = [Z^0_{ab}, X^{\pm}_{ij}] = 0,\\
&[Y^{+}_{pq}, Y^{+}_{rs}] = 0, \quad [Y^{+}_{pq}, Y^{-}_{rs}] = \delta_{qs} Z^0_{pr},\\
&[Y^{+}_{pq}, Z^{+}_{ij}] = -\delta_{qi} Y^{+}_{pj} + \delta_{qj} Y^{+}_{pi},\\
&[Y^{+}_{pq}, Z^{-}_{ij}] = -\delta_{qi} Y^{+}_{pj} - \delta_{qj} Y^{+}_{pi},\\
&[Y^{+}_{pq}, X^{+}_{ij}] = 0,\\
&[Y^{+}_{pq}, X^{-}_{ij}] = \delta_{qi} Y^{-}_{pj} + \delta_{qj} Y^{-}_{pi},\\
&[Y^{-}_{pq}, Y^{-}_{rs}] = 0,\\
&[Y^{-}_{pq}, Z^{+}_{ij}] = -\delta_{qi} Y^{-}_{pj} + \delta_{qj} Y^{-}_{pi},\\
&[Y^{-}_{pq}, Z^{-}_{ij}] = \delta_{qi} Y^{-}_{pj} + \delta_{qj} Y^{-}_{pi},\\
&[Y^{-}_{pq}, X^{+}_{ij}] = \delta_{qi} Y^{+}_{pj} + \delta_{qj} Y^{+}_{pi},\\
&[Y^{-}_{pq}, X^{-}_{ij}] = 0,
\end{aligned}$$

$$\begin{aligned}
[Z^+_{ij}, Z^+_{kl}] &= \delta_{ik} Z^+_{jl} - \delta_{il} Z^+_{jk} - \delta_{jk} Z^+_{il} + \delta_{jl} Z^+_{ik},\\
[Z^+_{ij}, Z^-_{kl}] &= \delta_{ik} Z^-_{jl} - \delta_{il} Z^-_{jk} + \delta_{jk} Z^-_{il} - \delta_{jl} Z^-_{ik},\\
[Z^+_{ij}, X^\pm_{kl}] &= \delta_{ik} X^\pm_{jl} - \delta_{jk} X^\pm_{il} + \delta_{il} X^\pm_{jk} - \delta_{jl} X^\pm_{ik},\\
[Z^-_{ij}, Z^-_{kl}] &= -\delta_{ik} Z^+_{jl} - \delta_{il} Z^+_{jk} - \delta_{jk} Z^+_{il} - \delta_{jl} Z^+_{ik},\\
[Z^-_{ij}, X^+_{ij}] &= \delta_{ik} X^+_{jl} + \delta_{il} X^+_{jk} + \delta_{jk} X^+_{il} + \delta_{jl} X^+_{ik},\\
[Z^-_{ij}, X^-_{ij}] &= -\delta_{ik} X^-_{jl} - \delta_{il} X^-_{jk} - \delta_{jk} X^-_{il} - \delta_{jl} X^-_{ik},\\
[X^+_{ij}, X^+_{kl}] &= [X^-_{ij}, X^-_{kl}] = 0,\\
[X^+_{ij}, X^-_{kl}] &= -\frac{1}{2}(\delta_{ik} Z^+_{jl} + \delta_{il} Z^+_{jk} + \delta_{jk} Z^+_{il} + \delta_{jl} Z^+_{ik})\\
&\quad + \frac{\sqrt{-1}}{2}(\delta_{ik} Z^-_{jl} + \delta_{il} Z^-_{jk} + \delta_{jk} Z^-_{il} + \delta_{jl} Z^-_{ik}).
\end{aligned}$$

Proof. It follows from Lemma 9.2. □

Corollary 9.6. *The set*

$$\left\{ Z^0_{ab},\, Z^+_{ij},\, Z^-_{kl} \,\middle|\, 1 \le a \le b \le m,\ 1 \le i < j \le n,\ 1 \le k < l \le n \right\}$$

form a basis of the complexification $\mathfrak{k}^J_{\mathbb{C}}$ *of* $\mathfrak{k}^J$ *and the set*

$$\left\{ X^\pm_{ij},\, Y^\pm_{pq} \,\middle|\, 1 \le i \le j \le n,\ 1 \le p \le m,\ 1 \le q \le n \right\}$$

form a basis of $\mathfrak{p}^J_{\mathbb{C}}$*. And* $\left\{ X^+_{ij},\, Y^+_{pq} \,\middle|\, 1 \le i \le j \le n,\ 1 \le p \le m,\ 1 \le q \le n \right\}$ *form a basis of* $\mathfrak{p}^J_+$ *and* $\left\{ X^-_{ij},\, Y^-_{pq} \,\middle|\, 1 \le i \le j \le n,\ 1 \le p \le m,\ 1 \le q \le n \right\}$ *form a basis of* $\mathfrak{p}^J_-$*. Both* $\mathfrak{p}^J_+$ *and* $\mathfrak{p}^J_-$ *are all abelian subalgebras of* $\mathfrak{g}^J_{\mathbb{C}}$*. We have the relation*

$$[\mathfrak{k}^J_{\mathbb{C}},\, \mathfrak{p}^J_+] \subset \mathfrak{p}^J_+, \qquad [\mathfrak{k}^J_{\mathbb{C}},\, \mathfrak{p}^J_-] \subset \mathfrak{p}^J_-.$$

$\mathfrak{g}_{\mathbb{C}}$ *is a subalgebra of* $\mathfrak{g}^J_{\mathbb{C}}$ *and* $\mathfrak{h}_{\mathbb{C}}$*, the complexification of* $\mathfrak{h}$*, is an ideal of* $\mathfrak{g}^J_{\mathbb{C}}$*.*

Proof. It follows immediately from Lemma 9.5. □

9.3. Jacobi Forms

Let ρ be a rational representation of $GL(n,\mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbb{R}^{(m,m)}$ be a symmetric half-integral semi-positive definite matrix of degree m. Let $C^\infty(H_{n,m},V_\rho)$ be the algebra of all C^∞ functions on $H_{n,m}$ with values in V_ρ. For $f\in C^\infty(H_{n,m},V_\rho)$, we define

$$\begin{aligned}&(f|_{\rho,\mathcal{M}}[(M,(\lambda,\mu,\kappa))])(Z,W)\\ &:= e^{-2\pi i\sigma(\mathcal{M}[W+\lambda Z+\mu](CZ+D)^{-1}C)}\times e^{2\pi i\sigma(\mathcal{M}(\lambda Z^t\lambda+2\lambda^tW+(\kappa+\mu^t\lambda)))}\\ &\quad\times\rho(CZ+D)^{-1}f(M<Z>,(W+\lambda Z+\mu)(CZ+D)^{-1}),\end{aligned}\tag{9.28}$$

where $M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in Sp(n,\mathbb{R}),\ (\lambda,\mu,\kappa)\in H_{\mathbb{R}}^{(n,m)}$ and $(Z,W)\in H_{n,m}$.

Definition 9.7. Let ρ and $\mathcal{M}$ be as above. Let

$$H_{\mathbb{Z}}^{(n,m)}:=\{(\lambda,\mu,\kappa)\in H_{\mathbb{R}}^{(n,m)}\,|\,\lambda,\mu\in\mathbb{Z}^{(m,n)},\ \kappa\in\mathbb{Z}^{(m,m)}\ \}.$$

A *Jacobi form* of index $\mathcal{M}$ with respect to ρ on Γ_n is a holomorphic function $f\in C^\infty(H_{n,m},V_\rho)$ satisfying the following conditions (A) and (B):

(A) $f|_{\rho,\mathcal{M}}[\tilde\gamma]=f$ for all $\tilde\gamma\in\Gamma_n^J:=\Gamma_n\ltimes H_{\mathbb{Z}}^{(n,m)}$.

(B) f has a Fourier expansion of the following form :

$$f(Z,W)=\sum_{\substack{T\geq 0\\ \text{half-integral}}}\ \sum_{R\in\mathbb{Z}^{(n,m)}}c(T,R)\cdot e^{2\pi i\,\sigma(TZ)}\cdot e^{2\pi i\sigma(RW)}$$

with $c(T,R)\neq 0$ only if $\begin{pmatrix}T & \frac12 R\\ \frac12\,{}^tR & \mathcal{M}\end{pmatrix}\geq 0$.

If $n\geq 2$, the condition (B) is superfluous by Köcher principle (cf. [110] Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma_n)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ_n. Ziegler (cf. [110] Theorem 1.8 or [26] Theorem 1.1) proves that the vector space $J_{\rho,\mathcal{M}}(\Gamma_n)$ is finite dimensional. For more results on Jacobi forms with $n>1$ and $m>1$, we refer to [61], [103]-[107] and [110].

Definition 9.8. A Jacobi form $f\in J_{\rho,\mathcal{M}}(\Gamma)$ is said to be a *cusp* (or *cuspidal*) form if $\begin{pmatrix}T & \frac12 R\\ \frac12\,{}^tR & \mathcal{M}\end{pmatrix}>0$ for any T, R with $c(T,R)\neq 0$. A Jacobi form $f\in J_{\rho,\mathcal{M}}(\Gamma)$ is said to be *singular* if it admits a Fourier expansion such that a Fourier coefficient $c(T,R)$ vanishes unless $\det\begin{pmatrix}T & \frac12 R\\ \frac12\,{}^tR & \mathcal{M}\end{pmatrix}=0$.

Example 9.9. Let $S\in\mathbb{Z}^{(2k,2k)}$ be a symmetric, positive definite unimodular even integral matrix and $c\in\mathbb{Z}^{(2k,m)}$. We define the theta series

$$\vartheta_{S,c}^{(g)}(Z,W):=\sum_{\lambda\in\mathbb{Z}^{(2k,n)}}e^{\pi i\{\sigma(S\lambda Z\,{}^t\lambda)+2\sigma({}^tcS\lambda,{}^tW)\}},\quad Z\in H_n,\ W\in\mathbb{C}^{(m,n)}.$$

We put $\mathcal{M}:=\frac12\,{}^tcSc$. We assume that $2k<g+\mathrm{rank}\,(\mathcal{M})$. Then it is easy to see that $\vartheta_{S,c}^{(g)}$ is a singular form in $J_{k,\mathcal{M}}(\Gamma_g)$ (cf. [110] p. 212).

9.4. Characterization of Jacobi Forms as Functions on the Jacobi Group G^J

In this section, we lift a Jacobi form $f \in J_{\rho,\mathcal{M}}(\Gamma_n)$ to a smooth function Φ_f on the Jacobi group G^J and characterize the lifted function Φ_f on G^J.

We recall that for given ρ and $\mathcal{M}$, the canonical automorphic factor $J_{\mathcal{M},\rho}$: $G^J \times H_{n,m} \longrightarrow GL(V_\rho)$ is given by

$$J_{\mathcal{M},\rho}(g,(Z,W)) = e^{-2\pi i\sigma(\mathcal{M}[W+\lambda Z+\mu](CZ+D)^{-1}C)} \times e^{2\pi i\sigma(\mathcal{M}(\lambda Z\,{}^t\lambda+2\lambda\,{}^tW+\kappa+\mu\,{}^t\lambda))}\rho(CZ+D)^{-1},$$

where $g = (M,(\lambda,\mu;\kappa)) \in G^J$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$. It is easy to see that the automorphic factor $J_{\mathcal{M},\rho}$ satisfies the cocycle condition:

$$J_{\mathcal{M},\rho}(g_1g_2,(Z,W)) = J_{\mathcal{M},\rho}(g_2,(Z,W))\, J_{\mathcal{M},\rho}(g_1, g_2\cdot(Z,W)) \tag{9.29}$$

for all $g_1,\ g_2 \in G^J$ and $(Z,W) \in H_{n,m}$.

Since the space $H_{n,m}$ is diffeomorphic to the homogeneous space G^J/K^J, we may lift a function f on $H_{n,m}$ with values in V_ρ to a function Φ_f on G^J with values in V_ρ in the following way. We define the lifting

$$\varphi_{\rho,\mathcal{M}} : \mathcal{F}(H_{n,m},V_\rho) \longrightarrow \mathcal{F}(G^J,V_\rho), \quad \varphi_{\rho,\mathcal{M}}(f) := \Phi_f \tag{9.30}$$

by

$$\begin{aligned}\Phi_f(g) :&= (f|_{\rho,\mathcal{M}}[g])(iE_n,0)\\ &= J_{\mathcal{M},\rho}(g,(iE_n,0))\, f(g\cdot(iE_n,0)),\end{aligned}$$

where $g \in G^J$ and $\mathcal{F}(H_{n,m},V_\rho)$ (resp. $\mathcal{F}(G^J,V_\rho)$) denotes the vector space consisting of functions on $H_{n,m}$ (resp. G^J) with values in V_ρ.

For brevity, we set $\Gamma := \Gamma_n = Sp(n,\mathbb{Z})$ and $\Gamma^J = \Gamma \ltimes H_{\mathbb{Z}}^{(n,m)}$. We let $\mathcal{F}^{\Gamma}_{\rho,\mathcal{M}}$ be the space of all functions f on $H_{n,m}$ with values in V_ρ satisfying the transformation formula

$$f|_{\rho,\mathcal{M}}[\gamma] = f \quad \text{for all } \gamma \in \Gamma^J. \tag{9.31}$$

And we let $\mathcal{F}^{\Gamma}_{\rho,\mathcal{M}}(G^J)$ be the space of functions $\Phi : G^J \longrightarrow V_\rho$ on G^J with values in V_ρ satisfying the following conditions (9.32) and (9.33):

$$\Phi(\gamma g) = \Phi(g) \qquad \text{for all} \quad \gamma \in \Gamma^J \quad \text{and} \quad g \in G^J. \tag{9.32}$$

$$\Phi(g\ \ r(k,\kappa)) = e^{2\pi i\sigma(\mathcal{M}\kappa)}\rho(k)^{-1}\Phi(g), \qquad \forall\ \ r(k,\kappa) := [k,(0,0;\kappa)] \in K^J. \tag{9.33}$$

Lemma 9.10. *The space $\mathcal{F}^{\Gamma}_{\rho,\mathcal{M}}$ is isomorphic to the space $\mathcal{F}^{\Gamma}_{\rho,\mathcal{M}}(G^J)$ via the lifting $\varphi_{\rho,\mathcal{M}}$.*

Proof. Let $f \in \mathcal{F}^{\Gamma}_{\rho,\mathcal{M}}$. If $\gamma \in \Gamma^J,\ g \in G^J$ and $r(k,\kappa) = [k,(0,0;\kappa)] \in K^J$, then we have

$$\begin{aligned}\Phi_f(\gamma g) &= (f|_{\rho,\mathcal{M}}[\gamma g])(iE_n,0)\\ &= ((f|_{\rho,\mathcal{M}}[\gamma])|_{\rho,\mathcal{M}}[g])(iE_n,0)\\ &= (f|_{\rho,\mathcal{M}}[g])(iE_n,0) \qquad\qquad (\text{ since } f \in \mathcal{F}^{\Gamma}_{\rho,\mathcal{M}}\ )\\ &= \Phi_f(g)\end{aligned}$$

and

$$\begin{aligned}\Phi_f(g\,r(k,\kappa)) &= J_{\mathcal{M},\rho}(g\,r(k,\kappa),(iE_n,0))\,f(g\,r(k,\kappa)\cdot(iE_n,0))\\ &= J_{\mathcal{M},\rho}(r(k,\kappa),(iE_n,0))\,J_{\mathcal{M},\rho}(g,(iE_n,0))\,f(g\cdot(iE_n,0))\\ &= e^{2\pi i\sigma(\mathcal{M}\kappa)}\rho(k)^{-1}\Phi_f(g).\end{aligned}$$

Here we identified $k = \begin{pmatrix} A & -B \\ B & A\end{pmatrix} \in K$ with $A+iB \in U(n)$.

Conversely, if $\Phi \in \mathcal{F}^{\Gamma}_{\rho,\mathcal{M}}(G^J),\ G^J$ acting on $H_{n,m}$ transitively, we may define a function f_Φ on $H_{n,m}$ by

$$f_\Phi(g\cdot(iE_n,0)) := J_{\mathcal{M},\rho}(g,(iE_n,0))^{-1}\Phi(g). \tag{9.34}$$

Let $\gamma \in \Gamma^J$ and $(Z,W) = g\cdot(iE_n,0)$ for some $g \in G^J$. Then using the cocycle condition (9.29), we have

$$\begin{aligned}(f_\Phi|_{\rho,\mathcal{M}}[\gamma])(Z,W) &= J_{\mathcal{M},\rho}(\gamma,(Z,W))f_\Phi(\gamma\cdot(Z,W))\\ &= J_{\mathcal{M},\rho}(\gamma,g\cdot(iE_n,0))\,f_\Phi(\gamma g\cdot(iE_n,0))\\ &= J_{\mathcal{M},\rho}(\gamma,g\cdot(iE_n,0))J_{\mathcal{M},\rho}(\gamma g,(iE_n,0))^{-1}\Phi(\gamma g)\\ &= J_{\mathcal{M},\rho}(\gamma,g\cdot(iE_n,0))\,J_{\mathcal{M},\rho}(\gamma,g\cdot(iE_n,0))^{-1}\\ &\quad J_{\mathcal{M},\rho}(g,(iE_n,0))^{-1}\Phi(g)\\ &= J_{\mathcal{M},\rho}(g,(iE_n,0))^{-1}\Phi(g)\\ &= f_\Phi(g\cdot(iE_n,0)) = f_\Phi(Z,W).\end{aligned}$$

This completes the proof. □

Now we have the following two algebraic representations $T_{\rho,\mathcal{M}}$ and $\dot{T}_{\rho,\mathcal{M}}$ of G^J defined by

$$T_{\rho,\mathcal{M}}(g)f := f|_{\rho,\mathcal{M}}[g^{-1}], \quad g \in G^J,\ f \in \mathcal{F}^{\Gamma}_{\rho,\mathcal{M}} \tag{9.35}$$

and

$$\dot{T}_{\rho,\mathcal{M}}(g)\Phi(g') := \Phi(g^{-1}g'), \quad g,\, g' \in G^J, \quad \Phi \in \mathcal{F}^{\Gamma}_{\rho,\mathcal{M}}(G^J). \tag{9.36}$$

Then it is easy to see that these two models $T_{\rho,\mathcal{M}}$ and $\dot{T}_{\rho,\mathcal{M}}$ are intertwined by the lifting $\varphi_{\rho,\mathcal{M}}$.

Proposition 9.11. The vector space $J_{\rho,\mathcal{M}}(\Gamma_n)$ is isomorphic to the space $A_{\rho,\mathcal{M}}(\Gamma^J)$ of smooth functions Φ on G^J with values in V_ρ satisfying the following conditions:

(1a) $\Phi(\gamma g) = \Phi(g)$ for all $\gamma \in \Gamma^J$.
(1b) $\Phi(gr(k,\kappa)) = e^{2\pi i \sigma(\mathcal{M}\kappa)}\rho(k)^{-1}\Phi(g)$ for all $g \in G^J,\ r(k,\kappa) \in K^J$.
(2) $X^-_{ij}\Phi = Y^-_{ij}\Phi = 0, \quad 1 \le i, j \le n$.
(3) For all $M \in Sp(n,\mathbb{R})$, the function $\psi : G^J \longrightarrow V_\rho$ defined by

$$\psi(g) := \rho(Y^{-\frac{1}{2}})\,\Phi(Mg), \quad g \in G^J$$

is bounded in the domain $Y \ge Y_0$. Here $g\cdot(iE_n, 0) = (Z, W)$ with $Z = X + iY,\ Y > 0$.

Corollary 9.12. $J^{\mathrm{cusp}}_{\rho,\mathcal{M}}(\Gamma_n)$ is isomorphic to the subspace $A^0_{\rho,\mathcal{M}}(\Gamma^J)$ of $A_{\rho,\mathcal{M}}(\Gamma^J)$ with the condition (3') the function $g \longmapsto \Phi(g)$ is bounded.

9.5. Unitary Representations of the Jacobi Group G^J

In this section, we review some results of Takase (cf. [82]-[84]) on the unitary representations of the Jacobi group G^J. We follow the notations in the previous sections.

First we observe that G^J is not reductive because the center of G^J is given by

$$\mathcal{Z} = \left\{ [E_{2n}, (0,0;\kappa)] \in G^J \,\middle|\, \kappa = {}^t\kappa \in \mathbb{R}^{(m,m)} \right\} \cong \mathrm{Sym}^2(\mathbb{R}^m).$$

Let $d_K(k)$ be a normalized Haar measure on K so that $\int_K d_K(k) = 1$ and $d_{\mathcal{Z}}(\kappa) = \prod_{i \le j} d\kappa_{ij}$ a Haar measure on $\mathcal{Z}$. We let $d_{K^J} = d_K \times d_{\mathcal{Z}}$ be the product measure on $K^J = K \times \mathcal{Z}$. The Haar measure d_{G^J} on G^J is normalized so that

$$\int_{G^J} f(g) d_{G^J}(g) = \int_{G^J/K^J} \left(\int_{K^J} f(gh) d_{K^J}(h) \right) d_{G^J/K^J}(\dot{g})$$

for all $f \in C_c(G^J)$.

From now on, we will fix a real positive definite symmetric matrix $S \in \mathrm{Sym}^2(\mathbb{R}^m)$ of degree m. For any fixed $Z = X + iY \in H_n$, we define a measure $\nu_{S,Z}$ on $\mathbb{C}^{(m,n)}$ by

$$d\nu_{S,Z}(W) = (\det 2S)^n\,(\det Y)^{-m}\,\kappa_S(Z,W)\,dU dV, \tag{9.37}$$

where $W = U + iV \in \mathbb{C}^{(m,n)}$ with $U, V \in \mathbb{R}^{(m,n)}$ and

$$\kappa_S(Z, W) = e^{-4\pi\sigma({}^tVSVY^{-1})}. \tag{9.38}$$

Let $H_{S,Z}$ be the complex Hilbert space consisting of all $\mathbb{C}$-valued holomorphic functions φ on $\mathbb{C}^{(m,n)}$ such that $\int_{\mathbb{C}^{(m,n)}} |\varphi(W)|^2 d\nu_{S,Z} < +\infty$. The inner product on $H_{S,Z}$ is given by

$$(\varphi, \psi) = \int_{\mathbb{C}^{(m,n)}} \varphi(W)\overline{\psi(W)} d\nu_{S,Z}(W), \qquad \varphi,\, \psi \in H_{S,Z}.$$

We put

$$\eta_S = J_{S,\delta}^{-1} \quad \text{(see subsection 9.4)},$$

where δ denotes the trivial representation of $GL(n, \mathbb{C})$. Now we define a unitary representation $\Xi_{S,Z}$ of $H_{\mathbb{R}}^{(n,m)}$ by

$$\left(\Xi_{S,Z}(h)\varphi\right)(W) = \eta_S(h^{-1}, (Z, W))^{-1} \cdot \varphi(W - \lambda Z - \mu), \tag{9.39}$$

where $h = (\lambda, \mu, \kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $\varphi \in H_{S,Z}$. It is easy to see that $(\Xi_{S,Z}, H_{S,Z})$ is irreducible and $\Xi_{S,Z}(0, 0, \kappa) = e^{-2\pi i\sigma(S\kappa)}$.

Let

$$\mathcal{X} = \left\{ T \in \mathbb{C}^{(n,n)} \,\middle|\, T = {}^tT, \ \ \mathrm{Re}\, T > 0 \right\}$$

be a connected simply connected open subset of $\mathbb{C}^{(n,n)}$. Then there exists uniquely a holomorphic function $\det^{\frac{1}{2}}$ on $\mathcal{X}$ such that

$$\begin{aligned} &(1) \qquad \left(\det^{\frac{1}{2}} T\right)^2 = \det T \qquad \text{for all } T \in T \in \mathcal{X}, \\ &(2) \qquad \det^{\frac{1}{2}} T = (\det T)^{\frac{1}{2}} \qquad \text{for all } T \in \mathcal{X} \cap \mathbb{R}^{(n,n)}. \end{aligned}$$

For any integer $k \in \mathbb{Z}$, we set

$$\det^{\frac{k}{2}} T = \left(\det^{\frac{1}{2}} T\right)^k, \quad T \in \mathcal{X}. \tag{9.40}$$

For any $g = (\sigma, h) \in G^J$ with $\sigma \in G$, we define an integral operator $T_{S,Z}(g)$ from $H_{S,Z}$ to $H_{S,\sigma<Z>}$ by

$$\left(T_{S,Z}(g)\varphi\right)(W) = \eta_S(g^{-1}, (Z, W))^{-1}\varphi(W'), \tag{9.41}$$

where $(Z', W') = g^{-1} \cdot (Z, W)$ and $\varphi \in H_{S,Z}$. And for any fixed Z and Z' in H_n, we define a unitary mapping

$$U_{Z',Z}^S : H_{S,Z} \longrightarrow H_{S,Z'} \tag{9.42}$$

by

$$\left(U^S_{Z',Z}\varphi\right)(W') = \gamma(Z',Z)^m \cdot \int_{\mathbb{C}^{(m,n)}} \kappa_S((Z',W'),(Z,W))^{-1}\varphi(W)\,d\nu_{S,Z}(W),$$

where

$$\gamma(Z',Z) = \det{}^{-\frac{1}{2}}\left(\frac{Z'-\bar{Z}}{2i}\right)\cdot \det(\operatorname{Im} Z')^{\frac{1}{4}} \cdot \det(\operatorname{Im} Z)^{\frac{1}{4}}$$

and

$$\kappa_S((Z',W'),(Z,W)) = e^{2\pi i\sigma(S[W'-\overline{W}]\cdot(Z'-\overline{Z})^{-1})}.$$

For any $g=(\sigma,h)\in G^J$, we define a unitary operator $T_S(g)$ of H_{S,iE_n} by

$$T_S(g) = T_{S,\sigma^{-1}<iE_n>}(g)\circ U^S_{\sigma^{-1}<iE_n>,iE_n}. \tag{9.43}$$

We put, for any $\sigma_1,\sigma_2\in G$,

$$\beta(\sigma_1,\sigma_2) = \frac{\gamma(\sigma_1^{-1}<iE_n>,iE_n)}{\gamma(\sigma_2^{-1}\sigma_1^{-1}<iE_n>,\sigma_1^{-1}<iE_n>)}\quad . \tag{9.44}$$

Then the function $\beta(\sigma_1,\sigma_2)$ satisfies the cocycle condition

$$\beta(\sigma_2,\sigma_3)\beta(\sigma_1\sigma_2,\sigma_3)^{-1}\beta(\sigma_1,\sigma_2\sigma_3)\beta(\sigma_1,\sigma_2)^{-1}=1$$

for all $\sigma_1,\sigma_2,\sigma_3\in G$. Thus $\beta(\sigma_1,\sigma_2)$ defines a group extension $G\ltimes\mathbb{C}_1$ by $\mathbb{C}_1=\{z\in\mathbb{C}\,|\,|z|=1\,\}$. Precisely, $G\ltimes\mathbb{C}_1$ is a topological group with multiplication

$$(\sigma_1,\epsilon_1)\cdot(\sigma_2,\epsilon_2) = (\sigma_1\sigma_2,\beta(\sigma_1,\sigma_2)\epsilon_1\epsilon_2)$$

for all $(\sigma_i,\epsilon_i)\in G\times\mathbb{C}_1$ $(i=1,2)$. If we put

$$\epsilon(\sigma) = \frac{\det J(\sigma^{-1},iE_n)}{|\det J(\sigma^{-1},iE_n)|}\quad ,\qquad \sigma\in G, \tag{9.45}$$

then we have the relation

$$\beta(\sigma_1,\sigma_2)^2 = \epsilon(\sigma_1)\cdot\epsilon(\sigma_1\sigma_2)^{-1}\cdot\epsilon(\sigma_2),\quad \sigma_1,\,\sigma_2\in G.$$

Therefore we have a closed normal subgroup

$$G_2 = \left\{(\sigma,\epsilon)\in G\ltimes\mathbb{C}_1\,\middle|\,\epsilon^2=\epsilon(\sigma)^{-1}\right\} \tag{9.46}$$

of $G_2\ltimes\mathbb{C}_1$ which is a connected two-fold covering group of G. Since G_2 acts on the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$ via the projection $p: G_2\longrightarrow G$, we may put

$$G_2^J = G_2\ltimes H_{\mathbb{R}}^{(n,m)}.$$

Now we define the unitary representation ω_S of G_2^J by

$$\omega_S(g) = \epsilon^m \cdot T_S(\sigma, h), \quad g = ((\sigma, \epsilon), h) \in G_2^J. \tag{9.47}$$

It is easy to see that (ω_S, H_{S,iE_n}) is irreducible and the restriction of ω_S to G_2 is the m-fold tensor product of the Weil representation. ω_S is called the *Weil representation* of the Jacobi group G^J.

We set

$$\begin{aligned} &p : G_2 \longrightarrow G, \quad p(\sigma, \epsilon) = \sigma, \\ &p^J : G_2^J \longrightarrow G^J, \quad p^J((\sigma, \epsilon), h) = (\sigma, h), \\ &q : G^J \longrightarrow G, \quad q(\sigma, h) = \sigma, \\ &q^J : G_2^J \longrightarrow G_2, \quad q^J((\sigma, \epsilon), h) = (\sigma, \epsilon). \end{aligned}$$

Proposition 9.13. *Let χ_S be the character of $\mathcal{Z} \cong Sym^2(\mathbb{R}^m)$ defined by $\chi_S(\kappa) = e^{2\pi i \sigma(S\kappa)}$, $\kappa \in \mathcal{Z}$. We denote by $\hat{G}_2^J(\bar{\chi}_S)$ the set of all equivalence classes of irreducible unitary representations τ of G_2^J such that $\tau(\kappa) = \chi_S(\kappa)^{-1}$ for all $\kappa \in \mathcal{Z}$. We put $\tilde{\pi} = \pi \circ q^J \in \hat{G}_2^J$ for any $\pi \in \hat{G}_2$. The correspondence*

$$\pi \longmapsto \tilde{\pi} \otimes \omega_S$$

is a bijection from $\hat{G}_2$ to $\hat{G}_2^J(\bar{\chi}_S)$. And $\tilde{\pi} \otimes \omega_S$ is square-integrable modulo $\mathcal{Z}$ if and only if π is square integrable.

Proof. See [82], Proposition 11.8. □

Proposition 9.14. *Let m be even. We put $\check{\pi} = \pi \circ q \in \hat{G}^J$ for any $\pi \in \hat{G}$. Then the correspondence*

$$\pi \longmapsto \check{\pi} \otimes \omega_S$$

is a bijection of $\hat{G}$ to $\hat{G}^J$. And $\check{\pi} \otimes \omega_S$ is square integrable modulo A if and only if π is square integrable.

Proof. See [82]. □

The above proposition was proved by Satake [75] or by Berndt [6] in the case $m = 1$.

Let (ρ, V_ρ) be an irreducible representation of $K = U(n)$ with highest weight $l = (l_1, l_2, \cdots, l_n) \in \mathbb{Z}^n$, $l_1 \geq \cdots \geq l_n \geq 0$. Then ρ is extended to a rational representation of $GL(n, \mathbb{C})$ which is also denoted by ρ. The representation space V_ρ of ρ has an hermitian inner product $(\ , \ )$ such that $(\rho(g)u, v) = (u, \rho(g^*)v)$ for all $g \in GL(n, \mathbb{C})$, $u, v \in V_\rho$, where $g^* = {}^t\bar{g}$. We let the mapping $J : G \times H_n \longrightarrow GL(n, \mathbb{C})$ be the automorphic factor defined by

$$J(\sigma, Z) = CZ + D, \quad \sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G.$$

We define a unitary representation τ_l of K by

$$\tau_l(k) = \rho(J(k, iE_n)), \quad k \in K. \tag{9.48}$$

We set $J_{\rho,S} = J_{S,\rho}^{-1}$ (cf. subsection 9.4). According to the definition, we have

$$J_{\rho,S}(g,(Z,W)) = \eta_S(g,(Z,W))\, \rho(J(\sigma,Z))$$

for all $g = (\sigma, h) \in G^J$ and $(Z,W) \in H_{n,m}$. For any $g = (\sigma, h) \in G^J$ and $(Z,W) \in H_{n,m}$, we set

$$\begin{aligned}
\overline{J_{\rho,S}(g,(Z,W))} &= \overline{\eta_S(g,(Z,W))}\, \rho(\overline{J(\sigma,Z)}), \\
{}^t J_{\rho,S}(g,(Z,W)) &= \eta_S(g,(Z,W))\, \rho({}^t J(\sigma,Z)), \\
J_{\rho,S}(g,(Z,W))^* &= {}^t\overline{J_{\rho,S}(g,(Z,W))}.
\end{aligned}$$

Then for all $g \in G^J$, $(Z,W) \in H_{n,m}$ and $u, v \in V_\rho$, we have

$$(J_{\rho,S}(g,(Z,W))u, v) = (u, J_{\rho,S}(g,(Z,W))^* v)$$

We denote by $E(\rho, S)$ the Hilbert space consisting of V_ρ-valued measurable functions φ on $H_{n,m}$ such that

$$|\varphi|^2 = \int_{H_{n,m}} (\rho(Im\, Z)\, \varphi(Z,W), \varphi(Z,W))\ \kappa_S(Z,W)\, d(Z,W),$$

where

$$d(Z,W) = (\det Y)^{-(m+n+1)} dX\, dY\, dU\, dV, \quad Z = X + iY,\ W = U + iV$$

denotes a G^J-invariant volume element on $H_{n,m}$. The induced representation $\mathrm{Ind}_{K^J}^{G^J}(\rho \otimes \bar{\chi}_S)$ is realized on $E(\rho, S)$ as follows: For any $g \in G^J$ and $\varphi \in E(\rho, S)$, we have

$$\left(\mathrm{Ind}_{K^J}^{G^J}(\rho \otimes \bar{\chi}_S)(g)\varphi\right)(Z,W) = J_{\rho,S}(g^{-1},(Z,W))^{-1} \varphi(g^{-1} \cdot (Z,W)).$$

We recall that χ_S is the unitary character of A defined by $\chi_S(\kappa) = e^{2\pi i \sigma(S\kappa)}$, $\kappa \in \mathcal{Z}$. Let $H(\rho, S)$ be the subspace of $E(\rho, S)$ consisting of $\varphi \in E(\rho, S)$ which is holomorphic on $H_{n,m}$. Then $H(\rho, S)$ is a closed G^J-invariant subspace of $E(\rho, S)$. Let $\pi^{\rho,S}$ be the restriction of the induced representation $\mathrm{Ind}_{K^J}^{G^J}(\rho \otimes \bar{\chi}_S)$ to $H(\rho, S)$.

Takase (cf. [83], Theorem 1.1) proved the following

Theorem 9.15. *Suppose $l_n > n + \frac{m}{2}$. Then $H(\rho, S) \neq 0$ and $\pi^{\rho,S}$ is an irreducible unitary representation of G^J which is square integrable modulo $\mathcal{Z}$. The multiplicity of ρ_l in $\pi^{\rho,S}|_K$ is equal to one.*

We put

$$K_2 = p^{-1}(K) = \left\{ (k,\epsilon) \in K \times \mathbb{C}_1 \,\middle|\, \epsilon^2 = \det J(k, iE_n) \right\}.$$

The Lie algebra $\mathfrak{k}$ of K_2 and its Cartan algebra are given by

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \,\middle|\, A + {}^tA = 0,\ B = {}^tB \right\}$$

and

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix} \in \mathbb{R}^{(2n,2n)} \,\middle|\, C = \operatorname{diag}(c_1, c_2, \cdots, c_n) \right\}.$$

Here $\operatorname{diag}(c_1, c_2, \cdots, c_n)$ denotes the diagonal matrix of degree n. We define $\lambda_j \in \mathfrak{h}_{\mathbb{C}}^*$ by $\lambda_j \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix} := \sqrt{-1}c_j$. We put

$$M^+ = \left\{ \sum_{j=1}^n m_j \lambda_j \,\middle|\, m_j \in \frac{1}{2}\mathbb{Z},\ m_1 \geq \cdots \geq m_n,\ m_i - m_j \in \mathbb{Z} \text{ for all } i,j \right\}.$$

We take an element $\lambda = \sum_{j=1}^n m_j \lambda_j \in M^+$. Let ρ be an irreducible representation of K with highest weight $l = (l_1, \cdots, l_n) \in \mathbb{Z}^n$, where $l_j = m_j - m_n \, (1 \leq j \leq n-1)$. Let $\rho_{[\lambda]}$ be the irreducible representation of K_2 defined by

$$\rho_{[\lambda]}(k,\epsilon) = \epsilon^{2m_n} \cdot \rho(J(k, iE_n)), \quad (k,\epsilon) \in K_2. \tag{9.49}$$

Then $\rho_{[\lambda]}$ is the irreducible representation of K_2 with highest weight $\lambda = (m_1, \cdots, m_n)$ and $\lambda \longmapsto \rho_{[\lambda]}$ is a bijection from M^+ to $\hat{K}_2$, the unitary dual of K_2.

The following proposition is a special case of [44], Theorem 7.2.

Proposition 9.16. *We have an irreducible decomposition*

$$\omega_S \Big|_{K_2} = \oplus_\lambda \, m(\lambda) \rho_{[\lambda]},$$

where λ runs over

$$\lambda = \sum_{j=1}^{\nu} l_j \lambda_j + \frac{m}{2} \sum_{j=1}^{n} \lambda_j \in M^+ \quad (\nu = min\{m,n\}),$$

$$\lambda_j \in \mathbb{Z} \text{ such that } l_1 \geq l_2 \cdots \geq l_\nu \geq 0$$

and the multiplicity $m(\lambda)$ is given by

$$m(\lambda) = \prod_{1 \leq i < j \leq m} \left(1 + \frac{l_i - l_j}{j - i} \right),$$

where $l_j = 0$ *if* $j > \nu$. *Let* $\hat{G}_{2,d}$ *be the set of all the unitary equivalence classes of square integrable irreducible unitary representations of* G_2. *The correspondence*

$$\pi \longmapsto \textit{Harish-Chandra parameter of } \pi$$

is a bijection from $\hat{G}_{2,d}$ *to* Λ^+, *where*

$$\Lambda^+ = \left\{ \sum_{j=1}^{n} m_j \lambda_j \in M^+ \;\middle|\; m_1 > \cdots > m_n,\ m_i - m_j \neq 0 \textit{ for all } i,j,\ i \neq j \right\}.$$

See [95], Theorem 10.2.4.1 for the details.

We take an element $\lambda = \sum_{j=1}^{n} m_j \lambda_j \in M^+$. Let $\pi^\lambda \in \hat{G}_{2,d}$ be the representation corresponding to the Harish-Chandra parameter

$$\sum_{j=1}^{n} (m_j - j)\lambda_j \in \Lambda^+.$$

The representation π^λ is realized as follows (see [56], Theorem 6.6): Let (ρ, V_ρ) be the irreducible representation of K with highest weight $l = (l_1, \cdots, l_n)$, $l_i = m_i - m_n$ ($1 \leq j \leq n$). Let H^λ be a complex Hilbert space consisting of the V_ρ-valued holomorphic functions φ on H_n such that

$$|\varphi|^2 = \int_{H_n} (\rho(\mathrm{Im}\, Z)\, \varphi(Z), \varphi(Z)) \cdot (\det \mathrm{Im}\, Z)^{m_n}\, dZ < +\infty,$$

where dZ is the usual G_2-invariant measure on H_n. Then π^λ is defined by

$$\left(\pi^\lambda(g)\varphi\right)(Z) = J_\lambda(g^{-1}, Z)^{-1} \varphi(g^{-1} < Z >)$$

for all $g = (\sigma, \epsilon) \in G_2$ and $\varphi \in H^\lambda$. Here

$$J_\lambda(g, Z) = \rho(J(\sigma, Z)) \cdot J_{\frac{1}{2}}(g, Z)^{m_n},$$

where

$$J_{\frac{1}{2}}(g, Z) = \frac{\gamma(\sigma < Z >, \sigma < iE_n >)}{\gamma(Z, iE_n)} \cdot \beta(\sigma, \sigma^{-1}) \cdot \epsilon \cdot |det\, J(\sigma, Z)|^{\frac{1}{2}}\ .$$

Proposition 9.17. *Suppose* $l_n > n + \frac{m}{2}$. *We put* $\lambda = \sum_{j=1}^{n} (l_j - \frac{m}{2})\lambda_j \in M^+$. *Then* $\pi^{\rho,S}$ *is an irreducible unitary representation of* G^J *and we have a unitary equivalence*

$$(\pi^\lambda \circ q^J) \otimes \omega_S \longrightarrow \pi^{\rho,S} \circ p^J$$

via the intertwining operator $\Lambda_{\rho,S} : H^\lambda \otimes H_{S,iE_n} \longrightarrow H(\rho, S)$ *defined by*

$$(\Lambda_{\rho,S}(\varphi \otimes \psi))(Z, W) = (\, det 2S\,)^n (\, det\, Im\, Z\,)^{-\frac{m}{4}} \varphi(Z)(U^S_{Z,iE_n}\psi)(W)$$

for all $\varphi \in H^\lambda$ *and* $\psi \in H_{S,iE_n}$.

9.6. Duality Theorem for G^J

In this subsection, we state the duality theorem for the Jacobi group G^J.

Let E_{ij} denote a square matrix of degree $2n$ with entry 1 where the i-th row and the j-th column meet, all other entries being 0. We put

$$H_i = E_{ii} - E_{n+i,n+i}\,(1 \le i \le n), \quad \mathfrak{h} = \sum_{i=1}^{n} \mathbb{C} H_i.$$

Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $e_j : \mathfrak{h} \longrightarrow \mathbb{C}\,(1 \le j \le n)$ be the linear form on $\mathfrak{h}$ defined by

$$e_j(H_i) = \delta_{ij},$$

where δ_{ij} denotes the Kronecker delta symbol. The roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ are given by

$$\pm 2e_i\ (1 \le i \le n),\ \pm e_k \pm e_l\,(1 \le k < l \le n).$$

The set Φ^+ of positive roots is given by

$$\Phi^+ = \{2e_i\,(1 \le i \le n),\ e_k + e_l\,(\,1 \le k < l \le n\,)\}.$$

Let

$$\mathfrak{g}_\alpha = \{\, X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\,\}$$

be the root space corresponding to a root α of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We put $\mathfrak{n} = \sum_{\Phi^+} \mathfrak{g}_\alpha$. We define

$$N^J = \left\{ [\exp X, (0, \mu, 0)] \in G^J \,\middle|\, X \in \mathfrak{n} \right\},$$

where $\exp : \mathfrak{g} \longrightarrow G$ denotes the exponential mapping from $\mathfrak{g}$ to G. A subgroup N^g of G^J is said to be *horosherical* if it is conjugate to N^J, that is, $N^g = gN^Jg^{-1}$ for some $g \in G$. A horospherical subgroup N^g is said to be *cuspidal* for $\Gamma^J = \Gamma_n \ltimes H_{\mathbb{Z}}^{(n,m)}$ in G^J if $(N^g \cap \Gamma^J)\backslash N^g$ is compact. Let $L^2(\Gamma^J\backslash G^J, \rho)$ be the complex Hilbert space consisting of all Γ^J-invariant V_ρ-valued measurable functions Φ on G^J such that $||\Phi|| < \infty$, where $||\ \ ||$ is the norm induced from the norm $|\ \ |$ on $E(\rho, \mathcal{M})$ by the lifting from $H_{n,m}$ to G^J. We denote by $L_0^2(\Gamma^J\backslash G^J, \rho)$ the subspace of $L^2(\Gamma^J\backslash G^J, \rho)$ consisting of functions φ on G^J such that $\varphi \in L^2(\Gamma^J\backslash G^J, \rho)$ and

$$\int_{N^g \cap \Gamma^J \backslash N^g} \varphi(ng_0)dn = 0$$

for any cuspidal subgroup N^g of G^J and almost all $g_0 \in G^J$. Let R be the right regular representation of G^J on $L_0^2(\Gamma^J\backslash G^J, \rho)$.

Now we state the duality theorem for the Jacobi group G^J.

Duality Theorem. *Let ρ be an irreducible representation of K with highest weight $l=(l_1,\cdots,l_n)\in\mathbb{Z}^n$, $l_1\geq l_2\geq\cdots\geq l_n$. Suppose $l_n>n+\frac{1}{2}$ and let $\mathcal{M}$ be a half integrable positive definite symmetric matrix of degree m. Then the multiplicity $m_{\rho,\mathcal{M}}$ of $\pi^{\rho,\mathcal{M}}$ in the right regular representation R of G^J in $L^2_0(\Gamma^J\backslash G^J,\rho)$ is equal to the dimension of $J^{cusp}_{\rho,\mathcal{M}}(\Gamma_n)$, that is,*

$$m_{\rho,\mathcal{M}}=\dim_{\mathbb{C}} J^{\mathrm{cusp}}_{\rho,\mathcal{M}}(\Gamma_n).$$

We may prove the above theorem following the argument of [10] in the case $m=n=1$. So we omit the detail of the proof.

9.7. Coadjoint Orbits for the Jacobi Group G^J

We observe that the Jacobi group G^J is embedded in $Sp(n+m,\mathbb{R})$ via

$$(M,(\lambda,\mu,\kappa))\mapsto\begin{pmatrix}A & 0 & B & A^t\mu-B^t\lambda\\ \lambda & E_m & \mu & \kappa\\ C & 0 & D & C^t\mu-D^t\lambda\\ 0 & 0 & 0 & E_m\end{pmatrix}, \tag{9.50}$$

where $(M,(\lambda,\mu,\kappa))\in G^J$ with $M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in Sp(n,\mathbb{R})$. The Lie algebra $\mathfrak{g}^J$ of G^J is given by

$$\mathfrak{g}^J=\Big\{(X,(P,Q,R))\mid X\in\mathfrak{g},\ P,Q\in\mathbb{R}^{(m,n)},\ R={}^tR\in\mathbb{R}^{(m,m)}\Big\} \tag{9.51}$$

with the bracket

$$[(X_1,(P_1,Q_1,R_1)),\,(X_2,(P_2,Q_2,R_2))]=(\tilde{X},(\tilde{P},\tilde{Q},\tilde{R})), \tag{9.52}$$

where

$$X_1=\begin{pmatrix}a_1 & b_1\\ c_1 & -{}^ta_1\end{pmatrix},\quad X_2=\begin{pmatrix}a_2 & b_2\\ c_2 & -{}^ta_2\end{pmatrix}\in\mathfrak{g}$$

and

$$\begin{aligned}\tilde{X}&=X_1X_2-X_2X_1,\\ \tilde{P}&=P_1a_2+Q_1c_2-P_2a_1-Q_2c_1,\\ \tilde{Q}&=P_1b_2-Q_1\,{}^ta_2-P_2b_1+Q_2\,{}^ta_1,\\ \tilde{R}&=P_1\,{}^tQ_2-Q_1\,{}^tP_2-P_2\,{}^tQ_1+Q_2\,{}^tP_1.\end{aligned}$$

Indeed, an element $(X,(P,Q,R))$ in $\mathfrak{g}^J$ with $X=\begin{pmatrix}a & b\\ c & -{}^ta\end{pmatrix}\in\mathfrak{g}$ may be identified with the matrix

$$\begin{pmatrix}a & 0 & b & {}^tQ\\ P & 0 & Q & R\\ c & 0 & -{}^ta & -{}^tP\\ 0 & 0 & 0 & 0\end{pmatrix},\quad b={}^tb,\ c={}^tc,\ R={}^tR \tag{9.53}$$

in $\mathfrak{sp}(n+m,\mathbb{R})$.

Let us identify $\mathfrak{g}_{n+m} := \mathfrak{sp}(n+m,\mathbb{R})$ with its dual $\mathfrak{g}^*_{n+m}$ (see Proposition 6.1.3. (6.5)). In fact, there exists a G-equivariant linear isomorphism

$$\mathfrak{g}^*_{n+m} \longrightarrow \mathfrak{g}_{n+m}, \quad \lambda \mapsto X_\lambda$$

characterized by

$$\lambda(Y) = \mathrm{tr}(X_\lambda Y), \quad Y \in \mathfrak{g}_{n+m}. \tag{9.54}$$

Then the dual $(\mathfrak{g}^J)^*$ of $\mathfrak{g}^J$ consists of matrices of the form

$$\begin{pmatrix} x & p & y & 0 \\ 0 & 0 & 0 & 0 \\ z & q & -\,{}^t x & 0 \\ {}^t q & r & -{}^t p & 0 \end{pmatrix}, \quad y = {}^t y,\ z = {}^t z,\ r = {}^t r. \tag{9.55}$$

There is a family of coadjoint orbits Ω_δ which have the minimal dimension $2n$, depending on a nonsingular $m \times m$ real symmetric matrix parameter δ and are defined by the equation

$$\delta = r, \quad XJ_n = \begin{pmatrix} p \\ q \end{pmatrix} \delta^{-1}\, {}^t\!\begin{pmatrix} p \\ q \end{pmatrix}, \tag{9.56}$$

where $X = \begin{pmatrix} x & y \\ z & -\,{}^t x \end{pmatrix}$ with $y = {}^t y$ and $z = {}^t z$ in (9.55). Let us denote by $\mathfrak{h}_{n,m}$ the Lie algebra of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$. Then the family Ω_δ ($\delta = {}^t\delta,\ \delta \in GL(m,\mathbb{R})$) have the following properties ($\Omega1$)-($\Omega2$):

($\Omega1$) Under the natural projection on $\mathfrak{h}^*_{n,m}$, the orbit Ω_δ goes to the orbit which corresponds to the irreducible unitary representation $U(\delta)$ of the Heisenberg group $H_{\mathbb{R}}^{(n,m)}$, namely, the Schrödinger representation of $H_{\mathbb{R}}^{(n,m)}$ (cf. (8.19)).

($\Omega2$) Under the projection on $\mathfrak{g}^* = \mathfrak{sp}(n,\mathbb{R})^*$, the orbit Ω_δ goes to $\Omega_{\mathrm{sign}(\det(\delta))}$.

In fact, there is an irreducible unitary representation π_δ ($\delta = {}^t\delta,\ \delta \in GL(m,\mathbb{R})$) of G^J (or its universal cover) with properties

$$\mathrm{Res}^{G^J}_{H_{\mathbb{R}}^{(n,m)}} \pi_\delta \cong U(\delta), \quad \mathrm{Res}^{G^J}_{G} \pi_\delta \cong \pi_{\mathrm{sign}(\det(\delta))}, \tag{9.57}$$

where $\pi_\pm$ are some representations of G (or its universal cover) corresponding to the minimal orbits $\Omega_\pm \subset \mathfrak{g}^*$. Indeed, $\pi_\pm$ are two irreducible components of the Weil representation of G and π_δ is one of the irreducible components of the Weil representation of G^J (cf. (9.47)). These are special cases of the so-called *unipotent* representations of G^J. We refer to [90]-[91], [93] for a more detail on unipotent representations of a reductive Lie group.

Now we consider the case $m = n = 1$. If

$$g^{-1} = \begin{pmatrix} a & 0 & b & a\mu - b\lambda \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & c\mu - d\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is an element of the Jacobi group G^J, then its inverse is given by

$$g = \begin{pmatrix} d & 0 & -b & -\mu \\ c\mu - d\lambda & 1 & \lambda b - \mu a & -\kappa \\ -c & 0 & a & \lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We put

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then according to (9.55), X, Y, Z, P, Q, R form a basis for $(\mathfrak{g}^J)^*$. By an easy computation, we see that the coadjoint orbits Ω_X, Ω_Y, Ω_Z, Ω_P, Ω_Q, Ω_R of X, Y, Z, P, Q, R respectively are given by

$$\Omega_X = \left\{ \begin{pmatrix} ad+bc & 0 & -2ab & 0 \\ 0 & 0 & 0 & 0 \\ 2cd & 0 & -(ad+bc) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; ad - bc = 1,\ a, b, c, d \in \mathbb{R} \right\},$$

$$\Omega_Y = \left\{ \begin{pmatrix} bd-ac & 0 & a^2-b^2 & 0 \\ 0 & 0 & 0 & 0 \\ d^2-c^2 & 0 & ac-bd & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; ad - bc = 1,\ a, b, c, d \in \mathbb{R} \right\},$$

$$\Omega_Z = \left\{ \begin{pmatrix} -(ac+bd) & 0 & a^2+b^2 & 0 \\ 0 & 0 & 0 & 0 \\ -(c^2+d^2) & 0 & ac+bd & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; ad - bc = 1,\ a, b, c, d \in \mathbb{R} \right\},$$

$$\Omega_P = \left\{ \begin{pmatrix} (2ad-1)\lambda - 2ac\mu & a & 2ab\lambda - 2a^2\mu & 0 \\ 0 & 0 & 0 & 0 \\ 2c^2\mu - 2cd\lambda & c & (1-2ad)\lambda + 2ac\mu & 0 \\ c & 0 & -a & 0 \end{pmatrix} \;\middle|\; \begin{matrix} ad - bc = 1, \\ a, b, c, d, \lambda, \mu \in \mathbb{R} \end{matrix} \right\},$$

$$\Omega_Q = \left\{ \begin{pmatrix} (2ad-1)\mu - 2bd\lambda & b & 2b^2\lambda - 2ab\mu & 0 \\ 0 & 0 & 0 & 0 \\ 2cd\mu - 2d^2\lambda & d & (1-2ad)\mu + 2bd\lambda & 0 \\ d & 0 & -b & 0 \end{pmatrix} \,\middle|\, \begin{matrix} ad - bc = 1, \\ a, b, c, d, \lambda, \mu \in \mathbb{R} \end{matrix} \right\}$$

and

$$\Omega_R = \left\{ \begin{pmatrix} (a\mu - b\lambda)(c\mu - d\lambda) & a\mu - b\lambda & -(a\mu - b\lambda)^2 & 0 \\ 0 & 0 & 0 & 0 \\ (c\mu - d\lambda)^2 & c\mu - d\lambda & -(a\mu - b\lambda)(c\mu - d\lambda) & 0 \\ c\mu - d\lambda & 1 & b\lambda - a\mu & 0 \end{pmatrix} \,\middle|\, \begin{matrix} ad - bc = 1, \\ a, b, c, d, \lambda, \mu \in \mathbb{R} \end{matrix} \right\}.$$

Moreover we put

$$S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the coadjoint orbits Ω_S and Ω_T of S and T are given by

$$\Omega_S = \left\{ \begin{pmatrix} -ab & 0 & a^2 & 0 \\ 0 & 0 & 0 & 0 \\ -b^2 & 0 & ab & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$$

and

$$\Omega_T = \left\{ \begin{pmatrix} ab & 0 & -a^2 & 0 \\ 0 & 0 & 0 & 0 \\ b^2 & 0 & -ab & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}.$$

For an element of $(\mathfrak{g}^J)^*$, we write

$$\begin{pmatrix} x & p & y+z & 0 \\ 0 & 0 & 0 & 0 \\ y-z & q & -x & 0 \\ q & r & -p & 0 \end{pmatrix} = xX + yY + zZ + pP + qQ + rR. \tag{9.58}$$

The coadjoint orbit Ω_X is represented by the one-sheeted hyperboloid

$$x^2 + y^2 - z^2 = 1 > 0, \quad p = q = r = 0. \tag{9.59}$$

The coadjoint orbit Ω_Y is also represented by the one-sheeted hyperboloid (9.59). The coadjoint orbit Ω_Z is represented by the two-sheeted hyperboloids

$$x^2 + y^2 = z^2 - 1 > 0, \quad p = q = r = 0. \tag{9.60}$$

The coadjoint G^J-orbit Ω_S of S is represented by the the cone

$$x^2 + y^2 = z^2 > 0, \quad z > 0, \quad p = q = r = 0. \tag{9.61}$$

On the other hand, the coadjoint G^J-orbit Ω_T of T is represented by the cone

$$x^2 + y^2 = z^2 > 0, \quad z < 0, \quad p = q = r = 0. \tag{9.62}$$

The coadjoint orbit Ω_P is represented by the variety

$$2pqx + (q^2 - p^2)y + (p^2 + q^2)z = 0, \quad (p, q) \in \mathbb{R}^2 - \{(0,0)\}, \quad r = 0 \tag{9.63}$$

in $\mathbb{R}^6$. The coadjoint orbit orbit Ω_Q is represented by the variety (9.63) in $\mathbb{R}^6$. particular, we are interested in the coadjoint orbits orbits $\Omega_{hR}\,(h \in \mathbb{R},\ h \neq 0)$ of hR which are represented by

$$x^2 + y^2 = z^2,\ x = h^{-1}pq,\ y + z = -h^{-1}p^2,\ y - z = h^{-1}q^2 \text{ and } r = h. \tag{9.64}$$

For a fixed $h \neq 0$, we note that Ω_{hR} is two dimensional and satisfies the equation (9.56). Indeed, from the above expression of Ω_{hR} and (9.58), we have $X = \begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix}$ and

$$\begin{aligned} x &= h(a\mu - b\lambda)(c\mu - d\lambda), \\ y + z &= -h(a\mu - b\lambda)^2, \\ y - z &= h(c\mu - d\lambda)^2, \\ p &= h(a\mu - b\lambda), \quad q = h(c\mu - d\lambda), \quad r = h. \end{aligned}$$

Hence these satisfy the equation (9.56). An irreducible unitary representation π_h that corresponds to a coadjoint orbit Ω_{hR} satisfies the properties (9.57). In fact, π_h is one of the irreducible components of the so-called (Schrödinger-)Weil representation of G^J (cf. (9.47)). A coadjoint orbit $\Omega_{mR+\alpha X}$ or $\Omega_{mR+\alpha Y}\,(m \in \mathbb{R}^\times,\ \alpha \in \mathbb{R})$ is corresponded to a principal series $\pi_{m,\alpha,\frac{1}{2}}$, the coadjoint orbit $\Omega_{mR+kZ}\,(m \in \mathbb{R}^\times,\ k \in \mathbb{Z}^+)$ of $mR + kZ$ is attached to the discrete series $\pi^{\pm}_{m,k}$ of G^J. There are no coadjoint G^J-orbits which correspond to the complimentary series $\pi_{m,\alpha,\nu}\,(m \in \mathbb{R}^\times,\ \alpha \in \mathbb{R},\ \alpha^2 < \frac{1}{2},\ \nu = \pm\frac{1}{2})$. See [11], pp. 47-48. There are no unitary representations of G^J corresponding to the G^J-orbits of $\alpha P_* + \beta Q_*$ with $(\alpha, \beta) \neq (0,0)$.

Finally we mention that the coadjoint orbit $\Omega_{mR+\alpha X}$ or $\Omega_{mR+\alpha Y}\,(m \in \mathbb{R}^\times,\ \alpha \in \mathbb{R})$ is characterized by the variety

$$x^2 + y^2 - (z^2 + \alpha^2) = \frac{2}{m}pqx + \frac{1}{m}(q^2 - p^2)y + \frac{1}{m}(p^2 + q^2)z, \quad r = m. \tag{9.65}$$

and the coadjoint orbit $\Omega_{mR+kZ}(m \in \mathbb{R}^\times, k \in \mathbb{Z}^+)$ of $mR + kZ$ is represented by the variety

$$x^2 + y^2 - (z^2 - k^2) = \frac{2}{m}pqx + \frac{1}{m}(q^2 - p^2)y + \frac{1}{m}(p^2 + q^2)z,\ z > 0,\ r = m. \tag{9.66}$$

or

$$(9.67)\quad x^2 + y^2 - (z^2 - k^2) = \frac{2}{m}pqx + \frac{1}{m}(q^2 - p^2)y + \frac{1}{m}(p^2 + q^2)z,\ z < 0,\ r = m$$

depending on the sign $\pm$.

References

[1] Arthur J., *Unipotent Automorphic Representation,: Conjectures*, Société France, Astérique, **171-172**(1989), 13-71.

[2] Arnold V.I. and Givental A.B., Symplectic geometry, Encyclopaedia of Mathematical Sciences, Dynamical systems IV, Springer,**4**(1990), 1-136.

[3] Auslander L. and Kostant B.,*Polarization and unitary representations of solvable Lie groups* , Invent. Math., **14**(1971), 255-354.

[4] Bargmann V., *Irreducible unitary representations of the Lorentz group*, Ann. of Math., **48**(1947), 568-640.

[5] Bernat P., Conze N., Duflo M. et al., Représentations des groupes de Lie résolubles, Dunod Paris, (1972), 1-272.

[6] Berndt R., *Die Jacobigruppe und die Wärmeleitungsgleichung*, Math. Z., **191**(1986), 351-361.

[7] Berndt R., *Some remarks on automorphic forms for the Jacobi group*, IHES/M, (1989).

[8] Berndt R., *The Continuous Part of $L^2(\Gamma^J\backslash G^J)$ for the Jacobi Group*, Abh. Math. Sem. Univ. Hamburg., **60**(1990), 225-248.

[9] Berndt R., *On Automorphic Forms for the Jacobi Group*, Jb. d. Dt. Math.-Verein., **97**(1995), 1-18.

[10] Berndt R. and Böcherer S., *Jacobi Forms and Discrete Series Representations of the Jacobi Group*, Math. Z., **204**(1990), 13-44.

[11] Berndt R. and Schmidt R., Elements of the Representation Theory of the Jacobi Group, Birkhäuser, **163**(1998).

[12] Borel A. and Jacquet H., *Automorphic forms and automorphic representations*, Proc. Symposia in Pure Math., **XXXIII, Part 1**(1979), 189-202.

[13] Brylinski, R. and Kostant, B., *Minimal representations of E_6, and E_8 and the generalized Capelli identity*, Proc. Nat. Acad. Sci. U.S.A., **91, No.7**(1994), 2469-2472.

[14] Brylinski, R. and Kostant, B., *Minimal representations, geometric quantization, and unitarity*, Proc. Nat. Acad. Sci. U.S.A., **91, No.13**(1994), 6026-6029.

[15] MathbbRown I.D., *Dual topology of a nilpotent Lie group*, Ann. Sci. École Sup., (1973), 407-411.

[16] Bump D., Automorphic Forms and Representations, Cambridge University Press, (1997).

[17] Busyatskaya I.K., *Representations of exponential Lie groups*, Funct. Anal. and Appl., **7, No.2**(1973), 151-152.

[18] Brylinski R.K. and Kostant B., *Minimal representations, geometric quantization and unitarity*, Proc. Natl. Acad. Sci. USA, **91**(1994), 6026-6029.

[19] Dixmier J., *Sur les représentation unitaires des groupes de Lie nilpotents, III* , Canad. J. Math., **10**(1958), 321-348.

[20] Dixmier J., Enveloping algebras, North Holland, Amsterdam, (1977).

[21] Dooley A.H. and Wildberger N.J., *Harmonic analysis and the global exponential map for compact Lie groups*, Funct. Anal. and Appl., **27**(1993), 25-32.

[22] Duflo M., *Fundamental-series representations of a semisimple Lie group*, Funct. Anal. and Appl., **4**(1970), 122-126.

[23] Duflo M., *Opérateurs différentiels bi-invariants sur un groupe de Lie*, Ann. Sci. École Norm. Sup., **IV 10 Ser. 3**(1977), 265-283.

[24] Duflo M., *Représentations unitaires irréducibles des groupes semi-simples complexes de rang deux*, Bull. Soc. Math. Fr., **107**(1979), 55-96.

[25] Duflo M., *Théorie de Mackey pour les groupes de Lie algèbriques*, Acta Math., **149**(1982), 153-213.

[26] M. Eichler and D. Zagier, The Theory of Jacobi Forms, Progress in Math., Birkhäuser, Boston-Basel- Stuttgart, **55**(1985).

[27] Enright T., Howe R. and Wallach N., *A Classification of Unitary Highest weight Modules*, in: Representation Theory of Reductive Groups, Birkhäuser, Boston (1983), 97-144.

[28] Faltings G. and Chai C.-L., Degeneration of Abelian Varieties, EMG, Band, Springer-Verlag, New York/Berlin, **22**(1990).

[29] Fell J.M.G., *Weak containment and induced representations of groups*, Canad. J. Math., **14**(1962), 237-268.

[30] Ginzburg V.A., *The method of orbits in the representation theory of complex Lie groups*, Funct. Anal. and Appl., **15**(1981), 18-28.

[31] Ginzburg V.A., $\mathfrak{g}$*-modules, Springer's representations and bivariant Chern classes*, Adv. in Math., **61**(1986) , 1-48.

[32] Gross B. and Wallach N., *A distinguished family of unitary representations for the exceptional groups of real rank=4*, in: Lie theory and geometry, in honor of B. Kostant, Birkhäuser, (1994), 289-304.

[33] Guillemin V. and Sternberg S., *Quantization and multiplicities of group representations*, Invent. Math., **67**(1982), 515-538.

[34] Guillemin V. and Sternberg S., Symplectic techniques in physics, Cambridge Univ. Press, (1994), MR 86f: 58054

[35] Harish-Chandra, *Discrete series for semisimple Lie groups II*, Acta Math., **116**(1966), 1-111.

[36] Heckman G.J., *Projections of orbits and asymptotic behavior of multiplicities for compact connected Lie groups*,Invent. Math., **67**(1983), 333-356.

[37] Howe R., *On Frobenius reciprocity for unipotent algebraic groups over* $\mathbb{Q}$, American Jour. Math., **93**(1971), 163-172.

[38] Howe R., *θ series and invariant theory* , Automorphic forms, representations, and L-functions, Proc. Symp. Pure Math., **33**(1979), 275-285.

[39] Howe R., *Reciprocity laws in the theory of dual pairs*, Representation Theory of Reductive Groups (P. Trombi, ed.), Progress in Math., Birkhäuser, Boston, **40**(1983).

[40] Howe R., *Transcending Classical Invariant Theory* , Journal American Math. Soc., **2**(1989), 535-552.

[41] Huang J.-S. and Li J.-S., *Unipotent representations attached to spherical nilpotent orbits* , Amer. Jour. Math., **121**(1999), 497-517.

[42] Huang J.-S., Pandzic P. and Savin G., *New dual pair correspondences*, Duke Math. J., **82**(1996), 447-471.

[43] Joseph A., *The minimal orbit in a simple simple Lie algemathbbRa and its associated matrix ideal* , Ann. Scient. Ec. Norm. Sup., **4**(1976), 1-30.

[44] Kashiwara M. and Vergne M., *On the Segal-Shale-Weil representations and harmonic polynomials*, Invent. Math., **44**(1978), 1-47.

[45] Kashiwara M. and Vergne M., *The Campbell-Hausdorff formula and invariant hyperfunctions*, Invent. Math., **47**(1978), 249-272.

[46] Kazhdan D. and Savin G., *The smallest representation of simply laced groups*, Israel Math. Conf. Proceedings, **2**(1990), 209-223.

[47] Kirillov A.A., *Unitary representations of nilpotent Lie groups*, Russian Math. Surveys, **17**(1962), 57-110.

[48] Kirillov A.A., Elements of the theory of representations, Springer, (1976), (Russian editions 1972, 1978).

[49] Kirillov A.A., Introduction to the theory of representations and noncommutative harmonic analysis, Encyclopaedia of Mathematical Sciences, Representation theory and noncommutative harmonic analysis I, Springer, **22**(1994), 1-156, CMP95: 06.

[50] Kirillov A.A., (Adams J., Kleppner A., Li J.-S., Lipsman R., Rosenberg J. , eds), Introduction to the orbit method. I, II., Progress in Math., **145**(1992).

[51] Kirillov A.A., *Geometric quantization*, Encyclopaedia of Mathematical Sciences, Dynamical systems, Springer, **4**(1990) (1997, the second edition), 137-172.

[52] Kirillov A.A., *Variations on the triangular theme*, AMS Translations, AMS, **169**(1995), 43-74.

[53] Kirillov A.A., *The characters of unitary representations of Lie groups*, Funct. Anal. and Appl., **2, No.2**(1968), 133-146.

[54] Kirillov A.A., *Geometric approach to Discrete Series of Unirreps for Vir*, Comm. Pures Appl. Math., **77**(1998), 735-746.

[55] Kirillov A.A., *Merits and Demerits of the Orbit Method*, Bull. AMS, **36, No.4**(1999), 433-488.

[56] Knapp, A.W., *Representation Theory of Semisimple Groups*, Princeton Univ. Press, Princeton, (1986).

[57] Kostant B., *Quantization and unitary representations*, Lecture Notes in Math., Springer, **170**(1970), 87-208.

[58] Kostant B., *Graded manifolds, graded Lie theory and prequantization*, Lecture Notes in Math., Springer, **570**(1970), 177-306.

[59] Kostant B., *On the existence and irreducibility of certain series of representations*, Lie groups and their representations, Proceedings of the Summer School of the Bolyai Janos Math. Soc., I.M. Gelfand, Editor, John Wiley and Sons, (1975), 231-331.

[60] Kostant B., *The solution to a generalized Toda lattice and representation theory*, Adv. in Math., **34**(1979), 195-338.

[61] Kramer J., *An arithmetic theory of Jacobi forms in higher dimensions*, J. reine angew. Math., **458**(1995), 157-182.

[62] Lang S., $SL_2(R)$, Springer-Verlag, (1985).

[63] Leptin H. and Ludwig J., *Unitary representation theory of exponential Lie groups*, Walter de Gruyter & Co., (1994), X + 200.

[64] Li J.-S., *The correspondence of infinitesimal characters for reductive dual pairs in simple Lie groups*, Duke Math. J., **97**(1999), 347-377.

[65] Li J.-S., Minimal Representations and Reductive Dual Pairs in Representation Theory of Lie Groups, IAS/PARK CITY, Math. Series, AMS and IAS, **8**(2000), 179-238.

[66] Loke H.Y., *Dual pairs correspondences of $E_{8,4}$ and $E_{7,4}$*, Thesis, Harvard University, (1997).

[67] Mackey G.W., *Induced Representations of Locally Compact Groups I*, Ann. of Math., **55**(1952), 101-139.

[68] Moore C.C.paper, *Decomposition of unitary representations defined by a discrete subgroup of nilpotent groups*, Ann. of Math., **82**(1965), 146-182.

[69] Namikawa Y., *Toroidal Compactification of Siegel Spaces*, Lect. Notes in Math., Springer-Verlag, New York /Berlin, **812**(1980).

[70] Pukanszky L., *On the theory of exponential groups*, Trans. Amer. Math. Soc., **126** (1967), 487-507.

[71] Pukanszky L., *Unitary representations of solvable Lie groups*, Ann. Sci. Ec. Norm. Sup., **IV, Ser. 4**(1971), 457-608.

[72] Rossmann W., *Kirillov's character formula for reductive Lie groups*, Invent. Math., **48**(1978), 207-220.

[73] Runge B., *Theta functions and Siegel-Jacobi functions*, Acta Math., **175**(1995), 165-196.

[74] I. Satake, *Fock Representations and Theta Functions*, Ann. Math. Studies, **66**(1971), 393-405.

[75] I. Satake, *Unitary representations of a semi-direct products of Lie groups on $\bar{\partial}$-cohomology spaces*, Math. Ann., **190**(1971), 177-202.

[76] Schmid W. and Vilonen K., *On the geometry of nilpotent orbits*, Asian J. Math., **3, No.1**(1999), 233-274.

[77] Schmid W. and Vilonen K., *Characteristic cycles and wave front cycles of representations of reductive Lie groups*, Ann. of Math.(2), **3, No.3**(2000), 1071-1118.

[78] Schwartz J., *The determination of the admissible orbits in the real classical groups*, Ph.D. dissertation, MIT.

[79] Sekiguchi J., *Remarks on nilpotent orbits of a symmetric pair*, Jour. Math. Soc. Japan, **39**(1987), 127-138.

[80] Shchepochkina I.M., *On representations of solvable Lie groups*, Funct. Anal. and Appl., **11**(1977), 93-94.

[81] Shchepochkina I.M, *Orbit method for the restriction-induction problems for normal subgroup of a solvable Lie group*, C.R. de l'Acad. bulgare des Sci., **33**(1980), 1039-1042.

[82] Takase K., *A note on automorphic forms*, J. reine angew. Math., **409**(1990), 138-171.

[83] Takase K., *On unitary representations of Jacobi groups*, J. reine angew. Math., **430**(1992), 130-149.

[84] Takase K., *Correction to the paper "On unitary representations of Jacobi groups"*, J. reine angew. Math., **441**(1993), 215-216.

[85] Torasso P., *Quantification géométrique, opérateurs d'entrelacement et représentations unitaires de* $\tilde{SL}_3(\mathbb{R})$, Acta Math., **150**(1983), 153-242.

[86] Torasso P., *Méthod des orbites de Kirillov-Duflo et représentations minimales des groupes simples sur un corps local de caractéristique nulle*, Duke Math. J., **90**(1997), 261-377.

[87] Vergne M., *Instantons et correspondence de Kostant-Sekiguchi*, C.R. Acad. Sci. Paris, **320**(1995), 901-906.

[88] Vogan D., *Gelfand-Kirillov dimension for Harish-Chandra modules*, Invent. Math., **48**(1978), 75-98.

[89] Vogan D., *Singular unitary representations*, Non-commutative Harmonic Analysis and Lie Groups, Springer Lecture Notes, **880**(1980), 506-534.

[90] Vogan D., *Associated Varieties and Unipotent Representations*, Harmonic Analysis on Reductive Groups, Progr. Math., Birkhäuser, Boston **101**(1991), 315-388.

[91] Vogan D., Unitary representations of reductive Lie groups, Princeton Univ. Press, (1987).

[92] Vogan D., *The unitary dual of* G_2, Invent. Math., **116**(1994), 677-791.

[93] Vogan D., The Methods of Coadjoint Orbits for Real Reductive Groups in Representation Theory of Lie Groups, IAS/PARK CITY, Math. Series, AMS and IAS, **8**(2000), 179-238.

[94] Waldspurger, J.L., *Démonstration d'une conjecture de dualité de Howe dans le cas p-adique,* $p \neq 2$, Festschrift in honor of I.I. Piatetski-Shapiro on the occasion of his sixties birthday, Part 1, Israel Math. Conf. Proc., **2**(1989), 267-324.

[95] G. Warner, Harmonic Analysis on Semisimple Lie Groups, I, II, Springer-Verlag, Berlin-Heidelberg-New York, (1972).

[96] Wildberger N.J., *On a relationship between adjoint orbits and conjugacy classes of a Lie group*, Canad. Math. Bull., **33, No.3**(1990), 297-304.

[97] Yang, J.-H., Represntation Theory of Lie Groups (Korean), Minumsa, (1998).

[98] Yang, J.-H., *Theta Functions and Heisenberg groups*, Pyungsan Institute for Math. Sci., PIMS, **99, No.5**(2000), 1-68.

[99] Yang, J.-H., *Harmonic Analysis on the Quotient Spaces of Heisenberg Groups*, Nagoya Math. J., **123**(1991), 103-117.

[100] Yang, J.-H., *Harmonic Analysis on the Quotient Spaces of Heisenberg Groups, II*, J. Number Theory, **49**(1994), 63-72.

[101] Yang, J.-H., *A decomposition theorem on differential polynomials of theta functions of high level*, Japanese J. Math., Math. Soc. Japan, New Series, **22**(1996), 37-49.

[102] Yang, J.-H., *Lattice representations of* $H_{\mathbb{R}}^{(g,h)}$, Math. Ann., **317**(2000), 309-323.

[103] Yang, J.-H., *The Siegel-Jacobi Operator*, Abh. Math. Sem. Univ. Hamburg, **63**(1993), 135-146.

[104] Yang, J.-H., *Remarks on Jacobi forms of higher degree*, Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, PIMS, (1993), 33-58.

[105] Yang, J.-H., *Singular Jacobi Forms*, Trans. Amer. Math. Soc., **347**(1995), 2041-2049.

[106] Yang, J.-H., *Construction of Vector-Valued Modular Forms from Jacobi Forms*, Canadian J. Math., **47**(1995), 1329-1339.

[107] Yang, J.-H., *A geometrical theory of Jacobi forms of higher degree*, Kyungpook Math. J., **40**(2000), 209-237.

[108] Yang J.-H. and Son J.-W., *Unitary representations of Heisenberg groups*, Proc. of Japan-Korea Joint Seminar, (1997), 143-157.

[109] Zelevinsky A.V., *Representations of finite classical groups*, In: Lecture Notes in Math., Springer, **869**(1981), 1-184.

[110] Ziegler C., *Jacobi Forms of Higher Degree*, Abh. Math. Sem. Univ. Hamburg, **59**(1989), 191-224.